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Transfer Theorems on Tautological Modules of Hilbert Schemes of Nodal Curves and de Jonquieres' Formulas

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Transfer Theorems on Tautological Modules of Hilbert Schemes of Nodal Curves and de Jonquieres' Formulas

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ABSTRACT OF THE DISSERTATION

Transfer Theorems on Tautological Modules of Hilbert Schemes of Nodal Curves and de Jonquieres' Formulas

by

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Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, September 2012
Dr. Ziv Ran , Chairperson

Given a linear system (L, V), where $L \in Pic^d(X)$ and $V \in G(r+1, H^0(L))$, on a smooth algebraic curve X, the classical de Jonquieres' formula gives the number of divisors of degree n of the form $D = a_1D_1 + \cdots + a_kD_k$, where $degD_i = n_i$ and $\sum a_in_i = n$, contained in this system, provided this number is finite. In this dissertation we verify the de Jonquieres' formula for a curve and get some de Jonquieres' formulas for a family of nodal curves using Module theorem, Splitting principle, and Transfer theorems.

Contents

Chapter 1

Introduction

Counting hyperplanes multi-tangent to a curve is well known as a particular case of the classical formula of de Jonquieres. With the complete understanding of the Chow ring of $X^{(n)}$, the symmetric product of a smooth algebraic curve, MacDonald[Mac] reduces many questions of enumerative geometry on the curve X to simple computations, e.g. *de Jonquieres' formula.*

The aim of enumerative geometry is to count how many geometric figures satisfy given conditions. One of the typical enumerative problem is: How many lines in $\mathbb{P}^3$, in general, intersect four given lines? To see this, one can degenerate the arrangement of four lines so that the first intersect the second and the third intersect the fourth. Then there are two lines: the line joining the two points of intersection and the line of intersection of the two planes. Now Schubert's *principle of conservation of number* asserts that this is the general result.

David Hilbert asked in his 15th problem for solid mathematical foundations and a systematic approach to enumerative geometry. A fundamental breakthrough in this direction was to develop a theory of *parameter spaces* or *moduli spaces*, i.e., parameterizing the geometric objects to be studied. Imposing geometric conditions corresponds to cutting appropriate subspaces in the moduli space. Thus enumerative geometry is reduced to intersection theory on moduli spaces. For example, by Schubert calculus on Grassmannian $G(2,4)$, there are 2 lines intersecting general four given lines $L_1, \cdots, L_4$: the fourfold self-intersection $\sigma_1^4 = 2$, where σ_1 is the Schubert cycle, i.e. the cohomology class of Schubert variety of lines in $\mathbb{P}^3$ meeting L_1[GH p.206].

One of the first proofs of the existence of special divisors(one of the main results of Brill-Noether theory[H1],[ACGH]) was based on intersection theory on symmetric products, the *Hilbert schemes* of a smooth algebraic curve, developed by MacDonald[Mac] using the *Porteous' formula.*

Theorem 1 *(Kempf[Ke], Kleiman-Laksov[KL]) When the Brill-Noether number $\rho \geq 0$, every curve of genus g possesses a g_d^r.*

In [Mum] Mumford defined certain cohomology classes called tautological classes on the moduli space of smooth curves of genus g, $\mathcal{M}_g$: κ_i, $1 \leq i \leq 3g-3$ and λ_l, $1 \leq l \leq g$ classes. He suggested studying the moduli space of curves in the same way of Grassmannian $G(k,n)$ parametrizing k-plaines in $\mathbb{C}^n$; there is a universal bundle E on $G(k,n)$ of rank k, and this induces Chern classes $c_l(E), 1 \leq l \leq k$, in Chow ring. Then this Chow ring is generated as ring by $\{c_l(E)\}$ with tautological relations

$$(1 + c_1(E) + \cdots + c_k(E))_l^{-1} = 0, l > n-k.$$

For $\mathcal{M}_g$ he considered the tautological subring of Chow or cohomology ring generated by tautological classes κ_i's, λ_l's; any geometric calculation can be translated to the Chow ring will require only knowledge of the tautological subring and this subring is much smaller than the Chow or cohomology ring.

Theorem 2 *(Mumford) The tautological subring $R^*(\mathcal{M}_g)$ is generated by the $g-2$ classes $\kappa_1, \cdots, \kappa_{g-2}$.*

The objects in this paper are the flat families of nodal curves

$$\begin{array}{c} X \\ \pi \big\downarrow \\ B, \end{array}$$

where $\pi^{-1}(b)$ is an nodal curve. By nodal curve, we mean a curve that has only nodes as singularities. We want to take B itself projective, which means one must allow some singular fiber. One example is $B = \overline{\mathcal{M}_g}$, the moduli space of *Deline-Mumford stable curves*, a nodal curve with only finitely many automorphisms. Note that by *semistable reduction*, any family can be modified so as to have node singularity without changing the general fiber. For enumerative geometry, however, we may loose some characters of the family.

Many questions in the classical projective and enumerative geometry of this family ([Mum], [Kon]) can be phrased in the context of the relative Hilbert scheme $X_B^{[m]} = Hilb_m(X/B)$. This is a universal parameter space for length-m subschemes of X contained in fibers of π, and carries the natural tautological vector bundles $\Lambda_m(E)$, associated to any vector bundle E on X. Enumerative questions for a family of curves contain relative multiple points and multisecants formulas whose solutions involves Chern numbers of the tautological bundles. Thus, turning these formal solution into meaningful ones requires computing the Chern numbers in question.

For the enumerative geometry of Hilbert schemes one uses the induction procedure that allows one to compare the geometric properties of $X_B^{[m]}$ and $X_B^{[m-1]}$ by flag schemes, i.e. schemes parametrizing flags of subschemes. For a family of nodal curves we also consider flag relative Hilbert scheme. In this paper we will verify the classical de Jonquieres' formula of low degree for a single smooth curve and get some de Jonquiere's formula for a family of nodal curves.

Chapter 2

Tautological module

This chapter contains the results that are relevant to our work. The most theoretical results in this chapter are in [R5].

2.1 Blowup theorem

For a flat family of curve, we consider the relative Hilbert schemes of points of X contained in fibers of π. Hence we have a constant *Hilbert polynomial*, say $P = m$. It is well-known that the absolute Hilbert scheme of a smooth algebraic curve X is isomorphic to the symmetric product. It has been proved that the variety of divisors of degree n and the n-fold symmetric product are isomorphic[S]. For the symmetric product or more generally quotient varieties we refer to [H2]. For the isomorphism of symmetric product and Hilbert scheme of a smooth algebraic curve, we may associate a point $[Z] \in X^{[m]}$ a formal sum $\sum_{x \in X} length(\mathcal{O}_{Z,x}) \cdot x$ a point in symmetric product $X^{(m)}$. Note that since Z is an 0-dimensional subscheme $H^0(Z, \mathcal{O}_Z)$ is an Artinian $\mathbb{C}$-algebra and the $length(\mathcal{O}_{Z,x}) = length(H^0(Z, \mathcal{O}_Z) = dim_{\mathbb{C}} H^0(\mathcal{O}_Z)$. This defines the

Hilbert-Chow morphism

$$\rho : X^{[m]} \to X^{(m)},$$

at least set-theoretically. ρ is indeed a morphism[Leh].

Theorem 3 *The Hilbert-Chow morphism for a smooth algebraic curve X*

$$c_m : X^{[m]} \to X^{(m)}, Z \mapsto \sum_{p \in supp(Z)} length(\mathcal{O}_{Z,p})[p]$$

is an isomorphism.

Proof. As the local ring of X at a point p is a discrete valuation ring, all ideals in $\mathcal{O}_{X,p}$ are powers of the maximal ideal m_p. Thus for all $[Z] \in X^{[m]}$ we have

$$\mathcal{O}_Z = \otimes_i \mathcal{O}_{X,p_i}/m_{p_i}^{m_i}, \sum_i m_i = m.$$

Then c_m sends Z to $\sum_i m_i[p_i]$, hence c_m is bijective. As it is also birational, it is an isomorphism by Zariski's main theorem. ■

For a smooth surface, Forgarty[Fo] showed that the Hilbert scheme is the resolution of singularities of symmetric product.

Remark 4 *Note that the main difference of Hilbert schemes and Chow varieties is that the Hilbert scheme has a natural scheme structure whereas the Chow variety does not[Kol].*

For the family of nodal curves

$$\begin{array}{c} X \\ \pi \big\downarrow \\ B, \end{array}$$

we have the

Theorem 5 *(Blowup Theorem)[R4] The cycle map*

$$c_m : X_B^{[m]} \to X_B^{(m)}$$

is equivalent to the blowing up of the big diagonal $D^m \subset X_B^{(m)}$.

Proof. (sketch) The theorem is the statement that the natural birational correspondence between $X_B^{[m]}$ and $Bl_{D^m}(X^{(m)})$ projects isomorphically both ways. By GAGA, it suffices to prove for the corresponding anayltic spaces. Then the statement is local over $X_B^{(m)}$ and by splitting argument we may reduced the theorem to the case where X/B is the standard family $xy = t$. We let U denote any neighborhood of the origin in X. Then the relative cartesian product U_B^m as a subscheme of $U^m \times B$ is given locally by

$$x_1 y_1 = \cdots = x_m y_m = t.$$

Letting σ_i^x, σ_i^y be the elementary symmetric functions in $x_1, \cdots, x_m$ and $y_1, \cdots, y_m$, respectively, where $\sigma_0 = 1$, we have an embedding near mp

$$\sigma : U_B^{(m)} \to \mathbb{A}^{2m} \times B,$$

where p is the node.

Since the fiber $c_m^{-1}(mp)$ is the union of C_i^m, $i = 1, \cdots, m-1$, it is reasonable to try to model the cycle map on the 1-parameter of curves specializing to a chain of $m-1$ lines. Let $C_1, \cdots, C_{m-1}$ be copies of $\mathbb{P}^1$ with homogeneous coordinates u_i, v_i on the i-th copy. Let

$$\widetilde{C} \subset C_1 \times \cdots \times C_{m-1} \times B/B$$

be the subscheme over B defined by

$$v_1 u_2 = t u_1 v_2, \cdots, v_{m-2} u_{m-1} = t u_{m-2} v_{m-1}.$$

Note that $\widetilde{C}$ is smooth and specializes to its unique singular fibre $\widetilde{C}_0$, the union of $m-1$-copies of $\mathbb{P}^1$. To construct our model $\widetilde{H}$, define $\widetilde{H} \subset \widetilde{C} \times \mathbb{A}^{2m}$ be the subscheme defined by

$$a_0 u_1 = t v_1, d_0 v_{m-1} = t u_{m-1}$$

$$a_1 u_1 = d_{m-1} v_1, \cdots, a_{m-1} u_{m-1} = d_1 v_{m-1}.$$

Then we have an isomorphism

$$\Phi : \widetilde{H} \to U_B^{[m]}.$$

Now we need to show that $c_m^{-1}(D^m) = 2\Gamma^m$ is Cartier. For this, consider ordered Hilbert scheme

$$\begin{array}{ccc} X_B^{\lceil m \rceil} & \xrightarrow{\varpi_m} & X_B^{[m]} \\ \downarrow{\scriptstyle oc_m} & & \downarrow{\scriptstyle c_m} \\ X_B^m & \xrightarrow{\omega_m} & X_B^{(m)}, \end{array}$$

where $X_B^{\lceil m \rceil} = X_B^{[m]} \times_{X_B^{(m)}} X_B^m$. Now it suffices to show that $\omega_m^*(D^m) = 2OD^m$, where $OD^m = \sum_{i<j} p_{i,j}^{-1}(OD^2)$, is Cartier. Indeed if this is the case then the natural map $X_B^{\lceil m \rceil} \to Bl_{2OD^m} X_B^m$ is an isomorphism. Then so is the S_m-equivariant map

$$f : X_B^{\lceil m \rceil} \to (Bl_{D^m} X_B^{(m)}) \times_{X_B^{(m)}} X_B^m,$$

which is just the pullback of the natural map

$$c'_m : X_B^{[m]} \to Bl_{D^m} X_B^{(m)}$$

by the finite flat surjective map ϖ_m, therefore so is c'_m. That $\omega_m^*(D^m) = 2OD^m$ is Cartier follows from the following lemma. ■

Lemma 6 (R4) *G_i generates $\mathcal{O}(-O\Gamma^{(m)})$ over $\widetilde{U}_i$, where*

$$G_i = \pm det(V_i^m).$$

In particular, $O\Gamma^{(m)}$ is Cartier.

This allow us to study the relative Hilbert schemes of family of nodal curves. In the proof we considered the geometry of special fiber of c_m over the maximal singular point $mp \in X(m)$ which is the union of $m-1$ copies of $\mathbb{P}^1$ as in the

Theorem 7 (R3) *The punctual Hilbert scheme of the analytic neighborhood of a node is a union of $m-1$ copies of $\mathbb{P}^1$*

$$C_1^m \cup_{Q_2^m} C_2^m \cup \cdots \cup_{Q_{m-1}^m} C_{m-1}^m$$

normally crossing at Q_i^m and smooth elsewhere.

For the case of the affine line and small m, we refer to [Leh].

2.2 Tautological module

In this section we define the *tautological module* and consider the module structure on the small diagonal. By $T^m(X/B)$ we mean, as a group, generated by the followings:

1. the diagonal loci $\Gamma_\mu^{(m)}$, where $\mu = (n_1, \cdots, n_k)$ any partition of m: this locus is the closure of the set of schemes of the form $n_1p_1 + \cdots + n_kp_k$, where p_i are distinct smooth points of the same fiber. More generally, we will consider twisted classes $\Gamma_\mu^{(m)}[\alpha.]$, where $\alpha.$ are the base classes, i.e. $\alpha. \in H^*(sym^m(X))$.

2. the node classes. First, the node scrolls $F_j^{n,m}(\theta)$: $\mathbb{P}^1$-bundles over a diagonal locus of the boundary family of curves. Moreover the node sections $-\Gamma^{(m)}F_j^{n,m}(\theta)$. Similarly as above these can be considered as operators on $H^*(sym^m(X))$.

Notation: $\Gamma^{(m)} := \frac{1}{2}c_m^{-1}(D^{(m)})$, the *discriminant polarization*, where $D^{(m)}$ is the big diagonal on the relative symmetric product $X_B^{(m)}$.
By the module theorem below this is a $\mathbb{Q}[\Gamma^{(m)}]$-module.

Remark 8 *Node classes*

1. *The node scroll $F_j^{n,m}(\theta)$ is the closure of the set of schemes of the form $n\theta + D$, where θ is a node in a fiber and in the same fiber D is in the diagonal class $\Gamma_\nu^{(m-n)} \subset (X_T^\theta)_T^{[m-n]}$, where ν is a partition of $m-n$, T is a boundary component*

of B with a diagram

$$\begin{array}{ccc} & & X \\ & \overset{\theta}{\nearrow} & \downarrow \pi \\ T & \overset{\delta}{\hookrightarrow} & B, \end{array}$$

and X_T^θ is the blowup of the relative node in X. That is, $(X_T^\theta)_T^{[m-n]}$ is $(m-n)$-th relative Hilbert scheme of the family of nodal curves X_T^θ and $\Gamma_\nu^{(m-n)}$ is a diagonal class of this relative Hilbert scheme. Note that the fibers of this family have 1 less genus of that of the family X/B.

2. *Since the fiber is union of $\mathbb{P}^1$ by the Theorem 7, we see that $F_j^{n,m}(\theta)$ is $\mathbb{P}^1$-bundle over $(X_T^\theta)_T^{[m-n]}$.*

3. *Now we have node section $-\Gamma^{(m)} F_j^{n,m}(\theta)$ of this $\mathbb{P}^1$-bundle.*

In this section we consider the small diagonal locus and this is the heart of the matter, indeed, the intersection of any diagonal loci with $\Gamma^{(m)}$ is determined by reduction to the small diagonal locus. Let $\Gamma_{(m)} \subset X_B^{[m]}$ be the small diagonal, i.e. the closure of the subschemes of mp in a fiber or equivalently the pullback of the small diagonal

$$D_{(m)} \simeq X \subset X_B^{(m)}.$$

The restriction of the cycle map is a birational morphism

$$c_m : \Gamma_{(m)} \to X$$

which is an isomorphism except over the nodes of X/B. Recall that for the family of nodal curves X/B we have the relative dualizing sheaf of the family $\omega_{X/B}$; if X is smooth, e.g. versal family, then $\omega_{X/B} = K_X \otimes \pi^* K_B^\vee$([HM]). Note that for a family of curves, the dualizing sheaf ω exists and if $\omega^2 = 0$, then X/B is trivial family. As a corollary of the blowup theorem above we have the

Proposition 9 *$c_m : \Gamma_{(m)} \to X$ is equivalent to the blowup of $J^{\theta.}_m$, where $J^{\theta.}_m$ is the ideal of nodes. If $\mathcal{O}_{\Gamma_{(m)}}(1)_J$ denotes the canonical blowup polarization, we have*

$$\mathcal{O}_{\Gamma_{(m)}}(-\Gamma^{(m)}) = \omega_{X/B}^{\otimes\binom{m}{2}} \otimes \mathcal{O}_{\Gamma_{(m)}}(1)_J.$$

Proof. We may work with the ordered Hilbert scheme $X_B^{\lceil m \rceil}$, then pass to S_m-invariants. Note that $X_B^{\lceil m \rceil}$ is the blowup of $OD^m := \sum_{i<j} D_{i,j}$, where $D_{i,j}$ is the pullback of the diagonal from the i, j factors([R4]). Because blowup and Hilbert scheme are both compatible with base-change, we may then assume X/B is given by $xy = t$. Then the ideal of OD^m is generated by $G_1, \cdots, G_m$, so restrict this on the small diagonal $OD_{(m)} \simeq X$. To this end, consider the natural map

$$\mathcal{I}_{OD^m} \to \omega^{\binom{m}{2}}, \omega := \omega_{X/B}.$$

Note that since $OD^m := \sum_{i<j} D_{i,j}$, this is well-defined. To identify the image, note that

$$(x_i - x_j)|_{OD_{(m)}} = dx = x\frac{dx}{x}$$

and $\eta = \frac{dx}{x} = -\frac{dy}{y}$ is a local generator of ω along θ. Therefore

$$G_1|_{OD_{(m)}} = x^{\binom{m}{2}}\eta^{\binom{m}{2}}.$$

By the formula of G_i we have

$$G_i|_{OD_{(m)}} = x^{\binom{m-i+1}{2}} y^{\binom{i}{2}} \eta^{\binom{m}{2}}, i = 1, \cdots, m.$$

Now over a neighborhood of θ, we have

$$I_{OD^m}.OD_{(m)} \simeq J^{\theta}_m \otimes \omega^{\binom{m}{2}}.$$

This being true for each node, it is also true globally and by passing to S_m-quotients, we also have

$$I_{D^m}.D_{(m)} \simeq J^{\theta}_m \otimes \omega^{\binom{m}{2}}.$$

Pulling back to $X_B^{[m]}$ we get the proposition. ■

Proposition 10 *(i) The pullback ideal of* J^θ_m *on* $\Gamma_{(m)}$ *defines a Cartier divisor of the form*

$$e^\theta_m = \sum_{i=1}^{m-1} \frac{i(m-i)m}{2} C^m_i(\theta).$$

(ii) Each C^m_i *is a* $\mathbb{Q}$*-Cartier divisor on* $\Gamma_{(m)}$*;* mC^m_i *is Cartier.*

Now we have the intersection of small diagonal with $\Gamma^{(m)}$.

Proposition 11

$$\Gamma^{(m)}.\Gamma_{(m)} = \sum_{\theta,i=1}^{m-1} \frac{i(m-i)m}{2} C^m_i(\theta) - \binom{m}{2}\omega.$$

Proof. Consider the exact sequence

$$0 \to \mathcal{O}_{\Gamma_{(m)}}(-\Gamma^{(m)}) \to \mathcal{O}_{\Gamma_{(m)}} \to \mathcal{O}_{\Gamma_{(m)}}|_{\Gamma^{(m)}} \to 0.$$

Now it follows from the exact sequence and Propositions. ∎

2.3 Polyblocks

For the ordered relative Hilbert scheme $X^{\lceil m \rceil}_B$, we have the

Proposition 12 *We have an equality of divisor classes on* Γ_I*:*

$$\Gamma^{\lceil m \rceil}.\Gamma_I = \sum_{i<j\notin I} \Gamma_{I|\{i,j\}} + |I| \sum_{i \notin I} \Gamma_{I\cup\{i\}} - \binom{|I|}{2} p^*_{min(I)}\omega + \sum_s \frac{1}{deg(\delta_s)} \sum_{j=1}^{|I|-1} \nu_{|I|,j}\delta^I_{s,j^*} OF^I_j(\theta_s),$$

where $I|\{i,j\}$ *and* $I \cup \{i\}$ *denote uniting blocks.*

Proof. Since the asserted equality trivially holds away from the exceptional locus(the locus at where the map is not isomorphism) of oc_m, the first, second and third summands come from components $\Gamma_{i,j}$ of $\Gamma^{\lceil m \rceil}$ having $|I \cap \{i,j\}| = 0, 1, 2$, respectively. Next, both sides being divisors on Γ_I, it will suffice to check equality away from codimension 2, e.g. over a generic point of each boundary locus $(X^\theta_T)^{K-I|K^c-I}$. But there, our cycle map oc_m is locally just $oc_r \times iso$, $r = |I|$, with

$$\Gamma^{\lceil m\rceil} \sim \Gamma^{\lceil r\rceil} + \sum_{\{i,j\}\not\subseteq I} \Gamma_{i,j}.$$

We are then reduced to the case of the small diagonal. ■

Passing to (unordered) relative Hilbert scheme $X_B^{[m]}$ by S_m-quotient we have

Proposition 13 *For a partition μ of weight m, we have an equality of operators of $Hom(TS_\mu(R), A.(X_B^{[m]}))$:*

$$\Gamma^{(m)}.\Gamma_\mu[] = \Gamma_\mu \circ (Dsc^{(m)} - U_\omega) + \sum_\theta \sum_{\mu(n)>0} \sum_{j=1}^{n-1} \frac{j(n-j)n}{2} F^{n,m}_{j,\mu-1_n}(\theta)[] \circ u_{n,\theta^*_\mu}.$$

Notations: 1. Dsc_μ is an operator on base cohomology classes; for a base class $\alpha. = (\alpha_1, \cdots, \alpha_{wt(\mu)}) \in Sym^{wt(\mu)}(H^\cdot(X))$, $Dsc_\mu(\alpha.) = \sum_{n_1 \geq n_1} n_1 n_2 u_{n_1,n_2,\mu}$, where u is uniting operator uniting n_1 and n_2 blocks. In particular, $Dsc^{(m)} := Dsc_{(1,1,\cdots,1)}$.

2. $U_{\omega,\mu}(\alpha.) := \sum_n \binom{n}{2} u_{n,\omega,\mu}(\alpha.)$, where $u_{n,\omega,\mu}(\alpha.)$ is an operator multiplying ω on n block.

Now we know that for any diagonal classes and node classes $\Gamma_\mu, F_j^{n,m}(\theta) \in T^m(X/B)$, the intersections with discriminant $\Gamma^{(m)}$ lie in $T^m(X/B)$. To finish the Module theorem, we need to show that this is true for node sections $-\Gamma^{(m)}F_j^{n,m}(\theta)$ which follows from the

Theorem 14 (R5) *For any twisted node scroll class $F_j^{n,m}(\theta)[\beta]$, we have*

$$\begin{aligned}&(-\Gamma^{(m)})^l F_j^{n,m}(\theta)[\beta] \\ &= (-\Gamma^{(m)}) F_j^{n,m}(\theta)[s_{l-1}(e_j^{n,m}, e_{j+1}^{n,m})\beta] - F_j^{n,m}(\theta)[e_j^{n,m} e_{j+1}^{n,m} s_{l-2}(e_j^{n,m}, e_{j+1}^{n,m})\beta].\end{aligned}$$

Now we have the

Theorem 15 *(Module Theorem)[R5] Compatibly with intersection product, $T^m(X/B)$ is a $\mathbb{Q}[\Gamma^{(m)}]$-module.*

Proof. For any twisted diagonal class $\Gamma_\mu[\alpha.]$, by Proposition 13, $\Gamma^{(m)}.\Gamma_\mu[\alpha.]$ can be written by generators. For the node classes, it is clear. Finally Theorem 14 implies that this is true for the node sections. ■

2.4 Transfer theorem and Splitting principles

For the enumerative geometry of Hilbert schemes one uses the induction procedure that allows one to compare the geometric properties of $X_B^{[m]}$ and $X_B^{[m-1]}$ by flag schemes, i.e. schemes parametrizing flags. For a family of nodal curves we also consider flag relative Hilbert scheme. Let

$$X_B^{[m,m-1]} \subset X_B^{[m]} \times_B X_B^{[m-1]}$$

denote the flag Hilbert scheme, parametrizing pairs of schemes (z_1, z_2) satisfying $z_1 \supset z_2$ and z_1 lies in some fiber. This comes equipped with a (flag) cycle map

$$c_{m,m-1} : X_B^{[m,m-1]} \to X_B^{(m,m-1)},$$

where $X_B^{(m,m-1)} \subset X_B^{(m)} \times_B X_B^{(m-1)}$ is the subvariety parametrizing cycle pairs $(c_m \geq c_{m-1})$. Note that this is a blowup of the sheaf of ideals $\mathcal{I}_{D^{m-1}}.\mathcal{I}_{D^m}$ on $X_B^{(m,m-1)}$([R4]). By the construction of the flag Hilbert scheme, we have natural projections and annhilator map a

$$\begin{array}{ccccc} & & X_B^{[m,m-1]} \xrightarrow{a} X & & \\ & \swarrow^{p_m} & & \searrow^{p_{m-1}} & \\ X_B^{[m]} & & & & X_B^{[m-1]} \end{array} \quad , \text{ where } a(z_1, z_2) = ann(z_1/z_2),$$

identifying X with the Hilbert scheme of colength-1 ideals.

Now for the enumerative geometry we consider a transfer from $X_B^{[m-1]}$ to $X_B^{[m]}$ allowing twisting by base classes. Indeed $\Gamma^{(m)}\Gamma_{(m)} = -\binom{m}{2}\Gamma_{(m)}[\omega] + \sum_{\theta,i=1}^{m-1} \frac{i(m-i)m}{2} C_i^m(\theta)$, i.e.

we have to allow twists. Precisely define the twisted transfer map τ_m by

$$\tau_m = p_{m*}(p_{m-1}^* \otimes a^*) : A.(X_B^{[m-1]}) \otimes A.(X) \to A.(X_B^{[m]})_{\mathbb{Q}}.$$

For the definition of τ_m: by *exterior product* of Chow groups we have $A.(X_B^{[m-1]}) \otimes A.(X) \to A.(X_B^{[m-1]} \times_B X)$ and then from the flat morphism $p_{m-1} \times a : X_B^{[m,m-1]} \to X_B^{[m-1]} \times_B X$ we have pullback from $A.(X_B^{[m-1]} \times_B X) \to A.(X_B^{[m,m-1]})$. Finally by projection(proper) p_m we have the push-forward to $A.(X_B^{[m]})$.

Note that for a smooth algebraic curve X,

$$A.(X^{(m-1)}) \otimes A.(X) \xrightarrow{\times} A.(X^{(m)})$$

is an isomorphism.

Remark 16 *Recall the exterior product([Fu])*

For algebraic schemes X, Y over B we have the fiber product $X \times_B Y$. Hence we have the exterior product

$$Z_k(X) \otimes Z_l(Y) \xrightarrow{\times} Z_{k+l}(X \times_B Y)$$

by $[V] \times [W] \mapsto [V \times_B W]$. Since this map preserves the rational equivalence, we have the following exterior product

$$A_k(X) \otimes A_l(Y) \xrightarrow{\times} A_{k+l}(X \times_B Y).$$

Properties:

1. *the exterior product is associative:*

$$(\alpha \times \beta) \times \gamma = \alpha \times (\beta \times \gamma) \text{ for } \alpha \in A_*X, \beta \in A_*Y, \gamma \in A_*Z.$$

2. *If $Y = \mathbb{A}^n$, then*

$$A.(X) \otimes A.(Y) \xrightarrow{\times} A.(X \times_B Y)$$

is an isomorphism for any X.

Theorem 17 (R4)

$$c_{m-1,1} : X_B^{[m,m-1]} \to X_B^{[m-1]} \times_B X$$

defined by $p_{m-1} \times a$ *is the blowup of the incidence variety* $D^{(m-1,1)} = \{(z,x) : x \in z\}$.

Proposition 18 (R5) *(i) The projection* p_{m-1} *is flat, with* 1*-dimensional fibers;*

(ii) Let $z \in X_B^{[m-1]}$ *be a subscheme of a fiber* X_s*, and let* z_0 *be the part of* z *supported on nodes of* X_s*, if any. Then if* z_0 *is principal(i.e. Cartier) on* X_s*, the fiber* $p_{m-1}^{-1}(z)$ *is birational to* X_s *and its general members are equal to* z_0 *locally at the nodes.*

Now we have the

Theorem 19 *(Tautological Transfer)* τ_m *takes tautological classes on* $X_B^{[m-1]}$ *to tautological classes on* $X_B^{[m]}$*. More specifically we have, for any class* $\beta \in A.(X)$*:*

1. $$\tau_m(\Gamma_\mu[\alpha.]\beta_{(m)}) = \Gamma_{\mu+1_1}[\alpha.\beta],$$

   where 1_1 *is the partition of weight* 1 *and support* $\{1\}$.

2. *for* $F_j^{n,m-1}(\theta)[\alpha.], \alpha. \in T^{m-n-1}(X_T^\theta)$,

   $$\tau_m(F_j^{n,m-1}[\alpha.]\beta_{(m)}) = F_j^{n,m}(\theta)[\tau_{m-n,X_T^\theta/T}(\alpha. \otimes (\beta|_{X_T^\theta}))].$$

3. $$\begin{aligned}&\tau_m(-\Gamma^{(m-1)}F_j^{n,m-1}[\alpha.]\beta_{(m)}) = \\ &\theta^*(\beta)F_j^{n+1,m}(\theta)[\alpha.] + (-\Gamma^{(m)})F_j^{n,m}(\theta)[\tau_{m-n,X_T^\theta/T}(\alpha.\beta|_{X_T^\theta})] \\ &- F_j^{n,m}(\theta)[e_{j+1}^{n,m}(\theta)(\tau_{m-n,X_T^\theta/T}(\alpha.\beta|_{X_T^\theta}))] \\ &+ F_j^{n,m}(\theta)[\tau_{m-n,X_T^\theta/T}(e_{j+1}^{n,m-1}(\theta)(\alpha.).\beta|_{X_T^\theta}))].\end{aligned}$$

Proof. 1 is obvious. The flatness of p_{m-1} allows us to work over general $z \in F$ and by Proposition 18 we may assume that the added point is a general point on the fibre X_s. For 3, by [R4]

$$-\Gamma^{(m-1)} \sim Q_j^{n,m-1} + e_{j+1}^{n,m-1}.$$

Hence suffice to prove that

$$\tau_m(Q_{j+1}^{n,m-1}[\alpha.]\beta_{(m)}) = \theta^*(\beta)F_{j+1}^{n+1,m}(\theta)[\alpha.] + Q_{j+1}^{n,m}[\alpha.\beta].$$

To this end, note that, with $Q = Q_{j+1}^{n,m-1}$, p_{m-1}^*Q splits in two parts, depending on whether the point w added to a scheme $z \in Q$ is in the off-node or nodebound portion of z. The first part gives rise to the 2nd term in the RHS of the equality.

For the other case we may assume that $m = n+1$, i.e. F is just C_j^{m-1}, a $\mathbb{P}^1$. For this case we refer to [R5]. ■

Now the last tool for our enumerative geometry is the *Splitting principle.* Let

$$W^m(X/B) \xrightarrow{\pi^{(m)}} B$$

denote the relative flag Hilbert scheme of X/B, parametrizing flags of subschemes

$$z. = (z_1 < \cdots < z_m)$$

where z_i has length i and z_m is contained in some fiber of X/B. Let

$$a_i : W^m \to X$$

be the canonical map sending a flag $z.$ to the 1-point support of z_i/z_{i-1}. Let

$$\mathcal{I}_m < \mathcal{O}_{X_B^{[m]}\times_B X}$$

be the universal ideal of colength m. For any vector bundle E on X, set

$$\Lambda_m(E) = p_{X_B^{[m]}*}(p_X^*(E) \otimes (\mathcal{O}_{X_B^{[m]}\times_B X}/\mathcal{I}_m));$$

this is called the *tautological bundle* on $X_B^{[m]}$ of rank $m + rkE$ and more generally this is defined for each coherent sheaf on X. This is a secant bundle which was first introduced by [S] and for a smooth algebraic curve this is just *symmetrization* $\mathcal{E}_m(E)$ in [Mat] in which the total Chern class of it was computed. Note that at $z \in X_B^{[m]}$, $\Lambda_m(E)|_z = H^0(z, E \otimes \mathcal{O}_z)$. Set

$$\Delta^{(m)} = \Gamma^{(m)} - \Gamma^{(m-1)}.$$

The various tautological sheaves form a flag of quotients on W^m:

$$\cdots \twoheadrightarrow \Lambda_{m,i}(E) \twoheadrightarrow \Lambda_{m,i-1}(E) \twoheadrightarrow \cdots .$$

This gives the

Theorem 20 *(Splitting principle)[R2] On $W^m(X/B)$*

$$c(\Lambda_m(E)) = \prod_{i=1}^{m} c(a_i^*(E)(-\Delta^{(i)})).$$

Moreover, in $A.(X_B^{[m,m-1]})_{\mathbb{Q}}$

$$c(\Lambda_m(E)) = c(\Lambda_{m-1}(E))c(a_m^*(E)(-\Delta^{(m)})).$$

Proof. For the first we refer to [R2]. For the second they both pull back to the same class in W^m. As the projection $W^m \to X_B^{[m,m-1]}$ is generically finite, they agree mod torsion. ■

The following theorem makes us to compute the Chern numbers using Tautological module and transfer theorem.

Theorem 21 (R5) *There is a computable inclusion*

$$TC_R^m \to T_R^m,$$

where TC_R^m is the R-subalgebra of $A.(X_B^{[m]})_{\mathbb{Q}}$ generated by the Chern classes of $\Lambda_m(E)$ and the discriminant class $\Gamma^{(m)}$.

2.5 Enumerative geometry of a family of nodal curves

We restrict to the small diagonal and the 1-parameter families of nodal curves. In this case the (punctual) transfer $\tau_m^0 : T_R^{m-1,0}(X/B) \to T_R^{m,0}(X/B)$ that fits in the diagram

$$\begin{array}{ccc} T_R^{m-1,0}(X/B) & \longrightarrow & T_R^{m,0}(X/B) \\ \downarrow & & \downarrow \\ A^{\cdot}(\Gamma_{(m-1)}) & \longrightarrow & A.(\Gamma_{(m)}) \end{array}$$

is given by the

Proposition 22 *For each node θ,*

$$\tau_m(C_i^{m-1}(\theta)) = \frac{m-i}{m-1} C_i^m(\theta) + \frac{i+1}{m-1} C_{i+1}^m(\theta).$$

$$\begin{aligned} \tau_m(-\Gamma^{(m-1)} C_i^{m-1}(\theta)) = & -\Gamma^{(m)} C_{i+1}^m(\theta) - C_{i+1}^m(\theta)[\frac{m-i-1}{m}\psi_{i+2}^m + \frac{i+1}{m}\psi_{i+1}^m] \\ & + \frac{m-i}{m-1} C_i^m(\theta)[\psi_i^{m-1}] + \frac{i+1}{m-1} C_{i+1}^m(\theta)[\psi_i^{m-1}]. \end{aligned}$$

Convention: For two line bundle L and M, let LM denote the degree of $c_1(L)\cdot c_1(M) \in H^4(X,\mathbb{Z})$.

Example 23 *Given a family X/B and a map*

$$f : X \to \mathbb{P}^n, n < m,$$

$c_{m-n}(\Lambda_m(L)|_{\Gamma_{(m)}})$*, where $L = f^*(\mathcal{O}(1))$, represents the locus of points in X where the fiber admits an m-*contact hyperplane*, e.g. if $n = 1$, this is the locus of $(m-1)$-st order ramification points. Note that if dim$B = m-n-1$, we have Chern number. For example,*

$$c_2(\Lambda_m(L)|_{\Gamma_{(m)}}) = \binom{m}{2} L^2 + (3\binom{m+1}{4} - \binom{m}{3})\omega^2 + (3\binom{m+1}{3} - 2\binom{m}{2})L\omega - \binom{m+1}{4}\sigma,$$

where σ is the number of nodes.

Chapter 3

Classical de Jonquieres' formula

3.1 Degeneracy Loci and Porteous' formula

Porteous formula expresses the class of the locus where the rank of a map between vector bundles is less than or equal to a given bound. One of the applications of this formula is the first proof of the existence of special linear series on an arbitrary curve whenever the *Brill-Noether number* $\rho \geq 0$.

Let $\sigma : E \to F$ be a homomorphism of vector bundles of ranks e and f on an n-dimensional variety X. For $k \leq min(e, f)$, set

$$D_k(\sigma) = \{x \in X | rank(\sigma(x)) \leq k\}.$$

On an affine open set U where E and F are trivial, σ is defined by a matrix of elements in the coordinate ring of U, which generate the ideal of $Z(\sigma)$ on U. More generally, for a non-negative integer $k \leq min(e, f)$, we have the *k-th degeneracy locus*

$$D_k(\sigma) = \{x \in X | rank(\sigma(x)) \leq k\} = Z(\bigwedge^{k+1}(\sigma)).$$

Hence this degeneracy locus has a natural scheme structure on $D_k(\sigma)$, locally defined by the vanishing of $(k+1)$-minors of a matrix representation of σ. One expects $D_k(\sigma)$

to be m dimensional, where

$$m = n - (e-k)(f-k),$$

but in general one can only state that each irreducible component of $D_k(\sigma)$ has dimension at least m.

For any formal series $c_t = \sum_i c_i t^i$, any integer a and any positive integer b, we define $M_{a,b}(c_t)$ to be the $b \times b$ matrix whose (i,j)-th entry is c_{a+j-i}. Finally, we set $\Delta_{a,b}(c_t) = det(M_{a,b}(c_t))$. In these terms, Porteous formula is the

Theorem 24 *(Porteous formula) Let $\sigma : E \to F$ be a homomorphism between vector bundles of respective ranks e and f on a smooth variety X. Let*

$$D_k(\sigma) = \{x \in X | rank(\sigma_x) \leq k\}$$

and let $[D_k(\sigma)] \in A_m(X)$ be the fundamental class of $D_k(\sigma)$. If $D_k(\sigma)$ is either empty, or of the expected codimension $(e-k)(f-k)$, then

$$[D_k(\sigma)] = \Delta_{e-k,f-k}((c_t(F-E))).$$

Example 25 *The locus D_0 is the zero locus of σ, considered as a section of $Hom(E,F)$, so that*

$$[D_0(\sigma)] = c_{ef}(E^* \otimes F);$$

and, in case $e = f$, we have that D_{e-1} is the zero locus of $\bigwedge^e \sigma$, so that

$$\begin{aligned}[D_{e-1}] =& c_1(\bigwedge^e E^* \otimes \bigwedge^e F) \\ =& c_1(F) - c_1(E).\end{aligned}$$

Remark 26 *We may describe the degeneracy loci zero sections by the Grassmann bundle of $(e-k)$-planes in the fibers of E with universal subbundle and quotient bundle.*

Proof. (sketch proof of Theorem 1) By the Riemann-Roch a divisor D of degree d moves in a linear series of dimension at least r if and only if the rank of the evaluation map

$$H^0(K_X) \to H^0(K_X/K_X(D))$$

is $d-r$ or less. As D varies, the target and domain spaces of this map give vector bundles over the symmetric product $X^{(d)}$, and applying Porteous' formula to the corresponding bundle map we arrive at a formula for the class of the locus in $X^{(d)}$ of divisors D such that $r(D) \geq r$. In particular, observing that this class is nonzero (when its codimension is $d-r$ or less) gives the first proof of the existence of special linear series on an arbitrary curve whenever the Brill-Noether number $\rho \geq 0$. ■

For more details, we refer to [Fu],[ACGH].

3.2 de Jonquieres' formula for a smooth curve

Given a linear system (L, V), where $L \in Pic^d(X)$ and $V \in G(r+1, H^0(L))$, the classical de Jonquieres' formula gives the number of divisors of degree n of the form $D = a_1D_1 + \cdots + a_kD_k$, where $degD_i = n_i$ and $\sum a_in_i = n$, contained in this system, provided this number is finite, i.e. $\sum n_i = n - r$ and that they intersect properly.

For a given g^r_d on a smooth curve X, i.e. a pair (L, V), where $L \in Pic^d(X)$ and $V \in G(r+1, H^0(L))$, the evaluation map $V \otimes \mathcal{O}_X \xrightarrow{eval_V} L$ induces a morphism of vector bundles $V \otimes \mathcal{O}_{X^{(d)}} \xrightarrow{\phi} \Lambda_d L$ on $X^{(d)}$. Since $X^{(d)}$ is a parameter space of effective divisors of degree d on X we may see a point $z \in X^{(d)}$ as an effective divisor on X. Over a point $z \in X^{(d)}$, this bundle morphism becomes a map $V \to H^0(z, L \otimes \mathcal{O}_z)$. Hence the r-th degeneracy locus of this bundle morphism is enumerating the effective divisors of degree d on X. Indeed the r-th degeneracy locus is $\{z \in X^{(d)}$: there is a section $s \in$

V s.t. $s|_z = 0\}$.

By Porteous' formula, the fundamental class of r-th degeneracy locus of this morphism is $\triangle_{d-r,1}(c_t(\Lambda_d(L) - V \otimes \mathcal{O}_{X^{(d)}})) = \triangle_{d-r,1}(c_t(\Lambda_d(L))) = c_{d-r}(\Lambda_d(L))$. So the de Jonquieres' formula is the formula $c_{d-r}(\Lambda_d(L)|_{\Gamma_{(a_1,\cdots,a_n)}})$ when $d - r = n$.

Remark 27 *Note that $r(D) \geq r$ if and only if there is a divisor in $|D|$ containing any r given points of the curve. Indeed consider the exact sequence, for any p,*

$$0 \to k(p) \to \mathcal{L}(D) \to \mathcal{L}(D - p) \to 0.$$

Remark 28 Pascal's identity*:* $\binom{z+1}{r} - \binom{z}{r} = \binom{z}{r-1}$.

Then by integrating we have $\sum_{k=r-1}^{z} \binom{k}{r-1} = \binom{z+1}{r}$.

Example 29 *For a single block $\Gamma_{(d)}$, by recursion we have*

$$c(\Lambda_d(L)|_{\Gamma_{(d)}}) = \prod_{i=1}^{d}(1 + L_i + (i-1)\omega) = 1 + dL + \binom{d}{2}\omega,$$

in particular, $c_1 = d(deg(L)) + \binom{d}{2}(2g-2) = d(dg - g + 1)$.

Indeed let $c_d = c(\Lambda_d(L)|_{\Gamma_{(d)}}) = \alpha_d$. *Then* $\alpha_d = \tau_d(\alpha_{d-1}(1 + L_d + \Gamma^{(d-1)}) + (-\Gamma^{(d)})\tau_d(\alpha_{d-1})$ $= \alpha_{d-1} + \alpha_{d-1}L - \alpha_{d-1}\binom{d-1}{2}\omega\Gamma_{(d)} + \alpha_{d-1}\binom{d}{2}\omega = \alpha_{d-1}(1 + L + (d-1)\omega)$, *hence* $\alpha_d = \prod_{i=1}^{d}(1 + L_i + (i-1)\omega)$.

Remark 30 *Recall the bundle of* principal parts*(or the* Jet bundle*)[Fu](2.5.6)*

Let C be a non-singular projective curve of genus g, and let $C(r) \subset C \times C$ be the subscheme defined by the ideal sheaf $\mathcal{I}^{r+1}$, where $\mathcal{I}$ is the ideal sheaf of the diagonal; let p and q be the first and second projections from $C(r)$ to C. For a line bundle L on C, the bundle of principal parts $P^r(L)$ is the sheaf on C defined by:

$$P^r(L) = p_*q^*L = p_*(q^*L \otimes \mathcal{O}_{C\times C}/\mathcal{I}^{r+1}).$$

Then $P^0(L) = L$*, and for* $r > 0$ *there is an exact sequence*

$$0 \to (\Omega^1_C)^{\otimes r} \otimes L \to P^r(L) \to P^{r-1}(L) \to 0.$$

Indeed on $C \times C$*, there is an exact sequence*

$$0 \to \mathcal{I}^r/\mathcal{I}^{r+1} \to \mathcal{O}_{C\times C}/\mathcal{I}^{r+1} \to \mathcal{O}_{C\times C}/\mathcal{I}^r \to 0.$$

Since q^*L *is locally free, we have*

$$0 \to \mathcal{I}^r/\mathcal{I}^{r+1} \otimes q^*L \to \mathcal{O}_{C\times C}/\mathcal{I}^{r+1} \otimes q^*L \to \mathcal{O}_{C\times C}/\mathcal{I}^r \otimes q^*L \to 0.$$

Since p *is homeomorphism on* $C(r)$*, we have*

$$0 \to \mathcal{I}^r/\mathcal{I}^{r+1} \otimes L \to P^r(L) \to P^{r-1}(L) \to 0.$$

Since C *is smooth,* Ω^1_C *is locally free and*

$$Sym^*_{\mathcal{O}_C}(\Omega^1_C) \to \bigoplus_{r=0}^{\infty} \mathcal{I}^r/\mathcal{I}^{r+1}$$

is an isomorphism, so $\mathcal{I}^r/\mathcal{I}^{r+1} \cong (\Omega^1_C)^{\otimes r}$*. Therefore* $P^r(L)$ *is locally free of rank* $r+1$*, and we have*

$$0 \to \bigwedge^{r} P^{r-1}(L) \otimes (\Omega^1_C)^{\otimes r} \otimes L \to \bigwedge^{r+1} P^r(L) \to \bigwedge^{r+1} P^{r-1}(L) \to 0,$$

where $\bigwedge^{r+1} P^{r-1}(L) = 0$*. Hence*

$$\begin{aligned} c_1(P^r(L)) =& c_1(\bigwedge^{r+1} P^r(L)) \\ =& c_1(\bigwedge^{r} P^{r-1}(L) \otimes (\Omega^1_C)^{\otimes r} \otimes L) \\ =& c_1(\bigwedge^{r} P^{r-1}(L)) + rc_1(\Omega^1_C) + c_1(L). \end{aligned}$$

Integrating this, we have

$$c_1(\bigwedge^{r+1} P^r(L)) = (r+1)c_1(L) + \binom{r+1}{2} c_1(\Omega^1_C).$$

If $V \subset H^0(C,L)$ is a subspace, there are canonical homomorphisms of vector bundles on C,

$$\sigma : C \times V \to P^r(L).$$

If $dimV = r+1$, i.e., the linear system determined by V has dimension r, then $det(\sigma)$ is a section of $\bigwedge^{r+1} P^r(L)$, well-defined up to scalars. If $det(\sigma) \neq 0$, its divisor of zeros, denoted δ_V, measures the osculation of the linear system. Then

$$deg(\delta_V) = (r+1)deg(L) + \binom{r+1}{2}(2g-2).$$

Remark 31 Plücker formulas *are the formulas relating the ramification indices of a linear system and the degrees of the associated maps, e.g. for g^1_d on a curve X Plücker formula is just the* Riemann-Hurwitz formula.

We can derive the Plücker formulas by the lemma and example 29.

Lemma 32

$$P^r(L) \cong \Lambda_{r+1}(L)|_{\Gamma_{(m)}}.$$

Remark 33 *By definition* Weierstrass points *are the points $p \in C$ for which gp is a special divisor. Hence the locus of Weierstrass points is the degeneracy locus of*

$$H^0(C,K) \otimes \mathcal{O}_{C^{(2g-2)}} \to \Lambda_{2g-2}(K).$$

So if $L = K, V = H^0(K)$, then $c_1 = (g+1)g(g-1)$ is the number of Weierstrass points on a curve genus g.

Remark 34 *Moreover $c_1(\Lambda_{r+1}L|_{\Gamma_{(r+1)}}) = (r+1)deg(L) + \binom{r+1}{2}(2g-2)$ is the* Brill-Segre formula*[Lak],[EH] enumerating the* strictly $(r+1)$-tuple points *of the complete linear system $H^0(L)$ over the ground field k, where $char(k) = p$ with $p = 0$ or $p > d$.*

In the rest of this section we verify the classical de Jonquieres' fromula for low degrees with technics in Ch2.

Theorem 35 *(de Jonquieres' formula[ACGH],[Mac],[V]) Let $a_1, \cdots, a_k, n_1, \cdots, n_k, d$ be positive integers. Let r be a non-negative integer. Suppose the $a_i's$ are distinct, $\Sigma n_i = d - r$, and $\Sigma a_i n_i = d$. Set $a = (a_1, \cdots, a_k), n = (n_1, \cdots, n_k)$. Then the virtual number $\mu_{a,n}$ of divisors having n_i points of multiplicity a_i in a given linear series of dimension r and degree d is*

$$\mu_{a,n} = [R_a(t)^g P_a(t)^{d-r-g}]_{t_1^{n_1} \cdots t_k^{n_k}}.$$

This formula is valid in arbitrary characteristic, as proved in Mattuck[Mat].

Let B be a point, i.e. X/B is a smooth curve of genus g. By splitting principle,

$$c(\Lambda_d(L)) = c(\Lambda_{d-1}(L))(1 + L_d + \Gamma^{(d-1)} - \Gamma^{(d)}) = \prod_{i=1}^{d}(1 + L_i + \Gamma^{(i-1)} - \Gamma^{(i)}),$$

where $L_i = p_i^*(L), p_i : W^d \to X$ is the i-th projection and $W^d = X_B^{[d,d-1,\cdots,1]}$ is the full-flag Hilbert scheme. Now for

$$\begin{aligned}
&c(\Lambda_d(L)|_{\Gamma_{(a_1,\cdots,a_k)}}) \\
&= \prod_{i_1=1}^{a_1}(1 + L_{i_1} + \Gamma^{(i_1-1)} - \Gamma^{(i_1)}) \prod_{i_2=a_1+1}^{a_1+a_2}(1 + L_{i_2} + \Gamma^{(i_2-1)} - \Gamma^{(i_2)}) \cdots \\
&\quad \cdots \prod_{i_k=a_{k-1}+1}^{d}(1 + L_{i_k} + \Gamma^{(i_k-1)} - \Gamma^{(i_k)}),
\end{aligned}$$

consider the polynomial in $t_1, \cdots, t_k$

$$(*) \quad \begin{aligned}
&\prod_{i_1=1}^{a_1}((L_{i_1} + \Gamma^{(i_1-1)} - \Gamma^{(i_1)})t_1) \prod_{i_2=a_1+1}^{a_1+a_2}((L_{i_2} + \Gamma^{(i_2-1)} - \Gamma^{(i_2)})t_2) \cdots \\
&\qquad \cdots \prod_{i_k=a_{k-1}+1}^{d}((L_{i_k} + \Gamma^{(i_k-1)} - \Gamma^{(i_k)})t_k).
\end{aligned}$$

Then we have $c(\Lambda_d(L)|_{\Gamma_{(a_1,\cdots,a_k)}}) = 1 + c_1 + \cdots + c_k$, where $c_i = [*]_{t_1^{\alpha_1} \cdots t_k^{\alpha_k}}$ with $\alpha_1 + \cdots + \alpha_k = i$ and $\alpha_l \leq l$ for any $1 \leq l \leq k$ and by $[*]_{t_1^{\alpha_1} \cdots t_k^{\alpha_k}}$ we mean the coefficient of the monomial $t_1^{\alpha_1} \cdots t_k^{\alpha_k}$. For the de Jonquiere's question for a smooth curve we have to compute the top Chern number c_k. This can be computed by recursion via transfer theorems.

For the single block the example 29 verifies the de Jonquieres' formula. For multiblocks we need a general transfer theorem as in next section. For a single smooth curve, i.e. $B = pt$, we don't have boundary families and nodes. Thus we have simple transfer theorems from Theorems 42 and 43:

Theorem 36 *For $\sum_{i=1}^{k} a_i = d - 1$ and with notations $\tau_{d,f}, \tau_{d,p}$ for free and punctual transfers,*

$$\tau_{d,f}(\Gamma_{(a_1,\cdots,a_k)}) = \Gamma_{(a_1,\cdots,a_k,1)},$$

$$\tau_{d,p}(\Gamma_{(a_1,\cdots,a_k)}) = \Gamma_{(a_1,\cdots,a_k+1)}.$$

Example 37 *By splitting principle, we have*

$$c(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}}) = \prod_{i=1}^{a_1}(1 + L_i + \Gamma^{(i-1)} - \Gamma^{(i)}) \prod_{j=a_1+1}^{d}(1 + L_j + \Gamma^{(j-1)} - \Gamma^{(j)}),$$

where $a_1 + a_2 = d$. Considering a polynomial in t_1, t_2, for $c_2(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}})$, we have to compute the coefficients of $t_1 t_2$ and t_2^2, i.e. the followings:

1. $\sum_{i=1}^{a_1}(L_i + \Gamma^{(i-1)} - \Gamma^{(i)})(a_2 L_d + \Gamma^{(a_1)} - \Gamma^{(d)})$
2. $\sum_{j=a_1+1}^{d-1}(L_j + \Gamma^{(j-1)} - \Gamma^{(j)})((d-j)L_d + \Gamma^{(j)} - \Gamma^{(d)})$.

Since

$$\sum_{i=1}^{a_1}(L_i + \Gamma^{(i-1)} - \Gamma^{(i)})(a_2 L_d + \Gamma^{(a_1)} - \Gamma^{(d)}) = c_1(\Lambda_{a_1} L|_{\Gamma_{(a_1)}})(a_2 L_2 - \Gamma^{(d)}),$$

the first sum

$$\begin{aligned}(a_1 L_1 + \binom{a_1}{2}\omega_1)(a_2 L_2 - \Gamma^{(d)}) =& a_1 a_2 L^2 + (a_1\binom{a_2}{2} + a_2\binom{a_1}{2})L\omega \\ &+ \binom{a_1}{2}\binom{a_2}{2}\omega^2 - a_1^2 a_2 L - a_1 a_2\binom{a_1}{2}\omega.\end{aligned}$$

Moreover, for $a_1 < i < j$,

$$
\begin{aligned}
&(L_j + \Gamma^{(j-1)} - \Gamma^{(j)})((d-j)L_d + \Gamma^{(j)} - \Gamma^{(d)}) \\
&\qquad = (j - a_1 - a_2)a_1 L_j - a_1 L_d - (j - a_1 - 1)\binom{a_1}{2}\omega_1\omega_2 \\
&\qquad + (j - a_1 - 1)(j - a_1)a_1\omega_2 + \binom{j}{2} a_1\omega\Gamma_{(d)} + (j - a_1 - 1)\binom{a_1}{2}\omega_1\omega_2 \\
&\qquad - (j - a_1 - 1)a_1 a_2\omega_2\Gamma_{(d)} - a_1\binom{d}{2}\omega\Gamma_{(d)}.
\end{aligned}
$$

Integrating the second

$$
\begin{aligned}
&\sum_{j=a_1+1}^{d-1} (j - a_1 - a_2)a_1 d - (d-j)a_1 d + (j - a_1 - 1)(j-d)a_1\omega - a_1(\binom{d}{2} - \binom{j}{2})\omega \\
&= -2a_1\binom{a_2}{2}d - a_1\binom{a_2}{2}(d-1)\omega \\
&= a_1\binom{a_2}{2}(-2d - (d-1)\omega).
\end{aligned}
$$

Hence

$$
\begin{aligned}
&c_2(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}}) \\
&= a_1 a_2 d + d\omega(a_1\binom{a_2}{2} + a_2\binom{a_1}{2}) - \binom{a_1}{2}a_1 a_2\omega + \binom{a_1}{2}\binom{a_2}{2}\omega^2 - a_1\binom{a_2}{2}(d-1)\omega.
\end{aligned}
$$

Alternatively, we may compute $c_2(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}})$ as follows.

Example 38 *Consider $c_2(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}})$. Then*

$$
\begin{aligned}
c_{2,(a_1,a_2)} &:= c_2(\Lambda_d(L)|_{\Gamma_{(a_1,a_2)}}) \\
&= \tau_{d,p}(c_{2,(a_1,a_2-1)}) + (a_1 L_1 + (a_2 - 1)L_2 - \Gamma^{(d-1)})(L_d + \Gamma^{(d-1)} - \Gamma^{(d)}),
\end{aligned}
$$

hence

$$
\begin{aligned}
c_{2,(a_1,a_2)} - c_{2,(a_1,a_2-1)} =& a_1 L^2 + a_1 L_1(\Gamma^{(d-1)} - \Gamma^{(d)}) + (a_2 - 1)L_2(\Gamma^{(d-1)} - \Gamma^{(d)}) \\
& - \Gamma^{(d-1)}L_d - \Gamma^{(d-1)}(\Gamma^{(d-1)} - \Gamma^{(d)}) \\
=& a_1 L^2 + (a_1(a_2 - 1) + \binom{a_1}{2})L\omega - a_1(a_1 + 2a_2 - 2)L \\
& + \binom{a_1}{2}(a_2 - 1)\omega^2 - a_1(\binom{a_1}{2} + \binom{a_2 - 1}{2} + (a_2 - 1)(d-1))\omega.
\end{aligned}
$$

By integrating this, we have

$$c_{2,(a_1,a_2)} = a_1a_2L^2 + (a_1\binom{a_2}{2} + a_2\binom{a_1}{2})L\omega - a_1(a_1a_2 + 2\binom{a_2}{2})L + \binom{a_1}{2}\binom{a_2}{2}\omega^2$$
$$- (a_1a_2\binom{a_1}{2} + a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} + a_1\binom{a_2}{2}\frac{2a_2-1}{3})\omega.$$

Example 39 *Consider* $c_{3,(a_1,a_2,a_3)} := c_3(\Lambda_d(L)|_{\Gamma_{(a_1,a_2,a_3)}})$*, where* $a_1 + a_2 + a_3 = d$*.*

$$c_3(\Lambda_d(L)|_{\Gamma_{(a_1,a_2,a_3)}}) = \tau_{d,p}(c_{3,(a_1,a_2,a_3-1)}) + c_{2,(a_1,a_2,a_3-1)}(L_d - \triangle^{(d)}).$$

Now for $c_{2,(a_1,a_2,a_3-1)}$*, we have*

$$c_{2,(a_1,a_2,a_3-1)} - \tau_{d-1,p}(c_{2,(a_1,a_2,a_3-2)}) = c_{1,(a_1,a_2,a_3-2)}(L_{d-1} - \triangle^{(d-1)})$$

$$=(a_1L_1 + a_2L_2 - \Gamma^{(d-2)})(L_{d-1} + \Gamma^{(d-2)} - \Gamma^{(d-1)})$$

$$= \sum_{i=1,2} a_iL_iL_3 + a_i(a_3-2)L_i\omega_3 + \binom{a_i}{2}L_3\omega_i + \binom{a_i}{2}(a_3-2)\omega_i\omega_3$$

$$- a_i(a_1L_1 + a_2L_2 + (a_3-2)L_3 + (a_3-2)(a_i + a_3 - 2)\omega_3)\Gamma_{(a_i+a_3-1,a_{3-i})}$$

$$- a_1a_2(L_3 + (a_3-2)\omega_3)\Gamma_{(a_1+a_2,a_3-1)} + a_1a_2(a_1 + a_2 + 2(a_3-2))\Gamma_{(d-1)}.$$

Integrating this we get,

$$c_{2,(a_1,a_2,a_3-1)} = c_{2,(a_1,a_2,1)} + \sum_{i=1,2} a_i(a_3-2)L_iL_3 + a_i\binom{a_3-1}{2}L_i\omega_3 + \binom{a_i}{2}(a_3-2)L_3\omega_i$$
$$+ \binom{a_i}{2}\binom{a_3-1}{2}\omega_i\omega_3 - a_i((a_3-2)(a_1L_1 + a_2L_2) + \binom{a_3-1}{2}L_3$$
$$+ ((a_i-1)\binom{a_3-1}{2} + 2\binom{a_3}{3})\omega_3)\Gamma_{(a_i+a_3-1,a_{3-i})} - a_1a_2((a_3-2)L_3$$
$$+ \binom{a_3-1}{2}\omega_3)\Gamma_{(a_1+a_2,a_3-1)} + a_1a_2((a_1+a_2)(a_3-2)$$
$$+ 2\binom{a_3-1}{2})\Gamma_{(d-1)},$$

where

$$c_{2,(a_1,a_2,1)} = \tau(c_{2,(a_1,a_2)}) + \sum_{i=1,2}(a_iL_i + \binom{a_i}{2}\omega_i)L_3 + (-a_iL_i + \binom{a_i}{2}\omega_i)(a_1\Gamma_{(a_1+a_3-1,a_2)}$$
$$+ a_2\Gamma_{(a_2+a_3-1,a_1)}) - a_1a_2L_3\Gamma_{(a_1+a_2,a_3-1)} + a_1a_2(a_1+a_2)\Gamma_{(d-1)}$$

and $c_{2,(a_1,a_2)}$ *is given in above example. Therefore*

$$
\begin{aligned}
& c_{2,(a_1,a_2,a_3-1)} \\
=& \tau(c_{2,(a_1,a_2)}) + \sum_{i=1,2} a_i(a_3-1)L_iL_3 + a_i\binom{a_3-1}{2}L_i\omega_3 + \binom{a_i}{2}(a_3-1)L_3\omega_i \\
& + \binom{a_i}{2}\binom{a_3-1}{2}\omega_i\omega_3 - a_i((a_3-1)(a_1L_1+a_2L_2) + \binom{a_3-1}{2}L_3 \\
& - \binom{a_i}{2}\omega_i + ((a_i-1)\binom{a_3-1}{2} + 2\binom{a_3}{3})\omega_3)\Gamma_{(a_i+a_3-1,a_{3-i})} \\
& - a_1a_2((a_3-1)L_3 + \binom{a_3-1}{2}\omega_3)\Gamma_{(a_1+a_2,a_3-1)} \\
& + a_1a_2(a_3-1)(a_1+a_2+a_3)\Gamma_{(d-1)}.
\end{aligned}
$$

Then $c_{2,(a_1,a_2,a_3-1)}(L_3+\Gamma^{(d-1)}-\Gamma^{(d)})$

$$
\begin{aligned}
=& a_1a_2L^3 + (a_1\binom{a_2}{2} + a_2\binom{a_1}{2} + a_1a_2(a_3-1))L^2\omega + (\binom{a_1}{2}\binom{a_2}{2} + a_1\binom{a_2}{2}(a_3-1) \\
& + a_2\binom{a_1}{2}(a_3-1))L\omega^2 + \binom{a_1}{2}\binom{a_2}{2}(a_3-1)\omega^3 - a_1(2a_1a_2 + a_2^2 + 2\binom{a_2}{2})L^2 \\
& - (a_1a_2\binom{a_1}{2} + a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} + a_1\binom{a_2}{2}\frac{2a_2-1}{3} + a_1^2a_2(a_3-1) \\
& + 2a_1\binom{a_2}{2}(a_3-1) + (a_1+a_2)(a_1\binom{a_2}{2} + a_2\binom{a_1}{2}))L\omega - ((a_3-1)(a_1a_2\binom{a_1}{2} \\
& + a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} + a_1\binom{a_2}{2}\frac{2a_2-1}{3}) + (a_1+a_2)\binom{a_1}{2}\binom{a_2}{2} + a_1\binom{a_2}{2}\binom{a_3-1}{2} \\
& + a_2\binom{a_1}{2}\binom{a_3-1}{2})\omega^2 + a_1(a_1+a_2)(a_1a_2 + 2\binom{a_2}{2})L + (a_1+a_2)(a_1a_2\binom{a_1}{2} \\
& + a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} + a_1\binom{a_2}{2}\frac{2a_2-1}{3})\omega.
\end{aligned}
$$

Finally integrating this, we have

$$c_{3,(a_1,a_2,a_3)} =$$

$$a_1a_2a_3L^3 + (a_1a_3\binom{a_2}{2} + a_2a_3\binom{a_1}{2} + a_1a_2\binom{a_3}{2})L^2\omega + (a_1\binom{a_2}{2}\binom{a_3}{2} + a_2\binom{a_1}{2}\binom{a_3}{2}$$
$$+ a_3\binom{a_1}{2}\binom{a_2}{2})L\omega^2 + \binom{a_1}{2}\binom{a_2}{2}\binom{a_3}{2}\omega^3 - a_1(2a_1a_2a_3 + a_2^2a_3 + 2a_3\binom{a_2}{2}$$
$$+ 4a_2\binom{a_3}{2})L^2 - (a_1a_2a_3\binom{a_1}{2} + a_1a_3\binom{a_2}{3} + a_1^2a_3\binom{a_2}{2} + a_1a_3\binom{a_2}{2}\frac{2a_2-1}{3}$$
$$+ a_1^2a_2\binom{a_3}{2} + 2a_1\binom{a_2}{2}\binom{a_3}{2} + (a_1+a_2)a_3(a_1\binom{a_2}{2} + a_2\binom{a_1}{2}) + 2a_1a_2\binom{a_3}{2}$$
$$+ a_1\binom{a_2}{2}\binom{a_3}{2} + a_2\binom{a_1}{2}\binom{a_3}{2} + (a_1+a_2-2)a_1a_2\binom{a_3}{2} + 4a_1a_2\binom{a_3+1}{3})L\omega$$
$$- (a_1a_2\binom{a_1}{2}\binom{a_3}{2} + a_1\binom{a_2}{3}\binom{a_3}{2} + a_1^2\binom{a_2}{2}\binom{a_3}{2} + a_1\binom{a_2}{2}\binom{a_3}{2}\frac{2a_2-1}{3}$$
$$+ (a_1+a_2)a_3\binom{a_1}{2}\binom{a_2}{2} + a_1\binom{a_2}{2}\binom{a_3}{3} + a_2\binom{a_1}{2}\binom{a_3}{3})\omega^2 + (3a_1^2a_2\binom{a_3}{2})$$
$$+ 3a_1a_2^2\binom{a_3}{2} + 4a_1a_2\binom{a_3}{2} + a_1^2a_2^2\binom{a_3}{2} + a_1a_3(a_1+a_2)(a_1a_2 + 2\binom{a_2}{2}))L$$
$$+ (2a_1\binom{a_2}{2}\binom{a_3}{3} + 2a_2\binom{a_1}{2}\binom{a_3}{3} + 4a_1a_2\binom{a_3+1}{4} - a_1a_2a_3\binom{a_1}{2} - a_1a_2a_3\binom{a_2}{2}$$
$$+ a_1^2a_2^2\binom{a_3}{3} + 2a_1a_2\binom{a_3}{3}(a_1+a_2-1) + 6a_1a_2\binom{a_3+1}{4} + (a_1+a_2)a_3(a_1a_2\binom{a_1}{2}$$
$$+ a_1\binom{a_2}{3} + a_1^2\binom{a_2}{2} + a_1\binom{a_2}{2}\frac{2a_2-1}{3}))\omega.$$

Remark 40

1. *These are polynomials in d and g.*
2. *For a smooth surface S, the de Jonquieres' formula is a polynomial in $L^2, L\omega, \omega^2$, and $c_2(S)$. More generally Göttsche conjectured(recently proved by Y. Tzeng [T] and M. Kool, V. Shende, R.P. Thomas [KST]) that for every r, the numbers of r-nodal curves are given by universal polynomials of these four topological numbers.*

Chapter 4

de Jonquieres' formula for pencil

4.1 Transfer theorems

For the de Jonquieres' formula for a family of curves X/B, we need to compute a polynomial in Chern classes of $\Lambda_m(L)$ over Γ_μ for a partition μ of m, $P(c(\Lambda_m(L)_{\Gamma_\mu}))$ for a line bundle L on X/B. More generally, our object is to compute all polynomials in the Chern classes of tautological bundle $\Lambda_m(L)$. The idea is to use the splitting principle and transfer theorem via flag Hilbert scheme, $X_B^{[m,m-1]}$, recursively. Indeed, for any polynomial in the Chern classes of $\Lambda_m(L)$, we can write this as a sum of monomials of the form $P_1 \cdots P_m$, where P_i comes from $X_B^{[i]}$ by splitting principle. Inductively we can compute any such polynomials by free and punctual transfer theorems for any partition μ. In this section we will give these transfers generalizing the transfers in Ch2.

Let $\pi = (a_1, \cdots, a_k)$ with $wt(\pi) := \sum_{i=1}^{k} a_i = m$. By a consolidation of π we mean any partition μ obtained from π by repeating the operation of uniting two distinct blocks. Now we define a tautological group T_π^m associated to a partition π generated by

1. Γ_μ, where μ is any consolidation of π,

2. polyscrolls $F^{(n.;m)}_{(j.,\mu')}(\theta.;X/B)$, where $n.=(n_1,\cdots,n_r), j.=(j_1,\cdots,j_r)$, $\theta.=(\theta_1,\cdots,\theta_r)$, and μ' is a partition such that $\mu'\coprod(n_1,\cdots,n_r)=\mu$,

3. polysections $(\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}$, where each $e_i=0$ or $m-\sum_{j=1}^{i-1}n_j$ and μ' is a partition such that $\mu'\coprod(n_1,\cdots,n_r)=\mu$.

Note that if $e_i=0$ we have scroll for the node θ_i otherwise we have section.

Convention: By $e_i=0$ or 1 we mean $e_i=0$ or $m-\sum_{j=1}^{i-1}n_j$.

Now then we have the

Corollary 41 *T^m_π is $\mathbb{Q}[\Gamma^{(m)}]$-module under the intersection with discriminant $\Gamma^{(m)}$.*

Proof. This can be proved similarly as module theorem 15. ∎

Let $\tau_{m,f}$ and $\tau_{m,p}$ be m-th free transfer and m-th punctual transfer, respectively. For a free transfer calculus, we need to see how the map $\tau_{m,f}:T^{m-1}_\pi\to T^m_{\pi+1_1}$ sends tautological classes on T^{m-1}_π.

Theorem 42 *(Free transfer) $\tau_{m,f}$ takes tautological classes on T^{m-1}_π to tautological classes on $T^m_{\pi+1_1}$ as follows:*

1. *for any twisted polyblock diagonal class $\Gamma_\mu[\alpha.], \alpha.\in TS_\mu(H^\cdot(X)), wt(\mu)=m-1$,*

$$\tau_{m,f}(\Gamma_\mu[\alpha.]\beta_{(m)})=\Gamma_{\mu+1_1}[\alpha.\beta].$$

2. *for any twisted polyscroll class $F^{n.,m-1}_{j.,\nu}(\theta.)[\alpha.], \alpha.\in T^{m-n.-1}(X^{\theta.}_T(\theta.))$,*

$$\tau_{m,f}(F^{n.,m-1}_{j.,\nu}(\theta.)[\alpha.]\beta_{(m)})=F^{n.,m}_{j.,\nu+1_1}(\theta.)[\tau_{m-n.,X^{\theta.}_T}(\alpha.\beta|_{X^{\theta.}_T})],$$

where $\tau_{m-n.,X^{\theta.}_T}$ is transfer on the tautological module of the boundary family $X^{\theta.}_T$.

3. for any twisted nodesection $(-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)[\alpha.], \alpha. \in T^{m-n-1}(X^\theta_T(\theta))$,

$$\begin{aligned}\tau_{m,f}((-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)[\alpha.]\beta_{(m)}) =& \theta^*(\beta)F^{n+1,m}_{j,\nu}(\theta)[\alpha.] \\ &+ (-\Gamma^{(m)})F^{n,m}_{j,\nu+1_1}(\theta)[\tau_{m-n,X^\theta_T}(\alpha.\beta|_{X^\theta_T})] \\ &- F^{n,m}_{j,\nu+1_1}(\theta)[e^{n,m}_{j+1}(\tau_{m-n,X^\theta_T}(\alpha.\beta|_{X^\theta_T}))] \\ &+ F^{n,m}_{j,\nu+1_1}(\theta)[\tau_{m-n,X^\theta_T}(e^{n,m-1}_{j+1}(\alpha.)\beta|_{X^\theta_T})].\end{aligned}$$

4. more generally, for any twisted polysection $(-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}[\alpha.], \alpha. \in T^{m-n.-1}(X^\theta_T(\theta.))$*, where each* $e_i = 0$ *or* 1 *and* $\mu' \coprod (n_1, \cdots, n_r) = \mu$,

$$\begin{aligned}&\tau_{m,f}((-\Gamma^{(e_1)}F^{n_1,m-1}_{j_1}(\theta_1))\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}[\alpha.\beta_{(m)}]) \\ =&(-\Gamma^{(e_1)}F^{n_1,m}_{j_1}(\theta_1))(\sum_{i=2}^r(-\Gamma^{(e_2)}F^{n_2,m-n_1}_{j_2}(\theta_2))\cdots\theta^*_i(\beta)F^{n_i+1}_{j_i}(\theta_i)\cdots \\ &\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}[\alpha.]) \\ &+ \tau_{m,f}(-\Gamma^{(e_1)}F^{n_1,m}_{j_1}(\theta_1))(\tau_{m-n_1,f}(-\Gamma^{(e_2)}F^{n_2,m-1-n_1}_{j_2}(\theta_2)\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}[\alpha.\beta]) \\ &- \sum_{i=2}^r(-\Gamma^{(e_2)}F^{n_2,m-n_1}_{j_2}(\theta_2))\cdots\theta^*_i(\beta)F^{n_i+1}_{j_i}(\theta_i)\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}[\alpha.]).\end{aligned}$$

Proof. Part 1 is obvious. For 2, we will use induction on r, the number of node blocks. First consider nodescroll $F^{n,m-1}_{j,\nu}(\theta; X/B)$. Recall that $F^{n,m-1}_{j,\nu}(\theta)$ is defined by the fiber square

$$\begin{array}{ccc} F^{n,m-1}_{j,\nu}(\theta) & \longrightarrow & F^{n,m-1}_{j}(\theta) \\ \downarrow & & \downarrow p_{[m-1-n]} \\ \Gamma_{\nu,X^\theta_T} & \xrightarrow{g} & (X^\theta_T)^{[m-1-n]}, \end{array}$$

where $p_{[m-1-n]}$ is $\mathbb{P}^1$-bundle projection and g is generically finite onto the locus of schemes of type ν on the boundary family X^θ_T. Then since the free transfer $\tau_{m,f}$ of $F^{n,m-1}_{j,\nu}(\theta; X/B)$ is equivalent to the free transfer of the base $\Gamma_{\nu,X^\theta_T} \xrightarrow{g} (X^\theta_T)^{[m-1-n]}$, by the transfer of the base the fiber square reduces to

$$\begin{array}{ccc} \tau_{m,f}(F^{n,m-1}_{j,\nu}(\theta)) & \longrightarrow & F^{n,m}_{j}(\theta) \\ \downarrow & & \downarrow p_{[m-n]} \\ \Gamma_{\nu+1_1,X^\theta_T} & \longrightarrow & (X^\theta_T)^{[m-n]}, \end{array}$$

hence $\tau_{m,f}(F^{n,m-1}_{j,\nu}(\theta)) = F^{n,m}_{j,\nu+1_1}(\theta)$. For polyscroll/sections, we may use induction on the number of nodes. Since nodes are disjoint it suffices to consider 2-blocks $F = F^{n_1,n_2;m-1}_{j_1,j_2;\nu}(\theta_1,\theta_2;X/B)$, where $\nu + 1_{n_1} + 1_{n_2} = \mu$. Recall that by the construction of polyscroll, we have the following fiber product

$$\begin{array}{ccc} F & \longrightarrow & F^{n_1,m-1}_{j_1}(\theta_1;X/B) \\ \downarrow & & \downarrow \\ F' & \longrightarrow & (X^{\theta_1}_{T(\theta_1)})^{[m-1-n_1]}, \end{array}$$

where $F' = F^{n_2,m-1-n_1}_{j_2,\nu+1_{j_1}}(\theta_2;X^{\theta_1}_{T(\theta_1)})$. Note that the right vertical map is $\mathbb{P}^1$-bundle projection and the bottom horizontal map is generically finite map. Now the base is transfered to $F^{n_2,m-n_1}_{j_2,\nu+1_{j_1}+1}(\theta_2;X^{\theta_1}_{T(\theta_1)}) \to (X^{\theta_1}_{T(\theta_1)})^{[m-n_1]}$. Hence we have

$\tau_{m,f}(F^{n_1,n_2;m-1}_{j_1,j_2;\nu}(\theta_1,\theta_2;X/B)) = F^{n_1,n_2;m}_{j_1,j_2;\nu+1_1}(\theta_1,\theta_2;X/B)$. Now by induction we have part 2.

For a nodesection $(-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)$, consider the fiber square

$$\begin{array}{ccc} (-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta) & \longrightarrow & F^{n,m-1}_{j}(\theta;X/B) \\ \downarrow & & \downarrow \\ \Gamma_\nu & \longrightarrow & (X^{\theta}_{T(\theta)})^{[m-1-n_1]}. \end{array}$$

Note that on $F^{n,m-1}_{j}(\theta)$, we have $-\Gamma^{(m-1)} \sim Q^{n,m-1}_j + e^{n,m-1}_{j+1}$. $p^*_{[m-1]}Q^{n,m-1}_j$ splits in two parts, depending on whether the point w added to a scheme $z \in Q^{n,m-1}_j$ is in the off-node or nodebound portion of z. The first case we have $\Gamma_{\nu+1_1}$ and the second case we have $F^{n+1,m}_j(\theta)$ by the base transfer of the square, hence we have part 3.

For more general polysections we use induction on the number of nodes. For a polysection $(-\Gamma^{(e_1)}F^{n_1,m-1}_{j_1}(\theta_1))(-\Gamma^{(e_2)}F^{n_2,m-1-n_1}_{j_2}(\theta_2))_{\mu'}[\alpha.], \alpha. \in T^{m-n.-1}(X^{\theta.}_T(\theta.))$, consider the fiber square

$$\begin{array}{ccc} (-\Gamma^{(e_1)}F^{n_1,m-1}_{j_1}(\theta_1))(-\Gamma^{(e_2)}F^{n_2,m-1-n_1}_{j_2}(\theta_2))_{\mu'} & \longrightarrow & -\Gamma^{(e_1)}F^{n_1,m-1}_{j_1}(\theta_1) \\ \downarrow & & \downarrow \\ -\Gamma^{(e_2)}F^{n_2,m-1-n_1}_{j_2}(\theta_2)_{\mu'} & \longrightarrow & (X^{\theta_1}_{T(\theta_1)})^{[m-1-n_1]}. \end{array}$$

Let's consider the transfer of the base, i.e. $-\Gamma^{(e_2)}F_{j_2}^{n_2,m-1-n_1}(\theta_2)_{\mu'}$. The node part(i.e. θ_2) transfers this to $F_{j_2}^{n_2+1,m-n_1}(\theta_2)$ and hence the corresponding right vertical $\mathbb{P}^1$ bundle is $(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))$, so we have

$$(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))F_{j_2}^{n_2+1,m-n_1}(\theta_2).$$

Now the off-node part transfers the base to

$$\begin{aligned}&-\Gamma^{(e_2)}F_{j_2}^{n_2,m-n_1}(\theta_2)[\tau_{m-n_1-n_2,X_T^{\theta.}}(\cdot)]-F_{j_2,\mu+1_1}^{n_2,m-n_1}(\theta_2)[e_{j_2+1}^{n_2,m-n_1}(\tau_{m-n.,X_T^{\theta.}}(\cdot))]\\&+F_{j_2,\mu+1_1}^{n_2,m-n_1}(\theta_2)[\tau_{m-n.,X_T^{\theta.}}(e_{j_2+1}^{n_2,m-n_1-1}(\cdot))]\end{aligned}$$

and hence the corresponding right vertical $\mathbb{P}^1$ bundle is $\tau_{m,f}(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))$, so we have

$$\begin{aligned}&\tau_{m,f}(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))(-\Gamma^{(e_2)}F_{j_2}^{n_2,m-n_1}(\theta_2)[\tau_{m-n_1-n_2,X_T^{\theta.}}(\cdot)]\\&-F_{j_2,\mu+1_1}^{n_2,m-n_1}(\theta_2)[e_{j_2+1}^{n_2,m-n_1}(\tau_{m-n.,X_T^{\theta.}}(\cdot))]+F_{j_2,\mu+1_1}^{n_2,m-n_1}(\theta_2)[\tau_{m-n.,X_T^{\theta.}}(e_{j_2+1}^{n_2,m-n_1-1}(\cdot))]).\end{aligned}$$

Combining these we have

$$\begin{aligned}&\tau_{m,f}((-\Gamma^{(e_1)}F_{j_1}^{n_1,m-1}(\theta_1))(-\Gamma^{(e_2)}F_{j_2}^{n_2,m-1-n_1}(\theta_2))_{\mu'})\\=&(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))F_{j_2}^{n_2+1,m-n_1}(\theta_2)\\&+\tau_{m,f}(-\Gamma^{(e_1)}F_{j_1}^{n_1,m}(\theta_1))(\tau_{m-n_1,f}(-\Gamma^{(e_2)}F_{j_2}^{n_2,m-1-n_1}(\theta_2))-F_{j_2}^{n_2+1,m-n_1}(\theta_2)).\end{aligned}$$

Now inductively we have part 4. ■

Next, for a punctual transfer calculus, we need to see how the map $\tau_{m,p}:T_\pi^{m-1}\to T_{\pi'}^m$ defined by the restriction of $p_{[m]*}p_{[m-1]}^*$ to Γ_π, where $\pi=(a_1,\cdots,a_k),\pi'=(a_1,\cdots,a_k+1)$ sends tautological classes on T_π^{m-1}.

Theorem 43 *(Punctual transfer) $\tau_{m,p}$ takes tautological classes on T_π^{m-1} to tautological classes on $T_{\pi'}^m$ as follows:*

1. *for any polyblock diagonal class* $\Gamma_{\mu}, wt(\mu) = m-1$,

$$\tau_{m,p}(\Gamma_{\mu}) = \Gamma_{\mu'},$$

where μ' *is the corresponding partition of* μ *under* $\pi \to \pi'$.

2. *for any nodescroll/section* $F^{n,m-1}_{j,\nu}(\theta), -\Gamma^{(m-1)}F^{n,m-1}_{j,\nu}(\theta)$, *where* a_k *belongs to the* ν,

$$\tau_{m,p}(F^{n,m-1}_{j,\nu}(\theta)) = F^{n,m}_{j,\nu'}(\theta),$$

and

$$\tau_{m,p}((-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)) = \tau_{m,f}(-\Gamma^{(m-1)}F^{n,m-1}_{j,\nu'}(\theta)).$$

More generally for any polyscroll/sections $(-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\nu}$, *where each* $e_i = 0$ *or* 1,

$$\tau_{m,p}((-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\nu}) = \tau_{m,f}((-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\nu'}),$$

where ν' *is the corresponding partition of* ν *under* $\pi \to \pi'$.

(Note that this is free transfer corresponding to $\nu \to \nu'$*.)*

3. *for any polyscroll* $F^{n.;m-1}_{j.;\nu}(\theta.; X/B)$, *where* a_k *is one of the summands making up some* n_i,

$$\tau_{m,p}(F^{n.;m-1}_{j.;\nu}(\theta.)) = \frac{n_i+1-j_i}{n_i}F^{n_1,\cdots,n_i+1,\cdots,n_r,m}_{j_1,\cdots,j_i,\cdots,j_r,\nu}(\theta.)+\frac{j_i+1}{n_i}F^{n_1,\cdots,n_i+1,\cdots,n_r,m}_{j_1,\cdots,j_i+1,\cdots,j_r,\nu}(\theta.).$$

4. *for any nodesection* $(-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)$, *where* a_k *is one of the summands making up* n,

$$\begin{aligned}&\tau_{m,p}((-\Gamma^{(m-1)})F^{n,m-1}_{j,\nu}(\theta)) = \\ &(-\Gamma^{(m)})F^{n+1,m}_{j+1,\nu}(\theta) - F^{n+1,m}_{j+1,\nu}(\theta)[(\frac{n-j-1}{n}\psi^{n}_{j+2} + \frac{j+1}{n}\psi^{n}_{j+1})] \\ &+ \frac{n-j}{n-1}F^{n+1,m}_{j,\nu}(\theta)[\psi^{n-1}_{j}\alpha.] + \frac{j+1}{n-1}F^{n+1,m}_{j+1,\nu}(\theta)[\psi^{n-1}_{j}\alpha.].\end{aligned}$$

5. *more generally, for any polysection* $(-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}$, *where each* $e_i =$ 0 *or* 1, *where* a_k *is one of the summands making up some* n_i,

$$\tau_{m,p}((-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}) =$$

$$(-\Gamma^{(e_1)}F^{n_1}_{j_1})\cdots\tau_{m-\sum_{j=1}^{i-1}n_j,p}((-\Gamma^{(e_i)})F^{n_i,m-1}_{j_i}(\theta_i))\cdots(-\Gamma^{(e_r)}F^{n_r}_{j_r})_{\mu'}).$$

To prove this theorem we use the following lemma

Lemma 44

$$\tau_{m,f}\circ\tau_{m-1,p} = \tau_{m,p}\circ\tau_{m-1,f},$$

where the punctual transfers on both are considered on the same block.

Proof. Obvious. ∎

Now we prove the theorem.

Proof. 1 is obvious. 2 is from free transfer theorem above and note that when we think $p^*_{m-1}Q$ has only off-node point which belongs to some block of ν.

For 3, we use induction on r, the number of node blocks. First we prove

$$\tau_{m,p}(F^{n,m-1}_{j,\nu}(\theta)) = \frac{n+1-j}{n}F^{n+1,m}_{j,\nu}(\theta) + \frac{j+1}{n}F^{n+1,m}_{j+1,\nu}(\theta).$$

Recall that $\tau_{m,p}(C^{m-1}_j(\theta)) = \frac{m-j}{m-1}C^m_j(\theta) + \frac{j+1}{m-1}C^m_{j+1}(\theta)$.

Note that $F^{n,m-1}_j(\theta) = \tau_{m-1,f}\tau_{m-2,f}\cdots\tau_{n+1,f}(C^n_j(\theta))$, $(m-1-n)$-free transfers. Now consider $\tau_{m,p}(F^{n,m-1}_j(\theta))$ punctual transfer. Since $(m-1-n)$-free and then 1 punctual is 1 punctual and then $(m-1-n)$-free by lemma,

$$\begin{aligned}
\tau_{m,p}(F^{n,m-1}_j(\theta)) =&\tau_{m,p}(\tau_{m-1,f}\tau_{m-2,f}\cdots\tau_{n+1,f}(C^n_j(\theta))) \\
=&\tau_{m,f}\cdots\tau_{n+2,f}(\tau_{n+1,p}(C^n_j(\theta))) \\
=&\tau_{m.,f}(\frac{n+1-j}{n}C^{n+1}_j(\theta) + \frac{j+1}{n}C^{n+1}_{j+1}(\theta)) \\
=&\frac{n+1-j}{n}F^{n+1,m}_j(\theta) + \frac{j+1}{n}F^{n+1,m}_{j+1}(\theta).
\end{aligned}$$

Now inductively we have part 3 with the fact that

$F_{j_1}^{n_1,m-1}F_{j_2}^{n_2,m-1-n_1} = F_{j_2}^{n_2,m-1}F_{j_1}^{n_1,m-1-n_2}$.

For 4, recall the transfer,i.e. the restriction of $p_{[m]*}p^*_{[m-1]}$ to Γ_π when a_k is one of the summands making up n. Letting $\widetilde{\Gamma_\pi} := p^{-1}_{[m-1]}(\Gamma_\pi)$, this is based on the correspondence

$$\begin{array}{ccc} & \widetilde{\Gamma_\pi} & \\ {\scriptstyle p_m}\swarrow & & \searrow{\scriptstyle p_{m-1}} \\ \Gamma_{\pi'} & & \Gamma_\pi. \end{array}$$

We may consider the ordered flag Hilbert scheme and pass to unordered one. Consider

$$\begin{array}{ccc} & X_B^{\lceil m,m-1 \rceil} & \\ {\scriptstyle op_m}\swarrow & & \searrow{\scriptstyle op_{m-1}} \\ X_B^{\lceil m \rceil} & & X_B^{\lceil m-1 \rceil}. \end{array}$$

Now for the nodesection $-\Gamma^{(m-1)}F_{j,\nu}^{n,m-1}(\theta)$ of $\Gamma_{1_n+\nu}$, the ordered locus is a nodesection of $\Gamma_{I|J}$, where $I = 1_n, J = \nu$. Then since a_k is one of the summands making up n, $op_{m-1} = op_I \times iso$. Now op_I and the unordered $p_{[n]}$ is understood by Proposition 22, hence we have part 4.

Now 5 follows from the induction and the fact that

$$(-\Gamma^{(e_1)}F_{j_1}^{n_1,m-1})(-\Gamma^{(e_2)}F_{j_2}^{n_2,m-1-n_1}) = (-\Gamma^{(e_2)}F_{j_2}^{n_2,m-1})(-\Gamma^{(e_1)}F_{j_1}^{n_1,m-1-n_2}).$$

■

4.2 de Jonquieres' problems for a family of curves

Let's start with the definition of a family of g^r_d's.

Definition 45 *By* a family of g^r_d's on C parametrized by an analytic space S *we mean the datum of*

1. *A family L of degree d line bundles on C, parametrized by S.*

2. *A locally free, rank $(r+1)$ subsheaf F of ϕ_*L, where ϕ is the projection of $C \times S$ onto S, with the property that, for each $s \in S$, the homomorphism*

$$F \otimes k(s) \to H^0(\phi^{-1}(s), L \otimes \mathcal{O}_{\phi^{-1}(s)})$$

is injective.

At least when S is reduced, a family of g^r_d's on C parametrized by S can be thought of as a holomorphically varying family

$$L_s \to C, s \in S,$$

of degree d line bundles on C, together with a holomorphically varying family $\mathcal{D}_s$ of g^r_d's

$$\mathcal{D}_s \subset |L_s|, s \in S.$$

If we are given a family of g^r_d's, $\mathcal{G} = (L, F)$ on C parametrized by S, and a morphism

$$f : T \to S,$$

we can define the pull-back

$$f^*(\mathcal{G}) = ((1_C \times f)^*L, f^*(F)),$$

which is a family of g^r_d's on C parametrized by T.

Two families $(L, F), (L', F')$ of g^r_d's on C parametrized by S are said to be equivalent if there exist a line bundle $\mathcal{R}$ on S and an isomorphism

$$L' \cong L \otimes \phi^*\mathcal{R}$$

such that F' is identified with $F \otimes \mathcal{R}$. Then

$$G^r_d(C) := \{g^r_d\text{'s on } C\}$$

is the universal parametrizing space;

Theorem 46 *([ACGH]) For any analytic space S and any family $\mathcal{G}$ of g_d^r's on C parametrized by S, there is a unique morphism from S to $G_d^r(C)$ such that the pull-back of the universal family parametrized by $G_d^r(C)$ is equivalent to $\mathcal{G}$.*

For a family of smooth curves genus $g > 1$, $\pi : X \to B$, we have *Brill-Noether varieties*, $\mathcal{C}_d^r$ and $\mathcal{W}_d^r$ which coincide with the Brill-Noether varieties C_d^r and W_d^r for a smooth curve, i.e. $B =$ a point. For $\pi : X \to B$, define

$$supp(\mathcal{C}_d^r) = \{(b, D) : b \in B, D \in (X_b)^{(d)} \text{ such that } h^0(X_b, \mathcal{O}_{X_b}(D)) \geq r+1\}$$

and

$$supp(\mathcal{W}_d^r) = \{(b, L) : b \in B, L \in Pic^d(X_b) \text{ such that } h^0(X_b, L) \geq r+1\}.$$

Then they have a scheme structure by showing that they are determinantal varieties[ACG]. Moreover we have another Brill-Noether variety $\mathcal{G}_d^r$ parameterizing all g_d^r's on the fiber X_b of the family π.

Proposition 47 (ACG) *When $g \geq 2$ every component of $\mathcal{G}_d^r$ has dimension at least $3g-3+\rho$. Similarly, when $r \geq 0$ and $r \geq g-d$, every component of $\mathcal{W}_d^r$ has dimension at least $3g-3-\rho$, and every component of $\mathcal{C}_d^r$ has dimension at least $3g-3+\rho+r$, where ρ is the Brill-Noether number.*

For a family of nodal curves X/B consider a line bundle L on X and a vector bundle E on B such that $E \subset \pi_* L$. Note that for any $b \in B$, $E_b \subset H^0(X_b, L|_{X_b})$, i.e. a g_d^r on X_b, hence we have a family of g_d^r. Let $\pi^{[m]} : X_B^{[m]} \to B$. As the de Jonquieres' problem for a single smooth curve, for a family of nodal curves we have to find a certain degeneracy locus of the morphism of vector bundles $\phi_m : (\pi^{[m]})^* E \to \Lambda_m L$ on $X_B^{[m]}$; first consider the composition $(\pi^* E \hookrightarrow \pi^* \pi_* L) \circ (\pi^* \pi_* L \to L)$, i.e. $\pi^* E \to L$. By pulling back via p_1, restriction to the universal subscheme of the relative Hilbert scheme Z, and pushing forward via p_2, we have $p_{2_*}p_1^*\pi^*E \to p_{2_*}(p_1^*L \otimes \mathcal{O}_Z) = \Lambda_m L$. Now by composing with a natural morphism $(\pi^{[m]})^*E \to p_{2_*}p_1^*\pi^*E$ we get the morphism ϕ. Note that over $X_b^{[m]}$, $(\phi_m)|_{X_b^{[m]}} : \mathcal{O}_{X_b^{[m]}} \otimes E_b \to (\Lambda_m L)|_{X_b^{[m]}}$. Further over $z \in X_b^{[m]}$, $\phi_z : E_b \to H^0(L|_z)$. Now suppose $degL = m$, then since $H^0(L) \leq degL$, $rk(E) = e \leq m$.

Now $(e-1)$-th degeneracy locus is $\{z \in X_B^{[m]}$: there is a section $s \in E_b \subset H^0(L|_{X_b})$ s.t. $s|_z = 0\}$. Now $(e-1)$-degeneracy locus of ϕ is $\triangle_{m-e+1,1}(\Lambda_m L - \pi^{[m]*}E)$ by Porteous' formula.

Note that the expected dimension is $m+1-(m-e+1)$. Hence on Γ_μ, where $wt(\mu) = k$, the expected dimension of the degeneracy locus is 0, i.e. finite set of divisors when $m-e = k$. Therefore for the de Jonquieres' problem for a 1-parameter family we need to find $c_{k+1}(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_\mu}$, where $\mu = (a_1, \cdots, a_k)$. For example $c_2(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(m)}}$ and $c_3(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(a_1,a_2)}}$, where $a_1 + a_2 = m$. Here since $dimB = 1$ we have

$$c_2(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(m)}} = c_2(\Lambda_m L|_{\Gamma_{(m)}}) - c_1(\Lambda_m L|_{\Gamma_{(m)}})x,$$

$$c_3(\Lambda_m L - \pi^{[m]*}E)|_{\Gamma_{(a_1,a_2)}} = c_3(\Lambda_m L|_{\Gamma_{(a_1,a_2)}}) - c_2(\Lambda_m L|_{\Gamma_{(a_1,a_2)}})y,$$

where $x = (\pi^{[m]})^* c_1(E) \cap [\Gamma_{(m)}]$ and $y = (\pi^{[m]})^* c_1(E) \cap [\Gamma_{(a_1,a_2)}]$. We will get these formulas in this section.

Lemma 48 $\tau_{m,p} \cdots \tau_{i+1,p}(C_j^i) = \sum_{k=0}^{m-i} \frac{\binom{m-j-k}{m-i-k}\binom{j+k}{k}}{\binom{m-1}{m-i}} C_{j+k}^m$.

Proof. Note that

$$\tau_{i+1,p}(C_j^i) = \frac{i+1-j}{i} C_j^{i+1} + \frac{j+1}{i} C_{j+1}^{i+1}.$$

To use induction suppose that $\tau_{m-1,p}\cdots\tau_{i+1,p}(C^i_j)=\sum_{k=0}^{m-1-i}\frac{\binom{m-1-j-k}{m-i-k}\binom{j+k}{k}}{\binom{m-2}{m-1-i}}C^{m-1}_{j+k}$. Then

$$\begin{aligned}&\tau_{m,p}(\sum_{k=0}^{m-1-i}\frac{\binom{m-1-j-k}{m-i-k}\binom{j+k}{k}}{\binom{m-2}{m-1-i}}C^{m-1}_{j+k})\\=&\sum_{k=0}^{m-1-i}\frac{\binom{m-1-j-k}{m-i-k}\binom{j+k}{k}}{\binom{m-2}{m-1-i}}(\frac{m-j-k}{m-1}C^m_{j+k}+\frac{j+k+1}{m-1}C^m_{j+k+1}).\end{aligned}$$

By simple algebra the lemma is proved. ■

Similarly we have the

Lemma 49 *For i, where $a_1+1\leq i\leq a_2$ and $a_1+a_2=m$,*

$$\tau_{m,p}\cdots\tau_{k+1,p}(F^{i,k}_{j,a_1}(\theta_2))=\sum_{l=0}^{a_2-i}\frac{\binom{a_2-j-l}{a_2-i-l}\binom{j+l}{l}}{\binom{a_2-1}{a_2-i}}F^{a_2,m}_{j+l,a_1}(\theta_2).$$

Remark 50 *We also have*

1. $\tau_{m,p}\cdots\tau_{m-i+2,p}(\sum_{k=1}^{a_2-i}k(a_2-i+1-k)F^{a_2-i+1,m-i+1}_{k,a_1})=\sum_{k=1}^{a_2-1}k(a_2-k)\frac{a_2-i}{a_2-1}F^{a_2,m}_{k,a_1}$.

2. $\tau_{m,p}\cdots\tau_{i+1,p}(\sum_{j=1}^{i-1}\frac{j(i-j)i}{2}C^i_j)=\sum_{j=1}^{m-1}\frac{j(m-j)\binom{i}{2}}{m-1}C^m_j$.

Let $\mu=(a_1,\cdots,a_k),\mu-i=(a_1,\cdots,a_k-i)$ for $1\leq i\leq a_k$ and consider the de Jonquieres' formula; that is, we have to compute the Chern classes $c_{k+1,\mu}:=c_{k+1}(\Lambda_mL|_{\Gamma_\mu})$ and $c_{k,\mu}:=c_k(\Lambda_mL|_{\Gamma_\mu})$ for a pencil case. Indeed, the formula is $c_{k+1,\mu}-c_{k,\mu}x$, where $x=(\pi^{[m]})^*c_1(E)|_{\Gamma_\mu}$.

Note that we have

$$\begin{aligned}c_{k+1,\mu}-c_{k+1,\mu-1}=&c_{k,\mu-1}(L_m-\Delta^{(m)})\\c_{k,\mu-1}=&\tau_p(c_{k,\mu-2})+c_{k-1,\mu-2}(L_{m-1}-\Delta^{(m-1)}).\end{aligned}$$

By integrating the second, we have

$$\begin{aligned}c_{k,\mu-1}=&c_{k-1,\mu-2}(L_{m-1}-\Delta^{(m-1)})+\cdots+(\tau_p)^i(c_{k-1,\mu-(i+2)}(L_{m-(i+1)}-\Delta^{(m-(i+1))}))\\&+\cdots+(\tau_p)^{a_k-1}(c_{k,\mu-(a_k)}),\end{aligned}$$

where $(\tau_p)^i := (\tau_{m,p})(\tau_{m-1,p})\cdots(\tau_{a_k+1,p})$ and $\tau_{j,p}$ is the j-th punctual transfer from $(a_1, \cdots, a_{k-1}, j-1)$ to $(a_1, \cdots, a_{k-1}, j)$. Now

$$c_{k,\mu} = \tau_{m,p}(c_{k,\mu-1}) + c_{k-1,\mu-1}(L_m - \Delta^{(m)}).$$

Inductively, $c_{k,\mu-1}$ and $c_{k-1,\mu-1}$ are given, hence we can derive the de Jonquieres' formula, $c_{k+1,\mu} - c_{k,\mu}x$. Note that for a single block this is a polynomial in $L^2, L\omega, \omega^2$, and σ the number of nodes. For two blocks this is a polynomial in $L^3, L^2\omega, L\omega^2, \omega^3$, and σ.

Example 51 *We compute* $c_{2,m} := c_2(\Lambda_m L|_{\Gamma_{(m)}})$.

By Splitting principle, we have

$$\begin{aligned}
&c_{2,m} - c_{2,m-1}\\
=&((m-1)L - \Gamma^{(m-1)})(L + \Gamma^{(m-1)} - \Gamma^{(m)})\\
=&(m-1)L^2 + (m-1)L(\Gamma^{(m-1)} - \Gamma^{(m)}) - \Gamma^{(m-1)}L - \Gamma^{(m-1)}(\Gamma^{(m-1)} - \Gamma^{(m)})\\
=&(m-1)L^2 + (m-1)L(\tau_{m,p}(-\binom{m-1}{2}\omega + \sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2}C_i^{m-1})\\
&+\binom{m}{2}\omega - \sum_{i=1}^{m-1}\frac{i(m-i)m}{2}C_i^m) + \binom{m-1}{2}L\omega + (\binom{m-1}{2}\omega\\
&-\sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2}C_i^{m-1})(\Gamma^{(m-1)} - \Gamma^{(m)})\\
=&(m-1)L^2 + (m-1)L(-\binom{m-1}{2}\omega + \binom{m}{2}\omega) + \binom{m-1}{2}L\omega\\
&+(\binom{m-1}{2}\omega)(\Gamma^{(m-1)}\\
&-\Gamma^{(m)}) - (\sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2}C_i^{m-1})(\Gamma^{(m-1)} - \Gamma^{(m)})\\
=&(m-1)L^2 + \frac{(m-1)(3m-4)}{2}L\omega + (m-1)\binom{m-1}{2}\omega^2\\
&+\sigma(\sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2} - \sum_{i=1}^{m-1}\frac{i(m-i)(m-2)}{2})\\
=&(m-1)L^2 + \frac{(m-1)(3m-4)}{2}L\omega + (m-1)\binom{m-1}{2}\omega^2 - \binom{m}{3}\sigma.
\end{aligned}$$

NB.

$$\tau_{m,p}(\sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2}C_i^{m-1})$$
$$=\sum_{i=1}^{m-2}\frac{i(m-1-i)(m-1)}{2}(\frac{m-i}{m-1}C_i^m+\frac{i+1}{m-1}C_{i+1}^m)$$
$$=\sum_{i=1}^{m-2}\frac{i(m-i)(m-2)}{2}C_i^m+\frac{(m-1)(m-2)}{2}C_{m-1}^m.$$

Now integrating this, we have

$$c_{2,m}=\binom{m}{2}L^2+(m-1)\binom{m}{2}L\omega+(3\binom{m+1}{4}-\binom{m}{3})\omega^2-\binom{m+1}{4}\sigma.$$

Example 52 *Hence for a single block, de Jonquieres' formula is*

$$c_2(\Lambda_m L-(\pi^{[m]})^*E)\cap[\Gamma_{(m)}]=c_2(\Lambda_m L|_{\Gamma_{(m)}})-c_1(\Lambda_m L|_{\Gamma_{(m)}})x$$
$$=\binom{m}{2}L^2+(m-1)\binom{m}{2}L\omega+(3\binom{m+1}{4}-\binom{m}{3})\omega^2-\binom{m+1}{4}\sigma$$
$$-(mL+\binom{m}{2}\omega-\sigma\sum_{i=1}^{m-1}\frac{i(m-i)m}{2}C_i^m)x,$$

where $x=(\pi^{[m]})^*c_1(E)\cap[\Gamma_{(m)}]$.

Example 53 *By Splitting principle,*

$c_2(\Lambda_m L|_{(a_1,a_2)})=\prod_{i=1}^{a_1}(1+L_i+\Gamma^{(i-1)}-\Gamma^{(i)})\prod_{j=a_1+1}^{m}(1+L_j+\Gamma^{(j-1)}-\Gamma^{(j)})$, *where* $a_1+a_2=m$. *Writing*

$$\prod_{i=1}^{a_1}((L_i+\Gamma^{(i-1)}-\Gamma^{(i)})t_1)\prod_{j=a_1+1}^{m}((L_j+\Gamma^{(j-1)}-\Gamma^{(j)})t_2), \qquad (*)$$

$c_2(\Lambda_m(L)|_{\Gamma_{(a_1,a_2)}})=[*]_{t_1^2}+[*]_{t_1t_2}+[*]_{t_2^2}$, *i.e., we have to compute the followings:*

1. $\sum_{i=1}^{a_1-1}(L_i+\Gamma^{(i-1)}-\Gamma^{(i)})((a_1-i)L_{a_1}+\Gamma^{(i)}-\Gamma^{(a_1)})$,
2. $\sum_{j=a_1+1}^{m-1}(L_j+\Gamma^{(j-1)}-\Gamma^{(j)})((m-j)L_m+\Gamma^{(j)}-\Gamma^{(m)})$,
3. $(a_1L_1-\Gamma^{(a_1)})(a_2L_2+\Gamma^{(a_1)}-\Gamma^{(m)})$.

Since the first sum is

$$\tau_{m,f}\cdots\tau_{a_1+1,f}(c_2(\Lambda_{a_1}L|_{\Gamma_{(a_1)}})),$$

we have

$$\begin{aligned}
&\tau_{m,f}\cdots\tau_{a_1+1,f}(c_2(\Lambda_{a_1}L|_{\Gamma_{(a_1)}}))\\
=&\binom{a_1}{2}L_1^2+(a_1-1)\binom{a_1}{2}L_1\omega_1+(3\binom{a_1+1}{4}-\binom{a_1}{3})\omega_1^2\\
&-\sigma\binom{a_1+1}{4}(-\Gamma^{(m)}F_i^{a_1,m}(\theta_1)+\sum_{k=0}^{a_2-1}\frac{\binom{m-i-k}{a_1}\binom{i+k}{k}}{\binom{a_2-1}{a_1}}C_{i+k}^m),
\end{aligned}$$

where i is any $1\le i\le a_1-1$.

Note that $\tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1}))=\sum_{k=0}^{a_2-1}\frac{\binom{m-i-k}{a_1}\binom{i+k}{k}}{\binom{a_2-1}{a_1}}C_{i+k}^m$ *by Lemma 48.*

For the second summand, note that

$$\text{(i)}\qquad L_2(\Gamma^{(j)}-\Gamma^{(m)})=-a_1(m-j)L\Gamma_{(m)}+(\binom{a_2}{2}-\binom{j-a_1}{2})L_2\omega_2$$

$$\text{(ii)}\qquad L_2(\Gamma^{(j-1)}-\Gamma^{(j)})=(j-a_1-1)L_2\omega_2-a_1L$$

$$\text{(iii)}\qquad\begin{aligned}
&(\Gamma^{(j-1)}-\Gamma^{(j)})\Gamma^{(j)}\\
=&((j-a_1-1)\omega_2-a_1\Gamma_{(j)}-\sum_{k=1}^{j-1-a_1}k(j-a_1-k)F_{k,a_1}^{j-a_1,j})\Gamma^{(j)}\\
=&-\binom{a_1}{2}(j-a_1-1)\omega_1\omega_2-\binom{j-a_1}{2}(j-a_1-1)\omega_2^2\\
&+a_1(2\binom{j-a_1}{2}+\binom{j}{2})\omega+\sigma((j-a_1-1)\sum_{k=1}^{a_1-1}\frac{k(a_1-k)a_1}{2}F_{k,j-a_1}^{a_1,j}[\omega_2]\\
&-a_1\sum_{k=1}^{j-1}\frac{k(j-k)j}{2}C_k^j-\sum_{k=1}^{j-1-a_1}k(j-a_1-k)\Gamma^{(j)}F_{k,a_1}^{j-a_1,j})
\end{aligned}$$

$$
\begin{aligned}
&(\Gamma^{(j-1)}-\Gamma^{(j)})\Gamma^{(m)}\\
&=((j-a_1-1)\omega_2-a_1\Gamma_{(j)}-\sum_{k=1}^{j-1-a_1}k(j-a_1-k)F_{k,a_1}^{j-a_1,j})\Gamma^{(m)}\\
&=-\binom{a_1}{2}(j-a_1-1)\omega_1\omega_2-\binom{a_2}{2}(j-a_1-1)\omega_2^2+a_1(a_2(j-a_1-1)\\
&\quad+\binom{m}{2})\omega+\sigma((j-a_1-1)\sum_{k=1}^{a_1-1}\frac{k(a_1-k)a_1}{2}F_{k,a_2}^{a_1,m}[\omega_2]\\
&\quad-a_1\sum_{k=1}^{m-1}\frac{k(m-k)m}{2}C_k^m-\sum_{k=1}^{a_2-1}k(a_2-k)\frac{j-1-a_1}{a_2-1}\Gamma^{(m)}F_{k,a_1}^{a_2,m}).
\end{aligned}
\tag{iv}
$$

Hence for $a_1+1\le j\le m-1$, we have

$$
\begin{aligned}
&(L_j+\Gamma^{(j-1)}-\Gamma^{(j)})((m-j)L_m+\Gamma^{(j)}-\Gamma^{(m)})\\
=&(m-j)L_2^2+L_2(\Gamma^{(j)}-\Gamma^{(m)})+(m-j)L_2(\Gamma^{(j-1)}-\Gamma^{(j)})+(\Gamma^{(j-1)}-\Gamma^{(j)})\Gamma^{(j)}\\
&-(\Gamma^{(j-1)}-\Gamma^{(j)})\Gamma^{(m)}\\
=&(m-j)L_2^2+(\binom{a_2}{2}-\binom{j-a_1}{2}-2\binom{j-a_1}{2}+a_2(j-a_1-1))L_2\omega_2\\
&+(j-a_1-1)(\binom{a_2}{2}-\binom{j-a_1}{2})\omega_2^2-2a_1(m-j)L+a_1(\binom{j}{2}-\binom{m}{2}+2\binom{j-a_1}{2}\\
&-a_2(j-a_1-1))\omega-\sigma(a_1\sum_{k=1}^{m-1}\frac{k(m-k)\binom{j}{2}}{m-1}C_k^m-a_1\sum_{k=1}^{m-1}\frac{k(m-k)m}{2}C_k^m\\
&+\sum_{k=1}^{j-1-a_1}k(j-a_1-k)\Gamma^{(m)}F_{k+m-j,a_1}^{a_2,m}-\sum_{k=1}^{a_2-1}k(a_2-k)\frac{j-1-a_1}{a_2-1}\Gamma^{(m)}F_{k,a_1}^{a_2,m}).
\end{aligned}
$$

By taking $\sum_{j=a_1+1}^{m-1}$, the second sum

$$
\begin{aligned}
=&\binom{a_2}{2}L_2^2+\binom{a_2}{2}(a_2-1)L_2\omega_2+(\binom{a_2}{2}\binom{a_2-1}{2}-3\binom{a_2+1}{4}+2\binom{a_2}{3})\omega_2^2\\
&-2a_1\binom{a_2}{2}L+a_1(\binom{m}{3}-\binom{a_1+1}{3}-\binom{m}{2}(a_2-1)-\binom{a_2}{3})\omega\\
&-\sigma(a_1\sum_{k=1}^{m-1}k(m-k)(\binom{m}{3}\\
&-\binom{a_1+1}{3}-\binom{m}{2}(a_2-1))C_k^m-\sum_{k=1}^{a_2-1}k\binom{a_2-k}{2}\Gamma^{(m)}F_{k,a_1}^{a_2,m}).
\end{aligned}
$$

Using

$$L_1(\Gamma^{(a_1)} - \Gamma^{(m)}) = \binom{a_2}{2} L_1\omega_2 - a_1a_2L - \sigma \sum_{i=1}^{a_2-1} \frac{i(a_2-i)a_2}{2} F_{i,a_1}^{a_2,m}[L_1]$$
$$\omega_1(\Gamma^{(a_1)} - \Gamma^{(m)}) = \binom{a_2}{2} \omega_1\omega_2 - a_1a_2\omega - \sigma \sum_{i=1}^{a_2-1} \frac{i(a_2-i)a_2}{2} F_{i,a_1}^{a_2,m}[\omega_1],$$

the last sum

$$\begin{aligned}
&(a_1L_1 + \binom{a_1}{2}\omega_1 - \sigma \sum_{i=1}^{a_1-1} \frac{i(a_1-i)a_1}{2} C_i^{a_1}(\theta_1))(a_2L_2 + \Gamma^{(a_1)} - \Gamma^{(m)}) \\
&= a_1a_2L_1L_2 + \binom{a_1}{2} a_2\omega_1L_2 + a_1\binom{a_2}{2} L_1\omega_2 - a_1^2a_2L\Gamma_{(m)} \\
&- \sigma a_1 \sum_{i=1}^{a_2-1} \frac{i(a_2-i)a_2}{2} F_{i,a_1}^{a_2,m}[L_1] + \binom{a_1}{2}\binom{a_2}{2}\omega_1\omega_2 - \binom{a_1}{2} a_1a_2\omega\Gamma_{(m)} \\
&- \sigma\binom{a_1}{2} \sum_{i=1}^{a_2-1} \frac{i(a_2-i)a_2}{2} F_{i,a_1}^{a_2,m}[\omega_1] \\
&- \sigma a_2 \sum_{i=1}^{a_1-1} \frac{i(a_1-i)a_1}{2} F_{i,a_2}^{a_1,m}[L_2] + \sigma\frac{a_1}{2}\binom{a_1+1}{3} \tau_{m,p}\cdots\tau_{a_1+2}(C_i^{a_1+1}),
\end{aligned}$$

since

$$\begin{aligned}
-\sigma \sum_{i=1}^{a_1-1} \frac{i(a_1-i)a_1}{2} C_i^{a_1}(\theta_1))(\Gamma^{(a_1)} - \Gamma^{(m)}) &= \sigma\tau_{m,p}\cdots\tau_{a_1+1,p}(\sum_{i=1}^{a_1-1} \frac{i(a_1-i)a_1}{2} C_i^{a_1}(\theta_1)) \\
&= \sigma\frac{a_1}{2}\binom{a_1+1}{3} \tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1}),
\end{aligned}$$

where $\tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1})) = \sum_{k=0}^{a_2-1}\frac{\binom{m-i-k}{a_1}\binom{i+k}{k}}{\binom{a_2-1}{a_1}}C_{i+k}^m$ *by Lemma 48. Hence we have*

$$
\begin{aligned}
&c_2(\Lambda_m L|_{(a_1,a_2)})\\
=&\binom{a_1}{2}L_1^2+\binom{a_2}{2}L_2^2+a_1a_2L_1L_2+(a_1-1)\binom{a_1}{2}L_1\omega_1+\binom{a_2}{2}(a_2-1)L_2\omega_2\\
&+\binom{a_1}{2}a_2\omega_1L_2+a_1\binom{a_2}{2}L_1\omega_2+(3\binom{a_1+1}{4}-\binom{a_1}{3})\omega_1^2+\binom{a_1}{2}\binom{a_2}{2}\omega_1\omega_2\\
&+(3\binom{a_2+1}{4}-\binom{a_2}{3})\omega_2^2-a_1a_2(m-1)L\\
&+a_1(\binom{m}{3}-\binom{a_1+1}{3}-\binom{m}{2}(a_2-1)-\binom{a_2}{3}-\binom{a_1}{2}a_2)\omega\\
&-\sigma(a_1\sum_{k=1}^{m-1}k(m-k)(\binom{m}{3}-\binom{a_1+1}{3}-\binom{m}{2}(a_2-1))C_k^m\\
&-\sum_{k=1}^{a_2-1}k\binom{a_2-k}{2}\Gamma^{(m)}F_{k,a_1}^{a_2,m}+a_1\sum_{i=1}^{a_2-1}\frac{i(a_2-i)a_2}{2}F_{i,a_1}^{a_2,m}[L_1]\\
&+a_2\sum_{i=1}^{a_1-1}\frac{i(a_1-i)a_1}{2}F_{i,a_2}^{a_1,m}[L_2]+\binom{a_1}{2}\sum_{i=1}^{a_2-1}\frac{i(a_2-i)a_2}{2}F_{i,a_1}^{a_2,m}[\omega_1]\\
&+\binom{a_1+1}{4}(-\Gamma^{(m)}F_{i,a_2}^{a_1,m})-\binom{a_1+2}{4}\sum_{k=0}^{a_2-1}\frac{\binom{m-i-k}{a_1}\binom{i+k}{k}}{\binom{a_2-1}{a_1}}C_{i+k}^m).
\end{aligned}
$$

Example 54 *Let's compute* $c_3(\Lambda_m L|_{(a_1,a_2)})$*, where* $a_1+a_2=m$*. Note that*

$$c_{3,\mu}-c_{3,\mu-1}=c_{2,\mu-1}(L_2+\Gamma^{(m-1)}-\Gamma^{(m)}),$$

where $c_{2,\mu-1}$ *is given above example.*

$$
\begin{aligned}
c_{2,\mu-1}L_2=&\binom{a_1}{2}L_1^2L_2+a_1(a_2-1)L_1L_2^2+(a_1-1)\binom{a_1}{2}L_1L_2\omega_1+\binom{a_1}{2}(a_2-1)\omega_1L_2^2\\
&+a_1\binom{a_2-1}{2}L_1L_2\omega_2+(3\binom{a_1+1}{4}-\binom{a_1}{3})\omega_1^2L_2+\binom{a_1}{2}\binom{a_2-1}{2}\omega_1\omega_2L_2\\
&-a_1(a_2-1)(m-2)L^2+a_1(\binom{m-1}{3}\\
&-\binom{a_1+1}{3}-\binom{m-1}{2}(a_2-2)-\binom{a_2-1}{3}-\binom{a_1}{2}(a_2-1))L\omega.
\end{aligned}
$$

Note that $(\Gamma^{(m-1)} - \Gamma^{(m)}) = (a_2-1)\omega_2 - a_1\Gamma_{(m)} - \sum_{j=1}^{a_2-1} j(a_2-j)F^{a_2,m}_{j,a_1}$.

$$
\begin{aligned}
&c_{2,\mu-1}(\Gamma^{(m-1)} - \Gamma^{(m)})\\
=&\binom{a_1}{2}(a_2-1)L_1^2\omega_2 + a_1(a_2-1)^2L_1L_2\omega_2 + (a_1-1)\binom{a_1}{2}(a_2-1)L_1\omega_1\omega_2\\
&+\binom{a_1}{2}(a_2-1)^2\omega_1\omega_2L_2 + a_1(a_2-1)\binom{a_2-1}{2}L_1\omega_2^2 + (a_2-1)(3\binom{a_1+1}{4}\\
&-\binom{a_1}{3})\omega_1^2\omega_2 + \binom{a_1}{2}\binom{a_2-1}{2}(a_2-1)\omega_1\omega_2^2 - a_1(a_2-1)(m-1)(m-2)L\omega\\
&+a_1(m-1)(\binom{m-1}{3} - \binom{a_1+1}{3} - \binom{m-1}{2}(a_2-2) - \binom{a_2-1}{3}\\
&-\binom{a_1}{2}(a_2-1))\omega^2 - a_1(\binom{a_1}{2} + \binom{a_2-1}{2} + a_1(a_2-1))L^2\\
&-a_1((a_1-1)\binom{a_1}{2} + \binom{a_2-1}{2}(a_2-2) + \binom{a_1}{2}(a_2-1) + a_1\binom{a_2-1}{2})L\omega\\
&-a_1(3\binom{a_1+1}{4} - \binom{a_1}{3} + \binom{a_1}{2}\binom{a_2-1}{2} + 3\binom{a_2}{4} - \binom{a_2-1}{3})\omega^2\\
&-\sigma(a_1\frac{(m-2)(m-3)}{3}(\binom{m-1}{3} - \binom{a_1+1}{3} - \binom{m-1}{2}(a_2-2))\\
&-a_1\binom{a_2}{3}\Gamma^{(m)}F^{a_2,m}_{i,a_1}[L_1] - \binom{a_1}{2}\binom{a_2}{3}\Gamma^{(m)}F^{a_2,m}_{i,a_1}[\omega_1]\\
&+\binom{a_1+2}{4}\frac{2}{m-1}\Gamma^{(m-1)}\tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1})),
\end{aligned}
$$

where note that $(\tau_{m-1,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1}))(\Gamma^{(m-1)} - \Gamma^{(m)}) = \frac{2}{m-1}\Gamma^{(m)}\tau_{m,p}\cdots\tau_{a_1+2}(C_i^{a_1+1})$.

Now we have

$$
\begin{aligned}
&c_{3,\mu} - c_{3,\mu-1} = c_{2,\mu-1}(L_2 + \Gamma^{(m-1)} - \Gamma^{(m)}) \\
=&\binom{a_1}{2}L_1^2L_2 + a_1(a_2-1)L_1L_2^2 + \binom{a_1}{2}(a_2-1)L_1^2\omega_2 + \binom{a_1}{2}(a_2-1)\omega_1L_2^2 \\
&+ (a_1-1)\binom{a_1}{2}L_1L_2\omega_1 + a_1(\binom{a_2-1}{2} + (a_2-1)^2)L_1L_2\omega_2 \\
&+ a_1(a_2-1)\binom{a_2-1}{2}L_1\omega_2^2 \\
&+ (3\binom{a_1+1}{4} - \binom{a_1}{3})\omega_1^2L_2 + \binom{a_1}{2}(\binom{a_2-1}{2} + (a_2-1)^2)\omega_1\omega_2L_2 \\
&+ (a_1-1)\binom{a_1}{2}(a_2-1)L_1\omega_1\omega_2 + (a_2-1)(3\binom{a_1+1}{4} - \binom{a_1}{3})\omega_1^2\omega_2 \\
&+ \binom{a_1}{2}\binom{a_2-1}{2}(a_2-1)\omega_1\omega_2^2 - a_1(2a_1(a_2-1) + \binom{a_1}{2} + 3\binom{a_2-1}{2})L^2 \\
&+ a_1(\binom{m}{3} - \binom{m-1}{2}(a_2-1) - 4\binom{a_1+1}{3} - 2\binom{a_1}{2}(a_2-2) - \binom{a_2}{3} \\
&- \binom{a_2-1}{2}(m-3))L\omega + a_1(m-1)(\binom{m-1}{3} - \binom{a_1+1}{3} - \binom{m-1}{2}(a_2-2) \\
&- \binom{a_2-1}{3} - \binom{a_1}{2}(a_2-1))\omega^2 - a_1(3\binom{a_1+1}{4} - \binom{a_1}{3} + \binom{a_1}{2}\binom{a_2-1}{2} \\
&+ 3\binom{a_2}{4} - \binom{a_2-1}{3})\omega^2 - \sigma(a_1\frac{(m-2)(m-3)}{3}(\binom{m-1}{3} - \binom{a_1+1}{3} \\
&- \binom{m-1}{2}(a_2-2)) - a_1\binom{a_2}{3}\Gamma^{(m)}F_{i,a_1}^{a_2,m}[L_1] \\
&- \binom{a_1}{2}\binom{a_2}{3}\Gamma^{(m)}F_{i,a_1}^{a_2,m}[\omega_1] \\
&+ \binom{a_1+2}{4}\frac{2}{m-1}\Gamma^{(m)}\tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1})).
\end{aligned}
$$

Integrating this, we have

$$
\begin{aligned}
& c_{3,\mu} \\
=& \binom{a_1}{2} a_2 L_1^2 L_2 + a_1 \binom{a_2}{2} L_1 L_2^2 + \binom{a_1}{2}\binom{a_2}{2} L_1^2\omega_2 + \binom{a_1}{2}\binom{a_2}{2}\omega_1 L_2^2 \\
&+ (a_1-1)a_2\binom{a_1}{2} L_1L_2\omega_1 + (a_2-1)a_1\binom{a_2}{2}L_1L_2\omega_2 + a_1(3\binom{a_2+1}{4} - \binom{a_2}{3})L_1\omega_2^2 \\
&+ (3\binom{a_1+1}{4}a_2 - \binom{a_1}{3}a_2)\omega_1^2L_2 + (a_2-1)\binom{a_1}{2}\binom{a_2}{2}\omega_1\omega_2L_2 \\
&+ (a_1-1)\binom{a_1}{2}\binom{a_2}{2}L_1\omega_1\omega_2 + \binom{a_2}{2}(3\binom{a_1+1}{4} - \binom{a_1}{3})\omega_1^2\omega_2 \\
&+ \binom{a_1}{2}(3\binom{a_2+1}{4} - \binom{a_2}{3})\omega_1\omega_2^2 - a_1a_2\binom{m-1}{2}L^2 \\
&+ a_1(-2\binom{m+1}{4} - \binom{a_1+1}{4} + (a_1+1)\binom{m}{3} - 4\binom{a_1+1}{3}a_2 - 2\binom{a_1}{2}\binom{a_2-1}{2} \\
&- \binom{a_2}{2}(m-2))L\omega + a_1(4\binom{m+1}{5} - 4\binom{a_1+1}{5} - \binom{m}{4} + \binom{a_1}{4} - \binom{m}{2}\binom{a_1+1}{3} \\
&+ \binom{a_1}{2}\binom{a_1+1}{3} - 12\binom{m+2}{5} + 12\binom{a_1+2}{5} + 12\binom{m+1}{4} - 12\binom{a_1+1}{4} \\
&- \binom{m}{3}(a_1+2) + \binom{a_1}{3}(a_1+2) - (a_1-1)\binom{a_2}{4} - 4\binom{a_2+1}{5} - 2\binom{a_1}{2}\binom{a_2}{3} \\
&- 3\binom{a_1+1}{3}\binom{a_2}{2})\omega^2 \\
&- a_1(3\binom{a_1+1}{4}a_2 - \binom{a_1}{3}a_2 + \binom{a_1}{2}\binom{a_2}{3} + 3\binom{a_2+1}{5} - \binom{a_2}{4})\omega^2 \\
&- \sigma(\frac{a_1}{3}(20\binom{m+2}{6} - 20\binom{a_1+2}{6} - 12\binom{m+1}{5} + 12\binom{a_1+1}{5} - 2\binom{m-1}{3}\binom{m+1}{3} \\
&+ 2\binom{a_1-1}{3}\binom{m+1}{3} - 60\binom{m+2}{6} + 60\binom{a_1+2}{6} + 12\binom{m+1}{5}(a_1+1) \\
&- 12\binom{a_1+1}{5}(a_1+1) + 24\binom{m+1}{5} - 24\binom{a_1+1}{5} - 6\binom{m}{4}(a_1+1) + 6\binom{a_1}{4}(a_1+1) \\
&+ \sigma(a_1\binom{a_2+1}{4}\Gamma^{(m)}F_{i,a_1}^{a_2,m}[L_1] + \binom{a_1}{2}\binom{a_2+1}{4}\Gamma^{(m)}F_{i,a_1}^{a_2,m}[\omega_1]) \\
&- \sigma\binom{a_1+2}{4}\sum_{i=a_1}^{m-1}\frac{2}{i}\Gamma^{(m)}\tau_{m,p}\cdots\tau_{a_1+2,p}(C_i^{a_1+1}).
\end{aligned}
$$

With examples 53 and 54 we have the de Jonquieres' formula on $\Gamma_{(a_1,a_2)}$, where $a_1+a_2=m$, i.e. $c_{3,(a_1,a_2)} - c_{2,(a_1,a_2)}x$, where $x = \pi^{[m]*}c_1(E)\cap[\Gamma_{(a_1,a_2)}]$.

Example 55 *Inductively we compute the total Chern class $c(\Lambda_m(L)|_{\Gamma_{(n_1,n_2)}})$ for pencil,* *where* $n_1 + n_2 = m$

Note that $T^m_{(n_1,n_2)}$ *is generated by*

1. $\Gamma_{(n_1,n_2)}, \Gamma_{(m)}$,

2. $F_j^{n_1,m}, F_j^{n_2,m}$, *and* C_j^m,

3. $-\Gamma^{(m)}F_j^{n_1,m}, -\Gamma^{(m)}F_j^{n_2,m}$, *and* $-\Gamma^{(m)}C_j^m = Q_j^m$.

Assume that $n_2 \geq 1$, *hence* $m \geq n_1 + 1$.

Now write recursively

$$\begin{aligned}
&c_{(n_1,n_2)} := c(\Lambda_m(L)|_{\Gamma_{(n_1,n_2)}}) \\
=&a_{n_2}\Gamma_{(n_1,n_2)} + b_m\Gamma_{(m)} + \sum_\theta(\sum_{j=1}^{n_1-1} c_m^j F_j^{n_1,m} + \sum_{k=1}^{n_2-1} d_{n_2}^k F_k^{n_2,m} + \sum_{l=1}^{m-1} e_m^l C_l^m \\
&+ \sum_{j=1}^{n_1-1} f_m^j(-\Gamma^{(m)}F_j^{n_1,m}) + \sum_{k=1}^{n_2-1} g_{n_2}^k(-\Gamma^{(m)}F_k^{n_2,m})) + A_m.
\end{aligned}$$

By splitting principle we have

$$c_{(n_1,n_2)} = \tau_{m,p}(c_{(n_1,n_2-1)}(1 + L_2 + \Gamma^{(m-1)})) + (-\Gamma^{(m)})\tau_{m,p}(c_{(n_1,n_2-1)}).$$

$$\begin{aligned}
&c_{(n_1,n_2-1)}L_2 \\
=&a_{n_2-1}L_2 + b_{m-1}L_2 + \sum_\theta(\sum_{j=1}^{n_1-1} c_{m-1}^j F_j^{n_1,m-1}[L_2] + \sum_{j=1}^{n_1-1} f_{m-1}^j(-\Gamma^{(m-1)}F_j^{n_1,m-1}[L_2]))
\end{aligned}$$

$$\begin{aligned}
&c_{(n_1,n_2-1)}\Gamma^{(m-1)} \\
=&a_{n_2-1}(n_1(n_2-1)\Gamma_{(m-1)} - \binom{n_1}{2}\omega_1 - \binom{n_2-1}{2}\omega_2 + \sum_{j=1}^{n_1-1}\frac{j(n_1-j)n_1}{2}F_j^{n_1,m-1} \\
&+ \sum_{k=1}^{n_2-2}\frac{k(n_2-1-k)(n_2-1)}{2}F_k^{n_2-1,m-1}) + b_{m-1}(\sum_{l=1}^{m-2}\frac{l(m-1-l)(m-1)}{2}C_l^{m-1} \\
&- \binom{m-1}{2}\omega\Gamma_{(m-1)}) + \sum_{k=1}^{n_2-2}\frac{k(n_2-1-k)(n_2-1)}{2}F_k^{n_2-1,m-1}) \\
&+ \sum_\theta(\sum_{j=1}^{n_1-1} c_{m-1}^j\Gamma^{(m-1)}F_j^{n_1,m} + \sum_{k=1}^{n_2-2} d_{n_2-1}^k\Gamma^{(m-1)}F_k^{n_2-1,m-1} - \sum_{l=1}^{m-2} e_{m-1}^l \\
&+ \sum_{j=1}^{n_1-1} f_{m-1}^j(-\Gamma^{(m-1)^2}F_j^{n_1,m-1}) + \sum_{k=1}^{n_2-2} g_{n_2-1}^k(-\Gamma^{(m-1)^2}F_k^{n_2-1,m-1})).
\end{aligned}$$

By transfer theorems, the first summand is

$$
\begin{aligned}
& a_{n_2-1}(1+L_2-\binom{n_1}{2}\omega_1-\binom{n_2-1}{2}\omega_2)\Gamma_{(n_1,n_2)}+(b_{m-1}(1+L-\binom{m-1}{2}\omega) \\
& + a_{n_2-1}n_1(n_2-1))\Gamma_{(m)}+\sum_\theta(\sum_{j=1}^{n_1-1}(c^j_{m-1}(1-L_2)+a_{n_2-1}\frac{j(n_1-j)n_1}{2})F^{n_1,m}_j \\
& + \sum_{k=1}^{n_2-1}(d^k_{n_2-1}\frac{n_2-k}{n_2-1}+d^{k-1}_{n_2-1}\frac{k}{n_2-1} \\
& + \frac{k(n_2-k)(n_2-2)}{2}a_{n_2-1})F^{n_2,m}_k+\sum_{l=1}^{m-2}(e^l_{m-1}\frac{m-l}{m-1}+e^{l-1}_{m-1}\frac{l}{m-1})C^m_l \\
& + \sum_{j=1}^{n_1-1}(f^j_{m-1}+c^{m-1}_j)(-\Gamma^{(m)}F^{n_1,m}_j)+\sum_{k=1}^{n_2-2}(g^k_{n_2-1}+d^k_{n_2-1})(-\Gamma^{(m)}F^{n_2,m}_{k+1})+A_{m-1} \\
& + \sum_{j=1}^{n_1-1}f^j_{m-1}(-\Gamma^{(m-1)}F^{n_1,m-1}_j[L_2])-\sum_{l=1}^{m-2}e^l_{m-1}+\sum_{j=1}^{n_1-1}f^j_{m-1}(-\Gamma^{(m-1)^2}F^{n_1,m-1}_j) \\
& + \sum_{k=1}^{n_2-2}g^k_{n_2-1}(-\Gamma^{(m-1)^2}F^{n_2-1,m-1}_k)).
\end{aligned}
$$

By transfer theorem the second summand is

$$
\begin{aligned}
& a_{n_2-1}\Gamma_{(n_1,n_2)}+(b_{m-1}\Gamma_{(m)}+\sum_\theta(\sum_{j=1}^{n_1-1}c^j_{m-1}F^{n_1,m}_j \\
& + \sum_{k=1}^{n_2-1}(d^k_{n_2-1}\frac{n_2-k}{n_2-1}+d^{k-1}_{n_2-1}\frac{k}{n_2-1})F^{n_2,m}_k \\
& + \sum_{l=1}^{m-2}(e^l_{m-1}\frac{m-l}{m-1}+e^{l-1}_{m-1}\frac{l}{m-1})C^m_l+\sum_{j=1}^{n_1-1}f^j_{m-1}(-\Gamma^{(m)}F^{n_1,m}_j) \\
& + \sum_{k=1}^{n_2-2}g^k_{n_2-1}(-\Gamma^{(m)}F^{n_2,m}_{k+1})+A_{m-1}.
\end{aligned}
$$

So

$$
\begin{aligned}
&-\Gamma^{(m)}\tau^0_{m,p}(c_{n_1,n_2-1})\\
&= a_{n_2-1}(\binom{n_1}{2}\omega_1 + \binom{n_2}{2}\omega_2 - n_1n_2\Gamma_{(m)} - \sum_\theta(\sum_{j=1}^{n_1-1}\frac{j(n_1-j)n_1}{2}F_j^{n_1,m}\\
&- \sum_{k=1}^{n_2-1}\frac{k(n_2-k)n_2}{2}F_k^{n_2,m}))\\
&+ b_{m-1}(\binom{m}{2}\omega - \sum_{\theta,l=1}^{m-1}\frac{l(m-l)m}{2}C_l^m)\\
&+ \sum_\theta(\sum_{j=1}^{n_1-1}c^j_{m-1}(-\Gamma^{(m)}F_j^{n_1,m}) + \sum_{k=1}^{n_2-1}(d^k_{n_2-1}\frac{n_2-k}{n_2-1} + d^{k-1}_{n_2-1}\frac{k}{n_2-1})(-\Gamma^{(m)}F_k^{n_2,m})\\
&+ \sum_{l=1}^{m-2}e^l_{m-1}\frac{m+1}{m-1} + e^{m-2}_{m-1} + \sum_{j=1}^{n_1-1}f^j_{m-1}(\Gamma^{(m)2}F_j^{n_1,m}) + \sum_{k=1}^{n_2-2}g^k_{n_2-1}(\Gamma^{(m)2}F_{k+1}^{n_2,m})).
\end{aligned}
$$

Hence, we have

$$a_{n_2} = a_{n_2-1}(1 + L_2 + (n_2-1)\omega_2)$$

$$b_m = b_{m-1}(1 + L + (m-1)\omega) - n_1a_{n_2-1}$$

$$c^j_m = c^j_{m-1}(1 - L_2)$$

$$d^k_{n_2} = d^k_{n_2-1}\frac{n_2-k}{n_2-1} + d^{k-1}_{n_2-1}\frac{k}{n_2-1} - k(n_2-k)a_{n_2-1}$$

$$e^l_m = e^l_{m-1}\frac{m-l}{m-1} + e^{l-1}_{m-1}\frac{l}{m-1} - \frac{l(m-l)m}{2}b_{m-1}$$

$$f^j_m = f^j_{m-1}$$

$$g^k_{n_2} = g^{k-1}_{n_2-1} + \frac{n_2-k}{n_2-1}d^k_{n_2-1} + \frac{n_2-1-k}{n_2-1}d^{k-1}_{n_2-1}$$

$$A_m = A_{m-1} + \sum_{j=1}^{n_1-1}f^j_{m-1}(-\Gamma^{(m-1)}F_j^{n_1,m-1}[L_2]) - \sum_{l=1}^{m-2}e^l_{m-1} + \sum_{l=1}^{m-2}e^l_{m-1}\frac{m+1}{m-1} + e^{m-2}_{m-1}.$$

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