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The 2-sphere is Wecken for n-valued maps

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Abstract. We prove the theorem of the title. Every n-valued map $\phi\colon S^2 \multimap S^2$ of the 2-sphere has the Wecken property for n-valued maps, that is, it is n-valued homotopic to a map with $N(\phi)$ fixed points, where $N(\phi)$ is the Nielsen number of ϕ.

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In 1942, Wecken proved [10] that if X is a compact, connected triangulated manifold, with or without boundary, of dimension three or greater and $f\colon X \to X$ is a map, then f is homotopic to a map with exactly $N(f)$ fixed points, where $N(f)$ is the Nielsen number of f. Subsequently, spaces for which all self-maps have this property became known as *Wecken spaces*. One-dimensional manifolds are obviously Wecken. Jiang proved in [6] that no hyperbolic two-manifold, that is a two-manifold with a negative Euler characteristic, is Wecken. However, the seven non-hyperbolic two-manifolds are all Wecken, see [7].

In this paper, we explore the Wecken property in the setting of n-valued maps. An n-valued map is a lower semi-continuous and hence also upper semi-continuous (see [2]) set-valued function $\phi\colon X \multimap Y$ such that $\phi(x)$ is n points of Y for each $x \in X$. Schirmer defined the Nielsen number $N(\phi)$ for n-valued maps $\phi\colon X \multimap X$ of finite polyhedra in [8] and proved in [9] that if X is a compact, connected triangulated manifold of dimension three or greater, then ϕ is n-valued homotopic to a map $\psi\colon X \multimap X$, that is, there exists an n-valued map $H\colon X \times [0,1] \multimap X$ such that $H(x,0) = \phi(x)$ and $H(x,1) = \psi(x)$ for all $x \in X$, such that ψ has exactly $N(\phi)$ fixed points. Thus, X is *Wecken for n-valued maps*. For one-dimensional manifolds, the interval is obviously Wecken for n-valued maps and the same property was proved for the circle in [1]. The Wecken property fails for hyperbolic two-manifolds since, by Jiang's result, it fails when $n = 1$. It is then a natural problem to inquire whether the Wecken property holds for n-valued maps of the non-hyperbolic two-manifolds if $n > 1$.

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The disc D^2 possesses this property: for $\phi\colon D^2 \multimap D^2$ define $H\colon D^2 \times [0,1] \multimap D^2$ by $H(x,t) = \phi(tx)$, then $N(\phi) = n$ by Corollary 7.3 of [8], so ϕ has the Wecken property. Gonçalves and Guaschi established the Wecken property for n-valued maps of the projective plane in [4]. As its title states, the present paper proves that the two-sphere is also Wecken for n-valued maps. It is not known whether the remaining non-hyperbolic two-manifolds are Wecken for n-valued maps.

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If $f\colon S^2 \to S^2$ is a single-valued map, then it has long been known that it has the Wecken property. If f is of degree -1 then, by the Hopf Classification Theorem, f is homotopic to the antipodal map which is fixed point free. Otherwise, the same theorem implies that f is homotopic to the suspension of a self-map of the equator and thus homotopic to a map that fixes only the poles. A neighborhood of an arc connecting the poles has the structure of a cone with its vertex at one of the poles. Using the cone structure, the map may then be homotoped to a map with a single fixed point at that vertex, thus completing the proof.[1]

Since S^2 is simply connected, if $\phi\colon S^2 \multimap S^2$ is an n-valued map then, by the Splitting Lemma (see [2]), there are single-valued maps $f_0, \ldots, f_{n-1}\colon S^2 \to S^2$ such that $\phi(x) = \{f_0(x), \ldots, f_{n-1}(x)\}$ for all $x \in S^2$.

Lemma 1. *Let $f, g\colon S^2 \to S^2$ be maps such that $f(x) \neq g(x)$ for all $x \in S^2$, then their degrees are related by $\deg(f) = -\deg(g)$. Consequently, if $\phi = \{f_0, \ldots, f_{n-1}\}\colon S^2 \multimap S^2$ is an n-valued map for $n \geq 3$, then all the $f_i\colon S^2 \to S^2$ are inessential maps and, therefore, $N(\phi) = n$.*

Proof. Since $f(x) \neq g(x)$ for all $x \in S^2$, then the Lefschetz coincidence number

$$L(f,g) = \sum_{q=0}^{2} (-1)^q \operatorname{trace}(D_q g^* D_q^{-1} f_*) = \deg(f) + \deg(g) = 0,$$

where $D_q\colon H^{2-q}(S^2) \to H_q(S^2)$ is the Poincaré Duality isomorphism. Therefore, for f_i, f_j, f_k we have

$$\deg(f_i) = -\deg(f_j) = \deg(f_k) = -\deg(f_i)$$

so $\deg(f_i) = \deg(f_j) = \deg(f_k) = 0$. It follows that $N(\phi) = n$ by Corollary 7.3 of [8]. □

Proposition 3 below is Proposition 4(c)(i) of [4]. We include a proof so that the proof of the Wecken property for all n-valued maps of S^2 will be self-contained in this paper. Our proof is quite different from that in [4].

Lemma 2. *Let $\phi\colon S^2 \multimap \mathbb{R}^2$ be an $(n-1)$-valued map, then ϕ is $(n-1)$-valued homotopic to a constant map.*

[1]The definition of f_0 in Proposition 5 of this paper presents another, more explicit, construction of the required single-valued maps of S^2.

Proof. As in [2], we view ϕ as the single-valued map $\phi\colon S^2 \to D_{n-1}(\mathbb{R}^2)$, where $D_{n-1}(\mathbb{R}^2)$ is the configuration space of unordered $(n-1)$-tuples of distinct points of $\mathbb{R}^2$. By Chapter IV, Theorem 1.1 of [3], $D_{n-1}(\mathbb{R}^2)$ is aspherical, that is, $\pi_q(D_{n-1}(\mathbb{R}^2)) = 0$ for all $q > 1$. Thus, in particular, the map $\phi\colon S^2 \to D_{n-1}(\mathbb{R}^2)$ is homotopic to a constant map. □

Proposition 3. (Gonçalves–Guaschi) *Let $\phi\colon S^2 \multimap S^2$ be an n-valued map, $n \geq 3$, then ϕ is n-valued homotopic to a constant map and, thus, a map with $n = N(\phi)$ fixed points. Therefore, ϕ has the Wecken property for n-valued maps.*

Proof. Let
$$W = \{(u, v) \in S^2 \times S^2 \colon v \neq -u\}.$$
Define $r\colon W \to SO(3)$ by letting $r(u, u)$ be the identity and for $(u, v) \in W$ with $v \neq u$, let $r(u, v)$ be the rotation about the axis perpendicular to u and v sending u to v, that is, $r(u, v)(u) = v$ for all $(u, v) \in W$.

Let $\phi = \{f_0, \ldots, f_{n-1}\}\colon S^2 \multimap S^2$. Since $n \geq 3$, then by Lemma 1 there exists $h\colon S^2 \times [0, 1] \to S^2$ such that $h(x, 0) = f_0(x)$ and $h(x, 1) = c_0$ for all $x \in S^2$. A map $\rho\colon S^2 \times [0, 1] \to SO(3)$ such that $\rho(x, 0) = x$ and $h(x, t) = \rho(x, t)(f_0(x))$ for all $x \in S^2$ and $t \in [0, 1]$ may be constructed as follows. Since $S^2 \times [0, 1]$ is compact, there exists an integer k such that if $|s - t| \leq \frac{1}{k}$, then $\delta(h(x, s), h(x, t)) < 2$ for all $x \in S^2$, where δ denotes the metric of $\mathbb{R}^3$ and S^2 is the unit sphere in $\mathbb{R}^3$. In particular, if $\frac{j-1}{k} \leq s \leq \frac{j}{k}$, then $h(x, s) \neq -h(x, \frac{j-1}{k})$ for all $x \in S^2$, so there is a rotation $r(h(x, \frac{j-1}{k}), h(x, s))$. Now, for $\frac{j-1}{k} \leq s \leq \frac{j}{k}$, let
$$\rho(x, s) = r\left(h\left(x, \frac{j-1}{k}\right), h(x, s)\right) \cdot r\left(h\left(x, \frac{j-2}{k}\right), h\left(x, \frac{j-1}{k}\right)\right) \cdots r\left(h\left(x, \frac{1}{k}\right), h\left(x, \frac{2}{k}\right)\right) \cdot r\left(h\left(x, \frac{1}{k}\right), h(x, 0)\right)$$
which defines the required map $\rho\colon S^2 \times [0, 1] \to SO(3)$.

For $i = 0, \ldots, n-1$, define $h_i(x, t) = \rho(x, t)(f_i(x))$ and let
$$H(x, t) = \{h_0(x, t), h_1(x, t). \ldots, h_{n-1}(x, t)\}.$$
Then, H is an n-valued homotopy because
$$h_i(x, t) = \rho(x, t)(f_i(x)) \neq \rho(x, t)(f_j(x)) = h_j(x, t)$$
for $i \neq j$. Now ϕ is homotopic to $\psi = \{g_0, \ldots, g_{n-1}\}$ defined by $\psi(x) = H(x, 1)$ such that $g_0(x) = c_0$ for all $x \in S^2$. Since ψ is an n-valued map, $g_i(x) \neq c_0$ for $i = 1, \ldots, n-1$ and, therefore, we have the $(n-1)$-valued map
$$\psi' = \{g_1, \ldots, g_{n-1}\}\colon S^2 \multimap S^2 \setminus \{c_0\} = \mathbb{R}^2.$$
By Lemma 2, there is an $(n-1)$-valued homotopy $K'\colon S^2 \times [0, 1] \multimap S^2 \setminus \{c_0\}$ such that $K'(x, 0) = \psi'(x)$ and $K'(x, 1) = \{c_1, \ldots, c_{n-1}\}$ for all $x \in S^2$. The homotopy H followed by the n-valued homotopy $K\colon S^2 \times [0, 1] \multimap S^2$ defined by $K(x, t) = \{c_0\} \cup K'(x, t)$ is an n-valued homotopy of ϕ to a map which is constant and, therefore, has n fixed points. We conclude that ϕ has the Wecken property for n-valued maps with $n \geq 3$. □

It remains to prove that 2-valued maps of S^2 satisfy the Wecken property. Let $\phi = \{f_0, f_1\}\colon S^2 \multimap S^2$ then, by Lemma 1, $\deg(f_0) = -\deg(f_1)$. We define the *degree* of ϕ to be $\deg(\phi) = |d|$; which is well-defined letting d be the degree of either f_0 or f_1.

The Hopf Classification Theorem, that is used to prove the Wecken property for single-valued maps of the sphere, can also be applied to understand 2-valued maps, as follows.

Proposition 4. *Let $\phi, \psi\colon S^2 \multimap S^2$ be 2-valued maps. If $\deg(\phi) = \deg(\psi)$, then ϕ and ψ are 2-valued homotopic.*

Proof. By Theorem 8 of [5] applied to maps of $S^2 \times [0,1]$ to the configuration space $D_2(S^2)$, the 2-valued homotopy classes of 2-valued maps of S^2 are $[S^2, D_2(S^2)]$, the unbased homotopy classes of maps. There is a homotopy equivalence $RP^2 = S^2/\{\pm 1\} \to D_2(S^2)$ given by sending $x \in S^2$ to $\{x, -x\}$. The reason is that there is a homeomorphism from the open unit disc bundle $B(H^\perp)$ in the orthogonal complement of the Hopf line bundle H in $\mathbb{R}^3$ to $D_2(S^2)$. It is defined by sending $(\pm x, y)$, where $y \in \mathbb{R}^3$ is orthogonal to x and $\|y\| < 1$ to $\{(tx, y), (-tx, y)\}$ where $t^2 + \|y\|^2 = 1$. A map $a\colon S^2 \to RP^2$ lifts to two maps $\tilde{a}, -\tilde{a}\colon S^2 \to S^2$. Suppose $H\colon S^2 \times [0,1] \to RP^2$ such that $H(x,0) = a(x)$ and $H(x,1) = b(x)$ for all $x \in S^2$ and b lifts to $\tilde{b}, -\tilde{b}\colon S^2 \to S^2$. If H is lifted to $\tilde{H}\colon S^2 \times [0,1] \to S^2$ such that $\tilde{H}(x,0) = \tilde{a}(x)$, then either $\tilde{H}(x,1) = \tilde{b}(x)$ for all $x \in S^2$ or $\tilde{H}(x,1) = -\tilde{b}(x)$ for all $x \in S^2$. Since, by the Hopf Classification Theorem, the homotopy classes of single-valued self-maps of S^2 are determined by the degree, the homotopy class of a is determined by $|\deg(\tilde{a})|$. It follows that $[S^2, D_2(S^2)]$ is classified by the degree $|d|$. □

Having established Proposition 4, to complete the proof of the Wecken property for n-valued maps of the 2-sphere, for each integer d we will exhibit a 2-valued map $\phi_d\colon S^2 \multimap S^2$ of degree $|d|$ that has $N(\phi_d)$ fixed points. If $\phi = \{f_0, f_1\}\colon S^2 \multimap S^2$, then $N(\phi) = N(f_0) + N(f_1)$ by Corollary 7.2 of [8]. For $f\colon S^2 \to S^2$, we have $N(f) = 1$ except that $N(f) = 0$ if $\deg(f) = -1$ so $N(\phi) = 1$ if $\deg(\phi) = 1$ and $N(\phi) = 2$ otherwise.

A constant 2-valued map $\phi_0\colon S^2 \multimap S^2$ has two fixed points. For $|d| = 1$, let $\phi_1 = \{f_0, f_1\}$ where f_0 is a small deformation of the identity map with one fixed point and f_1 is the antipodal map, so ϕ_1 has one fixed point.

Proposition 5. *For each integer d with $|d| \geq 2$, there exists a 2-valued map $\phi_d = \{f_0, f_1\}\colon S^2 \multimap S^2$ such that $\deg(\phi_d) = |d|$ and ϕ_d has $N(\phi_d) = 2$ fixed points.*

Proof. Choose a map $q\colon S^1 \to S^1$ of degree $d+1$. Let τ be the tangent bundle of S^2, then $\mathbb{R} \oplus \tau$ is the trivial bundle with fiber $\mathbb{R} \oplus V$ where $V = \mathbb{R}^2$. We can think of selfmaps of S^2 as sections of the trivial sphere bundle $S(\mathbb{R} \oplus V)$ with the identity map corresponding to the constant section $(1, \mathbf{0})$, where $\mathbf{0} \in V$ is the zero vector. Therefore, for $\phi_d = \{f_0, f_1\}$, the maps $f_0, f_1\colon S^2 \to S^2$ will be specified by sections s_0 and s_1 with $s_0(x) \neq s_1(x)$ for all $x \in S^2$.

Consider the hemispheres

$$S_+ = \{(t, \mathbf{u}) \in S(\mathbb{R} \oplus V) \,|\, t \geq 0\}, \quad S_- = \{(t, \mathbf{u}) \in S(\mathbb{R} \oplus V) \,|\, t \leq 0\}$$

and choose orthogonal trivializations $\theta_\pm\colon \tau|S_\pm \to V$ over the hemispheres such that θ_+ restricts to the identity at the north pole $(1,\mathbf{0})$ and θ_- is the identity at the south pole $(-1,\mathbf{0})$. Let $\kappa\colon S(V)\to O(V)$, where $S(V)$ denotes the unit sphere in V and $O(V)$ the orthogonal transformations of V, be the clutching map on the equator: $\kappa(v)(\theta_+)_v = (\theta_-)_v$ for $v\in S(V)$.

We define the sections s_0 and s_1 as follows. For $v\in S(V)$ and $0\le t\le 1$, let

$$(1\oplus\theta_+)s_0(t,(1-t^2)^{1/2}v) = (t,(1-t^2)^{1/2}q(v))$$
$$(1\oplus\theta_+)s_1(t,(1-t^2)^{1/2}v) = (-t,-(1-t^2)^{1/2}q(v))$$

and for $-1\le t\le 0$ let

$$(1\oplus\theta_-)s_0(t,(1-t^2)^{1/2}v) = (t,(1-t^2)^{1/2}\kappa(v)q(v))$$
$$(1\oplus\theta_-)s_1(t,(1-t^2)^{1/2}v) = (-t,-(1-t^2)^{1/2}\kappa(v)q(v)).$$

By construction, $s_0(x)\neq s_1(x)$ for all $x\in S(\mathbb{R}\oplus V)$. Furthermore, $s_0(x)=(1,\mathbf{0})$ if and only if $x=(1,\mathbf{0})$ and $s_1(x)=(-1,\mathbf{0})$ if and only if $x=(-1,\mathbf{0})$. Thus, the north pole $(1,\mathbf{0})$ is the single fixed point of f_0 and the south pole $(-1,\mathbf{0})$ is the fixed point of f_1.

In a neighborhood of its fixed point $(1,\mathbf{0})$, the map f_0 can be described in suitable coordinates by the self-map of V that takes rv, for $r\ge 0$ and $v\in S(V)$, to $rv+rq(v)$ so that its fixed point is $\mathbf{0}\in V$. The Lefschetz index of f_0 is equal to the degree of the map $-q\colon S(V)\to S(V)$, that is, $\deg(q)$. We conclude that $1+\deg(f_0)=\deg(q)$ so, since $\deg(q)=d+1$, then $\deg(f_0)=d$ and consequently $\deg(f_1)=-d$ by Lemma 1; thus $\deg(\phi_d)=|d|$.[2] □

Propositions 4 and 5 complete the proof of

Theorem 6. *The 2-sphere S^2 has the Wecken property for n-valued maps, that is, every n-valued map $\phi\colon S^2 \multimap S^2$ is n-valued homotopic to a map with $N(\phi)$ fixed points.*

References

[1] Brown, R.: Fixed points of n-valued multimaps of the circle. Bull. Polish Acad. Sci. Math. **54**, 153–162 (2006)

[2] Brown, R., Gonçalves, D.: On the topology of n-valued maps. Adv. Fixed Point Theory **8**, 205–220 (2018)

[3] Fadell, E., Husseini, S.: Geometry and Topology of Configuration Spaces, Springer Monographs in Mathematics (2000)

[2] The same construction, with $V=\mathbb{R}^m, m\ge 2$, and q of degree $d+(-1)^m$, produces an explicit 2-valued map $\{f_0,f_1\}\colon S^m \multimap S^m$ with $\deg f_0 = d$ and $\deg f_1 = (-1)^{m+1}d$. For spheres S^m with $m>2$ this construction may be used to realize the general result of Schirmer [9].

[4] Gonçalves, D., Guaschi, J.: Fixed points of n-valued maps on surfaces and the Wecken property—a configuration space approach. Sci. China Math. **60**, 1561–1574 (2017)

[5] Gonçalves, D., Guaschi, J.: Fixed points of multimaps, the fixed point property and the case of surfaces—a braid approach. Indag. Math. **29**, 91–124 (2018)

[6] Jiang, B.: Fixed points and braids, II. Math. Ann. **272**, 249–256 (1985)

[7] Jiang, B.: The Wecken property of the projective plane. Nielsen Theory Reidemeister Torsion Banach Center Publ. **49**, 223–225 (1999)

[8] Schirmer, H.: An index and Nielsen number for n-valued multifunctions. Fund. Math. **121**, 201–219 (1984)

[9] Schirmer, H.: A minimum theorem for n-valued multifunctions. Fund. Math. **126**, 83–92 (1985)

[10] Wecken, F.: Fixpunktklassen, III. Math. Ann. **118**, 544–577 (1942)

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