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ABSTRACT

If the dilatation charge annihilates the vacuum presence of zero mass states (or complete scale invariance) follows.

Dilatations [1] involve rescaling of the energy and thus in a dilatation invariant theory the dilatation charge $Q_D(x_o)$ and the Hamiltonian do not commute; one has [2,3] instead

$$[Q_D(x_o), H] = i H \qquad (1)$$

(as also seen below). Since the dilatation current [4]

$$S_\mu(x) = T_{\mu\nu}(x)\, x^\nu - \frac{1}{2} \sum_i \partial_\mu\, \phi_i^{\,2}(x) - F_\mu(x) \qquad (2)$$

explicitly involves x usual arguments may not be applied to give $Q_D(x_o)|\Omega\rangle \neq 0$. On the contrary we will derive below the following analog to Coleman's theorem [5-7] by formal reasoning:

Statement: Define S_μ by Eqn. (2) and define [8]

$$Q_D(x_o) = \int d^3x \{S_o(x) - \langle S_o(x)\rangle_o\} . \qquad (3)$$

Assume [9] $\sigma(x) \equiv \partial^\mu S_\mu(x)$ to be scalar. Then (for x fixed)

$$\sigma(x) = 0 \qquad (4)$$

implies (for y_o arbitrary)

$$Q_D(y_o)|\Omega\rangle = 0 . \qquad (5)$$

Vice versa, Eqn. (5) (for y_o fixed) implies Eqn. (4) (for x arbitrary). Furthermore [10] [from Eqns. (2) and (3) alone]

$$[Q_D(x_o), H] = i H + i \int d^3x\, \sigma(x) . \qquad (6)$$

We assume absence of zero mass states in deriving (4) from (5). More generally, Eqn. (5) implies presence of some zero mass states (or complete scale invariance).

Due to the somewhat surprising result that the theorem [6] "the invariance of the vacuum is the invariance of the world" also holds for [11] scale invariance, we stress that this might be due to the assumptions implicit in the formal reasoning (interchange of limits etc.). Physical understanding of the role of these assumptions (and the assumed absence of zero mass states) in the derivation seems necessary to obtain physical understanding of the result.--We will have nothing (except Ref. 11) to say about this in the present note and proceed with the argument.

We remark that Eqn. (4) [Eqn. (5)] holds for any x (y_o) if it holds for a certain one. For Eqn. (4) this follows since σ may be written as [4]

$$\sigma(x) = T_\mu{}^\mu(x) - \frac{1}{2} \sum_i \Box\, \phi_i{}^2(x) - \partial^\mu F_\mu(x) \tag{7}$$

and thus transforms covariantly under translations. For Eqn. (5) we note that [due to Eqn. (2)] all the explicit y_o-dependence of $Q_D(y_o)$ comes from Hy_o (since F_μ as given in Ref. [4] contains no explicit x-dependence) which cannot contribute to Eqn. (5).

We now derive Eqn. (6) using

$$\partial^\mu T_{\mu\nu}(x) = 0 \tag{8}$$

by performing partial integrations

$$[Q_D(x_o),H] = i \int d^3x \{T_{om}(x)\, x^m - \frac{1}{2} \sum_i \partial_o{}^2\, \phi_i{}^2(x) - F_o(x)\}$$

$$= i \int d^3x \{T_\mu{}^\mu(x) - \frac{1}{2} \sum_i \Box\, \phi_i{}^2(x) - \partial^\mu F_\mu(x)\} - i \int d^3x\, T_{oo}(x) \; . \tag{9}$$

Upon comparison of this Eqn. with (7) we arrive at (6).

The equivalence of Eqns. (4) and (5) is now easily shown. Assume first that

$$\partial^{\mu} S_{\mu}(x) = 0 . \tag{10}$$

Then from Eqn. (6)

$$H\, Q_D(x_o)|\Omega\rangle = 0 \tag{11}$$

and thus

$$Q_D(x_o)|\Omega\rangle = \alpha|\Omega\rangle \tag{12}$$

with $\alpha = 0$ due to Eqn. (3).--This is in accordance with the assumption made in the appendix of Ref. [4].

Next assume Eqn. (5) for fixed y_o. Due to the remarks made in connection with Eqn. (7) we have (for any y_o)

$$\partial^{o}\, Q_D(y_o)|\Omega\rangle = 0 . \tag{13}$$

Thus, for any state with 3-momentum $\underset{\sim}{p}$

$$\langle \underset{\sim}{p}| \int d^3x\, \partial^{o}\, S_o(x)|\Omega\rangle = 0 . \tag{14}$$

By partial integration

$$\langle \underset{\sim}{p}| \int d^3x\, \sigma(x)|\Omega\rangle = 0 . \tag{15}$$

This is almost the desired result. Using the fact that σ is translation covariant we may perform the $\underset{\sim}{x}$-integration in Eqn. (15)

and obtain (integrating the result over $\underset{\sim}{p}$)

$$\langle \underset{\sim}{p} = \underset{\sim}{0} | \sigma(0) | \Omega \rangle = 0 \; . \qquad (16)$$

Since σ is scalar we have for any state $|z\rangle$ (by Lorentz transformations) [12]

$$\langle z | \sigma(0) | \Omega \rangle = 0 \; , \qquad (17)$$

i.e., $\sigma(x)$ annihilates the vacuum. Since $\sigma(x)$ is local the desired result $[\sigma(x) = 0]$ follows.

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FOOTNOTES AND REFERENCES

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1. Dilatations have been frequently discussed in the literature. We mention Refs. 2-4 (which we will use) and refer to the further citations given therein.

2. H. A. Kastrup, Nucl. Phys. 58, 561 (1964).

3. G. Mack, Nucl. Phys. B5, 491 (1968).

4. D. J. Gross and J. Wess, "Scale Invariance, conformal invariance and the high energy behavior of scattering amplitudes," CERN-Preprint TH. 1076.--Our notation is that of this ref. except that we denote Θ by T.

5. S. Coleman, Phys. Letters 19, 144 (1965).

6. S. Coleman, J. Math. Phys. 7, 787 (1966).

7. Our proof is partly an adaption of that in Ref. 6 to the present case.--If one derives (H. Genz, "On the Vacuum expectation value of the σ-term," Preprint, II. Inst. f. Theor. Phys. d. Universität, Hamburg and Z. Physik, to be published) Coleman's theorem using the spectral representation for $\langle [J_\mu(x), J_\nu(y)] \rangle_0$ one implicitly assumes translation covariance of J_μ [not shared by Eqn. (2)].

8. If there were a contributing vacuum expectation value it should be subtracted out (elsewise the charge would diverge linearly). Independent of this subtraction Eqn. (4) will later formally follow from Eqn. (5).

9. Since F_μ (as given in Ref. 4) does not contain any explicit x-dependence, the assumption that σ is scalar comes down to

assuming a suitable definition of the products of fields at a point involved.

10. Note that dilatation invariance which we defined by $\sigma(x) = 0$ [due to Eqn. (6)] implies $Q_D(x_o) = Q_D(0)$.

11. Arguments favoring scale invariance of high energy scattering in our application seem to favor $Q_D(y_o)|\Omega\rangle = 0$ also. If one avoids to conclude scale invariance from Eqn. (5) by allowing for states of vanishing mass having the quantum numbers of σ then the situation appears to be $(Q_D(y_o)|\Omega\rangle = 0$ and $\partial^\mu S_\mu \neq 0)$ opposite to the Goldstone situation $(Q|\Omega\rangle \neq 0$ and $\partial^\mu j_\mu = 0)$ for internal transformations (for a review see C. A. Orzalesi, Lectures on Field-Theoretic Aspects of Current Algebra, Maryland Technical Report No. 833, p.96).

12. For this conclusion absence of zero mass states has to be assumed.

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