Title

Characterizations of the Boolean Prime Ideal Theorem in Models Without Choice

ISBN

9798186337430

Author

Ransom, Brian

Publication Date

2026-08-11

UNIVERSITY OF CALIFORNIA,
IRVINE

Characterizations of the Boolean Prime Ideal Theorem in Models Without Choice

DISSERTATION

submitted in partial satisfaction of the requirements
for the degree of

DOCTOR OF PHILOSOPHY

in Mathematics

by

Brian Ransom

Dissertation Committee:
Professor Isaac Goldbring, Chair
Professor Manuel Reyes
Professor Martin Zeman

2026

DEDICATION

To my mom and dad, thank you for everything.

TABLE OF CONTENTS

LIST OF FIGURES

ACKNOWLEDGMENTS

I would like to thank my academic mentors, whose support, guidance, patience, encouragement, and kindness has been exceptional: my co-advisors at UCI, Isaac Goldbring and Martin Zeman, Toby Meadows, my REU mentors Rollie Trapp and Corey Dunn, and Richard Bertram.

Thank you to my family: mom, dad, Amy, Bobby, Trinka, Bill, Marilyn, and to all the extended family members who have been a part of my life. Your love and support have meant everything to me.

Thank you to my incredible friends, from all stages of my life: Amy Meltzer, Anthony "Callue" Callue, Danny Geleroff, German Acosta Quiros, Jacob Britton, Kawan Amelung, Naveed Nikpour, Nick Grossi, and Patrick Martin from high school; Addison Mathis, Amanda Cowell, Anna Wuest, Courtney Scott, David Grossman, David Kroonen, Devin Michaels, Elihu Diaz, Ellie Cooper, Fabio Rivero Ochoa, Gabe Lopez, Gabby Konyndyk, Hannah Olson, Jack Fox, Jacob Goldstein-Greenwood, Jacob Kalinowski (my emergency contact), Jared Ifland, Kaelin Fabian, Kirsty Scott, Maia Houck, Maxine Calle, Maya Klaib, Molly McQueeney, Monica Woods, Nic McIver, Noah Bartfield, Paul Zammit, Reef Landrum, Ryan Rhoades, Sean Scott, Sofía Medina, Taegan Dennis, Tanner Bloom, Wil Coenen, and Wilson Archibald from college; Ainsley Pullen, Anna Cusenza, Arghya Rakshit, Ari Rosenfield, Anubhav Nanavaty, Christie Barrera, Dhiya Ram, Fabian Ramirez, Jenny Pi, Josh Jordan, Maureen Zhang, Michael Womack, Mia Villegas, Omar Hurtado, Sebastian Weaver, Sidhanth Raman, Sireesh Vinnakota, So Nakamura, Suhasiny Naik, Sydney Schmeltz, Temoc Ramirez, Vignesh Iyer, and Yasmeen Baki from grad school. I have been surrounded by the best people.

To my thoughtful, kind, funny, and supportive partner Winnie Tran: thank you for making my life better. Thank you also to To Lien Nguyen, Than Tran, and Tien Tran – my second family, who have welcomed me with love.

VITA

Brian Ransom

EDUCATION

Doctor of Philosophy in Mathematics **2026**
University of California, Irvine *Irvine, CA*

Master of Science in Mathematics **2022**
University of California, Irvine *Irvine, CA*

Bachelor of Science in Mathematics **2019**
Florida State University *Tallahassee, FL*

RESEARCH EXPERIENCE

Graduate Research Assistant **2020–2026**
University of California, Irvine *Irvine, CA*

TEACHING EXPERIENCE

Teaching Assistant **2009–2010**
University of California, Irvine *Irvine, CA*

PREPRINTS

On BPI in Symmetric Extensions Part I **November 2025**

ABSTRACT OF THE DISSERTATION

Characterizations of the Boolean Prime Ideal Theorem in Models Without Choice

By

Brian Ransom

Doctor of Philosophy in Mathematics

University of California, Irvine, 2026

Professor Isaac Goldbring, Chair

This thesis aims to address three main objectives: to better understand when, and how BPI holds in permutation models and in symmetric extensions; to develop a unifying relationship between variants of the Halpern-Läuchli theorem and BPI in symmetric extensions; to develop and expose simple dynamical perspectives on BPI in models without choice. Our approach to these questions is to introduce a characterization program, where we aim to provide conditions that are equivalent to BPI in permutation models and in symmetric extensions. By reproving known results, reworking known proofs, and proving new ones, we show how each of these characterizing properties offer novel perspectives on BPI, and how they address the three objectives listed above.

The case of BPI in permutation models is studied in chapter 3. This work reproves and extends several results from [Bla87], [KPT05], and [Bla11]. These characterization theorems, and the proofs we offer to relate them, can be seen as simpler, motivating versions of the main theorems of this thesis, in chapter 4.

In chapter 4, we characterize BPI in symmetric extensions. The characterizing properties that we introduce are based on the various properties studied in chapter 3 for permutation models, and on our new proof of BPI in the generalized Cohen model. This new proof is given

in section 4.2, and we adapt methods from Harrington's forcing proof of the Halpern-Läuchli theorem.

In chapter 5, we extend results from [KS20], and thereby answer a question asked in that paper. In the context of our characterization program, these results allow us to infer the general case of BPI in the generalized Cohen model from the particular case that Harrington's forcing proof can be adapted to handle.

A concise description of the main finding in this work is that to extend symmetric filters to symmetric ultrafilters, one can consult an *arbitrary* (and not necessarily symmetric) ultrafilter in an outer model of choice. One can dynamically identify, inside this particular ultrafilter, all the information that is needed to extend the symmetric filter to a symmetric ultrafilter. We show that BPI holds in permutation models if and only if it can be proven in this manner.

Chapter 1

Introduction

A central tension in the study of infinite objects is our limited ability to describe them. We can illustrate this by considering three classical questions about real numbers:

(i) How many real numbers are there?

(ii) Is it possible to wellorder the real numbers?

(iii) Is there a way to (simultaneously) choose between every subset $y \subseteq \mathbb{R}$ and its complement $\mathbb{R} \setminus y$?

The third of these questions is essentially taken from the introductory chapter of [Jec08], and we can repeat his comment: try to come up with concrete sets that witness a positive resolution to these questions, i.e. a bijection between $\mathbb{R}$ and a cardinal, an explicit wellordering of $\mathbb{R}$, or choice function on the set of two-set partitions of $\mathbb{R}$. One can quickly get the sense that these questions cannot be answered with concrete examples of sets, and that axioms beyond ZF are needed to resolve them.

To justify this intuition, is not hard to imagine that there are decimal sequences that are "random" beyond our ability to say anything concrete about more than finitely many of their digits. This is approximately demonstrated by the ability to force Cohen-generic reals over a model $M \vDash \mathrm{ZFC}$. Here, we take the sets that exist in M to be a proxy for the sets that we can concretely describe. In the forcing extension $M[c]$ that adds a Cohen-generic real c, M only has access to the finite subsequences of digits of the decimal expansion of c.

There are subtleties to the overall discussion of concretely describing infinite objects; see, for instance, [Ham25]. However, we wish to emphasize the heuristic that there is no reason to believe that we can fully describe the details of an arbitrary infinite object in finite time, and with finitely many symbols. In the case of the reals, this amounts to the fact that we can only answer the question of *which* real numbers there are with some degree of vagueness: they are all the infinite decimal expansions, or all the Cauchy sequences, or all the Dedekind cuts, etc., but we are limited in our ability to describe which particular decimal expansions, Cauchy sequences, or Dedekind cuts there are. This vagueness with respect to particular, arbitrary real numbers explains why our three questions about reals are challenging to answer – or, more formally, why their answers differ between models of ZF.

Over ZF(A), the axiom of choice (AC) and its consequences are non-effective axioms that allow for a controlled degree of vagueness in asserting which sets exist.[1] The result is a great strengthening of our ability to reason about infinite objects, without requiring fully concrete details of their properties. For instance, although AC cannot address question (i), it can positively answer questions (ii) and (iii), without the need to specify any actual details of the given wellordering or choice function.

Since we use consequences of AC to augment our ability to describe infinite objects, a natural

[1] ZFA denotes the theory ZF with the existence of "atoms;" see subsection 2.1 of this paper or [Jec08] for more details.

and motivating question is raised: to what extent can these non-effective axioms be used to describe one another? The program to study the (lack of) implications between consequences of AC is a response to this central question. It is known, for instance, that our question (*iii*) can be positively answered by the axiom of choice for sets of pairs, which does not imply the existence of a wellordering of the reals, as in our question (*ii*).[2] Using the technique of symmetric extensions, as exposited in [Jec08], is a standard exercise to give a model in which the reals can be wellordered, so (*ii*) can be answered positively, but in which AC fails, whereby a positive answer to (*ii*) does not imply AC.

In this work, we revisit the relationship between the axiom of choice (AC) and the ultrafilter lemma – that every filter, on every set, can be extended to an ultrafilter. The ultrafilter lemma, which is implied by AC, has many known equivalents over ZF(A), two of which are the compactness theorem for first-order logic, and the Boolean prime ideal theorem (BPI). By historical convention, we use the label BPI throughout the thesis. Informally, BPI expresses the ability to make arbitrarily many *finitely bounded* choices that are altogether *locally compatible* with one another. For instance, an ultrafilter is a choice function on the set of finite partitions of a given set, with a notion of local compatibility given by the finite intersection property.

The ability to make locally compatible choice is fundamental, and as a result, BPI is both a strong choice principle and central to mathematics. The role of ultrafilters in mathematics, for instance, is ubiquitous – see [Gol22] – and its equivalent form of the first-order compactness theorem is a staple tool of model theory. It is thus of interest that such a choice principle can hold while the full axiom of choice fails.

[2]In [HL64] it was announced that every set in the Cohen model can be linearly ordered, which implies the axiom of choice for sets of pairs; in the Cohen model, $\mathbb{R}$ is not wellorderable. Proofs of these standard theorems can be found in [Jec08].

Historically, it has taken significant effort to prove BPI in models without AC. It was first shown in Halpern's thesis [Hal62] that ZFA + BPI is strictly weaker than ZFCA; there it is shown in particular that BPI holds in the ordered Mostwoski model of ZFA, introduced in [Mos39].[3,4] Then, in [HL71], it was first shown that ZF + BPI is strictly weaker than ZFC; in particular, they show that BPI holds in the Cohen model, and more generally that BPI holds in any model of a certain theory that abstracts properties of the Cohen model. In the former work [Hal62], Halpern introduced a contradiction framework to prove BPI. In [HL71], the main argument applies this same contradiction framework, and critically uses the original version of the Halpern-Läuchli theorem from [HL66] to derive the contradiction.

In terms of our informal characterization of BPI as a choice principle, the results from [Hal62] and [HL71] may seem surprising: locally compatible choices do not need to be made "one at a time" via some wellordering by AC.[5] Put another way, these results may seem expected: one can get the sense that first-order compactness should not be able to wellorder an arbitrary set, since compactness can only exert local control over the construction. This tension captures much of the typical intrigue and difficulty of questions in the study of fragments of AC.

Halpern's contradiction framework, as applied in [Hal62] and [HL71], also appears, to the author's knowledge, in every other proof of BPI in a choiceless model to date, including both permutation models of ZFA and symmetric extensions of ZF.[6] While this contradiction framework has been widely influential, the underlying forcing used to define symmetric extensions has often made its application quite technical.[7] To simplify these arguments would

[3]This portion of Halpern's thesis was later rewritten as the paper [Hal64].

[4]The ordered Mostowski model and Halpern's proof of BPI are exposited in [Jec08].

[5]This is a prime example of the usefulness of AC: choices in a construction can be made in some wellordered infinite sequence, with perfect hindsight for previous choices.

[6]In [Pin97], Pincus made these same observations on the known proofs of BPI in models without choice.

[7]Comments along these lines are made, for instance, in [Pin77b] and in [Jec08].

be of inherent interest, but it would also help to address several contemporary questions relating to BPI in choiceless models. The question of preserving BPI in iterated symmetric extensions, for instance, is raised in [Kar19]. This is related to longstanding open conjectures from [Pin77a]: that BPI holds in a candidate model for the independence of AC from ZF + BPI+DC (an independence which Pincus later proved in [Pin77b]), and that a certain related variant of the Halpern-Läuchli theorem holds.[8] Questions like these seem to have remained open because generalizing the existing techniques would be exceedingly complicated.

The broad, primary goal of this thesis is thus to better understand when, and how BPI holds in permutation models and in symmetric extensions. We carry out this goal by studying properties that equivalently characterize BPI in permutation models and in symmetric extensions, and that are primarily based around a new, direct framework of proof.

A secondary goal is to better understand the abstract relationship between variants of the Halpern-Läuchli theorem and BPI in symmetric extensions. Various connections along these lines have been exposed in the literature, for instance in [HL71], [Pin77a], and [Ste23] in the manner described above. This suggests that a unifying relationship is nascent.

A tertiary goal is to develop and expose simple dynamical perspectives on BPI in models without choice. We will see that one such perspective for permutation models is mostly known, from the work in [Bla11], based on the work in [Bla87] and [KPT05]. To our knowledge, however, the dynamics of BPI in symmetric extensions have not yet been considered.

After the preliminaries chapter 2, the remaining chapters can be read roughly independently of one another, although they are ordered to build upon and motivate one another. We will very briefly outline chapters 3, 4, and 5 below.[9] The introductory section of each chapter

[8] In [Pin77a], Pincus proved his conjectured Halpern-Läuchli theorem to be sufficient for BPI in this candidate model. His argument follows the approach from [HL71].

[9] Chapter 2 states the preliminaries, and we aim to only assume knowledge of forcing.

provides a more thorough outline of said chapter, while still remaining concise.

In chapter 3, we study characterizations of BPI in permutation models. Several of these characterizing properties are fully or partially known (see [Bla87], [Bla11], and [KPT05]). In the case of known results, we introduce new proofs, or adapt known proofs to the context of choiceless set theory, when doing so provides a new perspective. Our work in chapter 3 serves as a simpler, and perhaps more clearly motivated version of the work in chapter 4.

In chapter 4 we study analogous versions of the properties in chapter 3, and we prove that they characterize BPI in symmetric extensions. Each of these properties and characterization theorems are new. Several of these characterizations are made possible by analyzing a new proof of BPI in the generalized Cohen model $N(I, Q)$, which is based on Harrington's proof of the Halpern-Läuchli theorem; this new proof is given in section 4.2.[10]

In chapter 5, we strengthen a result from [KS20], and thereby answer a question posed in that same paper. This is essentially done to show that the index set I in the generalized Cohen model $N(I, Q)$ can be taken to be arbitrarily large, without loss of generality.[11] This is of independent interest, but we originally pursued this problem for its application to the material in chapter 4. Specifically, our adaptation of Harrington's proof takes the set I, in $N(I, Q)$, to be large enough for a certain Ramsey-theoretic argument. The work in chapter 5 shows that this assumption is inessential.

With these brief outlines of the chapters in mind, we will take more care to outline the characterizations of BPI that we will study, and the perspectives they offer in models without choice. These are the main topics of the thesis, along with the new proof that $N(I, Q) \vDash$ BPI

[10]The generalized Cohen model $N(I, Q)$ adds an infinite, Dedekind-finite set of mutually Q-generic filters. This is done using a poset that adds I-many mutually Q-generics filters, then moving to the inner model $N(I, Q)$ that strips away their wellordering. See the preliminaries section 2.5 for complete details.

[11]This result is shown for the Cohen model $N(\omega, 2^{<\omega})$ in [KS20].

in section 4.2. In the outline below, we will group together the analogous properties in the cases of permutation models and symmetric extensions. In each setting, the perspectives that these analogous characterizing properties offer are essentially similar. We will refer to basic defining features of permutation models and symmetric extensions, which the reader can consult sections 2.1 and 2.2 to review.

The Filter Extension Property

In a model of ZFC(A), the standard proof of BPI involves showing that

(1) an arbitrary filter F, on an arbitrary set x, can be extended to include an arbitrary subset $y \subseteq x$ or its complement $x \setminus y$, and

(2) Zorn's lemma can be applied to produce a maximal filter $U \supseteq F$ on x; by (1), the maximality of U ensures that it "completes" the extension of F to an ultrafilter on x.

We find this method of proof to be exceptionally clear, and we wish to use a coarse version of these two points to prove BPI in permutation models and in symmetric extensions. We call the appropriate coarsening of (1) the filter extension property; we state versions of this property for permutation models in chapter 3 and for symmetric extensions in chapter 4.

In the case of permutation models, given a filter F on a set x, both in the permutation model $M(A, \mathcal{G}, \mathcal{F})$, we aim to find a $K \in \mathcal{F}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, so that the following extension is possible:

$$F \cup \mathrm{Orb}(y_0', K) \cup \mathrm{Orb}(y_1', K) \cup \mathrm{Orb}(y_2', K) \cup ... = U,$$

where each $y_i' \in \{y_i, x \setminus y_i\}$ is a choice of a subset of x in $M(A, \mathcal{G}, \mathcal{F})$ or its complement. If this sequence can be extended to consider every subset of x in $M(A, \mathcal{G}, \mathcal{F})$, all while preserving the finite intersection property (FIP), then $U \in M(A, \mathcal{G}, \mathcal{F})$ is the desired ultrafilter on x that extends F, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. Here, the filter extension property (for permutation models) is a statement that allows for some uniform manner to extend F by a single orbit $\mathrm{Orb}(y', K)$ while retaining the FIP; this can then be iterated to produce the line above.

In the case of a symmetric extension N, based on the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, we would like to similarly characterize when $p \Vdash_N$ BPI.[12] We do so by stating the filter extension property below p. Given names $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$, so that $1 \Vdash$ "$\dot{F} \subseteq \dot{x}$ has the FIP," we aim to find a $K \in \mathcal{F}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, so that the following extension is possible:

$$\dot{F} \cup \mathrm{Orb}(\langle r_0, y'_0 \rangle, K) \cup \mathrm{Orb}(\langle r_1, y'_1 \rangle, K) \cup \mathrm{Orb}(\langle r_2, y'_2 \rangle, K) \cup \ldots = \dot{U}.$$

In this case, the names $y'_i \in \mathrm{HS}_{\mathcal{F}}$ are forced by 1 to be a subset of $\dot{x}$; for a given such name y' and the name for its complement, which we denote $\dot{x} \setminus y'$, the conditions r_i that tag either y' or $\dot{x} \setminus y'$ accumulate to be dense below some condition $d \leq p$. If this sequence of extensions can be made, while at each step retaining that 1 forces the FIP, then said condition d forces that resulting name $\dot{U} \supseteq \dot{F}$ is the desired symmetric ultrafilter on $\dot{x}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$. We will ensure that this works for a dense set of d below p.

Here, the filter extension property below p allows for a uniform manner to extend $\dot{F}$ by a single K-orbit $\mathrm{Orb}(\langle r, y' \rangle, K)$ while retaining the FIP; again by iterating, we can produce the line above. There is some additional subtlety to the picture, but this captures the essential features of the filter extension property below p, which we prove exactly characterizes the fact that $p \Vdash_N$ BPI.

[12]Note that in this work, we will always define N with respect to a ground model of ZFC.

The Ramsey Property and the Halpern-Läuchli Property

For a permutation model $M(A, \mathcal{G}, \mathcal{F})$, the Ramsey property for the normal filter $\mathcal{F}$ was introduced in [Bla87]. In this same paper, Blass proved that $M(A, \mathcal{G}, \mathcal{F}) \vDash$ BPI if and only if $\mathcal{F}$ has the Ramsey property. To prove that the Ramsey property on $\mathcal{F}$ implied BPI in $M(A, \mathcal{G}, \mathcal{F})$, Blass followed Halpern's contradiction framework. In chapter 3, we give a new, direct proof of this implication. Specifically, given $F, x, y \in M(A, \mathcal{G}, \mathcal{F})$, with F a filter on x and $y \subseteq x$, let $U_0 \supseteq F$ be any (not necessarily symmetric) ultrafilter on x in the outer model $M \vDash$ ZFCA. We show that the Ramsey property implies that U_0 contains enough information, up to translation by elements of some $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, to check the instance of the filter extension property that, for some $y' \in \{y, x \setminus y\}$,

$$F \cup \mathrm{Orb}(y', K) \text{ has the FIP.}$$

In this sense, the Ramsey property is a dynamical condition that allows ultrafilters in an outer model of choice to coordinate the extension of symmetric filters to symmetric ultrafilters. By Blass' equivalence theorem, BPI holds in a permutation model if and only if it can be explained in this manner.

For a symmetric extension N, based on the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, we would like to introduce an analogous property to characterize when $p \Vdash_N$ BPI. We call this the Halpern-Läuchli property below p. Given $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$, with

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

let $\dot{U}_0 \supseteq \dot{F}$ be any $\mathbb{P}$-name for an ultrafilter on $\dot{x}$. The Halpern-Läuchli property below p allows one to identify information in $\dot{U}_0$ that, up to translation by elements of some $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, checks the following instance of the filter extension property: for some $r \leq d \leq p$ (with $d \leq p$ as in our sketch of the filter extension property above) and some $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$,

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

We prove that the Halpern-Läuchli property below p is necessary for $p \Vdash_N$ BPI by generalizing Blass' proof from [Bla87] that the Ramsey property on $\mathcal{F}$ is necessary for $M(A, \mathcal{G}, \mathcal{F}) \models$ BPI. Thus, BPI holds in a symmetric extension if and only if this dynamical argument can be used to prove it.

We can also note that we chose the name of the Halpern-Läuchli property because it can be seen as an abstract form of the Halpern-Läuchli theorem. In fact, the Halpern-Läuchli property generalizes features of a new proof of the BPI in the Cohen model from the Halpern-Läuchli theorem, which we recently developed, and which will appear in forthcoming work (but not in this thesis). In this sense, the Halpern-Läuchli property gives our desired abstract unifying relationship between variants of the Halpern-Läuchli theorem and BPI in symmetric extensions. This connection will also be described in greater detail in forthcoming work.

Extreme Amenability

In [KPT05], seminal connections are developed between structural Ramsey theory, extremely amenable topological groups, and Fraïssé limits. In [Bla11], Blass expresses a partial connection between extreme amenability and the Ramsey property for $\mathcal{F}$. To do so, he combines his characterization of BPI in permutation models in terms of the Ramsey property on $\mathcal{F}$, from [Bla87], and a characterization of certain extremely amenable groups from [KPT05].

In chapter 3, we develop this partial connection from [Bla11], to show that BPI holds in a permutation model $M(A, \mathcal{G}, \mathcal{F})$ if and only if some group $K \in \mathcal{F}$ is extremely amenable (with the topology generated by subgroups of K in $\mathcal{F}$). Our main proofs modify two proofs from [KPT05] to the context of BPI in permutation models. This exposes an essentially dynamical perspective on BPI in permutation models.

The version of this property that characterizes BPI in symmetric extensions will appear in forthcoming work. One of the main goals of the work in this thesis is to develop preliminary ideas that will allow us to state a generalization of extreme amenability to characterizes BPI in symmetric extensions.

The Search-and-Shift Property

The search-and-shift property for permutation models captures convergent features of our proof of the filter extension property from the Ramsey property, and our proof of BPI from the extreme amenability of some $K \in \mathcal{F}$. Let $F, x \in M(A, \mathcal{G}, \mathcal{F})$, with F a filter on x, and let $U_0 \supseteq F$ be a maximal filter on x containing only subsets of x in $M(A, \mathcal{G}, \mathcal{F})$.[13] The search-and-shift property (for permutation models) ensures the existence of some $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ and some symmetric ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on x, with $U \supseteq F$ and $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, so that U is an ultralimit of the space $K \cdot U_0$.[14] Expressing the desired symmetric ultrafilter U as an ultralimit of $K \cdot U_0$ is a compressed way to say that U_0 contains enough information, up to translation by K, to fully determine U.

For a symmetric extension N, based on the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, we would again like to analogously characterize when $p \Vdash_N$ BPI. We call the resulting property the search-and-shift property below p. Let $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$, with

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

[13] We call such a U_0 an $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter on x.

[14] Here the action of K on U_0 is taken from the definition of the permutation model, and the topology on $K \cdot U_0$ is the standard topology on a space of ultrafilters.

Let $\dot{U}_0 \supseteq \dot{F}$ have $\mathrm{rng}(\dot{U}_0) \subseteq \mathrm{HS}_{\mathcal{F}}$, but otherwise let 1 force $\dot{U}_0$ be a maximal filter on $\dot{x}$ up to this constraint.[15] The search-and-shift property below p guarantees some $d \leq p$, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, and $\dot{U} \supseteq \dot{F}$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, so that

$$d \Vdash \text{“}\dot{U} \text{ is an ultralimit of } K \cdot \dot{U}_0\text{”} \wedge \text{“}\dot{U} \text{ is an ultrafilter on } \dot{x}.\text{”}$$

This is again a compressed way to say that $\dot{U}_0$ contains enough information, up to translation by K, to fully determine the desired symmetric ultrafilter $\dot{U}$.

By introducing these versions of the search-and-shift property, we aim to capture an essential dynamical feature of BPI in models without choice. This work is also intended to suggest the right generalization of extreme amenability to characterize BPI in symmetric extensions. For permutation models, we noted that the search-and-shift property captures convergent features of our characterizations of BPI with the Ramsey property and with extreme amenability. In forthcoming work, aim to use the version of the search-and-shift property for symmetric extensions to state the intended generalization of extreme amenability to this context.

[15] We call this an $\mathrm{HS}_{\mathcal{F}}$-valued ultrafilter on $\dot{x}$.

Chapter 2

Preliminaries

2.1 Permutation Models

Permutation models of ZFA are the precursors to symmetric extensions. They were introduced to give methods by which models without choice could be produced before the introduction of forcing. A comprehensive treatment of this topic can be found in [Jec08], including a proof of the standard results that we list below.

The language of ZF(C)A includes the binary predicates $=, \in$, the constant symbol 0 used to denote the empty set, and the constant symbol A used to denote the set of atoms. There are several standard modifications that ZFCA makes to ZFC. The primary adjustment is to include the axiom A, which states that the "atoms" in A have no elements, but are distinct from one another and from the empty set. The rest of the ZFC axioms are adjusted accordingly to form the theory ZFCA; ZFA is this theory without choice.

Definition 2.1.1. Given a model $M \vDash$ ZFCA, with A its set of atoms and ON the class of ordinals, for $\alpha \in$ ON, one can recursively define

(i) $V_0(A) = A$,

(ii) $V_{\alpha+1}(A) = 2^{V_\alpha(A)}$,[1]

(iii) $V_\alpha(A) = \bigcup_{\beta<\alpha} V_\beta(A)$ if α is a limit ordinal.

Theorem 2.1.2. *Given a model $M \vDash$ ZFCA, with A its set of atoms,*

$$M = \bigcup_{\alpha \in \mathrm{ON}} V_\alpha(A).$$

Definition 2.1.3. For $M \vDash$ ZFCA and $x \in M$, define $\mathrm{rnk}(x) = \inf\{\alpha \mid x \in V_\alpha\}$.

Note that we will write $H \preceq K$ to denote that H is a subgroup of K.

Definition 2.1.4. For a model $M \vDash$ ZFCA, let $\mathrm{Aut}(A)$ be the group of permutations of the set A of atoms in M. Given $\mathcal{G} \preceq \mathrm{Aut}(A)$, recursively define the action of $\pi \in \mathcal{G}$ on M by

$$\pi x = \{\pi y \mid y \in x\}.$$

Lemma 2.1.5. *Every $\pi \in \mathrm{Aut}(A)$, acting on M as above, is an $\in$-respecting automorphism of M.*

Definition 2.1.6. Let $M \vDash$ ZFCA, $A \in M$ be the set of atoms, and $\mathcal{G} \preceq \mathrm{Aut}(A)$. $\mathcal{F}$ is called a *normal filter of subgroups* of $\mathcal{G}$ if

(i) Every $K \in \mathcal{F}$ is a subgroup of $\mathcal{G}$,

(ii) $\mathcal{G} \in \mathcal{F}$,

(iii) if $K \in \mathcal{F}$ and $K \preceq H \preceq \mathcal{G}$, then $H \in \mathcal{F}$,

(iv) if $K, H \in \mathcal{F}$ then $K \cap H \in \mathcal{F}$,

[1] Throughout the thesis, we use the convention that 2^x denotes the powerset of a set x.

(v) if $\pi \in \mathcal{G}$ and $K \in \mathcal{F}$ then $\pi K \pi^{-1} \in \mathcal{F}$ (this is called the "normality" condition).

Definition 2.1.7. For $x \in M \vDash$ ZFCA, let $\mathrm{sym}_{\mathcal{G}}(x) = \{\pi \in \mathcal{G} \mid \pi x = x\}$.

Definition 2.1.8. For any $y \in M \vDash$ ZFCA and $K \preceq \mathrm{Aut}(A)$, let

$$\mathrm{Orb}(y, K) = \{\pi y \mid \pi \in K\}.$$

Definition 2.1.9. Given a model $M \vDash$ ZFCA, a group $\mathcal{G} \preceq \mathrm{Aut}(A)$, and $\mathcal{F}$ a normal filter of subgroups of $\mathcal{G}$, $x \in M$ is called *$\mathcal{F}$-symmetric* if $\mathrm{sym}_{\mathcal{G}}(x) \in \mathcal{F}$. Recursively, $x \in M$ is called and *hereditarily $\mathcal{F}$-symmetric* if x is $\mathcal{F}$-symmetric and every $y \in x$ is hereditarily $\mathcal{F}$-symmetric.

Intuitively, a set $x \in M$ is $\mathcal{F}$-symmetric if it is fixed by an $\mathcal{F}$-large subgroup of $\mathcal{G}$. The normality condition ensures that if x is $\mathcal{F}$-symmetric, then so is πx for every $\pi \in \mathcal{G}$. The intent is that $\mathcal{G}$ and $\mathcal{F}$ can be selected so that hereditarily $\mathcal{F}$-symmetric sets cannot witness particular consequences of AC, because they cannot sufficiently tell atoms apart. This targets the rigidity of sets witnessing consequences of AC. For instance, for any wellordering f of A, $\mathrm{sym}_{\mathcal{G}}(f)$ is the trivial group.

Definition 2.1.10. Let $M \vDash$ ZFCA, $A \in M$ be the set of atoms, $\mathcal{G}$ be a group of permutations of A, and $\mathcal{F}$ be a normal filter on $\mathcal{G}$. Define $M(A, \mathcal{G}, \mathcal{F}) = \{x \in M \mid x$ is hereditarily $\mathcal{F}-$ symmetric$\}$.

Theorem 2.1.11. *$M(A, \mathcal{G}, \mathcal{F}) \vDash$ ZFA.*

We also state two normalizations, which Blass notes in [Bla87] and [Bla11].

Lemma 2.1.12 (See Blass, [Bla87])**.** *If $M(A, \mathcal{G}, \mathcal{F})$ is a permutation model, and if*

$$\mathcal{F}' = \{H \in \mathcal{F} \mid (\exists x \in M(A, \mathcal{G}, \mathcal{F}))\ \mathrm{sym}_{\mathcal{G}}(x) = H\},$$

then $M(A, \mathcal{G}, \mathcal{F}) = M(A, \mathcal{G}, \mathcal{F}')$.

Unless otherwise stated, we will *always* assume that the normal filter $\mathcal{F}'$ has the property described above.

Lemma 2.1.13 (See Blass, [Bla11])**.** *If $M \vDash$ ZFCA and $M(A, \mathcal{G}, \mathcal{F})$ is a permutation model, then $N = \bigcap_{K \in \mathcal{F}} K$ is a normal subgroup of $\mathcal{G}$, $\mathcal{F}/N = \{K/N \mid K \in \mathcal{F}\}$ is a normal filter of subgroups of $\mathcal{G}/N$, and*

$$M(A, \mathcal{G}, \mathcal{F}) = M(A, \mathcal{G}/N, \mathcal{F}/N).$$

2.2 Symmetric Extensions

A symmetric extension is an inner model of a $\mathbb{P}$-extension $M[G]$, defined analogously to a permutation model by using a group $\mathcal{G} \preceq \mathrm{Aut}(\mathbb{P})$ and a normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$. By considering automorphisms of the underlying forcing $\mathbb{P}$, symmetric extensions avoid the need to use atoms. This results in a model of ZF, rather than ZFA. The reader can again refer to [Jec08] for a complete exposition of this topic, including proofs of the standard results listed below.

Definition 2.2.1. A *symmetric system* is a tuple $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, where $\mathbb{P}$ is a poset, $\mathcal{G} \preceq \mathrm{Aut}(\mathbb{P})$ is a subgroup of automorphisms of $\mathbb{P}$, and $\mathcal{F}$ is a normal filter on $\mathcal{G}$ (as in definition 2.1.6).

Definition 2.2.2. For a poset $\mathbb{P} \in M \vDash$ ZFC, let $M^{\mathbb{P}}$ denote the class of $\mathbb{P}-$names in M.

Definition 2.2.3. Given $\pi \in \mathcal{G} \preceq \mathrm{Aut}(\mathbb{P})$, recursively define the action of π on $M^{\mathbb{P}}$ by

$$\pi \dot{x} = \{\langle \pi p, \pi \dot{z} \rangle \mid \langle p, \dot{z} \rangle \in \dot{x}\}.$$

Lemma 2.2.4 (Symmetry Lemma)**.** *For $\pi \in \mathcal{G} \preceq \mathrm{Aut}(\mathbb{P})$,*

$$p \Vdash \phi(\dot{x}_0, ..., \dot{x}_n) \iff \pi p \Vdash \phi(\pi \dot{x}_0, ..., \pi \dot{x}_n).$$

Definition 2.2.5. For a $\mathbb{P}$-name $\dot{x}$ and a group $\mathcal{G} \preceq \mathrm{Aut}(\mathbb{P})$, let $\mathrm{sym}_{\mathcal{G}}(\dot{x}) = \{\pi \in \mathcal{G} \mid \pi \dot{x} = \dot{x}\}$.

Definition 2.2.6. For any $\mathbb{P}$-name $\dot{y}$, any condition $p \in \mathbb{P}$, and any $K \preceq \mathrm{Aut}(\mathbb{P})$, let

$$\mathrm{Orb}(\langle p, \dot{y}\rangle, K) = \{\langle \pi p, \pi \dot{y}\rangle \mid \pi \in K\}.$$

Definition 2.2.7. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, a $\mathbb{P}$-name $\dot{x}$ is *$\mathcal{F}$-symmetric* if $\mathrm{sym}_{\mathcal{G}}(\dot{x}) \in \mathcal{F}$. Recursively, we say that $\dot{x}$ is *hereditarily $\mathcal{F}$-symmetric* if $\mathrm{sym}_{\mathcal{G}}(\dot{x}) \in \mathcal{F}$ and every $\dot{y} \in \mathrm{rng}(\dot{x})$ is hereditarily $\mathcal{F}$-symmetric.

Similar to the motivation for permutation models, the intent is that symmetric systems can be defined so that hereditarily $\mathcal{F}$-symmetric names cannot be forced to witness certain consequences of AC.

Definition 2.2.8. Given a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, let $\mathrm{HS}_{\mathcal{F}}$ be the class of hereditarily $\mathcal{F}$-symmetric $\mathbb{P}$-names (with $\mathbb{P}$, $\mathcal{G}$, and the ground model M understood from context).

Definition 2.2.9. Given a ground model $M \vDash \mathrm{ZF(C)}$, a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, and an $(M, \mathbb{P})$-generic filter G, let $\mathrm{HS}_{\mathcal{F}}^G$ denote the class of interpretations of the names in $\mathrm{HS}_{\mathcal{F}}$ by G. We call $\mathrm{HS}_{\mathcal{F}}^G$ a *symmetric extension* of M, presented by $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$.

Unless otherwise specified, we will take the convention that symmetric extensions are defined over ground models of ZFC.

Theorem 2.2.10. *If $M \vDash \mathrm{ZF(C)}$, $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ is a symmetric system, and G is a $\mathbb{P}$-generic filter, then $M \subseteq \mathrm{HS}_{\mathcal{F}}^G$ and*

$$\mathrm{HS}_{\mathcal{F}}^G \vDash \mathrm{ZF}.$$

Remark 2.2.11. For a symmetric extension N based on the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, one can define the forcing relation $\Vdash_{\mathrm{HS}_{\mathcal{F}}}$, or $\Vdash_N$, by relativizing the forcing relation for $\mathbb{P}$ over M to the class $\mathrm{HS}_{\mathcal{F}}$. In this work, we will often use $\Vdash$ to mean either, and in any such context the correct formalization will be clear.

2.3 Ultralimits

Definition 2.3.1. If X is a topological space, $\{x_i \mid i \in I\} \subseteq X$, and U is an ultrafilter on I, then an *ultralimit* $\lim_{i,U} x_i = x$ is a point in X so that for every open neighborhood $W \subseteq X$,

$$\{i \in I \mid x_i \in W\} \in U.$$

Often, one takes $I = X$ in the definition above, and takes U to be an ultrafilter on X itself; then $\lim_{z,U} z$ denotes the corresponding ultralimit. The following facts are standard, and are proven in [Gol22].

Lemma 2.3.2. *If X is Hausdorff, all ultralimits in X that exist are unique.*

Theorem 2.3.3. *A topological space X is compact if and only if it is closed under ultralimits.*

2.4 The Finite Intersection Property

Throughout this work, we refer to the finite intersection property (FIP): F has the finite intersection property if the intersection of finitely many elements of F is infinite. It is well-known that "is infinite" is not a first-order definable concept, but in a model $M \vDash \mathrm{ZFC}$ there are many known and equivalent definitions of infinite that reference the natural numbers $\omega \in M$.

In a model $N \vDash \mathrm{ZF}$, many of these definitions of infinite are known to no longer be equivalent. Because the models we study in this paper do not have choice, we need to specify what is meant by the FIP. In both permutation models and in symmetric extensions, note that ω is still definable in the model. We can give the following definition for "infinite."

Definition 2.4.1. In a model $M \vDash \mathrm{ZF(A)}$, we say that a set x is *infinite* if there are no bijections between x and any $n \in \omega$.

Note that this definition of "infinite" is absolute between a permutation model or symmetric extension and their relevant outer model of ZFC(A).

2.5 The Generalized Cohen Model

Our presentation follows [Ste23], where the generalized Cohen model was introduced.

Definition 2.5.1. A set $x \in M \vDash \mathrm{ZF(A)}$ is called *Dedekind-finite* if there are no injections from ω into x in M. In this paper, we will further require that Dedekind-finite sets are infinite, in the sense of definition 2.4.1.

The Cohen model primarily introduces a Dedekind-finite set of mutually generic Cohen reals; the generalized Cohen model, denoted $N(I, Q)$ for some index set I and poset Q, primarily introduces a Dedekind-finite set of mutually Q-generic filters, which is defined with respect to an index set I in the ground model. We will first define a poset that adds I-many mutually Q-generic filters.

Definition 2.5.2. For a poset Q and an index set I, let $\mathbb{P}(I, Q)$ denote the partial order of finite functions from I to Q ordered by

$$(\forall p, q \in \mathbb{P}(I, Q))(q \leq p) \iff (\mathrm{dom}(p) \subseteq \mathrm{dom}(q) \wedge (\forall i \in \mathrm{dom}(p)) q(i) \leq_Q p(i))).$$

Note that the coordinatewise strengthening $q(i) \leq_Q p(i)$ refers to the order $\leq_Q$ on the poset Q. We will just write $q(i) \leq p(i)$ when the context is clear.

Definition 2.5.3. Given a set X, let $[X]^\kappa = \{A \subseteq X \mid |A| = \kappa\}$ and $[X]^{<\kappa} = \bigcup_{\alpha<\kappa}[X]^\alpha$.

Definition 2.5.4. For a group $\mathcal{G} \preceq \mathrm{Aut}(I)$ and $E \in [I]^{<\omega}$, let $\mathrm{fix}_{\mathcal{G}}(E) = \{\pi \in \mathcal{G} \mid \pi \text{ fixes } E \text{ pointwise}\}$.

Definition 2.5.5. Let $M \vDash \mathrm{ZFC}$, and define the symmetric system $\langle \mathbb{P}(I, Q), \mathcal{G}, \mathcal{F} \rangle$ by setting

(i) $\mathbb{P}(I, Q)$ as in definition 2.5.2, with I an infinite index and Q a nontrivial poset,

(ii) $\mathcal{G}$ the group of finite permutations of I, acting on $p \in \mathbb{P}(I,Q)$ by $\mathrm{dom}(\pi p) = \pi\mathrm{dom}(p)$ and

$$(\forall i \in \mathrm{dom}(p))\pi p(\pi i) = p(i),^{2}$$

(iii) the normal filter $\mathcal{F}$ on $\mathcal{G}$ generated by subgroups of the form $\mathrm{fix}_{\mathcal{G}}(E)$, for $E \in [I]^{<\omega}$.[3]

The corresponding symmetric extension of M is a generalized Cohen model, denoted $N(I,Q)$. If we wish to emphasize the index set I, we will write $\langle \mathbb{P}(I,Q), \mathcal{G}_I, \mathcal{F}_I\rangle$, and $\mathrm{HS}_{\mathcal{F}_I}$. If we wish to emphasize the $\mathbb{P}(I,Q)$-generic filter G used to define $N(I,Q)$, we will write $N^G(I,Q)$.

Throughout the paper, whenever we refer to $\langle \mathbb{P}(I,Q), \mathcal{G}, \mathcal{F}\rangle$, we will mean the symmetric system from definition 2.5.5.

Definition 2.5.6. For $n \in I$, let $\dot{a}_n = \{\langle p, \check{q}\rangle \mid p(n) = q\}$, and let $\dot{A} = \{\langle 1, \dot{a}_n\rangle \mid n \in I\}$.

Theorem 2.5.7. *For a generalized Cohen model $N^G(I,Q)$ over a ground model $M \vDash \mathrm{ZFC}$, $\dot{a}_n^G \in N^G(I,Q)$ is Q-generic over M and $\dot{A}^G \in N^G(I,Q)$ and is Dedekind-finite in $N^G(I,Q)$.*

Definition 2.5.8. For $D \in [I]^{<\omega}$, disjoint sets $H_i \subseteq I$ (for $i \in D$), $\vec{\alpha} = \{\langle i, \alpha_i\rangle \mid i \in D\} \in \prod_{i\in D} H_i$, and $\mathcal{G}$ as in definition 2.5.5, define the permutation $\pi_{\vec{\alpha}} \in \mathcal{G}$ by

$$\pi_{\vec{\alpha}} = \prod_{i\in D}(i\ \alpha_i).$$

Note the convention that elements $\vec{\alpha} \in \prod_{i\in D} H_i$ are functions from D into $\bigcup_{i\in D} H_i$, with $\vec{\alpha}(i) = \alpha_i \in H_i$. This will be convenient for tracking the supports of sets, as defined below.

Lemma 2.5.9. *For the symmetric system $\langle \mathbb{P}(I,Q), \mathcal{G}, \mathcal{F}\rangle$, if $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$, there is an $X \in [I]^{<\omega}$ so that $\mathrm{fix}_{\mathcal{G}}(X) \preceq \mathrm{sym}_{\mathcal{G}}(\dot{x})$, and $X \subseteq X'$ for every other $X' \in [I]^{<\omega}$ with $\mathrm{fix}_{\mathcal{G}}(X') \preceq \mathrm{sym}_{\mathcal{G}}(\dot{x})$.*

[2]By this group action, $\mathcal{G}$ can be formally identified with a subgroup of $\mathrm{Aut}(\mathbb{P}(I,Q))$, as is required in definition 2.2.1.

[3]For every $K \in \mathcal{F}$, this guarantees that we can find some $E \in [I]^{<\omega}$ for which $\mathrm{fix}_{\mathcal{G}}(E) \preceq K \preceq \mathcal{G}$. In general, however, $K \in \mathcal{F}$ may not equal $\mathrm{fix}_{\mathcal{G}}(E)$ for any $E \in [I]^{<\omega}$.

Proof. See [Jec08] for the proof of "least supports" in "support models." $\square$

Definition 2.5.10. $X \in [I]^{<\omega}$ as in lemma 2.26 is *the support* of $\dot{x}$, denoted $\mathrm{supp}(\dot{x}) = X$.

Definition 2.5.11. For $p \in \mathbb{P}(I, Q)$, let $\mathrm{supp}(p) = \mathrm{dom}(p)$.

Definition 2.5.12. Given $p \in \mathbb{P}(I, Q)$ and $X \subseteq I$, define the *restriction of* p *to* X to be the (possibly empty) function $p \restriction_X = \{\langle i, p(i)\rangle \mid i \in X\}$.

The following lemma states an essential property that is used throughout the paper: when forcing a property, one only needs to consider the behavior of conditions on the relevant supports of names.

Lemma 2.5.13. *For the symmetric system* $\langle \mathbb{P}(I, Q), \mathcal{G}, \mathcal{F}\rangle$*, if* $\dot{x}_0, ..., \dot{x}_n \in HS_{\mathcal{F}}$ *have supports* $X_0, ..., X_n$*, and* $p \Vdash \phi(\dot{x}_0, ..., \dot{x}_n)$*, then for* $X \supseteq \bigcup_{i \le n} X_i$*,*

$$p \restriction_X \Vdash \phi(\dot{x}_0, ..., \dot{x}_n).$$

Proof. By the symmetry lemma 2.2.4, if $\pi \in \mathrm{fix}_{\mathcal{G}}(X)$, then $\pi p \Vdash \phi(\dot{x}_0, ..., \dot{x}_n)$. Because the set of conditions $D = \{\pi p \mid \pi \in \mathrm{fix}_{\mathcal{G}}(X)\}$ is dense below $p \restriction_X$ in $\mathbb{P}(I, Q)$, we get that $p \restriction_X \Vdash \phi(\dot{x}_0, ..., \dot{x}_n)$. $\square$

Definition 2.5.14. Given $\mathcal{G}$ as in definition 2.5.5, $\pi \in \mathcal{G}$, and $X \subseteq I$, define the *restriction of* π *to* X to be the (possibly empty) function $\pi \restriction_X = \{\langle i, \pi(i)\rangle \mid i \in X\}$. Note that $\pi \restriction_X$ may no longer be an element of $\mathcal{G} = \mathrm{Aut}(I)$.

Lemma 2.5.15. *Let* $\langle \mathbb{P}(I, Q), \mathcal{G}, \mathcal{F}\rangle$ *be as in definition 2.5.5,* $p \in \mathbb{P}(I, Q)$*, and* $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$*. If* $\mathrm{supp}(p), \mathrm{supp}(\dot{x}) \subseteq D \in [I]^{<\omega}$*, then for every* $\pi, \tau \in \mathcal{G}$ *with* $\pi \restriction_D = \tau \restriction_D$*,*

$$\pi p = \tau p \wedge \pi \dot{x} = \tau \dot{x}.$$

Proof. Because $\pi \restriction_D = \tau \restriction_D$, we have that $\pi^{-1}\tau \in \mathrm{fix}_{\mathcal{G}}(D)$. Then,

$$\pi p = \pi(\pi^{-1}\tau p) = \tau p \wedge \pi\dot{x} = \pi(\pi^{-1}\tau\dot{x}) = \tau\dot{x}.$$

$\square$

Theorem 2.5.16 (Halpern, Lévy, [HL71], Stefanović, [Ste23][4])**.** $N(I, Q) \vDash \mathrm{BPI}$.

[4]This is proven for the Cohen model in [HL71]; in [Ste23], the proof from [HL71] is shown to generalize $N(I, Q)$.

Chapter 3

Characterizations of BPI in Permutation Models

The main goals of this chapter are to study conditions that are equivalent to BPI in permutation models, and to give simple proofs of the equivalence of these conditions. It is of independent interest to do so, but this will also motivate our analogous characterizations of BPI in symmetric extensions in chapter 4, which are the main topics of this thesis.

As noted in the preliminaries section 2.1, we will always make two assumptions about a permutation model $M(A, \mathcal{G}, \mathcal{F})$. First, we will take $M(A, \mathcal{G}, \mathcal{F})$ to be defined with respect to an outer model $M \models \mathrm{ZFCA}$. Second, we will always assume that for every $H \in \mathcal{F}$, there is an $x \in M(A, \mathcal{G}, \mathcal{F})$ with $\mathrm{sym}_{\mathcal{G}}(x) = H$. As in lemma 2.1.12, this second assumption can be made without affecting which sets appear in the permutation model. These two assumptions match assumptions that Blass often makes, for instance in [Bla79], [Bla87], and [Bla11].

Theorem 3.4.4 and its uniform version, theorem 3.4.5, are the main results of this chapter. We can note that the second normalizing assumption listed above is *only* used in the proof of the implications $(i) \Rightarrow (iii)$ and $(i) \Rightarrow (iv)$ in both of these main theorems. In the case of the other implications, one can ignore this extra assumption.

There are several known results relating to BPI in permutation models, notably from Halpern, Blass, and from joint work by Kechris, Pestov, and Todorčević. Several of our arguments will be new proofs of known implications, or known proofs reworked into a new context, relating particularly to work from [Bla87], [KPT05], and [Bla11]. We will survey these known results before outlining each section of this chapter.

In Halpern's thesis [Hal62], later published as [Hal64], it was shown that BPI holds in the ordered Mostowski model of ZFA. This gave the first proof that AC is independent of BPI over ZFA. In this proof, Halpern introduced a contradiction framework that appears, to the author's knowledge, in every other known proof of BPI in a model of ZF(A) without choice.[1]

By analyzing Halpern's proof, and a similar proof of Johnstone in [Joh84], Blass was able to isolate a dynamical property that exactly characterizes BPI in permutation models, which he called the Ramsey property. His proof that the Ramsey property implies BPI closely follows Halpern's contradiction proof of BPI in the ordered Mostowski model. We state Blass' Ramsey property and characterization theorem below. We will use the equivalent form of the Ramsey property given in [Bla11].[2]

Definition 3.0.1 (Blass, [Bla87], [Bla11])**.** Let $\mathcal{G}$ be a group.

(i) An action of $\mathcal{G}$ on a set A has the Ramsey property if for every coloring $c : A \to 2$ and for every $\Delta \in [A]^{<\omega}$, there is a $\sigma \in G$ so that the restriction $c \restriction_{\sigma\Delta}$ is monochromatic.

[1]Pincus has made this same observation 29 years ago, in [Pin97].

[2]If the reader is unfamiliar with this definition, or finds it to be technical, note that it can be motivated as the apparent and exact device needed in our new proof of BPI from the Ramsey property in section 3.2.

(ii) A subgroup $H \preceq \mathcal{G}$ is called a Ramsey subgroup of $\mathcal{G}$ if the action of $\mathcal{G}$ on the cosets $\mathcal{G}/H$ (by left multiplication) has the Ramsey property.

(iii) A normal filter $\mathcal{F}$ of subgroups of $\mathcal{G}$ has the Ramsey property if $\mathcal{F}$ has a base $\mathcal{B}$, so that for every $K \in \mathcal{B}$, every subgroup H of K in $\mathcal{F}$ is a Ramsey subgroup of K.

Theorem 3.0.2 (Blass, [Bla87])**.** *If $M \vDash$ ZFCA, given our normalizing assumptions for a permutation model $M(A, \mathcal{G}, \mathcal{F})$, the following are equivalent:*

(i) $M(A, \mathcal{G}, \mathcal{F}) \vDash$ BPI,

(ii) $\mathcal{F}$ has the Ramsey property.

Note that Blass' Ramsey property is defined for normal filters $\mathcal{F}$ of subgroups of some group $\mathcal{G}$, as are seen in the definition of a permutation model $M(A, \mathcal{G}, \mathcal{F})$. We would like to draw the distinction between Blass' Ramsey property from [Bla87], and the related (structural) Ramsey property studied in [KPT05]. In this latter paper, connections are developed between this structural Ramsey property, extremely amenable topological groups, and Fraïssé limits.

In response to [KPT05], Blass examined the connection between these two versions of the Ramsey property in [Bla11]. He focused on the connection between his characterization of BPI in permutation models and a characterization of certain extremely amenable topological groups in proposition 4.2 of [KPT05]. We state below several necessary definitions, and then Blass' theorem from [Bla11].

Definition 3.0.3. A group $\mathcal{G}$ is a *topological group* if it is equipped with a topology so that the group operation corresponds to a continuous map from $\mathcal{G} \times \mathcal{G}$ (with the product topology) to $\mathcal{G}$, and the mapping of elements of $\mathcal{G}$ to their inverses in $\mathcal{G}$ is continuous.

Definition 3.0.4. The action of a topological group $\mathcal{G}$ on a topological space X is continuous if the associated map from $\mathcal{G} \times X$ (with the product topology) to X is continuous.

Informally, for a continuous group action of $\mathcal{G}$ on X, if $g, h \in \mathcal{G}$ are close in $\mathcal{G}$ and $x, y \in X$ are close in X, then $gx, hy \in X$ are also close in X. The following formalization of this idea makes assumptions that are relevant to this chapter. This is standard to prove, but we include its proof regardless for the reader's benefit.

Lemma 3.0.5. *Let K be a topological group acting continuously on a topological space X, let $\{\pi_i \mid i \in I\} \subseteq K$, let $\{x_i \mid i \in I\}, \{\pi_i x_i \mid i \in I\} \subseteq X$, and let U be an ultrafilter on I. Then if the limits below exist, they satisfy the relationship*

$$\lim_{i,U} \pi_i x_i = (\lim_{i,U} \pi_i)(\lim_{i,U} x_i).$$

Proof. Let $\pi = \lim_{i,U} \pi_i$ and let $x = \lim_{i,U} x_i$. Let $W \subseteq X$ be an open neighborhood of $\pi x \in X$. We aim to show that

$$\{i \in I \mid \pi_i x_i \in W\} \in U,$$

from which the lemma follows. By the continuity of the action of K on X, there are open neighborhoods $W_\pi \subseteq K$ of π and $W_x \subseteq X$ of x, so that

$$W_\pi \cdot W_x = \{\tau z \mid \tau \in W_\pi,\ z \in W_x\} \subseteq W.$$

By the definition of the ultralimits $\pi = \lim_{i,U} \pi_i$ and $x = \lim_{i,U} x_i$,

$$\{i \in I \mid \pi_i \in W_\pi\} \in U \quad \text{and} \quad \{i \in I \mid x_i \in W_x\} \in U,$$

so

$$V = \{i \in I \mid \pi_i \in W_\pi\} \cap \{i \in I \mid x_i \in W_x\} \in U.$$

Since

$$\{\pi_i x_i \mid i \in V\} \subseteq W_\pi \cdot W_x \subseteq W,$$

we have that

$$V \subseteq \{i \in I \mid \pi_i x_i \in W\} \in U.$$

$\square$

Definition 3.0.6. A topological group $\mathcal{G}$ is *extremely amenable* if every continuous action of $\mathcal{G}$ on a compact, Hausdorff space X has a fixed point.

We find it startling that a group can enjoy such a strong fixed-point property; the authors of [KPT05] refer to the existence of non-trivial extremely amenable groups as "remarkable." A connection between extreme amenability and BPI in permutation models is thus of interest to further clarify each property. This is the main topic of [Bla11].

Note that the group $\mathcal{G} \preceq \mathrm{Aut}(A)$ in a permutation model $M(A, \mathcal{G}, \mathcal{F})$ is not already assumed to be a topological group. We can repeat several observations that Blass makes in [Bla11], to see that the notion of "small open subgroups" of a topological group can be used to naturally associate topological groups with the groups used to define permutation models.

Definition 3.0.7. For a topological group $\mathcal{G}$, an *open subgroup* is a subgroup of $\mathcal{G}$ that is also an open set in the topology on $\mathcal{G}$.

Definition 3.0.8. A topological group $\mathcal{G}$ is said to have *small open subgroups* if every open neighborhood of the identity element of $\mathcal{G}$ contains an open subgroup of $\mathcal{G}$.

For a permutation model $M(A, \mathcal{G}, \mathcal{F})$, there is a natural way to promote $\mathcal{G} \preceq \mathrm{Aut}(A)$ to be a topological group with small open subgroups: take the topology on $\mathcal{G}$ generated by the base of open sets $\mathcal{B} = \{gK \mid g \in \mathcal{G}, K \in \mathcal{F}\} = \{Kg \mid g \in \mathcal{G}, K \in \mathcal{F}\}$, so that the normal filter $\mathcal{F}$

contains exactly the small open subgroups of $\mathcal{G}$.[3]

Similarly, given a topological group $\mathcal{G} \preceq \mathrm{Aut}(A)$ with small open subgroups, the set $\mathcal{F}$ of small open subgroups of $\mathcal{G}$ is a normal filter $\mathcal{F}$. This can then be used to define a permutation model $M(A, \mathcal{G}, \mathcal{F})$. Note that $\mathcal{F}$ may not meet our normalizing assumption that for every $H \in \mathcal{F}$, there is an $x \in M(A, \mathcal{G}, \mathcal{F})$ so that $\mathrm{sym}_{\mathcal{G}}(x) = H$. This isn't a concern, because in every proof that starts by considering a permutation model based on a topological group $\mathcal{G}$, we will never use this normalizing assumption anyway.

In each case, the topology on $\mathcal{G}$ is Hausdorff if and only if $\bigcap_{K \in \mathcal{F}} K$ is the trivial group. We can assume that this intersection is trivial without loss of generality, by lemma 2.1.13. For a topological group $\mathcal{G}$, we can note that $\mathcal{G}$ is extremely amenable if and only if $\mathcal{G}/\bigcap_{K \in \mathcal{F}} K$ is, since the relevant actions of $\mathcal{G}$ are assumed to be continuous. There is thus minimal cost to move between the groups used to define permutation models and (Hausdorff) topological subgroups of $\mathrm{Aut}(A)$ with small open subgroups.

Theorem 3.0.9 (Blass, [Bla11])**.** *Let $\mathcal{G}$ be a topological group with small open subgroups, and let $\mathcal{F}$ be the normal filter of its small open subgroups.*[4] *The following are equivalent:*

(i) $\mathcal{G}$ is extremely amenable,

(ii) every subgroup H of $\mathcal{G}$ in $\mathcal{F}$ is a Ramsey subgroup of $\mathcal{G}$.

In [Bla11], Blass notes the difference between point *(ii)* in his above theorem and the full Ramsey property on $\mathcal{F}$, where the latter of which calls for a base of subgroups in $\mathcal{F}$ with the property from *(ii)*. To make this point, he gives the following example.

Example 3.0.10 (Blass, [Bla11])**.** Let $\mathcal{G}$ be extremely amenable. The product $\mathcal{G} \times \mathbb{Z}/2$ is

[3] The equality in this line follows from the normality of $\mathcal{F}$.

[4] Blass takes the convention that the topology on $\mathcal{G}$ is Hausdorff. By his observations that we repeat above the theorem, he justifies that this is inessential for part *(i)*. In section 3.3, we will see that this is also inessential for part *(ii)*.

not extremely amenable, because it can act continuously without fixed points on a set with 2 elements. The normal filter $\mathcal{F}$ of open subgroups of $\mathcal{G}$, identified with $\mathcal{G}\times\{0\}$, has the Ramsey property by theorem 3.0.9. He concludes by writing: "[s]o extreme amenability cannot correspond exactly to Ramseyness of the filter of open subgroups; the latter is preserved from $\mathcal{G}$ to $\mathcal{G} \times \mathbb{Z}/2$, and the former is not."

With this survey of the literature in mind, we can outline the sections of this chapter. Again, we aim to further develop a list of properties that are equivalent to BPI in permutation models, along with the perspectives on BPI offered by each of these properties. Theorem 3.4.4 and its uniform version, theorem 3.4.5, collect the various conditions that we will prove to be equivalent to BPI in permutation models; these are again the main theorems of this chapter. Across the sections, we prove various implications between the conditions in these main theorems. Several of these proofs are formally redundant, as they follow from linking together other implications, but we include them when they offer novel perspectives on the relationships between these equivalent conditions that characterize BPI. The proofs in this chapter can again be recognized as simpler, motivating versions of the more involved proofs from chapter 4.

In section 3.1, we develop means to directly prove BPI in permutation models. We intend for this approach to clarify the existing proofs in the literature, all of which follow Halpern's contradiction framework from [Hal62]. To do so, we wish to formalize the naïve approach to extend filters to ultrafilters (thus proving BPI) by repeatedly extending filters by minimal increments. Accordingly, we introduce the filter extension property (for permutation models) and prove that it is sufficient for BPI in permutation models.[5] More interestingly, we show that the filter extension property is also necessary for BPI in permutation models.

[5]In chapter 4, we introduce a more general version of the filter extension property for symmetric systems. We will refer to either as the filter extension property, and the correct version will be clear from context.

Thus our naïve, direct approach to extend filters to ultrafilters in permutation models fully characterizes BPI.

In section 3.2, we prove that the filter extension property follows directly from Blass' Ramsey property. This gives a new, direct proof of Blass' theorem that the Ramsey property implies BPI in permutation models. We call this a search-and-shift argument: we show that the Ramsey property is exactly what allows one to search through an arbitrary ultrafilter from an outer model of choice, to identify certain isomorphic (shifted) copies of certain substructures. These substructures are then used to justify the filter extension property, i.e. the ability to extend a given hereditarily symmetric filter by a "minimal" symmetric increment. This proof is inspired by methods used in Harrington's proof of the Halpern-Läuchli theorem, which we adapt further in chapter 4 to prove BPI in certain symmetric extensions.

In section 3.3, we further develop the connection to extreme amenability given in [Bla11], which is in turn based on previous work from [Bla87] and [KPT05]. We first observe that, depite Blass' example 3.0.10, his theorem 3.0.9 can be extended to give a full characterization of BPI in terms of extreme amenability: $\mathcal{F}$ has the Ramsey property if and only if there is some subgroup $K \in \mathcal{F}$ that is extremely amenable.[6] We then rework, in terms of symmetric ultrafilters, the two directions of the proof of proposition 4.2 from [KPT05] – these are the main arguments of the section. As an application, we use this work to give a short proof of a uniform version of our main characterization theorem (a preliminary version of theorem 3.4.5), which answers a natural question that arises from section 3.2.

In section 3.4, we compare the proof of BPI from extreme amenability in section 3.3 to the search-and-shift proof of BPI from the Ramsey property in section 3.2. This suggests a way to formalize the idea that BPI holds in a permutation model if and only if it can be

[6] Again, with the topology on K generated by the subgroups of K in $\mathcal{F}$.

proven using a search-and-shift argument. We thus introduce the search-and-shift property (for permutation models), and prove that it also characterizes BPI.[7] Doing so also lets us explicitly connect our proofs of BPI from the Ramsey property and from extreme amenability. The main theorems 3.4.4 and 3.4.5 are given in this section. Our characterization of BPI using the search-and-shift property, and the fact that it can be generalized to symmetric extensions in chapter 4, are two central ideas of the thesis.

3.1 The Filter Extension Property for Permutation Models

In a model of ZFC(A), the standard proof of BPI involves showing that

(1) an arbitrary filter F, on an arbitrary set x, can be extended to include an arbitrary subset $y \subseteq x$ or its complement $x \setminus y$, and

(2) Zorn's lemma can be applied to produce a maximal filter $U \supseteq F$ on x; by (1), the maximality of U ensures that it "completes" the extension of F to an ultrafilter on x.

While the full axiom of choice generally fails in permutation models, it can still be found to hold up to a coarse degree – certain choice functions, wellorderings, etc. are also hereditarily symmetric. In this section, we will introduce a coarsening of (1) called the filter extension property for permutation models (definition 3.1.1), which can be used to prove BPI by a coarse version of (2) (theorem 3.1.2). We will also show that the filter extension property is necessary for BPI to hold in permutation models (theorem 3.1.11).

To motivate the filter extension property, consider $F, x \in M(A, \mathcal{G}, \mathcal{F})$, with F a filter on x.

[7] We will later introduce a version of the property for symmetric systems, and call each the search-and-shift property when the correct version is clear from context.

To mimic (1), we consider how to extend F by some minimal, symmetric increment while retaining the finite intersection property (FIP). If this can be done in a manner that is stable under iteration, we can then apply a coarse version of (2) to produce an ultrafilter $U \supseteq F$.

To determine what should be meant by extending F by a "minimal, symmetric increment," we can note that for an arbitrary $w \in M(A, \mathcal{G}, \mathcal{F})$,

$$w = \bigcup_{y \in w} \mathrm{Orb}(y, \mathrm{sym}_{\mathcal{G}}(w)).$$

This is a partition of w, up to the fact that some orbits may be repeated in the union above. One can view these orbits as the building blocks of (hereditarily) symmetric sets. This also gives the sense in which choice holds coarsely in a permutation model: if $f \in M \models \mathrm{ZFCA}$ is a wellordering of the set $\{\mathrm{Orb}(y, \mathrm{sym}_{\mathcal{G}}(w)) \mid y \in w\}$ of orbits contained in w, then $\mathrm{sym}_{\mathcal{G}}(f) = \mathrm{sym}_{\mathcal{G}}(w)$, and so $f \in M(A, \mathcal{G}, \mathcal{F})$. While w itself may not be wellorderable in $M(A, \mathcal{G}, \mathcal{F})$, it is still possible to wellorder the partition of w given by the $\mathrm{sym}_{\mathcal{G}}(w)$-orbits.

Thus, to extend the filter F by a minimal, symmetric increment involves selecting a subset $y \in M(A, \mathcal{G}, \mathcal{F})$ of x, a group $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, and extending to

$$F \cup \mathrm{Orb}(y, K) \text{ or } F \cup \mathrm{Orb}(x \setminus y, K),$$

while ensuring that the finite intersection property holds in the chosen extension.[8] If this process can be iterated, using the same group $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ at every stage of the iteration, the union of these extensions is an ultrafilter $U \supseteq F$ on x in $M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}(U)$. Definition 3.1.1 expresses that the hypotheses in this sketch hold, and theorem 3.1.2 shows that BPI follows.

[8]Here we mean "minimal" more in a figurative than a literal sense, as it may be possible to choose a smaller group K with this same property.

Definition 3.1.1. A permutation model $M(A, \mathcal{G}, \mathcal{F})$ has the *filter extension property* (for permutation models) if $\mathcal{F}$ has a base $\mathcal{B}$, so that the following property holds for every $K \in \mathcal{B}$: for every $F, x, y \in M(A, \mathcal{G}, \mathcal{F})$ such that $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ and

$$M(A, \mathcal{G}, \mathcal{F}) \models \text{“}F \subseteq 2^x \text{ has the FIP”} \wedge y \subseteq x,$$

there is a $y' \in \{y, x \setminus y\}$ such that

$$M(A, \mathcal{G}, \mathcal{F}) \models \text{“}F \cup \mathrm{Orb}(y', K) \subseteq 2^x \text{ has the FIP.”}$$

Theorem 3.1.2 uses the fact that the filter extension property guarantees groups $K \in \mathcal{F}$ that are stable under iterations of filter extensions, in the sense described above.

Theorem 3.1.2. *If* $M \models \mathrm{ZFCA}$ *and* $M(A, \mathcal{G}, \mathcal{F})$ *has the filter extension property, then*

$$M(A, \mathcal{G}, \mathcal{F}) \models \mathrm{BPI}.$$

Proof. Let $F, x \in M(A, \mathcal{G}, \mathcal{F})$ so that

$$M(A, \mathcal{G}, \mathcal{F}) \models \text{“}F \subseteq 2^x \text{ has the FIP,”}$$

and let $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ witness the filter extension property. To prove BPI in $M(A, \mathcal{G}, \mathcal{F})$, we will show that there is an ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on x that extends F. At the cost of lengthening a standard application of Zorn's lemma, we would like our proof to emphasize the notion that K can be used to extend F, by minimal increments, in a manner that is stable under iteration.

In the outer model $M \models \mathrm{ZFCA}$, use choice to wellorder the subsets $y \subseteq x$ that appear in

$M(A, \mathcal{G}, \mathcal{F})$; denote this wellordering as $\{y_\alpha | \alpha < \beta\}$. Using choice again in M, define the sequence F_α, for $\alpha \leq \beta$, in the following manner.

Let $F_0 = F$. At successor stages $\alpha + 1$, choose between

$$F_{\alpha+1} = F_\alpha \cup \mathrm{Orb}(y_\alpha, K)$$

and

$$F_{\alpha+1} = F_\alpha \cup \mathrm{Orb}(x \setminus y_\alpha, K),$$

so that $F_{\alpha+1}$ has the FIP and $K \preceq \mathrm{sym}_{\mathcal{G}}(F_{\lambda+1}) \cap \mathrm{sym}_{\mathcal{G}}(x)$. If we suppose inductively that F_α meets these constraints, then the fact that K witnesses the filter extension property ensures that $F_{\alpha+1}$ can be chosen to meet these constraints as well.

At limit stages α, let

$$F_\alpha = \bigcup_{\gamma < \alpha} F_\gamma.$$

If we again suppose inductively that for every $\gamma < \gamma' < \alpha$, F_γ has the FIP, $K \preceq \mathrm{sym}_{\mathcal{G}}(F_\gamma) \cap \mathrm{sym}_{\mathcal{G}}(x)$, and $F_\gamma \subseteq F_{\gamma'}$, then F_α has the FIP and $K \preceq \mathrm{sym}_{\mathcal{G}}(F_\alpha) \cap \mathrm{sym}_{\mathcal{G}}(x)$ as well. To prove that F_α has the FIP, note that we use the fact that $F_\gamma \subseteq F_{\gamma'}$, if $\gamma < \gamma'$, to conclude that every finite subset of the limit F_α appears as a finite subset of some F_γ, with $\gamma < \alpha$.

Then $U = F_\kappa \supseteq F$ is an ultrafilter on x in $M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. $\square$

Note that theorem 3.2 is essentially the same as the step in Halpern's contradiction framework to consider a filter $U \in M(A, \mathcal{G}, \mathcal{F})$ on x that extends F and is maximal with respect to having some specified $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. The next step in Halpern's contradiction framework, to produce a contradiction from the assumption that there is a subset $y \in M(A, \mathcal{G}, \mathcal{F})$ of x for which $y \notin U$ and $x \setminus y \notin U$, can be rephrased as a contradiction proof of the filter extension property.

We will now prove that the filter extension property is necessary for BPI to hold in a permutation model, whereby this relationship to Halpern's contradiction framework is inevitable. Note that the filter extension property ensures that filters can be extended in a manner based *uniformly* on witness groups $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, irrespective of the filter F or the set x. In other words, regardless of which new $K-$orbits of subsets of x are chosen to extend F, there is no need to consider a less restrictive group $K_0 \preceq K$ to extend the filter further. This uniformity can be explained in terms of the uniform method given in [Bla79] to define ultrafilters in models of SVC + BPI.

Definition 3.1.3 (Blass, [Bla79])**.** Let SVC (*small violations of choice*) be the axiom that there is a set S so that, for every set a, there is an ordinal α and a function from a subset of $S \times \alpha$ onto a.[9]

Theorem 3.1.4 (Blass, [Bla79])**.** *A model of* ZF *satisfies* SVC *if and only if some generic extension is a model of* ZFC*.*

Note that forcing has also been studied over models of ZFA, and we can cite the relevant version of the above theorem in this context.

Theorem 3.1.5 (Hall, [Hal07])**.** *A model of* ZFA *satisfies* SVC *if and only if some generic extension is a model of* ZFCA*.*

Theorem 3.1.6 (Blass, [Bla79])**.** *Permutation models satisfy* SVC*.*

For our purposes, it is important to check that there is a witness $S \in M(A, \mathcal{G}, \mathcal{F})$ with $\mathrm{sym}_{\mathcal{G}}(S) = \mathcal{G}$. This can be observed in Blass' original proof, which we sketch below.

Proof sketch, based on the proof of theorem 4.2 from [Bla79]. Consider the permutation model

[9] In [Bla79], the original statement is that there is a function from the *entire* set $S \times \alpha$ onto a. Blass notes that this is equivalent to the form stated here, since all points outside a subset of $S \times \alpha$ can be mapped to a single element.

$M(A,\mathcal{G},\mathcal{F})$. Recall that we take the normalizing assumption that for every $K \in \mathcal{F}$, there is a $x \in M(A,\mathcal{G},\mathcal{F})$ so that $K = \mathrm{sym}_{\mathcal{G}}(x)$.[10]

For $H, K \in \mathcal{F}$, with $H \preceq K$, let $y_{H,K}$ be a set in $M(A,\mathcal{G},\mathcal{F})$ so that $\mathrm{sym}_{\mathcal{G}}(y_{H,K}) = H$. Ensure also that if $H \neq H'$ or $K \neq K'$, then $\mathrm{Orb}(y_{H,K},\mathcal{G}) \cap \mathrm{Orb}(y_{H',K'},\mathcal{G}) = \emptyset$. Let

$$S = \bigcup_{H,K \in \mathcal{F}, H \preceq K} \mathrm{Orb}(y_{H,K},\mathcal{G}),$$

so that $\mathrm{sym}_{\mathcal{G}}(S) = \mathcal{G}$, and thus $S \in M(A,\mathcal{G},\mathcal{F})$. For every $x \in M(A,\mathcal{G},\mathcal{F})$, we have that

$$x = \bigcup_{y \in x} \mathrm{Orb}(y, \mathrm{sym}_{\mathcal{G}}(x)),$$

which is a partition of x up to the repetition of orbits. For every $y \in x$, there is a natural bijection $b : \mathrm{Orb}(y_{H,K}, K) \to \mathrm{Orb}(y, \mathrm{sym}_{\mathcal{G}}(x))$, with $H = \mathrm{sym}_{\mathcal{G}}(y)$, $K = \mathrm{sym}_{\mathcal{G}}(x)$, and $\mathrm{sym}_{\mathcal{G}}(b) = K$. Let $\alpha = |\{\mathrm{Orb}(y, \mathrm{sym}_{\mathcal{G}}(x)) \mid y \in x\}|$, so that there is a bijection f between a subset $s \subseteq S \times \alpha$ and x. Here, we can take s to be a union of sets of the form $\mathrm{Orb}(y_{H,K}, K) \times \{\beta\}$, with $K = \mathrm{sym}_{\mathcal{G}}(x)$ and $\beta < \alpha$, so that $\mathrm{sym}_{\mathcal{G}}(s) = \mathrm{sym}_{\mathcal{G}}(f) = K$. Because $s, f \in M(A,\mathcal{G},\mathcal{F})$, S witnesses SVC. □

We can isolate two features of the construction given above, that will be useful in later arguments.

Definition 3.1.7. The set $S \in M(A,\mathcal{G},\mathcal{F})$ is a *normal witness of* SVC if it witnesses SVC for $M(A,\mathcal{G},\mathcal{F})$, if $\mathrm{sym}_{\mathcal{G}}(S) = \mathcal{G}$, and if for every $K \in \mathcal{F}$, there is an $s \in S$ with $\mathrm{sym}_{\mathcal{G}}(s) \preceq K$.

Corollary 3.1.8. *There is a normal witness of* SVC *in every permutation model, given our normalizing assumption from lemma 2.1.12.*

[10] This will be convenient to assume, but it is not needed in this argument.

Proof. Given our normalizing assumption, our proof of theorem 3.1.6 proves the corollary.

□

For our purposes, it is also important that SVC gives a uniform characterization of BPI.

Definition 3.1.9. For a set S, the *fine filter* on $S^{<\omega}$ is the filter generated by the sets $\{p \in S^{<\omega} | s \in \text{rng}(p)\}$, for $s \in S$. An ultrafilter on $S^{<\omega}$ is *fine* if it extends the fine filter.

Theorem 3.1.10 (Blass, [Bla79])**.** *If $N \vDash$ ZF(A) and S witnesses* SVC*, then $N \vDash$ BPI if and only if there is a fine ultrafilter on $S^{<\omega}$.*

We will need to use certain definability features that can be observed in Blass' proof, which we sketch below.

Proof sketch, based on the proof of theorem 4.1(b) from [Bla79]. The "only if" direction is clear, so we will prove the "if" direction. Let B be a boolean algebra, let S witness SVC, let $f : S \times \alpha \to B$ be a surjection (as is guaranteed in footnote 9), and let U' be an ultrafilter as in the theorem statement. We claim that there is a prime ideal I of B, definable with respect to the parameters f, S and U'. After proving this claim, we will show how to modify the argument to get the same result if the domain of f is an arbitrary subset of $S \times \alpha$.

Every $p \in S^{<\omega}$ is itself a wellordering of $\text{rng}(p)$. Based on this, wellorder $\text{rng}(p) \times \alpha$ by the lexicographical order; this is definable with respect to the parameter p. Then, wellorder $f(\text{rng}(p) \times \alpha)$ by sending points to the least element of their preimage in $\text{rng}(p) \times \alpha$; this is definable with respect to the parameters p and f. Let B_p be the subalgebra of B generated by $f(\text{rng}(p) \times \alpha)$; this is also definable with respect to the parameters p and f. Define a prime ideal I_p of B_p based on the wellorder of $f(\text{rng}(p) \times \alpha)$, again with respect to the parameters p and f. One can then define, with respect to the parameters f, S, and U', the prime ideal of B:

$$I = \{b \in B \mid \{p \in S^{<\omega} \mid b \in I_p\} \in U'\}.$$

If we had taken $\mathrm{dom}(f) \subsetneq S \times \alpha$, the same proof works to define a prime ideal I of B from f, S, and U', if one takes the convention that

$$f(\mathrm{rng}(p) \times \alpha) = \{f(y) \mid y \in (\mathrm{rng}(p) \times \alpha) \cap \mathrm{dom}(f)\}.$$

$\square$

We can now extend theorem 3.1.2 to an equivalence.

Theorem 3.1.11. *If* $M \vDash \mathrm{ZFCA}$*, the following are equivalent:*

(i) $M(A, \mathcal{G}, \mathcal{F}) \vDash \mathrm{BPI}$*,*

(ii) $M(A, \mathcal{G}, \mathcal{F})$ *has the filter extension property.*

Proof. The direction $(ii) \Rightarrow (i)$ is given in theorem 3.1.2, so we will prove the remaining direction $(i) \Rightarrow (ii)$. Fix a permutation model $M(A, \mathcal{G}, \mathcal{F}) \vDash \mathrm{BPI}$. By theorem 3.1.10, there is some $S \in M(A, \mathcal{G}, \mathcal{F})$ that witnesses SVC. In $M(A, \mathcal{G}, \mathcal{F})$, let B be a Boolean algebra, let f be a surjection from a subset of $S \times \alpha$ to B, and let U' be the ultrafilter on $S^{<\omega}$ from the statement of theorem 3.1.10. By the proof sketch of theorem 3.1.6, we can assume that $\mathrm{sym}_{\mathcal{G}}(S) = \mathcal{G}$ and $\mathrm{sym}_{\mathcal{G}}(f) = \mathrm{sym}_{\mathcal{G}}(B)$. By the proof sketch of theorem 3.1.10, we have a formula $\phi(w_0, w_1, w_2, w_3)$ so that the following is a prime ideal in B:

$$I = \{b \mid \phi(b, f, S, U')\}.$$

Given a filter F on a set x in $M(A, \mathcal{G}, \mathcal{F})$, one can let $B = 2^x/F$ to get a first-order formula $\varphi(w_0, w_1, w_2, w_3)$ for which

$$U = \{y \mid \varphi(y, f, S, U')\} \tag{3.1.0.1}$$

is an ultrafilter on x extending F. By this definition and by lemma 2.1.5, we have that

$$\mathrm{sym}_{\mathcal{G}}(f) \cap \mathrm{sym}_{\mathcal{G}}(S) \cap \mathrm{sym}_{\mathcal{G}}(U') \preceq \mathrm{sym}_{\mathcal{G}}(U).$$

Because $\mathrm{sym}_{\mathcal{G}}(S) = \mathcal{G}$ and $\mathrm{sym}_{\mathcal{G}}(f) = \mathrm{sym}_{\mathcal{G}}(2^x/F) = \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, we then have

$$\mathrm{sym}_{\mathcal{G}}(x) \cap \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(U') \preceq \mathrm{sym}_{\mathcal{G}}(U). \tag{3.1.0.2}$$

Now let $K = \mathrm{sym}_{\mathcal{G}}(U')$. The normal filter $\mathcal{F}$ has a base of subgroups $\{H \cap K | H \in \mathcal{F}\}$; we will show that this base witnesses the filter extension property. Let $H \in \mathcal{F}$, and let $F \in M(A, \mathcal{G}, \mathcal{F})$ be a filter on a set $x \in M(A, \mathcal{G}, \mathcal{F})$, so that $H \cap K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$. Let $U \supseteq F$ be the ultrafilter on x in $M(A, \mathcal{G}, \mathcal{F})$, defined by the formula φ as in line (3.1.0.1). For every $y \subseteq x$ in $M(A, \mathcal{G}, \mathcal{F})$, there is a $y' \in \{y, x \setminus y\}$ so that

$$F \cup \mathrm{Orb}(y', \mathrm{sym}_{\mathcal{G}}(U)) \subseteq U.$$

By line (3.1.0.2), and since $H \cap K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ and $K = \mathrm{sym}_{\mathcal{G}}(U')$, we have that

$$H \cap K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x) \cap \mathrm{sym}_{\mathcal{G}}(U') \preceq \mathrm{sym}_{\mathcal{G}}(U),$$

and so

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \cup \mathrm{Orb}(y', H \cap K) \subseteq F \cup \mathrm{Orb}(y', \mathrm{sym}_{\mathcal{G}}(U)) \subseteq U \subseteq 2^x \text{ has the FIP.”}$$

Thus, $H \cap K$ is a witness of the filter extension property. Because $H \cap K$ was an arbitrary group in the specified base for $\mathcal{F}$, $M(A, \mathcal{G}, \mathcal{F})$ has the filter extension property. $\square$

3.2 The Ramsey Property

We can now turn our attention to developing direct methods to prove the filter extension property. As mentioned in the introduction to this chapter, Blass introduced the Ramsey property for normal filters of subgroups in [Bla87], and used it to exactly characterize BPI in permutation models; his proof that the Ramsey property on $\mathcal{F}$ implies BPI in $M(A, \mathcal{G}, \mathcal{F})$ uses Halpern's contradiction framework.

We will directly prove this implication, by showing that the Ramsey property on $\mathcal{F}$ implies that $M(A, \mathcal{G}, \mathcal{F})$ has the filter extension property. The style of argument we use, which we call a search-and-shift argument, is generalized several times in chapter 4 to address BPI in symmetric extensions. The version of this argument for permutation models is simple enough to be faithfully captured by the doodle in figure 3.1 at the end of this section.

Before giving our direct proof, note that the main theorem of this section is already proven:

Theorem 3.2.1. *If $M \vDash$ ZFCA, the following are equivalent:*

(i) $M(A, \mathcal{G}, \mathcal{F}) \vDash$ BPI*,*

(ii) $M(A, \mathcal{G}, \mathcal{F})$ *has the filter extension property (for permutation models),*

(iii) $\mathcal{F}$ *has the Ramsey property.*

Proof. Combine the equivalences from theorems 3.1.11 and 3.0.2. $\square$

The filter extension property and its relationship to the Ramsey property are not identified explicitly in [Bla87], but the implications $(iii) \Rightarrow (ii) \Rightarrow (i)$ in the theorem above are implicit in Blass' use of Halpern's contradiction framework, for the reasons sketched after our proof of theorem 3.1.2; Blass proves the implication $(i) \Rightarrow (iii)$ explicitly. We can now present a new, direct proof of the implication $(iii) \Rightarrow (ii)$.

Proof of $(iii) \Rightarrow (ii)$. Let $K \in \mathcal{F}$ witness the Ramsey property. We will show that K also witnesses the filter extension property; then, the base of subgroups in $\mathcal{F}$ witnessing the Ramsey property also witnesses the filter extension property. Let $F, x, y \in M(A, \mathcal{G}, \mathcal{F})$, $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, and

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \subseteq 2^x \text{ has the FIP”} \wedge y \subseteq x.$$

Let $H = K \cap \mathrm{sym}_{\mathcal{G}}(y) = K \cap \mathrm{sym}_{\mathcal{G}}(x \setminus y)$. Because K is a witness of the Ramsey property and H is a subgroup of K in $\mathcal{F}$, H is a Ramsey subgroup of K.

Let $U \in M$ be an ultrafilter on x extending F; U itself is not necessarily hereditarily $\mathcal{F}$-symmetric, but exists in $M \vDash$ ZFCA. Define the coloring $c : K/H \to \{\text{green, red}\}$ by setting

$$c(\pi H) = \begin{cases} \text{green} & \text{if } \pi y \in U \\ \text{red} & \text{if } \pi(x \setminus y) \in U. \end{cases}$$

Note that $c(\pi H)$ is well-defined, since every $h \in H$ fixes both y and $x \setminus y$. For every $\Delta = \{\pi_0 H, ..., \pi_n H\} \in [K/H]^{<\omega}$, the fact that H is a Ramsey subgroup of K guarantees a $\sigma \in K$ so that $c \restriction \sigma\Delta$ is monochromatic. If c colors the elements of $\sigma\Delta = \{\sigma\pi_0 H, ..., \sigma\pi_n H\}$ green, then

$$(F \cup \{\sigma\pi_i y \mid i \leq n\}) \subseteq U. \tag{3.2.0.1}$$

Because $\sigma, \sigma^{-1} \in K \subseteq \text{fix}(F) \cap \text{fix}(x)$, we can apply σ^{-1} to conclude that

$$(F \cup \{\pi_i y \mid i \leq n\}) \subseteq \sigma^{-1} U.$$

By symmetry (lemma 2.1.5), $\sigma^{-1}U$ is still an ultrafilter on $\sigma^{-1}x = x$ in M, so we can conclude

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \cup \{\pi_i y \mid i \leq n\} \subseteq 2^x \text{ has the FIP.”}$$

If c colors the elements of $\sigma\Delta = \{\sigma\pi_0 H, ..., \sigma\pi_n H\}$ red, the same argument concludes

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \cup \{\pi_i (x \setminus y) \mid i \leq n\} \subseteq 2^x \text{ has the FIP.”}$$

Now define a coloring $C : [K/H]^{<\omega} \to \{\text{green, red}\}$ by

$$C(\Delta) = \begin{cases} \text{green} & \text{if } (\exists \sigma \in K) c \restriction \sigma\Delta \text{ is green.} \\ \text{red} & \text{otherwise.} \end{cases}$$

Ordering $[K/H]^{<\omega}$ by reverse-inclusion, $[K/H]^{<\omega}$ is downward-directed by unions. This ensures that C colors a dense subset of this ordering on $[K/H]^{<\omega}$ to be either mutually green or mutually red. In the former case, we claim that $F \cup \text{Orb}(y, K)$ has the FIP; in the latter case, we claim that $F \cup \text{Orb}(x \setminus y, K)$ has the FIP. Without loss of generality, we prove the former claim.

For $\pi_0, ..., \pi_n \in K$ arbitrary, we aim to show that $F \cup \{\pi_i y \mid i \leq n\}$ has the FIP. Let $\Delta = \{\pi_0 H, ..., \pi_n H\} \in [K/H]^{<\omega}$. By the assumption that C colors a dense subset of $[K/H]^{<\omega}$ green, we can find some $\Delta' \supseteq \Delta$ in $[K/H]^{<\omega}$ so that $C(\Delta') = \text{green}$. In other words, there is a $\sigma \in K$ so that $c \restriction \sigma\Delta'$ is green, and so

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \cup \{\pi y \mid \pi H \in \Delta'\} \text{ has the FIP.”}$$

Since $\Delta \subseteq \Delta'$, this gives the desired result that

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \cup \{\pi_i y \mid i \leq n\} \text{ has the FIP.”}$$

Since $\pi_0, ..., \pi_n \in K$ were arbitrary, we can conclude that $F \cup \mathrm{Orb}(y, K)$ has the FIP. Because $F, x, y \in M(A, \mathcal{G}, \mathcal{F})$ arbitrarily meet the hypothesis of the filter extension property, with $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, we can conclude that K witnesses the filter extension property. $\square$

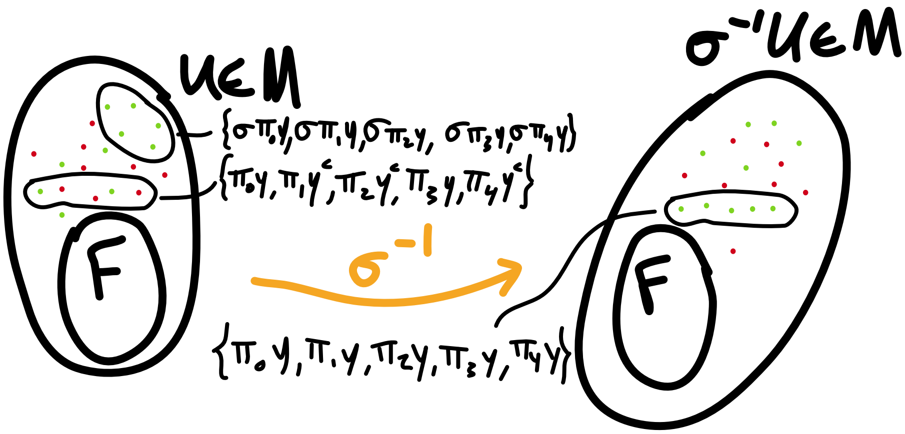

Figure 3.1: The main step in our proof of theorem 3.2.1$(iii) \Rightarrow (ii)$.

Remark 3.2.2. We see that the Ramsey property is a dynamical condition that allows ultrafilters in the outer model $M \vDash \mathrm{ZFCA}$ to coordinate symmetric filter extension in the inner model $M(A, \mathcal{G}, \mathcal{F})$. By the Blass' theorem 3.0.2, whenever BPI holds in $M(A, \mathcal{G}, \mathcal{F})$ it can be attributed to this mechanism of inheritance from the outer model M.

We call this method of proof a “search-and-shift” argument, since we search through an arbitrary ultrafilter $U \supseteq F$ on x, with U from an outer model of choice, then shift the results of this search to verify instances of the filter extension property.

3.3 Extreme Amenability

To relate BPI in permutation models to extreme amenability, we will first show how minor observations can be made to strengthen theorem 3.0.9.

Theorem 3.3.1. *Let $M \models \mathrm{ZFCA}$ and let $M(A, \mathcal{G}, \mathcal{F})$ be a permutation model. The following are equivalent:*

(i) $M(A, \mathcal{G}, \mathcal{F}) \models \mathrm{BPI}$,

(ii) $M(A, \mathcal{G}, \mathcal{F})$ *has the filter extension property (for permutation models),*

(iii) $\mathcal{F}$ *has the Ramsey property,*

(iv) there is a $K \in \mathcal{F}$ *that is extremely amenable, with topology generated by the subgroups of* K *in* $\mathcal{F}$.

Proof. To extend theorem 3.2.1, we will show that $(iv) \Rightarrow (i)$, and that $(iii) \Rightarrow (iv)$, under the assumption that $\bigcap_{K \in \mathcal{F}}$ is the trivial group, and so the topology on $\mathcal{G}$ is Hausdorff. After proving the result in this case, we show that the general case follows.

$\underline{(iii) \Rightarrow (iv)}$ If $K \in \mathcal{F}$, then K is a topological group with the subspace topology from $\mathcal{G}$, and we can define the normal filter of open subgroups of K

$$\mathcal{F}_K = \{H \cap K \mid H \in \mathcal{F}\}.$$

By theorem 3.0.9, K is extremely amenable if and only if every $H \in \mathcal{F}_K$ is a Ramsey subgroup of K. Equivalently, we have that:

Observation 3.3.2. $K \in \mathcal{F}$ is extremely amenable if and only if K is a witness of the Ramsey property for $\mathcal{F}$.

From this observation, it is immediate that $(iii) \Rightarrow (iv)$.

$\underline{(iv) \Rightarrow (i)}$ Suppose that $K \in \mathcal{F}$ is extremely amenable, with the subspace topology from $\mathcal{G}$. By observation 5.2.2, $K \in \mathcal{F}$ is a witness of the Ramsey property. By our proof of theorem 3.2.1$(iii) \Rightarrow (ii)$, if $K \in \mathcal{F}$ witnesses the Ramsey property, then it also witnesses the filter extension property.

Recall theorem 3.1.10: $M(A, \mathcal{G}, \mathcal{F}) \vDash$ BPI if and only if there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$, where S witnesses SVC. By corollary 3.1.8, for the fine filter F on $S^{<\omega}$, we can take $\mathrm{sym}_{\mathcal{G}}(S^{<\omega}) = \mathrm{sym}_{\mathcal{G}}(F) = \mathcal{G}$, so that $K \preceq \mathrm{sym}_{\mathcal{G}}(S^{<\omega}) \cap \mathrm{sym}_{\mathcal{G}}(F) = \mathcal{G}$. Then, since K witnesses the filter extension property, by our proof of theorem 3.1.2, there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. Thus (i) holds, so $(iv) \Rightarrow (i)$.

Now, suppose we do not assume that $N = \bigcap_{K \in \mathcal{F}} K$ is trivial, nor equivalently that the topology on $\mathcal{G}$ is Hausdorff.

To show $(iv) \Rightarrow (iii)$, note that Blass' observations, which we repeat before theorem 3.0.9, justify that K is extremely amenable if and only if K/N is, and that K/N is Hausdorff. The above argument then proves BPI in $M(A, \mathcal{G}/N, \mathcal{F}/N)$, which is the same as $M(A, \mathcal{G}, \mathcal{F})$ by lemma 2.1.13. By theorem 3.0.2, (iii) then holds.

To show $(i) \Rightarrow (iv)$, we similarly note that BPI in $M(A, \mathcal{G}/N, \mathcal{F}/N)$ is equivalent to (i), that $\mathcal{G}/N$ is Hausdorff and so our arguments above justify that $\mathcal{G}/N$ is extremely amenable, which is then equivalent to (iv). This settles the general case of the theorem. $\square$

Remark 3.3.3. Note how the above theorem addresses Blass' example 3.0.10, where the subgroup $\mathcal{G} \times \{0\} \preceq \mathcal{G} \times \mathbb{Z}/2\mathbb{Z}$ is extremely amenable.

The main idea of this proof is observation 5.2.2; this is essentially theorem 3.0.9, which Blass notes is essentially proposition 4.2 of [KPT05]. It is worthwhile to rework the proof of proposition 4.2 from [KPT05] to the language of BPI in permutation models. Doing so results in proofs of the implications $(i) \Rightarrow (iv)$ and $(iv) \Rightarrow (i)$ from the theorem above, which avoid mention of the points (ii) and (iii). These are the two main proofs of this section.

Proof of theorem 3.3.1$(i) \Rightarrow (iv)$, based on the proof of proposition 4.2$(ii) \Rightarrow (i)$ from [KPT05]. Let S be a normal witness of SVC in $M(A, \mathcal{G}, \mathcal{F})$.[11] Given that $M(A, \mathcal{G}, \mathcal{F}) \vDash \mathrm{BPI}$, there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$.[12] We will show that, if $K \in \mathcal{F}$ and $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, then K is extremely amenable with the subspace topology from $\mathcal{G}$.

Towards this end, let fix a $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$ in $\mathcal{F}$, let X be an arbitrary compact, Hausdorff space, and let K act continuously on X. Our aim is to find a fixed point $x \in X$ of the action. We prove a motivating lemma below.

Lemma 3.3.4. *Let K be a topological group that acts continuously on a compact Hausdorff space X. Let V be an ultrafilter on X, with $\lim_{z,V} z = x \in X$. If $\pi W \in V$ for every open neighborhood $W \subseteq X$ of x and every $\pi \in K$, then x is a fixed point of the action.*

Proof. Let $\pi \in K$ be arbitrary. We aim to show that $\lim_{z,V} \pi z = x$. Then, by the continuity of the action,

$$\pi x = (\lim_{z,V} \pi)(\lim_{z,V} z) = \lim_{z,V} \pi z = x.$$

[11] Here we rely on our normalizing assumption for permutation models, from lemma 2.1.12, in the form of corollary 3.1.8.

[12] Recall theorem 3.1.10, that the existence of a fine ultrafilter U on $S^{<\omega}$ is equivalent to BPI in $M(A, \mathcal{G}, \mathcal{F})$.

Let $W \subseteq X$ be an arbitrary open neighborhood of x. Since $\lim_{z,V} z = x$, we have that $W \in V$. Since $\pi^{-1} \in K$, by assumption we have that

$$\{z \in X \mid \pi z \in W\} = \{\pi^{-1} z \in X \mid z \in W\} = \pi^{-1} W \in V.$$

Since $W \subseteq X$ was an arbitrary open neighborhood of x, we can conclude that $\lim_{z,V} \pi z = x$, and that $\pi x = x$. Since $\pi \in K$ was arbitrary, x is a fixed point of the action. $\square$

Thus, the overall proof is completed by identifying such an ultrafilter V on X. The next lemma shows provides a method to approximate the action of K on X with the action of K on $S^{<\omega} \in M(A, \mathcal{G}, \mathcal{F})$, where the latter action is from the definition of the permutation model. Instead of the particular case for $S^{<\omega}$, with $\mathrm{sym}_{\mathcal{G}}(S^{<\omega}) = \mathcal{G}$, we phrase the lemma in general terms of a set $Z \in M(A, \mathcal{G}, \mathcal{F})$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(Z)$.

Lemma 3.3.5. *Let $Z \in M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(Z)$. In $M \vDash \mathrm{ZFCA}$, there is a function $f : Z \to X$ so that for every $z \in Z$, and every $\pi \in K$,*

$$f(\pi z) \in \pi(\mathrm{sym}_{\mathcal{G}}(z) \cap K) f(z) = \{\pi \tau f(z) \mid \tau \in \mathrm{sym}_{\mathcal{G}}(z) \cap K\},$$

where πz is given by the action of K on Z and $\pi \tau f(z)$ is given by the action of K on X.

Proof. It suffices to define the function $f : Z \to X$ on a single orbit $\mathrm{Orb}(z, K) \subseteq Z$. The entire function is then the union of its definition over these orbits. Accordingly, let $z_0 \in Z$. Fix $f(z_0) \in X$ arbitrarily, and for every (distinct) $\pi z_0 \in \mathrm{Orb}(z_0, K)$, choose some $\tau \in \mathrm{sym}_{\mathcal{G}}(z_0) \cap K$ and set $f(\pi z_0) = \pi \tau f(z_0)$.

To show that f satisfies the lemma, let $z \in \mathrm{Orb}(z_0, K)$ be arbitrary, and fix some $\sigma \in K$ so that $z = \sigma z_0$. Let $\pi \in K$ be arbitrary. Then there are $\tau_0, \tau_1 \in \mathrm{sym}_{\mathcal{G}}(z) \cap K$, so that

$$f(z) = f(\sigma z_0) = \sigma\tau_0 f(z_0) \quad \text{and} \quad f(\pi z) = f(\pi\sigma z_0) = \pi\sigma\tau_1 f(z_0).$$

Then

$$f(\pi z) = \pi\sigma\tau_1\tau_0^{-1}\sigma^{-1}f(z).$$

Thus $f(\pi z) = \pi\tau f(z)$, for $\tau = \sigma\tau_1\tau_0^{-1}\sigma^{-1} \in \sigma(\mathrm{sym}_{\mathcal{G}}(z_0) \cap K)\sigma^{-1} = \mathrm{sym}_{\mathcal{G}}(z) \cap K$. □

We can think of f as approximating the action of K on X with the action of K on $Z \in M(A, \mathcal{G}, \mathcal{F})$. We can only approximate, because although we have $\tau z = z$ for every $\tau \in \mathrm{sym}_{\mathcal{G}}(z) \cap K$, it may not be the case that $\tau f(z) = f(z)$. Instead of exactly relating the actions by $f(\pi z) = \pi f(z)$, we can think of $\mathrm{sym}_{\mathcal{G}}(z) \cap K$ as an error term in the approximation.

If the error term $\mathrm{sym}_{\mathcal{G}}(z) \cap K$ tends to the identity subgroup $\{e\}$ as z ranges over Z, the error term vanishes in the limit. This is the use of taking $Z = S^{<\omega}$, for a normal witness S of SVC. Recall that S is a normal witness of SVC if $\mathrm{sym}_{\mathcal{G}}(S) = \mathrm{sym}_{\mathcal{G}}(S^{<\omega}) = \mathcal{G}$ and if for every $H \in \mathcal{F}$, there is an $s_0 \in S$ so that $\mathrm{sym}_{\mathcal{G}}(s_0) \preceq H$.

Fix $f : S^{<\omega} \to X$, as in lemma 3.3.5. Note that the fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$ is not generally an ultrafilter in $M \vDash \mathrm{ZFCA}$, since it only evaluates the subsets of $S^{<\omega}$ in $M(A, \mathcal{G}, \mathcal{F})$. Fix some $U' \supseteq U$ that is an ultrafilter on $S^{<\omega}$ in M. Consider the pushforward

$$V = f(U') = \{y \subseteq X \mid f^{-1}(y) \in U'\},$$

which is an ultrafilter on X. We claim that V meets the criteria of lemma 3.3.4, and thus that $\lim_{z,V} z = x \in X$ is a fixed point of the action.

Note first that the limit $\lim_{z,V} z = x$ exists, since X is compact, and is unique, because X is Hausdorff. To prove that lemma 3.3.4 is satisfied, let $W \subseteq X$ be an arbitrary open neighborhood of x, and let $\pi \in K$ be arbitrary. We aim to show that $\pi W \in V$.

By the continuity of the action, there are open neighborhoods $W_e \subseteq K$ of the identity $e \in K$ and $W_x \subseteq X$ of x, so that

$$W_e \cdot W_x = \{\tau z \mid \tau \in W_e,\ z \in W_x\} \subseteq W.$$

Let $H \subseteq W_e$ be an open subgroup of K, and fix $s_0 \in S$ so that $\mathrm{sym}_{\mathcal{G}}(s_0) \preceq H \subseteq W_e$. Then for every $s \in S^{<\omega}$ with $s_0 \in \mathrm{rng}(s)$, we have that

$$\mathrm{sym}(s) \preceq \mathrm{sym}_{\mathcal{G}}(s_0) \preceq H \subseteq W_e.$$

Since $W_x \subseteq X$ is an open neighborhood of $x = \lim_{z,V} z$, we have that $W_x \in V$, and in particular that

$$\{s \in S^{<\omega} \mid f(s) \in W_x\} \in U'.$$

Since U' extends U,

$$\{s \in S^{<\omega} \mid s_0 \in \mathrm{rng}(s),\ f(s) \in W_x\} \in U'.$$

Then, since $\{s \in S^{<\omega} \mid s_0 \in \mathrm{rng}(s),\ f(s) \in W_x\} \subseteq \{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\}$,

$$\{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\} \in U'. \tag{3.3.0.1}$$

Recall that $U' \in M$ extends $U \in M(A, \mathcal{G}, \mathcal{F})$, so it may be the case that $\{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\} \notin M(A, \mathcal{G}, \mathcal{F})$, and thus that $\{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\} \notin U$.

To account for this, let $y = \bigcup_{h \in H} h\{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\}$, so that $\mathrm{sym}_{\mathcal{G}}(y) \preceq H$, and $y \in M(A, \mathcal{G}, \mathcal{F})$. Then because $\{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\} \subseteq y$, we have that $y \in U'$. Since $y \in U' \supseteq U$, U is an ultrafilter on $S^{<\omega}$ in $M(A, \mathcal{G}, \mathcal{F})$, and $y \in M(A, \mathcal{G}, \mathcal{F})$, we have that $y \in U$.

Next, note that an arbitrary element of y is of the form hs, with $h \in H$ and $s \in \{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\}$. By the defining property of the approximating map $f : S^{<\omega} \to X$,

$$f(hs) \in h(\mathrm{sym}_{\mathcal{G}}(s) \cap K)f(s) \subseteq Hf(s) \subseteq W_e \cdot W_x \subseteq W.$$

Since $\pi \in K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, we also have that $\pi y \in U$. Then, for $\pi hs \in \pi y$, with $h \in H$ and $s \in \{s \in S^{<\omega} \mid \mathrm{sym}_{\mathcal{G}}(s) \preceq H,\ f(s) \in W_x\}$, we have that

$$f(\pi hs) \in \pi h(\mathrm{sym}_{\mathcal{G}}(s) \cap K)f(s) \subseteq \pi Hf(s) \subseteq \pi W_e \cdot W_x \subseteq \pi W.$$

This shows that $\pi y \subseteq f^{-1}(\pi W)$, so $f^{-1}(\pi W) \in U'$. Thus, $\pi W \in V$, which show that lemma 3.3.4 is satisfied, and consequently that $\lim_{z,V} z = x \in X$ is a fixed point of the action. $\square$

Before giving the corresponding proof of theorem 3.3.1$(iv) \Rightarrow (i)$, we can derive two consequences from the above proof. The first can be motivated by the following question.

Question 3.3.6. In section 3.2, we proved that if $K \in \mathcal{F}$ witnesses the Ramsey property, then it also witnesses the filter extension property. Is this implication reversible?

Theorem 3.2.1 establishes that if $\mathcal{F}$ has a base of subgroups $\mathcal{B}$ witnessing the filter extension property, then it also has a base of subgroups $\mathcal{B}'$ witnessing the Ramsey property – but this does not justify whether one can take $\mathcal{B}' = \mathcal{B}$. We can answer this question by proving a uniform version of theorem 3.3.1.

Theorem 3.3.7. *Let $M \vDash$ ZFCA, let $\mathcal{G} \preceq \mathrm{Aut}(A)$ be a topological group with small open* *subgroups, let $\mathcal{F}$ be the normal filter of its small open subgroups, fix $K \in \mathcal{F}$, and fix $S \in M(A, \mathcal{G}, \mathcal{F})$ that is a normal witness of* SVC. *The following are equivalent:*

(i) there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$,

(ii) K witnesses the filter extension property (for permutation models)

(iii) K witnesses the Ramsey property

(iv) K is extremely amenable, with the subspace topology from $\mathcal{G}$,

and the existence of such a $K \in \mathcal{F}$ is equivalent to BPI *in $M(A, \mathcal{G}, \mathcal{F})$.*

Proof. The implication $(iv) \Rightarrow (iii)$ is a result of observation 5.2.2; $(iii) \Rightarrow (ii)$ is shown in our direct proof of theorem 3.2.1$(iii) \Rightarrow (ii)$; $(ii) \Rightarrow (i)$ is shown in our proof of theorem 3.1.2, since S is a normal witness of SVC, and so $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(S^{<\omega}) = \mathcal{G}$; $(i) \Rightarrow (iv)$ is shown in the version of the proof of 3.3.1$(i) \Rightarrow (iv)$ given above. □

We next prove a corollary, which prepares us to adapt the other direction of the proof of proposition 4.2 from [KPT05].

Corollary 3.3.8. *Let $M \vDash$* ZFCA, *let $\mathcal{G} \preceq \mathrm{Aut}(A)$ be a topological group with small open subgroups, let $\mathcal{F}$ be the normal filter of its small open subgroups. If $K \in \mathcal{F}$ witnesses any point of theorem 3.3.7, then so does every subgroup H of K in $\mathcal{F}$.*

Proof. If K satisfies any of the equivalent points from theorem 3.3.7, then there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$, with S a normal witness of SVC, and with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. Then, for every $H \preceq K$ with $H \in \mathcal{F}$, we have that $H \preceq K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, so H also satisfies all of the equivalent points from theorem 3.3.7. □

Recall that we saw part of the above corollary in our proof of 3.1.11, when we showed that if K witnessed theorem 3.3.7(i), then every subgroup of K in $\mathcal{F}$ witnessed the filter extension property. Note also that conclusion of the corollary for extreme amenability is the same as lemma 13 from [BPT13], where it is shown that open subgroups of an extremely amenable group are also extremely amenable. The corollary could also be proven by citing their lemma.

Definition 3.3.9. Let $M(A, \mathcal{G}, \mathcal{F})$ be a permutation model of $M \vDash$ ZFCA, and let $x \in M(A, \mathcal{G}, \mathcal{F})$. We say that $U \in M$ is a $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter on x if U is a maximal filter in M, subject to the constraint that $U \subseteq M(A, \mathcal{G}, \mathcal{F})$.

Definition 3.3.10. For $F, x \in M(A, \mathcal{G}, \mathcal{F})$, where $F \subseteq 2^x$ has the FIP, let $\beta x/F$ be the space of $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilters on x that extend F, with the usual topology generated by the open sets $\{U \in \beta x/F \mid y \in U\}$, for $y \subseteq x$.

It is standard that $\beta x/F$ is closed under ultralimits, and is thus compact by theorem 2.3.3.

Proof of theorem 3.3.1(iv) $\Rightarrow$ (i), based on the proof of proposition 4.2(i) $\Rightarrow$ (ii) from [KPT05]. Suppose that $K \in \mathcal{F}$ is extremely amenable, with the subspace topology from $\mathcal{G}$, so that K also has small open subgroups. Let $F, x \in M(A, \mathcal{G}, \mathcal{F})$, with F a filter on x in $M(A, \mathcal{G}, \mathcal{F})$. Let $K' = K \cap \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$. Since the groups in this interesection are are small open subgroups of K, so is K'. Then, by corollary 3.3.8, K' is extremely amenable.

Take care to note that this application of corollary 3.3.8 uses the theorem 3.3.7(iv) $\Rightarrow$ (i). To avoid circularity in the current proof, one has two options. First, one could draw this same conclusion that K' is extremely amenable from lemma 13 of [BPT13]. Second, one could first run this argument for the fine filter F on $S^{<\omega}$, with S a normal witness of SVC, so that $K' = K \cap \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(S^{<\omega}) = K \cap \mathcal{G} \cap \mathcal{G} = K$ is extremely amenable without cost; this would prove the theorem 3.3.7(iv) $\Rightarrow$ (i) from scratch. One would then use this to derive corollary 3.3.8 and proceed with the general case, as is given below.

Returning to the main argument, consider the same action of K' on $\beta x/F$ as is used to define the permutation model $M(A, \mathcal{G}, \mathcal{F})$. To see that this action on $\beta x/F$ is continuous, note that K' fixes both F and x, and so the basic open set $\{U \in \beta x/F \mid y \in U\}$, for $y \subseteq x$, is fixed by the small open subgroup $\mathrm{sym}_{\mathcal{G}}(y) \cap K' \preceq K'$.

Fixing an arbitrary $U_0 \in \beta x/F$, note that the closure $\overline{U_0 \cdot K} \subseteq \beta x/F$ is also compact, and that the action of K on $\overline{U_0 \cdot K}$ is also continuous. Then, since K is extremely amenable, there is a fixed point $U \in \overline{U_0 \cdot K}$. This is our desired ultrafilter $U \supseteq F$ on x in $M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. Since $F, x \in M(A, \mathcal{G}, \mathcal{F})$ were arbitrary, we then have $M(A, \mathcal{G}, \mathcal{F}) \vDash \mathrm{BPI}$. $\square$

3.4 The Search-and-Shift Property for Permutation Models

We can conclude this chapter by observing a fundamental connection between our search and shift proof of the filter extension property from the Ramsey property in section 3.2 and the use of extreme amenability to prove BPI in section 3.3. After the former proof, we noted in remark 3.2.2 that *every* $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter $U_0 \in M$ on x that extends F contains enough information to check (*individual*) instances of the filter extension property. As in our proof that the filter extension property implies BPI, one would use U_0 to extend F to some $F_1 = F \cup \mathrm{Orb}(y_0, K)$, then use some $U_1 \supseteq F_1$ to extend F_1 to some $F_2 = F_1 \cup \mathrm{Orb}(y_1, K)$, and so on. By this method, one gets a sequence of $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilters $U_0, U_1, U_2, ... \in \beta x/F$ that converge to U.[13] In other words, to extend F to U with a search-and-shift argument that uses the Ramsey property, the method given in sections 3.1 and 3.2 references infinitely many ultrafilters $U_0, U_1, U_2, ... \in \beta x/F$.

[13] One can check that U is the ultralimit of this sequence, for any ultrafilter on the indices $0, 1, 2, ...$ that avoids initial segments.

It is not immediately apparent that a search-and-shift argument can succeed while using only one ultrafilter $U_0 \in \beta x/F$. In section 3.3, however, our proof of BPI from extreme amenability gives $U \supseteq F$ as an ultralimit of $K \cdot U_0$ – so it must be possible for a single U_0 to do the job of the entire sequence $U_0, U_1, U_2, ... \in \beta x/F$. We introduce a final characterization of BPI that captures the notions that

(i) search-and-shift arguments exactly characterize BPI in permutation models,

(ii) an arbitrary $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter $U_0 \supseteq F$ contains enough information to extend the filter $F \in M(A, \mathcal{G}, \mathcal{F})$ to an ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$, and

(iii) our search-and-shift proof of BPI from the Ramsey property in section 3.2 can be adapted to explicitly identify the desired ultralimit U of $K \cdot U_0$ from section 3.3.

Point (*iii*) offers a notable connection between our proofs of BPI using the Ramsey property (and filter extension property) in section 3.2 and extreme amenability in section 3.3.

Definition 3.4.1. A permutation model $M(A, \mathcal{G}, \mathcal{F})$ has the *search-and-shift property* (for permutation models) if $\mathcal{F}$ has a base $\mathcal{B}$, so that the following property holds for every $K \in \mathcal{B}$: for every $F, x, y \in M(A, \mathcal{G}, \mathcal{F})$ such that $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ and

$$M(A, \mathcal{G}, \mathcal{F}) \vDash \text{“}F \subseteq 2^x \text{ has the FIP,”}$$

and for every $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter $U_0 \supseteq F$ on x, there is a $U \in M(A, \mathcal{G}, \mathcal{F})$ that is an ultralimit of the subspace $K \cdot U_0 \subseteq \beta x/F$, so that $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$ and

$$M(A, \mathcal{G}, \mathcal{F}) \vDash U \supseteq F \text{ is an ultrafilter on } x.\text{”}$$

To justify how the above property is related to search-and-shift proofs, the definition below isolates a feature of our search-and-shift proof of the filter extension property from the Ramsey property. Note that this is exactly the feature captured in figure 3.1.

Definition 3.4.2. For $y \in M(A, \mathcal{G}, \mathcal{F})$, with $y \subseteq x$, and for $K \in \mathcal{F}$, we say that $U_0 \in \beta x/F$ *checks* y (with respect to K) if for every $\delta \in [K]^{<\omega}$, there is some $\sigma \in K$ so that

$$\{\pi y_i \mid \pi \in \delta\} \subseteq \sigma U_0.$$

When $K \in \mathcal{F}$ is clear from context, we will just say that U_0 checks y.

Lemma 3.4.3. *Let $F, x \in M(A, \mathcal{G}, \mathcal{F})$ with $F \subseteq 2^x$, let $K \in \mathcal{F}$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$, and let $U_0, U \in \beta x/F$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$. Then U_0 checks every $y \in U$ (with respect to K) if and only if U is an ultralimit of the subspace $K \cdot U_0 \subseteq \beta x/F$.*

Proof. Suppose that U is an ultralimit of $K \cdot U_0$. Let $y \in U$ and let $\delta \in [K]^{<\omega}$; since $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, we have that

$$\{\pi y \mid \pi \in \delta\} \subseteq U.$$

Since U is an ultralimit of $K \cdot U_0$, there is at least one $\sigma U_0 \in K \cdot U_0$, with $\sigma \in K$, so that $\{\pi y \mid \pi \in \delta\} \subseteq \sigma U_0$. Since $\delta \in [K]^{<\omega}$ was arbitrary, U_0 checks y; since $y \in U$ was arbitrary, U_0 checks every $y \in U$.

Now suppose that U_0 checks every $y \in U$. To express U as an ultralimit of $K \cdot U_0$, let $y \in U$ and let $W = \{U' \in \beta x/F \mid y \in U'\}$ be a basic open neighborhood of U in $\beta x/F$. If $W \cap K \cdot U_0$ is nonempty, it follows that U is an ultralimit of the subspace $K \cdot U \subseteq \beta x/F$. Because U_0 checks y, for $\delta = \{e\} \in [K]^{<\omega}$, where $e \in K$ is the identity, there is a $\sigma \in K$ so that $y \in \sigma U_0$. Then $\sigma U_0 \in W \cap K \cdot U_0$, which completes the proof. $\square$

We state the main theorem of the chapter below.

Theorem 3.4.4. *Let $M \models$ ZFCA, let $\mathcal{G} \preceq \mathrm{Aut}(A)$ be a topological group with small open subgroups, and let $\mathcal{F}$ be the normal filter of its small open subgroups. The following are equivalent:*

(i) $M(A, \mathcal{G}, \mathcal{F}) \models \mathrm{BPI}$,

(ii) $M(A, \mathcal{G}, \mathcal{F})$ has the filter extension property (for permutation models),

(iii) $\mathcal{F}$ has the Ramsey property,

(iv) there is a $K \in \mathcal{F}$ that is extremely amenable, with the subspace topology from $\mathcal{G}$,

(v) $M(A, \mathcal{G}, \mathcal{F})$ has the search-and-shift property (for permutation models).

Proof. The equivalences of $(i) - (iv)$ have been established, it is straightforward that $(v) \Rightarrow (i)$, and the details of our proof of theorem 3.3.1$(iv) \Rightarrow (i)$ prove that $(iv) \Rightarrow (v)$. $\square$

We can also state the final uniform version of this theorem.

Theorem 3.4.5. *Let $M \models$ ZFCA, let $\mathcal{G} \preceq \mathrm{Aut}(A)$ be a topological group with small open subgroups, let $\mathcal{F}$ be the normal filter of its small open subgroups, fix $K \in \mathcal{F}$, and fix $S \in M(A, \mathcal{G}, \mathcal{F})$ that is a normal witness of* SVC. *The following are equivalent:*

(i) there is a fine ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ on $S^{<\omega}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$,

(ii) K witnesses the filter extension property (for permutation models)

(iii) K witnesses the Ramsey property

(iv) K is extremely amenable, with the subspace topology from $\mathcal{G}$,

(v) K witnesses the search-and-shift property (for permutation models),

and the existence of such a $K \in \mathcal{F}$ *is equivalent to* BPI *in* $M(A, \mathcal{G}, \mathcal{F})$.

Proof. The equivalences of $(i) - (iv)$ have been established, it is straightforward that $(v) \Rightarrow (i)$, and the details of our proof of theorem 3.3.1$(iv) \Rightarrow (i)$ again prove that $(iv) \Rightarrow (v)$. $\square$

We can now give the main proof of this section: a search-and-shift proof of theorem 3.4.5$(iii) \Rightarrow (v)$; theorem 3.4.4$(iii) \Rightarrow (v)$ then follows from its uniform version.

Proof of theorem 3.4.5$(iii) \Rightarrow (v)$. Fix $F, x \in M(A, \mathcal{G}, \mathcal{F})$ as in the hypotheses of the search-and-shift property, fix $U_0 \in \beta x/F$, and let $K \preceq \mathrm{sym}_{\mathcal{G}}(F) \cap \mathrm{sym}_{\mathcal{G}}(x)$ witness the Ramsey property for $\mathcal{F}$. We aim to show that K also witnesses the search-and-shift property. By lemma 3.4.3, it suffices to use the Ramsey property to construct an ultrafilter $U \supseteq F$ on x in $M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$, so that U_0 checks every $y \in U$ (with respect to K). Note that this construction addresses the question raised in the beginning of this section, by showing that a search-and-shift argument can succeed to extend F to U while using only one ultrafilter $U_0 \in \beta x/F$.

First, we prove a useful lemma. In section 3.2, we proved that the Ramsey property implies that U_0 checks at least one set in every partition $x = y \sqcup (x \backslash y)$, so long as $y, x \backslash y \in M(A, \mathcal{G}, \mathcal{F})$. We can generalize this result to arbitrary finite partitions of x, with all sets in $M(A, \mathcal{G}, \mathcal{F})$.

Lemma 3.4.6 (Blass, [Bla11])**.** *Let $k < \omega$. The Ramsey property, as stated in definition 3.0.1 with respect to colorings $c : A \to 2$ in part (i), can equivalently be stated with respect to colorings $c : A \to k$.*

Lemma 3.4.7. *With respect to K, x is hereditarily checked by U_0.*

Proof. Suppose that $x = y_0 \sqcup ... \sqcup y_n$, with $y_0, ..., y_n \in M(A, \mathcal{G}, \mathcal{F})$. We aim to show that there is some $i \leq n$ so that U_0 checks y_i. Our main proof in section 3.2 gave a $\{\text{green}, \text{ red}\}$-coloring of $K/\mathrm{sym}_{\mathcal{G}}(y)$ based on how U_0 made decisions on the images πy and $\pi(x \setminus y)$.[14] There we showed that if K witnessed the Ramsey property, and if a dense subset of $\delta \in [K]^{<\omega}$ were colored monochromatically green (resp. red), then U_0 checks y (resp. $x \setminus y$). This proof can be modified to consider the partition $x = y_0 \sqcup ... \sqcup y_n$, and one can give an analogous $n+1$-coloring of $K/(\bigcap_{i \leq n} \mathrm{sym}_{\mathcal{G}}(y_i))$, based on how U_0 makes decisions on the partition of images $x = \pi y_0 \sqcup ... \sqcup \pi y_n$, for $\pi \in K/(\bigcap_{i \leq n} \mathrm{sym}_{\mathcal{G}}(y_i))$. The version of the Ramsey property for $k = n+1$, as in lemma 3.4.6, can be applied to find a dense subset of $\delta \in [K]^{<\omega}$ that are colored monochromatically by some $i \leq n$. The same proof shows that U_0 checks y_i. $\square$

Now define the set

$$\begin{aligned} F_0 = \{y \subseteq x \mid\ & y \in M(A, \mathcal{G}, \mathcal{F}) \wedge \\ & \text{“}U_0 \text{ checks } y \text{ (with respect to } K)\text{”} \wedge \\ & \text{“}U_0 \text{ does not check } x \setminus y \text{ (with respect to } K)\text{”}\rangle\}. \end{aligned}$$

Claim 3.4.8. We claim that $F \subseteq F_0$, that $K \preceq \mathrm{sym}_{\mathcal{G}}(F_0)$, and that F_0 has the FIP.

We first apply the claim to complete the overall proof. Since $F_0 \subseteq M(A, \mathcal{G}, \mathcal{F})$ and $K \preceq \mathrm{sym}_{\mathcal{G}}(F_0)$, we have $F_0 \in M(A, \mathcal{G}, \mathcal{F})$. Because K witnesses the Ramsey property, it also witnesses the filter extension property. Then because $F_0 \subseteq 2^x$ has the FIP and $K \preceq \mathrm{sym}_{\mathcal{G}}(F_0) \cap \mathrm{sym}_{\mathcal{G}}(x)$, there is an ultrafilter $U \supseteq F_0 \supseteq F$ on x in $M(A, \mathcal{G}, \mathcal{F})$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(U)$.

Let $y \in U$ be arbitrary. If $y \in F_0$, then by definition U_0 checks y. By lemma 3.4.7, U_0 must check at least one $y' \in \{y, x \setminus y\}$. Then, if $y \in U \setminus F_0$, because $y \notin F_0$, it must be the case that U_0 checks both y and $x \setminus y$. We now complete the proof by establishing the claim.

[14]See figure 3.1.

Proof of claim 3.4.8. Because $F \subseteq U_0$ and $K \preceq \mathrm{sym}_{\mathcal{G}}(F)$, for every $\sigma \in K$, we have that $F \subseteq \sigma U_0$. From this, we have that U_0 checks every $y \in F$ and does not check any $x \setminus y$, for $y \in F$. Thus, $F \subseteq F_0$.

Suppose that $y \in F_0$ and $\pi \in K$. Then $y \in M(A, \mathcal{G}, \mathcal{F})$, U_0 checks y with respect to K, and U_0 does not check $x \setminus y$ with respect to K. It is immediate that $\pi y \in M(A, \mathcal{G}, \mathcal{F})$. By the definition of being checked by U_0 with respect to K, because $\pi \in K$, it is also the case that U_0 checks πy with respect to K and that U_0 does not check πy with respect to K. Then $\pi y \in F_0$. Since $\pi \in K$ was arbitrary, $K \preceq \mathrm{sym}_{\mathcal{G}}(F_0)$.

From the definition of being checked by U_0, we can observe that if y is checked by U_0, then every $y' \supseteq y$ (with $y' \subseteq x$) is also checked by U_0. Suppose that $y_0, ..., y_n \in F_0$. For $i \leq n$, let $z_i \in \{y_i, x \setminus y_i\}$ by arbitrary, and consider the intersection $z = \bigcap_{i \leq n} z_i$. If any $z_i = x \setminus y_i$, then $z \subseteq x \setminus y_i$. In this case, because U_0 does not check $x \setminus y_i$, it also does not check z. By lemma 3.4.7, it must then be the case that U_0 checks $\bigcap_{i \leq n} y_i$. In particular, this shows that $\bigcap_{i \leq n} y_i$ is infinite. □

This proof of the claim completes the overall proof, in the manner described above. □

In the context of chapter 4, this route to prove the search-and-shift property from the Ramsey property will be valuable. In this thesis, we do not provide a generalization of extreme amenability to characterize BPI in symmetric extensions – this will be the topic of future work. In section 4.3, we introduce a generalization of the Ramsey property to symmetric extensions. In section 4.4, we introduce a generalization of the search-and-shift property to symmetric extensions. By adapting the main proof in this section, we will still be able to prove that these two generalized properties are equivalent to one another, and to BPI, without reference to a generalized version of extreme amenability.

Chapter 4

Characterizations of BPI in Symmetric Extensions

It is our eventual goal to generalize the four properties considered in chapter 3 to also characterize BPI in symmetric extensions. In this chapter, we establish such generalizations of the filter extension property, the Ramsey property, and the search-and-shift property. Within the scope of this broader characterization program, the work in this chapter addresses several other goals. We outline these below, along with the sections of this chapter, and a survey of the relevant literature.

We note again that the known proofs of BPI in symmetric extensions all follow the contradiction framework introduced in [Hal62], where BPI was proven in the ordered Mostowski model of ZFA. The first proof of BPI in a symmetric extension (or in any model of ZF without choice) was given in [HL71], where BPI is proven in the Cohen model (and more generally in models of a certain theory that abstracts properties of the Cohen model).[1] The combi-

[1]While [HL71] was published in 1971, this work was originally given in the proceedings of a conference at UCLA in 1967.

natorial engine of the contradiction argument in [HL71] was the original Halpern-Läuchli theorem, which was proven in [HL66] for the purpose of its application in [HL71].

Both the main result and the proof method in [HL71] have had great influence over the literature surrounding BPI in symmetric extensions – many of the known models of ZF+BPI without choice are clever modifications of the Cohen model. A recent example of interest can be found in [Ste23], where the application of the Halpern-Läuchli theorem in [HL71] is generalized to prove BPI in the generalized Cohen model $N(I, Q)$.

While Halpern's contradiction framework has been widely influential, the underlying forcing used to define symmetric extensions has often made its application quite technical.[2] To simplify these arguments would be of inherent interest, but it would also help to address several contemporary questions relating to BPI in choiceless models. The question of preserving BPI in iterated symmetric extensions, for instance, is raised in [Kar19]. This is related to longstanding open conjectures from [Pin77a]: that BPI holds in a candidate model for the independence of AC from ZF + BPI + DC (an independence which Pincus later proved in [Pin77b]), and that a certain related variant of the Halpern-Läuchli theorem holds.[3] Questions like these seem to have remained open because generalizing the existing techniques would be exceedingly complicated.

A primary goal of this chapter is thus to provide a simpler, direct framework to prove BPI in symmetric extensions. In terms of generalizing the characterizations of BPI from chapter 3, we begin to achieve this goal by introducing the filter extension property for symmetric systems in section 4.1, and by proving its equivalence with BPI in symmetric extensions.[4]

[2] Comments along these lines are made, for instance, in [Pin77b] and in [Jec08].

[3] In [Pin77a], Pincus proved his conjectured Halpern-Läuchli theorem to be sufficient for BPI in this candidate model. His argument follows the approach from [HL71].

[4] For each of the characterizations of BPI that we give in this chapter, note the technical detail that we introduce versions of these properties that hold below some condition $p \in \mathbb{P}$, and prove that this is equivalent to p forcing BPI in the symmetric extension.

In section 4.2, we give a new proof of BPI in the generalized Cohen model $N(I, Q)$, by showing that the filter extension property holds. Recall from section 2.5 that the model $N(I, Q)$ primarily introduces a Dedekind-finite set of mutually Q-generic filters, as an inner model of a forcing extension $M[G]$ that adds I-many mutually Q-generic filters.

In subsection 4.2.1, we prove this result in the case that the index set I is large. To do so, we adapt Harrington's forcing proof of the Halpern-Läuchli theorem – a proof which is celebrated for its conceptual simplicity. Our adaptation of Harrington's proof is a search-and-shift argument, in an essentially similar way to our proof from section 3.2.[5]

In chapter 5 of this thesis, we generalize work from [KS20] to show that one can take the index set I in $N(I, Q)$ to be arbitrarily large, without loss of generality; we apply these theorems in section 4.2.2 to complete our new proof of BPI in $N(I, Q)$, when I is arbitrary.[6]

In section 4.3, we generalize features of our adaptation of Harrington's proof to state the Halpern-Läuchli property.[7] One can view this property as stating the most general dynamical conditions needed to run a search-and-shift proof of BPI; this is the desired generalization of the Ramsey property from chapter 3. We prove that the Halpern-Läuchli property is equivalent to BPI in symmetric extensions, and thus that search-and-shift arguments are a fully comprehensive method to prove BPI in symmetric extensions.

In chapter 3, one purpose of the search-and-shift property was to capture convergent features of our proofs of BPI from the Ramsey property and from extreme amenability. While we leave a generalization of extreme amenability for future work, in section 4.4 we use the Halpern-Läuchli property to state a version of the search-and-shift property for symmetric

[5]Actually, this adaptation of Harrington's proof was the first result of this thesis that was proven; the proofs in chapter 3 are modeled after this one.

[6]Note again that this result is originally due to Stefanović in [Ste23], based on the proof from [HL71].

[7]We use the name "Halpern-Läuchli property" in reference to forthcoming work, where we show that this property can be viewed as an abstract form of the Halpern-Läuchli theorem.

extensions. We prove that this search-and-shift property characterizes BPI in symmetric extensions. This emphasizes the fact that the Halpern-Läuchli property captures the ability to prove BPI using an abstract search-and-shift argument. The intent of this characterization is to prepare for the future work of generalizing extreme amenability to characterize BPI in symmetric extensions.

4.1 The Filter Extension Property

We will motivate the filter extension property by making similar observations as in section 3.1. Note again that in a model of ZFC(A), the standard proof of BPI involves showing that

(1) an arbitrary filter F, on an arbitrary set x, can be extended to include an arbitrary subset $y \subseteq x$ or its complement $x \setminus y$, and

(2) Zorn's lemma can be applied to produce a maximal filter $U \supseteq F$ on x; by (1), the maximality of U ensures that it "completes" the extension of F to an ultrafilter on x.

We would again like to identify some coarse version of (1) and (2) that can be applied in the context of a symmetric extension. Definition 4.1.3 – the filter extension property – is the appropriate modification of (1) above, and can be compared to definition 3.1.1 for permutation models; theorem 4.1.5 is the approapriate modification of (2) above, and can be compared to theorem 3.1.2 for permutation models. It is again the case that the filter extension property exactly characterizes BPI in symmetric extensions, which we prove in theorem 4.1.17; this theorem and its proof are analogous to theorem 3.1.11, where we characterized BPI using the filter extension property for permutation models. We will misuse notation slightly by writing $2^{\dot{x}}$ to denote a name for the powerset of $\dot{x}$ and $\dot{x} \setminus \dot{y}$ to denote a name for the complement of $\dot{y}$ in $\dot{x}$.

To state the filter extension property for the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, we will consider arbitrary $\mathbb{P}$-names $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$, for which

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

with the goal of extending $\dot{F}$ to choose between $\dot{y}$ and $\dot{x} \setminus \dot{y}$.

Lemma 4.1.1 justifies that we can assume, without loss of generality, that 1 forces the above line. Although this assumption is not strictly necessary, it helps to streamline our definition.

Lemma 4.1.1. *If $M \models \mathrm{ZFC}$, $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ is a symmetric system, G is $\mathbb{P}$-generic (over M), and N is the corresponding symmetric extension of M, then for every $F, x, y \in N$ for which*

$$N \models \text{“}F \subseteq 2^{x} \text{ has the FIP”} \wedge y \subseteq x,$$

there are names $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ so that $\dot{F}^G = F, \dot{x}^G = x, \dot{y}^G = y$, and

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x}.$$

Proof. Fix $F, x, y \in N$, $\dot{F}', \dot{x}, \dot{y}' \in \mathrm{HS}_{\mathcal{F}}$, and $p \in G$ so that $\dot{F}'^G = F, \dot{x}^G = x, \dot{y}'^G = y$, and $p \Vdash$ “$\dot{F}' \subseteq 2^{\dot{x}}$ has the FIP” $\wedge\, \dot{y}' \subseteq \dot{x}$. Without loss of generality, take $\langle q, \dot{z} \rangle \in \dot{F}' \iff q \Vdash \dot{z} \in \dot{F}$, and do the same for $\dot{y}'$. We aim to find $\dot{F}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ so that $p \Vdash \dot{F} = \dot{F}' \wedge \dot{y} = \dot{y}'$ and so that $1 \Vdash$ “$\dot{F} \subseteq 2^{\dot{x}}$ has the FIP” $\wedge\, \dot{y} \subseteq \dot{x}$. Since we assume $p \in G$, this would establish that $\dot{F}^G = \dot{F}'^G = F$ and $\dot{y}^G = \dot{y}'^G = y$; we have already assumed that $\dot{x}^G = x$. Thus, the lemma follows from identifying such names $\dot{F}$ and $\dot{y}$.

Let $\dot{y}^p = \{\langle q, \dot{z} \rangle \mid q \leq p \wedge \langle q, \dot{z} \rangle \in \dot{y}'\}$, so that $p \Vdash \dot{y}^p = \dot{y}'$. Let $K = \mathrm{sym}_{\mathcal{G}}(\dot{x}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{y}')$ and let $\dot{y} = \bigcup_{\pi \in K} \pi \dot{y}^p$, so that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{y})$ and $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$. Because $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{y}')$, we have that, as $\mathbb{P}$-names, $\dot{y}^p \subseteq \dot{y} \subseteq \dot{y}'$, and so $1 \Vdash \dot{y}^p \subseteq \dot{y} \subseteq \dot{y}'$. Since $p \Vdash \dot{y}^p = \dot{y}'$ and $1 \Vdash \dot{y}^p \subseteq \dot{y} \subseteq \dot{y}'$, we have that $p \Vdash \dot{y}^p = \dot{y} = \dot{y}'$.

To check that $1 \Vdash \dot{y} \subseteq \dot{x}$, we first prove $1 \Vdash \dot{y}^p \subseteq \dot{x}$. If $q \leq p$, recall that $p \Vdash \dot{y}^p = \dot{y}'$ and (by assumption) $p \Vdash \dot{y}' \subseteq \dot{x}$, so $q \Vdash \dot{y}^p = \dot{y}' \subseteq \dot{x}$. If $p, q \in \mathbb{P}$ are incompatible, then $q \Vdash \dot{y}^p = \emptyset \subseteq \dot{x}$. Thus, $1 \Vdash \dot{y}^p \subseteq \dot{x}$ as claimed. Because $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{x})$, for every $\pi \in K$ we then have that $1 \Vdash \pi\dot{y}^p \subseteq \dot{x}$, and so $1 \Vdash \dot{y} \subseteq \dot{x}$.

Similarly, let $\dot{F}^p = \{\langle q, \dot{z}\rangle \mid q \leq p \wedge \langle q, \dot{z}\rangle \in \dot{F}'\}$, let $K = \mathrm{sym}_{\mathcal{G}}(\dot{x}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{F}')$, and let $\dot{F} = \bigcup_{\pi \in K} \pi\dot{F}^p$. By similar reasoning as above, we can conclude that $p \Vdash \dot{F}^p = \dot{F} = \dot{F}'$.

To check that $1 \Vdash$ "$\dot{F} \subseteq 2^{\dot{x}}$ has the FIP," we again first prove that this is the case for $\dot{F}^p$. If $q \leq p$, then $q \leq p \Vdash \dot{F}^p = \dot{F}'$; if $p, q \in \mathbb{P}$ are incompatible, then $q \Vdash \dot{F}^p = \emptyset$. Thus,

$$1 \Vdash \text{“}\dot{F}^p = \check{\emptyset} \text{ has the FIP”} \vee \text{“}\dot{F}^p = \dot{F}' \text{ has the FIP.”}$$

Because $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{x}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{F}')$, for every $\pi \in K$ we have that

$$1 \Vdash \text{“}\pi\dot{F}^p = \check{\emptyset} \subseteq 2^{\dot{x}} \text{ has the FIP”} \vee \text{“}\pi\dot{F}^p = \dot{F}' \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

From this, we can conclude that

$$1 \Vdash \text{“}\dot{F} = \check{\emptyset} \subseteq 2^{\dot{x}} \text{ has the FIP”} \vee \text{“}\dot{F} = \dot{F}' \subseteq 2^{\dot{x}} \text{ has the FIP,”}$$

and thus

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

$\square$

Remark 4.1.2. Lemma 4.1.1 worked for names for filters because $\emptyset$ is trivially a filter, but the lemma will not generally work for names for ultrafilters. If N is a symmetric extension and $N \vDash$ "U is an ultrafilter on x," in general one can only identify $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$ and $d \in \mathbb{P}$ so that $d \Vdash$ "$\dot{U}$ is an ultrafilter on $\dot{x}$." The standard trick to "glue" these names for ultrafilters together, indexed by a maximal antichain of conditions, may no longer be a symmetric name.

Given $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ as above, we would again like to consider how to extend $\dot{F}$ by some minimal increment to match (1), in a manner that is stable under iteration to produce a name $\dot{U}$ that is forced to be an ultrafilter on $\dot{x}$. Note that if $\dot{w} \in \mathrm{HS}_{\mathcal{F}}$, then

$$\dot{w} = \bigcup_{\langle q, \dot{y} \rangle \in \dot{w}} \mathrm{Orb}(\langle q, \dot{y} \rangle, \mathrm{sym}_{\mathcal{G}}(\dot{w})),$$

which can again be expressed as a partition up to some repetition of orbits. We can draw the same conclusions as in the case for permutation models: these orbits are the building blocks for (hereditarily) symmetric names, and one can identify a name $\dot{f}$ for a wellordering of the $\mathrm{sym}_{\mathcal{G}}(\dot{w})$-orbits that partition $\dot{w}$, so that $\mathrm{sym}_{\mathcal{G}}(\dot{f}) = \mathrm{sym}_{\mathcal{G}}(\dot{w})$.

With remark 4.1.2 in mind, to extend $\dot{F}$ to a name $\dot{U}$ for an ultrafilter, we will first fix two conditions $d \leq p$, with the intention of reaching the conclusions that $p \Vdash_N$ BPI and $d \Vdash_N$ "$\dot{U} \supseteq \dot{F}$ is an ultrafilter on $\dot{x}$." In this case, extending $\dot{F}$ by some minimal increment involves selecting some condition $r \leq d$, some name $\dot{y}$ for a subset of $\dot{x}$, and some group $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, then extending to either

$$\dot{F} \cup \mathrm{Orb}(\langle r, \dot{y} \rangle, K) \quad \text{or} \quad \dot{F} \cup \mathrm{Orb}(\langle r, \dot{x} \setminus \dot{y} \rangle, K),$$

so that 1 forces that the FIP is still met.[8] We would again like to use the same group K at every stage of an iteration that terminates in a name $\dot{U}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, and so that

$$d \Vdash_{\mathrm{HS}_{\mathcal{F}}} \text{“}\dot{U} \supseteq \dot{F} \text{ is an ultrafilter on } \dot{x}.\text{”}$$

Doing so will involve extending to include a dense set of conditions r below d that force whether a given name $\dot{y}$ or its complement $\dot{x} \setminus \dot{y}$ appears in $\dot{U}$. Definition 4.1.3 expresses that the hypotheses in this sketch hold in a uniform way.

Definition 4.1.3. A symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ has the *filter extension property below* $p \in \mathbb{P}$ (over a ground model M) if there is a set $D \subseteq \mathbb{P}$ that is dense below p, so that for every $d \in D$, $\mathcal{F}$ has a base of subgroups $\mathcal{B}_d$, and so that the following holds for every $K \in \mathcal{B}_d$: for every $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ such that $K \preceq \mathrm{sym}(\dot{F}) \cap \mathrm{sym}(\dot{x})$ and

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

and for every $q \leq d$, there is an $r \leq q$ and a $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ so that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \text{ has the FIP.”}$$

When proving BPI from the filter extension property, we would like our proof to emphasize that definition 4.1.3 allows us to iterate the extension of names for filters in $\mathrm{HS}_{\mathcal{F}}$ by minimal symmetric increments. This again comes at the cost of lengthening a standard application of Zorn's lemma. Lemma 4.1.4 shows that the filter extension property implies the ability to extend a name for a filter $\dot{F} \in \mathrm{HS}_{\mathcal{F}}$ on a set $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$ to choose between an arbitrary subset $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$ and its complement in $\dot{x}$.

[8] Again, we mean “minimal” more figuratively than literally, as it may be possible to choose a smaller group K with this same property.

Lemma 4.1.4. *If $M \models$ ZFC, and if $d \in D \subseteq \mathbb{P}$ and $K \in \mathcal{B}_d \subseteq \mathcal{F}$ witness the filter extension property below $p \in \mathbb{P}$ (over M), for the symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, then for every $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ such that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$ and*

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

there is a name $\dot{F}_{\dot{y}} \supseteq \dot{F}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_{\dot{y}}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, so that

$$(1 \Vdash \text{“}\dot{F}_{\dot{y}} \subseteq 2^{\dot{x}} \text{ has the FIP”}) \wedge (d \Vdash \dot{y} \in \dot{F}_{\dot{y}} \vee \dot{x} \setminus \dot{y} \in \dot{F}_{\dot{y}}).$$

Proof. Fix $d \in D \subseteq \mathbb{P}$, $K \in \mathcal{F}$, and $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ as in the statement of the lemma. In the ground model $M \models$ ZFC, let $\{q_\alpha \mid \alpha < \beta\}$ denote a wellordering of the set $\{q \in \mathbb{P} \mid q \leq d\}$. Again in $M \models$ ZFC, inductively define the sequence $\dot{F}_\alpha$, for $\alpha \leq \beta$, as follows.

Let $\dot{F}_0 = \dot{F}$. For $\alpha \leq \beta$, assume inductively that for every $\gamma < \gamma' < \alpha$, we have $\dot{F}_\gamma \subseteq \dot{F}_{\gamma'}$, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_\gamma) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, and

$$1 \Vdash \text{“}\dot{F}_\gamma \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

At stages $\alpha + 1$, use filter extension property to choose $r \leq q_\alpha$ and $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ so that

$$1 \Vdash \text{“}F_\alpha \cup \mathrm{Orb}(\langle r, y' \rangle, K) \subseteq 2^{\dot{x}} \text{ has the FIP,”}$$

and set

$$\dot{F}_{\alpha+1} = F_\alpha \cup \mathrm{Orb}(\langle r, y' \rangle, K).$$

By the inductive hypothesis on $\dot{F}_\alpha$ and our application of the filter extension property, $\dot{F}_{\alpha+1}$ also meets the inductive hypothesis.

At limit stages α, set

$$\dot{F}_\alpha = \bigcup_{\gamma<\alpha} \dot{F}_\gamma.$$

By the inductive hypothesis for $\dot{F}_\gamma$ with $\gamma < \alpha$, particularly the fact that $\dot{F}_\gamma \subseteq \dot{F}_{\gamma'}$ if $\gamma < \gamma'$, we can see that $\dot{F}_\alpha$ also meets the inductive hypothesis.

Because we have added a dense set of conditions below d that decide between $\dot{y}$ and $\dot{x} \setminus \dot{y}$, the name $\dot{F}_{\dot{y}} = \dot{F}_\beta$ has the desired properties to satisfy the lemma. $\square$

Theorem 4.1.5. *If $M \vDash$ ZFC and $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ has the filter extension property below $p \in \mathbb{P}$ (over M), and N is the corresponding symmetric extension of M, then $p \Vdash_N$ BPI.*

Proof. Fix $\dot{F}, \dot{x} \in \mathrm{HS}_\mathcal{F}$ so that $1 \Vdash$ "$\dot{F} \subseteq 2^{\dot{x}}$ has the FIP" – lemma 4.1.1 shows that this restriction can be made without loss of generality. We will prove that p forces that $\dot{F}$ extends to an ultrafilter on $\dot{x}$; since $\dot{F}, \dot{x} \in \mathrm{HS}_\mathcal{F}$ are arbitrary, this shows that $p \Vdash_N$ BPI.

Fix a dense set $D \subseteq \mathbb{P}$, a $d \in D$, and a $K \preceq \mathrm{sym}_\mathcal{G}(\dot{F}) \cap \mathrm{sym}_\mathcal{G}(\dot{x})$ that witness the filter extension property. In the ground model $M \vDash$ ZFC, fix a wellordering $\{\dot{y}_\alpha \mid \alpha < \beta\}$ of the names in $\mathrm{HS}_\mathcal{F}$ that are forced by $1 \in \mathbb{P}$ to be subsets of $\dot{x}$ – again, lemma 4.1.1 shows that this restriction can be made without loss of generality. In a manner similar to the proof of lemma 4.1.4, one can inductively define a sequence $\dot{F}_\alpha$, for $\alpha \leq \beta$, with $\dot{F}_0 = \dot{F}$, and so that for every $\alpha \leq \beta$, $K \preceq \mathrm{sym}_\mathcal{G}(\dot{F}_\alpha) \cap \mathrm{sym}_\mathcal{G}(\dot{x})$,

$$(1 \Vdash \text{“}\dot{F}_\alpha \subseteq 2^{\dot{x}} \text{ has the FIP”}) \wedge (d \Vdash \dot{y}_\alpha \in \dot{F}_{\alpha+1} \vee \dot{x} \setminus \dot{y}_\alpha \in \dot{F}_{\alpha+1}),$$

and so that $\dot{F}_\lambda \subseteq \dot{F}_{\lambda'}$ if $\lambda < \lambda'$. Lemma 4.1.4 justifies the ability to extend from $\dot{F}_\alpha$ to $\dot{F}_{\alpha+1}$ in such a manner; at limit stages, unions are taken. We can then see that for $\dot{U} = \dot{F}_\beta$, we have that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, so $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, and furthermore that

$$d \Vdash_{\mathrm{HS}_{\mathcal{F}}} \dot{U} \supseteq \dot{F} \text{ is an ultrafilter on } \dot{x}.\text{''}$$

Because $D \subseteq \mathbb{P}$ is dense below p, we have that

$$p \Vdash \text{``}\dot{F} \text{ extends to an ultrafilter on } \dot{x}.\text{''}$$

□

Remark 4.1.6. One might wonder whether it is possible to prove theorem 4.1.5 while only assuming a ground model $M \models \mathrm{ZF} + \mathrm{BPI}$. This would not work; it is consistent that one can force with a poset $\mathbb{P}$ over a model $M \models \mathrm{ZF} + \mathrm{BPI}$ to add a filter that does not extend to any ultrafilter in the extension $M[G]$. Forcing with $\mathbb{P}$ can be expressed in terms of a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ in which $\mathcal{G}$ and $\mathcal{F}$ are trivial, and which trivially has the filter extension property. In [HKT13], for instance, the authors force over the Cohen model (a known model of $\mathrm{ZF} + \mathrm{BPI}$), and identify a filter on ω in the resulting extension that does not extend to an ultrafilter. The main obstacle that prevents a proof of theorem 4.1.5, using only BPI, is that there are infinitely many options to take $r \leq q$ that witnesses an instance of the filter extension property. This exceeds the ability of BPI to make finitely bounded choices.

For the same reason as the case for permutation models, a proof of BPI in a symmetric extension by Halpern's contradiction framework can be rephrased as a contradiction proof that filter extension property holds. With the same approach as in theorem 3.1.11, we can prove that the filter extension property is necessary for BPI in symmetric extensions, so that this connection is inevitable.

Note that the filter extension property below $p \in \mathbb{P}$ expresses a strong uniformity. For $d \in D$

and $K \in \mathcal{B}_d$, so long as $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, *every* filter extension of $\dot{F}$ by K-orbits can continue to be extended by K-orbits. We are never made to use a less restrictive group $K' \preceq K$. This can again be seen as a consequence of Blass' uniform classification of BPI in models of SVC. For the reader's convenience, we repeat the definition of SVC below, along with its relationships to AC and BPI.

Definition 4.1.7 (Blass, [Bla79])**.** Let SVC (*small violations of choice*) be the axiom that there is a set S so that, for every set a, there is an ordinal α and a function from a subset of $S \times \alpha$ onto a.[9]

Theorem 4.1.8 (Blass, [Bla79])**.** *A model of* ZF *satisfies* SVC *if and only if some generic extension is a model of* ZFC.

Definition 4.1.9. For a set S, the *fine filter* on $S^{<\omega}$ is the filter generated by the sets $\{p \in S^{<\omega} | s \in \mathrm{rng}(p)\}$, for $s \in S$. An ultrafilter on $S^{<\omega}$ is *fine* if it extends the fine filter.

Theorem 4.1.10 (Blass, [Bla79])**.** *If* $N \vDash \mathrm{ZF(A)}$ *and* S *witnesses* SVC*, then* $N \vDash \mathrm{BPI}$ *if and only if there is a fine ultrafilter on* $S^{<\omega}$*.*

Recall that the above theorem is listed as theorem 3.1.10 in chapter 3, where we sketch Blass' proof with special attention to certain definability features. These features will be useful here as well.

Theorem 4.1.11 (Blass, [Bla79])**.** *Symmetric extensions satisfy* SVC.[10]

In [Bla79], theorem 4.1.11 is proven using the theorem that a symmetric extension N, over a ground model M, can be expressed as the definable closure $M(x)$, for some $x \in M[G]$.[11] It will be useful for us to sketch a new proof, in terms of the orbits $\mathrm{Orb}(\langle p, \dot{x} \rangle, \mathcal{G})$, which we

[9] In [Bla79], the original statement is that there is a function from the *entire* set $S \times \alpha$ onto a. Blass notes that this is equivalent to the form stated here, since all points outside a subset of $S \times \alpha$ can be mapped to a single element.

[10] Recall that we take symmetric extensions to be based on set-sized symmetric systems and ground models of ZFC; this theorem fails for symmetric extensions that use proper class forcings in a nontrivial way.

[11] This was proven in [Gri75], and was recently reproven in terms of a simpler, unified framework in [KS26].

model after the use of the orbits $\mathrm{Orb}(y_{H,K}, \mathcal{G})$ in Blass' proof of SVC in permutation models (see our proof sketch of theorem 3.1.6).

In the case of permutation models, the orbit $\mathrm{Orb}(y_{H,K}, \mathcal{G})$, with $\mathrm{sym}_{\mathcal{G}}(y_{H,K}) = H$, is isomorphic to the cosets $\mathcal{G}/H$. There are then only set-many such orbits, up to isomorphism. In the proof sketch of theorem 3.1.6, this allows us to define S as the union of one orbit of each isomorphism type.

In the case of symmetric extensions, however, it may not be the case that the interpretation $\mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})^G$ is isomorphic to the cosets $\mathcal{G}/\mathrm{sym}_{\mathcal{G}}(\dot{y})$. For some $\pi \in \mathcal{G}$, it could be the case that $\pi\dot{y}^G$ is not forced to appear in $\mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})^G$ at all; for distinct $\pi\dot{y}$ and $\tau\dot{y}$, it may be the case that $\pi\dot{y}^G = \tau\dot{y}^G$. Thus, the interpretation of the orbit $\mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$ may look like some collapsed version of the cosets $\mathcal{G}/\mathrm{sym}_{\mathcal{G}}(\dot{y})$, instead of being simply isomorphic.

What is needed to model SVC is that, up to isomorphism, there are only set-many ways to collapse orbits from the set-many possible quotients $G/\mathrm{sym}(\dot{y})$, as these can then be gathered in a name $\dot{S}$ to witness SVC. We develop this formally below, and start by defining a strong notion that the interpretation of two $\mathcal{G}$-orbits collapse in an isomorphic way.

Definition 4.1.12. Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ be a symmetric system, let $p \in \mathbb{P}$, and let $\dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$. Let $\dot{f} \in \mathrm{HS}_{\mathcal{F}}$ be the name for a relation on $\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}) \times \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$, defined by the property that $\mathrm{sym}_{\mathcal{G}}(\dot{f}) = \mathcal{G}$, and that for every $\pi \in \mathcal{G}$,

$$\pi p \Vdash \dot{f}(\pi\dot{x}) = \pi\dot{y}.$$

We say that the names $\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}), \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G}) \in \mathrm{HS}_{\mathcal{F}}$ are *$\mathcal{G}$-orbit equivalent*, written

$$\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}) \sim_{\mathcal{G}} \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G}),$$

if it is the case that

$$1 \Vdash \text{“}\dot{f} : \mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}) \to \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G}) \text{ is a bijection.”}$$

We now define tools to prove that there can only be set-many $\mathcal{G}$-orbit equivalence classes.

Definition 4.1.13. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, a condition $p \in \mathbb{P}$, and a $\mathbb{P}$-name $\dot{x}$, define the equivalence relation $\sim_{p,\dot{x}}$ on $\mathcal{G}$ by

$$\pi \sim_{p,\dot{x}} \tau \iff p \Vdash \pi\dot{x} = \tau\dot{x}.$$

Definition 4.1.14. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ and a $\mathbb{P}$-name $\dot{x}$, the $(\mathbb{P}, \mathcal{G})$-*profile* of $\dot{x}$ is the function $\mathcal{P}_{\dot{x}} : \mathbb{P} \to 2^{\mathcal{G}}$, so that for every $p \in \mathbb{P}$, $\mathcal{P}_{\dot{x}}(p)$ is the set of $\sim_{p,\dot{x}}$-classes of $\mathcal{G}$.

Lemma 4.1.15. *For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ and a condition $p \in \mathbb{P}$, if $\dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ have the same $(\mathbb{P}, \mathcal{G})$-profile, then $\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}) \sim_{\mathcal{G}} \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$.*

Proof. Let $\dot{f}$ be as in definition 4.1.12. We can observe that 1 forces the domain and codomain of $\dot{f}$ (as a relation), respectively, to be $\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G})$ and $\mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$. To see that 1 forces $\dot{f}$ to be a well-defined function, suppose that there is a $q \in \mathbb{P}$ and $\pi, \tau \in \mathcal{G}$ so that

$$q \Vdash \pi\dot{x} = \tau\dot{x}.$$

Since $\mathcal{P}_{\dot{x}}(q) = \mathcal{P}_{\dot{y}}(q)$, we then also have that

$$q \Vdash \pi\dot{y} = \tau\dot{y},$$

and so 1 forces $\dot{f}$ to be a name for a well-defined function.

We can now check that 1 forces $\dot{f}$ to be injective. This is proven in the same way as above: if there is a $q \in \mathbb{P}$ and $\pi, \tau \in \mathcal{G}$ so that

$$q \Vdash \pi\dot{y} = \tau\dot{y},$$

then again since $\mathcal{P}_{\dot{x}}(q) = \mathcal{P}_{\dot{y}}(q)$, we can conclude that

$$q \Vdash \pi\dot{x} = \tau\dot{x}.$$

We conclude the proof by showing that 1 forces $\dot{f}$ to be surjective. If $q \Vdash \pi\dot{y} \in \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$, in general there must be some $\tau \in \mathcal{G}$ so that $q \leq \tau p$ – in which case $q \Vdash \tau\dot{y} \in \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G})$ – and that $q \Vdash \pi\dot{y} = \tau\dot{y}$. Since $\mathcal{P}_{\dot{x}}(q) = \mathcal{P}_{\dot{y}}(q)$, we also have that $q \Vdash \pi\dot{x} = \tau\dot{x}$, and since $q \leq \tau p$, we have that $q \Vdash \pi\dot{x} = \tau\dot{x} \in \mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G})$. Then, $q \Vdash \dot{f}(\pi\dot{x}) = \pi\dot{y}$, and so $q \Vdash \pi\dot{y} \in \mathrm{rng}(\dot{f})$. We conclude that 1 forces that $\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G})$ is the range of $\dot{f}$, so 1 forces that $\dot{f}$ is surjective, and thus bijective, ensuring that

$$\mathrm{Orb}(\langle p, \dot{x}\rangle, \mathcal{G}) \sim_{\mathcal{G}} \mathrm{Orb}(\langle p, \dot{y}\rangle, \mathcal{G}).$$

□

Lemma 4.1.16. *For a ground model $M \vDash \mathrm{ZFC}$ and a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, there is a set in M with one representative from each $\mathcal{G}$-orbit equivalence class.*

Proof. For every $p \in \mathbb{P}$, by lemma 4.1.15 there are at most $|(2^{\mathcal{G}})^{\mathbb{P}}|$ many $\mathcal{G}$-orbit equivalence classes of the form $\mathrm{Orb}(\langle p, \cdot\rangle, \mathcal{G})$. Thus, there are at most $|\mathbb{P} \times (2^{\mathcal{G}})^{\mathbb{P}}|$ many $\mathcal{G}$-orbit equivalence classes. Choice can be used in M to produce a set with an element of each class. □

We can now sketch a proof that SVC holds in symmetric extensions.

Proof sketch of theorem 4.1.11. Fix a ground model $M \vDash \mathrm{ZCF}$ and a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$. Let $S_0 \in M$ be a set with one representative of every $\mathcal{G}$-orbit equivalence class in $\mathrm{HS}_{\mathcal{F}}$. Ensure further that the representatives of these classes are forced by 1 to be disjoint (for instance, by tagging with distinct ordinals as in a disjoint union), and define the $\mathbb{P}$-name $\dot{S} = \bigcup S_0$. Then $\mathrm{sym}_{\mathcal{G}}(\dot{S}) = \mathcal{G}$, $\dot{S} \in \mathrm{HS}_{\mathcal{F}}$, and we will show that 1 forces $\dot{S}$ to witness SVC.

For every $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$, we have the identity

$$\dot{x} = \bigcup_{\langle p, \dot{z}\rangle \in \dot{x}} \mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{sym}_{\mathcal{G}}(\dot{x})),$$

which is a partition of $\dot{x}$ up to the repetition of orbits. By the construction of $\dot{S}$, for every $\langle p, \dot{z}\rangle \in \dot{x}$, there is a subset of $\dot{S}$ that is $\mathcal{G}$-orbit equivalent to $\mathrm{Orb}(\langle p, \dot{z}\rangle, \mathcal{G})$ – call this $\mathrm{Orb}(\langle p, \dot{w}\rangle, \mathcal{G}) \subseteq \dot{S}$. By the definition of $\mathcal{G}$-orbit equivalence, there is a name $\dot{f}$ for a bijection from $\mathrm{Orb}(\langle p, \dot{w}\rangle, \mathcal{G})$ to $\mathrm{Orb}(\langle p, \dot{z}\rangle, \mathcal{G})$, as in definition 4.1.12, with $\mathrm{sym}_{\mathcal{G}}(\dot{f}) = \mathcal{G}$. Fix a subset $\dot{g} \subseteq \dot{f}$ with $\mathrm{sym}_{\mathcal{G}}(\dot{g}) = \mathrm{sym}_{\mathcal{G}}(\dot{x}) = K$, so that for every $\pi \in K$,

$$\pi p \Vdash \dot{g}(\pi \dot{w}) = \pi \dot{z},$$

and so that 1 forces that $\dot{g}$ is a bijection between $\mathrm{Orb}(\langle p, \dot{w}\rangle, K) \subseteq \dot{S}$ and $\mathrm{Orb}(\langle p, \dot{z}\rangle, K)$. Letting $\alpha = |\{\mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{sym}_{\mathcal{G}}(\dot{x})) \mid \langle p, \dot{z}\rangle \in \dot{x}\}|$, it is straigthforward to define a name $\dot{s} \subseteq \dot{S}$ and a name $\dot{b}$ that 1 forces to be a bijection between a subset of $\dot{s} \times \check{\alpha}$ and $\dot{x}$, with $\mathrm{sym}_{\mathcal{G}}(\dot{x}) = \mathrm{sym}_{\mathcal{G}}(\dot{s}) = \mathrm{sym}_{\mathcal{G}}(\dot{b}) = K$; this is done in the same manner as in the proof of theorem 3.1.6. Thus 1 forces that $\dot{S}$ witnesses SVC. $\square$

Theorem 4.1.17. *For a ground model $M \vDash \mathrm{ZFC}$, a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$, and the corresponding symmetric extension $N \supseteq M$, the following are equivalent:*

(i) $p \Vdash_N \mathrm{BPI}$,

(ii) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the filter extension property below* p.

Proof. Part $(ii) \Rightarrow (i)$ is given in theorem 4.1.5. Given theorem 4.1.11, that symmetric extensions also model SVC, the proof of $(i) \Rightarrow (ii)$ is the same argument as our proof of theorem 3.1.11$(i) \Rightarrow (ii)$. Let $p \vDash_N$ BPI, and let 1 force that $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$ meet the hypothesis of the filter extension property. Let $\dot{S} \in \mathrm{HS}_{\mathcal{F}}$ be a witness of SVC in N, and let $\dot{b} \in \mathrm{HS}_{\mathcal{F}}$ be a name for a surjection from a subset of $\dot{S} \times \check{\alpha}$ onto the Boolean algebra $2^{\dot{x}}/\dot{F}$, with $\mathrm{sym}_{\mathcal{G}}(\dot{f}) = \mathrm{sym}_{\mathcal{G}}(2^{\dot{x}}/\dot{F})$.[12] By the proof of theorem 4.1.11, we are justified in assuming these symmetry properties for $\dot{S}$ and $\dot{f}$.

Since $p \Vdash_N$ BPI, let $D \subseteq \mathbb{P}$ be dense below p, so that each $d \in D$ forces that $\dot{U}'_d \in \mathrm{HS}_{\mathcal{F}}$ is a name for an ultrafilter witnessing 4.1.10 (the same as theorem 3.1.10 in chapter 3) in N. By our proof sketch of theorem 3.1.10, and since SVC holds in N, the first-order formula $\varphi(w_0, w_1, w_2, w_3)$ from our proof of theorem 3.1.11 part $(i) \Rightarrow (ii)$ can be used to define ultrafilters extending filters in N. In other words, for the name

$$\dot{U} = \{\langle q, \dot{y} \rangle | q \Vdash \varphi(\dot{y}, \dot{f}, \dot{S}, \dot{U}'_d)\},$$

since $d \leq 1$ forces that $\dot{f}, \dot{S}$ and $\dot{U}'_d$ play their respective roles in the definition of $\dot{U}$ by φ, we have

$$d \Vdash_N \text{“}\dot{U} \supseteq \dot{F} \text{ is an ultrafilter on } \dot{x}.\text{”}$$

By the symmetry lemma 2.2.4, we have that

$$\mathrm{sym}_{\mathcal{G}}(\dot{f}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{S}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{U}'_d) \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U}).$$

[12] The correct formalizations of these names should be clear.

Since $\mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x}) \preceq \mathrm{sym}_{\mathcal{G}}(2^{\dot{x}}/\dot{F}) = \mathrm{sym}_{\mathcal{G}}(f)$ and $\mathrm{sym}_{\mathcal{G}}(\dot{S}) = \mathcal{G}$, it follows that

$$\mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{U}'_d) \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U}).$$

As in the proof of lemma 4.1.1, since d forces $\dot{U}$ to be the name for a filter, we can modify $\dot{U}$ further to ensure that $1 \Vdash \dot{U} \subseteq 2^{\dot{x}}$ has the FIP," without changing $\mathrm{sym}_{\mathcal{G}}(\dot{U})$, and while retaining the fact that $d \Vdash_N$ "$\dot{U} \supseteq \dot{F}$ is an ultrafilter on $\dot{x}$."

Now, for $d \in D$, and letting $K_d = \mathrm{sym}_{\mathcal{G}}(\dot{U}'_d)$, we will show that the base of subgroups $\mathcal{B}_d = \{H \cap K_d \mid H \in \mathcal{F}\}$ of $\mathcal{F}$ witnesses the filter extension property. Let $H \in \mathcal{F}$, and consider $\dot{F}, \dot{x}, \dot{y}$ meeting the hypotheses of the filter extension property, with $H \cap K_d \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$. Let $\dot{U}$ be the name, defined by φ, that d forces to be an ultrafilter on $\dot{x}$ extending $\dot{F}$, and that 1 forces to have the FIP. For every $q \leq d$, there is an $r \leq q$ and a $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ so that

$$\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, \mathrm{sym}_{\mathcal{G}}(\dot{U})) \subseteq \dot{U}.$$

Because $H \cap K_d \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{U}'_d) \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, in particular

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle q, y' \rangle, H \cap K_d) \text{ has the FIP.”}$$

Thus the set $D \subseteq \mathbb{P}$ that is dense below p, and the bases $\mathcal{B}_d$, for $d \in D$, witness the filter extension property for $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$. □

4.2 BPI in the Generalized Cohen Model $N(I, Q)$

Having stated the filter extension property in the context of symmetric extensions, and proven its relationship to BPI, the next step in generalizing the material from chapter 3 would be to generalize the Ramsey property. Our first step in addressing this problem is to instead focus on giving a new, direct proof of BPI in the generalized Cohen model $N(I,Q)$.[13] In this section, we adapt Harrington's forcing proof of the Halpern-Läuchli theorem to give a search-and-shift proof of the filter extension property for the symmetric system $\langle \mathbb{P}(I,Q), \mathcal{G}, \mathcal{F} \rangle$, which is used to define $N(I,Q)$.

Recall that we gave a search-and-shift proof of the filter extension property for permutation models in section 3.2, and that figure 3.1 faithfully captures this method. Recall also that our main theorems 3.4.4 and 3.4.5 from chapter 3 express, in various ways, that BPI holds in a permutation model if and only if it can be proven by extending filters to ultrafilters using search-and-shift arguments. In fact, this style of proof was inspired by Harrington's proof of the Halpern-Läuchli theorem, and the manner in which we adapt Harrington's proof in the present section.[14]

In Harrington's proof, one searches through a $\mathbb{P}(\oplus_{i<d} T_i, \theta)$-name $\dot{U}$ for an ultrafilter on ω, and identifies "shifted" copies of certain substructures of $\dot{U}$, whose compatibility (in terms of the FIP from $\dot{U}$) is used to prove the Halpern-Läuchli theorem for the level product $\otimes_{i<d} T_i$.[15] The search through $\dot{U}$ is ensured to be successful by a Ramsey-theoretic argument, given that one takes θ to be sufficiently large, so that θ-many mutually generic branches through the each of the trees T_i are added to the extension. For more details, see the full exposition given in chapter 6 of [TF95].

The similarity to our search-and-shift proof in section 3.2 is apparent: there, we also searched through an ultrafilter U (on a particular set x, extending a particular filter F) in an outer model $M \vDash$ ZFCA, to identify shifted copies of substructures whose compatibility we checked

[13]Recall from section 2.5 that the generalized Cohen model, denoted $N(I,Q)$ for some index set I and poset Q, primarily introduces a Dedekind-finite set of mutually Q-generic filters, which is defined with respect to an index set I in the ground model.

[14]In the timeline of our work, we proved the results in this section before any other material in this thesis – along with early versions of some of the theorems in section 4.1.

[15]Here we take $\oplus$ to be the lottery sum, as in [Ham00].

using the FIP in U. In subsection 4.2.1, we show that Harrington's proof can be adapted more closely to handle the underlying forcing in the symmetric system $\langle \mathbb{P}(\theta, Q), \mathcal{G}, \mathcal{F} \rangle$ that is used to present the model $N(I, Q)$ – when θ is assumed to be large enough to support the Ramsey-theoretic part of Harrington's proof. Carrying this out will result in the desired search-and-shift proof of the filter extension property for $\langle \mathbb{P}(I, Q), \mathcal{G}, \mathcal{F} \rangle$, which refers to $\mathbb{P}(\theta, Q)$-names for ultrafilters in the outer model $M[G] \vDash \mathrm{ZFC}$. In subsection 4.2.2, we show that the index set I in $N(I, Q)$ can be assumed to be arbitrarily large, without loss of generality, from which we conclude that our adaptation of Harrington's proof addresses the general case of BPI in the model $N(I, Q)$.

More precisely, in both subsections, we will work to reprove the following theorem and corollary below. In [HL71], a contradiction proof was given for that BPI holds in the Cohen model; in [Ste23], it was shown that this contradiction proof can be modified to prove BPI in the generalized Cohen model $N(I, Q)$. We slightly reformat these results below to be stated in terms of the filter extension property:

Theorem 4.2.1 (Hapern, Lévy, [HL71], Stefanović, [Ste23]). *Let $\langle \mathrm{P}(I, Q), \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system for a Generalized Cohen model and let $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$, with $\mathrm{supp}(\dot{F}) = F$, $\mathrm{supp}(\dot{x}) = X$, and $K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$, such that*

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x}.$$

Then, for every $q \in \mathbb{P}(I, Q)$, there is an $r \leq q$ and a $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ so that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \text{ has the FIP.”}$$

Corollary 4.2.2 (Halpern, Lévy, [HL71], Stefanović, [Ste23]). *The symmetric system $\langle \mathbb{P}(I, Q), \mathcal{G}, \mathcal{F} \rangle$ has the filter extension property below 1, and thus $N(I, Q) \vDash \mathrm{BPI}$.*

Proof. We will show that, for every $d \in D = \mathbb{P}(I, Q)$, the base of subgroups $\mathcal{B}_d = \{\mathrm{fix}_{\mathcal{G}}(E) \mid E \in [I]^{<\omega}\}$ of $\mathcal{F}$ witnesses the filter extension property.

To see why, let $K = \mathrm{fix}_{\mathcal{G}}(E)$, let $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$ and

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

and let $q \leq d \in \mathbb{P}(I, Q)$ both be arbitrary. By theorem 4.2.1, for $\mathrm{supp}(\dot{F}) = F$ and $\mathrm{supp}(\dot{x}) = X$, there is an $r \leq q$ and $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ for which

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, \mathrm{fix}_{\mathcal{G}}(F \cup X)) \text{ has the FIP.”}$$

Because $K = \mathrm{fix}_{\mathcal{G}}(E) \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, by lemma 2.5.9 we can conclude that $F, X \subseteq E$, so $F \cup X \subseteq E$, and thus that $K = \mathrm{fix}_{\mathcal{G}}(E) \preceq \mathrm{fix}_{\mathcal{G}}(F \cup X)$. We can then conclude

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \text{ has the FIP.”}$$

□

4.2.1 Adapting Harrington's Proof

In Harrington's proof, given a level product $\otimes_{i<d} T$ and a coloring $c : \otimes_{i<d} T \to \{\text{green}, \text{ red}\}$, one searches through a $\mathbb{P}(\theta, T)$-name $\dot{U}$ for an ultrafilter on ω, and takes θ large enough to identify a node $\vec{t} \in \otimes_{i<d} T$ and a color $c' \in \{\text{green}, \text{ red}\}$ that are regulated by $\dot{U}$ to witness the Halpern-Läuchli theorem for $\otimes_{i<d} T$ and c.[16]

Suppose that $\dot{F}, \dot{x}$, and $\dot{y}$ meet the hypotheses of the filter extension property. The main

[16]This represents the case of the Halpern-Läuchli theorem in which $T_i = T$ for every tree in the level product $\otimes_{i<d} T_i$. See [TF95] for more details.

idea of this subsection is that one can search through a $\mathbb{P}(\theta, Q)$-name $\dot{U}$ for an ultrafilter on $\dot{x}$ that extends $\dot{F}$, and take θ large enough to identify a condition $r \leq q$ in $\mathbb{P}(\theta, Q)$ and a $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$, so that $\dot{U}$ can regulate that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, \mathrm{fix}_{\mathcal{G}}(F \cup X)) \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

This is made precise in our proof of theorem 4.2.11. For the rest of the subsection, we will write θ instead of I, to indicate that we wish to take $I = \theta$ to be sufficiently large. This also matches the notation from the exposition of Harrington's proof in chapter 6 of [TF95]; we have written our main proof in style similar to the exposition from [TF95], for ease of comparison between our proof and Harrington's.

The lemmas and definitions preceding theorem 4.2.11 establish several basic combinatorial features that are needed in our adaptation of Harrington's proof. Lemma 4.2.3, definition 4.2.4, and lemma 4.2.5 can be seen to be simple modifications made to the exposition preceding Harrington's proof in [TF95]. Note that our definition of a condition "type" differs slightly from theirs, and that our lemma 4.2.5 is essentially their lemma 6.8. See Appendix A for the proof of lemma 4.2.5 as a corollary of their lemma 6.8.

Lemma 4.2.3. *For every pair of cardinals κ, λ, there is a cardinal θ so that for every $d \in \omega$, and every function of the form*

$$c : \theta^d \to \kappa,$$

there are disjoint sets $\mathcal{H}_0, ..., \mathcal{H}_{d-1} \subseteq \theta$ of cardinality λ with f is constant on $\prod_{i<d} \mathcal{H}_i$.

Proof. This is essentially due to the Erdős-Rado theorem, as stated in line 17.32 of [EMHR11]. On page 280 of [Tod07], the cardinals Θ_d are defined, which satisfy the lemma for fixed d, in the case that $\kappa = \lambda = \omega$. It is noted in the remark on page 52 of [TF95] that the Erdős-Rado theorem implies that $\Theta_d \leq \beth_{d-1}^{+}$. This idea is then applied in theorem 11 of [Ste23] to prove that there is a θ_d satisfying the lemma for arbitrary κ, λ and fixed d; this is a pidgeonhole proof, by a finite induction on d that relies on the fact that powersets grow in cardinality. Taking $\theta = \bigcup_{d<\omega} \theta_d$ then satisfies the lemma. $\square$

Definition 4.2.4. Two conditions $p, q \in \mathbb{P}(\theta, Q)$ are said to be of the same *type* if there is an order-preserving map π from $\mathrm{dom}(p)$ to $\mathrm{dom}(q)$, so that $p(\xi) = q(e(\xi))$. We use the term *type* to refer to an equivalence class in $\mathbb{P}(\theta, Q)$ under this relation.

Lemma 4.2.5. *For every type t of a condition in $\mathbb{P}(\theta, Q)$ and integers m and d, there is an integer $M = M(t, m, d)$ such that for every set $\{p_{\vec{\alpha}} \mid \vec{\alpha} \in M^d\}$ of elements of $\mathbb{P}(\theta, Q)$ of type t, there exist $H_i \subseteq M$ (for $i < d$) of size m, so that the conditions in $\{p_{\vec{\alpha}} \mid \vec{\alpha} \in \prod_{i<d} H_i\}$ are pairwise compatible.*

Proof. See Appendix A, where this is derived from the analogous lemma 6.8 of [TF95]. $\square$

When choosing an $r \leq q$ to witness the filter extension property (in the form of theorem 4.2.1), one can strengthen r to refine which images of $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ can mutually appear in an interpretation of $\mathrm{Orb}(\langle r, y' \rangle, K)$. If $r(i) \perp r(j)$ for some $i, j \in \mathrm{supp}(r)$, then for $\pi = (i\ j)$, r and πr are incompatible and $y', \pi y'$ will never simultaneously appear in an interpretation of $\mathrm{Orb}(\langle r, y' \rangle, K)$. We introduce the definition below to capture this important property.

Definition 4.2.6. A condition $p \in \mathbb{P}(I, Q)$ *distinguishes its support* if for every distinct $i, j \in \mathrm{supp}(p)$, $p(i) \perp p(j)$.

Taking conditions to be of the same type and to distinguish their supports will also help manage the calculation of certain restrictions and images of conditions that are relevant to our proof; the following lemmas express these properties.

Lemma 4.2.7. *Let $\pi_0, ..., \pi_n \in \mathcal{G}$; if $r \in \mathbb{P}(I, Q)$ distinguishes its support, and if $\pi_0 r, ..., \pi_n r$ are compatible, then the union $\bigcup_{i \leq n} \pi_i r$ is a well-defined condition in $\mathbb{P}(I, Q)$.*

Proof. If each r distinguishes its support, then $\mathrm{rng}(r)$ is a finite antichain A in Q; by the definition of the action of $\mathcal{G}$ on $\mathbb{P}(\theta, Q)$, we also have that $\mathrm{rng}(\pi_0 r) = ... = \mathrm{rng}(\pi_n r) = A \subseteq Q$. Since the conditions $\pi_0 r, ..., \pi_n r$ are compatible, for every $i, j \leq n$ and every $k \in \mathrm{supp}(\pi_i r) \cap \mathrm{supp}(\pi_j r)$, it must be the case that $r_i(k) \in A$ and $r_j(k) \in A$ are compatible in Q. Since A is an antichain, this can only happen if $r_i(k) = r_j(k)$. For this, the union $\bigcup_{i\leq n} r_i$ is well-defined as a finite function from $\bigcup_{i\leq n} \mathrm{supp}(\pi_i r)$ to Q. $\square$

Corollary 4.2.8. *If $r_0, ..., r_n \in \mathbb{P}(\theta, Q)$ are compatible conditions of the same type, and that distinguish their supports, then the union $\bigcup_{i\leq n} r_i$ is a well-defined condition in $\mathbb{P}(\theta, Q)$.*

Proof. If $r_0, ..., r_n$ are of the same type, then there are (order-preserving) $\pi_1, ..., \pi_n \in \mathcal{G}$ so that $\pi_i r_0 = r_i$, for $1 \leq i \leq n$. Since r_0 distinguishes its support, and the conditions $r_0, \pi_1 r_0, ..., \pi_n r_0$ are mutually compatible, lemma 4.2.7 can be applied. $\square$

Lemma 4.2.9. *Let $r \in \mathbb{P}(\theta, Q)$, $K \in \mathcal{F}$, let $\pi_0, ..., \pi_n \in K$, and let $r_0 \leq \pi_0 r, ..., r_n \leq \pi_n r$ be compatible conditions of the same type, that distinguish their supports, and so that $r_i \restriction_{\mathrm{supp}(\pi_i r)} = \pi_i r$ for every $i \leq n$. Then $\bigcup_{i\leq n} r_i$ restricts to $\bigcup_{i\leq n} \pi_i r$ over $\mathrm{supp}(\bigcup_{i\leq n} \pi_i r)$.*

Proof. By our proof of the previous lemma, for every $i, j \leq n$ and $k \in \mathrm{supp}(r_i) \cap \mathrm{supp}(r_j)$, we have that $r_i(k) = r_j(k)$. In particular, since $r_j \restriction_{\mathrm{supp}(\pi_j r)} = \pi_j r$, for every $i, j \leq n$ and $k \in \mathrm{supp}(r_i) \cap \mathrm{supp}(\pi_j r)$, we have that $r_i(k) = \pi_j r(k)$. From this we can conclude that $\bigcup_{i\leq n} \pi_i r$ is well-defined, and that for every $k \in \mathrm{supp}(\bigcup_{i\leq n} \pi_i r)$, $(\bigcup_{i\leq n} r_i)(k) = (\bigcup_{i\leq n} \pi_i r)(k)$. $\square$

Lemma 4.2.10. *Suppose that $r \in \mathbb{P}(\theta, Q)$ distinguishes its support, and that for $\delta \in [\mathcal{G}]^{<\omega}$, the conditions in $\{\pi r \mid \pi \in \delta\}$ are mutually compatible. Then for $i \in \mathrm{supp}(r)$, the sets*

$$J_i = \{\pi(i) \mid \pi \in \delta\} \subseteq \theta$$

are mutually disjoint.

Proof. For each $\pi(i) \in J_i$, $\pi r(\pi(i)) = r(i)$ – in this sense, J_i collects the indices over which the conditions πr, for $\pi \in \delta$, copy the output of $r(i)$. Thus, if $k \in J_i \cap J_j$, with $i \neq j$, then for some $\pi, \tau \in \delta$ we have that $\pi r(k) = r(i)$ and $\tau r(k) = r(j)$. Because πr and τr are compatible, it must be the case that $\pi r(k) = r(i)$ and $\tau r(k) = r(j)$ are compatible in Q. This contradicts the fact that r distinguishes its support, and so $J_i \cap J_j = \emptyset$. □

We now prove the main theorem of this section. Recall that for ease of comparison, the language and formatting of our proof are modeled after the exposition of Harrington's proof from [TF95].

Theorem 4.2.11 (Halpern, Lévy, [HL71], Stefanović, [Ste23])**.** *Let Q be a poset, and let θ be large enough to satisfy lemma 4.2.3 with $\kappa = |Q|$. Then theorem 4.1 holds for $\langle \mathbb{P}(\theta, Q), \mathcal{G}, \mathcal{F} \rangle$.*

Proof. Let $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$, $K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$, and $q \in \mathbb{P}(\theta, Q)$ be as in the statement of theorem 4.1, with $F, X, Y \in [\theta]^{<\omega}$ the supports of $\dot{F}, \dot{x}$, and $\dot{y}$. Without loss of generality, we may assume that $\mathrm{supp}(q) \subseteq F \cup X \cup Y$. To see why, note that for $q' = q \restriction_{F \cup X \cup Y}$, our proof will produce witnesses $r' \leq q'$ and $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$, with $\mathrm{supp}(r') = F \cup X \cup Y$, so that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r', y' \rangle, K) \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

In this case, $r = r' \cup p$ is well-defined, and

$$1 \Vdash \mathrm{Orb}(\langle r, y' \rangle, K) \subseteq \mathrm{Orb}(\langle r', y' \rangle, K).$$

As such, $r \leq q$ and

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

We can thus assume without loss of generality that $\mathrm{supp}(q) \subseteq F \cup X \cup Y$, and produce witnesses $r \leq q$ and $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ to theorem 4.1, with $\mathrm{supp}(r) = F \cup X \cup Y$. This will simplify the calculation of supports throughout the proof.

For $i \in Y \setminus (F \cup X)$, let $\theta_i \subseteq \theta \setminus (F \cup X)$ be pairwise disjoint sets of cardinality θ. Recall from definition 2.5.8 that for

$$\vec{\alpha} = \{\langle i, \alpha_i \rangle | i \in Y \setminus (F \cup X)\} \in \prod_{i \in Y \setminus (F \cup X)} \theta_i$$

we define

$$\pi_{\vec{\alpha}} = \prod_{i \in Y \setminus (F \cup X)} (i \ \alpha_i) \in K = \mathrm{fix}_{\mathcal{G}}(F \cup X).$$

The membership $\pi_{\vec{\alpha}} \in K$ follows from the fact that each transposition $(i \ \alpha_i)$ cycles $i \in Y \setminus (F \cup X)$ with $\alpha_i \in \theta_i \subseteq \theta \setminus (F \cup X)$, and thus fixes $F \cup X$.

Let $\dot{U}$ be a (not necessarily symmetric) $\mathbb{P}(\theta, Q)$-name for an ultrafilter on $\dot{x}$ extending $\dot{F}$. For every $\vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} \theta_i$, either

1. $\pi_{\vec{\alpha}} q \Vdash \pi_{\vec{\alpha}}(\dot{x} \setminus \dot{y}) \in \dot{U}$, or

2. there is some $r_{\vec{\alpha}} \leq q$ so that $\pi_{\vec{\alpha}} r_{\vec{\alpha}} \Vdash \pi_{\vec{\alpha}} \dot{y} \in \dot{U}$.

In the second case, extend $r_{\vec{\alpha}}$ if necessary so that $F \cup X \cup Y \subseteq \mathrm{supp}(r_{\vec{\alpha}})$ and so that $r_{\vec{\alpha}}$ distinguishes its support. Recall that we assume θ to be large enough to satisfy lemma 4.2.3 for $\kappa = |Q|$. Then, for $i \in Y \setminus (F \cup X)$, there are disjoint infinite sets $\mathcal{H}_i \subseteq \theta_i$ so that either

(a) the first alternative above happens for all $\vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} \mathcal{H}_i$, or

(b) there is a type t and a condition r with $\mathrm{supp}(r) = F \cup X \cup Y$ that distinguishes its support, so that for all $\vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} \mathcal{H}_i$, $\pi_{\vec{\alpha}} r_{\vec{\alpha}}$ satisfies the second alternative above, is of type t, and we have the restriction $r_{\vec{\alpha}}|_{F \cup X \cup Y} = r$.[17]

[17]In the ground model $M \vDash \mathrm{ZFC}$, there are $\kappa = |Q|$ many possibilities specified in the outcome (b).

Note that $r \leq q$, as is required by theorem 4.2.1. Without loss of generality, we assume the second case and argue that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, \dot{y}\rangle, K) \subseteq 2^{\dot{x}} \text{ has the FIP.”} \tag{4.2.1.1}$$

By lemma 4.2.5, for every $m \in \omega$ and $i \in Y \setminus (F \cup X)$, we can find subsets $H_i^m \subseteq \mathcal{H}_i$ of cardinality m, so that for $\vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} H_i^m$, the conditions $\pi_{\vec{\alpha}} r_{\vec{\alpha}}$ are mutually compatible. We claim that for every $S \subseteq \prod_{i \in Y \setminus (F \cup X)} H_i^m$,

$$1 \Vdash \text{“}\dot{F} \cup \{\langle \pi_{\vec{\alpha}} r, \pi_{\vec{\alpha}} \dot{y}\rangle | \vec{\alpha} \in S\} \subseteq 2^{\dot{x}} \text{ has the FIP.”} \tag{4.2.1.2}$$

Since we have taken the conditions $\pi_{\vec{\alpha}} r_{\vec{\alpha}}$ to all be of the same type and distinguish their supports, and since the conditions $\pi_{\vec{\alpha}} r_{\vec{\alpha}}$, for $\vec{\alpha} \in S$, are also mutually compatible, we can apply lemma 4.2.7 to conclude that their union $\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r_{\vec{\alpha}}$ is a well-defined condition in $\mathbb{P}(\theta, Q)$. Then because the conditions $r_{\vec{\alpha}}$, for $\vec{\alpha} \in S$ satisfy case (b) above, we have that

$$\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r_{\vec{\alpha}} \Vdash (\forall \vec{\alpha} \in S) \pi_{\vec{\alpha}} \dot{y} \in \dot{U}.$$

Because $1 \Vdash \dot{F} \subseteq \dot{U}$, we can conclude in particular that

$$\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r_{\vec{\alpha}} \Vdash \text{“}\dot{F} \cup \{\langle 1, \pi_{\vec{\alpha}} \dot{y}\rangle | \vec{\alpha} \in S\} \subseteq 2^{\dot{x}} \text{ has the FIP.”} \tag{4.2.1.3}$$

We would like to use lemma 2.5.13 to restrict the condition in line (4.2.1.3) to the support of the names that appear in the statement being forced. We can first calculate this union of supports to be

$$\mathrm{supp}(\dot{F}) \cup \mathrm{supp}(\dot{x}) \cup \bigcup_{\vec{\alpha} \in S} \mathrm{supp}(\pi_{\vec{\alpha}} \dot{y}) =$$

$$F \cup X \cup \bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} Y =$$

$$\bigcup_{\vec{\alpha} \in S} \mathrm{supp}(\pi_{\vec{\alpha}} r) =$$

$$\mathrm{supp}(\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r).$$

One can verify that the hypothesis of lemma 4.2.9 are met by the conditions $\pi_{\vec{\alpha}} r_{\vec{\alpha}} \leq \pi_{\vec{\alpha}} r$, for $\vec{\alpha} \in S$. One can then apply the lemma to conclude that $\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r_{\vec{\alpha}}$ restricts to $\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r$ over the latter condition's support. Given that $\mathrm{supp}(\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r)$ is the support of the names in line (4.2.1.3), we can use lemma 2.5.13 to conclude that

$$\bigcup_{\vec{\alpha} \in S} \pi_{\vec{\alpha}} r \Vdash \text{“}\dot{F} \cup \{\langle 1, \pi_{\vec{\alpha}} \dot{y} \rangle | \vec{\alpha} \in S\} \subseteq 2^{\dot{x}} \text{ has the FIP.”} \tag{4.2.1.4}$$

Because $S \subseteq \prod_{i \in Y \setminus (F \cup X)} H_i^m$ was arbitrary, line (4.2.1.4) holds for every $S' \subseteq S$, which allows us to conclude line (4.2.1.2).

To complete our proof of line (4.2.1.1), let $\Delta \in [\mathrm{Orb}(\langle q, \dot{y} \rangle, K)]^{<\omega}$ be arbitrary, and take $\delta \in [K]^{<\omega}$ so that $\Delta = \{\langle \pi q, \pi \dot{y} \rangle \mid \pi \in \delta\}$. We may assume without loss of generality that the conditions in the set $\{\pi q \mid \pi \in \delta\}$ are mutually compatible, so that the names in $\mathrm{rng}(\Delta)$ can mutually appear in an interpretation of $\mathrm{Orb}(\langle q, \dot{y} \rangle, K)$. If we can derive that

$$1 \Vdash \text{“}\dot{F} \cup \Delta \subseteq 2^{\dot{x}} \text{ has the FIP,”} \tag{4.2.1.5}$$

then the desired conclusion in line (4.2.1.1) follows from the fact that $\Delta \in [\mathrm{Orb}(\langle q, \dot{y} \rangle, K)]^{<\omega}$ was taken to be arbitrary.

We will show that, essentially, the lines (4.2.1.2) and (4.2.1.5) are the same up to an application of the symmetry lemma by some $\sigma \in K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$. We claim that $m \in \omega$ can be taken large enough to define a $\sigma \in K$, for which

$$\sigma\delta = \{\sigma\pi \mid \pi \in \delta\} \subseteq \{\pi_{\vec{\alpha}} | \vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} H_i^m\}. \tag{4.2.1.6}$$

If this can be done, let $S = \sigma\delta$ and apply line (4.2.1.2) to get

$$1 \Vdash \text{“}\dot{F} \cup \sigma\Delta \subseteq 2^{\dot{x}} \text{ has the FIP.”} \tag{4.2.1.7}$$

Because $\sigma \in K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$, we can apply the symmetry lemma 2.2.4 with σ^{-1} to line (4.2.1.7), while fixing $\dot{F}$ and $\dot{x}$, to get line (4.2.1.5), which completes the proof.

We will now justify the existence of such a $\sigma \in K$. Fix an enumeration $\delta = \{\pi_j \mid j \leq n\}$, and for $j \leq n$, define

$$\vec{\beta}_j = \{\langle i, \pi_j(i)\rangle | i \in Y \setminus (F \cup X)\},$$

so that $\pi_j \restriction_{F \cup X \cup Y} = \pi_{\vec{\beta}_j}$. Because $\mathrm{supp}(\dot{y}) \subseteq \mathrm{supp}(q) = F \cup X \cup Y$, lemma 2.5.15 shows that $\pi_j r = \pi_{\vec{\beta}_j} r$ and $\pi_j \dot{y} = \pi_{\vec{\beta}_j} \dot{y}$. Without loss of generality, we can thus assume that $\delta = \{\pi_{\vec{\beta}_j} \mid j \leq n\}$. For $i \in Y \setminus (F \cup X)$, define the sets

$$J_i = \{\pi(i) | \pi \in \delta\}.$$

Because we assume that the conditions in $\{\pi q \mid \pi \in \delta\}$ are mutually compatible, by lemma 4.2.10 the sets J_i, for $i \in Y \setminus (F \cup X)$, are mutually disjoint. For every $i \in Y \setminus (F \cup X)$, because every $\pi \in \delta \subseteq K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$, J_i is also disjoint from $F \cup X$. For every $j \leq n$, $\vec{\beta}_j \in \prod_{i \in Y \setminus (F \cup X)} J_i$, which gives the inclusion

$$\delta = \{\pi_{\vec{\beta}_j} | j \leq n\} \subseteq \{\pi_{\vec{\gamma}} | \vec{\gamma} \in \prod_{i \in Y \setminus (F \cup X)} J_i\}.$$

Now let $m = \max_{i \in Y \setminus (F \cup X)} |J_i|$ and for $i \in Y \setminus (F \cup X)$, consider the sets $H_i^m \subseteq \mathcal{H}_i \subseteq \theta_i \subseteq \theta \setminus (F \cup X)$ defined above. For $i \in Y \setminus (F \cup X)$, let $f_i : J_i \to H_i^m$ be an injection. Define $\sigma \in K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$ by

$$\sigma = \prod_{i \in Y \setminus (F \cup X)} \prod_{a \in J_i} (a \ f_i(a)).$$

Because the sets J_i are mutually disjoint, and the functions f_i are injections into mutually disjoint codomains, σ is well-defined as an automorphism of the index set θ, an in particular it is an injection from $\prod_{i \in Y \setminus (F \cup X)} J_i$ into $\prod_{i \in Y \setminus (F \cup X)} H_i^m$. Moreover, for $i \in Y \setminus (F \cup X)$, the sets J_i and H_i^m are disjoint from $F \cup X$, so $\sigma \in K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$. We can then conclude that

$$\sigma\delta = \{\sigma\pi_j | j \leq n\} = \{\sigma\pi_{\vec{\beta}_j} | j \leq n\} \subseteq \{\sigma\pi_{\vec{\gamma}} | \vec{\gamma} \in \prod_{i \in Y (\setminus (F \cup X)} J_i\} \subseteq \{\pi_{\vec{\alpha}} | \vec{\alpha} \in \prod_{i \in Y \setminus (F \cup X)} H_i^m\},$$

which satisfies line (4.2.1.6) and completes the proof. □

Remark 4.2.12. As in our search-and-shift argument from section 3.2, we search through *any* name $\dot{U}$ for an ultrafilter on $\dot{x}$ that extends $\dot{F}$, then shift the outcomes of this search (stated in line (4.2.1.2)) to verify that for every relevant $\Delta \in [\mathrm{Orb}(\langle q, y' \rangle, K)]^{<\omega}$,

$$1 \Vdash \text{“}\dot{F} \cup \Delta \subseteq 2^{\dot{x}} \text{ has the FIP,”}$$

In this sense, BPI in $N(\theta, Q)$ can is inherited from the outer model $M[G] \models$ ZFC, with arbitrary names for ultrafilters coordinating the extension of names for symmetric filters.

4.2.2 The General Case of BPI in $N(I, Q)$

The previous subsection assumes that the index set $I = \theta$ is large enough to run our adaptation of Harrington's proof. In chapter 5 of this thesis, we will generalize results from [KS20] to show that one can assume that I is arbitrarily large, without loss of generality. We state these results from [KS20] below.

Definition 4.2.13 (Karagila, Schlicht, [KS20])**.** For sets X and Y, let $\mathrm{Col}_{\mathrm{inj}}(X, Y)$ be the poset of well-orderable partial injections from X into Y, ordered by reverse-inclusion.

Theorem 4.2.14 (Karagila, Schlicht, [KS20])**.** *Let $M \vDash \mathrm{ZFC}$, and $I, J \in M$ be infinite. Let G be $\mathbb{P}(I, 2^{<\omega})$-generic, and let $A \in N(I, 2^{<\omega})$ be the standard Dedekind-finite set of Cohen-generic reals. If F is $\mathrm{Coll}_{\mathrm{inj}}(J, A)$-generic over $N^G(I, Q)$, then F is $\mathbb{P}(J, 2^{<\omega})$-generic over M.*

Corollary 4.2.15 (Karagila, Schlicht, [KS20])**.** *Let $M \vDash \mathrm{ZFC}$ and let $I, J \in M$ be infinite. For every $\mathbb{P}(I, 2^{<\omega})$-generic G, there is a $\mathbb{P}(J, 2^{<\omega})$-generic F (as in theorem 5.0.2), so that*

$$N^G(I, 2^{<\omega}) = N^F(J, 2^{<\omega}).$$

Informally, one can view the generics F and G from theorem 5.0.2 as essentially containing two pieces of information: a new set A of mutually generic Cohen reals, and a bijection witnessing the cardinality of A. In theorem 5.0.2, one takes the generic filters F and G to have the same set A of Cohen reals, but to disagree on the bijection from A to a cardinal. Because the inner models $N^G(I, Q) \subseteq M[G]$ and $N^F(J, Q) \subseteq M[F]$ retain A but lose its bijection to any cardinal, we arrive at the same model $N^G(I, 2^{<\omega}) = N^F(J, 2^{<\omega})$ by removing the point of disagreement between $M[G]$ and $M[F]$.

In chapter 5, we justify that the same reasoning works to compare the models $N(I, Q)$ and $N(J, Q)$; this resolves a question posed in [KS20]. We state our theorem below.

Theorem 4.2.16. *Let $M \vDash \mathrm{ZFC}$, let $I, J \in M$ be infinite, and let $Q \in M$ be a poset. For every $\mathbb{P}(I,Q)$-generic G, there is a $\mathbb{P}(J,Q)$-generic F so that*

$$N^G(I,Q) = N^F(J,Q).$$

Corollary 4.2.17 (Stefanović, [Ste23])**.** *For every set I and every poset Q, $N(I,Q) \vDash \mathrm{BPI}$.*

Proof. Fix an arbitrary index set I, and a $\mathbb{P}(I,Q)$-generic G, for $N^G(I,Q)$. Let θ be large enough for theorem 4.2.11, so that for every $\mathbb{P}(\theta,Q)$-generic F, $N^F(\theta,Q) \vDash \mathrm{BPI}$. By corollary 5.0.3, there is a $\mathbb{P}(\theta,Q)$-generic F, so that

$$N^G(I,Q) = N^F(\theta,Q) \vDash \mathrm{BPI}.$$

□

While this justifies BPI in $N(I,Q)$, and thus the filter extension property below $1 \in \mathbb{P}(I,Q)$ as well, it leaves open two questions. First, we have not justified theorem 4.2.1, i.e. that the filter extension property holds for the base of subgroups $\mathrm{fix}_{\mathcal{G}}(E)$, with $E \in [I]^{<\omega}$. More broadly, we can ask how witnesses to the filter extension property, in the case when $I = \theta$ is large, relate to witnesses in the arbitrary case. To resolve these loose ends, we cite another theorem that we will prove in chapter 5.

Recall the convention that we can emphasize the index set I with the notation $\langle \mathbb{P}(I,Q), \mathcal{G}_I, \mathcal{F}_I \rangle$ $\mathrm{HS}_{\mathcal{F}_I}$. Note also that in chapter 5, we will define the map $\Sigma : \mathrm{HS}_{\mathcal{F}_I} \to \mathrm{HS}_{\mathcal{F}_J}$. For now, the exact details are not needed, except for the fact that $\mathrm{supp}(\dot{x}) = \mathrm{supp}(\Sigma(\dot{x}))$.

Theorem 4.2.18. *Let $M \vDash \mathrm{ZFC}$, let $I \subseteq J \in M$ be infinite, and let $Q \in M$ be a poset. For every $p \in \mathbb{P}(I,Q) \subseteq \mathbb{P}(J,Q)$, every $\dot{x}_0, ..., \dot{x}_n \in \mathrm{HS}_{\mathcal{F}_I}$, and every formula ϕ,*

$$p \Vdash_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n) \iff p \Vdash_{N(J,Q)} \phi(\Sigma(\dot{x}_0), ..., \Sigma(\dot{x}_n)).$$

Corollary 4.2.19 (Halpern, Lévy, [HL71], Stefonović, [Ste23])**.** *Theorem 4.2.1 holds in general.*

Proof. Let I be arbitrary, let $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}_I}$ meet the hypotheses of theorem 4.2.1, with $\mathrm{supp}(\dot{F}) = F, \mathrm{supp}(\dot{x}) = X, \mathrm{supp}(\dot{y}) = Y$, and $K = \mathrm{fix}_{\mathcal{G}}(F \cup X)$. Let $q \leq 1 \in \mathbb{P}(I,Q)$ be arbitrary.

Take $\theta \supseteq I$ large enough for theorem 4.2.11. By theorem 4.2.18, $\Sigma(\dot{x}), \Sigma(\dot{F}), \Sigma(\dot{y}) \in \mathrm{HS}_{\mathcal{F}_\theta}$ also meet the hypotheses of theorem 4.2.1. By the property of Σ noted above, $\mathrm{supp}(\Sigma(\dot{F})) = F, \mathrm{supp}(\Sigma(\dot{x})) = X$ and $\mathrm{supp}(\Sigma(\dot{y})) = Y$. Since $I \subseteq \theta$, we have that $q \in \mathbb{P}(I,Q) \subseteq \mathbb{P}(\theta, Q)$. We can then apply theorem 4.2.11 to produce witnesses $r \leq q$ (with $\mathrm{supp}(r) \subseteq \mathrm{supp}(q) \cup F \cup X \cup Y \subseteq I$, so that $r \in \mathbb{P}(I,Q)$) and $\Sigma(y') \in \{\Sigma(\dot{y}), \Sigma(\dot{x} \setminus \dot{y})\}$. By theorem 4.2.18, $r \leq q$ and $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ witness theorem 4.2.1 for $\langle \mathbb{P}(I,Q), \mathcal{G}_I, \mathcal{F}_I \rangle$. □

4.3 The Halpern-Läuchli Property

We are now able to return to our goal of generalizing the Ramsey property from chapter 3. In the previous subsection, we gave a search-and-shift proof of BPI in the generalized Cohen model $N(I,Q)$, in the case that I is sufficiently large. In this section, we introduce the Halpern-Läuchli property (below p), which can be seen to express the most general conditions that allow one to apply a search-and-shift argument to prove BPI in a symmetric extension. We show that this fully characterizes BPI in symmetric extensions. In future work, we will show that the Halpern-Läuchli property also captures an abstract form of the Halpern-Läuchli theorem, and formalizes the general relationship between variants of the Halpern-Läuchli theorem and BPI in symmetric extensions.

Definition 4.3.1. The symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ has the Halpern-Läuchli property below $p \in \mathbb{P}$ if there is a set $D \subseteq \mathbb{P}$ that is dense below p, so that for every $d \in D$, $\mathcal{F}$ has a base $\mathcal{B}_d$, so that the following holds for every $K \in \mathcal{B}_d$: for every $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$, every $\mathbb{P}$-name $\dot{c}$ that 1 forces to be a $\{\text{green}, \text{ red}\}$-coloring of $\mathrm{Orb}(\langle 1, \dot{y} \rangle, K)$, and every $q \leq d$ in $\mathbb{P}$, there is an $r \leq q$ and a $c' \in \{\text{green}, \text{ red}\}$, so that for every $\delta \in [K]^{<\omega}$, and every $s \leq \{\pi r \mid \pi \in \delta\}$ below d, there is a $\sigma \in K$ so that

$$\sigma s \not\Vdash (\exists \pi \in \delta) \dot{c}(\sigma \pi \dot{y}) \neq c'.$$

Remark 4.3.2. As in our proof from subsection 4.3.2, the idea behind the Halpern-Läuchli property is that for every condition s that forces finitely many images of y' to appear in the union $\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K)$, one can shift by σ so that it is *not too late* for $\dot{U}$ to verify the desired σ-shifted instance of the FIP.

Theorem 4.3.3. *For a ground model* $M \models \mathrm{ZFC}$*, a symmetric system* $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$*,* $p \in \mathbb{P}$*, and the corresponding symmetric extension* $N \supseteq M$*, the following are equivalent:*

(i) $p \Vdash_N \mathrm{BPI}$*,*

(ii) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the filter extension property below* p*,*

(iii) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the Halpern-Läuchli property below* p*.*

Proof. Theorem 4.1.17 establishes $(i) \iff (ii)$; we will prove $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (iii)$.

$\underline{(iii) \Rightarrow (ii)}$ Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system for which the Halpern-Läuchli property below p holds, as witnessed by the set $D \subseteq \mathbb{P}$ that is dense below p, and the bases $\mathcal{B}_d$ of $\mathcal{F}$, for $d \in D$. We aim to show that the same D, and $\mathcal{B}_d$, for $d \in D$, witness the filter extension property below p. To do so, fix arbitrary $d \in D$ and $K \in \mathcal{B}_d$, and we will show that d and K witness the filter extension property below p. We use another search-and-shift argument.

Let $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ so that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$ and

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP”} \wedge \dot{y} \subseteq \dot{x},$$

and fix $q \leq d$. Recall that we aim to produce an $r \leq q$ and a $y' \in \{\dot{y}, \dot{x} \setminus \dot{y}\}$ so that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \text{ has the FIP.”} \tag{4.3.0.1}$$

Let $\dot{U} \supseteq \dot{F}$ be a $\mathbb{P}$ name that is forced by 1 to be an ultrafilter on $\dot{x}$, and let $\dot{c}$ be the name for a $\{\text{green, red}\}$-coloring of $\mathrm{Orb}(\dot{y}, K)$ so that for every $\pi \in K$

$$(t \Vdash \dot{c}(\pi\dot{y}) = \check{\text{green}}) \iff (t \Vdash \pi\dot{y} \in \dot{U}) \;\text{ and }\; (t \Vdash \dot{c}(\pi\dot{y}) = \check{\text{red}}) \iff (t \Vdash \pi(\dot{x} \setminus \dot{y}) \in \dot{U}).$$

For the coloring $\dot{c}$ and the $q \leq d$ that we fixed above, take $r \leq q$ and $c' \in \{\text{green, red}\}$ to witness the Halpern-Läuchli property. We then claim that if $c' = \text{green}$ that, then $r \leq q$ and $y' = \dot{y}$ satisfy line (4.3.0.1), and if $c' = \text{red}$, then $r \leq q$ and $y' = \dot{x} \setminus \dot{y}$ satisfy line 4.3.0.1. Since $d \in D$ and $K \in \mathcal{B}_d$ were arbitrary, proving this claim completes the proof of the implication $(iii) \Rightarrow (ii)$.

Without loss of generality, we prove this in the case that $c' = \text{green}$. Suppose that $\delta \in [K]^{<\omega}$, and that the conditions in $\{\pi r \mid \pi \in \delta\}$ are mutually compatible, and are also compatible with d. This way, when G is $\mathbb{P}$-generic and $d \in G$, the names in $\Delta = \{\pi\dot{y} \mid \pi \in \delta\}$ can mutually appear in an interpretation of $\mathrm{Orb}(\langle r, y' \rangle, K)$. In this case, we would like to find a set of conditions that are dense below $\{\pi r \mid \pi \in \delta\}$ and d, and that force

$$\text{“}\dot{F} \cup \{\langle 1, \pi\dot{y}\rangle \mid \pi \in \delta\} \text{ has the FIP.”}$$

Fix an arbitrary $s \leq \{\pi r \mid \pi \in \delta\}$ below d, and use use Halpern-Läuchli property to identify $\sigma \in K$ so that

$$\sigma s \not\Vdash (\exists \pi \in \delta)\dot{c}(\sigma\pi\dot{y}) \neq \text{grĕen}.$$

Then, there must be some $t \leq \sigma s$ so that

$$t \Vdash (\forall \pi \in \delta)\dot{c}(\sigma\pi\dot{y}) = \text{green}.$$

By the definition of $\dot{c}$, this is equivalent to

$$t \Vdash \dot{F} \cup \{\langle 1, \sigma\pi\dot{y}\rangle \mid \pi \in \delta\} \subseteq \dot{U}.$$

Recall that we took $\dot{U} \supseteq \dot{F}$ to be forced by 1 to be an ultrafilter on $\dot{x}$. Since $\sigma, \sigma^{-1} \in K \preceq \mathrm{fix}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, $\sigma^{-1}\dot{U} \supseteq \dot{F}$ is also forced by 1 to be an ultrafilter on $\dot{x}$. So $\sigma^{-1}t \leq s$ and

$$\sigma^{-1}t \Vdash \dot{F} \cup \{\langle 1, \pi\dot{y}\rangle \mid \pi \in \delta\} \subseteq \sigma^{-1}\dot{U},$$

in which case

$$\sigma^{-1}t \Vdash \dot{F} \cup \{\langle 1, \pi\dot{y}\rangle \mid \pi \in \delta\} \text{ has the FIP.”}$$

Since $s \leq \{\pi r \mid \pi \in \delta\}$ below d was arbitrary, we have that the above line is forced by a dense set of conditions below $\{\pi r \mid \pi \in \delta\}$. Since $\delta \in [K]^{<\omega}$ was arbitrary, up the the constraint that the conditions in $\{\pi r \mid \pi \in \delta\}$ were compatible, we can then conclude that for every $\delta \in [K]^{<\omega}$, every condition s below d that forces $\{\pi\dot{y} \mid \pi \in \delta\} \subseteq \mathrm{Orb}(\langle r, \dot{y}\rangle, K)$, also forces the above line. We can thus conclude, as desired, that

$$1 \Vdash \text{“}\dot{F} \cup \mathrm{Orb}(\langle r, y' \rangle, K) \text{ has the FIP,”}$$

and that $d \in D$, $K \in \mathcal{B}_d$ witness the filter extension property below p.

(ii) ⇒ (iii) We will generalize Blass' argument from [Bla87], which shows that BPI in a permutation model $M(A, \mathcal{G}, \mathcal{F})$ implies the Ramsey property for $\mathcal{F}$. We encourage the reader to review Blass' proof in the setting of permutation models. The main difference between the proofs can be explained as follows. Blass begins with a group $K \in \mathcal{F}$ and identifies a subgroup $G \cap K \preceq K$ that witnesses the Ramsey property. We will instead take K be a witness of the filter extension property, and show that K itself witnesses the Halpern-Läuchli property. This adjustment allows us to avoid a certain technical issue that arises when adapting Blass' proof more directly.

In section 4.4, it will be useful to have a version of the Halpern-Läuchli property that is stated with respect to $\mathbb{P}$-names $\dot{c}$ that 1 forces to be functions with arbitrarily large finite codomains, instead of just $\{\text{green}, \text{ red}\}$. We will thus assume in this proof that the relevant names $\dot{c}$ have arbitrarily large finite codomain. The $\{\text{green}, \text{ red}\}$-coloring case is subsumed.

Let $D \subseteq \mathbb{P}$ (which is dense below p), and for $d \in D$, let $\mathcal{B}_d$ (which is a base for $\mathcal{F}$) witness the filter extension property below p. We will show that these also witness the Halpern-Läuchli property below p; we will prove this, in particular, for an arbitrary $d \in D$ and $K \in \mathcal{B}_d$.

For d and K, we say that $(\dot{y}, n, \dot{c}, q)$ is a contradiction of the Halpern-Läuchli property below p if $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$, $n < \omega$, $\dot{c}$ is a $\mathbb{P}$-name so that $1 \Vdash \dot{c} : \mathrm{Orb}(\langle 1, \dot{y} \rangle, K) \to \check{n}$, $q \leq d$, and if for every

$r \leq q$, there is a $\delta_r \in [K]^{<\omega}$ and an $s_r \leq \{\pi r \mid \pi \in \delta\}$ below d, so that for every $\sigma \in K$,

$$\sigma s_r \Vdash (\exists \pi_0, ..., \pi_{n-1} \in \delta_r)(\forall i \leq n)\dot{c}(\sigma\pi_i\dot{y}) \neq \check{i}. \tag{4.3.0.2}$$

We aim to prove that, for d and K, there are no contradictions $(\dot{y}, n, \dot{c}, q)$ to the Halpern-Läuchli property below p.

If there were such a contradiction $(\dot{y}, n, \dot{c}, q)$, we can use it to define a name $\dot{\theta}$ for finitely satisfiable first-order theory, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{\theta})$ and $\dot{\theta} \in \mathrm{HS}_{\mathcal{F}}$. Since $d \in D$, $K \in \mathcal{B}_d$ witness the filter extension property, we claim that there is then a name $\dot{A} \in \mathrm{HS}_{\mathcal{F}}$ that d forces to be a model of the theory $\dot{\theta}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{A})$. The proof of this claim is essentially the same as the proof of theorem 4.1.5; instead of giving this proof here, we outline it in Appendix B. We will define the name $\dot{\theta}$ so that the existence of such a $\dot{A}$ produces a contradiction.

For every $r \leq q$, choose some $\delta_r \in [K]^{<\omega}$ and $s_r \leq \{\pi r \mid \pi \in \delta\}$ below d, so that for every $\sigma \in K$, the issue in line (4.3.0.2) occurs. Then define

$$\dot{\theta} = \{\langle 1, \mathrm{rng}(\dot{C}) \subseteq \check{n} \rangle\} \cup \bigcup_{r \leq q} \mathrm{Orb}(\langle s_r, \bigwedge_{i \leq n} \bigvee_{\pi \in \delta_r} \dot{C}(\pi\dot{y}) \neq \check{i} \rangle, K).$$

We can briefly specify the formal details of the above line. First, we let $\dot{C}$ be a name for a function symbol, for some arbitrary $\dot{C} \in \mathrm{HS}_{\mathcal{F}}$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{C})$; then, we have that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{\theta})$. Recall that q is taken from the contradiction $(\dot{y}, n, \dot{c}, q)$, and define a name for the set of constant symbols

$$\dot{X} = \{\langle 1, \check{i} \rangle \mid i < n\} \cup \mathrm{Orb}(\langle 1, \dot{y} \rangle, K).$$

By the sentences $\mathrm{rng}(\dot{C}) \subseteq \check{n}$ and $\bigwedge_{i\leq n} \bigvee_{\pi\in\delta} \dot{C}(\pi\dot{y}) \neq \check{i}$, we mean names for these sentences, so that for every $\sigma \in K$,

$$\sigma(\mathrm{rng}(\dot{C}) \subseteq \check{n}) = \mathrm{rng}(\dot{C}) \subseteq \check{n},$$

and

$$\sigma(\bigwedge_{i\leq n} \bigvee_{\pi\in\delta} \dot{C}(\pi\dot{y}) \neq \check{i}) = \bigwedge_{i\leq n} \bigvee_{\pi\in\delta} \dot{C}(\sigma\pi\dot{y}) \neq \check{i}.$$

For $\dot{c}$, as in the contradiction $(\dot{y}, n, \dot{c}, q)$, we claim that

$$1 \Vdash \text{“}\dot{c} \text{ is a model of } \dot{\theta}.\text{”}$$

To see why, note that by the definition of $\dot{\theta}$, 1 forces that $\dot{\theta}$ is a subset of the theory of $\dot{c}$ (when $\dot{c}$ is taken as a structure in the relevant language). It is then immediate that 1 forces that $\dot{c}$ models its own theory. Because there is a name in $\mathrm{HS}_{\mathcal{F}}$ for every finite subset of $\dot{c}$,

$$1 \Vdash_N \text{“}\dot{\theta} \text{ is finitely satsfiable.”}$$

Because d and K witness the filter extension property, we then have some $\dot{A} \in \mathrm{HS}_{\mathcal{F}}$ with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{A})$, so that

$$d \Vdash_N \text{“}\dot{A} \text{ is a model of } \dot{\theta}.\text{”}$$

However, this cannot be the case. By a slight misuse of notation, we will let $\dot{A}$ denote the intepretation of the function symbol $\dot{C}$ in this model of $\dot{\theta}$. Recall from the definition of the name $\dot{X}$ for the set of constants, $1 \Vdash \dot{y} \in \dot{X}$, for $\dot{y}$ as in the contradiction $(\dot{y}, n, \dot{c}, q)$. Then for $q \leq d$, with q as in the contradiction $(\dot{y}, n, \dot{c}, q)$, there must be an $r \leq q$ and $i < n$ so that

$$r \Vdash \dot{A}(\dot{y}) = \check{i}.$$

Then, because $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{A})$, for every $\pi \in K$,

$$\pi r \Vdash \dot{A}(\pi \dot{y}) = \check{i}.$$

Then, for the chosen $\delta_r \in [K]^{<\omega}$ and $s_r \leq \{\pi r \mid \pi \in \delta\}$ below d, it must be the case that

$$s_r \Vdash \bigwedge_{\pi \in \delta_r} \dot{A}(\pi \dot{y}) = \check{i}.$$

This contradicts the fact that $s_r \Vdash \bigwedge_{i \leq n} \bigvee_{\pi \in \delta_r} \dot{C}(\pi \dot{y}) \neq \check{i} \in \dot{\theta}$. Thus $(\dot{y}, n, \dot{c}, q)$ cannot be a contradiction to the Halpern-Läuchli property below p, for $d \in D$ and $K \in \mathcal{B}_d$. We conclude that $D \subseteq \mathbb{P}$ and $\mathcal{B}_d$, for $d \in D$, witness the Halpern-Läuchli property below p. $\square$

Corollary 4.3.4. *A symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ has the Halpern-Läuchli property below p if and only if it has the version of the Halpern-Läuchli property below p that results from changing the $\mathbb{P}$-names $\dot{c}$, from definition 4.3.1, to be forced by 1 to have arbitrary finite codomains. In particular, $d \in D \subseteq \mathbb{P}$ and $K \in \mathcal{B}_d \subseteq \mathcal{F}$ witness the Halpern-Läuchli property below p if and only if they witness this adjusted version of the property.*

Proof. The case where the $\mathbb{P}$-names $\dot{c}$ are forced by 1 to be $\{\text{green}, \text{ red}\}$-colorings is subsumed by the general case, which proves the "if" direction. In the first direction of the proof of the above theorem, we showed that if $d \in D \subseteq \mathbb{P}$ and $K \in \mathcal{B}_d \subseteq \mathcal{F}$ witness the Halpern-Läuchli property below p, then they also witness the filter extension property below p. In the second direction, we showed that this implies that $d \in D \subseteq \mathbb{P}$ and $K \in \mathcal{B}_d \subseteq \mathcal{F}$ also witness this finite-codomain version of the Halpern-Läuchli property below p. $\square$

4.4 The Search-and-Shift Property

At this respective point in chapter 3, we dealt with extreme amenability as a characterization of BPI in permutation models. In section 3.4, we then introduced the search-and-shift property for permutation models, in part to analyze the convergent features of our proofs of BPI from the Ramsey property and from extreme amenability. This gives us two avenues to develop a version of the search-and-shift property for symmetric extensions: using an appropriate generalization of extreme amenability, or using the Halpern-Läuchli property.

In the case of symmetric extensions, the appropriate generalization of extreme amenability is not immediately apparent. In this section, we will thus develop a version of the search-and-shift property with respect to the Halpern-Läuchli property. Our overall strategy is to use this characterization of BPI to motivate a viable generalization of extreme amenability to the context of symmetric extensions. This will appear in forthcoming work, but not in this thesis. We sketch this overall idea in figure 4.1 below.

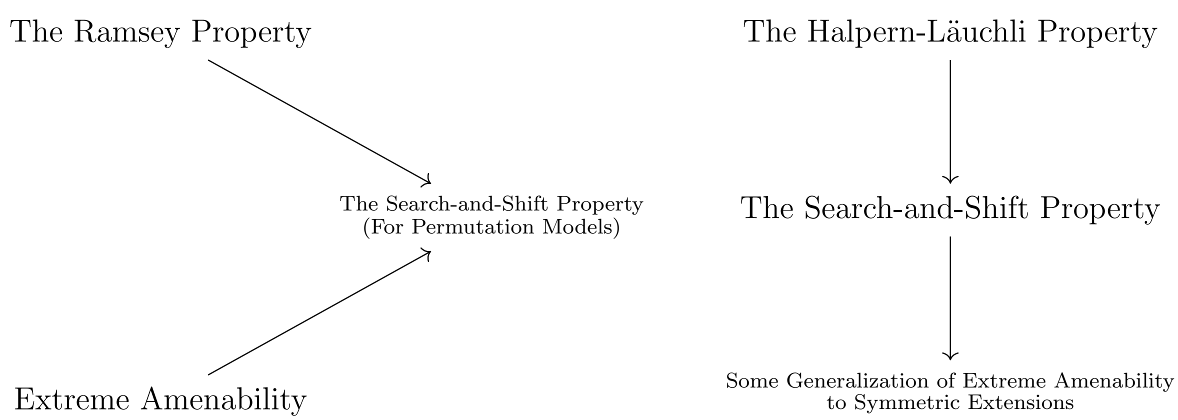

Figure 4.1: The flow of ideas in chapter 3, and its influence on this section and intended future work.

The Ramsey property and the Halpern-Läuchli property both express that search-and-shift arguments are a fully comprehensive method to prove BPI, respectively in permutation models and in symmetric extensions. This is the main idea that we will use to state a version of the search-and-shift property for symmetric extensions. We will briefly recall several details of the case for permutation models.

In section 3.4, we show that for $F, x \in M(A, \mathcal{G}, \mathcal{F})$ satisfying the hypotheses of the filter extension property, a single $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilter $U_0 \supseteq F$ on x could be used to determine an ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$ extending F on x. In this case, we showed that U_0 can be used in a search-and-shift argument to *check* each $y \in U$ (see definition 3.4.2), i.e. to verify that U has the FIP. This differed from our application of the Ramsey property to prove BPI in section 3.2, where a sequence $U_0, U_1, ...$ of $M(A, \mathcal{G}, \mathcal{F})$-valued ultrafilters is produced to guide the construction of the desired ultrafilter $U \in M(A, \mathcal{G}, \mathcal{F})$.

Similarly, by applying the Halpern-Läuchli property and the filter extension property to prove that $p \Vdash_N$ BPI, we start with names $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$ that satisfy the hypotheses of the filter extension property. We then produce a sequence

$$\dot{U}_0, \dot{F}_1 = \dot{F} \cup \mathrm{Orb}(\langle r_0, \dot{y}_0 \rangle, K), \dot{U}_1 \supseteq \dot{F}_1, \dot{F}_2 = \dot{F}_1 \cup \mathrm{Orb}(\langle r_1, \dot{y}_1 \rangle, K), ...$$

where each $\dot{U}_n \supseteq \dot{F}_n$ is forced by 1 to be an ultrafilter on $\dot{x}$, so that a search-and-shift argument using $\dot{U}_n$ and the Halpern-Läuchli property can be used to determine $\dot{F}_{n+1}$. Eventually, we have that $\dot{F}_\beta = \dot{U} \in \mathrm{HS}_{\mathcal{F}}$ is the desired name that some $d \leq p$ forces to be an ultrafilter on $\dot{x}$ extending $\dot{F}$. We can ask the same question as in section 3.4: can this entire process be completed by a single name for an ultrafilter $\dot{U}_0 \supseteq \dot{F}$?

To answer this question, we isolate the property that $\dot{U}_0$ can be used in a search-and-shift argument for a specific name $\dot{y}$ for a subset of $\dot{x}$ – this is the relevant generalization of definition 3.4.2. First, we introduce some basic terminology.

Definition 4.4.1. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, and $\dot{x} \in \mathrm{HS}_{\mathcal{F}}$, we say that a $\mathbb{P}$-name $\dot{U}$ is an $\mathrm{HS}_{\mathcal{F}}$-valued filter on $\dot{x}$ if $\mathrm{rng}(\dot{U}) \subseteq \mathrm{HS}_{\mathcal{F}}$, $1 \Vdash \dot{y} \subseteq \dot{x}$ for every $\dot{y} \in \mathrm{rng}(\dot{U})$, and if

$$1 \Vdash \dot{U} \subseteq 2^{\dot{x}} \text{ has the FIP.''}$$

If if is also the case that for every $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$ with $1 \Vdash \dot{y} \subseteq \dot{x}$,

$$1 \Vdash \dot{y} \in \dot{U} \vee \dot{x} \setminus \dot{y} \in \dot{U},$$

then we say that $\dot{U}$ is an *$\mathrm{HS}_{\mathcal{F}}$-valued ultrafilter on $\dot{x}$*.

Definition 4.4.2. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, and $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$ satisfying the hypotheses of the filter extension property, let $\beta\dot{x}/\dot{F}$ be the set of $\mathrm{HS}_{\mathcal{F}}$-valued ultrafilters $\dot{U}$ on $\dot{x}$ with $\dot{F} \subseteq \dot{U}$.

We can now state the generalization of definition 3.4.2.

Definition 4.4.3. For a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, and $\dot{F}, \dot{x}, \dot{y} \in \mathrm{HS}_{\mathcal{F}}$ satisfying the hypotheses of the filter extension property, $r \in \mathbb{P}$, and $K \in \mathcal{F}$, we say that $\dot{U}_0 \in \beta\dot{x}/\dot{F}$ *checks* $\langle r, \dot{y} \rangle$ (with respect to d and K) if for every $\delta \in [K]^{<\omega}$ and every $s \leq \{\pi r \mid \pi \in \delta\}$ below d, there is a $\sigma \in K$ so that

$$s \nVdash (\exists \pi \in \delta) \pi \dot{y} \notin \sigma \dot{U}_0.$$

When d and K are clear from context, we will just write that $\dot{U}_0$ checks $\langle r, \dot{y} \rangle$.

In lemma 3.4.3, we proved the central relationship between U_0 checking elements of a symmetric ultrafilter U, and U being an ultralimit of the space $K \cdot U_0$. Stating the appropriate generalization of this lemma is a key idea in this section.

Note that for $K \in \mathcal{F}$ and $\dot{U}_0 \in \beta\dot{x}/\dot{F}$, will write $K \cdot \dot{U}_0$ to formally mean the name $\{\langle 1, \sigma\dot{U}_0\rangle \mid \sigma \in K\}$ for a space with the standard topology of a space of ultrafilters.

Lemma 4.4.4. *Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F}\rangle$ be a symmetric system, let $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$ satisfy the hypotheses of the filter extension property, and let $\dot{U}_0 \in \beta\dot{x}/\dot{F}$. For $K \in \mathcal{F}$, suppose that $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, $\dot{F} \subseteq \dot{U}$, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, and $d \Vdash$ "$\dot{U}$ is an ultrafilter on $\dot{x}$." Then the following are equivalent:*

(i) $d \Vdash$ "$\dot{U}$ is an ultralimit of $K \cdot \dot{U}_0$,"

(ii) $\dot{U}_0$ checks every $\langle r, \dot{y}\rangle \in \dot{U}$, with $r \leq d$, with respect to d and K.

Proof. *(i) ⇒ (ii)* Suppose that $d \Vdash$ "$\dot{U}$ is an ultralimit of $K \cdot \dot{U}_0$." For $\dot{U}$ as in the statement of the lemma, let $r \leq d$ with $\langle r, \dot{y}\rangle \in \dot{U}$, $\delta \in [K]^{<\omega}$, and $s \leq \{\pi r \mid \pi \in \delta\}$ below d be arbitrary; we aim to show that $\dot{U}_0$ checks $\langle r, \dot{y}\rangle$ with respect to d and K. We have immediately that $r \Vdash \dot{y} \in \dot{U}$. Since we assume that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, for every $\pi \in K$, we also have that $\pi r \Vdash \pi\dot{y} \in \dot{U}$. Then

$$s \Vdash (\forall \pi \in \delta)\pi\dot{y} \in \dot{U}.$$

Since s is below d, we also have that

$$s \Vdash \text{"}\dot{U} \text{ is an ultralimit of } K \cdot \dot{U}_0\text{,"}$$

so there is at least one $t \leq s$ and $\sigma \in K$ with

$$t \Vdash (\forall \pi \in \delta)\pi\dot{y} \in \sigma\dot{U}_0.$$

Then for this σ,

$$s \not\Vdash (\exists \pi \in \delta)\pi\dot{y} \notin \sigma\dot{U}_0,$$

so $\dot{U}_0$ checks $\langle r, \dot{y}\rangle$ with respect to d and K.

(ii) $\Rightarrow$ (i) Suppose that $\dot{U}_0$ checks every $\langle r, \dot{y}\rangle \in \dot{U}$, with $r \leq d$, with respect to d and K. Note that $d \Vdash \dot{U} \in \beta\dot{x}/\dot{F}$. We will show that d forces that every basic open neighborhood of $\dot{U}$ in $\beta\dot{x}/\dot{F}$ meets $K \cdot \dot{U}_0$, which is equivalent to d forcing that $\dot{U}$ is an ultralimit of $K \cdot \dot{U}_0$.

Let $r \leq d$ force that $\dot{W}$ is a basic open neighborhood of $\dot{U}$ in the subspace $K \cdot \dot{U}_0 \subseteq \beta\dot{x}/\dot{F}$. Then there is some $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$ so that $r \Vdash \dot{y} \in \dot{U}$, and

$$r \Vdash \dot{W} = \{\langle s, \sigma\dot{U}_0\rangle \mid \sigma \in K,\ s \Vdash \dot{y} \in \sigma\dot{U}_0\}.$$

We complete the proof by showing that r also forces that $\dot{W}$ is nonempty.

Fix an arbitrary $s \leq r$. By the definition of the forcing relation, we can extend to some $t \leq s$ so that $\langle t, \dot{y}_t\rangle \in \dot{U}$ and $t \Vdash \dot{y} = \dot{y}_t$. Then, by assumption, U_0 checks $\langle t, \dot{y}_t\rangle$, so for every $u \leq t \leq d$, there is a $\sigma \in K$ so that

$$u \nVdash \dot{y} \notin \sigma\dot{U}_0.$$

In this case, there is a $v \leq u$ so that

$$v \Vdash \dot{y} \in \sigma\dot{U}_0,$$

and thus

$$v \Vdash \sigma\dot{U}_0 \in \dot{W}.$$

Because $s \leq r$ was arbitrary, and $v \leq s$ forces $\dot{W}$ to be nonempty, a dense set of conditions below r force that $\dot{W}$ is nonempty. Thus, r also forces that $\dot{W}$ is nonempty. Since $r \leq d$ was an arbitrary condition that forces that $\dot{W}$ is an open neighborhood of $\dot{U}$, d forces that $\dot{U}$ is an ultralimit of $K \cdot \dot{U}_0$. $\square$

Based on the connection in the lemma, we can define the search-and-shift property as follows.

Definition 4.4.5. The symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ has the search-and-shift property below $p \in \mathbb{P}$ if there is a set $D \subseteq \mathbb{P}$ that is dense below p, so that for every $d \in D$, $\mathcal{F}$ has a base $\mathcal{B}_d$, so that the following holds for every $K \in \mathcal{B}_d$: for every $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, and

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP,”}$$

and for every $\dot{U}_0 \in \beta\dot{x}/\dot{F}$, there is a $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, so that $\dot{F} \subseteq \dot{U}$, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$, and

$$d \Vdash \text{“}\dot{U} \text{ is an ultralimit of } K \cdot \dot{U}_0\text{”} \wedge \text{“}\dot{U} \text{ is an ultrafilter on } \dot{x}\text{”}$$

Theorem 4.4.6. *For a ground model $M \vDash \mathrm{ZFC}$, a symmetric system $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$, $p \in \mathbb{P}$, and the corresponding symmetric extension $N \supseteq M$, the following are equivalent:*

(i) $p \Vdash_N \mathrm{BPI}$,

(ii) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the filter extension property below p,*

(iii) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the Halpern-Läuchli property below p,*

(iv) $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ *has the search-and-shift property below p.*

Proof. The equivalences $(i) \iff (ii) \iff (iii)$ have been proven, and the implication $(iv) \Rightarrow (i)$ is immediate. We will thus prove $(iii) \Rightarrow (iv)$ to complete the theorem. We will follow the format of our proof that the Ramsey property implies the search-and-shift property for permutation models, in section 3.4.

Let $D \subseteq \mathbb{P}$ be dense below p, and for every $d \in D$, let $\mathcal{B}_d$ be a base of $\mathcal{F}$ that witnesses the

Halpern-Läuchli property below p. Let $d \in D$ and $K \in \mathcal{B}_d$ be arbitrary; we aim to show that d and K also witness the search-and-shift property below p.

Fix arbitrary $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$ for which $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$ and $1 \Vdash$ "$\dot{F} \subseteq 2^{\dot{x}}$ has the FIP," and let $\dot{U}_0 \in \beta\dot{x}/\dot{F}$ be arbitrary. By lemma 4.4.4, it suffices to show that there is a $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, so that $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$,

$$d \Vdash \text{“}\dot{U} \supseteq \dot{F} \text{ is an ultrafilter on } \dot{x}, \text{”}$$

and so that $\dot{U}_0$ checks every $\langle r, \dot{y}\rangle \in \dot{U}$, with $r \leq d$, with respect to d and K. To construct $\dot{U}$, we use following lemma.

Lemma 4.4.7. *For every $\dot{y}_0, ..., \dot{y}_n \in \mathrm{HS}_{\mathcal{F}}$ so that $1 \Vdash \dot{x} = \bigsqcup_{i \leq n} \dot{y}_i$, and for every $q \leq d$, there is an $r \leq q$ and an $i \leq n$ so that $\dot{U}_0$ checks $\langle r, \dot{y}_i\rangle$ with respect to d and K.*

Proof. We use the form of the Halpern-Läuchli property that references names for colorings with arbitrarily large finite codomains. By corollary 4.3.4, because d and K witness the Halpern-Läuchli property below p, they also witness this finite-codomain version of the Halpern-Läuchli property below p.

Let $\dot{y}_0, ..., \dot{y}_n \in \mathrm{HS}_{\mathcal{F}}$, and let $1 \Vdash \dot{x} = \bigsqcup_{i \leq n} \dot{y}_i$. Let $\dot{y} = \{\langle 1, \dot{y}_i\rangle \mid i \leq n\}$, and define an $n+1$-coloring of $\mathrm{Orb}(\langle 1, \dot{y}\rangle, K)$ by setting, for $r \leq d$,

$$(r \Vdash \dot{c}(\pi\dot{y}) = \check{i}) \iff (r \Vdash \pi\dot{y} \in \dot{U}_0).$$

The proof can then be completed in the same was as our proof that the Halpern-Läuchli property below p implies the filter extension property below p. ☐

Define the $\mathbb{P}$-name

$$\begin{aligned}\dot{F}_0 = \{\langle r, \dot{y}\rangle \mid r \Vdash{} & \dot{y} \subseteq \dot{x} \wedge \\ & \dot{y} \in \mathrm{HS}_{\mathcal{F}} \wedge \\ & \text{“}\dot{U}_0 \text{ checks } \langle r, \dot{y}\rangle \text{ (with respect to } d \text{ and } K)\text{”} \wedge \\ & \text{“}(\forall s \leq r)\ \dot{U}_0 \text{ does not check } \langle s, \dot{x} \setminus \dot{y}\rangle \text{ (with respect to } d \text{ and } K)\text{”}\rangle\}.\end{aligned}$$

Claim 4.4.8. We claim that $\dot{F} \subseteq \dot{F}_0$, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_0)$, and that $d \Vdash$ “$\dot{F}_0 \subseteq 2^{\dot{x}}$ has the FIP.”

Supposing the claim, we will show how to complete the proof. Because $d \in D$ and $K \in \mathcal{B}_d$ witness the Halpern-Läuchli property below p, they also witness the filter extension property below p. Because $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_0) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, by our proof of lemma 4.1.1, we can find a name $\dot{F}_0' \in \mathrm{HS}_{\mathcal{F}}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}')$ and so that

$$d \Vdash \dot{F}_0 = \dot{F}_0' \quad \text{and} \quad 1 \Vdash \text{“}\dot{F}_0' \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

Then, because $1 \Vdash$ “$\dot{F}_0' \subseteq 2^{\dot{x}}$ has the FIP” and $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_0) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, there is a $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$ and $\dot{F}_0' \subseteq \dot{U}$, so that

$$d \Vdash \dot{F}_0 = \dot{F}_0' \subseteq \dot{U} \wedge \text{“}\dot{U} \text{ is an ultrafilter on } \dot{x}.\text{”}$$

We can describe how to modify the name $\dot{U}$, so that $\dot{U}_0$ checks every $\langle r, \dot{y}\rangle \in \dot{U}$, with $r \leq d$, with respect to d and K. For every $\langle r, \dot{y}\rangle \in \dot{U}$, with $r \leq d$, there is a set $S \subseteq \mathbb{P}$ that is dense below r, so that for every $s \in S$,

(i) $s \Vdash \dot{y} \in \dot{F}_0$, or

(ii) $s \Vdash \dot{y} \notin \dot{F}_0$.

For every $s \in S$, we can extend further to some $t_s \leq s$, so that either

(i) there is a $\langle t_s, \dot{y}_{t_s} \rangle \in \dot{F}_0$ with $t_s \Vdash \dot{y} = \dot{y}_{t_s}$ (by the definition of $\dot{F}_0$, this would also mean that $\langle t_s, \dot{y} \rangle \in \dot{F}_0$), or

(ii) by using lemma 4.4.7, $\dot{U}_0$ checks both $\langle t_s, \dot{y} \rangle$ and $\langle t_s, \dot{x} \setminus \dot{y} \rangle$, with respect to d and K.

In either case, $\dot{U}_0$ checks $\langle t_s, \dot{y} \rangle$ with respect to d and K. By the claim, $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_0) = \mathrm{sym}_{\mathcal{G}}(\dot{U})$. Because of this, and the symmetry of the definition of being checked with respect to d and K, we have that for every $\pi \in K$, either

(i) $\langle \pi t_s, \pi \dot{y} \rangle \in \dot{F}_0$, or

(ii) $\dot{U}_0$ checks both $\langle \pi t_s, \pi \dot{y} \rangle$ and $\langle \pi t_s, \pi \dot{x} \setminus \dot{y} \rangle$ with respect to d and K.

Again, in each case $\dot{U}_0$ checks $\langle \pi t_s, \pi \dot{y} \rangle$ with respect to d and K.

To modify $\dot{U}$, we can remove the orbit $\mathrm{Orb}(\langle r, \dot{y} \rangle, K)$ from $\dot{U}$ and replace it with $\mathrm{Orb}(\langle t_s, \dot{y} \rangle, K)$, for every $s \in S$. For each $\langle \pi r, \pi \dot{y} \rangle \in \mathrm{Orb}(\langle r, \dot{y} \rangle, K)$ that we removed, we replaced with with $\langle \pi t_s, \pi \dot{y} \rangle$, for $s \in S$. Since $\{\pi t_s \mid s \in S\}$ is dense below πr, this adjustment does not affect the interpretation of $\dot{U}$. By making this modification for every $\langle r, \dot{y} \rangle \in \dot{U}$, with $r \leq d$, we can ensure that $\dot{U}_0$ checks every $\langle r, \dot{y} \rangle \in \dot{U}$, with $r \leq d$, with respect to d and K. Given the claim, this completes the proof.

Proof of claim 4.4.8. Because $\dot{F} \subseteq \dot{U}_0$ and $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F})$, we also have that for every $\sigma \in K$, $\dot{F} \subseteq \sigma \dot{U}_0$. Then for every $\langle r, \dot{y} \rangle \in \dot{F}$, $\dot{U}_0$ checks $\langle r, \dot{y} \rangle \in \dot{F}$ and fails to check every $\langle s, \dot{x} \setminus \dot{y} \rangle$, with $s \leq r$. Thus, $\langle r, \dot{y} \rangle \in \dot{F}_0$, and so $\dot{F} \subseteq \dot{F}_0$.

Suppose that $\langle r, \dot{y} \rangle \in \dot{F}_0$, so that $1 \Vdash \dot{y} \subseteq \dot{x}$, $\dot{y} \in \mathrm{HS}_{\mathcal{F}}$, and $\dot{U}_0$ checks $\langle r, \dot{y} \rangle$ and fails to check $\langle s, \dot{x} \setminus \dot{y} \rangle$, for every $s \leq r$. Let $\pi \in K$ be arbitrary. Because $\pi \in K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, we have that $1 \Vdash \pi\dot{y} \subseteq \dot{x}$. From the definition of the normal filter $\mathcal{F}$, $\pi\dot{y} \in \mathrm{HS}_{\mathcal{F}}$. By the definition of $\dot{U}_0$ checking $\langle r, \dot{y}\rangle$ with respect to d and K, for every $\pi \in K$, $\dot{U}_0$ checks $\pi\langle r, \dot{y}\rangle$ and fails to check $\pi\langle s, \dot{x} \setminus \dot{y}\rangle$, for every $s \leq r$. Then $\pi\langle r, \dot{y}\rangle \in \dot{F}_0$, so $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}_0)$.

Suppose that there are $\dot{y}_0, ..., \dot{y}_n \in \mathrm{HS}_{\mathcal{F}}$ and $q \leq d$, so that $q \Vdash \dot{y}_0, ..., \dot{y}_n \in \dot{F}_0$. There is then a dense set of conditions $R \subseteq \mathbb{P}$ below q, where for every $r \in R$, there are $\dot{y}_{0,r}, ..., \dot{y}_{n,r} \in \mathrm{HS}_{\mathcal{F}}$ so that $\langle r, \dot{y}_{0,r}\rangle, ..., \langle r, \dot{y}_{n,r}\rangle \in \dot{F}_0$ and

$$r \Vdash (\forall i \leq n)\dot{y}_i = \dot{y}_{i,r}.$$

By the definition of $\dot{F}_0$, for every $i \leq n$, $\dot{U}_0$ checks $\langle r, \dot{y}_i\rangle$ with respect to d and K, and for every $s \leq r$, $\dot{U}$ does not check $\langle s, \dot{x} \setminus \dot{y}_i\rangle$ with respect to d and K.

From this, we can observe that for every $i \leq n$, $s \leq r$, and $\dot{z} \in \mathrm{HS}_{\mathcal{F}}$ so that $1 \Vdash \dot{z} \subseteq \dot{x} \setminus \dot{y}_i$, $\dot{U}_0$ does not check $\langle s, \dot{z}\rangle$. Accordingly, for every $i \leq n$ $s \leq r$, and $\dot{z} \in \mathrm{HS}_{\mathcal{F}}$, $\dot{U}_0$ does not check $\langle s, \dot{z} \cap \dot{x} \setminus \dot{y}_i\rangle$.[18] Then by lemma 4.4.7, since $r \leq d$, there is an $s \leq r$ so that $\langle s, \bigcap_{i \leq n} \dot{y}_i\rangle$ is checked with respect to d and K, and consequently

$$s \Vdash \bigcap_{i \leq n} \dot{y}_i.$$

Thus, for every $r \in R$, there is an $s \leq r$ that forces the above line. Since $R \subseteq \mathbb{P}$ is dense below q, q also forces the above line. Since $q \leq d$ was chosen arbitrarily, we can conclude that $d \Vdash$ "$\dot{F}_0 \subseteq 2^{\dot{x}}$ has the FIP." □

The application of the claim described above then completes the argument.

□

[18] Where we take 1 to force $\dot{z} \cap \dot{x} \setminus \dot{y}_i$ to be the intended intersection.

Chapter 5

Elementary Embeddings Between Generalized Cohen Models

In section 4.2, we cited several results from chapter 5 to justify various ways in which the index set I in the model $N(I, Q)$ can be assumed to be arbitrarily large, without loss of generality.[1] This was of interest because I could then be taken to be large enough to apply certain Ramsey-theoretic arguments to prove BPI in $N(I, Q)$. We will now pay our debt and prove the theorems that we cited in section 4.2. Some of the citations and motivating comments below were already made in section 4.2, but we also include them here for the reader's convenience. The work in this chapter generalizes a theorem from [KS20], and answers a question posed in that paper.

In the standard presentation of the Cohen model, one forces countably many mutually Cohen-generic reals over a ground model of ZFC, then moves to an inner model of ZF in which the standard set of Cohen reals (denoted A) is Dedekind-finite. We use the notation $N(\omega, 2^{<\omega})$

[1] See section 2.5 for the definition of $N(I, Q)$.

for this construction. The main theorem of [KS20] is that one can think of the Cohen model as being truly agnostic about the size of A: one can force A to have an arbitrarily large cardinality in an extension of $N(\omega, 2^{<\omega})$. In a sense, this is the opposite process of forcing to collapse a cardinal. Another way to express their result is that, for any infinite index sets I and J, the models $N(I, 2^{<\omega})$ and $N(J, 2^{<\omega})$ – versions of the Cohen model that respectively add I and J many Cohen reals, before taking the relevant inner model – are the same, up to the right choice of generic filters. We state this work from [KS20] below.

Definition 5.0.1 (Karagila, Schlicht, [KS20])**.** For sets X and Y, let $\mathrm{Col}_{\mathrm{inj}}(X, Y)$ be the poset of well-orderable partial injections from X into Y, ordered by reverse-inclusion.

Theorem 5.0.2 (Karagila, Schlicht, [KS20])**.** *Let $M \vDash \mathrm{ZFC}$, let $I, J \in M$ be infinite, let G be $\mathbb{P}(I, 2^{<\omega})$-generic, and let $A \in N(I, 2^{<\omega})$ be the standard Dedekind-finite set of Cohen reals. If F is $\mathrm{Coll}_{\mathrm{inj}}(J, A)$-generic over $N^G(I, Q)$, then F is $\mathbb{P}(J, 2^{<\omega})$-generic over M.*

Corollary 5.0.3 (Karagila, Schlicht, [KS20])**.** *Let $M \vDash \mathrm{ZFC}$ and let $I, J \in M$ be infinite. For every $\mathbb{P}(I, 2^{<\omega})$-generic G, there is a $\mathbb{P}(J, 2^{<\omega})$-generic F (as in theorem 5.0.2), so that*

$$N^G(I, 2^{<\omega}) = N^F(J, 2^{<\omega}).$$

We can sketch a quick, informal account of theorem 5.0.2 and corollary 5.0.3. One can view the generics F and G from theorem 5.0.2 as essentially containing two pieces of information: a new set A of mutually generic Cohen reals, and a bijection witnessing the cardinality of A. In theorem 5.0.2, one takes the generic filters F and G to have the same set A of Cohen reals, but to disagree on the bijection from A to a cardinal. Because the inner models $N^G(I, Q) \subseteq M[G]$ and $N^F(J, Q) \subseteq M[F]$ retain A but lose its bijection to any cardinal, we arrive at the same model $N^G(I, 2^{<\omega}) = N^F(J, 2^{<\omega})$ by removing the point of disagreement between $M[G]$ and $M[F]$.

With this sketch in mind, one would expect that the proofs of theorem 5.0.2 and corollary 5.0.3 do not depend on the poset $2^{<\omega}$, and that the same results hold for $N(I,Q)$ – the result of forcing I-many mutually Q-generic filters over a model of ZFC, then moving to an inner model in which the standard set of Q-generic filters is Dedekind-finite. Question 5.5 from [KS20] essentially asks whether this intuition holds. Here, the authors note that their "proofs involved in a meaningful way the Cohen forcing [$\mathbb{P}(\omega, 2^{<\omega})$] itself," but that "on its face it seems that the proof uses more of the fact that we take a finite-support product of infinitely many copies of the same forcing, rather than the specific properties of the Cohen forcing." They then ask (up to a change in notation) whether the results in [KS20] hold for $N(I,Q)$. We will answer this question positively, in the case of corollary 5.0.3:

Theorem 5.0.4. *Let $M \vDash$ ZFC, let $I, J \in M$ be infinite, and let $Q \in M$ be a poset. For every $\mathbb{P}(I,Q)$-generic G, there is a $\mathbb{P}(J,Q)$-generic F so that*

$$N^G(I,Q) = N^F(J,Q).$$

This formalizes the notion that the index set I in $N(I,Q)$ can be taken to be arbitrarily large, without loss of generality. For the analysis given in section 4.2, we also used following the stronger theorem. Note that $\Sigma : \mathrm{HS}_{\mathcal{F}_I} \to \mathrm{HS}_{\mathcal{F}_J}$ will be defined in section 5.1.[2]

Theorem 5.0.5. *Let $M \vDash$ ZFC, let $I, J \in M$ be infinite, and let $Q \in M$ be a poset. For every $p \in \mathbb{P}(I,Q)$, every $\dot{x}_0, ..., \dot{x}_n \in \mathrm{HS}_{\mathcal{F}_I}$, and every formula ϕ,*

$$p \Vdash_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n) \iff p \Vdash_{N(J,Q)} \phi(\Sigma(\dot{x}_0), ..., \Sigma(\dot{x}_n)).$$

While the approach from [KS20] can likely be generalized to prove these two theorems, we

[2]As noted in definition 2.5.5, we can include the subscripts $\langle \mathbb{P}(I,Q), \mathcal{G}_I, \mathcal{F}_I \rangle$ and $\langle \mathbb{P}(J,Q), \mathcal{G}_J, \mathcal{F}_J \rangle$ to emphasize the difference in index sets I and J.

take a different approach that was inspired by a comment made in [KS20], in the paragraph immediately following their remark 4.4. There, the authors point out that their result does not imply a bijection between ω and κ (or, in our case, I and J) in the ground model M, only that κ is countable in is some ambient outer model of M. This suggests that one can study the results in [KS20] from the perspective of an outer model with a bijection between I and J. We will use M to denote our intended ground model, and V to denote an outer model of M with a bijection $b : I \to J$, and in which M is a class. In V, based on the bijection b, there are natural notions of isomorphism between $\mathbb{P}(I, Q)$ and $\mathbb{P}(J, Q)$, between $V^{\mathbb{P}(I,Q)}$ and $V^{\mathbb{P}(J,Q)}$, and thus between $\Vdash^V_{N(I,Q)}$ and $\Vdash^V_{N(J,Q)}$. We prove that, in V, the natural isomorphism between $V^{\mathbb{P}(I,Q)}$ and $V^{\mathbb{P}(J,Q)}$ restricts to an isomorphism between the relativized classes $(\mathrm{HS}_{\mathcal{F}_I})^M$ and $(\mathrm{HS}_{\mathcal{F}_J})^M$. From this, we show that the isomorphism in V between $\Vdash^V_{N(I,Q)}$ and $\Vdash^V_{N(J,Q)}$ restricts to an isomorphism between the relativized forcing relations $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$. Even though this isomorphism is not in M itself, the fact that the relations $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$ are relativized to M allows us to draw conclusions about the relationship between $\Vdash_{N(I,Q)}$ and $\Vdash_{N(J,Q)}$ in M. In particular, we can prove the two intended theorems listed above.

To carry out this approach in the outer model V, it will be useful to have a thorough understanding of the relationship between $\mathrm{HS}_{\mathcal{F}_I}$ and $\mathrm{HS}_{\mathcal{F}_J}$ that can be described in the ground model M. In section 5.1, we introduce the maps in M

$$\Sigma : \mathrm{HS}_{\mathcal{F}_I} \to \mathrm{HS}_{\mathcal{F}_J} \text{ and } \Gamma : \mathrm{HS}_{\mathcal{F}_J} \to \mathrm{HS}_{\mathcal{F}_I}.$$

These maps capture the ability of M to compare the classes $\mathrm{HS}_{\mathcal{F}_I}$ and $\mathrm{HS}_{\mathcal{F}_J}$, and hence to compare the symmetric systems $\langle \mathbb{P}(I, Q), \mathcal{G}_I, \mathcal{F}_I \rangle$ and $\langle \mathbb{P}(J, Q), \mathcal{G}_J, \mathcal{F}_J \rangle$. The main focus of section 5.1 is developing the basic algebraic properties of these maps, for use in section 5.2.

In section 5.2, we use the maps Σ and Γ to analyze the isomorphisms that extend from a bijection $b : I \to J$ in an outer model V of M, in which M is a class. We then carry out the details of the approach sketched above. We examine the extent to which the ground model M can detect the existence of an isomorphism between $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$ in the outer model V. We will show in theorem 5.2.11 that, in a sense, Σ is the residue in M of this isomorphism in V.

We note that, unavoidably, many of the details of the arguments in this chapter rely heavily on computation. While we have provided the full details of each proof, we have also divided the work in this chapter into definitions, lemmas, theorems, and corollaries that we hope are believably true and naturally build upon one another.

5.1 Spreading and Gathering Maps in the Ground Model

In the ground model $M \vDash \mathrm{ZFC}$, we will introduce the maps

$$\Sigma : \mathrm{HS}_{\mathcal{F}_I} \to \mathrm{HS}_{\mathcal{F}_J} \text{ and } \Gamma : \mathrm{HS}_{\mathcal{F}_J} \to \mathrm{HS}_{\mathcal{F}_I}$$

to capture the ability of M to compare $\mathrm{HS}_{\mathcal{F}_I}$ with $\mathrm{HS}_{\mathcal{F}_J}$, and hence to compare $\langle \mathbb{P}(I,Q), \mathcal{G}_I, \mathcal{F}_I \rangle$ with $\langle \mathbb{P}(J,Q), \mathcal{G}_J, \mathcal{F}_J \rangle$. We will assume without loss of generality that $I \subseteq J$ and $|I| < |J|$ in M. Given this assumption, we have that $\mathbb{P}(I,Q) \subseteq \mathbb{P}(J,Q)$. Our definition of the map Σ aims to provide a canonical sense in which a name $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$ can be modified to produce a corresponding name in $\mathrm{HS}_{\mathcal{F}_J}$ with the same support.

Definition 5.1.1. Let I, J be infinite, with $I \subseteq J$ and $|I| < |J|$. Define the "spreading" function

$$\Sigma : \mathrm{HS}_{\mathcal{F}_I} \to \mathrm{HS}_{\mathcal{F}_J}$$

by recursively setting

$$\Sigma(\dot{x}) = \bigcup_{\langle p,\dot{z}\rangle\in\dot{x}} \mathrm{Orb}(\langle p, \Sigma(\dot{z})\rangle, \mathrm{fix}_{\mathcal{G}_J}(\mathrm{supp}(\dot{x}))).$$

We can think of Σ as hereditarily "spreading" the name $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$ across the index set J, while fixing the part of $\dot{x}$ above its support. Ultimately, in theorem 5.2.11, we will show that Σ can be used to provide an elementary embedding in M between the relativized forcing relations $\Vdash_{N(I,Q)}$ and $\Vdash_{N(J,Q)}$. Lemma 5.1.2 justifies that $\mathrm{rng}(\Sigma) \subseteq \mathrm{HS}_{\mathcal{F}_J}$.

Lemma 5.1.2. *If* $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, *then* $\Sigma(\dot{x}) \in \mathrm{HS}_{\mathcal{F}_J}$ *and* $\mathrm{supp}(\Sigma(\dot{x})) \subseteq \mathrm{supp}(\dot{x}) \subseteq I$.[3]

Proof. Assume inductively that for every $\dot{z} \in \mathrm{rng}(\dot{x})$, $\Sigma(\dot{z}) \in \mathrm{HS}_{\mathcal{F}_J}$. From the definition of Σ, it is apparent that $\mathrm{fix}_{\mathcal{G}_J}(\mathrm{supp}(\dot{x})) \preceq \mathrm{sym}_{\mathcal{G}_J}(\Sigma(\dot{x}))$. Because $I \subseteq J$, note that $\mathrm{supp}(\dot{x}) \in [I]^{<\omega} \subseteq [J]^{<\omega}$, and so $\mathrm{fix}_{\mathcal{G}_J}(\mathrm{supp}(\dot{x})) \in \mathcal{F}_J$. Thus, $\Sigma(\dot{x})$ is $\mathcal{F}_J$-symmetric, and by the inductive assumption, $\Sigma(\dot{x}) \in \mathrm{HS}_{\mathcal{F}_J}$. By lemma 2.5.9, $\mathrm{supp}(\Sigma(\dot{x}))$ is minimal, and so $\mathrm{supp}(\Sigma(\dot{x})) \subseteq \mathrm{supp}(\dot{x}) \subseteq I$. $\square$

An important ingredient in the proofs of our main theorems is that every name in $\mathrm{HS}_{\mathcal{F}_J}$ can be described, in M, in terms of $\mathrm{HS}_{\mathcal{F}_I}$, Σ, and $\mathcal{G}_J$ (see corollary 5.1.8). To show this, it will be helpful to define a "gathering" function $\Gamma : \mathrm{HS}_{\mathcal{F}_J} \to \mathrm{HS}_{\mathcal{F}_I}$, and prove that the restriction of Γ to the class $\{\dot{w} \in \mathrm{HS}_{\mathcal{F}_J} \mid \mathrm{supp}(\dot{w}) \subseteq I\}$ is the inverse of Σ (see theorem 5.1.7).

Definition 5.1.3. Let I, J be infinite, with $I \subseteq J$ and $|I| < |J|$. Recursively define the "gathering" function

$$\Gamma : \mathrm{HS}_{\mathcal{F}_J} \to \mathrm{HS}_{\mathcal{F}_I}$$

by setting

$$\Gamma(\dot{w}) = \{\langle p, \Gamma(\dot{z})\rangle \mid \langle p, \dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I\}.$$

[3]In corollary 5.1.10, we strengthen this to show that $\mathrm{supp}(\dot{x}) = \mathrm{supp}(\Sigma(\dot{x}))$.

We can think of Γ as hereditarily "gathering" the subset of $\dot{w}$ that can be stated with respect to the index set $I \subseteq J$, and thus can be expressed as a $\mathbb{P}(I, Q)$-name.

Lemma 5.1.4. *For $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ and $\pi \in \mathcal{G}_I$, $\pi\Gamma(\dot{w}) = \Gamma(\pi\dot{w})$.*

Proof. Let $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ and inductively assume that for every $\dot{z} \in \mathrm{rng}(\dot{w})$ and every $\pi \in \mathcal{G}_I$, $\pi\Gamma(\dot{z}) = \Gamma(\pi\dot{z})$. We will first show that $\pi\Gamma(\dot{w}) \subseteq \Gamma(\pi\dot{w})$. Suppose that $\langle \pi p, \pi\Gamma(\dot{z})\rangle \in \pi\Gamma(\dot{w})$, with $\langle p, \Gamma(\dot{z})\rangle \in \Gamma(\dot{w})$, and $\langle p, \dot{z}\rangle \in \dot{w}$ with $\mathrm{supp}(p), \mathrm{supp}(\dot{w}) \subseteq I$. Then $\langle \pi p, \pi\dot{w}\rangle \in \pi\dot{w}$, and because $\pi \in \mathcal{G}_I$, $\mathrm{supp}(\pi p), \mathrm{supp}(\pi\dot{w}) \subseteq I$. From this, we see that $\langle \pi p, \Gamma(\pi\dot{z})\rangle \in \Gamma(\pi\dot{w})$, and by the inductive hypothesis $\langle \pi p, \pi\Gamma(\dot{z})\rangle \in \Gamma(\pi\dot{w})$.

To prove the containment $\Gamma(\pi\dot{w}) \subseteq \pi\Gamma(\dot{w})$, let $\langle \pi p, \Gamma(\pi\dot{z})\rangle \in \Gamma(\pi\dot{w})$, with $\langle \pi p, \pi\dot{z}\rangle \in \pi\dot{w}$ and $\mathrm{supp}(\pi p), \mathrm{supp}(\pi\dot{z}) \subseteq I$. Then $\langle p, \dot{z}\rangle \in \dot{w}$, and because $\pi, \pi^{-1} \in \mathcal{G}_I$, $\mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I$. By the definition of $\Gamma(\dot{w})$, $\langle p, \Gamma(\dot{z})\rangle \in \Gamma(\dot{w})$, and so $\langle \pi p, \pi\Gamma(\dot{z})\rangle \in \pi\Gamma(\dot{w})$. By the inductive hypothesis, $\langle \pi p, \Gamma(\pi\dot{z})\rangle \in \pi\Gamma(\dot{w})$, which completes the proof. $\square$

Lemma 5.1.5. *If $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ and $\mathrm{supp}(\dot{w}) \subseteq I$, then $\Gamma(\dot{w}) \in \mathrm{HS}_{\mathcal{F}_I}$ and $\mathrm{supp}(\Gamma(\dot{w})) \subseteq \mathrm{supp}(\dot{w})$.*[4]

Proof. Suppose inductively that for every $\dot{z} \in \mathrm{rng}(\dot{w})$ with $\mathrm{supp}(\dot{z}) \subseteq I$, $\Gamma(\dot{z}) \in \mathrm{HS}_{\mathcal{F}_I}$. From this, we can see that $\Gamma(\dot{w}) \in M^{\mathbb{P}(I,Q)}$, and that $\Gamma(\dot{z}) \in \mathrm{HS}_{\mathcal{F}_I}$ if $\mathrm{sym}_{\mathcal{G}_I}(\Gamma(\dot{w})) \in \mathcal{F}_I$. Let $\mathrm{supp}(\dot{w}) = W \in [I]^{<\omega}$; we will show that $\mathrm{fix}_{\mathcal{G}_I}(W) \preceq \mathrm{sym}_{\mathcal{G}_I}(\Gamma(\dot{w}))$, and thus that $\mathrm{sym}_{\mathcal{G}_I}(\Gamma(\dot{w})) \in \mathcal{F}_I$.

We claim that for every $\mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)) \subseteq \dot{w}$, it is possible to assume without loss of generality that $\mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I$. To see why, let $\mathrm{supp}(p) \cup \mathrm{supp}(\dot{z}) = S$; since $W \subseteq I$,

[4] In corollary 5.1.10, we strengthen this to show that $\mathrm{supp}(\dot{w}) = \mathrm{supp}(\Gamma(\dot{w}))$.

there is a $\pi \in \mathrm{fix}_{\mathcal{G}_J}(W)$ so that $\pi S \subseteq I$. Then $\mathrm{supp}(\pi p), \mathrm{supp}(\pi \dot{z}) \subseteq I$, and

$$\mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)) = \mathrm{Orb}(\langle \pi p, \pi \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)).$$

As a result, we can write

$$\dot{w} = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle q, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)). \tag{5.1.0.1}$$

Given that $\mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I$, note that

$$\{\langle q, \dot{y}\rangle \in \mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)) \mid \mathrm{supp}(q), \mathrm{supp}(\dot{y}) \subseteq I\} = \mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_I}(W)).$$

By this, and from the definition of Γ, we have that

$$\Gamma(\dot{w}) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \{\langle \pi p, \Gamma(\pi \dot{z})\rangle \mid \pi \in \mathrm{fix}_{\mathcal{G}_I}(W)\}.$$

By lemma 5.1.4, since $\pi \in \mathrm{fix}_{\mathcal{G}_I}(W) \preceq \mathcal{G}_I$, we have $\Gamma(\pi \dot{z}) = \pi \Gamma(\dot{z})$, so we rewrite this as

$$\Gamma(\dot{w}) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle p, \Gamma(\dot{z})\rangle, \mathrm{fix}_{\mathcal{G}_I}(W)). \tag{5.1.0.2}$$

From this, it is clear that $\mathrm{fix}_{\mathcal{G}_I}(W) \preceq \mathrm{sym}_{\mathcal{G}_I}(\Gamma(\dot{w}))$, which completes the proof. □

Remark 5.1.6. Generally, if $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$, then we have $\Gamma(\dot{w}) \in \mathrm{HS}_{\mathcal{F}_I}$, as the notation from definition 5.1.3 suggests. We will not use this in any proof, we leave this detail to the reader.

Theorem 5.1.7. *If $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, then $\Gamma(\Sigma(\dot{x})) = \dot{x}$; if $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ and $\mathrm{supp}(\dot{w}) \subseteq I$, then $\Sigma(\Gamma(\dot{w})) = \dot{w}$.*

Proof. To prove the first part of the theorem, let $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$ with $\mathrm{supp}(\dot{x}) = X$, and suppose inductively that for every $\dot{z} \in \operatorname{rng}(\dot{x})$, $\Gamma(\Sigma(\dot{z})) = \dot{z}$. From the definition of Σ and Γ, we can write

$$\Sigma(\dot{x}) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{x}} \operatorname{Orb}(\langle p, \Sigma(\dot{z})\rangle, \operatorname{fix}_{\mathcal{G}_J}(X)),$$

and

$$\Gamma(\Sigma(\dot{x})) = \{\langle \pi p, \Gamma(\pi\Sigma(\dot{z}))\rangle \mid \langle p, \dot{z}\rangle \in \dot{x} \wedge \pi \in \operatorname{fix}_{\mathcal{G}_J}(X) \wedge \operatorname{supp}(\pi p), \operatorname{supp}(\pi\Sigma(\dot{z})) \subseteq I\}.$$

To break this expression down further, we claim, for $\langle p, \dot{z}\rangle \in \dot{x}$ and for $\pi \in \mathcal{G}_J$, that $\operatorname{supp}(\pi p), \operatorname{supp}(\pi\Sigma(\dot{z})) \subseteq I$ if and only if there is a $\tau \in \operatorname{fix}_{\mathcal{G}_I}(X)$ so that $\pi p = \tau p$ and $\pi\Sigma(\dot{z}) = \tau\Sigma(\dot{z})$. To prove this, let $\langle \pi p, \pi\Sigma(\dot{z})\rangle \in \Sigma(\dot{x})$, with $\langle p, \dot{z}\rangle \in \dot{x}$ and $\pi \in \operatorname{fix}_{\mathcal{G}_J}(X)$. Then $\operatorname{supp}(p) \subseteq I$, and by lemma 5.1.2, $\operatorname{supp}(\Sigma(\dot{z})) \subseteq I$. The claim then holds by lemma 2.5.15. As a result, we can write

$$\Gamma(\Sigma(\dot{x})) = \{\langle \tau p, \Gamma(\tau\Sigma(\dot{z}))\rangle \mid \langle p, \dot{z}\rangle \in \dot{x} \wedge \tau \in \operatorname{fix}_{\mathcal{G}_I}(X)\}.$$

By lemma 5.1.4, since $\tau \in \mathcal{G}_I$, we can replace this with

$$\Gamma(\Sigma(\dot{x})) = \{\langle \tau p, \tau\Gamma(\Sigma(\dot{z}))\rangle \mid \langle p, \dot{z}\rangle \in \dot{x} \wedge \tau \in \operatorname{fix}_{\mathcal{G}_I}(X)\}.$$

By the inductive hypothesis, this can again be replaced with

$$\Gamma(\Sigma(\dot{x})) = \{\langle \tau p, \tau\dot{z}\rangle \mid \langle p, \dot{z}\rangle \in \dot{x} \wedge \tau \in \operatorname{fix}_{\mathcal{G}_I}(X)\} = \bigcup_{\langle p,\dot{z}\rangle \in \dot{x}} \operatorname{Orb}(\langle p, \dot{z}\rangle, \operatorname{fix}_{\mathcal{G}_I}(X)) = \dot{x},$$

which proves the first part of the theorem.

To prove the second part of the theorem, let $\operatorname{supp}(\dot{w}) = W$, and suppose inductively for every $\dot{z} \in \operatorname{rng}(\dot{w})$ with $\operatorname{supp}(\dot{z}) \subseteq I$, that $\Sigma(\Gamma(\dot{z})) = \dot{z}$. Because $W \subseteq I$, we meet the hypothesis of lemma 5.1.5; then, as in lines (5.1.0.1) and (5.1.0.2), we can express $\dot{w}$ and $\Gamma(\dot{w})$ as

$$\dot{w} = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle q, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)).$$

and

$$\Gamma(\dot{w}) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle p, \Gamma(\dot{z})\rangle, \mathrm{fix}_{\mathcal{G}_I}(W)).$$

From this, and the definition of Σ, we can see that

$$\Sigma(\Gamma(\dot{w})) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle p, \Sigma(\Gamma(\dot{z}))\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)).$$

To complete the proof, apply the inductive hypothesis to the names $\dot{z}$ in the subscript of the union, for which $\mathrm{supp}(\dot{z}) \subseteq I$, to get

$$\Sigma(\Gamma(\dot{w})) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{w} \wedge \mathrm{supp}(p), \mathrm{supp}(\dot{z}) \subseteq I} \mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(W)) = \dot{w}.$$

□

Corollary 5.1.8. *If $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$, $\pi \in \mathcal{G}_J$, and $\pi\mathrm{supp}(\dot{w}) \subseteq I$, then $\pi^{-1}\Sigma(\Gamma(\pi\dot{w})) = \dot{w}$.*

Proof. $\mathrm{supp}(\pi\dot{w}) = \pi\mathrm{supp}(\dot{w}) \subseteq I$, so by theorem 5.1.7, $\pi^{-1}(\Sigma(\Gamma(\pi\dot{w}))) = \pi^{-1}(\pi\dot{w}) = \dot{w}$. □

Corollary 5.1.8 proves our claim that M can describe the class $\mathrm{HS}_{\mathcal{F}_J}$ in terms of $\mathrm{HS}_{\mathcal{F}_I}, \Sigma$, and $\mathcal{G}_J$. The following corollary will also be useful in the section 5.2.

Corollary 5.1.9. *If $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$ and $\pi \in \mathcal{G}_I$, then $\Sigma(\pi\dot{x}) = \pi\Sigma(\dot{x})$.*

Proof. Let $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, and let $\Sigma(\dot{x}) = \dot{w}$. To prove the corollary, we will justify the line

$$\Sigma(\pi\dot{x}) = \Sigma(\pi\Gamma(\dot{w})) = \Sigma(\Gamma(\pi\dot{w})) = \pi\dot{w} = \pi\Sigma(\dot{x}).$$

By theorem 5.1.7, $\Gamma(\Sigma(\dot{x})) = \dot{x}$, so $\dot{x}$ can be replaced with $\Gamma(\dot{w})$ to show the first equality. Since we assume that $I \subseteq J$, we have that $\pi \in \mathcal{G}_I \preceq \mathcal{G}_J$; then, by lemma 5.1.4, we have that $\Gamma(\pi\dot{w}) = \pi\Gamma(\dot{w})$ to justify the second equality. We then claim that $\mathrm{supp}(\pi\dot{w}) \subseteq I$. We have that $\mathrm{supp}(\pi\dot{w}) = \mathrm{supp}(\pi\Sigma(\dot{x})) = \pi\mathrm{supp}(\Sigma(\dot{x}))$; by lemma 5.1.2, we have that $\pi\mathrm{supp}(\Sigma(\dot{x})) \subseteq \pi\mathrm{supp}(\dot{x})$; because $\pi \in \mathcal{G}_I$ and $\mathrm{supp}(\dot{x}) \subseteq I$, $\pi\mathrm{supp}(\dot{x}) \subseteq I$ and the claim holds. Thus, because $\mathrm{supp}(\pi\dot{w}) \subseteq I$ and by theorem 5.1.7, we have that $\Sigma(\Gamma(\pi\dot{w})) = \pi\dot{w}$. Lastly, we had set $\dot{w} = \Sigma(\dot{x})$ by definition. $\square$

Although our main theorems can be proven without using this next corollary, it is quick to prove and helps to fill out a mental image of the maps Σ and Γ.

Corollary 5.1.10. *If* $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$*, then* $\mathrm{supp}(\dot{x}) = \mathrm{supp}(\Sigma(\dot{x}))$*; if* $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ *and* $\mathrm{supp}(\dot{w}) \subseteq I$*, then* $\mathrm{supp}(\dot{w}) = \mathrm{supp}(\Gamma(\dot{w}))$*.*

Proof. Let $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$. By lemma 5.1.2, $\Sigma(\dot{x}) \in \mathrm{HS}_{\mathcal{F}_J}$ and $\mathrm{supp}(\Sigma(\dot{x})) \subseteq \mathrm{supp}(\dot{x})$. By lemma 5.1.5, $\mathrm{supp}(\Gamma(\Sigma(\dot{x}))) \subseteq \mathrm{supp}(\Sigma(\dot{x}))$, and by theorem 5.1.7, $\mathrm{supp}(\Gamma(\Sigma(\dot{x}))) = \mathrm{supp}(\dot{x})$. Thus $\mathrm{supp}(\dot{x}) \subseteq \mathrm{supp}(\Sigma(\dot{x}))$, and so $\mathrm{supp}(\dot{x}) = \mathrm{supp}(\Sigma(\dot{x}))$.

Let $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ with $\mathrm{supp}(\dot{w}) \subseteq I$. By lemma 5.1.5, $\Gamma(\dot{w}) \in \mathrm{HS}_{\mathcal{F}_I}$ and $\mathrm{supp}(\Gamma(\dot{w})) \subseteq \mathrm{supp}(\dot{w})$. By lemma 5.1.2, $\mathrm{supp}(\Sigma(\Gamma(\dot{w}))) \subseteq \mathrm{supp}(\Gamma(\dot{w}))$, and by theorem 5.1.7, $\mathrm{supp}(\Sigma(\Gamma(\dot{w}))) = \mathrm{supp}(\dot{w})$. Thus $\mathrm{supp}(\dot{w}) \subseteq \mathrm{supp}(\Gamma(\dot{w}))$, and so $\mathrm{supp}(\dot{w}) = \mathrm{supp}(\Gamma(\dot{w}))$. $\square$

5.2 An Isomorphism and an Elementary Embedding

We can now fix a bijection $b : I \to J$ in some outer model V of M, in which M is a class, and consider the natural isomorphisms in V between $V^{\mathbb{P}(I,Q)}$ and $V^{\mathbb{P}(J,Q)}$, and between $\Vdash^V_{N(I,Q)}$ and $\Vdash^V_{N(J,Q)}$. We will show that, in V, these maps restrict to isomorphisms between the relativizations $(\mathrm{HS}_{\mathcal{F}_I})^M$ and $(\mathrm{HS}_{\mathcal{F}_J})^M$, and between the relativizations $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$. We can first formally state the natural isomorphisms that b extends to in V.

Definition 5.2.1. Suppose that $b : I \to J$ is a bijection in $V \vDash \mathrm{ZFC}$. By a slight misuse of notation, define the map

$$b : \mathbb{P}(I,Q) \to \mathbb{P}(J,Q)$$

by sending $\mathrm{supp}(b(p)) = b(\mathrm{supp}(p))$ and for every $i \in \mathrm{supp}(b(p))$, setting

$$b(p)(b(i)) = p(i).$$

From this, recursively define the map $\varphi_b : V^{\mathbb{P}(I,Q)} \to V^{\mathbb{P}(J,Q)}$ by

$$\varphi_b(\dot{x}) = \{\langle b(p), \varphi_b(\dot{z})\rangle \mid \langle p, \dot{z}\rangle \in \dot{x}\}.$$

Observation 5.2.2. Notice that for $p \in \mathbb{P}(I,Q)$, the definition of $b(p)$ can equivalently be expressed by setting $b(p) = \pi p$, for any $\pi \in \mathcal{G}_J$ so that $\pi(i) = b(i)$ for every $i \in \mathrm{supp}(p)$.

It will be convenient to state the following lemma, which follows from this observation (or immediatly from definition 5.2.1).

Lemma 5.2.3. *Suppose that $b : I \to J$ is a bijection in $V \vDash \mathrm{ZFC}$, and that $\pi \in \mathcal{G}_J$. Then $b' : I \to J$, defined by*

$$b'(i) = \pi b(i),$$

is a bijection, and the consequent maps

$$b' : \mathbb{P}(I,Q) \to \mathbb{P}(J,Q),\ \ \varphi_{b'} : V^{\mathbb{P}(I,Q)} \to V^{\mathbb{P}(J,Q)},$$

from definition 5.2.1, are given by

$$b'(p) = \pi b(p), \;\; \varphi_{b'}(\dot{x}) = \pi \varphi_b(\dot{x}).$$

The following lemma is also standard to verify.

Lemma 5.2.4. *Given a bijection $b : I \to J$, the map $b : \mathbb{P}(I, Q) \to \mathbb{P}(J, Q)$ in definition 5.2.1 is a $\leq$-preserving isomorphism, and the map $\varphi_b : V^{\mathbb{P}(I,Q)} \to V^{\mathbb{P}(J,Q)}$ is an isomorphism in the sense that it is a bijection that preserves the relation*

$$\langle p, \dot{z}\rangle \in \dot{x} \iff \langle b(p), \varphi_b(\dot{z})\rangle \in \varphi_b(\dot{x}).$$

Moreover, $\langle b, \varphi_b\rangle$ is an isomorphism between $\Vdash^V_{N(I,Q)}$ and $\Vdash^V_{N(J,Q)}$ in the sense that for every formula ϕ, we have that for every $\dot{x}_0, ..., \dot{x}_n \in (\mathrm{HS}_{\mathcal{F}_I})^V$

$$p \Vdash^V_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n) \iff b(p) \Vdash^V_{N(J,Q)} \phi(\varphi_b(\dot{x}_0), ..., \varphi_b(\dot{x}_n)),$$

and for every $\dot{w}_0, ..., \dot{w}_n \in (\mathrm{HS}_{\mathcal{F}_J})^V$,

$$b^{-1}(p) \Vdash^V_{N(I,Q)} \phi(\varphi_b^{-1}(\dot{w}_0), ..., \varphi_b^{-1}(\dot{w}_n)) \iff p \Vdash^V_{N(J,Q)} \phi(\dot{w}_0, ..., \dot{w}_n).$$

We can now consider how the map φ_b behaves when restricted to $(\mathrm{HS}_{\mathcal{F}_I})^M$. Put informally, if $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, with $X = \mathrm{supp}(\dot{x})$, then φ_b alters $\dot{x}$ in two ways:

(i) it takes the part of $\dot{x}$ defined outside of its support X and spreads it across $J \setminus X$;

(ii) it copies the resulting name from (i), which still has the support X, and redefines it over the new support $b(X) = \{b(i) \mid i \in X\}$.

Point (i) can be expressed as taking $\dot{x}$ to $\Sigma(\dot{x})$; point (ii) can be expressed as taking $\Sigma(\dot{x})$ to $\pi\Sigma(\dot{x})$, for some π so that $\pi(i) = b(i)$ for every $i \in X$. Theorem 5.2.5 verifies that this intuition is correct, and establishes the central relationship between $\Sigma \in M$ and $\varphi_b \in V$.

Theorem 5.2.5. *Let $M \vDash \mathrm{ZFC}$, $I \subseteq J$ and $|I| < |J|$ in M; let $V \vDash \mathrm{ZFC}$ be an outer model of M, with M a class in V, and let $b \in V$ be a bijection from I to J. If $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, $\pi \in \mathcal{G}_J$, and $\pi(i) = b(i)$ for every $i \in \mathrm{supp}(\dot{x})$, then*

$$\varphi_b(\dot{x}) = \pi\Sigma(\dot{x}).$$

Proof. Let $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, $\mathrm{supp}(\dot{x}) = X$, and $\pi \in \mathcal{G}_J$ for which $\pi(i) = b(i)$ for every $i \in X$. Suppose inductively that for every bijection $b : I \to J$, every $\dot{z} \in \mathrm{rng}(\dot{x})$, and every $\tau \in \mathcal{G}_J$, that if $\tau(i) = b(i)$ for every $i \in \mathrm{supp}(\dot{z})$, then $\varphi_b(\dot{z}) = \tau\Sigma(\dot{z})$.

Let $b' = \pi^{-1}b$, so that $b'(i) = i$ for every $i \in X$. We will show that $\varphi_{b'}(\dot{x}) = \Sigma(\dot{x})$, in which case $\varphi_b(\dot{x}) = \pi\Sigma(\dot{x})$ by lemma 5.2.3. Recall that

$$\varphi_{b'}(\dot{x}) = \{\langle b'(p), \varphi_{b'}(\dot{z})\rangle \mid \langle p, \dot{z}\rangle \in \dot{x}\}$$

and

$$\Sigma(\dot{x}) = \bigcup_{\langle p,\dot{z}\rangle \in \dot{x}} \mathrm{Orb}(\langle p, \dot{z}\rangle, \mathrm{fix}_{\mathcal{G}_J}(X)).$$

To show that $\varphi_{b'}(\dot{x}) \subseteq \Sigma(\dot{x})$, let $\langle b'(p), \varphi_{b'}(\dot{z})\rangle \in \varphi_{b'}(\dot{x})$ be arbitrary, with $\langle p, \dot{z}\rangle \in \dot{x}$. By the inductive hypothesis, note that

$$\langle b'(p), \varphi_{b'}(\dot{z})\rangle = \langle b'(p), \tau\Sigma(\dot{z})\rangle$$

for some $\tau \in \mathcal{G}_J$ so that $b'(i) = \tau(i)$ for every $i \in \mathrm{supp}(\dot{z})$. We can also assume that

$\tau(i) = b'(i)$ for every $i \in \mathrm{supp}(p)$, so that $b'(p) = \tau p$ by observation 5.2.2. Because b' fixes X pointwise, we can further assume that $\tau \in \mathrm{fix}_{\mathcal{G}_J}(X)$. In this case, because $\langle p, \dot{z}\rangle \in \dot{x}$ and $\tau \in \mathrm{fix}_{\mathcal{G}_J}(X)$, we have that

$$\langle b'(p), \varphi_{b'}(\dot{z})\rangle = \langle \tau p, \tau\Sigma(\dot{z})\rangle \in \mathrm{Orb}(\langle p, \Sigma(\dot{z})\rangle, \mathrm{fix}_{\mathcal{G}_J}(X)) \subseteq \Sigma(\dot{x}),$$

so $\langle b'(p), \varphi_{b'}(\dot{z})\rangle \in \Sigma(\dot{x})$, and thus $\varphi_{b'}(\dot{x}) \subseteq \Sigma(\dot{x})$.

To show that $\Sigma(\dot{x}) \subseteq \varphi_{b'}(\dot{x})$, let $\langle \tau p, \tau\Sigma(\dot{z})\rangle \in \Sigma(\dot{x})$, with $\langle p, \dot{z}\rangle \in \dot{x}$ and $\tau \in \mathrm{fix}_{\mathcal{G}_J}(X)$. We wish to express $\tau = \rho\sigma$ for some $\sigma \in \mathrm{fix}_{\mathcal{G}_I}(X)$, and some $\rho \in \mathcal{G}_J$ so that $\rho(i) = b'(i)$ for every $i \in \mathrm{supp}(\sigma p) \cup \mathrm{supp}(\sigma\dot{z})$. Let $S = \mathrm{supp}(p) \cup \mathrm{supp}(\dot{z}) \subseteq I$, and fix $\sigma \in \mathcal{G}_J$ so that, for every $i \in S$, $\sigma(i) = b'^{-1}(\tau(i))$. Because b' fixes X pointwise and $\tau \in \mathrm{fix}_{\mathcal{G}_J}(X)$, it is possible to define such a σ in $\mathrm{fix}_{\mathcal{G}_J}(X)$. Because $S \subseteq I$ and $\mathrm{rng}(b'^{-1}) = I$, it is also possible to take $\sigma \in \mathrm{fix}_{\mathcal{G}_I}(X)$. Fix $\rho \in \mathcal{G}_J$ so that $\tau = \rho\sigma$, in which case for every $i \in S$, $\tau(i) = \rho(\sigma(i)) = b'(\sigma(i))$. Thus, $\rho(i) = b'(i)$ for every $i \in \mathrm{supp}(\sigma p) \cup \mathrm{supp}(\sigma\dot{z})$.

We can prove that $\langle \tau p, \tau\Sigma(\dot{z})\rangle \in \varphi_{b'}(\dot{x})$, and thus prove $\Sigma(\dot{x}) \subseteq \varphi_{b'}(\dot{x})$, by justifying the line

$$\langle \tau p, \tau\Sigma(\dot{z})\rangle = \langle \rho\sigma p, \rho\sigma\Sigma(\dot{z})\rangle = \langle \rho\sigma p, \rho\Sigma(\sigma\dot{z})\rangle = \langle b'(\sigma p), \varphi_{b'}(\sigma\dot{z})\rangle \in \varphi_{b'}(\dot{x}). \tag{5.2.0.1}$$

By observation 5.2.2, and because $\rho(i) = b'(i)$ for every $i \in \mathrm{supp}(\sigma p)$, $\tau p = \rho\sigma p = b(\sigma p)$; thus, equality holds on the first component across line (5.2.0.1). By corollary 5.1.9, because $\sigma \in \mathcal{G}_I$, $\tau\Sigma(\dot{z}) = \rho\sigma\Sigma(\dot{z}) = \rho\Sigma(\sigma\dot{z})$. By the inductive hypothesis, and because $\rho(i) = b'(i)$ for every $i \in \mathrm{supp}(\sigma\dot{z})$, $\rho\Sigma(\sigma\dot{z}) = \varphi_{b'}(\sigma\dot{z})$. Because $\sigma \in \mathrm{fix}_{\mathcal{G}_I}(X)$, $\langle \sigma p, \sigma\dot{z}\rangle \in \dot{x}$, and so $\langle b'(\sigma p), \varphi_{b'}(\sigma\dot{z})\rangle \in \varphi_{b'}(\dot{x})$. □

Remark 5.2.6. Although $\varphi_b \notin M$, by theorem 5.2.5 it is still the case that $\varphi_b(\dot{x}) \in M$ for every $\dot{x} \in (\mathrm{HS}_{\mathcal{F}_I})^M$. Theorem 5.2.5 does not imply that the entire function φ_b is in M, since this would require that M has a function assigning a $\pi \in \mathcal{G}_J$ to every $\dot{x} \in (\mathrm{HS}_{\mathcal{F}_I})^M$, so that $\pi(i) = b(i)$ for every $i \in \mathrm{supp}(\dot{x})$. It is straightforward to see that the existence of such a function is equivalent to the existence of the bijection b.

Lemma 5.2.7. *Let $M \vDash \mathrm{ZFC}$, $I \subseteq J$ and $|I| < |J|$ in M; let $V \vDash \mathrm{ZFC}$ be an outer model of M, with M a class in V, and let $b \in V$ be a bijection from I to J. The map φ_b restricts to an isomorphism from $(\mathrm{HS}_{\mathcal{F}_I})^M$ to $(\mathrm{HS}_{\mathcal{F}_J})^M$.*

Proof. We will prove that φ_b restricts to a bijection between $(\mathrm{HS}_{\mathcal{F}_I})^M$ and $(\mathrm{HS}_{\mathcal{F}_J})^M$. If this can be done, then because the map $\varphi_b : V^{\mathbb{P}(I,Q)} \to V^{\mathbb{P}(J,Q)}$ is an isomorphism, in the sense of lemma 5.2.4, the fact that $\varphi_b((\mathrm{HS}_{\mathcal{F}_I})^M) = (\mathrm{HS}_{\mathcal{F}_J})^M$ shows that the restriction of φ to $(\mathrm{HS}_{\mathcal{F}_I})^M$ will also be an isomorphism in the sense of lemma 5.2.4.

Theorem 5.2.5 shows, in particular, that $\varphi_b((\mathrm{HS}_{\mathcal{F}_I})^M) \subseteq (\mathrm{HS}_{\mathcal{F}_J})^M$; for $\dot{x} \in (\mathrm{HS}_{\mathcal{F}_I})^M$, theorem 5.2.5 gives us that $\varphi_b(\dot{x}) = \pi\Sigma(\dot{x})$, with $\pi \in \mathcal{G}_J \in M$ and $\Sigma(\dot{x}) \in (\mathrm{HS}_{\mathcal{G}_J})^M$ by lemma 5.1.2.

To prove $(\mathrm{HS}_{\mathcal{F}_J})^M \subseteq \varphi_b((\mathrm{HS}_{\mathcal{F}_I})^M)$, we will apply corollary 5.1.8. Let $\dot{w} \in (\mathrm{HS}_{\mathcal{F}_J})^M$ and take $\pi \in \mathcal{G}_J$ so that for every $i \in \mathrm{supp}(\dot{w})$, $\pi(i) = b^{-1}(i)$. In this case, $\mathrm{supp}(\pi\dot{w}) \subseteq I$, and by corollary 5.1.8,

$$\dot{w} = \pi^{-1}\Sigma(\Gamma(\pi\dot{w})).$$

Let $\dot{x} = \Gamma(\pi\dot{w})$. Because $\mathrm{supp}(\pi\dot{w}) \subseteq I$, by lemma 5.1.5, $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$. We can then calculate that, by corollary 5.1.10

$$\mathrm{supp}(\dot{x}) = \mathrm{supp}(\Gamma(\pi\dot{w})) = \mathrm{supp}(\pi\dot{w}) = \pi\mathrm{supp}(\dot{w}).$$

Because we took π so that for every $i \in \operatorname{supp}(\pi\dot{w}) = \operatorname{supp}(\dot{x})$, $\pi^{-1}(i) = b(i)$, we can apply theorem 5.2.5 to conclude that

$$\dot{w} = \pi^{-1}\Sigma(\Gamma(\pi\dot{w})) = \pi^{-1}\Sigma(\dot{x}) = \varphi_b(\dot{x}).$$

Since $\dot{w} \in \mathrm{HS}_{\mathcal{F}_J}$ was arbitrary, $(\mathrm{HS}_{\mathcal{F}_J})^M \subseteq \varphi_b((\mathrm{HS}_{\mathcal{F}_I})^M)$, which completes the proof. $\square$

Remark 5.2.8. Note the important distinction that φ_b does *not* restrict to an $\in$-preserving isomorphism between the relativized classes $M^{\mathbb{P}(I,Q)}$ and $M^{\mathbb{P}(J,Q)}$. If $\dot{A}_I$ and $\dot{A}_J$ are the respective names for the sets of mutually Q-generic filters added by $\mathbb{P}(I,Q)$ and $\mathbb{P}(J,Q)$, as in definition 2.5.6, then $\varphi_b(\dot{A}_I) = \dot{A}_J$. However, there is a $\mathbb{P}(I,Q)$-name $\dot{g}$ for a bijection between $\dot{A}_I$ and $|\check{I}|$; $\varphi_b(\dot{g})$ would also be a name for a bijection between $\dot{A}_J$ and $|\check{I}|$. Since $|I| < |J|$ in M, $\varphi_b(\dot{g}) \in V^{\mathbb{P}(J,Q)} \setminus M^{\mathbb{P}(J,Q)}$.

Theorem 5.2.9. *Let $M \models \mathrm{ZFC}$, $I \subseteq J$ and $|I| < |J|$ in M; let $V \models \mathrm{ZFC}$ be an outer model of M, with M a class in V, and let $b \in V$ be a bijection from I to J. Then the map $\langle b, \varphi_b\rangle$ from lemma 5.2.4 restricts to an isomorphism in V between the relativized forcing relations $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$.*

Proof. We have that $\langle b, \varphi_b\rangle$ is an isomorphism in V between $\Vdash^V_{N(I,Q)}$ and $\Vdash^V_{N(J,Q)}$, in the sense of lemma 5.2.4. Because φ_b restricts to an isomorphism between $(\mathrm{HS}_{\mathcal{F}_I})^M$ and $(\mathrm{HS}_{\mathcal{F}_J})^M$, it is straightforward to check that $\langle b, \varphi_b \restriction_{(\mathrm{HS}_{\mathcal{F}_I})^M}\rangle$ is an isomorphism between the relativized forcing relations $\Vdash^M_{N(I,Q)}$ and $\Vdash^M_{N(J,Q)}$, in that for every $p \in \mathbb{P}(I,Q)$ and $\dot{x}_0, ..., \dot{x}_n \in (\mathrm{HS}_{\mathcal{F}_I})^M$,

$$p \Vdash^M_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_0) \iff b(p) \Vdash^M_{N(J,Q)} \phi(\varphi_b(\dot{x}_0), ..., \varphi_b(\dot{x}_n)),$$

and that for every $p \in \mathbb{P}(J,Q)$ and $\dot{w}_0, ..., \dot{w}_n \in (\mathrm{HS}_{\mathcal{F}_J})^M$,

$$p \Vdash^M_{N(J,Q)} \phi(\dot{w}_0, ..., \dot{w}_0) \iff b^{-1}(p) \Vdash^M_{N(I,Q)} \phi(\varphi_b^{-1}(\dot{w}_0), ..., \varphi_b^{-1}(\dot{w}_n)).$$

$\square$

Corollary 5.2.10. *Let $M \models \mathrm{ZFC}$ and let $I, J \in M$ be infinite. For every $\mathbb{P}(I,Q)$-generic G, there is a $\mathbb{P}(J,Q)$-generic F so that $N^G(I,Q) = N^F(J,Q)$.*

Proof. Let G be an arbitrary $\mathbb{P}(I,Q)$-generic filter over M. Let H be $\mathrm{Coll}(I,J)$-generic over M, and let V be any outer model for which both $G, H \in V$, and in which M is a class. Let $b \in V$ be an bijection from I to J. Extend b to the isomorphism between $\mathbb{P}(I,Q)$ and $\mathbb{P}(J,Q)$ from definition 5.2.1. From this isomorphism, it is straightforward to show that $b(G)$ is $\mathbb{P}(J,Q)$-generic over M.

We then claim that for every $\dot{x} \in \mathrm{HS}_{\mathcal{F}_I}$, $\dot{x}^G = \varphi_b(\dot{x})^F$. If this can be shown, the fact that $\varphi_b \restriction_{(\mathrm{HS}_{\mathcal{F}_I})^M}$ surjects onto $(\mathrm{HS}_{\mathcal{F}_J})^M$ shows the desired result that

$$N^G(I,Q) = N^F(J,Q).$$

Assume inductively that for every $\dot{z} \in \mathrm{rng}(\dot{x})$, $\dot{z}^G = \varphi_b(\dot{z})^F$. Using theorem 5.2.9, we have the following sequence of equivalences in V:

$$\begin{aligned}
\dot{z}^G \in \dot{x}^G &\iff (\exists p \in G) p \Vdash^M_{N(I,Q)} \dot{z} \in \dot{x} \\
&\iff (\exists b(p) \in b(G) = F) b(p) \Vdash^M_{N(J,Q)} \varphi_b(\dot{z}) \in \varphi_b(\dot{x}) \\
&\iff \varphi_b(\dot{z})^F \in \varphi_b(\dot{x})^F.
\end{aligned}$$

By the inductive hypothesis, $\dot{z}^G = \varphi_b(\dot{z})^F$. By the definition of $\varphi_b(\dot{x})$, every element of $\varphi_b(\dot{x})^F$ is of the form $\varphi_b(\dot{z})^F$, for some $\dot{z} \in \mathrm{rng}(\dot{x})$. We have thus shown that $\dot{x}^G$ and $\varphi_b(\dot{x})^F$ have the same elements, and so they are equal. $\square$

Theorem 5.2.11. *For every $p \in \mathbb{P}(I,Q)$, every $\dot{x}_0, ..., \dot{x}_n \in \mathrm{HS}_{\mathcal{F}_I}$, and every formula ϕ,*

$$p \Vdash_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n) \iff p \Vdash_{N(J,Q)} \phi(\Sigma(\dot{x}_0), ..., \Sigma(\dot{x}_n)).$$

Proof. Let G be $\mathrm{Coll}(I,J)$-generic, and let $V = M[G]$. Fix a $p \in \mathbb{P}(I,Q)$, $\dot{x}_0, ..., \dot{x}_n \in \mathrm{HS}_{\mathcal{F}_I}$, and a formula ϕ. Let $X = \mathrm{supp}(p) \cup \bigcup_{i \leq n} \mathrm{supp}(\dot{x})$. There is a bijection $b_X \in V$ that fixes X pointwise, in which case $b_X(p) = p$ and by theorem 5.2.5, for every $i \leq n$, $\varphi_{b_X}(\dot{x}_i) = \Sigma(\dot{x})$. Applying theorem 5.2.9 results in the conclusion of the theorem for $p, \dot{x}_0, ..., \dot{x}_n$, and ϕ, which were arbitrary. □

Corollary 5.2.12. *Let $M \vDash \mathrm{ZFC}$, let $I \subseteq J$ be infinite sets in M, and let F be $\mathbb{P}(J,Q)$-generic over M. Then $G = \{p \in F \mid \mathrm{supp}(p) \subseteq I\}$ is $\mathbb{P}(I,Q)$-generic over M, $N^G(I,Q), N^F(J,Q) \subseteq M[F]$, and the map*

$$\Sigma^* : N^G(I,Q) \to N^F(J,Q)$$

in $M[F]$, defined by

$$\Sigma^*(\dot{x}^G) = \Sigma(\dot{x})^F$$

is well-defined and an elementary embedding.

Proof. The claim that G is $\mathbb{P}(I,Q)$-generic over M is straightforward to verify. To prove that Σ^* is well-defined, let $\dot{x}_0, \dot{x}_1 \in (\mathrm{HS}_{\mathcal{F}_I})^M$ so that $\dot{x}_0^G = \dot{x}_1^G$. Then there is a $p \in G \subseteq \mathbb{P}(I,Q)$ so that

$$p \Vdash^M_{N(I,Q)} \dot{x}_0 = \dot{x}_1.$$

By theorem 5.2.11, we get that

$$p \Vdash^M_{N(J,Q)} \Sigma(\dot{x}_0) = \Sigma(\dot{x}_1).$$

Since $G \subseteq F$, $p \in F$, so in $M[F]$, $\Sigma^*(\dot{x}_0^G) = \Sigma(\dot{x}_0)^F = \Sigma(\dot{x}_1)^F = \Sigma^*(\dot{x}_1^G)$, making Σ^* well-defined.

Nearly the same argument works to show that Σ^* is an elementary embedding. Let $\dot{x}_0^G, ..., \dot{x}_n^G \in N^G(I,Q)$, and let ϕ be an arbitrary formula. The corollary then follows by justifying that

$$\begin{aligned} N^G(I,Q) \vDash \phi(\dot{x}_0^G, ..., \dot{x}_n^G) &\iff (\exists p \in G) p \Vdash^M_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n) \\ &\iff (\exists q \in F) q \Vdash^M_{N(J,Q)} \phi(\Sigma(\dot{x}_0), ..., \Sigma(\dot{x}_n)) \\ &\iff N^F(J,Q) \vDash \phi(\Sigma^*(\dot{x}_0)^F, ..., \Sigma^*(\dot{x}_n)^F). \end{aligned}$$

The second equivalence is the only one that needs further justification. To prove the direction ($\Rightarrow$), note that $p \in G \subseteq F$, so by theorem 5.2.11 we can take $p = q \in F$ so that the latter statement holds. To prove the direction ($\Leftarrow$), note that for every $i \leq n$, $\text{supp}(\Sigma(\dot{x}_i)) \subseteq I$ by lemma 5.1.2. By lemma 2.5.13, taking $p = q \restriction_I \in F$, we have that

$$p \Vdash^M_{N(J,Q)} \phi(\Sigma(\dot{x}_0), ..., \Sigma(\dot{x}_n)),$$

and that $p \in G$. By theorem 5.2.11, we then have that

$$p \Vdash^M_{N(I,Q)} \phi(\dot{x}_0, ..., \dot{x}_n).$$

$\square$

Bibliography

[Bla79] Andreas Blass. Injectivity, projectivity, and the axiom of choice. *Transactions of the American Mathematical Society*, 255:31–59, 1979.

[Bla87] Andreas Blass. Prime ideals yield almost maximal ideals. *Fundamenta Mathematicae*, 127:57–66, 1987.

[Bla11] Andreas Blass. Partitions and permutation groups. *Contemporary Mathematics*, 558:453–466, 2011.

[BPT13] Manuel Bodirsky, Michael Pinsker, and Todor Tsankov. Decidability of definability. *The Journal of Symbolic Logic*, 78(4):1036–1054, 2013.

[EMHR11] Paul Erdös, Attila Máté, Andras Hajnal, and P Rado. *Combinatorial set theory: partition relations for cardinals*, volume 106. Elsevier, 2011.

[Gol22] Isaac Goldbring. *Ultrafilters throughout mathematics*, volume 220. American Mathematical Society, 2022.

[Gri75] Serge Grigorieff. Intermediate submodels and generic extensions in set theory. *Annals of Mathematics*, 101(3):447–490, 1975.

[Hal62] James D. Halpern. *Contributions to the study of the independence of the axiom of choice*. University of California, Berkeley, 1962.

[Hal64] James D. Halpern. The independence of the axiom of choice from the boolean prime ideal theorem. *Fundamenta Mathematicae*, 55:57–66, 1964.

[Hal07] Eric J Hall. Permutation models and svc. *Notre Dame Journal of Formal Logic*, 48(2):229–235, 2007.

[Ham00] Joel David Hamkins. The lottery preparation. *Annals of Pure and Applied Logic*, 101(2-3):103–146, 2000.

[Ham25] Joel David Hamkins. Every countable model of arithmetic or set theory has a pointwise-definable end extension. *KRITERION – Journal of Philosophy*, 39(1-2):27–48, 2025.

[HKT13] Eric J Hall, Kyriakos Keremedis, and Eleftherios Tachtsis. The existence of free ultrafilters on ω does not imply the extension of filters on ω to ultrafilters. *Mathematical Logic Quarterly*, 59:258–267, 2013.

[HL64] James D Halpern and Azriel Levy. The ordering theorem does not imply the axiom of choice. *Notices of the American Mathematical Society*, 11:56, 1964.

[HL66] James D. Halpern and Hans Läuchli. A partition theorem. *Transactions of the American Mathematical Society*, 124:360–367, 1966.

[HL71] James D Halpern and Azriel Lévy. The boolean prime ideal theorem does not imply the axiom of choice. In *Axiomatic set theory*, volume 13, pages 84–134. Proceedings of Symposia in Pure Mathematics, 1971.

[Jec08] Thomas J Jech. *The axiom of choice.* Courier Corporation, 2008.

[Joh84] Peter Johnstone. Almost maximal ideals. *Fundamenta Mathematicae*, 123:197–209, 1984.

[Kar19] Asaf Karagila. Iterated failures of choice. *arXiv preprint arXiv:1911.09285*, 2019.

[KPT05] AS Kechris, VG Pestov, and S Todorcevic. Fraïssé limits, ramsey theory, and topological dynamics of automorphism groups. *Geometric & Functional Analysis GAFA*, 15(1):106–189, 2005.

[KS20] Asaf Karagila and Philipp Schlicht. How to have more things by forgetting how to count them. *Proceedings of the Royal Society A*, 476:782–793, 2020.

[KS26] Asaf Karagila and Jonathan Schilhan. Towards a theory of symmetric extensions. *arXiv preprint arXiv:2602.17338*, 2026.

[Mos39] Andrzej Mostowski. Über die unabhängigkeit des wohlordnungssatzes vom ordnungsprinzip. *Fundamenta mathematicae*, 32(1):201–252, 1939.

[Pin77a] David Pincus. Adding dependent choice. *Annals of Mathematical Logic*, 11:105–145, 1977.

[Pin77b] David Pincus. Adding dependent choice to the prime ideal theorem. In *Studies in Logic and the Foundations of Mathematics*, volume 87, pages 547–565. Elsevier, 1977.

[Pin97] David Pincus. The dense linear ordering principle. *The Journal of Symbolic Logic*, 62:438–516, 1997.

[Ste23] Nedeljko Stefanovic. Alternatives to the halpern-läuchli theorem. *Ann. Pure Appl. Log.*, 174(9):103313, 2023.

[TF95] Stevo Todorchevich and Ilijas Farah. *Some applications of the method of forcing.* Yenisei, 1995.

[Tod07] Stevo Todorcevic. *Walks on ordinals and their characteristics.* Springer, 2007.

Appendix A

A Proof of Lemma 4.2.5

The poset $\mathcal{C}_\theta$ and the $\mathcal{C}_\theta$-name $\dot{r}_\alpha$, defined in [TF95], can be compared to $\mathbb{P}(\theta, 2^{<\omega})$, as in definition 2.5.2, and the α^{th} Cohen real added by $\mathbb{P}(\theta, 2^{<\omega})$, which we denoted $\dot{a}_\alpha$ in definition 2.5.6.

Definition A.0.1 (Todorčević, Farah, [TF95])**.** Let $\mathcal{C}_\theta$ denote the poset of finite partial functions $p : \theta \to 2$, ordered by reverse-inclusion. For $\alpha < \theta$, define the $\mathcal{C}_\theta$-name $\dot{r}_\alpha$ so that

$$(p \Vdash \dot{r}_\alpha(m) = \check{n}) \iff (\omega\alpha + m \in \operatorname{dom}(p) \wedge p(\omega\alpha + m) = n).$$

In [TF95], the term "isomorphism type" or just "type" is used. We replace this with "TF-type" and so as not to conflict with our use of "type" in definition 4.5.

Definition A.0.2 (Todorčević, Farah, [TF95])**.** Two conditions $p, q \in \mathcal{C}_\theta$ are said to be of the same *TF-type* if there is an order-preserving map π from $\operatorname{dom}(p)$ to $\operatorname{dom}(q)$, so that $p(\xi) = q(e(\xi))$. We use the term *TF-type* to refer to an equivalence class in $\mathcal{C}_\theta$ under this relation.

Note that the notion of TF-type is distinct from the notion of type in definition 4.5, given the difference in the underlying posets $\mathcal{C}_\theta$ and $\mathbb{P}(\theta, Q)$, but that these capture the same idea.

Lemma A.0.3 (Lemma 6.8, [TF95]). *For every TF-type t of a condition in $\mathcal{C}_\theta$ and integers m and d, there is an integer $M = M(t, m, d)$ such that for every set $\{p_{\vec{\alpha}} \mid \vec{\alpha} \in M^d\}$ of elements of $\mathcal{C}_\theta$ of TF-type t, there exist $H_i \subseteq M$ (for $i < d$) of size m, so that the conditions in $\{p_{\vec{\alpha}} \mid \vec{\alpha} \in \prod_{i<d} H_i\}$ are pairwise compatible.*

Corollary A.0.4. *Lemma 4.6 holds.*

Proof. Let t be a type in $\mathbb{P}(\theta, Q)$ and notice that for every $p, r \in t$ (which is well-defined, since $t \subseteq \mathbb{P}(\theta, Q)$ is an equivalence class), $\operatorname{rng}(p) = \operatorname{rng}(r) \subseteq Q$. Let $\operatorname{rng}(t) \subseteq Q$ be the finite range of any condition in t. For every $q \in \operatorname{rng}(t) \subseteq Q$, fix some $t_q \in 2^\omega$, so that for every $q, q' \in \operatorname{rng}(t)$,

$$t_q \mid\mid t_{q'} \iff q \mid\mid q'.$$

Define the map

$$f : t \to \mathcal{C}_\theta$$

by setting $f(p)$ to be the finite partial function

$$f(p) : \bigcup_{\alpha \in \operatorname{supp}(p)} [\omega\alpha, \omega(\alpha + 1)) \to 2,$$

so that when $p(\alpha) = q$, $f(p) \restriction_{[\omega\alpha, \omega\alpha + \operatorname{ht}(t_q))]}$ is isomorphic to $t_q \in 2^{<\omega}$. This ensures that conditions $p, r \in t$ are compatible iff and only if $f(p), f(r) \in 2^{<\omega}$ are compatible in $2^{<\omega}$.

The map f allows us to translate between lemma A.0.3 for FT-types in $\mathcal{C}_\theta$ and lemma 4.2.5 for types in $\mathbb{P}(\theta, Q)$, which concludes the proof. $\square$

Appendix B

The Filter Extension Property and the Compactness Theorem

Let $\langle \mathbb{P}, \mathcal{G}, \mathcal{F} \rangle$ be a symmetric system, let $p \in \mathbb{P}$, and let $D \subseteq \mathbb{P}$ (which is dense below p) and, for $d \in D$, $\mathcal{B}_d$ (which is a base for $\mathcal{F}$) witness the filter extension property below p. Suppose that $\dot{F}, \dot{x} \in \mathrm{HS}_{\mathcal{F}}$, with

$$1 \Vdash \text{“}\dot{F} \subseteq 2^{\dot{x}} \text{ has the FIP.”}$$

In the proof of theorem 4.1.5, it was shown that for $d \in D$ and $K \in \mathcal{B}_d$, if $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{x})$, then there is a $\dot{U} \supseteq \dot{F}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{U})$ and $\dot{U} \in \mathrm{HS}_{\mathcal{F}}$, so that

$$d \Vdash \text{“}\dot{U} \supseteq \dot{F} \text{ is an ultrafilter on } \dot{x}.\text{”}$$

The aim of this section is to prove the corresponding fact for applications of the first-order compactness theorem.

Lemma B.0.1. *Suppose that $d \in D \subseteq \mathbb{P}$ and $K \in \mathcal{B}_d$ witness the filter extension property* *below p. Let $\dot{\mathcal{L}}, \dot{\theta} \in \mathrm{HS}_{\mathcal{F}}$ be name that* 1 *forces respectively to be a first-order language, and an associated finitely satisfiable first-order $\dot{L}$-theory, and let $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{\mathcal{L}}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{\theta})$. Then there is a name $\dot{A} \in \mathrm{HS}_{\mathcal{F}}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{A})$, so that*

$$d \Vdash \text{“}\dot{A} \text{ is a model of } \dot{\theta}.\text{”}$$

Proof sketch. Recall that the ultrafilter lemma is the ability to extend every filter on every set to an ultrafilter. The standard proof that the ultrafilter lemma implies the first-order compactness theorem can be found in chapter 2 of [Jec08]. For the reader's convenience, we provide a sketch of this proof here. Let θ in a first-order language $\mathcal{L}$. There is a standard process, referenced in [Jec08], to extend $\mathcal{L}$ and θ so that for every single-variable formula $\phi(x)$, there is a Skolem constant $e_\phi \in \mathcal{L}$, and the following Skolem sentence is in θ:

$$(\exists v)\phi(v) \Rightarrow \phi(e_\phi).$$

From this, one can consider the Lindenbaum (Boolean) algebra of $\mathcal{L}$-sentences (see [Gol22]), in which θ generates a filter.

The problem of extending names for Boolean filters to names Boolean ultrafilters can be addressed in the same manner as in our proof of theorem 4.1.5. In other words, suppose that 1 forces that $\dot{\theta} \in \mathrm{HS}_{\mathcal{F}}$ is a name for the above filter and that $\dot{B} \in \mathrm{HS}_{\mathcal{F}}$ is the name for the above Lindenbaum (Boolean) algebra, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{F}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{B})$. Then essentially the same proof as in theorem 4.1.5 can be used to produce the desired name $\dot{A} \in \mathrm{HS}_{\mathcal{F}}$, with $K \preceq \mathrm{sym}_{\mathcal{G}}(\dot{A})$, that satisfies the lemma.

The only remaining detail is to observe that the names for the language $\dot{\mathcal{L}}$ and theory $\dot{\theta}$, with $K \preceq \mathrm{sym}_{\mathcal{F}}(\dot{\mathcal{L}}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{\theta})$, can be extended to contain the relevant Skolem constants and sentences, while retaining the fact that $K \preceq \mathrm{sym}_{\mathcal{F}}(\dot{\mathcal{L}}) \cap \mathrm{sym}_{\mathcal{G}}(\dot{\theta})$. $\square$