

# Lawrence Berkeley National Laboratory

## Recent Work

### Title

ON COLEMAN'S THEOREM FOR SCALE INVARIANCE

### Permalink

<https://escholarship.org/uc/item/0b17r6dg>

### Author

Genz, H.

### Publication Date

1969-11-17

Submitted to Physics Letters B

UCRL-19411  
Preprint

*eg. 2*

**RECEIVED** ON COLEMAN'S THEOREM FOR SCALE INVARIANCE  
LAWRENCE  
RADIATION LABORATORY

DEC 19 1969

LIBRARY AND  
DOCUMENTS SECTION

H. Genz

November 17, 1969

AEC Contract No. W-7405-eng-48

**TWO-WEEK LOAN COPY**

*This is a Library Circulating Copy  
which may be borrowed for two weeks.  
For a personal retention copy, call  
Tech. Info. Division, Ext. 5545*

**LAWRENCE RADIATION LABORATORY**  
**UNIVERSITY of CALIFORNIA BERKELEY**

UCRL-19411

*eg. 2*

## **DISCLAIMER**

This document was prepared as an account of work sponsored by the United States Government. While this document is believed to contain correct information, neither the United States Government nor any agency thereof, nor the Regents of the University of California, nor any of their employees, makes any warranty, express or implied, or assumes any legal responsibility for the accuracy, completeness, or usefulness of any information, apparatus, product, or process disclosed, or represents that its use would not infringe privately owned rights. Reference herein to any specific commercial product, process, or service by its trade name, trademark, manufacturer, or otherwise, does not necessarily constitute or imply its endorsement, recommendation, or favoring by the United States Government or any agency thereof, or the Regents of the University of California. The views and opinions of authors expressed herein do not necessarily state or reflect those of the United States Government or any agency thereof or the Regents of the University of California.

ON COLEMAN'S THEOREM FOR SCALE INVARIANCE\*

H. Genz

Lawrence Radiation Laboratory  
University of California  
Berkeley, California

November 17, 1969

ABSTRACT

If the dilatation charge annihilates the vacuum presence of zero mass states (or complete scale invariance) follows.

Dilatations [1] involve rescaling of the energy and thus in a dilatation invariant theory the dilatation charge  $Q_D(x_0)$  and the Hamiltonian do not commute; one has [2,3] instead

$$[Q_D(x_0), H] = i H \quad (1)$$

(as also seen below). Since the dilatation current [4]

$$S_\mu(x) = T_{\mu\nu}(x) x^\nu - \frac{1}{2} \sum_i \partial_\mu \phi_i^2(x) - F_\mu(x) \quad (2)$$

explicitly involves  $x$  usual arguments may not be applied to give  $Q_D(x_0)|\Omega\rangle \neq 0$ . On the contrary we will derive below the following analog to Coleman's theorem [5-7] by formal reasoning:

Statement: Define  $S_\mu$  by Eqn. (2) and define [8]

$$Q_D(x_0) = \int d^3x \{S_0(x) - \langle S_0(x) \rangle_0\} \quad (3)$$

Assume [9]  $\sigma(x) \equiv \partial^\mu S_\mu(x)$  to be scalar. Then (for  $x$  fixed)

$$\sigma(x) = 0 \quad (4)$$

implies (for  $y_0$  arbitrary)

$$Q_D(y_0)|\Omega\rangle = 0 \quad (5)$$

Vice versa, Eqn. (5) (for  $y_0$  fixed) implies Eqn. (4) (for  $x$  arbitrary). Furthermore [10] [from Eqns. (2) and (3) alone]

$$[Q_D(x_0), H] = i H + i \int d^3x \sigma(x) \quad (6)$$

We assume absence of zero mass states in deriving (4) from (5). More generally, Eqn. (5) implies presence of some zero mass states (or complete scale invariance).

Due to the somewhat surprising result that the theorem [6] "the invariance of the vacuum is the invariance of the world" also holds for [11] scale invariance, we stress that this might be due to the assumptions implicit in the formal reasoning (interchange of limits etc.). Physical understanding of the role of these assumptions (and the assumed absence of zero mass states) in the derivation seems necessary to obtain physical understanding of the result.--We will have nothing (except Ref. 11) to say about this in the present note and proceed with the argument.

We remark that Eqn. (4) [Eqn. (5)] holds for any  $x$  ( $y_0$ ) if it holds for a certain one. For Eqn. (4) this follows since  $\sigma$  may be written as [4]

$$\sigma(x) = T_{\mu}^{\mu}(x) - \frac{1}{2} \sum_i \square \phi_i^2(x) - \partial^{\mu} F_{\mu}(x) \quad (7)$$

and thus transforms covariantly under translations. For Eqn. (5) we note that [due to Eqn. (2)] all the explicit  $y_0$ -dependence of  $Q_D(y_0)$  comes from  $H_{y_0}$  (since  $F_{\mu}$  as given in Ref. [4] contains no explicit  $x$ -dependence) which cannot contribute to Eqn. (5).

We now derive Eqn. (6) using

$$\partial^{\mu} T_{\mu\nu}(x) = 0 \quad (8)$$

by performing partial integrations

$$\begin{aligned} [Q_D(x_0), H] &= i \int d^3x \{ T_{0m}(x) x^m - \frac{1}{2} \sum_i \partial_0^2 \phi_i^2(x) - F_0(x) \} \\ &= i \int d^3x \{ T_{\mu}^{\mu}(x) - \frac{1}{2} \sum_i \square \phi_i^2(x) - \partial^{\mu} F_{\mu}(x) \} - i \int d^3x T_{00}(x) . \end{aligned} \quad (9)$$

Upon comparison of this Eqn. with (7) we arrive at (6).

The equivalence of Eqns. (4) and (5) is now easily shown.

Assume first that

$$\partial^\mu S_\mu(x) = 0. \quad (10)$$

Then from Eqn. (6)

$$H Q_D(x_0)|\Omega\rangle = 0 \quad (11)$$

and thus

$$Q_D(x_0)|\Omega\rangle = \alpha|\Omega\rangle \quad (12)$$

with  $\alpha = 0$  due to Eqn. (3).--This is in accordance with the assumption made in the appendix of Ref. [4].

Next assume Eqn. (5) for fixed  $y_0$ . Due to the remarks made in connection with Eqn. (7) we have (for any  $y_0$ )

$$\partial^0 Q_D(y_0)|\Omega\rangle = 0. \quad (13)$$

Thus, for any state with 3-momentum  $\underline{p}$

$$\langle \underline{p} | \int d^3x \partial^0 S_0(x) |\Omega\rangle = 0. \quad (14)$$

By partial integration

$$\langle \underline{p} | \int d^3x \sigma(x) |\Omega\rangle = 0. \quad (15)$$

This is almost the desired result. Using the fact that  $\sigma$  is translation covariant we may perform the  $\underline{x}$ -integration in Eqn. (15)

and obtain (integrating the result over  $p$ )

$$\langle p = 0 | \sigma(0) | \Omega \rangle = 0 . \quad (16)$$

Since  $\sigma$  is scalar we have for any state  $|z\rangle$  (by Lorentz transformations) [12]

$$\langle z | \sigma(0) | \Omega \rangle = 0 , \quad (17)$$

i.e.,  $\sigma(x)$  annihilates the vacuum. Since  $\sigma(x)$  is local the desired result  $[\sigma(x) = 0]$  follows.

#### ACKNOWLEDGMENT

The author would like to thank Professors G. F. Chew and J. D. Jackson for their kind hospitality at the Lawrence Radiation Laboratory. A NATO grant is gratefully acknowledged.



## FOOTNOTES AND REFERENCES

- \* Supported by the DAAD through a NATO grant.
1. Dilatations have been frequently discussed in the literature. We mention Refs. 2-4 (which we will use) and refer to the further citations given therein.
  2. H. A. Kastrup, Nucl. Phys. 58, 561 (1964).
  3. G. Mack, Nucl. Phys. B5, 491 (1968).
  4. D. J. Gross and J. Wess, "Scale Invariance, conformal invariance and the high energy behavior of scattering amplitudes," CERN-Preprint TH. 1076.--Our notation is that of this ref. except that we denote  $\ominus$  by T.
  5. S. Coleman, Phys. Letters 19, 144 (1965).
  6. S. Coleman, J. Math. Phys. 7, 787 (1966).
  7. Our proof is partly an adaption of that in Ref. 6 to the present case.--If one derives (H. Genz, "On the Vacuum expectation value of the  $\sigma$ -term," Preprint, II. Inst. f. Theor. Phys. d. Universität, Hamburg and Z. Physik, to be published) Coleman's theorem using the spectral representation for  $\langle [J_\mu(x), J_\nu(y)] \rangle_0$  one implicitly assumes translation covariance of  $J_\mu$  [not shared by Eqn. (2)].
  8. If there were a contributing vacuum expectation value it should be subtracted out (elsewise the charge would diverge linearly). Independent of this subtraction Eqn. (4) will later formally follow from Eqn. (5).
  9. Since  $F_\mu$  (as given in Ref. 4) does not contain any explicit x-dependence, the assumption that  $\sigma$  is scalar comes down to

assuming a suitable definition of the products of fields at a point involved.

10. Note that dilatation invariance which we defined by  $\sigma(x) = 0$  [due to Eqn. (6)] implies  $Q_D(x_0) = Q_D(0)$ .
11. Arguments favoring scale invariance of high energy scattering in our application seem to favor  $Q_D(y_0)|\Omega\rangle = 0$  also. If one avoids to conclude scale invariance from Eqn. (5) by allowing for states of vanishing mass having the quantum numbers of  $\sigma$  then the situation appears to be  $(Q_D(y_0)|\Omega\rangle = 0$  and  $\partial^\mu S_\mu \neq 0$ ) opposite to the Goldstone situation  $(Q|\Omega\rangle \neq 0$  and  $\partial^\mu j_\mu = 0$ ) for internal transformations (for a review see C. A. Orzalesi, Lectures on Field-Theoretic Aspects of Current Algebra, Maryland Technical Report No. 833, p.96).
12. For this conclusion absence of zero mass states has to be assumed.

## LEGAL NOTICE

*This report was prepared as an account of Government sponsored work. Neither the United States, nor the Commission, nor any person acting on behalf of the Commission:*

- A. Makes any warranty or representation, expressed or implied, with respect to the accuracy, completeness, or usefulness of the information contained in this report, or that the use of any information, apparatus, method, or process disclosed in this report may not infringe privately owned rights; or*
- B. Assumes any liabilities with respect to the use of, or for damages resulting from the use of any information, apparatus, method, or process disclosed in this report.*

*As used in the above, "person acting on behalf of the Commission" includes any employee or contractor of the Commission, or employee of such contractor, to the extent that such employee or contractor of the Commission, or employee of such contractor prepares, disseminates, or provides access to, any information pursuant to his employment or contract with the Commission, or his employment with such contractor.*

TECHNICAL INFORMATION DIVISION  
LAWRENCE RADIATION LABORATORY  
UNIVERSITY OF CALIFORNIA  
BERKELEY, CALIFORNIA 94720