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ABSTRACT

It is shown that form factors with the asymptotic behavior $F(t) \sim \exp(-|t|^{\frac{1}{2}})$ are a consequence of the nonlinear bound state equation for the wave function in the crossing-symmetric bootstrap model. This non-linear equation incorporates the notion that the constituent particles are, themselves, composite. The result is obtained both from the Schroedinger equation and the Bethe-Salpeter equation; in the latter case, it is argued that the result should be valid for a wide class of interaction kernels.

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I. INTRODUCTION

An increasing amount of theoretical speculation[1] and experimental evidence[2] suggests the possibility that the form factors of the strongly interacting particles are decreasing exponentially in the momentum transfer variable $|t|^{\frac{1}{2}}$. Here we show that a particular dynamical model, the crossing symmetric bootstrap, may provide the explanation for this behavior.

The crossing symmetry condition applied to the bound state equation in a bootstrap model results in a nonlinear integral equation. While this equation appears too difficult to solve numerically or analytically, we can study the self-consistency requirement imposed by this equation on the asymptotic form of the wave-function in momentum space. Our result, which we obtain in a variety of Schroedinger and Bethe-Salpeter equation models, is that form factors with an asymptotic behavior $F(t) \sim \exp(-|t|^{\frac{1}{2}})$ are the only possible solutions of the bootstrap equations.

A consequence of the nonlinearity of the equations which we study is that the constituent particles enter into the dynamical equations in a manner consistent with the requirement that they are themselves composite particles. It is then, perhaps, no surprise that we obtain our result since a similar result was obtained recently by Stack[1] in a different model but one which also incorporated this requirement on the constituent particles. Stack considers an infinite set of linear Schroedinger equations for the nucleon bound in the nucleon-pion channel; the n^{th} equation describes the binding of the $n\text{-}1^{st}$ pion to a "nucleon" which consists of a bound state of a nucleon core and n-2 pions. He solves the n^{th} equation in the limit $n \rightarrow \infty$ under the assumption that the wave function and the form of the potential between the

n^{th} pion and the remaining n-1 body system is independent of n. This latter assumption seems to us to be difficult to support, however, particularly because one expects that as n increases and the constituent particle becomes more and more composite then the form factor of that particle, and hence the potential which acts on it, should fall off asymptotically with an increasing power behavior in momentum space. One virtue of our more general nonlinear analysis is that this self-consistency between the potential and the wave function is automatically incorporated.

In order to illustrate our model, we discuss first a scalar meson theory with a $\lambda\varphi^3$ interaction. Let us consider the scattering amplitude for meson-meson scattering with incoming momenta p_1, p_2 and outgoing momenta q_1, q_2, and assume that there exists a composite scalar meson in the two-meson channel. We introduce a center-of-momentum (c.m.) variable, P, and relative momentum variables p,q, defined by

$$P = p_1 + p_2 = q_1 + q_2 \tag{1.1}$$

and

$$p = \frac{p_1 - p_2}{2} , \; q = \frac{q_1 - q_2}{2} \quad . \tag{1.2}$$

Then at the scalar meson pole in the direct channel ($P^2 = M^2$) the Bethe-Salpeter equation[3] in ladder approximation reads

$$G_0^{-1}(P,p)\chi(P,p) = \frac{\lambda^2}{(2\pi)^4} \int \frac{d^4k \; \chi(P,k)}{(p-k)^2 - \mu^2} \tag{1.3}$$

and is illustrated graphically in Fig. 1. Here χ is the Bethe-Salpeter wave function, G_0 is the product of the constituent particle propagators, λ is a coupling constant, and μ is the constituent meson mass. The bootstrap conditions can now be imposed by requiring that[4] $M^2 = \mu^2$ and that

$\lambda = (2\pi)^2 G_0^{-1}(P,p)\chi(P,p)|_{\text{on shell}}$.

Our model consists in the replacement of the coupling constant λ by a form factor which is determined from the wave function, thus yielding a nonlinear equation. This equation (which has been discussed most intensively by Cutkosky and co-workers[5]) reads

$$G_0^{-1}(P,p)\chi(P,p) = \int d^4k\chi(P,k)G_0^{-1}(P,k)$$

$$\times \ \chi(\tfrac{P}{2} -p,\tfrac{P}{2} +p-2k)[(p-k)^2-\mu^2]\chi(\tfrac{P}{2} +p,\tfrac{P}{2} -p+2k) \qquad (1.4)$$

and is illustrated in Fig. 2. The crossing symmetry is apparent from the diagram. Several variations of this equation will be studied in this paper.

It is instructive to see the connection between this nonlinear equation and the requirement that the constituent particles be composite. In Fig. 3 we show the first iteration of Eq. (1.4). The essential feature of Fig. 3 is that the exchanged particle in the kernel of the iterated integral equation couples to the constituent particles with a composite particle vertex function. Successive iterations of this equation generate "cobweb" graphs which describe a relativistic generalization of the notion of "infinitely composite particle" introduced by Stack.

A realistic bootstrap is complicated, of course, by the presence of many two-particle channels and by the possibility of many different particles which can be exchanged in the t and u channels. However, if all particles have form factors with a common asymptotic behavior, then our arguments which lead to an $\exp(-|t|^{\frac{1}{2}})$ behavior should still be valid independent of this complication.

We first study in Section II a simple non-relativistic model which incorporates some of the essential features of the more general equation discussed above and demonstrate that the asymptotic form factor $F(\underset{\sim}{p}) \sim \exp(-|\underset{\sim}{p}|)$ is a consequence of the bound state equation. Then, in Section III, we turn to a completely relativistic Bethe-Salpeter equation which contains the simplification that one constituent particle is taken to be elementary. Equation (1.4) is discussed in Section IV, where we argue that our result is still plausible in a model with no elementary particles. In Section V we discuss the effect of higher order interaction kernels in the Bethe-Salpeter model of Section III and demonstrate that for a wide class of kernels, our result is still valid. Finally, we conclude in Section VI by pointing out the difficulty in obtaining our result from dispersion relations.

II. NON-RELATIVISTIC "BOOTSTRAP"

We discuss in this section a Schroedinger equation which incorporates the nonlinearity of the Bethe-Salpeter equation described in the introduction but which is considerably more transparent both physically and mathematically. In fact, our model is reminiscent of the Hartree-Fock approximation. Let us suppose that we have a spinless, composite particle with mass m and a "charge" density $\rho(r)$ given by

$$\rho(\underset{\sim}{r}) = \psi^*(\underset{\sim}{r})\psi(\underset{\sim}{r}) \quad . \tag{2.1}$$

We wish to calculate the wave function of a bound state of this particle in a point-source Yukawa potential which couples to the "charge" of the particle. We require, however, that the bound state is spinless, has mass m, and, most important, has a "charge" distribution which is identical to that of the constituent particle. For our potential we therefore take

$$V(\underset{\sim}{r}) = \lambda\int d^3r'\rho(\underset{\sim}{r}-\underset{\sim}{r}')\,\frac{e^{-mr'}}{r'} \tag{2.2}$$

and the Schroedinger equation becomes

$$(\frac{p^2}{m} - E)\psi(\underset{\sim}{p}) = \int d^3k\psi(\underset{\sim}{k})V(\underset{\sim}{k}-\underset{\sim}{p}) = \lambda\int \frac{d^3k\psi(\underset{\sim}{k})}{(\underset{\sim}{p}-\underset{\sim}{k})^2+m^2}\int d^3q\psi^*(\underset{\sim}{q})\psi(\underset{\sim}{k}-\underset{\sim}{p}-\underset{\sim}{q}) . \qquad (2.3)$$

We will usually neglect factors of $(2\pi)^3$ etc. in this paper as they are irrelevant to our results. It is convenient to make the change of variable $\underset{\sim}{k}-\underset{\sim}{p} \rightarrow \underset{\sim}{k}$ on the r.h.s. of Eq. (2.3) which simplifies this equation to

$$(\frac{p^2}{m} - E)\psi(\underset{\sim}{p}) = \lambda\int \frac{d^3k\psi(\underset{\sim}{k}+\underset{\sim}{p})}{k^2+m^2}\int d^3q\psi^*(\underset{\sim}{q})\psi(\underset{\sim}{k}-\underset{\sim}{q}) . \qquad (2.4)$$

Our object now is to show that the self-consistent asymptotic behavior for the wave-function $\psi(\underset{\sim}{p})$ is given by

$$\psi(\underset{\sim}{p}) \sim e^{-a\sqrt{p^2}} . \qquad (2.5)$$

This will imply that the form factor, which is the Fourier transform of the "charge" density, has a similar asymptotic behavior. The procedure will be to substitute various asymptotic forms into the r.h.s. of Eq. (2.4) and then to integrate to determine whether the asymptotic behavior of the l.h.s. is reproduced. We shall refer to wave functions which do have this property as self-consistent wave functions. The question might now be raised as to whether the asymptotic behavior of the integrand determines the asymptotic behavior of the integrals in Eq. (2.4). We shall show in the course of our analysis that indeed it does.

Consider, first, the asymptotic behavior

$$\psi(\underset{\sim}{p}) \sim e^{-ap^2} . \qquad (2.6)$$

Substituting into Eq. (2.4) we obtain

$$p^2e^{-ap^2} \sim \int \frac{d^3ke^{-a(\underset{\sim}{k}+\underset{\sim}{p})^2}}{k^2+m^2}\int d^3qe^{-aq^2}e^{-a(\underset{\sim}{k}-\underset{\sim}{q})^2} . \qquad (2.7)$$

We denote

$$F(\underset{\sim}{k}) = \int d^3q e^{-aq^2} e^{-a(\underset{\sim}{k}-\underset{\sim}{q})^2} \tag{2.8}$$

and write

$$F(\underset{\sim}{r}) = f^2(\underset{\sim}{r}) \tag{2.9}$$

where

$$F(\underset{\sim}{r}) = \int d^3k e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}} F(\underset{\sim}{k}) \tag{2.10}$$

and

$$f(\underset{\sim}{r}) = \int d^3q e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}} e^{-aq^2} = (\frac{\pi}{a})^{3/2} e^{-r^2/4a} . \tag{2.11}$$

Inverting the Fourier transform we obtain

$$F(\underset{\sim}{k}) \sim e^{-ak^2/2} . \tag{2.12}$$

Substituting back into Eq. (2.7) we then have the condition

$$p^2 e^{-ap^2} \sim \int \frac{d^3k e^{-a(\underset{\sim}{k}+\underset{\sim}{p})^2}}{k^2+m^2} e^{-ak^2/2} . \tag{2.13}$$

Now, if the asymptotic behavior on the left and right sides of the above equation were identical up to a polynomial, we could then go back and try a new asymptotic behavior $\psi(\underset{\sim}{p}) \sim p^n e^{-ap^2}$ which for some n might be self-consistent. This is not the case, however, as the integral in Eq. (2.13) can be evaluated and from this equation we obtain the condition

$$p^2 e^{-ap^2} \sim e^{-ap^2/3} . \tag{2.14}$$

Thus the left hand side is clearly not reproduced since the coefficient in the exponential is different.

Suppose, now, that we had replaced $\psi(\underset{\sim}{p}) \sim e^{-ap^2}$ by $\psi(\underset{\sim}{p}) \sim e^{-ap^2} + p^n e^{-bp^2}$ where b > a and n is arbitrary. This wave function has the same asymptotic behavior as that given by Eq. (2.6) but behaves quite differently for finite p. We have verified that the same asymptotic behavior is obtained for the integrals in Eq. (2.7) with this input function as was obtained before.

This can be understood, quite generally, by recalling that the large-p behavior of F(p) is determined from the small-r behavior of $f^2(\underset{\sim}{r})$ and that the large-k behavior of $\psi(\underset{\sim}{k})$ determines the small-r behavior of $f(\underset{\sim}{r})$ in the two Fourier transforms

$$F(\underset{\sim}{p}) = \int d^3r\ f^2(\underset{\sim}{r}) e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}$$

$$f(\underset{\sim}{r}) = \int d^3k\ e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\ \psi(\underset{\sim}{k}) \tag{2.15}$$

which arise in the evaluation of the faltung

$$F(\underset{\sim}{p}) = \int d^3k\ \psi(\underset{\sim}{k})\psi(\underset{\sim}{k}-\underset{\sim}{p}) \tag{2.16}$$

appearing in Eq. (2.4).

Now consider an asymptotic behavior of the form

$$\psi(\underset{\sim}{p}) \sim e^{-a\sqrt{p^2}} \tag{2.17}$$

We substitute into Eq. (2.4) and obtain the asymptotic integral equation,

$$p^2 e^{-a\sqrt{p^2}} \sim \int \frac{d^3k e^{-a\sqrt{(\underset{\sim}{k}+\underset{\sim}{p})^2}}}{k^2+m^2} \int d^3q\ e^{-a\sqrt{q^2}}\ e^{-a\sqrt{(\underset{\sim}{q}-\underset{\sim}{k})^2}}\ . \tag{2.18}$$

Defining

$$F(\underset{\sim}{k}) = \int d^3q\ e^{-a\sqrt{q^2}}\ e^{-a\sqrt{(\underset{\sim}{q}-\underset{\sim}{k})^2}} \tag{2.19}$$

we have

$$F(\underset{\sim}{k}) = \int d^3r\ e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}\ f^2(\underset{\sim}{r}) \tag{2.20}$$

where

$$f(\underset{\sim}{r}) = \int d^3q\ e^{i\underset{\sim}{q}\cdot\underset{\sim}{r}}\ e^{-a\sqrt{q^2}} \tag{2.21}$$

$$= \frac{a}{(r^2+a^2)^2}\ .$$

Hence

$$F(\underset{\sim}{k}) \propto \left(\frac{\partial}{\partial a^2}\right)^3 \int \frac{d^3r\ e^{i\underset{\sim}{k}\cdot\underset{\sim}{r}}}{r^2+a^2}$$

$$= \left(\frac{\partial}{\partial a^2}\right)^3 \frac{e^{-a\sqrt{k^2}}}{\sqrt{k^2}}$$

$$\sim k^2\ e^{-a\sqrt{k^2}} \quad . \tag{2.22}$$

The essential point is that when this expression is substituted into Eq. (2.18) then the resulting integral on the right hand side is again a faltung of the same form which we calculated in Eq. (2.19). Using our result, Eq. (2.22), and taking the asymptotic limit $p \to \infty$, we obtain for the right hand side of Eq. (2.18)

$$\int d^3k\ e^{-a\sqrt{k^2}}\ e^{-a\sqrt{(k+p)^2}}\ \frac{k^2}{k^2+m^2} \sim p^2\ e^{-a\sqrt{p^2}} \tag{2.23}$$

which agrees exactly with the leading asymptotic behavior for the left hand side of Eq. (2.18)! Thus the wave function with the asymptotic form given by Eq. (2.17) is self-consistent.

Finally, consider the possible asymptotic behavior

$$\psi(\underset{\sim}{p}) \sim (p^2+\mu^2)^{-n} \tag{2.24}$$

for arbitrary, finite $n \geq 1$. It is convenient to define the integrals

$$G_{mn}(\underset{\sim}{p}) = \int d^3k(k^2+\mu^2)^{-m}((\underset{\sim}{k}-\underset{\sim}{p})^2+\mu^2)^{-n} \tag{2.25}$$

which can be evaluated by again introducing Fourier transforms. We obtain the results

$$G_{mn}(\underset{\sim}{p}) = \left(\frac{\partial}{\partial a^2}\right)^{m-2} \left(\frac{\partial}{\partial b^2}\right)^{n-2} \frac{a+b}{(ab)(p^2+(a+b)^2)^2}\Bigg|_{a^2=b^2=\mu^2}$$

$$\sim (p^2)^{-\text{min.}(m,n)} \tag{2.26}$$

which, when substituted into Eq. (2.4), yields the asymptotic equation

$$p^2(p^2)^{-n} \sim (p^2)^{-n} \quad . \tag{2.27}$$

Hence the asymptotic behavior given by Eq. (2.24) is not self-consistent.

Since the mathematical complexities of Eq. (2.4) have forced us to employ a trial and error procedure for finding acceptable asymptotic solutions, our conclusions are not as general as one might hope. We can conclude that the asymptotic form $\psi(\underset{\sim}{p}) \sim e^{-a\sqrt{p^2}}$ is a possible asymptotic solution whereas the forms $\psi(\underset{\sim}{p}) \sim e^{-a\underset{\sim}{p}^2} p^n$ and $\psi(\underset{\sim}{p}) \sim (p^2+\mu^2)^{-n}$ are not. In the following section we demonstrate that a similar result holds when the bootstrap is described by a Bethe-Salpeter equation which is a generalization of Eq. (2.4).

III. RELATIVISTIC BOOTSTRAP

In this section we treat the relativistic problem of a composite scalar particle, called the S-meson, which is a bound state of itself and an elementary scalar meson, denoted by σ. Taking σ exchange for the binding mechanism, we have the following Bethe-Salpeter equation in ladder approximation

$$[p^2-m^2][(P-p)^2-m^2]\chi(p) = \frac{\lambda}{(2\pi)^2}\int d^4k\ \chi(k)[(P-k)^2+m^2]\chi(p-k) \tag{3.1}$$

which is illustrated graphically in Fig. 4. The variables P,p,k,r denote four-vectors throughout this section. This equation should have meaning only when the external S-meson momentum is evaluated on the mass shell. We note that the equation incorporates the physical requirement that the constituent S-meson is, itself, a composite particle; this is evident from Fig. 5 where we illustrate the first iteration of this equation. We have made the simplification in Eq. (3.1) of assuming that the wave function depends only on the square of the four-momentum of the σ-meson and not on the remaining invariant which is the square of the four-momentum of the off-shell S-meson.

We can proceed now as we did in the Schroedinger equation model, except for the additional complexity associated with the Minkowski metric in Eq. (3.1). When the asymptotic forms $\chi(p) \sim e^{-a\sqrt{p^2}}$ and $\psi(p) \sim (p^2+\mu^2)^{-n}$ are substituted into Eq. (3.1) the Kernel is sufficiently well behaved to allow a Wick rotation of the fourth component of the momentum variable which yields an integral equation in a Euclidean space. (Care must be taken to define the wave function $e^{-a\sqrt{p^2}}$ by $e^{-a(p^4)^{1/4}}$ and to carry out the Wick rotation on the appropriate sheet.) However, with the asymptotic behavior $\chi(p) \sim e^{-ap^2}$ the integral in Eq. (3.1) is no longer well-defined. This is reflected in the fact that one encounters essential singularities along the path of the Wick rotation. To avoid (but not solve!) this problem, we take as our definition of the integral in Eq. (3.1) the well-defined integral which one obtains by writing the equation from the outset in a Euclidean space.

Consider first the asymptotic behavior

$$\chi(p) \sim e^{-ap^2} \tag{3.2}$$

and substitute into the asymptotic limit of Eq. (3.1) which reads

$$p^4\chi(p) \sim \int d^4k\; k^2\chi(k)\chi(p-k) \quad . \tag{3.3}$$

We define

$$F(p) \equiv \int d^4k\; k^2 e^{-ak^2} e^{-a(p-k)^2} \quad . \tag{3.4}$$

Then

$$F(r) = f(r)g(r) \tag{3.5}$$

where

$$f(r) = \int d^4k\; e^{ik\cdot r} e^{-ak^2} = \left(\frac{\pi}{a}\right)^2 e^{-r^2/4a} \tag{3.6}$$

and

$$g(r) = -\frac{\partial}{\partial a} f(r) \ . \tag{3.7}$$

Hence we obtain

$$F(p) \sim \int d^4r \ e^{ip\cdot r} e^{-r^2/2a}(1-\frac{r^2}{2a}) \sim p^2 e^{-ap^2/2} \ . \tag{3.8}$$

Referring to Eq. (3.3), we have the condition

$$p^4 e^{-ap^2} \sim p^2 e^{-ap^2/2} \tag{3.9}$$

and we see that the wave function with an asymptotic behavior given by Eq. (3.2) is not self-consistent.

Now consider the possible solution

$$\chi(p) \sim e^{-a\sqrt{p^2}} \ . \tag{3.10}$$

Defining

$$F(p) \equiv \int d^4k \ e^{-a\sqrt{k^2}} e^{-a\sqrt{(p-k)^2}} k^2 \tag{3.11}$$

we have

$$F(r) = \int d^4p \ e^{ip\cdot r} F(p) = f(r) g(r) \tag{3.12}$$

where

$$f(r) = \int d^4k \ e^{ik\cdot r} e^{-a\sqrt{k^2}} = \frac{a}{(r^2+a^2)^{5/2}} \tag{3.13}$$

and

$$g(r) = \left(\frac{\partial}{\partial a}\right)^2 f(r) \ . \tag{3.14}$$

Taking the inverse Fourier transform of Eq. (3.12) we obtain the result

$$F(p) = \left[35a^2\left(\frac{\partial}{\partial a^2}\right)^6 - 15\left(\frac{\partial}{\partial a^2}\right)^5\right] \int \frac{d^4r e^{ip\cdot r}}{r^2+a^2} \ . \tag{3.15}$$

Thus $F(p)$ is given by the appropriate derivatives of an integral which is familiar as the Feynman propagator in configuration space but here written as a function of the momentum variable. Hence

$$F(p) \sim \left(\frac{\partial}{\partial a^2}\right)^6 \frac{aH_1^{(2)}(iap)}{p} \sim p^{9/2} e^{-a\sqrt{p^2}} \quad . \tag{3.16}$$

Comparing this expression with the l.h.s. of Eq. (3.3) we see that the exponential is self-consistent. If we assume the asymptotic wave function

$$\chi(p) \sim \frac{1}{\sqrt{|p|}}\, e^{-a\sqrt{p^2}} \tag{3.17}$$

then, in addition, the polynomial multiplying the exponential is self-consistent and we conclude that Eq. (3.10) provides an acceptable asymptotic form for the wave function. Finally we have examined the asymptotic behavior

$$\chi(p) \sim (p^2+a^2)^{-m} \quad . \tag{3.18}$$

Since the mathematical procedures and the analogies with the non-relativistic problem should be clear by now to the reader, we simply state the result that on the right hand side of Eq. (3.1) we obtain the asymptotic behavior $p^2(p^2+m^2)^{-m}$ which is inconsistent with the left hand side which behaves asymptotically like $p^4(p^2+m^2)^{-m}$. It may appear to be possible to find a self-consistent polynomial asymptotic behavior when spin complications are included. However, it is easy to show that if the S-meson had spin-1/2 then polynomial asymptotic behavior is still unacceptable. Since the inverse propagator of the S-meson appears on both the left and right hand sides of Eq. (3.1) it is unlikely that spin will affect our result.

IV. NO-ELEMENTARY-PARTICLE MODEL

It might be argued that the results we have obtained in the previous sections were a consequence of the artificiality of the point-source potential in the non-relativistic example and of the analogous elementary particle in the relativistic example. In these models, only the composite particle was treated crossing-symmetrically. So, here we investigate a model with no elementary particles. One such model, illustrated in Fig. 2, was discussed in the introduction. A practical difficulty with this model is that in the asymptotic region which we wish to study, the integral equation contains at least one wave function in the integrand with all three legs off the mass-shell. This is in contrast to the considerably simpler situation discussed in the previous section where we could take the external S-mesons on the mass shell and then determine the form factor as a function of the σ-meson momentum which was allowed to vary off-shell. Therefore, we shall look again at a Schroedinger equation model, but one in which the point-source of the potential in Eq. (2.2) is replaced by another composite particle.

Let us suppose that we have two identical, spinless particles with "charge" density

$$\rho(\underset{\sim}{r}) = \psi^*(\underset{\sim}{r})\psi(\underset{\sim}{r}) \tag{4.1}$$

which form a spinless bound state with the identical "charge" density. Again we assume a Yukawa potential which couples to the "charge." Thus

$$V(\underset{\sim}{r}_1,\underset{\sim}{r}_2) = \delta(\underset{\sim}{r}_1+\underset{\sim}{r}_2)\int d^3r_1'\int d^3r_2'\rho(\underset{\sim}{r}_2-\underset{\sim}{r}_2')\rho(\underset{\sim}{r}_1-\underset{\sim}{r}_1')\frac{e^{-m|\underset{\sim}{r}_1'-\underset{\sim}{r}_2'|}}{|\underset{\sim}{r}_1'-\underset{\sim}{r}_2'|} \; . \tag{4.2}$$

In momentum space, V(p) is given by

$$V(\underset{\sim}{p}) = \frac{1}{p^2+m^2}\int d^3k\int d^3q\psi^*(\underset{\sim}{k})\psi^*(\underset{\sim}{q})\psi[(\tfrac{1}{2}\underset{\sim}{p}+\underset{\sim}{k})^2]\psi[(\tfrac{1}{2}\underset{\sim}{p}+\underset{\sim}{q})^2] \tag{4.3}$$

and the Schroedinger equation takes the form

$$(\frac{p^2}{m}-m)\psi(\underset{\sim}{p}) = \int d^3k\psi(\underset{\sim}{k})V(\underset{\sim}{p}-\underset{\sim}{k}) \; . \tag{4.4}$$

The remarkable feature of Eqs. (4.3), (4.4) is the factor of 1/2 which multiplies the variable $\underset{\sim}{p}$ in $\psi[(\tfrac{1}{2}\underset{\sim}{p}+\underset{\sim}{k})^2]$ and $\psi[(\tfrac{1}{2}\underset{\sim}{p}+\underset{\sim}{k})^2]$ under the integral signs. To see the effect of this factor we first define

$$F(\underset{\sim}{p}) = \int d^3k\ \psi(\underset{\sim}{k})\psi(\underset{\sim}{p}/2+\underset{\sim}{k}) \ . \qquad (4.5)$$

We now substitute the asymptotic form $\psi(\underset{\sim}{k}) \sim e^{-a\sqrt{k^2}}$ and note from our previous result in Eq. (2.22) that

$$F(\underset{\sim}{p}) \sim p^2 e^{-\frac{a}{2}\sqrt{p^2}} \ . \qquad (4.6)$$

Therefore

$$V(\underset{\sim}{p}) = \frac{F^2(\underset{\sim}{p})}{p^2+m^2} \sim \frac{p^4}{p^2+m^2}\ e^{-a\sqrt{p^2}} \qquad (4.7)$$

and we recover the coefficient a in the exponential! The integral in Eq. (4.4) can then be performed and we see from our result in Section II that the exponentially falling form factor is self-consistent. A more detailed calculation shows that the correct self-consistent asymptotic behavior is given by $\psi(\underset{\sim}{p}) \sim e^{-a\sqrt{p^2}}/p^2$. In addition, wave functions with the asymptotic behavior $\psi(\underset{\sim}{p}) \sim e^{-ap^2}p^n$ or $(p^2+\mu^2)^{-n}$ can be easily shown to be inconsistent with Eq. (4.4); the arguments can be made analogously to those in Sections II and III.

It is seen then, that the replacement of the point-source potential in Eq. (2.2) by another composite particle does not affect the validity of our result. This makes it appear plausible to us that if the σ-meson in Section III is replaced by a composite S-meson and the point vertex in Fig. 4 replaced by a form factor, then our result should again be valid.

V. GENERALIZATION TO ARBITRARY INTERACTION KERNEL

We have found that the form factors of the strongly interacting particles should behave asymptotically as $F(t) \sim e^{-a\sqrt{|t|}}$. This result was shown to be the consequence of a bootstrap model with a crossing-symmetric three-point function and hence one which described the composite and constituent particles in an identical manner. It is natural to inquire, now, as to the generality of the result.

The bound state equations which we have considered up to this point were only approximations to a realistic theory because we treated the interaction kernel in ladder approximation. We give a plausibility argument now which suggests that the restriction to single-particle-exchange forces can be relaxed without destroying the self-consistency of the exponentially falling form factor. Let us return to the S and σ-meson model of Section III and consider the most general interaction kernel which contains one S-meson line and is of n^{th} order in the SSσ vertex (see Fig. 6). We assume the asymptotic behavior $\chi(p) \sim e^{-a\sqrt{p^2}}/\sqrt{p}$ as given by Eq. (3.17) and write the asymptotic Bethe-Salpeter equation which results from this interaction kernel as

$$p^4\chi(p) \sim \int d^4k \int d^4p_1 \ldots \int d^4p_n k^2 P_n^{-1}(k,p_1,\ldots,p_n)[\chi(k)\chi(p_1)\ldots\chi(p_n)]\delta^4(k+p_1+\ldots+p_n-p) \tag{5.1}$$

or

$$\frac{p^4 e^{-a\sqrt{p^2}}}{\sqrt{p}} \sim \int d^4k \int d^4p_1 \ldots \int d^4p_n k^2 P_n^{-1}(k,p_1,\ldots,p_n)$$

$$\times \left[\frac{e^{-a\sqrt{k^2}}}{\sqrt{|k|}} \cdot \frac{e^{-a\sqrt{p_1^2}}}{\sqrt{|p_1|}} \ldots \frac{e^{-a\sqrt{p_n^2}}}{\sqrt{|p_n|}}\right] \delta^4(k+p_1+\ldots+p_n-p) . \tag{5.2}$$

This equation is illustrated in Fig. 6. $P_n(k,p_1,\ldots,p_n)$ is a polynomial in the internal σ-meson momenta and consists of the product of the propagators for the internal σ-meson lines which begin and end within the shaded circle in Fig. 6. We will assume a $\lambda\sigma^3$ interaction at all vertices within the shaded circle and neglect those kernels with closed σ-meson loops since they lead to divergences which would be absent in a correct theory which treats the

σ-meson as composite. Then it can be shown that P_n consists of the product of n-1 propagators. Integrating out the δ-function in Eq. (5.2) we obtain for the r.h.s., denoted by $F(p)$, the expression

$$F(p) \equiv \int d^4k \int d^4p_1 \ldots \int d^4p_{n-1} k^2 P_n^{-1}(k,p_1,\ldots,p_{n-1})$$
$$\times \left[\frac{e^{-a\sqrt{k^2}}}{\sqrt{|k|}} \cdot \frac{e^{-a\sqrt{p_1^2}}}{\sqrt{|p_1|}} \cdots \frac{e^{-a\sqrt{p_{n-1}^2}}}{\sqrt{|p_{n-1}|}} \cdot \frac{e^{-a\sqrt{(k+p_1+\ldots+p_{n-1}-p)^2}}}{\sqrt{|k+p_1+\ldots+p_{n-1}-p|}}\right] \tag{5.3}$$

which is just a sequence of faltungs of essentially the same form as those we encountered in Section III. The first one, given by the integral around the furthest closed loop to the right in Fig. 6, is of the form

$$I_{n-1} \equiv \int d^4p_{n-1} \frac{e^{-a\sqrt{p_{n-1}^2}}\, e^{-a\sqrt{(k+p_1+\ldots+p_{n-1}-p)^2}}}{p_{n-1}^2\sqrt{|p_{n-1}|}\,\sqrt{|k+p_1+\ldots+p_{n-1}-p|}} \tag{5.4}$$

which integrates to give

$$I_{n-1} \sim \frac{e^{-a\sqrt{(k+p_1+\ldots+p_{n-2}-p)^2}}}{\sqrt{|k+p_1+\ldots+p_{n-2}-p|}} \quad . \tag{5.5}$$

Substituting back into Eq. (5.3) we obtain for the second integration

$$I_{n-2} \equiv \int d^4p_{n-2} \frac{e^{-a\sqrt{p_{n-2}^2}}}{p_{n-2}^2\sqrt{|p_{n-2}|}\,\sqrt{|k+p_1+\ldots+p_{n-2}-p|}} \sim \frac{e^{-a\sqrt{(k+p_1+\ldots+p_{n-2}-p)^2}}}{\sqrt{|k+p_1+\ldots+p_{n-3}-p|}} \; . \tag{5.6}$$

Continuing this procedure we finally come to the last integral which is of the form

$$F(p) = \int d^4k \frac{k^2}{\sqrt{|k|}} \; \frac{e^{-a\sqrt{k^2}}\, e^{-a\sqrt{(k-p)^2}}}{\sqrt{|k-p|}} \quad . \tag{5.7}$$

This is precisely the integral we encounter in the ladder approximation to the kernel and we obtain for the r.h.s. of Eq. (5.2) the asymptotic expression $F(p) \sim p^4 e^{-a\sqrt{p^2}}/\sqrt{|p|}$ which reproduces the behavior on the l.h.s.

We are tempted to speculate that the exponentially falling form factor is asymptotically self-consistent in the relativistic, no-elementary-particle bootstrap model with an arbitrary interaction kernel. However, even in the ladder approximation (see Eq. (1.4)), the absence of an elementary or bare vertex in the relativistic bound state problem forces one to treat a nonlinear equation in which the unknown wave function depends on three invariants. We defer speculation on this question until a further investigation.

VI. CONCLUDING REMARKS

Our discussion has been limited to off-shell dynamical models for composite particles. Can our results be obtained from dispersion relations? Consider the elastic-unitarity equation for the form factor

$$\mathrm{Im}F_\ell(s) = \rho(s)T_\ell^*(s)F_\ell(s) \tag{6.1}$$

where $\rho(s)$ is a kinematic factor and $T_\ell(s)$ is the partial wave scattering amplitude. When Eq. (6.1) is substituted into the dispersion relation

$$F_\ell(s) = \frac{1}{\pi}\int_4^\infty ds' \, \frac{\mathrm{Im}F_\ell(s')}{s'-s+i\epsilon} \tag{6.2}$$

we obtain the well-known result

$$F_\ell(s) = \exp\left[\frac{1}{\pi}\int_4^\infty \frac{ds'\delta_\ell(s')}{s'-s}\right] \tag{6.3}$$

where δ_ℓ is the elastic phase shift. In analogy to Eq. (1.4) it would appear natural to approximate $T_\ell(s)$ in Eq. (6.1) by the scattering amplitude in Born approximation but with the coupling constant replaced by the form factor appearing in Eq. (6.1). Thus we set

$$T_\ell(s) \sim \int dz \, \frac{P_\ell(z) F^2(t)}{t-m^2} \tag{6.4}$$

where $F(t) \sim e^{-a\sqrt{|t|}}$.

As emphasized by Mandelstam,[7] it is difficult to obtain a falling form factor on the l.h.s. of Eq. (6.3). The reason is that the asymptotic behavior of $F(s)$ is determined by the asymptotic value of the phase shift and exponentially falling form factors would result from Eq. (6.3) only if $\delta_\ell(\infty) = \infty$. Now, it is known that in potential scattering the condition $\delta_\ell(\infty) = \infty$ is generally possible only with a singular repulsive potential. Since the Fourier transform of $e^{-a\sqrt{|t|}}$ is $(r^2+a^2)^{-5/2}$, however, $T_\ell(s)$, as given by Eq. (6.4), certainly does not correspond to a singular potential. Thus we do not obtain self-consistency between the input form factor in Eq. (6.4) and the form factor obtained from Eq. (6.3) if the former is exponentially falling.

To conclude, we apologize to the reader for the rather crude mathematical methods which have been employed in this paper. We hope that the results which we have obtained here will inspire others to develop sharper tools which will allow a deeper exploration into the basic nonlinear equations of strong interaction physics.

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FOOTNOTES

1. G. Fast and R. Hagedorn, Nuovo Cimento 27, 208 (1963); G. Fast, R. Hagedorn and L. Jones, Nuovo Cimento 27, 856 (1963); T. T. Wu and C. N. Yang, Phys. Rev. 137, B708 (1965); A. Martin, Nuovo Cimento 37, 671 (1965); A. M. Jaffe, Phys. Rev. Letters 17, 661 (1966); J. Stack, University of Illinois preprint, June 1967, "Rapidly Decreasing Form Factors and Infinitely Composite Particles." S. Drell, A. Finn and Michael Goldhaber, Phys. Rev. 157, 1402 (1967).
2. See J. Stack, op. cit., for a summary of the experimental evidence for exponentially falling form factors.
3. M. Gell-Mann and F. Low, Phys. Rev. 84, 340 (1951); H. Bethe and E. Salpeter, Phys. Rev. 84, 1232 (1951).
4. See e.g. L.S. Liu, Phys. Rev. 125, 761 (1961); R. Cutkosky and M. Leon, Phys. Rev. 138, B667 (1965); W. Guttinger, Nuovo Cimento 36, 968 (1965); M. Broido and J. G. Taylor, Phys. Rev. 147, 993 (1966); J. Harte, Nuovo Cimento 45, 179 (1966).
5. R. Cutkosky and M. Leon, op. cit.; K. Lin and R. Cutkosky, Phys. Rev. 140, B205 (1965); R. Cutkosky, Phys. Rev. 154, 1375 (1967).
6. Suppose that at least one solution to this equation exists, $\psi_0(\underset{\sim}{p})$, that behaves like $\exp(-a|p|)$ as $|p| \to \infty$. Is this solution unique? Since the Frechèt derivative of ψ, evaluated at $\psi = \psi_0$ satisfies a homogeneous Fredholm equation, it follows from the Hildebrandt-Graves theorem that $\psi = \psi_0$ will almost always be locally unique. I am indebted to Dr. David Atkinson for discussions of uniqueness and existence theorems for nonlinear integral equations.
7. S. Mandelstam, Lawrence Radiation Laboratory Report UCRL 17250 (1966), (unpublished) "Recent Work on Regge Poles and the Three-Body Problem."

FIGURE CAPTIONS

Fig. 1. Bethe-Salpeter equation in ladder approximation with elementary constituents. The solid circle designates the Bethe-Salpeter wave function.

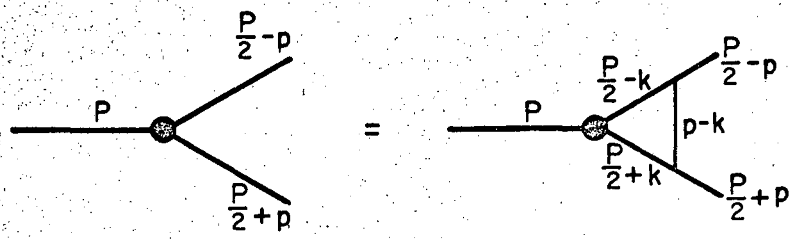
Fig. 2. Crossing-symmetric, nonlinear, Bethe-Salpeter equation in ladder approximation.

Fig. 3. First iteration of Eq. (1.4) illustrating that the constituents are, themselves, composite.

Fig. 4. Bethe-Salpeter equation in ladder approximation with a composite-constituent S-meson (solid line) and an elementary-constituent σ-meson (dotted line).

Fig. 5. First iteration of Eq. (3.1).

Fig. 6. Bethe-Salpeter equation with arbitrary interaction kernel of n^{th} order in the S-S-σ vertex and with one S-meson line. The shaded circle represents an arbitrary interaction among the $n+2$ σ-meson lines.



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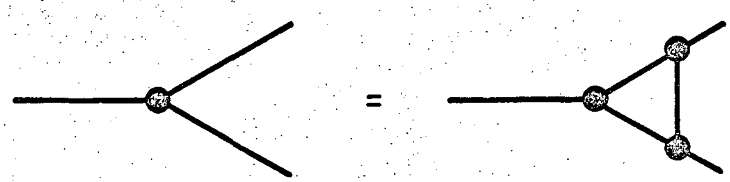

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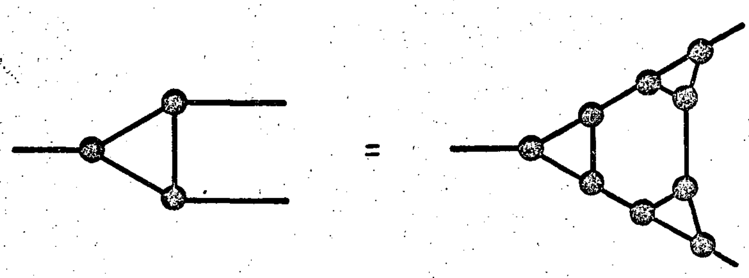

3.

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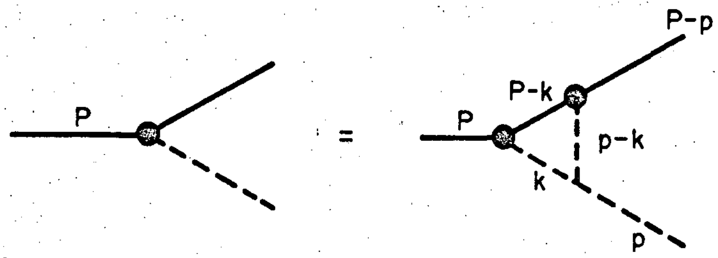
Figs. 1, 2, and 3

4.



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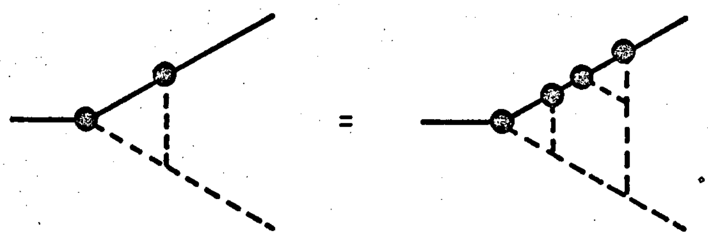

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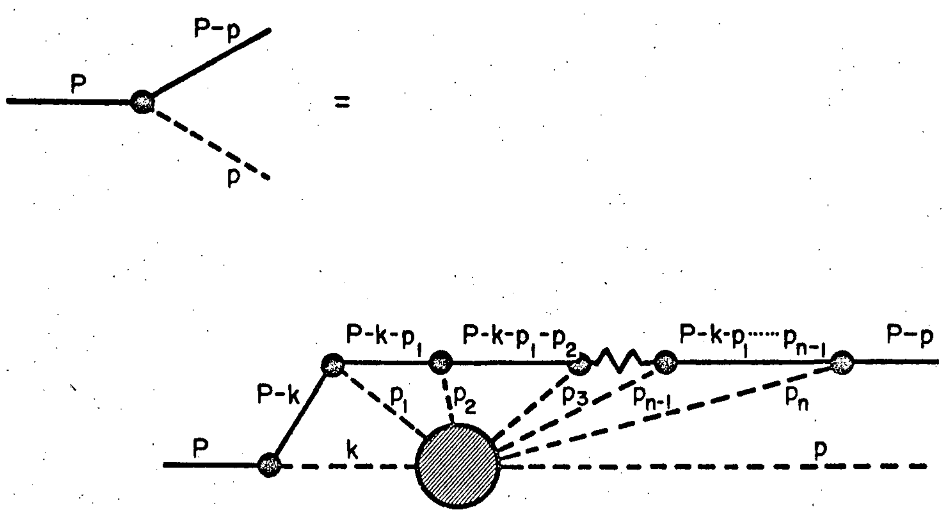

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Figs. 4, 5, and 6