Accurate Iterative Analysis of the K-V Equations

O.A. Anderson

Designers and experimentalists working with alternating-gradient (A-G) systems look for simple, accurate ways to analyze A-G performance for matched beams. The well-known K-V equations [1], based on an artificial beam model, are useful but their solution is not easy. The smooth approximation [2], [3], [4] is popular because it is simple and explicit. However, it becomes seriously inaccurate for applications with large focusing fields and large phase advances. Results of efforts to improve the accuracy [5], [6] have tended to be indirect or complex. Our generalizations presented previously [7] did improve the accuracy while retaining a simple explicit format. Unfortunately, the method used to derive our results (expansion in powers of a small parameter) was complex and hard to follow; furthermore, Ref. [7] only gave low-order correction terms.

The present paper uses a straightforward iteration method and obtains equations of higher order than those shown in our previous paper. As before, input quantities are A-G waveform and field strength; beam emittance; and beam charge or current. We solve for average radius, peak radius, and the phase advances. (Or, one can input the mean radius and solve for the beam current.)

Summary: We begin with the coupled K-V equations, then (Section 3) show how to decouple them. Section 4 expands the a(z) envelope about its mean value and splits this envelope equation into its average part A and its periodic part $\rho(z)$. The differential equation for $\rho(z)$ is solved (Section 5) by iteration. These results are combined to obtain a matching equation for the average radius A This equation is written in several versions (Section 6). The first (lowest) order case is usually called the smooth approximation. A second order term significantly improves the accuracy. The third order results are far more accurate over a wide range of parameters than any previously published.

Section 7 combines results from Sections 5 and 6 to give the maximum and minimum radii. The phase advances σ and σ_0 are given in Section 8. Appendices F, G, and H discuss how Fourier analysis of the A-G focusing waveform is used to facilitate the solution.

1. The K-V Equations and Symmetric Lattice Model

The K-V equations for the envelopes a(z) and b(z) are

$$a(z)'' = -K(z)\,a + \frac{\in^2}{a^3} + \frac{2\,Q}{a+b} \tag{1}$$

$$b(z)'' = +K(z)\,b + \frac{\in^2}{b^3} + \frac{2\,Q}{a+b} \tag{2}$$

with input parameters: normalized beam current Q; emittance ϵ; and A-G focus function K(z). The z origin is located at the midpoint of a quadrupole and K(z) is assumed 1to be symmetric about z = 0, periodic over a cell length 2L, and antisymmetric about L/2. Thus

$$K(z-2L) = K(z), \quad K(-z) = K(z), \quad K(L/2) = 0, \quad K(z-L) = -K(z). \tag{3}$$

We solve for the x and y beam envelopes a(z) and b(z), assumed to be matched to the lattice, i.e. periodic over 2L.

2. Operators, Functions, Parameters, and Derived Quantities

To aid the solution of Eqs. (1) and (2), we introduce in Eqs. (4)–(19) the operators on even periodic functions $\langle\ldots\rangle$, $\{\ldots\}$, $\int$, $\iint$; the even periodic functions h(z), g(z), δ(z), ρ(z); and the constants k, α, β, q, A, K^{eff}, Φ. In Eq. (19), h_1 is the first Fourier coefficient of h(z)—cf. App. F.

Table 1
Definitions to be used in this paper

$$\langle f\rangle \equiv (1/2L)\int_0^{2L} f(z)dz, \tag{4}$$

$$\{f\} \equiv f - \langle f\rangle. \tag{5}$$

For even $\psi(z)$ ∋ $\langle\psi\rangle = 0$:

$$\int\psi \equiv \int_0^z \psi(z')dz' \quad \text{and} \tag{6}$$

$$\iint\psi \equiv \left\{\int_0^z dz'\int_0^{z'} \psi(z'')dz''\right\}. \tag{7}$$

$$k \equiv K^{max}, \tag{8}$$

$$h(z) \equiv K(z)/k, \tag{9}$$

$$g \equiv \iint h, \tag{10}$$

$$\delta(z) \equiv \iint\{hg\}, \tag{11}$$

$$A \equiv \langle a(z)\rangle, \tag{12}$$

$$\rho(z) \equiv (a(z)-A)/A, \tag{13}$$

$$\rho_b(z) \equiv (b(z)-A)/A, \tag{14}$$

$$\alpha \equiv \frac{3\epsilon^2}{A^4}, \quad \beta \equiv \alpha\frac{L^2}{\pi^2}, \tag{15}$$

$$q \equiv Q/A^2, \tag{16}$$

$$K^{eff} \equiv k^2\langle[\int h]^2\rangle, \tag{17}$$

$$\Phi \equiv 3k^2\langle g^2\rangle, \tag{18}$$

$$\rho_m \equiv h_1 kL^2/\pi^2. \tag{19}$$

Most of these definitions will not be used immediately but are collected here for convenient reference. All the functions are even and periodic over 2L.

The operator $\langle\ldots\rangle$ averages over a cell length 2L while the operator $\{\ldots\}$ removes the average part of a periodic function: e.g., $2\{\cos^2 x\} = \{1+\cos 2x\} = \cos 2x$. The operator $\iint$ operates on periodic functions that have no average. It gives the repeated indefinite integral and removes the average part, if any, of the result. This removal can be implemented by constructing a suitable lower limit for the outer integral. For example,

Eq. (10) can be written

$$g \equiv \int_{L/2}^{z} dz' \int_{o}^{z'} h(z'')dz'',$$

which subtracts the value at L/2 so that $g(L/2)=0$. Alternatively, one can start both integrals at zero and then apply the operator $\{\ldots\}$, as in Eq. (7). A simple example is $\iint \cos x = \{\int \sin x\} = \{1-\cos x\} = -\cos x$.

According to Eq. (3), $K(0) \equiv k$ is the maximum value of the focusing strength so that $h(0)=1$. The defined quantities α, β, and q are not essential but are introduced to make some equations more compact and readable.

3. Decoupling the K-V Equations—Matched-beam Boundary Conditions

For the matched beam case, $\langle a \rangle = \langle b \rangle \equiv A$, and

$$a = A(1+\rho), \quad b = A(1+\rho_b). \tag{20}$$

The Q terms in Eqs. (1) and (2) can be expanded as

$$\frac{2Q}{a+b} = \frac{Q}{A}\left(1 - (\rho+\rho_b)/2 + \ldots\right) = \frac{Q}{A}\left(1 - k^2\delta(z) + \ldots\right), \tag{21}$$

since from Appendix A,

$$(\rho+\rho_b)/2 = k^2\delta(z) + \ldots . \tag{22}$$

From Eq. (11), $\delta(z)$ is a known function. Equation (21) decouples Eqs. (1) and (2)—but introduces A, a new unknown. See below.

After the decoupled version of Eq. (1) is solved for a(z), then b(z) may be found using the properties in Eq. (3). Therefore Eq. (2) is unnecessary from here on.

4. Expanding and Decomposing into Average and Periodic Parts

Substituting $a = A(1+\rho)$ in the first three terms of Eq. (1), expanding $1/a^3$, dividing by A, and using (21) and (15), the first K-V equation is equivalent to

$$\rho(z)'' = -kh(z) - kh(z)\rho + \frac{\alpha}{3}\left(1 - 3\rho + 6\rho^2 - 10\rho^3 + 15\rho^4 \ldots\right) + q\left(1 - k^2\delta(z)\ldots\right). \tag{23}$$

To solve for the ripple $\rho(z)$ and for the mean radius A (which is incorporated in the definition of α and q), we decompose Eq. (23) into two equations. Averaging Eq. (23),

$$0 = -k\langle h\rho\rangle + \frac{\alpha}{3} + 2\alpha\langle\rho^2\rangle - \frac{10}{3}\alpha\langle\rho^3\rangle + 5\alpha\langle\rho^4\rangle \ldots + q. \tag{24}$$

Subtracting,

$$\rho'' = -kh(z) - k\{h\rho\} - \alpha\rho + 2\alpha\{\rho^2\} - \frac{10}{3}\alpha\{\rho^3\} + 5\alpha\{\rho^4\} \ldots - qk^2\delta(z)\ldots , \tag{25}$$

with the $\{..\}$ operator defined by Eq. (5). We now have two equations, each containing A and $\rho(z)$. Equations (24) and (25) taken together with (21) have the same essence as the K-V equations (1) and (2).

5. Iterative Solution of the Envelope Equations

On the right of Eq. (25), the terms involving the unknown function $\rho(z)$ are dominated by kh(z); it is reasonable to omit them for the initial integrations, which give $\rho_{(0)}$. Then we insert $\rho_{(0)}$ into (25) and integrate again to get $\rho_{(1)}$. The process is repeated for $\rho_{(2)}$:

$$\rho_{(0)} = -kg, \tag{26a}$$

$$\rho_{(1)} = \rho_{(0)} + \alpha k \iint g + k^2 \delta + \frac{10}{3} \alpha k^3 \iint g^3, \tag{26b}$$

$$\rho_{(2)} = \rho_{(1)} - \alpha^2 k \iiint g - k^3 \iint h\delta - 2\alpha k^3 \iint g\delta. \tag{26c}$$

This $\rho(z)$ expansion is to be inserted into Eq. (24) to complete the solution of the K-V equations. Terms, such as $2\alpha k^2 \iint \{g^2\}$, that would make Eq. (24) higher than third order in the combined parameters k^2, α, q have been discarded. Some terms of this type might appear to form lower-order products, but the averages vanish by orthogonality: h(z), g(z), g^3, etc., possess only odd harmonics while $\delta(z)$, $\{g^2\}$, $\{g^4\}$ have only even.

A second-order term, $qk^2 \iint \delta(z)$, is also omitted. It involves multiple integrations of an already small function and would contribute less than 0.04% to the maximum radius a_{max} even at $\sigma_0 = 120°$. It would affect A by less than two parts in ten thousand.

Appendix B evaluates to third order the combination of Eqs. (26) and (24). The result, in simplified form, is shown in the next section.

6. Matching Equation: First, Second, and Third Order

Third Order: Inserting Eq. (26) into Eq. (24) yields seven terms [App. B, Eq. (B7)]. Some terms combine, resulting in

$$K_\dagger^{\,eff} - \frac{\epsilon_{III}^{\,2}}{A_{III}^{\,4}} - \frac{Q}{A_{III}^{\,2}} = 0, \tag{27}$$

where

$$K_\dagger^{\,eff} \equiv \langle [\int K(z)]^2 \rangle \left[1 + \frac{1}{24} \Phi \left(1 + \frac{20}{27} c_3 \right) \right]; \tag{28}$$

$$\epsilon_{III}^{\,2} \equiv \epsilon^2 \left[1 + \Phi \left(1 + \tfrac{1}{2}\Phi + 3\beta_I \right) \right]. \tag{29}$$

Here c_3 is of order unity [8]. Roman-numeral subscripts on A and ϵ signify the order of approximation—third order in this case. The subscript on $\beta \sim A^{-4}$ indicates that A_I [Eq. (33)] is used for A. The matching equation (27) is in the standard form of the smooth approximation, Eq. (33), and can be solved to find the third-order A:

$$A_{III}^2 = (Q/2K_\dagger^{eff}) + \left[(Q/2K_\dagger^{eff})^2 + \epsilon_{III}^2/K_\dagger^{eff}\right]^{1/2}. \tag{30}$$

If the input quantity is the mean radius A_{inp}, then Eq. (27) gives the allowable Q to third order,

$$Q_{III} = A_{inp}^2 K_\dagger^{eff} - \epsilon_{III}^2/A_{inp}^2 .$$

Second Order: Eq. (B7) has two second-order terms. One yields the correction to K^{eff} seen in Eq. (28). The other term is $\alpha k^2\langle g^2\rangle$, or, using definition (18), $\frac{\alpha}{3}\Phi$. We define

$$\epsilon_{II}^2 \equiv \epsilon^2(1+\Phi), \tag{31}$$

with the subscript signifying second order, and get

$$K_\dagger^{eff} - \frac{\epsilon_{II}^2}{A_{II}^4} - \frac{Q}{A_{II}^2} = 0. \tag{32}$$

Eq. (32) can be solved for A_{II} or Q_{II} in the same way as for the third order, giving useful approximations when K(z) and ϵ produce σ_0 and σ less than about 80°.

First Order: In Eq. (B7), the three terms of lowest order in α, q, k^2 produce what is called the first-order matching equation in this paper (Ref. [7] used another terminology). This is the classic smooth approximation. These terms give $k^2\langle[\int h]^2\rangle = \alpha/3 + q$, or, using the definitions (15), (16), and (17)

$$K^{eff} - \frac{\epsilon^2}{A_I^4} - \frac{Q}{A_I^2} = 0. \tag{33}$$

First, second, and third-order results for A, from (33), (32) and (30), are plotted in Fig. 1a, next page. The smooth approximation is somewhat inaccurate except near the point where its error curve happens to cross the 0 % line.

7. Explicit Third-Order Result for a_{max}

Knowing the matched mean radius A, one can complete the solution for the beam envelope $a(z) = A[1+\rho(z)]$ using $\rho(z)$ from Eq. (26); b(z) can be found by changing the sign of the terms that contain odd powers of k.

Some terms of Eq. (26) can be written in exact form [Appendix J] for models such

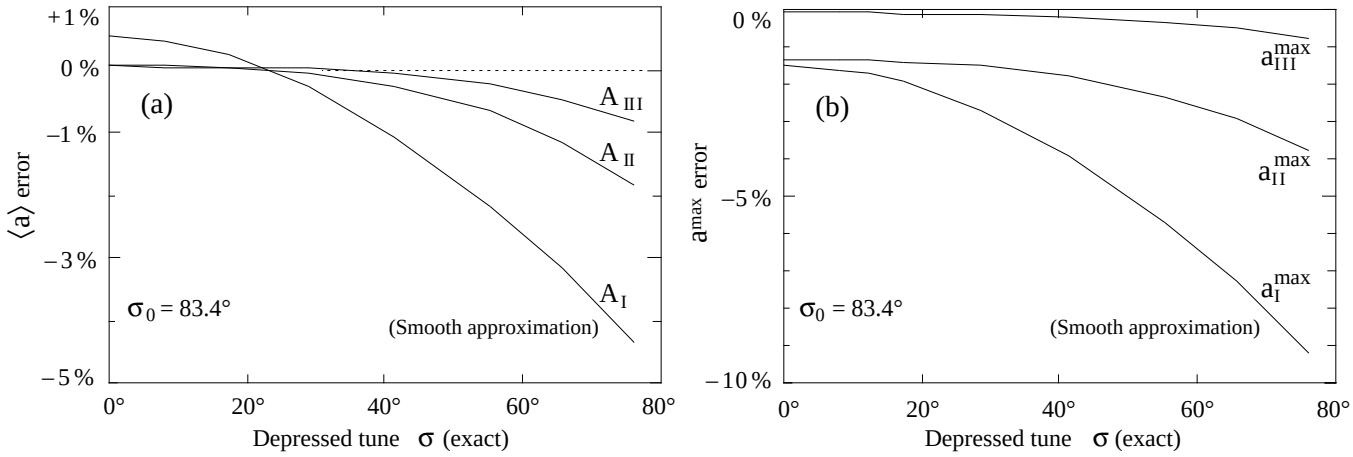

Fig. 1. Accuracy of: (a) mean radius A from Eqs. (30), (32), or (33); and (b) maximum radius a_{max} from Eqs. (36), (37), or (38). The quadrupole voltage V_Q, beam current I, and normalized emittance ϵ are input quantities. Other parameters are shown in Table 2. In this figure, V_Q is fixed at 20 kV, giving exact $\sigma_0 = 83.37°$; ϵ and I are varied so that σ ranges between 0° and 76.5°; these exact σ_0 and σ values are obtained numerically.

Table 2: Parameters Used in the Figures
(from the MFE ESQ accelerator [8] with $\eta = 0.5$)

Quad cell length	20 cm	Quad voltage (Figs. 1, 3a)	20 kV
Occupancy factor (η)	0.5	Quad voltage (Fig. 2)	25 kV
Quad radius (a_Q)	1.75 cm	Beam current (Fig. 2)	0.5 Amp
Beam energy	200 KeV	Norm emittance (Fig. 2)	0.3 πmrad-cm

Beam particles H^- ions

as FODO, but Fourier expansion is more useful in general:

$$h(z) = h_1\left[\cos\frac{\pi z}{L} + \frac{1}{3}c_3\cos 3\frac{\pi z}{L} + \frac{1}{5}c_5\cos 5\frac{\pi z}{L} + \ldots\right]. \tag{34}$$

Values (usually of order unity) of h_1 and c_n for both FODO and smooth profiles are given in Appendices G and H. With the definition

$$\beta_I \equiv 3\frac{L^2}{\pi^2}\frac{\epsilon^2}{A_I^4} \tag{35}$$

we have

$$a_{III}^{max} = A_{III}\left[1+\rho_m\left(1+\frac{1}{27}c_3+\frac{1}{125}c_5\right)+\frac{1}{8}\rho_m^2\left(1+\frac{25}{54}c_3\right)+\beta_I\rho_m\left(1+\frac{5}{2}\rho_m^2+\beta_I\right)\right] \tag{36}$$

using results from Appendix E. The accuracy of Eq. (36) is shown in Fig. 1b, along with that of the truncations

$$a_{II}^{max} = A_{II}\left[1+\rho_m\left(1+\frac{1}{27}c_3+\frac{1}{125}c_5\right)+\beta_I\rho_m\right] \tag{37}$$

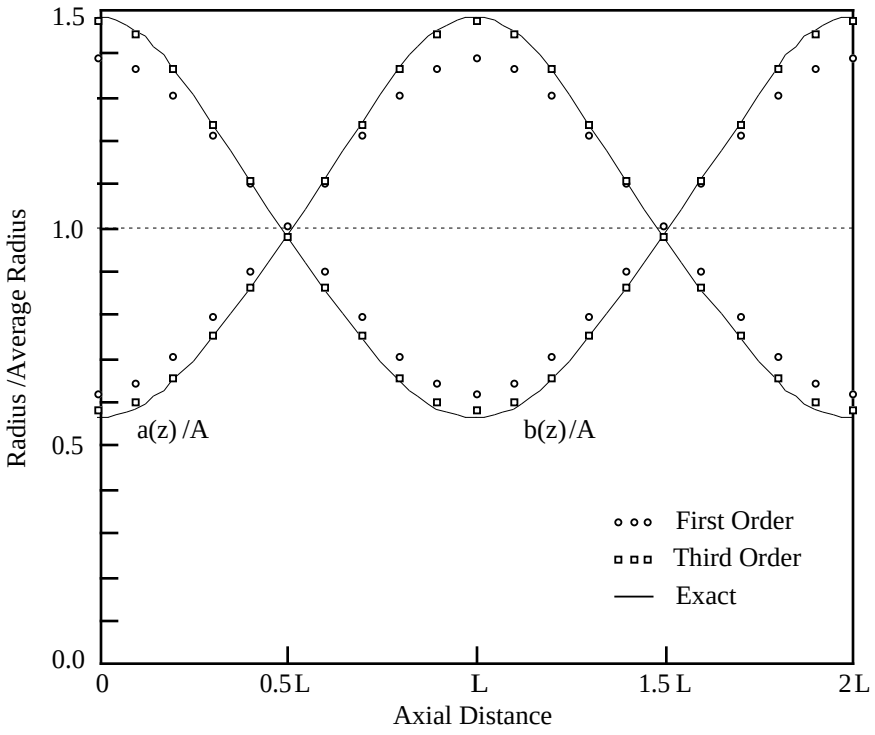

Fig. 2. Matched envelopes a(z) and b(z), normalized to mean radius A, with beam current 0.5 A, normalized emittance 1.55π mrad-cm, and quadrupole voltage 25 kV. Model: see Table 2. With these parameters, the tunes are σ_0 = 112.2° and σ = 86.9°. The exact envelopes (solid curves) were obtained numerically. The third-order results [Eqs. (E6)–(E10)] give an a_{max} error of –2.37%; in the smooth approximation [Eq. (E6) only] the error is –13.0%. Amplitude of half-period ripple = 5.6% of amplitude of full-period ripple.

and (the smooth approximation)

$$a_I^{max} = A_I\left[1+\rho_m\right]. \tag{38}$$

The time dependence of Eq. (36), from App. E, is plotted in Fig. 2 above.

8. Phase Advances

From the well-known phase-amplitude result [9], the phase advance per quadrupole cell of length 2L is

$$\sigma = \in\int_o^{2L}\frac{dz}{a^2} = 2L\in\langle a^{-2}\rangle,$$

We approximate a(z) by $A_{III}[1+\rho(z)]$ with A_{III} from Eq. (30) and $\rho(z)$ to third order from Eq. (26). (We omit subscripts for compactness.) Expanding a^{-2} and taking the average gives

$$\sigma = 2L\frac{\in}{A_{III}^2}\left[1+3\langle\rho^2\rangle-4\langle\rho^3\rangle+5\langle\rho^4\rangle-\cdots\right]. \tag{39}$$

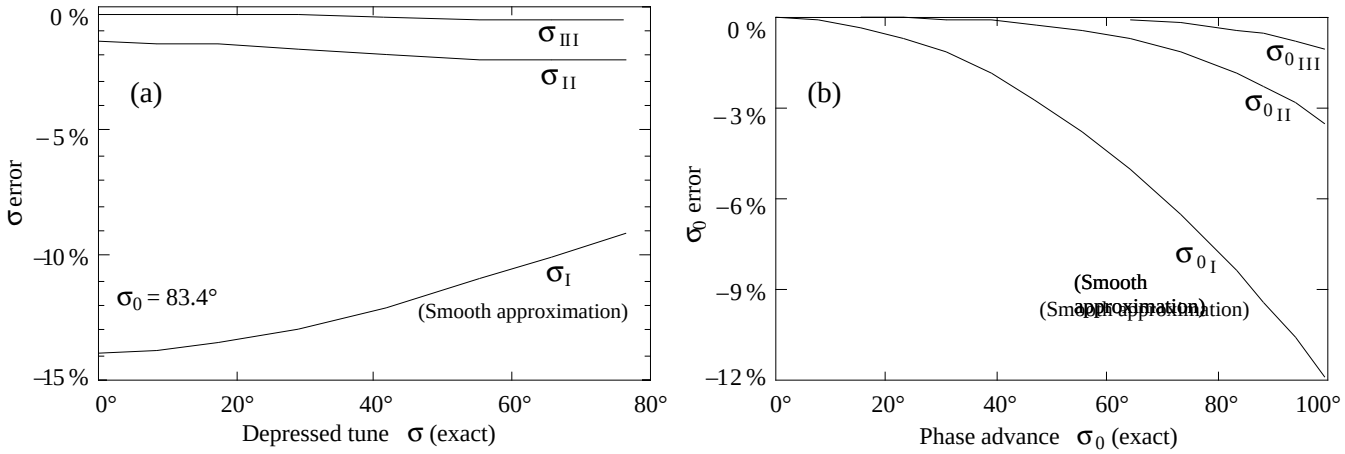

Fig. 3. (a) Accuracy of σ from Eqs. (40), (41), and (42). V_Q is fixed at 20 kV as in Fig. 1.
(b) Accuracy of σ_0 from Eqs. (43), (44), and (45). V_Q ranges from 5 kV to about 22 kV.

(The 2ρ term has zero average by definition.) Appendix D shows that to third-order accuracy

$$\sigma_{III} = 2L\frac{\epsilon}{A_{III}^2}\left[1+\Phi\left(1+\tfrac{3}{4}\Phi+2\beta_I\right)\right]. \tag{40}$$

Errors with respect to exact values from simulations are shown in Fig. 3a. Useful accuracy is retained after dropping two terms and using lower-order A_{II} from Eq. (32):

$$\sigma_{II} = 2L\frac{\epsilon}{A_{II}^2}(1+\Phi). \tag{41}$$

Figure 3a shows large errors for the first-order result (smooth approximation):

$$\sigma_I = 2L\frac{\epsilon}{A_I^2}. \tag{42}$$

The undepressed σ_0 is found by setting Q = 0 in Eq. (27), then eliminating ϵ from Eq. (40). Details are in Appendix D. The result is

$$\sigma_{0\,III} = 2L\left(K_\dagger^{eff}\right)^{1/2}\left[1+\tfrac{1}{2}\Phi+\tfrac{7}{8}\Phi^2\right]. \tag{43}$$

This equation is used to calculate σ_0 as a function of the strength of the quadrupole field gradient. Figure 3b shows its accuracy and also illustrates the second-order case

$$\sigma_{0\,II} = 2L\left(K_\dagger^{eff}\right)^{1/2}\left[1+\tfrac{1}{2}\Phi\right] \tag{44}$$

and the smooth approximation,

$$\sigma_{0\,I} = 2L\left(K^{eff}\right)^{1/2}. \tag{45}$$

In some cases it is more convenient to work with the squares of σ_0 and σ—see App. D.

Appendix A. Coupling Between a and b in the K-V Equations

This appendix calculates a(z)+b(z) for the Q-term denominators in the K-V equations. Dividing equations (1) and (2) by A and expanding gives

$$\rho_a'' = -kh(z) - kh\rho_a + \frac{\epsilon^2}{A^4}\left(1 - 3\rho_a + \ldots\right) + \frac{Q}{A^2}\left(1 - (\rho_a+\rho_b)/2 \ldots\right), \qquad \text{(A1)}$$

$$\rho_b'' = +kh(z) + kh\rho_b + \frac{\epsilon^2}{A^4}\left(1 - 3\rho_b + \ldots\right) + \frac{Q}{A^2}\left(1 - (\rho_a+\rho_b)/2 \ldots\right). \qquad \text{(A2)}$$

Subtracting the averages (using definitions from Table 1 except $\rho \to \rho_a$ here),

$$\rho_a'' = -kh - k\{h\rho_a\} - \alpha\rho_a - q(\rho_a+\rho_b)/2 \ \ldots, \qquad \text{(A3)}$$

$$\rho_b'' = +kh + k\{h\rho_b\} - \alpha\rho_b - q(\rho_a+\rho_b)/2 \ \ldots. \qquad \text{(A4)}$$

Taking sum and difference,

$$S'' = -k\{hD\} - (\alpha+q)S \ \ldots, \qquad \text{(A5)}$$

$$D'' = -kh - k\{hS\} - \alpha D \ \ldots, \qquad \text{(A6)}$$

where

$$S \equiv (\rho_a+\rho_b)/2, \qquad D \equiv (\rho_a-\rho_b)/2 \qquad \text{(A7)}$$

have periods L and 2L, respectively. The lowest iteration of (A6) is

$$D_{(0)} = -k\iint h = -kg. \qquad \text{(A8)}$$

Inserting this purely oscillatory D into (A5),

$$S_{(0)}'' = k^2\{hg\}. \qquad \text{(A9)}$$

Integrating,

$$S_{(0)} = k^2\iint\{hg\}. \qquad \text{(A10)}$$

Using Eq. (A7) and the definition of $\delta(z)$, this is

$$(\rho_a+\rho_b)/2 = k^2\delta(z) + \ldots \qquad \text{(A11)}$$

which yields Eq. (22). The expansion and iteration could be extended to produce more terms, but these would give even smaller corrections to our results.

Appendix B. Details of Derivation of Matching Equation

Moving the driving term to the left-hand side, Eq. (24) is

$$k\langle h\rho\rangle = \alpha\left[\tfrac{1}{3} + 2\langle\rho^2\rangle - \tfrac{10}{3}\langle\rho^3\rangle + 5\langle\rho^4\rangle \ldots\right] + q. \qquad \text{(B1)}$$

Inserting ρ from Eq. (26), the left side is

$$k\langle h\rho\rangle = k^2\left[-\langle hg\rangle + \alpha\langle h\iint g\rangle + \tfrac{10}{3}\alpha k^2\langle h\iint g^3\rangle - \alpha^2\langle h\iint\iint g\rangle - k^2\langle h\iint h\delta\rangle - 2\alpha k^2\langle h\iint g\delta\rangle \ \ldots\right]$$

where we have dropped the subscript on ρ. The orthogonal $k^2\delta$ term is absent. We simplify by changing the order of integrations, using the h(z) symmetries [Eq. (3)]. For example, $-\langle h\iint\iint g\rangle = -\langle g\iint g\rangle = +\langle[\int g]^2\rangle$. Applying this technique throughout gives

$$k\langle h\rho\rangle = +k^2\left[\langle[\int h]^2\rangle + \alpha\langle g^2\rangle + \alpha^2\langle[\int g]^2\rangle\right] + k^4\left[\langle[\int\{hg\}]^2\rangle + \tfrac{10}{3}\alpha\langle g^4\rangle + 2\alpha\langle g^2\delta\rangle\ldots\right]. \quad \text{(B2)}$$

For the right side of (B1),

$$\langle\rho^2\rangle = k^2\langle g^2\rangle - 2\alpha k^2\langle g\iint g\rangle + 2k^4\langle g\int\int h\delta\rangle\ldots, \quad \text{(B3)}$$

$$\langle\rho^3\rangle = 3k^4\langle g^2\delta\rangle\ldots, \quad \text{(B4)}$$

$$\langle\rho^4\rangle = k^4\langle g^4\rangle\ldots. \quad \text{(B5)}$$

The very small $k^4\langle\delta^2\rangle$ term was omitted from $\langle\rho^2\rangle$. Again changing the order of integrations, the right side of (B1) becomes

$$\text{rhs} = \alpha\left[\tfrac{1}{3} + k^2\left(2\langle g^2\rangle + 4\alpha\langle[\int g]^2\rangle\right) + k^4\left(4\langle g\int\int h\delta\rangle - 10\langle g^2\delta\rangle + 5\langle g^4\rangle\right)\right] + q\ldots. \quad \text{(B6)}$$

Four of the terms of (B6) combine with terms of (B2), so that

$$k^2\langle[\int h]^2\rangle + k^4\langle[\int\{hg\}]^2\rangle =$$
$$\alpha\left[\tfrac{1}{3} + k^2\left(\langle g^2\rangle + 3\alpha k^2\langle[\int g]^2\rangle\right) + k^4\left(4\langle g\int\int h\delta\rangle - 12\langle g^2\delta\rangle + \tfrac{5}{3}\langle g^4\rangle\right)\ldots\right] + q, \quad \text{(B7)}$$

the matching equation from Eq. (B1). Each term (except $\alpha/3$ and q) involves averages of functions of the focusing profile h(z). Given any h(z)—obtained from a model such as FODO or measured on an actual quadrupole cell—these averages can be calculated once and for all, being constant coefficients of the terms in α and k. Appendix C shows how to write Eq. (B7) in simple form [Eq. (C9)].

Appendix C. Simplification of Matching Equation

It is convenient to write the Fourier representation in the form

$$h(z) = h_1\left[\cos\frac{\pi z}{L} + \tfrac{1}{3}c_3\cos 3\frac{\pi z}{L} + \tfrac{1}{5}c_5\cos 5\frac{\pi z}{L} + \ldots\right] \quad \text{(C1)}$$

The axial profile of the quadrupole gradient determines h_1 and c_n. Tables G1 and H1 show that h_1 remains of the order of unity while c_3 and c_5 can change sign as the profile is varied. For the hard-edge quadrupole model (FODO) with occupancy $\eta = 0.5$, Table G1 shows that $c_3 = 1$. Because of multiple integrations, terms containing c_5 are usually negligible.

Right side of Eq. (B7). By definition, $k^2\langle g^2\rangle = \Phi/3$. For the factor $\langle[\int g]^2\rangle = \langle[\iint h]^2\rangle$, the third and higher harmonics make very small contributions because of the multiple integrations. Comparing leading terms for $\langle[\int g]^2\rangle$ and $\langle g^2\rangle$ gives

$$3k^2\langle[\int g]^2\rangle \approx \frac{L^2}{\pi^2}3k^2\langle g^2\rangle = \frac{L^2}{\pi^2}\Phi. \tag{C2}$$

The three k^4 terms on the rhs of Eq. (B7) are:

$$4\langle g\iint h\delta\rangle = \tfrac{1}{2}\langle g^2\rangle^2\left(1+\tfrac{19}{27}c_3+\ldots\right), \tag{C3}$$

$$-12\langle g^2\delta\rangle = -\tfrac{3}{2}\langle g^2\rangle^2\left(1+\tfrac{4}{9}c_3+\ldots\right), \tag{C4}$$

$$\tfrac{5}{3}\langle g^4\rangle = \tfrac{5}{2}\langle g^2\rangle^2\left(1+\frac{4}{81}c_3+\ldots\right), \tag{C5}$$

Adding (C3) through (C5) gives $\frac{3}{2}\langle g^2\rangle^2(1+(29/243)c_3\ldots)$, where the small c_3 correction can be neglected since it corrects a term which is already third order. Using all these results along with definition (15), the right side of (B7) (without the q term) becomes

$$\epsilon^2\left[1+\Phi+\tfrac{1}{2}\Phi^2+\frac{9}{\pi^2}\frac{\epsilon^2L^2}{A_I^4}\Phi\right] = \epsilon^2\left[1+\Phi+\tfrac{1}{2}\Phi^2+3\beta_I\Phi\right] \equiv \epsilon_{III}^2, \tag{C6}$$

as in the main text. In the last term,

$$\beta_I \equiv 3\frac{L^2}{\pi^2}\frac{\epsilon^2}{A_I^4} \tag{C7}$$

uses the lowest-order value for A because this term is already of the highest order that we retain.

Left side of Eq. (B7): the $\langle[\int\{hg\}]^2\rangle$ term is

$$\langle[\int\{hg\}]^2\rangle = \frac{1}{8}\langle g^2\rangle\left(1+\frac{20}{27}c_3+\frac{5^3}{3^6}c_3^2+\ldots\right)\langle[\int h]^2\rangle. \tag{C8}$$

Dropping the c_3^2 term in (C8) for simplicity, we define the LHS of (B7) as

$$K_\dagger^{\text{eff}} = k^2\langle[\int h]^2\rangle\left[1+\frac{1}{24}\Phi\left(1+\frac{20}{27}c_3\right)\right]. \tag{C9}$$

Altogether,

$$K_\dagger^{\text{eff}} = \frac{\epsilon_{III}^2}{A_{III}^4} + \frac{Q}{A_{III}^2} \tag{C10}$$

which is Eq. (27).

Appendix D. Depressed and undepressed tunes

Here we evaluate the expansion terms in Eq. (39). From Eqs. (B3-B5) in App. B,

$$3\langle\rho^2\rangle = 3k^2\langle g^2\rangle + 6\alpha k^2\langle[\int g]^2\rangle + 6k^4\langle g\iint h\delta\rangle\ldots, \tag{D1}$$

$$-4\langle\rho^3\rangle = -12k^4\langle g^2\delta\rangle\ldots, \tag{D2}$$

$$5\langle\rho^4\rangle = 5k^4\langle g^4\rangle\ldots . \tag{D3}$$

From App. C, Eqs. (C2)-(C5),

$$6\alpha k^2\langle[\int g]^2\rangle \approx \frac{L^2}{\pi^2}6\alpha k^2\langle g^2\rangle = 2\beta_I\Phi, \tag{D4}$$

$$6k^4\langle g\iint h\delta\rangle = \tfrac{3}{4}k^4\langle g^2\rangle^2\left(1+\tfrac{19}{27}c_3+\ldots\right), \tag{D5}$$

$$-12k^4\langle g^2\delta\rangle = -\tfrac{3}{2}k^4\langle g^2\rangle^2\left(1+\tfrac{4}{9}c_3+\ldots\right), \tag{D6}$$

$$5k^4\langle g^4\rangle = \tfrac{15}{2}k^4\langle g^2\rangle^2\left(1+\frac{4}{81}c_3+\ldots\right). \tag{D7}$$

When the last three are added, the c_3 coefficient is only $25/729$. Dropping this and using the definitions gives to third-order accuracy

$$\sigma = 2L\frac{\epsilon}{A_{III}^2}\left[1+\Phi+\tfrac{3}{4}\Phi^2+2\beta_I\Phi\right]. \tag{D8}$$

Undepressed tune:

Setting Q = 0, (27) is

$$K_\dagger^{\text{eff}} = \frac{\epsilon_{III}^2}{A_{III}^4} = \frac{\epsilon^2}{A_{III}^4}\left[1+\Phi+\tfrac{1}{2}\Phi^2+3\beta_I\Phi\right]^{1/2}. \tag{D9}$$

The factor ϵ^2/A_I^4 in the last term can be replaced by $K^{\text{eff}} \equiv k^2\langle[\int h]^2\rangle$, according to Eq. (33) with Q=0. Comparing with the definition of Φ and Fourier expanding as before, the last term, to lowest order, is $3\Phi^2$ for Q=0. Thus, altogether,

$$K_\dagger^{\text{eff}} = \frac{\epsilon^2}{A_{III}^4}\left[1+\Phi+\tfrac{7}{2}\Phi^2+\ldots\right]^{1/2}. \tag{D10}$$

Making a similar replacement in Eq. (40) for the case Q=0,

$$\sigma_0 = 2L\frac{\epsilon}{A_{III}^2}\left[1+\Phi+\tfrac{11}{2}\Phi^2+\ldots\right]. \tag{D11}$$

Using Eq. (D10) to eliminate ϵ/A_{III}^2,

$$\sigma_0 = 2L\left(K_\dagger^{\text{eff}}\right)^{1/2}\left[1+\Phi+\tfrac{11}{2}\Phi^2+\ldots\right]\left[1+\Phi+\tfrac{7}{2}\Phi^2+\ldots\right]^{-1/2}, \tag{D12}$$

or, finally,

$$\sigma_0 = 2L\left(K_\dagger^{eff}\right)^{1/2}\left[1 + \tfrac{1}{2}\Phi + \tfrac{7}{8}\Phi^2 + \ldots\right]. \qquad \text{(D13)}$$

to third order.

Sometimes it is convenient to work with the squares of σ_0 and σ, which are for third order

$$\sigma_0^2 = 4L^2 K_\dagger^{eff}\left[1 + \Phi + 2\Phi^2\right] \qquad \text{(D14)}$$

and

$$\sigma^2 = 4L^2 \frac{\in^2}{A_{III}^4}\left[1 + 2\Phi\left(1 + \tfrac{5}{4}\Phi^2 + 2\beta_I\right)\right]. \qquad \text{(D15)}$$

Appendix E. Calculation of a(z) and b(z).

Using Fourier expansion, written as in Appendix C,

$$kh(z) = kh_1\left[\cos\frac{\pi z}{L} + \tfrac{1}{3}c_3\cos 3\frac{\pi z}{L} + \tfrac{1}{5}c_5\cos 5\frac{\pi z}{L} \ldots\right],$$

and recalling $\rho_m \equiv h_1 kL^2/\pi^2$, the terms of Eq. (26) are

$$-kg = -k\iint h = \rho_m\left[\cos\frac{\pi z}{L} + \tfrac{1}{27}c_3\cos 3\frac{\pi z}{L} + \tfrac{1}{125}c_5\cos 5\frac{\pi z}{L} \ldots\right] \qquad \text{(E1)}$$

$$\alpha k\iint g = \beta\rho_m\left[\cos\frac{\pi z}{L} + \tfrac{1}{243}c_3\cos 3\frac{\pi z}{L} + \ldots\right] \qquad \text{(E2)}$$

$$k^2\delta(z) = k^2\iint\{hg\} = \tfrac{1}{8}\rho_m^2\left[\left(1 + \tfrac{10}{27}c_3\ldots\right)\cos 2\frac{\pi z}{L} + \tfrac{5}{54}c_3\cos 4\frac{\pi z}{L} \ldots\right] \qquad \text{(E3)}$$

$$\tfrac{10}{3}\alpha k^3\iint g^3 = \tfrac{5}{2}\beta\rho_m^3\left[\left(1 + \tfrac{1}{27}c_3\ldots\right)\cos\frac{\pi z}{L} + \tfrac{1}{27}\left(1 + \tfrac{2}{9}c_3\ldots\right)\cos 3\frac{\pi z}{L}\ldots\right] \qquad \text{(E4)}$$

$$-\alpha^2 k\iint\iint g = \beta^2\rho_m\cos\frac{\pi z}{L} + \ldots \qquad \text{(E5)}$$

The small final two terms from Eq. (26) have been omitted here for simplicity.

The Significant Terms

We drop small quantities in the above equations. The criterion is that they contribute less than 2 parts per thousand to the final result for a bad-case scenario: large focusing voltage (giving phase advance of 112°) and large β. This leaves

$$-kg = \rho_m\left(\cos\frac{\pi z}{L} + \tfrac{1}{27}c_3\cos 3\frac{\pi z}{L} + \tfrac{1}{125}c_5\cos 5\frac{\pi z}{L} \ldots\right) \qquad \text{(E6)}$$

$$\alpha k\iint g = \beta\rho_m\cos\frac{\pi z}{L} \ldots \qquad \text{(E7)}$$

$$k^2\delta(z) = \tfrac{1}{8}\rho_m^2\left(\cos 2\frac{\pi z}{L} + \tfrac{25}{54}c_3\cos 4\frac{\pi z}{L}\ldots\right) \qquad \text{(E8)}$$

$$\frac{10}{3}\alpha k^3 \iint g^3 = \frac{5}{2}\beta \rho_m^3 \cos\frac{\pi z}{L} \qquad \text{(E9)}$$

$$-\alpha^2 k \iiint g = \beta^2 \rho_m \cos\frac{\pi z}{L} \ \ldots \qquad \text{(E10)}$$

Adding all these terms gives $\rho(z)$ to third-order accuracy. A few small terms were omitted as mentioned earlier. Setting $z=0$ gives ρ_{max} and $a_{max} = A(1+\rho_{max})$ as presented in Section 7. Setting $z=L$ changes the sign of all terms, except the even term $k^2\delta(z)$, and yields a_{min}.

Results from Eqs. (E6) – (E10) are shown in Fig. 2 in the main text.

Appendix F. Fourier Representation for Arbitrary Symmetric Cases

Fourier Coefficients

Recall from Section 2 that the focusing force K(z) in the K-V equations is written as

$$K(z) = kh(z) \qquad \text{(F1)}$$

with $h(0) = 1$. Because of the assumed symmetries [Eq. (3)], there are only odd harmonics:

$$h(z) = \sum_{1,3,5\ldots} h_n \cos\frac{n\pi z}{L} \qquad \text{(F2)}$$

with the condition

$$\sum_{1,3,5\ldots} h_n = 1\,. \qquad \text{(F3)}$$

The Fourier coefficients are

$$h_n = \frac{1}{L}\int_o^{2L} h(z)\cos\frac{n\pi z}{L}\,dz. \qquad \text{(F4)}$$

It is often convenient to define

$$c_n \equiv n h_n / h_1, \qquad \text{(F5)}$$

where $c_1 = 1$ by definition and where $|c_3|$ usually turns out to be of order unity—see Tables G1 and H1. Then Eq. (F2) is written as

$$h(z) = h_1 \sum_{1,3,5\ldots} \frac{1}{n} c_n \cos n\frac{\pi z}{L}\,. \qquad \text{(F6)}$$

Solution of Envelope Equation

In the solution for $\rho(z)$, Eq. (26), the largest term is

$$\rho_{(0)}(z) = -kg(z) = -k\int_{L/2}^{z} dz' \int_o^{z'} h(z'')dz'',$$

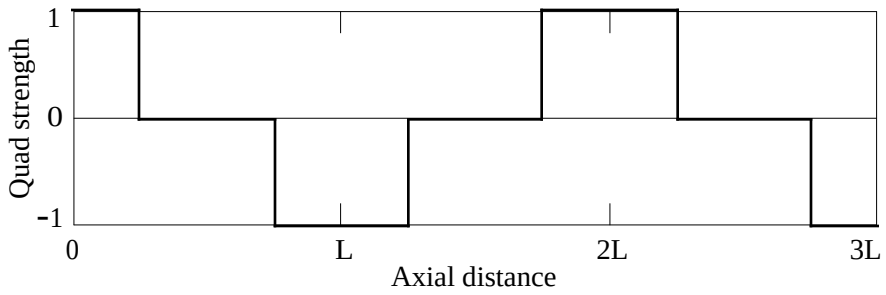

Fig. G1. Normalized quadrupole strength h(z) vs z for a FODO lattice having occupancy factor $\eta = 0.5$. The unit cell length is 2L.

which with Eqs. (F2) and (F5) is

$$-kg(z) = \frac{kL^2}{\pi^2} h_1 \sum_{1,3,5\ldots} \frac{c_n}{n^3} \cos\frac{n\pi z}{L}. \tag{F7}$$

The next largest term is

$$\alpha k\iint g = \alpha\frac{kL^4}{\pi^4} h_1 \sum_{1,3,5\ldots} \frac{c_n}{n^5} \cos\frac{n\pi z}{L}. \tag{F8}$$

To achieve 1% accuracy, the first three series elements of (F7) are usually required, whereas for Eq. (F8), only the fundamental is needed [cf. Eq. (E2)].

The additional terms of Eq. (26), shown in Eqs. (E3) and (E4) are found with the help of trigonometric identies.

The mean square of the integral of Eq. (F2) gives the effective force

$$K_{eff} = k^2\langle[\int h]^2\rangle = h_1^2 \frac{k^2L^2}{2\pi^2} \sum_{1,3,5\ldots} \frac{c_n^2}{n^4}, \tag{F9}$$

which is used in the matching equation and for calculating undepressed phase advance.

The correction term Φ (used in evaluating phase advances, average radius or transportable current, etc.) is

$$\Phi \equiv 3k^2\langle g^2\rangle = 3h_1^2\frac{k^2L^4}{2\pi^4} \sum_{1,3,5\ldots} \frac{c_n^2}{n^6} \rightarrow 3h_1^2\frac{k^2L^4}{2\pi^4} \tag{F10}$$

since the harmonics contribute practically nothing. Dividing this into Eq. (F8), we find $\Phi = 3K_{eff}(L^2/\pi^2)\left(1+ c_3^2/81 \ldots\right)$, which could be useful in certain calculations.

Appendix G. Fourier Solution for Case of FODO

Our first example is the FODO lattice (Fig. G1) where exact expressions are available for comparison (App. J). Given the occupancy factor η, the Fourier coefficients h_n are readily calculated from Eq. (F4):

$$h_n = \frac{4}{n\pi} \sin\frac{n\pi\eta}{2}, \tag{G1}$$

which satisfies Eq. (F3) for $0 < \eta \le 1$. From Eq. (F5),

$$c_n = n\,\frac{\sin(n\pi\eta/2)}{\sin(\pi\eta/2)} \tag{G2}$$

for FODO. All the results from App. F can be used for FODO by putting $h_1 = \sin\frac{\pi\eta}{2}$. Values of h_1 (normalized with $\pi/2$) and c_n are shown in Table G1 for various η.

Table G1: h_1 and $c_n \equiv nh_n/h_1$ for FODO case

η	$\pi h_1/2$	c_1	c_3	c_5	c_7	c_9	c_{11}
1/3	1	1	2	1	–1	-2	-1
1/2	$\sqrt{2}$	1	1	–1	–1	1	1
2/3	$\sqrt{3}$	1	0	–1	1	0	--1
1	2	1	-1	1	-1	1	1

Appendix H. Solution for Some Cases of Soft-Edge Profiles

Actual quadrupole fields are quite different from those in the so-called hard edge FODO model of Fig. G1. A few simple smooth models will be discussed.

Field Model 1: $K''(0) = 0$.

$$h_1 = \frac{9}{8}, \qquad c_3 = -\frac{1}{3}, \tag{H1}$$

with all the other coefficients zero. This choice gives a flat field at the midpoint of the quadrupoles, without the discontinuities of the hard edge model. From Eqs. (H1), (F7), (F9) and (F10) we get

$$\rho_{(0)}^{\max} = -kg(0) = \frac{9}{8}\frac{kL^2}{\pi^2}\left(1 - \frac{1}{9^2}\right), \tag{H2}$$

$$K_{\text{eff}} = \frac{9^2}{8^2}\,\frac{k^2L^2}{2\pi^2}\left(1 + \frac{1}{9^3}\right), \tag{H3}$$

$$\Phi = 3\,\frac{9^2}{8^2}\,\frac{k^2L^4}{2\pi^4}\left(1 + \frac{1}{9^4}\right), \tag{H4}$$

The third-harmonic correction for Φ can be neglected in most cases.. There are no higher terms.

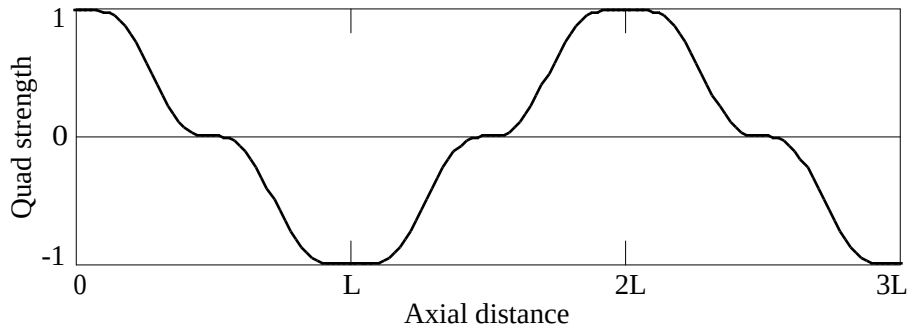

Fig. H1. Axial profile of normalized quadrupole strength h(z) for Field Model 3, which uses only the 3rd and 5th harmonics. The unit cell length is 2L.

Field Model 2: $K'(L/2) = 0$.

$$h_1 = \frac{3}{4}, \qquad c_3 = 1. \tag{H5}$$

This model is narrow, peaked at the quadrupole midpoints, with zero slope at the gap centers. It gives focusing strength equivalent to FODO having about 40% occupancy. The third-harmonic corrections to a^{max}, K^{eff} and Φ are 1/27, 1/81, and 1/243 respectively.

Field Model 3: $K''(0) = 0$ and $K'(L/2) = 0$.

$$h_1 = \frac{15}{16}, \qquad c_3 = \frac{1}{2}, \qquad c_5 = -\frac{1}{2}, \tag{H6}$$

which gives a fairly realistic profile (Fig. H1) and corresponds to FODO with $\eta \sim 53\%$. The third- and fifth-harmonic corrections are all less than 1% for this case.

Table H-1 summarizes the above results.

Table H1

h_1 and $c_n \equiv nh_n/h_1$ and η_{equiv} for smooth profiles

Model	h_1	c_1	c_3	c_5	c_7	c_9	η_{equiv}
#1	9/8	1	-1/3	–	–	–	0.69
#2	3/4	1	1	–	–	–	0.40
#3	15/16	1	1/2	-1/2	–	–	0.53

Appendix J. Some Exact Formulas for Case of FODO

Truncated Fourier representations for the hard-edge FODO may be compared with exact results of integration. (The FODO model is illustrated in Fig. G1.) Because of the

symmetries expressed in Eq. (3), the calculation of averages is simplified, requiring only integration over one-fourth of a cell. One finds for occupancy η

$$K_{eff} = k^2\langle[\int h]^2\rangle = \tfrac{1}{12}\eta^2(3-2\eta)k^2L^2, \tag{J1}$$

$$\Phi = 3k^2\langle[\iint h]^2\rangle = \tfrac{1}{16}\eta^2\left(1-\eta^2+\tfrac{2}{5}\eta^3\right)k^2L^4. \tag{J2}$$

One can do the integrals in the first two terms of $\rho(z)$, Eq. (26). For the integrations, we divide the cell into five zones:

Zone 0: $0 \le z \le \eta L/2$
Zone 1: $\eta L/2 < z \le L - \eta L/2$
Zone 2: $L - \eta L/2 < z \le L + \eta L/2$
Zone 3: $L + \eta L/2 < z \le 2L - \eta L/2$
Zone 4: $2L - \eta L/2 < z \le 2L$.

For even-numbered zones, the first integral is

$$-\iint h = P\left(\tfrac{n}{2}\right)\tfrac{1}{2}\left[\eta(2-\eta)\left(\tfrac{L}{2}\right)^2 - \left(z - n\tfrac{L}{2}\right)^2\right]; \tag{J3a}$$

for odd-numbered zones it is

$$-\iint h = P\left(\tfrac{n+1}{2}\right)\tfrac{1}{2}L\eta\left(z - n\tfrac{L}{2}\right), \tag{J3b}$$

with n the zone number and

$$P(m) \equiv +1 \text{ for even } m$$
$$P(m) \equiv -1 \text{ for odd } m.$$

As required by the definition of $\iint$, the average has been subtracted. The maximum value of $\rho_{(0)}$ (at z = o, where n = 0) is

$$\rho_{(0)}^{max} = -k\iint h\Big|_0 = \tfrac{1}{8}\eta(2-\eta)kL^2. \tag{J4}$$

The next term in Eq. (26) includes the integral $\iint g$. For even-numbered zones

$$\iint g = P\left(\tfrac{n}{2}\right)\left[\eta\left(1-\tfrac{\eta^2}{2}+\tfrac{\eta^3}{8}\right)\left(\tfrac{L}{2}\right)^4 - \tfrac{3}{4}\eta(2-\eta)\left(\tfrac{L}{2}\right)^2\left(z - n\tfrac{L}{2}\right)^2 + \tfrac{1}{8}\left(z - n\tfrac{L}{2}\right)^4\right], \tag{J5a}$$

and for odd-numbered zones

$$\iint g = P\left(\tfrac{n+1}{2}\right)\left[\left(\tfrac{3}{2}-\tfrac{\eta^2}{2}\right)\left(\tfrac{L}{2}\right)^3\left(z - n\tfrac{L}{2}\right) - \tfrac{L}{4}\left(z - n\tfrac{L}{2}\right)^3\right]. \tag{J5b}$$

The maximum value of $\iint g$ is

$$\left(\tfrac{L}{2}\right)^4\eta\left(1-\tfrac{\eta^2}{2}+\tfrac{\eta^3}{8}\right). \tag{J6}$$

These results, for any value of η, may be compared with those from Appendices G and H to determine the number of Fourier terms needed for a given accuracy in each case.

Acknowledgements

E.P. Lee and L.L. Lodestro offered helpful suggestions. Supported in part by the U.S. Department of Energy under Contract No. DE-AC03-76SF00098.

References

[1] I.M. Kapchinskij and V.V. Vladimirskij, *Proc. Int. Conf. on High Energy Accel. and Instrum.* (CERN Sci. Inf. Service, Geneva, 1959), p. 274.

[2] M. Reiser, "Periodic Focusing of Intense Beams," *Particle Accelerators* **8**, 167 (1978).

[3] J. Struckmeier and M. Reiser, *Particle Accelerators* **14**, 227 (1984).

[4] R.C. Davidson, *Physics of Nonneutral Plasmas*, Addison-Wesley, New York, 1990. R.C. Davidson and Q. Qian, *Phys. Plasmas* **1**, 3104 (1994).

[5] E.P. Lee, T.J. Fessenden, and L.J. Laslett, "Transportable charge in a periodic alternating gradient system," *IEEE Trans. Nuc. Sci.* **NS-32**, 2489 (1985).

[6] E.P. Lee, *Particle Accelerators* **52** (1996)

[7] O.A. Anderson, *Particle Accelerators* **52**, 133 (1996)
O.A. Anderson, *Lawrence Berkeley Laboratory report* LBL-261233 (Revised), 1995.

[8] O.A. Anderson, et al., "Applications of the Constant-Current Variable-Voltage DC Accelerator," *Nucl. Instrum. and Meth.* **B40/41**, 877 (1989).

[9] E.D. Courant and H.S. Snyder, "Theory of the Alternating-Gradient Synchrotron," *Ann. of Phys*. **3**, 1 (1958).