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ABSTRACT

The problem of simultaneous heat and mass transfer in free convection from a vertical, flat plate is investigated theoretically. An integral method is used to solve the resulting differential equations. Solutions are obtained for zero wall velocity and for a mass-transfer velocity at the wall, with the wall either insulated or held at a constant temperature. The zero-wall-velocity solution is compared with existing solutions and data for simultaneous heat and mass transfer, heat transfer alone, and mass transfer alone.

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INTRODUCTION

In many processes mass transfer and heat transfer occur simultaneously. In free convection these may either hinder or aid one another. For example, mass-transfer rates that are 100% greater than those predicted by considering only the effects of mass transfer were obtained by Mathers, Madden, and Piret for sublimation of spheres with simultaneous heat transfer.[2]

Two theoretical solutions have been carried out for this situation. Somers used an integral method for the problem of evaporation and condensation, taking into account the nonzero wall velocity arising from the mass transfer.[7] His solution, although perfectly correct, is complex and difficult to apply to specific problems. Mathers, Madded, and Piret used an analog computer to solve the problem for zero wall velocity.[2] However, in the process, they dropped the inertia terms in the momentum equation. Thus their solution strictly holds only for high Prandtl and Schmidt numbers. They also obtained data which shows that the general form of the solution is good for constant Prandtl and Schmidt numbers.

It is the purpose of this paper to derive simple results which will be expected to apply for all Prandtl and Schmidt numbers and for various situations.

DERIVATION OF BASIC EQUATIONS

The following assumptions are made in this derivation:

(a) The fluid has constant properties, except for a slight variation in density with temperature and composition.

(b) The heat generated by dissipation is negligible.

(c) The usual laminar boundary-layer assumptions are valid.[4]

With these assumptions the boundary-layer equations for laminar free convection on a vertical flat plate are:[2,5,9]

Continuity: $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ (1)

Momentum: $u \dfrac{\partial u}{\partial x} + v \dfrac{\partial u}{\partial y} = \nu \dfrac{\partial^2 u}{\partial y^2} + g\left(\dfrac{\rho_\infty - \rho}{\rho}\right)$ (2)

Energy: $u \dfrac{\partial T}{\partial x} + v \dfrac{\partial T}{\partial y} = \dfrac{k}{c_p \rho} \dfrac{\partial^2 T}{\partial y^2} = a \dfrac{\partial^2 T}{\partial y^2}$ (3)

Mass: $u \dfrac{\partial C}{\partial x} + v \dfrac{\partial C}{\partial y} = D \dfrac{\partial^2 C}{\partial y^2}$ (4)

Boundary conditions:

at $y = 0$: $T = T_0$, $C = C_0$, $u = 0$, $v = v_0$;

(5)

at $y = \infty$: $T = T_\infty$, $C = C_\infty$, $u = 0$.

Here the parameters are defined by:

$x =$ distance along the plate from the bottom edge up for flow up, and from the top edge down for flow down.

$y =$ distance out from the wall.

$u =$ velocity parallel to the plate in a positive x direction.

$v =$ velocity out from the wall.

$\nu =$ kinematic viscosity.

$g =$ acceleration due to gravity (taken as positive for flow up the plate and negative for flow down).

$\rho, \rho_\infty =$ density of the fluid at a point and in the bulk fluid, respectively.

k = thermal conductivity of the fluid.

c_p = heat capacity of the fluid.

T = temperature.

a = thermal diffusivity.

C = concentration of concerned component in fluid.

D = diffusivity of concerned component.

To solve this system, we must, in some way, relate the density term in Eq. (2) to the concentration and temperature. Therefore, let:

$$(\rho_\infty - \rho)/\rho = \alpha\,(C - C_\infty) + \beta\,(T - T_\infty)\,. \tag{6}$$

The validity of this substitution will be examined in detail later. Also, let:

$$\phi = \frac{T - T_\infty}{T_0 - T_\infty}\,, \quad \psi = \frac{C - C_\infty}{C_0 - C_\infty}$$

With these substitutions Eqs. (2) - (5) become:

Momentum: $u\dfrac{\partial u}{\partial x} + v\dfrac{\partial u}{\partial y} = \nu\dfrac{\partial^2 u}{\partial y^2} + \alpha\,g\,(C_0 - C_\infty)\,\psi + \beta\,g\,(T_0 - T_\infty)\,\phi$ (7)

Energy: $u\dfrac{\partial \phi}{\partial x} + v\dfrac{\partial \phi}{\partial y} = a\dfrac{\partial^2 \phi}{\partial y^2}$ (8)

Mass: $u\dfrac{\partial \psi}{\partial x} + v\dfrac{\partial \psi}{\partial y} = D\dfrac{\partial^2 \psi}{\partial y^2}$ (9)

Boundary Conditions:

at y = 0: $\phi = \psi = 1,\ u = 0,\ v = v_0$

at y = ∞: $\phi = \psi = u = 0.$

(10)

These are the basic differential equations and boundary conditions governing the process. To solve them, we use the usual integral method in which the equations are satisfied only on the average and not at every point.[7,9]

INTEGRAL EQUATIONS

Assume that u, ϕ, ψ are effectively zero at short distances from the plate, δ, δ', δ'', respectively. Equations (7), (8), and (9) are now integrated with respect to y from 0 to δ, δ', δ''. This integration will be shown in detail only for Eq. (9).

Let δ'' be the diffusion boundary-layer thickness. Then the integral of Eq. (9) is

$$\int_0^{\delta''} \left(u \frac{\partial \psi}{\partial x} + v \frac{\partial \psi}{\partial y}\right) dy = \int_0^{\delta''} D \left(\frac{\partial^2 \psi}{\partial y^2}\right) dy = D \left(\frac{\partial \psi}{\partial y}\right) \Bigg|_0^{\delta''} . \qquad (11)$$

As a condition of smoothness at the outer edge of the boundary layer, let

$$\left(\frac{\partial \psi}{\partial y}\right)_{\delta''} = 0.$$

This will be assured by the choice of the concentration profile later. Also note that from Eq. (1), we have

$$v = v_0 - \int_0^y \left(\frac{\partial u}{\partial x}\right) dy. \qquad (12)$$

Thus, Eq. (11) becomes

$$\int_0^{\delta''} \left[u \frac{\partial \psi}{\partial x} + v_0 \frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial y} \int_0^y \frac{\partial u}{\partial x} dy \right] dy = - D \left(\frac{\partial \psi}{\partial y}\right)_0 . \qquad (13)$$

Integrating by parts, however,

$$\int_0^{\delta''} \frac{\partial \psi}{\partial y} \int_0^y \frac{\partial u}{\partial x} dy = - \int_0^{\delta''} \psi \left(\frac{\partial u}{\partial x}\right) dy , \qquad (14)$$

and noting that

$$\int_0^{\delta''} v_0 \frac{\partial \psi}{\partial y} dy = - v_0 , \qquad (15)$$

we obtain, finally:

Mass: $\dfrac{d}{dx} \displaystyle\int_0^{\delta''} u\,\psi\,dy = v_0 - D\left(\dfrac{\partial\psi}{\partial y}\right)_0$,   (16)

and in a like manner:

Energy: $\dfrac{d}{dx} \displaystyle\int_0^{\delta'} u\,\phi\,dy = v_0 - a\left(\dfrac{\partial\phi}{\partial y}\right)$,   (17)

Momentum: $\dfrac{d}{dx} \displaystyle\int_0^{\delta} u^2\,dy = -\nu\left(\dfrac{\partial u}{\partial y}\right)_0 + \beta g\,(T_0-T_\infty)\displaystyle\int^{\delta'} \phi\,dy$

$+ \alpha g\,(C_0-C_\infty)\displaystyle\int^{\delta''} \psi\,dy$.   (17)

Note that in the momentum equation the limits of the last two integrals are δ' and δ'', not δ. This is done because the concentration and temperature profiles are the driving force for the velocity change. Thus we have $\delta \geq \delta'$, $\delta \geq \delta''$. Since $\phi = 0$ for $y \geq \delta'$, and $\psi = 0$ for $y \geq \delta''$, the limits on the integrals are correct as shown. The only way this would not be true is if either α or β were zero. If this were true, however, the integral concerned would drop out of the momentum equation, and so the limit is not important.

So that these equations may be solved, it is necessary to assume a form for the velocity, concentration, and temperature profiles. It is tacitly assumed that the shape of the velocity profile is independent of the ratio of the mass and heat transfer, while actually, of course, it is not. However, if the Schmidt and Prandtl numbers are not too different the result should be quite good anyway, because the equations will still be satisfied on the average.

SOLUTION FOR ZERO VELOCITY AT THE WALL

Strictly speaking, the condition of zero wall velocity is true only for equal-molal counterdiffusion. However, for low rates of mass transfer, the wall velocity is small enough to be safely neglected.

Assume that the velocity, concentration, and temperature profiles have the following forms:

$$\left.
\begin{aligned}
\frac{u}{u_x} &= \left(\frac{y}{\delta}\right) + b\left(\frac{y}{\delta}\right)^2 + c\left(\frac{y}{\delta}\right)^3 \\[2ex]
\phi &= 1 - a'\left(\frac{y}{\delta'}\right) - b'\left(\frac{y}{\delta'}\right)^2 - c'\left(\frac{y}{\delta'}\right)^3 \\[2ex]
\psi &= 1 - a''\left(\frac{y}{\delta''}\right) - b''\left(\frac{y}{\delta''}\right)^2 - c''\left(\frac{y}{\delta''}\right)^3
\end{aligned}
\right\} \qquad , \qquad (19)$$

where b, c, a', b', c', a'', b'', c'' are constants to be determined, and u_x is an arbitrary reference velocity which is a function of x. The constants are determined from the following conditions:

Velocity: at $y = 0$, $u = 0$
                 at $y = \delta$, $u = 0$, $\left(\frac{\partial u}{\partial y}\right)_\delta = 0$

Temperature: at $y = \delta'$, $\phi = \left(\frac{\partial \phi}{\partial y}\right)_{\delta'} = 0$
                 at $y = 0$, $\left(\frac{\partial^2 \phi}{\partial y^2}\right)_0 = 0.$

(This last condition follows from Eq. (8), since $u = v = 0$ at $y = 0$.)

Concentration: Entirely analogous to temperature.

These conditions yield the following profiles:

$$\left.
\begin{aligned}
\frac{u}{u_x} &= \left(\frac{y}{\delta}\right) - 2\left(\frac{y}{\delta}\right)^2 + \left(\frac{y}{\delta}\right)^3 \\[2ex]
\phi &= 1 - \frac{3}{2}\left(\frac{y}{\delta'}\right) + \frac{1}{2}\left(\frac{y}{\delta'}\right)^2 \\[2ex]
\psi &= 1 - \frac{3}{2}\left(\frac{y}{\delta''}\right) + \frac{1}{2}\left(\frac{y}{\delta''}\right)^3
\end{aligned}
\right\} \qquad . \qquad (20)$$

If these profiles are substituted into Eqs. (16), (17), and (18), the following equations result:

Momentum: $\quad \frac{1}{105} \frac{d}{dx} (\delta u_x^2) = - \frac{\nu u_x}{\delta} + \beta g (T_0 - T_\infty) \frac{3}{8} \delta' + \alpha g (C_0 - C_\infty) \frac{3}{8} \delta''$, (21)

Energy: $\quad \frac{4}{105} \kappa' \left[\frac{21}{8} - \frac{105}{48} \kappa' + \frac{63}{112} \kappa'^2 \right] \frac{d}{dx} (u_x \delta') = \frac{3a}{2\delta'}$, (22)

Mass: $\quad \frac{4}{105} \kappa'' \left[\frac{21}{8} - \frac{105}{48} \kappa'' + \frac{63}{112} \kappa''^2 \right] \frac{d}{dx} (u_x \delta'') = \frac{3D}{2\delta''}$, (23)

where $\quad \kappa' = \frac{\delta'}{\delta}$,

$\kappa'' = \frac{\delta''}{\delta}$.

To solve these equations, we must assume a relationship between δ, δ', and δ''. For $Sc > Pr$ the temperature profile must extend farther into the fluid than the concentration profile, i.e., $\delta' > \delta''$. Furthermore, the velocity must extend out to the end of the temperature profile, i.e., $\delta = \delta'$. This situation is here called "heat-transfer controlling," and is now solved in detail.

If u_x, δ, and δ'' are each assumed to be proportional to various powers of x, and it is further required that the terms in x drop out when these are substituted into Eqs. (21), (22), and (23), then we have

$$u_x = \alpha_1 x^{1/2}, \quad \delta = \delta' = \alpha_2 x^{1/4}, \quad \delta'' = \alpha_3 x^{1/4} ,$$

where α_1, α_2, α_3 are constants to be determined by substitution into Eqs. (21), (22), and (23). This substitution yields:

Momentum: $\quad \frac{5}{420} \alpha_1^2 \alpha_2 = - \nu \frac{\alpha_1}{\alpha_2} + \beta g (T_0 - T_\infty) \frac{3}{8} \alpha_2 + \alpha g (C_0 - C_\infty) \frac{3}{8} \alpha_3$, (24)

Energy: $\quad \frac{3}{105} \alpha_1 \alpha_2 = \frac{3a}{2\alpha_2}$, (25)

Mass: $\quad \frac{3}{105} \alpha_1 \alpha_3 r \left[\frac{21}{8} - \frac{105}{48} r + \frac{63}{112} r^2 \right] = \frac{3D}{2\alpha_3}$, (26)

where $r = \alpha_3 / \alpha_2 = \delta''/\delta'$.

From Eqs. (25) and (26), we obtain

$$\left(\frac{D}{a}\right) = \left(\frac{Pr}{Sc}\right) = r^3 \left[\frac{21}{8} - \frac{105}{48} \ r + \frac{63}{112} \ r^2\right] \quad , \tag{27}$$

which defines r vs Pr/Sc. This relationship will be calculated in detail later.

From Eqs. (24) and (25) and the definition of r, the value of α_2 is obtained. However, what is usually desired is the Nusselt numbers, $h_x x/k$ and $K_x x/D$. It can be shown that these definitions yield:[7]

$$Nu_x = - x \left(\frac{\partial \phi}{\partial y}\right)_0$$

$$* \ Nu'_x = - x \left(\frac{\partial \psi}{\partial y}\right)_0 \tag{28}$$

From the assumed velocity profiles and the form assumed for δ' and δ'', we obtain

$$Nu_x = \frac{3}{2} \ \frac{x^{3/4}}{\alpha_2}$$

$$Nu'_x = \frac{3}{2} \ \frac{x^{3/4}}{\alpha_3} = \frac{1}{r} \ Nu_x \tag{29}$$

Thus, for heat-transfer controlling, we have

$$Nu_x = \frac{0.435 \ (Pr)^{1/4} \left[Gr_x + r \ Gr'_x\right]^{1/4}}{\left(\frac{0.625}{Pr} + 1\right)^{1/4}} \quad , \tag{30}$$

where

Gr_x = local Grashof number for heat transfer $\left[= g \ x^3 \ \beta \ (T_0 - T_\infty)/\nu^2\right]$,

Gr'_x = local Grashof number for mass transfer $\left[= g \ x^3 \ \alpha \ (C_0 - C_\infty)/\nu^2\right]$.

---

* Primed dimensionless parameters are for mass transfer, unprimed for heat transfer.

Somers[7] and Mathers et al.[2] have found that $r \approx \sqrt{Pr/Sc}$, which will also be demonstrated later. With this fact there results:

$$Nu_x = \frac{0.435 \, (Pr)^{1/4} \left[Gr_x + \sqrt{Pr/Sc} \; Gr_x' \right]^{1/4}}{(\frac{0.625}{Pr} + 1)^{1/4}}$$

$$Nu'_x = \sqrt{Sc/Pr} \; Nu_x$$

$$(31)$$

By summation over the height of the plate, we obtain the over-all Nusselt numbers, hL/k and KL/D, for heat-transfer controlling:

$$Nu = 0.58 \frac{(Pr)^{1/4} \left[Gr + \sqrt{Pr/Sc} \; Gr' \right]^{1/4}}{(\frac{0.625}{Pr} + 1)^{1/4}}$$

$$Nu' = \sqrt{Sc/Pr} \; Nu$$

$$(32)$$

where

Gr = over-all Grashof number for heat transfer $\left[= g \, L^3 \, \beta \, (T_0 - T_\infty)/\nu^2 \right]$,

Gr' = over-all Grashof number for mass transfer $\left[= g \, L^3 \, \alpha \, (C_0 - C_\infty)/\nu^2 \right]$.

In a like manner, for mass-transfer controlling (Pr > Sc), we get

$$Nu = \sqrt{Pr/Sc} \; Nu'$$

$$Nu' = 0.58 \frac{(Sc)^{1/4} \left[Gr' + \sqrt{Sc/Pr} \; Gr \right]^{1/4}}{(\frac{0.625}{Sc} + 1)^{1/4}}$$

$$(33)$$

These results can also be obtained from Somers' solution by making the same simplifying assumptions and the same definitions as made here.[8]

A. Parabolic Temperature and Concentration Profiles

The constants in the equations just obtained are slightly different than those usually obtained by the integral method.[9] Thus, a solution will now be obtained using the usual parabolic temperature and concentration profiles; i.e., let:

$$\phi = (1 - y/\delta') , \quad \psi = (1 - y/\delta'')^2 . \tag{34}$$

Proceeding as before, we obtain a new defining equation for r:

$$\left(\frac{Pr}{Sc}\right) = r^3 \left[\frac{5}{2} - 2 r + \frac{1}{2} r^2 \right] . \tag{35}$$

Likewise, there is obtained for heat-transfer controlling:

$$\left. \begin{array}{l} Nu = \dfrac{0.677 \, (Pr)^{1/4} \left[Gr + \sqrt{Pr/Sc} \; Gr' \right]^{1/4}}{\left(\dfrac{0.952}{Pr} + 1\right)^{1/4}} \\[4ex] Nu' = \sqrt{Sc/Pr} \;\; Nu \end{array} \right\} , \tag{36}$$

and for mass-transfer controlling:

$$\left. \begin{array}{l} Nu = \sqrt{Pr/Sc} \;\; Nu' \\[3ex] Nu' = \dfrac{0.677 \, (Sc)^{1/4} \left[Gr' + \sqrt{Sc/Pr} \; Gr \right]^{1/4}}{\left(\dfrac{0.952}{Pr} + 1\right)^{1/4}} \end{array} \right\} . \tag{37}$$

B. Comparison of Results

In form, these results approach those of Mathers et al.,[2] as the Schmidt and Prandtl numbers grow large and as they approach one another in value. For these conditions, Eqs. (36) and (37) reduce to:

$$Nu = 0.677 \, (Pr)^{1/4} \left[Gr + \sqrt{Pr/Sc} \, Gr' \right]^{1/4}$$

$$Nu' = 0.677 \, (Sc)^{1/4} \left[Gr' + \sqrt{Sc/Pr} \, Gr \right]^{1/4}$$

(38)

At this point it is interesting to compare these various results, reduced to the case of heat transfer alone (Gr' = 0), with Ostrach's exact numerical solution for heat transfer.[3] This is done in Table I.

Table I

Comparison of theoretical results for heat transfer by free convection

Prandtl number	Calculated values of $Nu/Gr^{1/4}$			
	Ostrach[3] (numerical)	Eq. (32) (Gr' = 0)	Eq. (36) (Gr' = 0)	Eq. (38) (Gr' = 0)
0.01	0.0766	0.0650	0.0648	0.212
0.733	0.479	0.460	0.509	0.627
1	0.535	0.514	0.573	0.677
2	0.676	0.644	0.730	0.806
10	1.103	1.016	1.177	1.204
100	2.07	1.83	2.09	2.14
1000	3.74	3.26	3.81	3.81

Inspection of Table I shows that Eq. (32) is best for low Prandtl numbers, while Eq. (36) is best for high Prandtl numbers. Equation (38), and therefore Mather's analog solution, is suitable only for relatively high Prandtl numbers. It is expected that the same statements will hold true for simultaneous heat and mass transfer, with "Prandtl numbers" replaced in the foregoing by "Prandtl and Schmidt numbers."

C. Demonstration of Assumed Relation for r vs. Pr/Sc

Previously it was stated that $r \approx \sqrt{Pr/Sc}$. Table II shows the exact relation between r and Pr/Sc as given by Eqs. (27) and (35).

Table II

r	(Pr/Sc)			$\sqrt{Pr/Sc}$	
	Eq. (27)	Eq. (35)		Eq. (27)	Eq. (35)
1.0	1.0	1.0		1.0	1.0
0.8	0.623	0.632		0.790	0.795
0.6	0.320	0.327		0.566	0.572
0.4	0.114	0.118		0.337	0.343
0.2	0.0170	0.0196		0.130	0.140
0.0	0.0	0.0		0.0	0.0

This table shows that $r = \sqrt{Pr/Sc}$ is a reasonable approximation to the relation between r and Pr/Sc as given by the integral method. This approximation improves as Pr/Sc approaches 1.

D. Expansion Coefficients for an Ideal Gas

To solve the basic differential equations (1) to (5), we fount it necessary to assume a relation between temperature, concentration, and density. The validity of this relation, Eq. (6), for an ideal gas will now be examined in detail.

Let C, M_1 = concentration and molecular weight of the concerned component,

C_2, M_2 = concentration and average molecular weight of the rest of the mixture,

$$C_T = \text{total molar concentration, } C + C_2.$$

Then we have

$$\rho = CM_1 + C_2M_2 ; \tag{39}$$

but

$$C + C_2 = C_T = n/V = P/RT , \tag{40}$$

where

n = number of moles

V = volume

P = total pressure

R = ideal gas constant

T = absolute temperature.

Therefore, we have

$$\rho = C(M_1 - M_2) + PM_2/RT, \tag{41}$$

and so

$$\left(\frac{\rho_\infty - \rho}{\rho}\right) = \frac{(C_\infty - C)(M_1 - M_2) + \left(\frac{P}{R\,T_\infty} - \frac{P}{RT}\right)M_2}{C(M_1 - M_2) + (P/RT)M_2} . \tag{42}$$

To get this into the same form as Eq. (6), we must assume that the density varies only slightly, so that in the denominator an average temperature $\overline{T}$ and an average concentration $\overline{C}$ can be used to the first approximation. With this assumption, we obtain

$$\left(\frac{\rho_\infty - \rho}{\rho}\right) = \frac{(C_\infty - C)}{\overline{C} + \frac{P}{R\overline{T}}\left(\frac{M_1}{M_2} - 1\right)} + \frac{(T - T_\infty)}{T_\infty\left[1 + \frac{\overline{C}\,\overline{T}\,R}{P}\left(\frac{M_1}{M_2} - 1\right)\right]} . \tag{43}$$

Thus, by comparison with Eq. (6), we find

$$\alpha = -\frac{1}{\overline{C} + \frac{P}{R\,\overline{T}}\left(\frac{M_1}{M_2} - 1\right)} \tag{44}$$

and

$$\beta = \frac{1}{T_\infty \left[1 + \frac{\overline{C}\ \overline{T}\ R}{P} \left(\frac{M_1}{M_2} - 1\right) \right]} \ .$$

(45)

In general, therefore, for gases or liquids, α must be evaluated at constant temperature, and β at constant concentration. Both must be the value at some average value of both concentration and temperature.

FINITE WALL-VELOCITY SOLUTION

In many practical cases there is a bulk flow of material to or from the interface. This flow is caused by such processes as crystallization, dissolution, evaporation, and condensation. If only the component of interest partakes in these processes, i.e., it is dissolved in an inert solvent, the bulk flow is caused by diffusion. A material balance at the wall yields:[7]

$$v_0 = - \frac{D}{C_T - C_0} \left(\frac{\partial c}{\partial y}\right)_0 \quad , \tag{46}$$

where C_T = total molar concentration at the wall. In such processes the wall can either be held at some constant temperature or it can be insulated.

A. Specified Wall Temperature

This is the same problem as previously solved, except that v_0 in Eqs. (16) and (17) is given by Eq. (46), and is not zero. Assuming the parabolic profile, Eq. (34), and proceeding as before, we obtain, for heat-transfer controlling,

$$r \left(\frac{a}{D}\right) + \left[\frac{C_0 - C_\infty}{C_T - C_0}\right] = \frac{\left[\dfrac{C_T - C_\infty}{C_T - C_0}\right]}{r^2 \left[\dfrac{5}{2} - 2 r + \dfrac{1}{2} r^2\right]} \quad , \tag{47}$$

$$Nu_x = \frac{0.508 \, (Pr)^{1/2} \left[Gr_x + r \, Gr_x{}'\right]^{1/4}}{\left\{0.952 \left[1 + \dfrac{Pr}{80 \, r \, Sc} \left(\dfrac{C_0 - C_\infty}{C_T - C_0}\right)\right]^2 + Pr + \dfrac{Pr^2}{80 \, r \, Sc} \left(\dfrac{C_0 - C_\infty}{C_T - C_0}\right)\right\}^{1/4}} \quad ,$$

and $\quad Nu_x{}' = \frac{1}{r} \, Nu_x$ \hfill (48)

where Nu_x, $Nu_x{}'$ are defined by Eq. (28).

Actually, the parabolic profile is not entirely satisfactory in this case, because the profiles are not corrected for the effect of blowing at the wall. This effect could be quite large. It would be better to use a fourth-order polynomial for the velocity profile and third-order polynomials for concentration and temperature. Equations (7), (8), (9), (46) applied at the wall yield the extra conditions needed to specify these polynomials.

B. Adiabatic Wall

Occasionally convective mass-transfer processes may take place at an insulated wall.[6] In these cases the heat necessary for the phase transition is supplied or removed by the bulk fluid. A heat balance at the wall, expressing the fact that the heat conducted to the wall is used up as latent heat, is given by:

$$\kappa \left(\frac{\partial T}{\partial y}\right)_0 = v_0 \, \rho_0 \, \lambda \, , \tag{49}$$

where λ is the latent heat (taken as positive for melting, evaporation, etc.). Substituting for v_0 from Eq. (46), defining ϕ and ψ as before, and assuming the parabolic concentration and temperature profiles, we obtain

$$r = \left(\frac{D}{a}\right) \left(\frac{C_0 - C_\infty}{C_T - C_0}\right) \frac{\lambda}{C_p (T_\infty - T_0)} \, . \tag{50}$$

This, together with Eqs. (47) and (48) and the equilibrium relation between C_0 and T_0, is the solution to the adiabatic wall problem.

C. Example Calculation

The following example calculation illustrates the adiabatic wall solution.

Assume: water evaporating

no water vapor in bulk fluid

bulk temperature of 120°F

one atmosphere pressure

Problem: What is the wall temperature?

Solution: Note that we have:

$$\left(\frac{C_0 - C_\infty}{C_T - C_0}\right) \quad = \quad \left(\frac{p_s}{P - p_s}\right)$$

$$\left(\frac{C_T - C_\infty}{C_T - C_0}\right) \quad = \quad \left(\frac{P}{P - p_s}\right)$$

$$\qquad\qquad (51)$$

where p_s = vapor pressure of water at T_0. When the following physical properties are used:

$$c_p \quad = \quad 0.25 \text{ Btu/lb } {}^\circ\text{F (Perry[4])},$$
$$\lambda \quad = \quad 1060 \text{ Btu/lb (Keenan and Keyes[1])},$$
$$a/D \quad = \quad 0.81 \text{ (Sherwood and Pigford[6])},$$

Eqs. (47) and (50) become:

$$r \quad = \quad 5220 \; \left(\frac{p_s}{P - p_s}\right) \; \frac{1}{T_\infty - T_0} \qquad\qquad (52)$$

$$0 \quad = \quad \left(\frac{p_s}{P - p_s}\right) + 0.81 \, r - \frac{\left(\frac{P}{P - p_s}\right)}{r^2 \left[\frac{5}{2} - 2 \, r + \frac{1}{2} \, r^2\right]} = f . \qquad (53)$$

The method of solution is by trial and error. The temperature T_0 is assumed, and p_s is obtained from reference 1. With these values, r is calculated from Eq. (52). The correct result is recognized when f, as defined by Eq. (53), is zero. Table III summarizes the calculations for this problem. The result is that the calculated wall temperature is about 54°F, which is about 10°F lower than the temperature of adiabatic saturation given by Perry.[4]

v, v_0 = Velocity out from the wall, at a point and at the wall, respectively.

x = Distance along the plate, from the bottom edge up for flow up, and from the top edge down for flow down.

y = Distance from the plate.

α = Concentration densification coefficient.

$\alpha_1, \alpha_2, \alpha_3$ = Undetermined parameters used in expressing variation in u_x, δ, δ', and δ'', with x.

β = Temperature densification coefficient.

$\delta, \delta', \delta''$ = Velocity, temperature, and concentration boundary-layer thickness, respectively.

κ', κ'' = Ratios of thermal and of concentration boundary-layer thickness to the velocity boundary-layer thickness, respectively.

λ = Latent heat (taken as positive for melting, evaporation, etc.).

ν = Kinematic viscosity.

ρ, ρ_∞ = Density of the fluid at a point and in the bulk fluid, respectively.

$\phi = \left(\dfrac{T - T_\infty}{T_0 - T_\infty} \right)$.

$\psi = \left(\dfrac{C - C_\infty}{C_0 - C_\infty} \right)$.

Gr_x, Gr = Local and over-all Grashof numbers for heat transfer, respectively.

Respectively defined by $\dfrac{g\, x^3\, \beta\, (T_0 - T_\infty)}{\nu^2}$; $\dfrac{g\, L^3\, \beta\, (T_0 - T_\infty)}{\nu^2}$.

Gr_x', Gr' = Local and over-all Grashof numbers for mass transfer, respectively.

Respectively defined by $\dfrac{g\, x^3\, \alpha\, (C_0 - C_\infty)}{\nu^2}$; $\dfrac{g\, L^3\, \alpha\, (C_0 - C_\infty)}{\nu^2}$.

Nu_x, Nu = Local and over-all Nusselt numbers for heat transfer, respectively.

Respectively defined by $-x \left(\dfrac{\partial \phi}{\partial y} \right)_0$; $\dfrac{h\, L}{k}$.

$Nu'_x, Nu' =$ Local and over-all Nusselt numbers for mass transfer, respectively.

Respectively defined by $\quad - x \left(\dfrac{\partial \psi}{\partial y} \right)_0 \quad ; \quad \dfrac{K\,L}{D}$.

$Pr =$ Prandtl number (ν/a).

$Sc =$ Schmidt number (ν/D).

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