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# REPRESENTATION OF EXTENDIBLE BILINEAR FORMS

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ABSTRACT. We show that extendible bilinear forms can be represented in an integral form. The representation requires the use of bimeasures. We then study some properties of these extendible bilinear forms and see how they are related to the Grothendieck inequality.

## 1. INTRODUCTION

One of the cornerstones of modern functional analysis is the well-known Hahn-Banach extension theorem, a theorem which guarantees that a continuous linear functional on a subspace of a Banach space can be extended (not necessarily uniquely) to the Banach space containing it. It is easy to see that this theorem does not hold (in general) for bilinear functionals. For example, even the inner product on  $\ell^2$  does not extend to a continuous bilinear functional on  $\ell^\infty$ .

In the 1960s, Hayden found some conditions under which the Hahn-Banach extension theorem can be generalized to bilinear forms [11]. Since then, several authors have found various Hahn-Banach extension theorems, such as in [5, 12] and more recently in [4]. (See also the references therein.)

In this note, we show that a bounded bilinear functional  $\beta$  on  $E \times F$  can be extended to any superspace if and only if there exists a bimeasure  $\nu$  on  $B_{E^*} \times B_{F^*}$  such that  $\|\nu\|_{F_2} = \|\beta\|$  and

$$\beta(x, y) = \int_{B_{E^*} \times B_{F^*}} x^*(x) y^*(y) \nu(dx^*, dy^*), \quad (x, y) \in E \times F. \quad (1)$$

We recall that a bimeasure  $\nu$  on  $X \times Y$ , where  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  are measurable spaces, is a set function  $\nu : \mathcal{A} \times \mathcal{B} \rightarrow \mathbb{R}$  which is a measure in each argument separately, but need not have finite total variation. We remind the reader that, when  $\nu$  is actually a measure on  $X \times Y$ ,  $\beta$  is called an *integral bilinear form*. For this reason, when  $\nu$  is a bimeasure, we call  $\beta$  a *pseudo-integral bilinear form*.

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After showing that all extendible bilinear forms are pseudo-integral bilinear forms, we will discuss the question of integrability (with respect to a bimeasure). We then prove a Grothendieck-type inequality and investigate some consequences.

Throughout this note,  $\mathbb{R}$  is the scalar field, but the results can be adapted to  $\mathbb{C}$ , which some change in constants.

## 2. A REPRESENTATION THEOREM

Suppose that  $E$  and  $F$  are Banach spaces. A (real valued) bounded bilinear functional  $\beta$  on  $E \times F$  is said to be *extendible* if, whenever  $E'$  and  $F'$  are Banach spaces containing  $E$  and  $F$  (respectively) as subspaces, there exists a bounded bilinear functional  $\tilde{\beta} : E' \times F' \rightarrow \mathbb{R}$  such that  $\beta(x, y) = \tilde{\beta}(x, y)$  for all  $x \in E$  and  $y \in F$ .

Let  $E^*$  and  $F^*$  be the dual spaces for  $E$  and  $F$ , respectively. Endow  $B_{E^*}$  and  $B_{F^*}$  with the weak\* topology. Each of these spaces is a compact Hausdorff space ([17], Theorem 3.15). The spaces  $B_{E^*}$  and  $B_{F^*}$  play an important role in the study of extendible bilinear maps owing to the following result, due to Castilla, García, and Jaramillo [5, Proposition 2.1]:

**Proposition 1** (Castilla-García-Jaramillo). *Let  $E$  and  $F$  be Banach spaces. A bounded bilinear form  $\beta$  on  $E \times F$  is extendible if and only if  $\beta$  can be extended through the canonical embedding into  $C(B_{E^*}) \times C(B_{F^*})$ .*

Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. A set function  $\mu$  on  $\mathcal{A} \times \mathcal{B}$  is called a *bimeasure* on  $\mathcal{A} \times \mathcal{B}$  if  $\mu$  is a measure in each argument separately. Such a set function need not be a measure on the product space  $X \times Y$  and may not have finite total variation. If  $X$  and  $Y$  are compact Hausdorff spaces with Borel fields  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, and if  $\mu$  is a regular Borel measure in each argument separately, then we call  $\mu$  a *regular* bimeasure on  $\mathcal{A} \times \mathcal{B}$ .

Bimeasures (and their higher dimensional analogues) are variously known as Fréchet measures [2], polymasures [8], or multimeasures [9]. They have seen recent attention in such diverse areas as stochastic processes [15], quantum mechanics [13], and functional analysis [3].

We let  $\mathcal{F}_2(\mathcal{A}, \mathcal{B})$  denote the collection of all bimeasures on  $\mathcal{A} \times \mathcal{B}$ . (When there is no risk of confusion, we also write  $\mathcal{F}_2(X, Y)$  and say members of this set are bimeasures on  $X \times Y$ .) For a bimeasure  $\mu$ , the *Fréchet variation* of  $\mu$  is defined to be

$$\|\mu\|_{\mathcal{F}_2} = \sup \left| \sum_{m,n} \epsilon_m \delta_n \mu(A_m, B_n) \right|,$$

where the supremum is taken over all finite measurable partitions  $(A_m)$  of  $X$  and  $(B_n)$  of  $Y$  and all choices of signs  $(\epsilon_m)$  and  $(\delta_n)$ . When equipped with the Fréchet variation,  $\mathcal{F}_2(\mathcal{A}, \mathcal{B})$  becomes a Banach space. In the special case that  $X$

and  $Y$  are compact Hausdorff spaces with Borel fields  $\mathcal{A}$  and  $\mathcal{B}$ , respectively, we require elements in  $\mathcal{F}_2(\mathcal{A}, \mathcal{B})$  to be regular bimeasures. For a detailed discussion of bimeasures, see (for example) [2].

The next theorem, in its most basic form, can be traced back to the pioneering work of Fréchet in 1915 [10]. A proof of this version can be found in [2, Theorem VI.13].

**Theorem 2.** *If  $X$  and  $Y$  are compact Hausdorff spaces with respective Borel fields  $\mathcal{A}$  and  $\mathcal{B}$ , then  $\mathcal{F}_2(\mathcal{A}, \mathcal{B}) = (C(X) \hat{\otimes} C(Y))^*$ , where  $\hat{\otimes}$  denotes the projective tensor product. In particular, any  $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$  determines a bounded bilinear functional  $\eta$  on  $C(X) \times C(Y)$  and  $\|\mu\|_{\mathcal{F}_2} = \|\eta\|$ .*

We remark that, if the scalars are  $\mathbb{C}$ , then the most we can say about the norms is  $\|\mu\|_{\mathcal{F}_2} \leq \|\eta\| \leq 4\|\mu\|_{\mathcal{F}_2}$ . (The 4 results from the definition of the Fréchet variation and writing the integrands in terms of their real and imaginary parts.)

We are now prepared to prove our main result.

**Theorem 3.** *Let  $E$  and  $F$  be Banach spaces. If  $\beta$  is a bounded bilinear functional on  $E \times F$ , then  $\beta$  is extendible if and only if there exists a bimeasure  $\nu$  on  $B_{E^*} \times B_{F^*}$  such that  $\|\nu\|_{\mathcal{F}_2} = \|\beta\|$  and*

$$\beta(x, y) = \int_{B_{E^*} \times B_{F^*}} x^*(x) y^*(y) \nu(dx^*, dy^*), \quad (x, y) \in E \times F. \quad (2)$$

That is,  $\beta$  is extendible if and only if it is a pseudo-integral bilinear form.

*Proof.* First, suppose that  $\beta$  has the form of (2). That is, assume  $\beta$  is a pseudo-integral bilinear form. Define  $\tilde{\beta} : C(B_{E^*}) \times C(B_{F^*}) \rightarrow \mathbb{R}$  by

$$\tilde{\beta}(f, g) = \int_{B_{E^*} \times B_{F^*}} f(x^*) g(y^*) \nu(dx^*, dy^*), \quad (f, g) \in C(B_{E^*}) \times C(B_{F^*}). \quad (3)$$

By assumption,  $\nu$  is a member of the space  $\mathcal{F}_2(B_{E^*}, B_{F^*})$ . Consequently, by Theorem 2,  $\tilde{\beta}$  is a bounded bilinear functional on  $C(B_{E^*}) \times C(B_{F^*})$ . Because  $\tilde{\beta}(x, y) = \beta(x, y)$  for all  $(x, y) \in E \times F$ , we have that  $\beta$  can be extended from  $E \times F$  to  $C(B_{E^*}) \times C(B_{F^*})$ . Thus, by Proposition 1,  $\beta$  is extendible.

Now, let  $\beta$  be an extendible bounded bilinear functional on  $E \times F$ . Then, by assumption,  $\beta$  extends to a bounded bilinear functional  $\tilde{\beta}$  on  $C(B_{E^*}) \times C(B_{F^*})$  with the same norm. By Theorem 2, there exists a bimeasure  $\nu$  on  $B_{E^*} \times B_{F^*}$  such that  $\|\nu\|_{\mathcal{F}_2} = \|\tilde{\beta}\|$  and

$$\tilde{\beta}(f, g) = \int_{B_{E^*} \times B_{F^*}} f(x^*) g(y^*) \nu(dx^*, dy^*), \quad (f, g) \in C(B_{E^*}) \times C(B_{F^*}).$$

Pick  $f$  and  $g$  to be the functions defined by  $f(x^*) = x^*(x)$  for all  $x^* \in E^*$  and  $g(y^*) = y^*(y)$  for all  $y^* \in F^*$ . The result follows.  $\square$

Let  $E$  and  $F$  be two Banach spaces. The pair  $(E, F)$  is said to have BEP (the *Bilinear Extension Property*) if any bounded bilinear form on  $E \times F$  is extendible.

**Corollary 4.** *Let  $E$  and  $F$  be two Banach spaces. If  $(E, F)$  has BEP, then every bounded bilinear functional on  $E \times F$  is a pseudo-integral bilinear form.*

If  $E$  and  $F$  are Banach spaces, then their *injective tensor product* is denoted by  $E \hat{\otimes} F$  (e.g., [2, Section IV.6]). It is well-known (e.g., [6, Theorem 1.1.21]) that a bounded bilinear functional  $\beta$  on  $E \times F$  defines an element of  $(E \hat{\otimes} F)^*$  precisely when it has the form

$$\beta(x, y) = \int_{B_{E^*} \times B_{F^*}} x^*(x) y^*(y) \nu(dx^*, dy^*), \quad (x, y) \in E \times F,$$

where  $\nu$  is a *measure* on  $B_{E^*} \times B_{F^*}$ . Such bounded bilinear functionals are called *integral bilinear forms*. It is for this reason that, when we replace the measure  $\nu$  with a bimeasure, we call  $\beta$  a *pseudo-integral bilinear form*.

The projective tensor product of  $E$  and  $F$  is denoted by  $E \hat{\otimes} F$ . It is well-known that the space of bounded bilinear functionals on  $E \times F$  coincides with the space  $(E \hat{\otimes} F)^*$  (e.g., [6, Theorem 1.1.8]). Therefore,  $(E \hat{\otimes} F)^*$  is the space of integral bilinear forms and, if  $(E, F)$  has BEP,  $(E \hat{\otimes} F)^*$  is the space of pseudo-integral bilinear forms.

In certain situations, such as when  $E$  and  $F$  are cotype 2 spaces, the extendible and integral bilinear forms coincide. (See, for example, Proposition 2.5 in [5].) Since every pseudo-integral bilinear form is extendible, we conclude that (at least when  $E$  and  $F$  have cotype 2) every pseudo-integral bilinear form is integral, and hence every bimeasure on  $B_{E^*} \times B_{F^*}$  is a measure on  $B_{E^*} \times B_{F^*}$ .

### 3. INTEGRABILITY AND THE GROTHENDIECK INEQUALITY

Now let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and let both  $f : X \rightarrow E$  and  $g : Y \rightarrow F$  be weakly measurable bounded functions, where  $E$  and  $F$  are Banach spaces. (That is,  $u \circ f$  and  $v \circ g$  are scalar valued measurable functions for each  $u \in E^*$  and  $v \in F^*$ ; see [7].) Let  $\|f\|_\infty = \sup_{x \in X} \|f(x)\|_E$  and  $\|g\|_\infty = \sup_{y \in Y} \|g(y)\|_F$ .

Suppose  $\beta : E \times F \rightarrow \mathbb{R}$  is an extendible bilinear form on  $E \times F$ . By Theorem 3, there exists a bimeasure  $\nu$  on  $B_{E^*} \times B_{F^*}$  such that

$$\beta(f(x), g(y)) = \int_{B_{E^*} \times B_{F^*}} u(f(x)) v(g(y)) \nu(du, dv),$$

for all  $(x, y) \in X \times Y$ .

For ease of notation, we let  $E = F$  and denote the space  $B_{E^*} = B_{F^*}$  by  $K$ . Furthermore, denote the dual action of  $E^*$  on  $E$  by  $u(e) = \langle e, u \rangle$  for each  $e \in E$

and  $u \in E^*$ . Then, we may write

$$\beta(f(x), g(y)) = \int_{K \times K} \langle f(x), u \rangle \langle g(y), v \rangle \nu(du, dv),$$

for all  $(x, y) \in X \times Y$ .

Now let  $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$ . Define the integral of  $\beta(f, g)$  with respect to  $\mu$  as follows:

$$\begin{aligned} & \int_{X \times Y} \beta(f(x), g(y)) \mu(dx, dy) \\ &= \int_{K \times K} \left( \int_{X \times Y} \langle f(x), u \rangle \langle g(y), v \rangle \mu(dx, dy) \right) \nu(du, dv). \end{aligned} \quad (4)$$

We wish to show that this integral is well-defined. First, let  $\theta_\mu : K \times K \rightarrow \mathbb{R}$  be given by the formula

$$\theta_\mu(u, v) = \int_{X \times Y} \langle f(x), u \rangle \langle g(y), v \rangle \mu(dx, dy), \quad u, v \in K. \quad (5)$$

Observe that  $\theta_\mu$  is well-defined because  $f$  and  $g$  were assumed to be weakly measurable bounded functions. With the notation from (5), the definition in (4) becomes

$$\int_{X \times Y} \beta(f(x), g(y)) \mu(dx, dy) = \int_{K \times K} \theta_\mu(u, v) \nu(du, dv). \quad (6)$$

To show this integral is well-defined, it suffices to show that  $\theta_\mu \in C(K) \hat{\otimes} C(K)$ . We will prove this is the case when the Banach space  $E$  is separable.

We will make use of the following lemma:

**Lemma 5.** *Let  $K$  be a compact Hausdorff space and  $H$  a Hilbert space with inner product  $\langle \cdot, \cdot \rangle_H$ . Let  $A : K \rightarrow H$  and  $B : K \rightarrow H$  be continuous maps taking values in the unit ball of  $H$ . Then the function*

$$\langle A, B \rangle_H(u, v) = \langle A(u), B(v) \rangle_H, \quad (u, v) \in K \times K,$$

*is in  $C(K) \hat{\otimes} C(K)$  with norm bounded by  $K_G$ , the Grothendieck constant.*

*Proof.* We remind the reader that, since  $K$  is a compact Hausdorff space, the regular bimeasures on  $K \times K$  form the space dual to  $C(K) \hat{\otimes} C(K)$ .

Since  $A$  and  $B$  are continuous, they have separable ranges. Let  $(e_j)_{j \in \mathbb{N}}$  be an orthonormal set in  $H$  spanning the ranges of  $A$  and  $B$ , and let  $(A_j)_{j \in \mathbb{N}}$  and  $(B_j)_{j \in \mathbb{N}}$  be the respective coordinate functions of  $A$  and  $B$  with respect to the given orthonormal set. Then

$$\langle A(u), B(v) \rangle_H = \sum_{j=1}^{\infty} A_j(u) B_j(v), \quad (u, v) \in K \times K.$$

For each  $N \in \mathbb{N}$ , let  $\phi_N = \sum_{j=1}^N A_j \otimes B_j$ . Then  $\phi_N \in C(K) \otimes C(K)$  for each  $N$ . It suffices to show that  $(\phi_N)_{N \in \mathbb{N}}$  is a Cauchy sequence in  $C(K) \hat{\otimes} C(K)$ . Let  $\epsilon > 0$  be given. Suppose  $N \geq M$ . Then for any regular bimeasure  $\eta$  on  $K \times K$ ,

$$\int (\phi_N - \phi_{M-1}) d\eta = \sum_{j=M}^N \left( \int A_j \otimes B_j d\eta \right) = \sum_{j=M}^N \tilde{\eta}(A_j, B_j), \quad (7)$$

where  $\tilde{\eta}$  is the bilinear functional on  $C(K) \times C(K)$  induced by the bimeasure  $\eta$ . By a consequence of the Grothendieck inequality (see [1], Theorem 8.1.3), the right side of (7) is bounded by

$$K_G \|\eta\|_{\mathcal{F}_2} \sup_{u \in K} \left( \sum_{j=M}^N |A_j(u)|^2 \right)^{1/2} \sup_{v \in K} \left( \sum_{j=M}^N |B_j(v)|^2 \right)^{1/2}. \quad (8)$$

Because  $K$  is compact and  $A$  and  $B$  were assumed to be continuous, we have that  $\sum_{j=1}^{\infty} |A_j|^2$  and  $\sum_{j=1}^{\infty} |B_j|^2$  converge uniformly to  $\|A\|_H^2$  and  $\|B\|_H^2$ , respectively, by Dini's theorem. (See, for example, Theorem 7.13 of [16].) Thus, there exists an  $M \in \mathbb{N}$  such that  $\sum_{j=M}^{\infty} |A_j(u)|^2 < \epsilon$  and  $\sum_{j=M}^{\infty} |B_j(v)|^2 < \epsilon$ , for all  $u, v \in K$ . Therefore, by duality,  $\|\phi_N - \phi_{M-1}\|_{V_2} \leq K_G \epsilon$ , and the result follows.

To bound the projective tensor norm, argue as in (7) and (8) with  $\phi_{M-1}$  replaced by 0 and use  $\sup_{u \in K} \|A(u)\|_H \leq 1$  and  $\sup_{v \in K} \|B(v)\|_H \leq 1$ .  $\square$

**Remark 6.** Once again, let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces and let  $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$ . Define  $\Phi : L^\infty(X) \times L^\infty(Y) \rightarrow \mathbb{R}$  by

$$\Phi(h, k) = \int_{X \times Y} h(x) k(y) \mu(dx, dy).$$

By the Grothendieck Factorization Theorem [18], there exist probability measures  $\lambda_1$  on  $(X, \mathcal{A})$  and  $\lambda_2$  on  $(Y, \mathcal{B})$  such that

$$|\Phi(h, k)| \leq K_G \|\mu\|_{\mathcal{F}_2} \|h\|_{L^2(X, \lambda_1)} \|k\|_{L^2(Y, \lambda_2)}. \quad (9)$$

Therefore,  $\Phi$  extends to a bounded bilinear map  $\tilde{\Phi} : L^2(X, \lambda_1) \times L^2(Y, \lambda_2) \rightarrow \mathbb{R}$  with norm bounded by  $K_G \|\mu\|_{\mathcal{F}_2}$ . It follows that there exists a bounded linear operator  $T : L^2(Y, \lambda_2) \rightarrow L^2(X, \lambda_1)$  such that  $\|T\| \leq K_G \|\mu\|_{\mathcal{F}_2}$  and

$$\tilde{\Phi}(h, k) = \left\langle h, T(k) \right\rangle_{L^2(X, \lambda_1)}, \quad h \in L^2(X, \lambda_1), k \in L^2(Y, \lambda_2).$$

Now let  $K$  be a compact Hausdorff space. Suppose  $F : K \rightarrow L^2(X, \lambda_1)$  and  $G : K \rightarrow L^2(Y, \lambda_2)$  are continuous functions. Define  $\Theta : K \times K \rightarrow \mathbb{R}$  by

$$\Theta(u, v) = \tilde{\Phi}(F(u), G(v)) = \left\langle F(u), TG(v) \right\rangle_{L^2(X, \lambda_1)}, \quad u, v \in K.$$

By Lemma 5,  $\Theta \in C(K) \hat{\otimes} C(K)$  and

$$\begin{aligned} \|\Theta\|_{\hat{\otimes}} &\leq K_G \sup_{u \in K} \|F(u)\|_{L^2(X, \lambda_1)} \sup_{v \in K} \|TG(v)\|_{L^2(X, \lambda_1)} \\ &\leq K_G \|T\| \sup_{u \in K} \|F(u)\|_{L^2(X, \lambda_1)} \sup_{v \in K} \|G(v)\|_{L^2(Y, \lambda_2)}. \end{aligned}$$

This suggests the following proposition:

**Proposition 7.** *Let  $\theta_\mu$  be as in (5). If  $E$  is separable, then  $\theta_\mu$  is an element of  $C(K) \hat{\otimes} C(K)$  and  $\|\theta_\mu\|_{\hat{\otimes}} \leq K_G^2 \|\mu\|_{\mathcal{F}_2} \|f\|_\infty \|g\|_\infty$ , where  $\|\cdot\|_{\hat{\otimes}}$  denotes the projective tensor norm.*

*Proof.* Without loss of generality, assume  $f$  and  $g$  take values in  $B_E$ . We have the bimeasure  $\mu \in \mathcal{F}_2(\mathcal{A}, \mathcal{B})$  given in (5). Let  $\lambda_1$  and  $\lambda_2$  be probability measures on  $X$  and  $Y$  (respectively) guaranteed by the Grothendieck Factorization Theorem, as in (9). Define functions  $F : K \rightarrow L^2(X, \lambda_1)$  and  $G : K \rightarrow L^2(Y, \lambda_2)$  as follows: For  $u, v \in K$ , let

$$F(u)(x) = \langle f(x), u \rangle, \quad G(v)(y) = \langle g(y), v \rangle, \quad (x, y) \in X \times Y. \quad (10)$$

We claim  $F$  and  $G$  are continuous maps on  $K$ . We will show  $F$  is continuous; the arguments are similar for  $G$ . Let  $u_n \rightarrow u$  in  $K$ . Then for each  $x \in X$ ,  $\langle f(x), u_n \rangle \rightarrow \langle f(x), u \rangle$  (by the definition of the weak\* topology). Therefore  $F(u_n)(x) \rightarrow F(u)(x)$  for each  $x \in X$ , and so by the bounded convergence theorem  $F(u_n) \rightarrow F(u)$  in  $L^2(X, \lambda_1)$ . We conclude that  $F$  is sequentially continuous. Since  $E$  is assumed to be separable,  $K$  is metrizable, and hence  $F$  is a continuous function.

By Remark 6,

$$\theta_\mu(u, v) = \left\langle F(u), TG(v) \right\rangle_{L^2(X, \lambda_1)} \quad (11)$$

(for appropriately chosen  $T$  and  $\lambda_1, \lambda_2$ ), and so  $\theta_\mu$  is in  $C(K) \hat{\otimes} C(K)$ , as required. Lemma 5 and the representation in (11) provide a bound for the projective tensor norm: since  $\|T\| \leq K_G \|\mu\|_{\mathcal{F}_2}$ , we have  $\|\theta_\mu\|_{\hat{\otimes}} \leq K_G^2 \|\mu\|_{\mathcal{F}_2}$ .  $\square$

**Remark 8.** In Proposition 7, we assumed that  $E$  was separable in order to deduce continuity of the functions  $F$  and  $G$ . This assumption was stronger than required. We need only assume that  $K$  is a sequential space (i.e., that sequential continuity implies continuity).

We next show the definition in (4) is independent of choice of representing bimeasure  $\nu$ . Again assume that  $E$  is separable and let  $f$  and  $g$  be weakly measurable functions that take values in  $B_E$ . Suppose  $\nu$  and  $\nu'$  are two bimeasures on  $K \times K$  satisfying the equality in (2). We will show  $\int \theta_\mu d\nu = \int \theta_\mu d\nu'$ , where  $\theta_\mu$  is as defined in (5):

$$\theta_\mu(u, v) = \int_{X \times Y} \langle f(x), u \rangle \langle g(y), v \rangle \mu(dx, dy), \quad u, v \in K. \quad (12)$$

We recall that a function taking values in a Banach space is said to be *strongly measurable* if it is the pointwise norm limit of simple functions. (See [7, Section II.1].) The Pettis Measurability Theorem [7, Theorem II.1.2] guarantees that any weakly measurable function taking values in a separable Banach space is necessarily strongly measurable. Another consequence of the Pettis Measurability Theorem is that any strongly measurable function is the uniform limit (in norm) of a sequence of countably valued measurable functions. (See [7, Corollary II.1.3] for a proof in the case of a finite measure space. The case considered here can be proved with similar arguments.) Therefore, by Proposition 7, it suffices to prove  $\int \theta_\mu d\nu = \int \theta_\mu d\nu'$  for countably valued bounded measurable functions  $f$  and  $g$  in (12).

Suppose that  $f$  and  $g$  are countably valued bounded measurable functions taking values in  $B_E$ . (Since  $E$  is assumed to be separable, there is no ambiguity when calling a function “measurable,” since weak and strong measurability coincide.) Then

$$f = \sum_{j=1}^{\infty} a_j \mathbb{1}_{A_j} \quad \text{and} \quad g = \sum_{k=1}^{\infty} b_k \mathbb{1}_{B_k},$$

where  $a_j, b_k \in B_E$  for each  $j, k \in \mathbb{N}$  and  $(A_j)_{j=1}^{\infty}, (B_k)_{k=1}^{\infty}$  are sequences of pairwise disjoint measurable subsets of  $X$  and  $Y$ , respectively. (We use  $\mathbb{1}_A$  to denote the indicator function of the measurable set  $A$ .) For each  $n \in \mathbb{N}$ , let  $f_n = \sum_{j=1}^n a_j \mathbb{1}_{A_j}$  and  $g_n = \sum_{k=1}^n b_k \mathbb{1}_{B_k}$  and define

$$\theta_n(u, v) = \int_{X \times Y} \langle f_n(x), u \rangle \langle g_n(y), v \rangle \mu(dx, dy) = \sum_{j,k=1}^n \langle a_j, u \rangle \langle b_k, v \rangle \mu(A_j, B_k),$$

for all  $(u, v) \in K \times K$ .

We claim that  $\theta_n$  converges to  $\theta_\mu$  in  $C(K) \hat{\otimes} C(K)$ . In order to prove this claim, we introduce some notation. For each  $j, k \in \mathbb{N}$ , let  $F_j(u) = \langle a_j, u \rangle$  and  $G_k(v) = \langle b_k, v \rangle$  for  $u, v \in K$ . Then, for each  $j, k \in \mathbb{N}$ , the functions  $F_j$  and  $G_k$  are continuous on the compact Hausdorff space  $K = (B_E, \text{weak}^*)$ . Further, for all  $(j, k) \in \mathbb{N} \times \mathbb{N}$ , let  $\alpha_{jk} = \mu(A_j, B_k)$ . Then  $\alpha = (\alpha_{jk})_{j,k=1}^{\infty}$  is an infinite array with  $\|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \leq \|\mu\|_{\mathcal{F}_2(\mathcal{A}, \mathcal{B})}$ . With this notation,

$$\theta_n = \sum_{j,k=1}^n \alpha_{jk} F_j \otimes G_k.$$

Now, let  $\eta$  be an arbitrary member of  $\mathcal{F}_2(K, K)$  and let  $\xi_1$  and  $\xi_2$  be any Grothendieck measures on  $K$  associated to  $\eta$ . Then, for any  $n \in \mathbb{N}$ ,

$$\int \theta_n d\eta = \sum_{j,k=1}^n \alpha_{jk} \int F_j \otimes G_k d\eta = \sum_{j,k=1}^n \alpha_{jk} \langle F_j, TG_k \rangle_{L^2(\xi_1)},$$

for an appropriately chosen bounded linear operator  $T : L^2(\xi_2) \rightarrow L^2(\xi_1)$  with  $\|T\| \leq K_G \|\eta\|_{\mathcal{F}_2(K,K)}$ . (Refer to Remark 6.)

Let  $(e_i)_{i=1}^\infty$  be an orthonormal set in  $L^2(\xi_1)$  that spans a closed subspace containing the functions  $F_j$  and  $TG_k$  for all  $j, k \in \mathbb{N}$ . For each  $j, k \in \mathbb{N}$ , let  $(\phi_i(j))_{i=1}^\infty$  and  $(\psi_i(k))_{i=1}^\infty$  be the coordinate functions of  $F_j$  and  $TG_k$ , respectively, corresponding to  $(e_i)_{i=1}^\infty$ , so that  $F_j = \sum_{i=1}^\infty \phi_i(j) e_i$  and  $TG_k = \sum_{i=1}^\infty \psi_i(k) e_i$  and

$$\|F_j\|_{L^2(\xi_1)} = \left( \sum_{i=1}^\infty |\phi_i(j)|^2 \right)^{1/2} \quad \text{and} \quad \|TG_k\|_{L^2(\xi_1)} = \left( \sum_{i=1}^\infty |\psi_i(k)|^2 \right)^{1/2}.$$

Then, for each  $j$  and  $k$  in  $\mathbb{N}$ ,

$$\int F_j \otimes G_k d\eta = \sum_{i=1}^\infty \phi_i(j) \psi_i(k).$$

Let  $A$  and  $B$  be nonempty finite subsets of  $\mathbb{N}$ . Since  $\alpha$  is a bimeasure on  $\mathbb{N} \times \mathbb{N}$ , there exist (by the Grothendieck factorization theorem) sequences  $\sigma = (\sigma_i)_{i \in \mathbb{N}}$  and  $\tau = (\tau_i)_{i \in \mathbb{N}}$  in  $\ell^1$  having nonnegative terms and  $\ell^1$ -norm 1 such that

$$\left| \sum_{j \in A} \sum_{k \in B} \alpha_{jk} v_j \omega_k \right| \leq K_G \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \left( \sum_{j \in A} v_j^2 \sigma_j \right)^{1/2} \left( \sum_{k \in B} \omega_k^2 \tau_k \right)^{1/2},$$

whenever  $(v_j)_{j=1}^\infty$  and  $(\omega_k)_{k=1}^\infty$  are scalar sequences in  $\ell^2$ . That is,  $\sigma$  and  $\tau$  are Grothendieck probability measures on  $\mathbb{N}$  associated to  $\alpha$ . Consequently,

$$\begin{aligned} \left| \sum_{j \in A} \sum_{k \in B} \alpha_{jk} \int F_j \otimes G_k d\eta \right| &\leq \sum_{i=1}^\infty \left| \sum_{j \in A} \sum_{k \in B} \alpha_{jk} \phi_i(j) \psi_i(k) \right|, \\ &\leq K_G \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \sum_{i=1}^\infty \left( \sum_{j \in A} \phi_i(j)^2 \sigma_j \right)^{1/2} \left( \sum_{k \in B} \psi_i(k)^2 \tau_k \right)^{1/2}. \end{aligned}$$

Applying the Cauchy-Schwarz inequality, this is bounded by

$$\begin{aligned} &K_G \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \left( \sum_{i=1}^\infty \sum_{j \in A} \phi_i(j)^2 \sigma_j \right)^{1/2} \left( \sum_{i=1}^\infty \sum_{k \in B} \psi_i(k)^2 \tau_k \right)^{1/2} \\ &= K_G \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \left( \sum_{j \in A} \sum_{i=1}^\infty \phi_i(j)^2 \sigma_j \right)^{1/2} \left( \sum_{k \in B} \sum_{i=1}^\infty \psi_i(k)^2 \tau_k \right)^{1/2} \\ &\leq K_G \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \|T\| \left( \sum_{j \in A} \|F_j\|_{L^2(\xi_1)}^2 \sigma_j \right)^{1/2} \left( \sum_{k \in B} \|G_k\|_{L^2(\xi_2)}^2 \tau_k \right)^{1/2} \\ &\leq K_G^2 \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \|\eta\|_{\mathcal{F}_2(K,K)} \left( \sum_{j \in A} \sigma_j \right)^{1/2} \left( \sum_{k \in B} \tau_k \right)^{1/2}. \end{aligned}$$

Now, define a scalar valued array  $\gamma = (\gamma_{jk})_{j,k=1}^\infty$  by

$$\gamma_{jk} = \alpha_{jk} \int F_j \otimes G_k d\eta, \quad (13)$$

for  $(j, k) \in \mathbb{N} \times \mathbb{N}$ . We have established that, for any nonempty finite subsets  $A$  and  $B$  of  $\mathbb{N}$ ,

$$\left| \sum_{j \in A} \sum_{k \in B} \gamma_{jk} \right| \leq K_G^2 \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \|\eta\|_{\mathcal{F}_2(K, K)} \left( \sum_{j \in A} \sigma_j \right)^{1/2} \left( \sum_{k \in B} \tau_k \right)^{1/2}. \quad (14)$$

We comment here that, while the Grothendieck measures  $\xi_1$  and  $\xi_2$  were used to obtain this bound in (14), the bound itself is independent of the choice of Grothendieck measures and depends only on the bimeasure  $\eta$ . We also observe that the sequences  $\sigma$  and  $\tau$  in  $\ell^1$  are independent of  $\eta$  and depend only on the scalar array  $\alpha$ .

For any  $n \in \mathbb{N}$ ,

$$\int \theta_n d\eta = \sum_{j,k=1}^n \alpha_{jk} \int F_j \otimes G_k d\eta = \sum_{j,k=1}^n \gamma_{jk}.$$

Consequently, if  $m \in \mathbb{N}$  is chosen so that  $n > m$ , then

$$\int (\theta_n - \theta_{m-1}) d\eta = \sum_{j=m}^n \sum_{k=m}^n \gamma_{jk} + \sum_{j=1}^{m-1} \sum_{k=m}^n \gamma_{jk} + \sum_{j=m}^n \sum_{k=1}^{m-1} \gamma_{jk}. \quad (15)$$

Let  $\epsilon > 0$ . Because  $\|\sigma\|_{\ell^1} = 1$  and  $\|\tau\|_{\ell^1} = 1$ , there exists some  $M \in \mathbb{N}$  such that  $\sum_{j=M}^\infty \sigma_j < \epsilon^2$  and  $\sum_{k=M}^\infty \tau_k < \epsilon^2$ . Thus, if  $A$  and  $B$  are nonempty finite subsets of  $\mathbb{N}$  and either  $\min(A) > M$  or  $\min(B) > M$ , then

$$\left( \sum_{j \in A} \sigma_j \right)^{1/2} \left( \sum_{k \in B} \tau_k \right)^{1/2} < \epsilon. \quad (16)$$

(We note that if one sum is not bounded by  $\epsilon^2$ , then it is bounded by the  $\ell^1$ -norm, which is 1.) Choose  $n$  and  $m$  in  $\mathbb{N}$  such that  $n > m$  and  $m > M$ . Then, by (14), (15), and (16),

$$\left| \int (\theta_n - \theta_{m-1}) d\eta \right| \leq 3K_G^2 \|\alpha\|_{\mathcal{F}_2(\mathbb{N}, \mathbb{N})} \|\eta\|_{\mathcal{F}_2(K, K)} \epsilon.$$

This bound is uniform in  $\eta$ . Hence, taking the supremum over  $\eta$  in  $\mathcal{F}_2(K, K)$  with norm 1, we conclude that  $(\theta_n)_{n=1}^\infty$  is a Cauchy sequence in  $C(K) \hat{\otimes} C(K)$ . It follows that  $(\theta_n)_{n=1}^\infty$  has a limit in  $C(K) \hat{\otimes} C(K)$ . In order to see that this limit is indeed  $\theta_\mu$ , we observe that  $\theta_\mu$  is known to be the pointwise limit of  $(\theta_n)_{n=1}^\infty$ , by Theorem 2.3 in [9]. To apply this theorem, we use the fact that  $|\langle f_n(x), u \rangle| \leq 1$  and  $|\langle g_n(y), v \rangle| \leq 1$  for all  $(x, y) \in X \times Y$ ,  $(u, v) \in K \times K$ , and  $n \in \mathbb{N}$ .

Therefore,

$$\begin{aligned}
 \int \theta_\mu \, d\nu &= \lim_{n \rightarrow \infty} \int \theta_n \, d\nu = \lim_{n \rightarrow \infty} \int_{K \times K} \left( \sum_{j,k=1}^n \langle a_j, u \rangle \langle b_k, v \rangle \mu(A_j, B_k) \right) \nu(du, dv) \\
 &= \lim_{n \rightarrow \infty} \sum_{j,k=1}^n \left( \int_{K \times K} \langle a_j, u \rangle \langle b_k, v \rangle \nu(du, dv) \right) \mu(A_j, B_k) \\
 &= \lim_{n \rightarrow \infty} \sum_{j,k=1}^n \beta(a_j, b_k) \mu(A_j, B_k).
 \end{aligned}$$

The same equalities will hold if  $\nu$  is replaced by  $\nu'$ , which completes the proof. We summarize with the following theorem.

**Theorem 9.** *The integral in (4) is well-defined, provided that  $E$  is a separable Banach space.*

A consequence of Proposition 7 and Theorem 9 is the following Grothendieck-type inequality for extendible bilinear maps.

**Corollary 10** (Grothendieck-type inequality). *Let  $(X, \mathcal{A})$  and  $(Y, \mathcal{B})$  be measurable spaces. Let  $E$  and  $F$  be Banach spaces, let  $f : X \rightarrow E$  and  $g : Y \rightarrow F$  be bounded strongly measurable functions, and suppose  $\beta : E \times F \rightarrow \mathbb{R}$  is an extendible bilinear functional. If  $\mu \in F_2(\mathcal{A}, \mathcal{B})$ , then*

$$\left| \int_{X \times Y} \beta(f(x), g(y)) \mu(dx, dy) \right| \leq K_G^2 \|\beta\| \|\mu\|_{F_2} \|f\|_\infty \|g\|_\infty.$$

*Proof.* Since  $f$  and  $g$  are strongly measurable, they are weakly measurable and have separable range (by the Pettis Measurability Theorem [7, Theorem II.1.2]). The integral is well-defined by Theorem 9 and we can apply Proposition 7 to complete the proof.  $\square$

#### 4. FINAL COMMENTS

The representation in Theorem 3 allows for simple proofs of known results about extendible bilinear forms (given in, for example, Theorem 2.2 in [5]). As an example, we consider the following:

**Corollary 11.** *Let  $E$  and  $F$  be Banach spaces. If  $\beta$  is an extendible bounded bilinear functional on  $E \times F$ , there is a Hilbert space  $H$  and bounded linear maps  $U : E \rightarrow H$  and  $V : F \rightarrow H$  such that  $\beta(x, y) = \langle U(x), V(y) \rangle_H$  for all  $x \in E$  and  $y \in F$ .*

*Proof.* Let  $\nu$  be the bimeasure on  $B_{E^*} \times B_{F^*}$  corresponding to  $\beta$  and let  $\lambda_1$  and  $\lambda_2$  be associated Grothendieck measures. Then  $H = L^2(\lambda_1)$ ,  $U = j_E$ , and  $V = T \circ j_F$ , where  $j_E : E \rightarrow L^2(\lambda_1)$  and  $j_F : F \rightarrow L^2(\lambda_2)$  are the canonical injections and  $T : L^2(\lambda_2) \rightarrow L^2(\lambda_1)$  is the map defined in Remark 6.  $\square$

Corollary 10 is called a Grothendieck-type inequality for the following reason: Let  $X = Y = \mathbb{N}$  and  $\mathcal{A} = \mathcal{B} = 2^{\mathbb{N}}$ , the power set of  $\mathbb{N}$ . A bimeasure  $\mu$  on  $\mathbb{N} \times \mathbb{N}$  corresponds to an infinite array  $(a_{ij})_{i,j \in \mathbb{N}}$ , and so we have, for each  $N \in \mathbb{N}$ :

$$\left| \sum_{i,j=1}^N a_{ij} \beta(x_i, y_j) \right| \leq K_G^2 \|\beta\| \sup_{\substack{\epsilon_i, \delta_j = \pm 1 \\ n \in \mathbb{N}}} \left| \sum_{i,j=1}^n a_{ij} \epsilon_i \delta_j \right| \sup_{i \in \mathbb{N}} \|x_i\|_E \sup_{j \in \mathbb{N}} \|y_j\|_F, \quad (17)$$

where  $f : \mathbb{N} \rightarrow E$  and  $g : \mathbb{N} \rightarrow F$  are defined by  $f(i) = x_i \in E$  and  $g(j) = y_j \in F$  for all  $i, j \in \mathbb{N}$ . This inequality has the form of the celebrated Grothendieck inequality, as formulated by Lindenstrauss and Pełczyński in [14].

There is also a different, more readily obtained, Grothendieck-type inequality that holds for extendible bilinear maps:

**Proposition 12.** *Let  $E$  and  $F$  be Banach spaces and suppose  $\beta : E \times F \rightarrow \mathbb{R}$  is a bounded bilinear functional. If  $\beta$  is extendible, then*

$$\left| \sum_{i=1}^n \beta(x_i, y_i) \right| \leq K_G \|\beta\| \sup_{x^* \in B_{E^*}} \left( \sum_{i=1}^n |x^*(x_i)|^2 \right)^{1/2} \sup_{y^* \in B_{F^*}} \left( \sum_{i=1}^n |y^*(y_i)|^2 \right)^{1/2},$$

for all finite sequences  $(x_i)_{i=1}^n$  in  $E$  and  $(y_i)_{i=1}^n$  in  $F$ , where  $n \in \mathbb{N}$ .

*Proof.* We may extend  $\beta$  to a bounded bilinear map  $\tilde{\beta} : C(B_{E^*}) \times C(B_{F^*}) \rightarrow \mathbb{R}$ , by assumption. By a well-known consequence of Grothendieck's inequality,

$$\left| \sum_{i=1}^n \tilde{\beta}(f_i, g_i) \right| \leq K_G \|\beta\| \sup_{x^* \in B_{E^*}} \left( \sum_{i=1}^n |f_i(x^*)|^2 \right)^{1/2} \sup_{y^* \in B_{F^*}} \left( \sum_{i=1}^n |g_i(y^*)|^2 \right)^{1/2},$$

for  $(f_i)_{i=1}^n$  in  $C(B_{E^*})$  and  $(g_i)_{i=1}^n$  in  $C(B_{F^*})$  and  $n \in \mathbb{N}$ . (See, for example, Theorem 8.1.3 in [1].) The desired result follows by choosing  $f_i$  and  $g_i$  so that  $f_i(x^*) = x^*(x_i)$  and  $g_i(y^*) = y^*(y_i)$  for all  $i \in \{1, \dots, n\}$ .  $\square$

As a consequence of Proposition 12, we have the following:

**Corollary 13.** *Let  $E$  and  $F$  be Banach spaces. Suppose  $\beta : E \times F \rightarrow \mathbb{R}$  is a bounded bilinear functional and define a bounded linear map  $T_\beta : E \rightarrow F^*$  by  $T_\beta(x)(y) = \beta(x, y)$  for all  $x \in E$  and  $y \in F$ . If  $\beta$  is extendible, then  $T_\beta$  is absolutely 2-summing.*

*Proof.* Let  $(x_i)_{i=1}^n$  be a finite sequence in  $E$ . Then

$$\left( \sum_{i=1}^n \|T_\beta(x_i)\|_{F^*}^2 \right)^{1/2} = \left\| \left( T_\beta(x_i) \right)_{i=1}^n \right\|_{\ell_n^2(F^*)}.$$

Consequently,

$$\left\| \left( T_\beta(x_i) \right)_{i=1}^n \right\|_{\ell_n^2(F^*)} = \sup \left| \sum_{i=1}^n \langle y_i, T_\beta(x_i) \rangle \right| = \sup \left| \sum_{i=1}^n \beta(x_i, y_i) \right|,$$

where the supremum is taken over all  $(y_i)_{i=1}^n$  in  $\ell_n^2(F)$  with norm 1. The result now follows from Proposition 12.  $\square$

Corollary 13 leads immediately to a further corollary.

**Corollary 14.** *Let  $E$  and  $F$  be Banach spaces. If  $(E, F)$  has BEP, then every bounded linear operator  $T : E \rightarrow F^*$  is absolutely 2-summing.*

*Proof.* If  $\beta(x, y) = \langle y, T(x) \rangle$  for all  $x \in E$  and  $y \in F$ , then  $\beta$  is a bounded bilinear functional on  $X \times Y$  and  $T = T_\beta$ .  $\square$

Corollary 13 is already known (see, for example, Proposition 3.1 of [12]); although the current author has not seen it demonstrated using the approach given here.

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