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On Betti Tables, Monomial Ideals, and Unit Groups

by

Yi-Chang Chen

A dissertation submitted in partial satisfaction of the

requirements for the degree of

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in

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in the

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of the

University of California, Berkeley

Committee in charge:

Professor David Eisenbud, Chair

Professor Mark Haiman

Professor Kam-Biu Luk

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Abstract

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This thesis explores two topics in commutative algebra. The first topic is Betti tables, particularly of monomial ideals, and how these relate to Betti tables of arbitrary graded ideals. We systematically study the concept of mono , the largest monomial subideal of a given ideal, and for an Artinian ideal I , deduce relations in the last column of the Betti tables of I and $\text{mono}(I)$. We then apply this philosophy towards a conjecture of Postnikov-Shapiro, concerning Betti tables of certain ideals generated by powers of linear forms: by studying monomial subideals of the so-called power ideal, we deduce special cases of this conjecture.

The second topic concerns the group of units of a ring. Motivated by the question of when a surjection of rings induces a surjection on unit groups, we give a general sufficient condition for induced surjectivity to hold, and introduce a new class of rings, called semi-fields, in the process. As units are precisely the elements which avoid all the maximal ideals, we then investigate infinite prime avoidance in general, and in this direction, produce an example of a ring that is not a semi-field, for which surjectivity on unit groups still holds.

To my parents, Kuo-Cheng Chen and Shih-Ying Pan

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This thesis would also not be possible without the people who provided me inspiration for the questions which are dealt with here. I am indebted to Bill Heinzer for posing the question which would turn into the main subject of Chapter 3, on surjectivity of unit groups. Even more so, I owe a great deal of thanks to Madhusudan Manjunath, who first told me of the Postnikov-Shapiro conjecture (which is the main subject of Chapter 2), and generously enabled collaboration both at Berkeley and at Oberwolfach. Our joint discussions with Joe Kileel, despite not producing any concrete results at the time, directly inspired nearly all the results in Chapter 2.

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Introduction

In the first half of this thesis, we study monomial ideals and their Betti tables. Monomial ideals are a rich class of ideals for study: on one hand, they exhibit much more structure than arbitrary ideals – geometrically, they correspond to (thickened) unions of coordinate planes – and can be approached combinatorially, especially in the squarefree case (which can often be reduced to by polarization). On the other hand, for many purposes monomial ideals capture a great deal of the complexity of arbitrary ideals, most often via the procedure of Gröbner degeneration, which preserves Hilbert functions (and thus substantial geometric information, such as dimension and degree).

However, Betti tables – being finer invariants than the Hilbert function – are in general not preserved by Gröbner degenerations. With this in mind, a unifying theme for the first two chapters is the goal of relating Betti tables of arbitrary ideals to Betti tables of associated monomial ideals, which arise in a different manner than taking initial ideals. In the next chapter, we consider a special class of graded ideals coming from graph theory, and a similarly special class of associated monomial ideals. In the present chapter though, we examine a different (yet natural) way to associate a monomial ideal to any ideal – namely, by considering the largest monomial subideal of a given ideal.

Chapter 1

Mono

Let $R = k[x_1, \dots, x_n]$ be a polynomial ring over a field k in n variables. For any ideal $I \subseteq R$, let $\text{mono}(I)$ denote the largest monomial subideal of I , i.e. the ideal generated by all monomials contained in I . Geometrically, $\text{mono}(I)$ defines the smallest torus-invariant subscheme containing $V(I) \subseteq \text{Spec } R$ (the so-called *torus-closure* of $V(I)$).

The concept of *mono* has been relatively unexplored, despite the naturality of the definition. The existing work in the literature concerning *mono* has been essentially algorithmic and/or computational. For convenience, we summarize this in the following two theorems:

Theorem 1.0.1 ([23], Algorithm 4.4.2). *Let $I = (f_1, \dots, f_r)$. Fix new variables y_1, \dots, y_n , and let $\tilde{f}_i := f_i(\frac{x_1}{y_1}, \dots, \frac{x_n}{y_n}) \cdot \prod_{i=1}^n y_i^{\deg_{x_i}(f)}$ be the multi-homogenization of f_i with respect to y . Let $>$ be an elimination term order on $k[\underline{x}, y]$ satisfying $y_i > x_j$ for all i, j . If \mathcal{G} is a reduced Gröbner basis for $(\tilde{f}_1, \dots, \tilde{f}_r) : (\prod_{i=1}^n y_i)^\infty$ with respect to $>$, then the monomials in \mathcal{G} generate $\text{mono}(I)$.*

Cf. also [16] for a generalization computing the largest A -graded subideal of an ideal, for an integer matrix A (*mono* being the special case when A is the identity matrix). The next theorem gives an alternate description of *mono* for a particular class of ideals, involving the dual concept of *Mono*, which is the smallest monomial ideal containing a given ideal (notice that $\text{Mono}(I)$ is very simple to compute, being generated by all terms appearing in a generating set of I).

Theorem 1.0.2 ([20], Lemma 3.2). *Let I be an unmixed ideal, and suppose there exists a regular sequence $\underline{\beta} \subseteq I$ consisting of $\text{codim } I$ monomials. Then $\text{mono}(I) = (\underline{\beta}) : \text{Mono}((\underline{\beta}) : I)$.*

However, it appears that no systematic study of *mono* as a operation on ideals has yet been made. It is the goal of this note [2] to provide first steps in this direction; in particular exploring the relationship between I and $\text{mono}(I)$. By way of understanding *mono* as an algebraic process, we consider the following questions:

1. When is $\text{mono}(I) = 0$, or prime, or primary, or radical?
2. To what extent does taking mono depend on the ground field?
3. Which invariants are preserved by taking mono ? For instance, do I and $\text{mono}(I)$ have the same (Castelnuovo-Mumford) regularity?
4. How do the Betti tables of I and $\text{mono}(I)$ compare? Do they have the same shape?
5. To what extent is mono non-unique? E.g. which monomial ideals arise as mono of a non-monomial ideal?
6. What properties of I are preserved by $\text{mono}(I)$, and conversely, what properties of I are reflected by $\text{mono}(I)$?

1.1 Basic properties

We first give some basic properties of mono , which describe how mono interacts with various algebraic operations. As above, R denotes a polynomial ring $k[x_1, \dots, x_n]$, and $\mathfrak{m} = (x_1, \dots, x_n)$ denotes the homogeneous maximal ideal of R .

Proposition 1.1.1. *Let I be an R -ideal.*

1. mono is decreasing, inclusion-preserving, and idempotent.
2. mono commutes with radicals, i.e. $\text{mono}(\sqrt{I}) = \sqrt{\text{mono}(I)}$.
3. mono commutes with intersections, i.e. $\text{mono}(\bigcap I_i) = \bigcap \text{mono}(I_i)$ for any ideals I_i .
4. $\text{mono}(I_1)\text{mono}(I_2) \subseteq \text{mono}(I_1I_2) \subseteq \text{mono}(I_1) \cap \text{mono}(I_2)$.

Proof. 1. Each property – $\text{mono}(I) \subseteq I$, $I_1 \subseteq I_2 \implies \text{mono}(I_1) \subseteq \text{mono}(I_2)$, $\text{mono}(\text{mono}(I)) = \text{mono}(I)$ – is clear from the definition.

2. If $u \in \sqrt{\text{mono}(I)}$ is a monomial, say $u^m \in \text{mono}(I)$, then $u \in \sqrt{I} \implies u \in \text{mono}(\sqrt{I})$. Conversely, if $u \in \text{mono}(\sqrt{I})$ is monomial, then $u \in \sqrt{I}$, say $u^m \in I$ and hence $u^m \in \text{mono}(I) \implies u \in \sqrt{\text{mono}(I)}$.

3. $\bigcap I_i \subseteq I_i \implies \text{mono}(\bigcap I_i) \subseteq \text{mono}(I_i)$, hence $\text{mono}(\bigcap I_i) \subseteq \bigcap \text{mono}(I_i)$. On the other hand, an arbitrary intersection of monomial ideals is monomial, and $\bigcap \text{mono}(I_i) \subseteq \bigcap I_i$, hence $\bigcap \text{mono}(I_i) \subseteq \text{mono}(\bigcap I_i)$.

4. $\text{mono}(I_1)\text{mono}(I_2) \subseteq I_1I_2$, and a product of two monomial ideals is monomial, hence $\text{mono}(I_1)\text{mono}(I_2) \subseteq \text{mono}(I_1I_2)$. The second containment follows from applying (1) and (3) to the containment $I_1I_2 \subseteq I_1 \cap I_2$. \square

Next, we consider how prime and primary ideals behave under taking mono:

Proposition 1.1.2. *Let I be an R -ideal.*

1. *If I is prime resp. primary, then so is $\text{mono}(I)$.*
2. *$\text{Ass}(R/\text{mono}(I)) \subseteq \{\text{mono}(P) \mid P \in \text{Ass}(R/I)\}$.*
3. *$\text{mono}(I)$ is prime iff $\text{mono}(I) = \text{mono}(P)$ for some minimal prime P of I . In particular, $\text{mono}(I) = 0$ iff $\text{mono}(P) = 0$ for some $P \in \text{Min}(I)$.*

Proof. 1. To check that $\text{mono}(I)$ is prime (resp. primary), it suffices to check that if u, v are monomials with $uv \in \text{mono}(I)$, then $u \in \text{mono}(I)$ or $v \in \text{mono}(I)$ (resp. $v^n \in \text{mono}(I)$ for some n). But this holds, as I is prime (resp. primary) and u, v are monomials.

2. If $I = Q_1 \cap \dots \cap Q_r$ is a minimal primary decomposition of I , so that $\text{Ass}(R/I) = \{\sqrt{Q_i}\}$, then $\text{mono}(I) = \text{mono}(Q_1) \cap \dots \cap \text{mono}(Q_r)$ is a primary decomposition of $\text{mono}(I)$, so every associated prime of $\text{mono}(I)$ is of the form $\sqrt{\text{mono}(Q_i)} = \text{mono}(\sqrt{Q_i})$ for some i .

3. $\text{mono}(I) = \sqrt{\text{mono}(I)} = \text{mono}(\sqrt{I}) = \bigcap_{P \in \text{Min}(I)} \text{mono}(P)$. Since $\text{mono}(I)$ is prime and the intersection is finite, $\text{mono}(I)$ must equal one of the terms in the intersection. The converse follows from (1). \square

We now examine the sharpness of various statements in Propositions 1.1.1 and 1.1.2:

Example 1.1.3. Let $R = k[x, y]$, where k is an infinite field, and let I be an ideal generated by 2 random quadrics. Then R/I is an Artinian complete intersection of regularity 2, so $\mathfrak{m}^3 \subseteq I$. By genericity, I does not contain any monomials in degrees $\leq \text{reg}(R/I)$, so $\text{mono}(I) = \mathfrak{m}^3$. On the other hand, I^2 is a 3-generated perfect ideal of grade 2, so the Hilbert-Burch resolution of R/I^2 shows that $\text{reg } R/I^2 = 4$, hence $\mathfrak{m}^5 \subseteq I^2 \subseteq \mathfrak{m}^4$ as I^2 is generated by quartics (in fact, $\text{mono}(I^2) = \mathfrak{m}^5$). Thus for such $I = I_1 = I_2$, both containments in Proposition 1.1.1(4) are strict.

Similar to Proposition 1.1.1(4), the containment in Proposition 1.1.2(2) is also strict in general: take e.g. $I = I' \cap \mathfrak{m}^N$ where $\text{mono}(I') = 0$ and $N > 0$ is such that $I' \not\subseteq \mathfrak{m}^N$. However, combining these two statements yields:

Corollary 1.1.4. *Let I be an R -ideal.*

1. *Nonzerodivisors on R/I are also nonzerodivisors on $R/\text{mono}(I)$.*
2. *Let $u \in R$ be a monomial that is a nonzerodivisor on R/I . Then $\text{mono}((u)I) = (u)\text{mono}(I)$.*

Proof. 1.
$$\bigcup_{P \in \text{Ass}(R/\text{mono}(I))} P \subseteq \bigcup_{P \in \text{Ass}(R/I)} \text{mono}(P) \subseteq \bigcup_{P \in \text{Ass}(R/I)} P.$$

2. This follows from (1) and Proposition 1.1.1(4). □

Remark 1.1.5. Since monomial prime ideals are generated by (sets of) variables, if $P \subseteq \mathfrak{m}^2$ is a nondegenerate prime, then $\text{mono}(P) = 0$. It thus follows from Proposition 1.1.2(3) that “most” ideals I satisfy $\text{mono}(I) = 0$: namely, this is always the case unless each component of $V(I)$ is contained in some coordinate hyperplane in $\mathbb{A}^n = \text{Spec } R$. The case that $\text{mono}(I)$ is prime is analogous: if $\text{mono}(I) = (x_{i_1}, \dots, x_{i_r})$ for some $\{i_1, \dots, i_r\} \subseteq \{1, \dots, n\}$, then $V(I)$ becomes nondegenerate upon restriction to the coordinate subspace $V(x_{i_1}, \dots, x_{i_r}) \cong \mathbb{A}^r$ (i.e. $\text{mono}(\bar{I}) = \bar{0}$ where $\bar{(\cdot)}$ denotes passage to the quotient $R/(x_{i_1}, \dots, x_{i_r})$).

In contrast to the simple picture when $\text{mono}(I)$ is prime, the case where $\text{mono}(I)$ is primary is much more interesting, due to nonreducedness issues. The foremost instance of this case is when $\text{mono}(I)$ is \mathfrak{m} -primary, i.e. $\text{mono}(I)$ is Artinian. A first indication that this case is interesting is that under this assumption, $\text{mono}(I)$ is guaranteed not to be 0. For this and other reasons soon to appear, we will henceforth deal primarily with this case – the reader should assume for the remainder of this chapter,

Unless stated otherwise, I will henceforth denote an Artinian ideal.

1.2 Dependence on scalars

We now briefly turn to Question 2: to what extent does taking mono depend on the ground field k ? To make sense of this, let $S = \mathbb{Z}[x_1, \dots, x_n]$ be a polynomial ring over \mathbb{Z} . Then for any field k , the universal map $\mathbb{Z} \rightarrow k$ induces a ring map $S \rightarrow S_k := S \otimes_{\mathbb{Z}} k = k[x_1, \dots, x_n]$. Given an ideal $I \subseteq S$, one can consider the extended ideal IS_k . The question is then: as the field k varies, how does $\text{mono}(IS_k)$ change?

It is easy to see that if k_1 and k_2 have the same characteristic, then $\text{mono}(IS_{k_1})$ and $\text{mono}(IS_{k_2})$ have identical minimal generating sets. Thus it suffices to consider prime fields \mathbb{Q} and \mathbb{F}_p , for $p \in \mathbb{Z}$ prime. Another moment’s thought shows that mono can certainly change in passing between different characteristics; e.g. if all but one of the coefficients of some generator of I is divisible by a prime p . However, even excluding simple examples like this, by requiring that the generators of I all have unit coefficients, mono still exhibits dependence on characteristic. We illustrate this with a few examples:

Example 1.2.1. Let $S = \mathbb{Z}[x, y, z]$ be a polynomial ring in 3 variables.

(1) Set $I := (x^3, y^3, z^3, xy(x + y + z))$. Then $xyz^2 \in \text{mono}(IS_k)$ iff $\text{char } k = 2$ (consider $xy(x + y + z)^2 \in I$). Notice that I is equi-generated, i.e. all minimal generators of I have the same degree.

(2) For a prime $p \in \mathbb{Z}$, set $I_p := (x^p, y^p, x + y + z)$. Then $z^p \in \text{mono}(I_p S_k)$ iff $\text{char } k = p$ (consider $(x + y + z)^p \in I_p$). If the presence of the linear form is objectionable, one may increase the degrees, e.g. $(x^{2p}, y^{2p}, x^2 + y^2 + z^2)$.

From these examples we see that mono is highly sensitive to characteristic in general. However, this is not the whole story: cf. Remark 1.4.2 for one situation where taking mono is independent of characteristic.

1.3 Relations on Betti tables

We now consider how invariants of I behave when passing to $\text{mono}(I)$. As mentioned in Remark 1.1.5, although I and $\text{mono}(I)$ are typically quite different, for Artinian graded ideals there is a much closer relationship:

Proposition 1.3.1. *Let I be a graded R -ideal. Then I is Artinian iff $\text{mono}(I)$ is Artinian. In this case, $\text{reg}(R/I) = \text{reg}(R/\text{mono}(I))$.*

Proof. Since $I \subseteq \mathfrak{m}$ is graded, I is Artinian iff $\mathfrak{m}^s \subseteq I$ for some $s > 0$. This occurs iff $\mathfrak{m}^s \subseteq \text{mono}(I)$ for some $s > 0$ iff $\text{mono}(I)$ is Artinian.

Next, recall that if $M = \bigoplus M_i$ is Artinian graded, then the regularity of M is $\text{reg } M = \max\{i \mid M_i \neq 0\}$. The inclusion $\text{mono}(I) \subseteq I$ induces a (graded) surjection $R/\text{mono}(I) \rightarrow R/I$, which shows that $\text{reg}(R/\text{mono}(I)) \geq \text{reg}(R/I)$. Now if $u \in R$ is a standard monomial of $\text{mono}(I)$ of top degree ($= \text{reg}(R/\text{mono}(I))$), then $u \notin \text{mono}(I) \implies u \notin I$, hence $\text{reg}(R/I) \geq \text{deg } u = \text{reg}(R/\text{mono}(I))$. \square

A restatement of Proposition 1.3.1 is that for any Artinian graded ideal I , the graded Betti tables of I and $\text{mono}(I)$ have the same number of rows and columns (since any Artinian ideal has projective dimension $n = \dim R$). However, it is not true (even in the Artinian case) that the Betti tables of I and $\text{mono}(I)$ have the same shape ($=$ (non)zero pattern) – e.g. take an ideal I' with $\text{mono}(I') = 0$, and consider $I := I' + \mathfrak{m}^N$ for $N \gg 0$. Despite this, there is one positive result in this direction:

Proposition 1.3.2. *Let I be an Artinian graded R -ideal. Then $\beta_{n,j}(R/\text{mono}(I)) \neq 0 \implies \beta_{n,j}(R/I) \neq 0$, for any j .*

Proof. Notice that $\beta_{n,j}(R/I) \neq 0$ iff the socle of R/I contains a nonzero form of degree j . Let $m \in R$ be a monomial with $\bar{0} \neq \bar{m} \in \text{soc}(R/\text{mono}(I))$ and $\text{deg } m = j$. Then $m \in (\text{mono}(I) :_R \mathfrak{m}) \setminus \text{mono}(I)$, hence $m \in (I :_R \mathfrak{m}) \setminus I$ as well, i.e. $\bar{0} \neq \bar{m} \in \text{soc}(R/I)$. \square

Corollary 1.3.3. *Let I be an Artinian graded level R -ideal (i.e. $\text{soc}(R/I)$ is nonzero in only one degree). Then $\text{mono}(I)$ is also level, with the same socle degree as I .*

Proof. Follows immediately from Proposition 1.3.2. \square

We illustrate these statements with some examples of how the Betti tables of R/I and $R/\text{mono}(I)$ can differ:

Example 1.3.4. Let $R = k[x, y, z, w]$, $J = I_2 \begin{pmatrix} x & y^2 & yw & z \\ y & xz & z^2 & w \end{pmatrix}$ the ideal of the rational quartic curve in \mathbb{P}^3 , and $I = J + (x^2, y^4, z^4, w^4)$. Then the Betti tables of R/I and $R/\text{mono}(I)$ respectively in Macaulay2 format are:

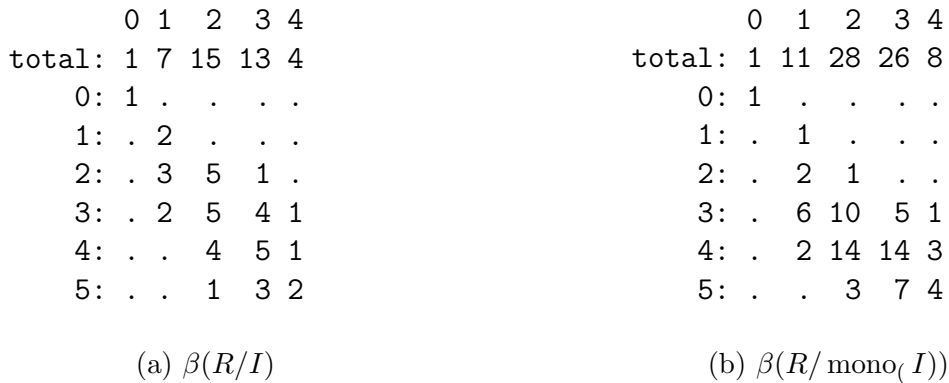


Figure 1.1: Artinian reduction of twisted quartic

Notice that $\beta_{1,5}(R/\text{mono}(I)) = 2 \neq 0 = \beta_{1,5}(R/I)$, and likewise $\beta_{3,5}(R/I) = 1 \neq 0 = \beta_{3,5}(R/\text{mono}(I))$. In addition, $\beta_{1,2}, \beta_{1,3}, \beta_{2,4}$ for $R/\text{mono}(I)$ are all strictly smaller than their counterpart for R/I (and still nonzero).

Example 1.3.5. Let $R = k[x, y, z]$, $\ell \in R_1$ a general linear form (e.g. $\ell = x + y + z$), and $I = (x\ell, y\ell, z\ell) + (x, y, z)^3$. Then $\text{mono}(I) = (x, y, z)^3$, and the Betti tables of R/I and $R/\text{mono}(I)$ respectively are:



Figure 1.2: $\text{mono}(I)$ level but I not level

Here $\text{mono}(I)$ is level, but I is not. This shows that the converse to Corollary 1.3.3 is not true in general.

Example 1.3.6. We revisit Example 1.1.3. Since $\text{mono}(I)$ is a power of the maximal ideal, $R/\text{mono}(I)$ has a linear resolution, whereas R/I has a Koszul resolution (with no linear

forms), so in view of Proposition 1.3.2, the Betti tables of R/I and $R/\text{mono}(I)$ have as disjoint shapes as possible. Thus no analogue of Proposition 1.3.2 can hold for β_i , $i < n$, in general.

From the examples above, one can see that the earlier propositions on Betti tables are fairly sharp. Another interesting pattern observed above is that even when the ideal-theoretic description of $\text{mono}(I)$ became simpler than that of I , the Betti table often grew worse (e.g. had larger numbers on the whole). This leads to some natural refinements of Questions 4 and 6:

7. Are the total Betti numbers of $\text{mono}(I)$ always at least those of I ?
8. Does $\text{mono}(I)$ Gorenstein imply I Gorenstein?

Notice that the truth of Question 7 implies the truth of Question 8. As it turns out, the answer to these will follow from the answer to Question 5.

1.4 Uniqueness and the Gorenstein property

Lemma 1.4.1. *Let M be a monomial ideal, and $u_1 \neq u_2$ standard monomials of M . Then $\text{mono}(M + (u_1 + u_2)) = M$ iff $M : u_1 = M : u_2$.*

Proof. \Rightarrow : By symmetry, it suffices to show that $M : u_1 \subseteq M : u_2$. Let m be a monomial in $M : u_1$. Then $mu_2 = m(u_1 + u_2) - mu_1 \in \text{mono}(M + (u_1 + u_2)) = M$, i.e. $m \in M : u_2$.

\Leftarrow : Passing to R/M , it suffices to show that $(\bar{u}_1 + \bar{u}_2)$ contains no monomials in R/M . Let $g \in R$ be such that $\bar{g}(\bar{u}_1 + \bar{u}_2) \neq \bar{0} \in R/M$, and write $g = g_1 + \dots + g_s$ as a sum of monomials. By assumption, $g_i u_1 \in M$ iff $g_i u_2 \in M$, so after removing some terms of g we may assume there exists g_i of top degree in g such that $g_i u_1, g_i u_2 \notin M$. But then $\bar{g}_i \bar{u}_1$ and $\bar{g}_i \bar{u}_2$ both appear as distinct terms in $\bar{g}(\bar{u}_1 + \bar{u}_2)$, so $\bar{g}(\bar{u}_1 + \bar{u}_2)$ is not a monomial in R/M . \square

Remark 1.4.2. Since colons of monomial ideals are characteristic-independent, the second condition in Lemma 1.4.1 is independent of the ground field k . Thus if I is an ideal defined over \mathbb{Z} which is “nearly” monomial (i.e. is generated by monomials and a single binomial), and $\text{mono}(I)$ is as small as possible in one characteristic, then $\text{mono}(I)$ is the same in all characteristics.

Remark 1.4.3. For any polynomial $f \in R$, it is easy to see that

$$\text{mono}(M + (f)) \supseteq M + \sum_{u \in \text{terms}(f)} \text{mono}(M : f - u)u$$

If $f = u_1 + u_2$ is a binomial, then this simplifies to the statement that $\text{mono}(M + (u_1 + u_2)) \supseteq M + (M : u_2)u_1 + (M : u_1)u_2$. However, equality need not hold: e.g. $M = (x^6, y^6, x^2 y^4)$, $u_1 = x^2 y$, $u_2 = x y^2$ (or even $M = (x^3, y^2)$, $u_1 = x$, $u_2 = y$ if one allows linear forms).

Theorem 1.4.4. *The following are equivalent for a monomial ideal M :*

- (1) *There exists a graded non-monomial ideal I such that $\text{mono}(I) = M$.*
- (2) *There exist $t \geq 2$ monomials u_1, \dots, u_t not contained in M and of the same degree, such that $M : u_i = M : u_j$ for all i, j .*
- (3) *There exist monomials $u_1 \neq u_2$ with $u_1, u_2 \notin M$, $\deg u_1 = \deg u_2$ and $M : u_1 = M : u_2$.*

Proof. (1) \implies (2): Fix $f \in I \setminus M$ graded of minimal support size t (so $t \geq 2$), and write $f = u_1 + \dots + u_t$ where u_i are standard monomials of M of the same degree. Fix $1 \leq i \leq t$, and pick a monomial $m \in M : u_i$. Then $m(f - u_i) \in I$ has support size $< t$, so minimality of t gives $m(f - u_i) = \sum_{j \neq i} mu_j \in M$. Since M is monomial, $mu_j \in M$ for each $j \neq i$, i.e. $m \in M : u_j$ for all j . By symmetry, $M : u_i = M : u_j$ for all i, j .

(2) \implies (3): Clear.

(3) \implies (1): Set $I := M + (u_1 + u_2)$, and apply Lemma 1.4.1. Notice that I is not monomial: if it were, then $u_1 + u_2 \in I \implies u_1 \in I \implies u_1 \in \text{mono}(I) = M$, contradiction. \square

Corollary 1.4.5. *Let I be an Artinian graded R -ideal. Then $\text{mono}(I)$ is a complete intersection iff $\text{mono}(I)$ is Gorenstein iff $I = \mathfrak{m}^{\mathbf{b}} := (x_1^{\mathbf{b}_1}, \dots, x_n^{\mathbf{b}_n})$ for some $\mathbf{b} = (\mathbf{b}_i) \in \mathbb{N}^n$.*

Proof. Any Artinian Gorenstein monomial ideal is irreducible, hence is of the form $\mathfrak{m}^{\mathbf{b}}$, which is a complete intersection. By Theorem 1.4.4, it suffices to show that for $M := \mathfrak{m}^{\mathbf{b}}$, no distinct standard monomials of M satisfy $M : u_1 = M : u_2$. To see this, note that since $u_1 \neq u_2$, there exists $j \in [n]$ such that x_j appears to different powers $a_1 \neq a_2$ in u_1 and u_2 , respectively. Taking $a_1 < a_2$ WLOG gives $x_j^{\mathbf{b}_j - a_2} \in (M : u_2) \setminus (M : u_1)$. \square

Combining the proofs above shows that an Artinian monomial ideal is not expressible as mono of *any* non-monomial ideal iff it is Gorenstein:

Corollary 1.4.6. *Let M be an Artinian monomial ideal. Then there exists a non-monomial R -ideal I with $\text{mono}(I) = M$ iff M is not Gorenstein (iff M is not of the form $\mathfrak{m}^{\mathbf{b}}$ for $\mathbf{b} \in \mathbb{N}^n$).*

Proof. \implies : If $M = \text{mono}(I)$ were Gorenstein, then by Corollary 1.4.5 I is necessarily of the form $\mathfrak{m}^{\mathbf{b}}$, contradicting the hypothesis that I is non-monomial.

\Leftarrow : Since M is not Gorenstein, there exist monomials $u_1 \neq u_2$ in the socle of R/M . Then $u_1 \in M : \mathfrak{m} \implies \mathfrak{m} \subseteq M : u_1 \implies \mathfrak{m} = M : u_1$, and similarly $\mathfrak{m} = M : u_2$. By Lemma 1.4.1, $I := M + (u_1 + u_2)$ is a non-monomial ideal with $\text{mono}(I) = M$. \square

As evidenced by Remark 1.4.3, finding $\text{mono}(M + (f))$ can be subtle, for arbitrary $f \in R$. There is one situation however which can be determined completely:

Theorem 1.4.7. *Let M be a monomial ideal, and let u_1, \dots, u_r be the socle monomials of R/M . Let $f_j := \sum_{i=1}^r a_{ij}u_i$, $1 \leq j \leq s$, be k -linear combinations of the u_i . Then $\text{mono}(M + (f_1, \dots, f_s)) = M$ iff no standard basis vector \mathbf{e}_i is in the column span of the matrix (a_{ij}) over k .*

Proof. Let $v \in \text{mono}(M + (f_1, \dots, f_r))$ be a monomial. Pick $g_i \in R$ and $m \in M$ such that $v = m + \sum_{j=1}^s g_j f_j$. Write $g_j = b_j + g'_j$ where $b_j \in k$ and $g'_j \in \mathfrak{m}$. Since $f_j \in \text{soc}(R/M)$, this is the same as saying $\bar{v} = \sum_{j=1}^s b_j \bar{f}_j$ in R/M . Since v is a monomial, it must appear as one of the terms in the sum, hence v must be a socle monomial of M . Then $v = u_i$ for some i , so v corresponds to a standard basis vector \mathbf{e}_i , and then writing v as a k -linear combination of f_j is equivalent to writing \mathbf{e}_i as a k -linear combination of the columns of (a_{ij}) . \square

Example 1.4.8. Let $R = k[x, y, z]$, and set $M := (x^2, xy, xz, y^2, z^2)$. Then R/M has a 2-dimensional socle $k\langle x, yz \rangle$, so $\text{mono}(M + (x + yz)) = M$ by Lemma 1.4.1 or Theorem 1.4.7. However, the only standard monomials of M of the same degrees are x, y, z , which have distinct colons into M . Thus by Theorem 1.4.4 there is no *graded* non-monomial I with $\text{mono}(I) = M$.

In general, even if there are u_1, u_2 of the same degree with $M : u_1 = M : u_2$, there may not be any such in top degree: e.g. $(x^2, y^2)\mathfrak{m} + (z^3)$ is equi-generated with symmetric Hilbert function $1, 3, 6, 3, 1$, but is not level (hence not Gorenstein).

Example 1.4.9. Fix $\mathbf{b} = (b_i) \in \mathbb{N}^n$ with $b_i \geq 2$ for all i . Then the irreducible ideal $\mathfrak{m}^{\mathbf{b}}$ has a unique socle element $x^{\mathbf{b}-1} := x_1^{b_1-1} \dots x_n^{b_n-1}$. Let $M := \mathfrak{m}^{\mathbf{b}} + (x^{\mathbf{b}-1})$, which is Artinian level with n -dimensional socle $\langle \frac{x^{\mathbf{b}-1}}{x_i} \mid 1 \leq i \leq n \rangle =: \langle s_1, \dots, s_n \rangle$. By setting all socle elements of M equal to each other we obtain a graded ideal $I := M + (s_1 - s_i \mid 2 \leq i \leq n)$. As all the non-monomial generators are in the socle of R/M , we may apply Theorem 1.4.7: the coefficient matrix (a_{ij}) is given by

$$\begin{bmatrix} 1 & 1 & \dots & 1 \\ -1 & 0 & \dots & 0 \\ 0 & -1 & \dots & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \dots & 0 & -1 \end{bmatrix}$$

which by inspection has no standard basis vectors in its column span, so by Theorem 1.4.7, $\text{mono}(I) = M$. For $\mathbf{b} = (2, 2, 3, 3)$, the Betti tables of R/I and $R/\text{mono}(I)$ respectively are:

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Figure 1.3: Betti numbers of $\text{mono}(I)$ less than those of I

Notice that the (total) Betti numbers of R/I are strictly greater than those of $R/\text{mono}(I)$. This shows that the answer to Question 7 is false in general.

Finally, we include a criterion for recognizing when a monomial subideal of I is equal to $\text{mono}(I)$, in terms of its socle monomials:

Proposition 1.4.10. *Let I be an R -ideal and $M \subseteq I$ an Artinian monomial ideal. Then the following are equivalent:*

- (a) $M = \text{mono}(I)$
- (b) I contains no socle monomials of M
- (c) $(M : \mathfrak{m}) \cap \text{mono}(I) \subseteq M$

Proof. (a) \implies (b), (a) \implies (c): Clear.

(b) \implies (a): Recall that for an Artinian monomial ideal M , a monomial $x^{\mathbf{b}}$ is nonzero in $\text{soc}(R/M)$ iff $\mathbf{m}^{\mathbf{b}+1}$ appears in the unique irreducible decomposition of M into irreducible monomial ideals (here $\mathbf{b} + 1 = (b_1 + 1, \dots, b_n + 1)$: cf. [17], Exercise 5.8). Let $x^{\mathbf{b}_1}, \dots, x^{\mathbf{b}_r}$ be the socle monomials of R/M . Then by assumption $x^{\mathbf{b}_1}, \dots, x^{\mathbf{b}_r}$ are also socle monomials of $R/\text{mono}(I)$, so $\mathbf{m}^{\mathbf{b}_1+1}, \dots, \mathbf{m}^{\mathbf{b}_r+1}$ all appear in the irreducible decomposition of $\text{mono}(I)$, hence $\text{mono}(I) \subseteq M$.

(b) \iff (c): Notice that (b) is equivalent to: any monomial $u \in (M : \mathfrak{m}) \setminus M$ is not in I ; or alternatively, any monomial in both $M : \mathfrak{m}$ and I is also in M ; i.e. $\text{mono}((M : \mathfrak{m}) \cap I) \subseteq M$. Now apply Proposition 1.1.1(3). \square

We now apply the results of mono to study a particular class of ideals, generated by powers of linear forms, or so-called power ideals. The particular class of power ideals under examination arise as invariants of a graph, in a natural way. Modifying the definition of these power ideals gives rise to an associated monomial ideal, which is of fundamental importance in chip-firing and divisor theory on graphs.

A remarkable conjecture of Postnikov-Shapiro asserts that for any graph, the associated monomial ideal and power ideal have identical Betti tables. In the same spirit as the previous chapter (i.e. relating Betti tables of graded ideals to Betti tables of monomial ideals), the goal of the next chapter is to study these power ideals, motivated by the Postnikov-Shapiro conjecture, by considering their monomial subideals.

Chapter 2

Power Ideals

Let $G = (V, E)$ be a(n undirected, loopless, multi-)graph on $n + 1$ vertices $\{0, \dots, n\}$. Fix, once and for all, a distinguished vertex, say 0, called the sink. For a field k of characteristic 0, let $R := k[V(G) \setminus \{\text{sink}\}] = k[x_1, \dots, x_n]$ be a polynomial ring, with variables indexed by non-sink vertices. In [21], Postnikov and Shapiro define an ideal from G which is generated by powers of linear forms:

$$J_G := \left(\left(\sum_{i \in S} x_i \right)^{d(S, \bar{S})} \mid \emptyset \neq S \subseteq V(G) \setminus \{\text{sink}\} \right) \subseteq R$$

where $d(S, T) := \sum_{i \in S, j \in T} a_{i,j}$ is the total degree between subsets $S, T \subseteq V(G)$ (here $[a_{i,j}] \in \mathbb{Z}_{\geq 0}^{(n+1) \times (n+1)}$ is the adjacency matrix of G), and $\bar{S} := V(G) \setminus S$ is the complement of S . Formally, this is an “additive” version of the G -parking function ideal

$$M_G := \left(\prod_{i \in S} x_i^{d(i, \bar{S})} \mid \emptyset \neq S \subseteq V(G) \setminus \{\text{sink}\} \right) \subseteq R$$

(note that $d(S, \bar{S}) = \sum_{i \in S} d(i, \bar{S})$, so the generators of J_G and M_G for a given subset S have the same degree). Also, G is disconnected iff J_G (or equivalently, M_G) is the unit ideal, so henceforth we assume G is connected.

The G -parking function ideal M_G is an Artinian monomial ideal of considerable intrinsic interest, being closely related to *chip-firing* and the *sandpile model*. Additionally, the degree, or multiplicity, of M_G counts the spanning trees of G :

Proposition 2.0.1 ([21], Corollary 2.2). *Let G be a connected graph. Then $\dim_k R/M_G$ is equal to the number of spanning trees of G .*

More surprising is the fact that J_G also counts the spanning trees of G : indeed, one of the main results of [21] is that J_G and M_G have the same Hilbert function:

Theorem 2.0.2 ([21], Theorem 3.1). *Let G be a connected graph. Then the standard monomials of R/M_G form a k -basis for R/J_G . In particular the Hilbert series of R/J_G and R/M_G coincide.*

This is the first evidence of a deep connection between homological properties of J_G and M_G . Indeed, Postnikov-Shapiro conjecture something much stronger, namely that the Betti tables of J_G and M_G are the same:

Conjecture 2.0.3 (Postnikov-Shapiro, [21]). *Let G be a connected graph. Then $\beta_{i,j}(R/J_G) = \beta_{i,j}(R/M_G)$ for all $i, j \in \mathbb{N}$.*

This conjecture would, in a strong sense, “explain” why J_G and M_G have the same Hilbert function, as knowing the Hilbert function is equivalent to knowing the alternating sums over diagonals of the Betti table (given that the number of variables of the ambient polynomial ring is known).

To date, the minimal free resolution (and thus the Betti numbers) of R/M_G is known, due independently to Manjunath-Schreyer-Wilmes [15], Mohammadi-Shokrieh [18], and Dochtermann-Sanyal [5]. However, much less is known about the power ideal J_G , and the Postnikov-Shapiro conjecture remains open in general.

Remark 2.0.4. One natural first attempt to tackle the Postnikov-Shapiro conjecture is to determine if M_G is a Gröbner degeneration of J_G , i.e. whether $M_G = \text{in}_>(J_G)$ for some monomial order $>$. After all, each monomial generator of M_G appears as a term in the corresponding generator of J_G , and the behavior of Betti tables under Gröbner degenerations is well-studied. However, this approach is doomed by the fact that in general, the point corresponding to M_G lies (strictly) in the interior of the Newton polytope of J_G . Thus, there will be no term order which can select the monomial of M_G for every generator of J_G .

One can see this e.g. in the complete graph K_4 on vertices $\{0, 1, 2, 3\}$: here $M_G = (x_i^3, x_i^2 x_j^2 \mid 1 \leq i, j \leq 3) + (x_1 x_2 x_3)$, and $J_G = (x_i^3, (x_i + x_j)^4 \mid 1 \leq i, j \leq 3) + ((x_1 + x_2 + x_3)^3)$. Any monomial order will choose a term of $(x_1 + x_2 + x_3)^3$ different from the generator $x_1 x_2 x_3$ of M_G .

Remark 2.0.5. Another natural proof strategy would be to show that the Betti tables of J_G and M_G obey some inductive relation, and the most common inductive procedure on graphs is deletion-contraction (of edges). However, this is also doomed to fail: first, any numerical invariant that is a deletion-contraction invariant is a specialization of the Tutte polynomial. Next, the computation of the Betti table of M_G by [18] shows that M_G has the same Betti table as the cocircuit ideal of G , i.e. the Stanley-Reisner ideal of the (independence complex of the) cographic matroid (= dual matroid of the matroid of G). But there exist 2 graphs on 8 vertices ([4], Example 3.2) with the same Tutte polynomial, but whose cocircuit ideals have distinct Betti tables, as shown in Figure 2.1.

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Figure 2.1: Betti table of M_G is not a deletion-contraction invariant

2.1 Generators of J_G

The computation of the Betti table of M_G by [15] shows that each graded Betti number of M_G has a combinatorial interpretation in the graph G , which we review now:

Definition 2.1.1. Let G be a connected graph. A connected i -partition Π of G is a partition of the vertex set $V(G)$ into i parts, such that the induced subgraph on each part is connected. Given a connected i -partition Π , one can consider the graph $G(\Pi)$ obtained by contracting each part of Π to a single vertex (and deleting loops: note that this is sensible since each part is connected) – so in particular $G(\Pi)$ has i vertices, and is connected since G is connected. We set the number of edges of a connected partition Π to be the number of edges of $G(\Pi)$. An orientation of $G(\Pi)$ is called 0-acyclic if it is acyclic (i.e. has no directed cycles), has a sink at the part containing the sink vertex 0, and has no other sinks.

Theorem 2.1.2 ([15], Theorem 1.1). *Let G be a connected graph, and $i, j \in \mathbb{N}$. Then $\beta_{i,j}(R/M_G)$ is equal to the number of 0-acyclic orientations of connected $(i + 1)$ -partitions of G with j edges.*

In particular, taking $i = 1$ gives that the (total) number of minimal generators of M_G is equal to the number of 0-acyclic orientations of connected 2-partitions of G . Since $G(\Pi)$ has a unique 0-acyclic orientation for any 2-partition Π , this says exactly that $\beta_1(R/M_G)$ is the number of subsets $S \subseteq V(G) \setminus \{\text{sink}\}$ such that the induced subgraphs on S and \bar{S} are both connected.

Remark 2.1.3. It is natural to ask about the dependence of J_G and M_G on the choice of sink vertex. In general, the ideals J_G and M_G certainly do depend on the choice of sink – indeed, whether or not J_G is monomial can vary with the sink. However, it follows from Theorem 2.1.2 that the Betti table of M_G does not depend on the sink – a bijection between i -acyclic orientations and j -acyclic orientations is given by reversing orientations of all edges in paths from i to j .

We now show that one can restrict to the same subsets of $V(G) \setminus \{\text{sink}\}$ which give minimal generators of M_G , to obtain generators of J_G :

Proposition 2.1.4. *Let G be a connected graph. Then*

$$J_G = \left(\left(\sum_{i \in S} x_i \right)^{d(S, \bar{S})} \mid S \subseteq V(G) \setminus \{\text{sink}\}, S, \bar{S} \text{ connected} \right).$$

Proof. Set

$$\begin{aligned} \Sigma &:= \{S \subseteq V(G) \setminus \{\text{sink}\} \mid S \text{ connected}\}, \\ \bar{\Sigma} &:= \{S \subseteq V(G) \setminus \{\text{sink}\} \mid \bar{S} \text{ connected}\}. \end{aligned}$$

It suffices to show that for any $T \subseteq V(G) \setminus \{\text{sink}\}$, the generator $(\sum_{i \in T} x_i)^{d(T, \bar{T})}$ is in $(\sum_{i \in S} x_i)^{d(S, \bar{S})} \mid S \in \Sigma \cap \bar{\Sigma}$. We first show that it is enough to consider $T \in \Sigma$. Suppose T has connected components C_1, \dots, C_r . Then $d(T, \bar{T}) = \sum_{j=1}^r d(C_j, \bar{C}_j)$, as any edge from C_j to \bar{C}_j must in fact be an edge from C_j to \bar{T} (else C_j would not be a component of T). This implies that

$$\begin{aligned} \left(\sum_{i \in T} x_i \right)^{d(T, \bar{T})} &= \left(\sum_{j=1}^r \sum_{i \in C_j} x_i \right)^{\sum_{j=1}^r d(C_j, \bar{C}_j)} \\ &\in \left(\left(\sum_{i \in C_1} x_i \right)^{d(C_1, \bar{C}_1)}, \dots, \left(\sum_{i \in C_r} x_i \right)^{d(C_r, \bar{C}_r)} \right). \end{aligned}$$

Since each $C_i \in \Sigma$, this shows that it is enough to consider only connected T . So suppose $T \subseteq V(G) \setminus \{\text{sink}\}$ is connected, and let D_0, \dots, D_s be the connected components of \bar{T} , with the sink in D_0 . Notice that \bar{D}_i is connected for each i : indeed, $\bar{D}_i = T \cup \left(\bigcup_{j \neq i} D_j \right)$, and since G is connected, each D_j is connected to T ; thus starting from any vertex in some D_j , $j \neq i$, one can first walk to T , and then (by connectedness of T) to any other $D_{j'}$. Moreover, the same reasoning as before shows that $d(T, \bar{T}) = d(\bar{T}, T) = \sum_{j=0}^s d(D_j, \bar{D}_j)$. Thus,

$$\left(\sum_{i \in T} x_i \right)^{d(T, \bar{T})} \in \left(\left(\sum_{i \in \bar{D}_0} x_i \right)^{d(\bar{D}_0, D_0)}, \left(\sum_{i \in \bar{D}_1} x_i \right)^{d(\bar{D}_1, D_1)}, \dots, \left(\sum_{i \in \bar{D}_s} x_i \right)^{d(\bar{D}_s, D_s)} \right)$$

(note that $\bar{D}_0 = T \cup D_1 \cup \dots \cup D_s \subseteq V(G) \setminus \{\text{sink}\}$), using the fact that $y^{a_0 + \dots + a_m} \in \left((y + z_1 + \dots + z_m)^{a_0}, z_1^{a_1}, \dots, z_m^{a_m} \right)$ in the “universal” ring $\mathbb{Z}[y, z_1, \dots, z_m]$, for any $a_i \in \mathbb{N}$. Since $\bar{D}_0, D_1, \dots, D_s \in \Sigma \cap \bar{\Sigma}$, this gives the desired result. \square

Thus, equality of the first column of the Betti tables of J_G and M_G is equivalent to minimality of the generating set in Proposition 2.1.4. Since the Hilbert functions of J_G and M_G agree (Theorem 2.0.2), it is true that J_G and M_G have the same number of minimal generators of lowest degree, i.e. for $j = \min\{j' \mid \beta_{1,j'}(R/J_G) \neq 0\}$, one has $\beta_{1,j}(R/J_G) = \beta_{1,j}(R/M_G)$.

One can also use Theorem 2.0.2 to draw analogous conclusions for the last column of the Betti tables:

Proposition 2.1.5. *Let G be a connected graph. Then*

- (i) M_G is level (i.e. has socle concentrated in a single degree),
- (ii) $\text{reg}(R/M_G) = \text{reg}(R/J_G)$,
- (iii) $\beta_n(R/M_G) \leq \beta_n(R/J_G)$, and equality holds iff J_G is level.

Proof. (i): It follows from the combinatorial interpretation of the Betti numbers of M_G (Theorem 2.1.2) that M_G is level: indeed, the last column of the Betti table $\beta_{n,j}(R/M_G)$ counts the number of 0-acyclic orientations of connected $(n+1)$ -partitions of G with j edges. Since G itself is the only $(n+1)$ -partition of G , this shows that $\beta_{n,j}(R/M_G) \neq 0$ iff $j = |E(G)|$.

(ii) As J_G, M_G are Artinian ideals with the same Hilbert function, $\text{reg}(R/J_G) = \max\{d \mid (R/J_G)_d \neq 0\} = \max\{d \mid (R/M_G)_d \neq 0\} = \text{reg}(R/M_G)$.

(iii): Note that $\beta_n(R/I)$ is the k -vector space dimension of the socle of R/I , for any Artinian graded ideal I . Setting $r := \text{reg}(R/M_G)$, it follows from (i) and (ii) that $\beta_n(R/M_G) = \dim_k(R/M_G)_r = \dim_k(R/J_G)_r$, and that r is also the top nonzero degree of R/J_G , so $(R/J_G)_r$ is contained in the socle of R/J_G , which gives the desired result. \square

Next, we recall the following important graph-theoretic invariant:

Definition 2.1.6. Let G be any graph (not necessarily connected). The following quantities are equal:

- $|E(G)| - |V(G)| + \#(\text{components}(G))$
- $\min\{r \mid \exists e_1, \dots, e_r \in E(G) \text{ with } G \setminus \{e_1, \dots, e_r\} \text{ acyclic}\}$
- $\beta_1(G) = \text{rank}_{\mathbb{Z}} H_1(G, \mathbb{Z})$
- $\text{rank } M(G)^*$ (= rank of cographic matroid)

This number is called the genus, or circuit rank, of G .

For example, the complete graph K_{n+1} has genus $g(K_{n+1}) = \binom{n+1}{2} - (n+1) + 1 = \binom{n}{2}$. Intuitively, the genus measures the number of “independent” cycles of G , which is the same as the minimum number of edges that must be deleted to remove all cycles. Proposition 2.1.5 shows that it has another interpretation for J_G and M_G :

Proposition 2.1.7. *Let G be a connected graph. Then $g(G) = \text{reg}(R/M_G) (= \text{reg}(R/J_G))$.*

Proof. As R/M_G is Cohen-Macaulay (being Artinian), the regularity of R/M_G is determined by the last column of the Betti table. Thus as in the proof of Proposition 2.1.5(i), $\text{reg}(R/M_G) = \max\{j-n \mid \beta_{n,j}(R/M_G) \neq 0\} = |E(G)| - n = |E(G)| - (n+1) + 1 = g(G)$. \square

2.2 The tree case

From the standpoint of commutative algebra it is natural to ask when the power ideal J_G satisfies good algebraic properties. Since J_G and M_G are Artinian, they are trivially Cohen-Macaulay; but one could still ask when J_G is Gorenstein, or a complete intersection. As M_G is Artinian monomial, thus is Gorenstein iff it is a complete intersection (cf. Corollary 1.4.5), it is natural – particularly in light of Conjecture 2.0.3 – to guess that J_G is Gorenstein iff it is a complete intersection as well. We prove this now, and characterize graph-theoretically when this occurs:

Proposition 2.2.1. *Let G be a connected simple graph on $n + 1$ vertices. The following are equivalent:*

- i) G is a tree
- ii) J_G is a complete intersection
- iii) J_G is Gorenstein
- iv) $\beta_n(J_G) = 1$.

Proof. i) \implies ii): View G as a rooted tree with root at n , the sink. Then a subset $S \subseteq V(G)$ satisfies S and \bar{S} connected iff S consists of all the descendants of a single vertex. Taking one such generator at each non-sink vertex and applying Proposition 2.1.4 gives a generating set of J_G with $|V(G) \setminus \{\text{sink}\}| = n$ elements. Since $\text{codim } J_G = n$, this implies that the generating set of Proposition 2.1.4 must be minimal (otherwise $\text{codim } J_G \leq n - 1$ by Krull's Altitude Theorem), and hence J_G is a complete intersection.

ii) \implies iii) \implies iv): Clear.

iv) \implies i): By Proposition 2.1.5(i) and (iii), $1 \leq \beta_n(M_G) \leq \beta_n(J_G) = 1 \implies \beta_n(M_G) = 1$. By Theorem 2.1.2, this means that there is a unique acyclic orientation of G with unique sink. By ([8], Theorem 1.2) (cf. also ([11], Theorem 7.3)), the coefficient of the linear term of the chromatic polynomial of G is 1, and by ([6], Corollary 2), this implies that G is acyclic, hence a tree. \square

As a consequence, we deduce Conjecture 2.0.3 for trees (notice that the implication i) \implies ii) in Proposition 2.2.1 still holds, with the same reasoning, if G is a multigraph).

Corollary 2.2.2. *Conjecture 2.0.3 holds if G is a (multi-)tree.*

Proof. By Proposition 2.2.1(i) \implies (ii) and Theorem 2.1.2, both J_G and M_G are complete intersections, hence are minimally resolved by Koszul complexes. Moreover, the degrees of the minimal generating sets of J_G and M_G are equal, so the two Koszul complexes have exactly the same degree shifts, and thus the same Betti tables. \square

2.3 A monomial subideal of J_G

In the case of a simple tree, $J_G = M_G = (x_1, \dots, x_n)$, the maximal homogeneous (= irrelevant) ideal of R . Although this is no longer the case for multitrees (which in any case has

also been dealt with), we now seek to generalize this approach, by studying which monomials J_G and M_G have in common. This will turn out to have a number of implications on Conjecture 2.0.3 for graphs of low genus.

Remark 2.3.1. It is a consequence of Theorem 2.0.2 that $\text{mono}(J_G) \subseteq M_G$: if $u \notin M_G$ is a monomial, then u is a standard monomial of M_G , hence part of a basis of R/J_G ; in particular \bar{u} cannot be $\bar{0}$ in $R/J_G \implies u \notin \text{mono}(J_G)$. In particular, this shows that J_G is monomial $\iff J_G = \text{mono}(J_G) \iff J_G \subseteq M_G \iff J_G = M_G$ (again by Theorem 2.0.2).

We return to our original setup – henceforth G is always a connected loopless undirected multigraph on $\{0, \dots, n\}$ with fixed sink 0. The following is a basic relation between the genus (Definition 2.1.6) of an induced subgraph and that of its complement:

Lemma 2.3.2. *Let G be a connected graph. For any $S \subseteq V(G)$,*

$$g(G) + 1 - g(\bar{S}) = g(S) + d(S, \bar{S}) + (1 - c(S)) + (1 - c(\bar{S}))$$

where $g(\cdot), c(\cdot)$ denotes the genus resp. number of components of the induced subgraph.

Proof. Any edge $e \in E(G)$ is either internal to S , or internal to \bar{S} , or connects S to \bar{S} (i.e. contributes to $d(S, \bar{S})$). Moreover $c(G) = 1$ by assumption. Thus

$$\begin{aligned} & g(G) + 1 - g(\bar{S}) \\ &= \left(|E(G)| - |V(G)| + 1 \right) + 1 - \left(|E(G)| - d(S, \bar{S}) - |E(S)| - (|V(G)| - |S|) + c(\bar{S}) \right) \\ &= |E(S)| - |S| + d(S, \bar{S}) + 2 - c(\bar{S}) \\ &= g(S) + d(S, \bar{S}) + (1 - c(S)) + (1 - c(\bar{S})). \quad \square \end{aligned}$$

We now define a monomial ideal, which (as we will see) is contained in J_G :

Definition 2.3.3. Let G be a connected graph. Define

$$F_G := \sum_{S \subseteq V(G) \setminus \{\text{sink}\}} (x_i \mid i \in S)^{g(G)+1-g(\bar{S})}$$

Note that F_G is a sum of powers of monomial prime ideals.

Remark 2.3.4. In the definition of F_G , it is not enough to consider only subsets S with S, \bar{S} connected. For example, for the bowtie graph in Figure 2.2, taking only those S with S, \bar{S} connected would miss the monomials x_3, x_1x_2, x_2^2 in F_G .

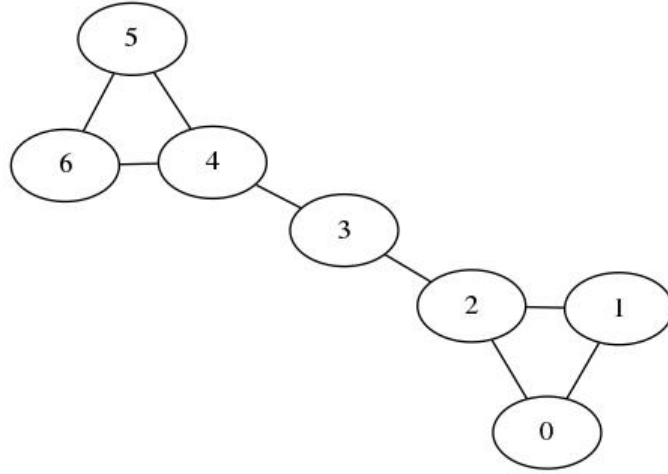


Figure 2.2: A bowtie graph

Theorem 2.3.5. *Let G be a connected graph. Then $F_G \subseteq J_G$.*

Proof. Fix $0 \neq S \subseteq V(G) \setminus \{\text{sink}\}$. Consider a graph G' obtained from G by contracting each component of \bar{S} to a vertex (and removing all loops). Explicitly, if \bar{S} has connected components C_0, C_1, \dots, C_r (where the sink lies in C_0), then G' consists of the induced subgraph on S , along with edges of total weight $d(S, C_j)$ from S to a new vertex v_{C_j} for each C_j (including C_0). Then G' is connected, since G is connected (note though that the v_{C_j} need not be leaves: indeed, if S has multiple components, then at least one C_j is connected to more than one component of S .)

Next, $g(G') = |E(G)| - |E(\bar{S})| - (|S| + r + 1) + 1 = |E(S)| + d(S, \bar{S}) - |S| + (1 - (r + 1)) = g(S) + d(S, \bar{S}) + (1 - c(\bar{S})) - c(S)$. Thus $J_{G'}$ is an Artinian ideal in $k[y_i, y_{C_j} \mid i \in S, 1 \leq j \leq r]$ of regularity $g(G')$, so $(y_i, y_{C_j})^{g(G')+1} \subseteq J_{G'}$. There is an injective ring map

$$\begin{aligned} \varphi_S : k[y_i, y_{C_j} \mid i \in S, 1 \leq j \leq r] &\hookrightarrow R = k[x_i \mid i \in V(G) \setminus \{\text{sink}\}] \\ y_i &\mapsto x_i \\ y_{C_j} &\mapsto \sum_{l \in C_j} x_l \end{aligned}$$

such that the extended ideal $\varphi_S(J_{G'})R \subseteq J_G$: indeed, the images of generators of $J_{G'}$ are precisely the generators of J_G corresponding to subsets $\emptyset \neq T \subseteq V(G) \setminus C_0$ such that for all $1 \leq j \leq r$, $T \cap C_j$ is either \emptyset or C_j . Then by Lemma 2.3.2,

$$(x_i \mid i \in S)^{g(G)+1-g(\bar{S})} = (x_i \mid i \in S)^{g(G')+1} = \varphi_S((y_i \mid i \in S)^{g(G')+1})R \subseteq \varphi_S(J_{G'})R \subseteq J_G. \quad \square$$

Having identified a large monomial subideal of J_G , we can now deduce the Postnikov-Shapiro conjecture for a certain class of graphs:

Corollary 2.3.6. *Let G be a connected graph, which is a cone over a forest, i.e. there exists $v_0 \in V(G)$ such that $G \setminus \{v_0\}$ is acyclic. If v_0 is chosen as the sink, then $M_G = J_G$. In particular Conjecture 2.0.3 holds for (G, v_0) .*

Proof. By hypothesis, $g(S) = 0$ for every $S \subseteq V(G) \setminus \{\text{sink}\}$. By Theorem 2.3.5, this implies that $(x_i \mid i \in S)^{d(S, \bar{S})} \subseteq J_G$, so in particular the generator $\prod_{i \in S} x_i^{d(i, \bar{S})}$ of M_G is in J_G . It follows that $M_G \subseteq J_G$, and since they have the same Hilbert function, $M_G = J_G$. \square

Corollary 2.3.7. *Let G be a connected graph of genus ≤ 1 . Then Conjecture 2.0.3 holds for G .*

Proof. If G has genus 0, then G is a tree, so one may use Corollary 2.2.2.

If G has genus 1, then after iteratively removing leaves (which does not affect the truth of Conjecture 2.0.3), G may be reduced to a cycle graph. But if G is a cycle graph, then for any $v \in V(G)$, $G \setminus \{v\}$ is a line graph, hence acyclic; so the result follows from Corollary 2.3.6. \square

Remark 2.3.8. In fact, the proof technique of Corollary 2.3.7 applies to many graphs of genus 2 as well – namely, the graphs which can be reduced to a union of 2 cycles by iteratively removing leaves. Moreover, there do exist graphs of arbitrarily large genus which are cones over forests. However, it is unlikely that the method of Corollary 2.3.6 will imply Conjecture 2.0.3 in much greater generality than already given above.

2.4 Alexander duality

We now examine the Alexander dual of the monomial ideal F_G defined above (cf. Definition 2.3.3). First, we review the notion of Alexander duality for arbitrary monomial ideals (following the exposition in [17], cf. Definition 5.20 in loc. cit.):

Definition 2.4.1. Let $\mathbf{a} = (a_i) \in \mathbb{Z}^n$. For $\mathbf{b} = (b_i) \in \mathbb{Z}^n$ with $\mathbf{b} \leq \mathbf{a}$ (i.e. $b_i \leq a_i$ for all $i = 1, \dots, n$), define

$$\mathbf{a} \setminus \mathbf{b} := \begin{cases} a_i + 1 - b_i, & b_i \neq 0 \\ 0, & b_i = 0 \end{cases}$$

Let $I \subseteq k[x_1, \dots, x_n]$ be a monomial ideal, with minimal generators $I = (x_1^{b_1}, \dots, x_n^{b_t})$. For $\mathbf{a} \in \mathbb{Z}^n$ with $\mathbf{b}_j \leq \mathbf{a}$ for $1 \leq j \leq t$, the Alexander dual of I with respect to \mathbf{a} is

$$I^{[\mathbf{a}]} := \bigcap_{i=1}^t \mathfrak{m}^{\mathbf{a} \setminus \mathbf{b}_i}$$

where as before $\mathfrak{m}^{\mathbf{b}}$ is the monomial complete intersection $\mathfrak{m}^{\mathbf{b}} = (x_1^{b_1}, \dots, x_n^{b_n})$ (note that $\mathfrak{m}^{\mathbf{b}}$ is Artinian iff all $b_i \neq 0$). We denote by I^* the Alexander dual of I with respect to the least such \mathbf{a} , i.e. $I^{[\mathbf{a}]}$ with $\mathbf{a} = (\max\{(\mathbf{b}_j)_i \mid 1 \leq j \leq t\})_{i=1}^n$ (= exponent of lcm of generators of I).

Thus just as in the squarefree case, Alexander duality interchanges minimal generators with irreducible components. In general, if I is any monomial ideal and $\mathbf{a} \in \mathbb{Z}^n$, then $(I^{[\mathbf{a}]})^{[\mathbf{a}]} = I$. However, it is not true in general that $(I^*)^* = I$, as the (exponent for the) lcm of generators of I need not be the same as that of I^* .

Remark 2.4.2. If I is an Artinian monomial ideal, then a pure power of each variable appears in the minimal generating set of I , say $x_i^{d_i} \in I$, $i = 1, \dots, n$. In this case the least common multiple of the exponent vectors of all generators of I is equal to $(d_1, \dots, d_n) = \sum_{i=1}^n d_i \mathbf{e}_i$ (where \mathbf{e}_i is the i^{th} standard basis vector of \mathbb{Z}^n). Moreover, $(d_1, \dots, d_n) \setminus d_i \mathbf{e}_i = \mathbf{e}_i$, so for every i , the principal prime ideal $(x_i) = \mathbf{m}^{\mathbf{e}_i}$ appears in the irreducible decomposition of I^* . In particular $I^* \subseteq (\prod_{i=1}^n x_i)$ has codimension 1.

For the complete graph on $n + 1$ vertices, we now give a combinatorial description of the Alexander dual of $F_{K_{n+1}}$:

Definition 2.4.3. Let $\mathbf{a} = (a_i)$, $\mathbf{b} \in \mathbb{N}^n$.

(1) The increasing rearrangement of \mathbf{a} is the vector $\mathbf{a}_{inc} = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$ where $\sigma \in S_n$ is any permutation satisfying $a_{\sigma(1)} \leq a_{\sigma(2)} \leq \dots \leq a_{\sigma(n)}$. Note that although σ need not be unique (i.e. when some a_i repeats), the vector \mathbf{a}_{inc} is always uniquely determined by \mathbf{a} .

(2) We say that $\mathbf{b} \preceq \mathbf{a}$ if \mathbf{a} can be transformed to \mathbf{b} by a finite sequence of moves of the form $\mathbf{a} \mapsto (a_1, \dots, a_i + 1, a_{i+1} - 1, \dots, a_n)$ for $i \in \{1, \dots, n - 1\}$. (Notice: this implies that $|\mathbf{a}| := \sum_{i=1}^n a_i = |\mathbf{b}|$).

Theorem 2.4.4. The following are equivalent for $\mathbf{b} = (b_i) \in \mathbb{N}^n$:

- (i) $x^{\mathbf{b}} \in (F_{K_{n+1}} : \mathbf{m}) \setminus F_{K_{n+1}}$, i.e. $x^{\mathbf{b}}$ is a socle monomial of $F_{K_{n+1}}$
- (ii) $|\mathbf{b}| = \binom{n}{2}$ and $\sum_{i=0}^j (\mathbf{b}_{inc})_{n-i} \leq \sum_{i=0}^j (n - 1 - i)$ for $j = 0, \dots, n - 1$
- (iii) $\mathbf{b}_{inc} \preceq (0, 1, \dots, n - 1)$.

Proof. (iii) \implies (ii): Note that $(0, 1, \dots, n - 1)$ satisfies the inequalities of (ii), and that satisfying the inequalities is preserved under the moves in Definition 2.4.3(2) of \preceq .

(ii) \implies (iii): It suffices to show that given \mathbf{b} satisfying (ii), one can apply “reverse” moves to \mathbf{b}_{inc} to obtain $(0, 1, \dots, n - 1)$. This can be done by induction on n : since $(\mathbf{b}_{inc})_n \leq n - 1$ but $|\mathbf{b}| = \binom{n}{2}$, there exists i such that $(\mathbf{b}_{inc})_i > i$, and one can apply a sequence of reverse moves to decrease $(\mathbf{b}_{inc})_i$ by 1 and increase $(\mathbf{b}_{inc})_n$ by 1. In this way one can successively increase $(\mathbf{b}_{inc})_n$ up to $n - 1$, at which point induction guarantees that, by applying reverse moves, the remaining $n - 1$ components of \mathbf{b}_{inc} can be turned into $(0, \dots, n - 2)$.

(ii) \implies (i): We first set up some notation, for convenience: given S (which in this proof always denotes a subset of $V(K_{n+1}) \setminus \{\text{sink}\}$), let $P_S := (x_i \mid i \in S)$ be the monomial prime ideal of variables in S , and write $d_S := g(G) + 1 - g(\overline{S})$, so that $F_{K_{n+1}} = \sum_S P_S^{d_S}$.

Next, notice that if $x^{\mathbf{a}} \in R$ is a monomial, then $x^{\mathbf{a}} \in F_{K_{n+1}}$ iff $x^{\mathbf{a}} \in P_S^{d_S}$ for some S (since a monomial in $F_{K_{n+1}}$ must be a multiple of a single generator – note that this fails dramatically for sums of ordinary homogeneous ideals). As $P_S^{d_S}$ is a power of a monomial

prime, this occurs iff there is a subset S such that $|\mathbf{a}|_S = \sum_{i \in S} a_i \geq d_S$. Now,

$$\begin{aligned} d_S &= g(S) + d(S, \overline{S}) \text{ (by Lemma 2.3.2, since } S, \overline{S} \text{ are connected)} \\ &= \binom{|S| - 1}{2} - |S| + 1 + |S|(n + 1 - |S|) \\ &= 1 + \sum_{i=0}^{|S|-1} (n - 1 - i). \end{aligned}$$

Thus, given a monomial $x^{\mathbf{b}}$ where \mathbf{b} satisfies (ii), one sees that for any $\{c_1, \dots, c_j\} \subseteq \{1, \dots, n\}$, $\sum_{i=1}^j \mathbf{b}_{c_i} \leq \sum_{i=0}^{j-1} (\mathbf{b}_{inc})_{n-i} \leq \sum_{i=0}^{j-1} (n - 1 - i)$. By the previous computation (as S ranges over all $\{c_1, \dots, c_j\}$), this says exactly that $x^{\mathbf{b}} \notin P_S^{d_S}$ for any S , so $x^{\mathbf{b}} \notin F_{K_{n+1}}$. Since $|\mathbf{b}| = \binom{n}{2}$ by assumption, and any monomial of degree $> \binom{n}{2}$ is in $F_{K_{n+1}}$ (by taking $S = \{1, \dots, n\}$, $d_S = \binom{n}{2}$), this shows that $x^{\mathbf{b}}$ is a socle monomial of $F_{K_{n+1}}$.

(i) \implies (ii): By the reasoning in (ii) \implies (i), it suffices to show that $F_{K_{n+1}}$ is level of socle degree $\binom{n}{2}$. Suppose $x^{\mathbf{b}} \in (F_{K_{n+1}} : \mathfrak{m}) \setminus F_{K_{n+1}}$. Then $x^{\mathbf{b}} \notin F_{K_{n+1}}$ implies, as above, that $|\mathbf{b}| \leq \sum_{i=0}^{n-1} (n - 1 - i) = \binom{n}{2}$.

To show that $|\mathbf{b}| \geq \binom{n}{2}$, choose i_0 such that \mathbf{b}_{i_0} is minimal among all components of \mathbf{b} . Then there exists S such that $x_{i_0} x^{\mathbf{b}} = x^{\mathbf{b} + \mathbf{e}_{i_0}} \in P_S^{d_S}$. Setting $r := n - |S|$ and applying the reasoning above yields $\sum_{i=0}^{n-r-1} ((\mathbf{b} + \mathbf{e}_{i_0})_{inc})_{n-1-i} \geq d_S = 1 + \sum_{i=0}^{n-r-1} (n - 1 - i)$. But since $x^{\mathbf{b}} \notin P_S^{d_S}$, we must have that \mathbf{b}_{i_0} is the smallest term in the sum on the right above, namely $\mathbf{b}_{i_0} = (\mathbf{b} + \mathbf{e}_{i_0})_{i_0} - 1 = n - 1 - (n - r - 1) - 1 = r - 1$. By choice of i_0 , the remaining r indices not in S contribute at least $r(r - 1)$ to the total degree of \mathbf{b} . Thus $1 + |\mathbf{b}| = |\mathbf{b} + \mathbf{e}_{i_0}| \geq 1 + \sum_{i=0}^{n-r-1} (n - 1 - i) + r(r - 1) \geq 1 + \binom{n}{2} \implies |\mathbf{b}| \geq \binom{n}{2}$. \square

2.5 Questions/conjectures

We conclude with some unresolved questions, discovered in the process of investigating J_G and $\text{mono}(J_G)$. Each of the conjectures below concerning graphs have been verified by computer for all graphs up to 6 vertices, using the computer algebra system Macaulay2 [10]:

Conjecture 2.5.1. *Let G be a connected graph. Then $F_G = \text{mono}(J_G)$.*

In other words, the reverse inclusion in Theorem 2.3.5 should also hold. In light of this, each of the conjectures below that mentions F_G is really a conjecture about $\text{mono}(J_G)$.

The next conjecture states that the Betti table of the Alexander dual of $\text{mono}(J_G)$ is remarkably simple, and also encodes some combinatorial data about the graph:

Conjecture 2.5.2. *Let G be a connected graph. Then:*

i) $\beta_n(R/F_G)$ is equal to the number of spanning forests F of $G \setminus \{\text{sink}\}$ such that $F \cup \{\text{sink}\}$ is connected.

ii) R/F_G^ has a linear free resolution.*

In other words, Conjecture 2.5.2 is equivalent to the statement that the Betti table of R/F_G^* is given by $\beta_{0,0} = 1$, $\beta_{1,g(G)+n} = n_G$, $\beta_{i,j} \neq 0$ for $i \geq 2$ and $j = i + g(G) + n - 1$, and all other $\beta_{i,j} = 0$ (where n_G is the number of special spanning forests as in Conjecture 2.5.2(i)). In fact, a much stronger statement than Conjecture 2.5.2(ii) seems to be true, namely:

Conjecture 2.5.3. *Let P_1, \dots, P_m be monomial prime ideals, $d_i \in \mathbb{N}$, and set $F := \sum_{i=1}^m P_i^{d_i}$. Then F^* has a linear free resolution iff F is level (i.e. F^* is generated in a single degree).*

Of course, being level is an obvious necessary condition for the Alexander dual to have a linear resolution, as being linear in the first step of the resolution already implies equi-generation. Since M_G is known to be level (Proposition 2.1.5(i)), Conjecture 2.0.3 would imply that J_G is also level, and then it would follow from Corollary 1.3.3 that $\text{mono}(J_G)$ is level. Thus Conjecture 2.5.3 implies Conjecture 2.5.2(ii) (given Definition 2.3.3, and assuming Conjecture 2.0.3).

In the special case of F_G , it seems to be true that (the minimal generating set of) F_G^* has linear quotients (cf. [13], Proposition 8.2.1), i.e. the polarization of F_G^* corresponds to a shellable simplicial complex. If true, this would be another strengthening of Conjecture 2.5.2(ii).

Finally, having linear resolution indicates that the graded Betti numbers have been “squashed”, in the sense that all graded Betti numbers in a given homological degree all have the same twist. It is reasonable to ask if there is a corresponding ideal whose graded Betti numbers have not been “squashed”:

Definition 2.5.4. Let M be a monomial ideal, with unique decomposition $M = \bigcap_{j=1}^s Q_j$ into irreducible monomial ideals. Define

$$M_{>i} = \bigcap_{\text{codim } Q_j > i} Q_j$$

i.e., $M_{>i}$ is the intersection of all irreducible components of M of $\text{codim} > i$.

Conjecture 2.5.5. *Let G be a connected graph. Then $\beta_i(R/(F_G^*)_{>1}) = \beta_i(R/F_G^*)$ for all i .*

In other words, Conjecture 2.5.5 states that the Betti table of $R/(F_G^*)_{>1}$ is the Betti table of R/F_G^* , but “pulled apart” to separate the graded Betti numbers into different degrees, while preserving the total Betti numbers.

Note that by Remark 2.4.2, $F_G^* = (\prod_{i=1}^n x_i) \cap (F_G^*)_{i>1}$. In general it is not true for monomial ideals I that the total Betti numbers of I are the same as those of $I \cap (u)$, where u is a monomial (which is a (non-zero) zerodivisor on R/I): e.g. $I = (x_1x_2, x_1x_3)$, $u = (x_1x_4)$. The crux of Conjecture 2.5.5 though is that this does hold, for ideals of the form $(F_G^*)_{>1}$.

The second half of this thesis has a rather different flavor than the first half – whereas the previous chapters had a large combinatorial emphasis (coming from monomial ideals and graph theory), the next two chapters are squarely within abstract commutative algebra, much of it applying to non-Noetherian rings and their ideals.

The main question that will occupy us in the next chapter is: given a surjective map of rings, when is the induced group homomorphism on units surjective? A moment's thought will show that this is usually not the case: in making this precise, we introduce notions of semi-inverses, semi-units, and semi-fields [3], which (to the best of our knowledge) represent new concepts in ring theory.

Chapter 3

Surjections of unit groups

3.1 Introduction

Let \mathbf{CRing} be the category of commutative rings with $1 \neq 0$, and \mathbf{Ab} the category of abelian groups. One of the most natural functors from \mathbf{CRing} to \mathbf{Ab} is the group of units functor, $(_)^\times$, associating to any (commutative) ring its (abelian) group of units. Functoriality follows from the fact that a ring homomorphism $\varphi : R \rightarrow S$ sends 1 to 1, hence units to units, and thus induces (by set-theoretic restriction) a group homomorphism $\varphi^\times : R^\times \rightarrow S^\times$. By definition as a set-theoretic restriction, one sees that φ injective implies φ^\times injective (i.e., $(_)^\times$ is “left exact”). The question we now consider is: when does φ surjective imply φ^\times surjective, i.e., how does $(_)^\times$ fail to be “right exact”?

Example 3.1.1. For any prime number p , the natural surjection $\mathbb{Z} \rightarrow \mathbb{Z}/p\mathbb{Z}$ induces a group homomorphism $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}^\times \rightarrow (\mathbb{Z}/p\mathbb{Z})^\times \cong \mathbb{Z}/(p-1)\mathbb{Z}$, which is a surjection iff $p = 2, 3$.

Example 3.1.2. For a field k , any ring surjection $\varphi : k \rightarrow R$ is necessarily injective, hence an isomorphism, so (by functoriality) φ^\times is also an isomorphism.

Example 3.1.3. For a field k , the surjection $\varphi_1 : k[x] \rightarrow k[x]/(x) \cong k$ induces a surjection on unit groups, but $\varphi_2 : k[x] \rightarrow k[x]/(x^2)$ does not, as $\varphi_2(1+x) \in (k[x]/(x^2))^\times$, but is not the image of any unit of $k[x]$ (= nonzero constant in k).

With these examples at hand, we make the following (non-vacuous) definition:

Definition 3.1.4. A ring surjection $\varphi : R \rightarrow S$ has $(*)$ if $\varphi^\times : R^\times \rightarrow S^\times$ is surjective. We say that the ring R has $(*)$ if every ring surjection $\varphi : R \rightarrow S$ (for any ring S) has $(*)$.

If $\varphi : R \rightarrow S$ is a ring surjection, then $S \cong R/I$ for some R -ideal I (namely $I = \ker \varphi$), so one may instead refer to an ideal I having $(*)$ (i.e. if the canonical surjection $R \rightarrow R/I$ has $(*)$). Thus R has $(*)$ iff I has $(*)$ for every R -ideal I , so in this way property $(*)$ for a ring becomes an ideal-theoretic statement. The examples above say that any field k has $(*)$, while \mathbb{Z} and $k[x]$ do not.

We begin with some characterizations of $(*)$. Recall that if W is a multiplicative set, the *saturation* of W is defined as $W^\sim := \{r \in R \mid \exists s \in R, sr \in W\}$, and W is called *saturated* if $W = W^\sim$.

Proposition 3.1.5. *Let R be a ring, I an R -ideal. The following are equivalent:*

- i) I has $(*)$
- ii) $R^\times + I$ is saturated
- iii) $R^\times + I = (1 + I)^\sim$
- iv) For any $a \in R$ such that $1 - ab \in I$ for some $b \in R$, there exists $u \in R^\times$ with $1 - au \in I$.

Proof. (ii) \iff (iii): follows from the containment $1 + I \subseteq R^\times + I \subseteq (1 + I)^\sim$ which holds for any ideal I , and the fact that saturation is a closure operation (in particular, is monotonic and idempotent).

(i) \implies (iii): Suppose that the canonical surjection $p : R \twoheadrightarrow R/I$ induces a surjection $p^\times : R^\times \twoheadrightarrow (R/I)^\times$, i.e. if $r \in R$ is such that $p(r)$ is a unit, then $p(r) = p(u)$ for some $u \in R^\times$. Then $r - u \in \ker p = I$, i.e. $r \in R^\times + I$. Thus the preimage of the units of R/I is contained in $R^\times + I$, but this preimage is exactly $(1 + I)^\sim$, since $p(r)$ is a unit $\iff 1 = p(1) = p(r)p(s)$ for some $s \in R \iff 1 - rs \in I \iff rs \in 1 + I$.

(iii) \implies (i): if $R^\times + I = (1 + I)^\sim$, then any preimage of a unit of R/I differs from a unit of R by an element of I , so every unit of R/I is the image of a unit of R .

(iii) \iff (iv): Notice that $a \in (1 + I)^\sim \iff 1 - ab \in I$ for some $b \in R$, and $a \in R^\times + I \iff v - a \in I$ for some $v \in R^\times \iff 1 - v^{-1}a \in I$. \square

3.2 Sufficient conditions for $(*)$

As a first application of Proposition 3.1.5, one has the following sufficient condition for an ideal to have $(*)$ (hereafter, the Jacobson radical of R is denoted by $\text{Rad}(R) := \bigcap_{\mathfrak{m} \in \text{mSpec}(R)} \mathfrak{m}$, the intersection of all maximal ideals of R).

Corollary 3.2.1. *Let R be a ring, I an R -ideal. If $I \subseteq \text{Rad}(R)$, then I has $(*)$.*

Proof. If $I \subseteq \text{Rad}(R)$, then $R^\times + I = R^\times = \{1\}^\sim$ is saturated, so Proposition 3.1.5(ii) applies. \square

In fact, rather than requiring I to be contained in every maximal ideal, one can allow finitely many exceptions:

Theorem 3.2.2. *Let R be a ring, I an R -ideal. If I is contained in all but finitely many maximal ideals of R (i.e. $|\text{mSpec}(R) \setminus V(I)| < \infty$), then I has $(*)$.*

Proof. Write $\text{mSpec}(R) \setminus V(I) := \{m_1, \dots, m_n\}$, so that $\{I, m_1, \dots, m_n\}$ are pairwise comaximal (the case $n = 0$ is Corollary 3.2.1). Let $p : R \twoheadrightarrow R/I$ be the canonical surjection, pick $v \in (R/I)^\times$, and write $v = p(r)$ for some $r \in R$. By Chinese Remainder, there exists $a \in R$

with $a \equiv 0 \pmod{I}$, $a \equiv 1 - r \pmod{m_i}$ for $i = 1, \dots, n$. Since r is not contained in any maximal ideal containing I , $r + a \in R^\times$, and $p(r + a) = p(r) = v$. \square

Corollary 3.2.3. *Let R be a semilocal ring, i.e. $|\text{mSpec}(R)| < \infty$. Then R has $(*)$.*

Proof. If R is semilocal, then for any R -ideal I , $\text{mSpec}(R) \setminus V(I)$ is finite. \square

Corollary 3.2.1 lends support to the idea that the Jacobson radical will not play a role in whether or not a ring has $(*)$. This is indeed true, as the following reduction to the J -semisimple case (i.e. $\text{Rad}(R) = 0$) will show.

Proposition 3.2.4. *Let R be a ring, I an R -ideal, $p : R \twoheadrightarrow R/I$ the canonical surjection, and $\bar{p} : R/\text{Rad}(R) \twoheadrightarrow R/(\text{Rad}(R) + I)$ the map obtained by applying $-\otimes_R R/\text{Rad}(R)$. Then p has $(*)$ iff \bar{p} has $(*)$. In particular, R has $(*)$ iff $R/\text{Rad}(R)$ has $(*)$.*

Proof. Consider the commutative diagram of natural maps

$$\begin{array}{ccc} R & \xrightarrow{p} & R/I \\ \alpha \downarrow & & \downarrow \beta \\ R/\text{Rad}(R) & \xrightarrow{\bar{p}} & R/(\text{Rad}(R) + I) \end{array}$$

If p^\times is surjective, then since β^\times is surjective (by Corollary 2, as $(\text{Rad}(R) + I)/I \subseteq \text{Rad}(R/I)$), so is \bar{p}^\times . Conversely, suppose \bar{p}^\times is surjective, and let $v \in (R/I)^\times$. Then $\beta(v) \in (R/(\text{Rad}(R) + I))^\times$, so there exists $\bar{u} \in (R/\text{Rad}(R))^\times$ with $\bar{p}(\bar{u}) = \beta(v)$. By Corollary 2, α^\times is surjective, hence $\bar{u} = \alpha(u)$ for some $u \in R^\times$. Then $\beta(p(u)) = \bar{p}(\alpha(u)) = \beta(v)$, so $v - p(u) \in \ker \beta$. But $\ker \beta = p(\text{Rad}(R))$, so $v - p(u) = p(r)$ for some $r \in \text{Rad}(R)$. Then $v = p(u + r)$, and $u + r \in R^\times + \text{Rad}(R) = R^\times$. \square

We can use Proposition 3.2.4 to give examples of rings with $(*)$ that are not semilocal. Although the following lemma should be well-known, we include a proof for completeness.

Lemma 3.2.5. *For an arbitrary direct product of rings, $\text{Rad}(\prod_i R_i) = \prod_i \text{Rad}(R_i)$.*

Proof. \supseteq : let $(a_i) \in \prod_i \text{Rad}(R_i)$. Then for each i and any $b_i \in R_i$, $1 - a_i b_i \in R_i^\times$, so every $b = (b_i) \in \prod_i R_i$ satisfies $1 - ab = (1 - a_i b_i) \in \prod_i R_i^\times = (\prod_i R_i)^\times$.

\subseteq : for any surjective ring map $\varphi : R \twoheadrightarrow S$, $\varphi(\text{Rad}(R)) \subseteq \text{Rad}(S)$, so applying this to each natural projection $\pi_j : \prod_i R_i \twoheadrightarrow R_j$ gives $\pi_j(\text{Rad}(\prod_i R_i)) \subseteq \text{Rad}(R_j)$. \square

Example 3.2.6. i) If $R = \prod_i R_i$ is an arbitrary product of semilocal rings, then R has $(*)$ (note that such a ring can have infinite Krull dimension, cf. [9]). To see this, note that by Proposition 3.2.4 and Lemma 3.2.5, it suffices to show that any product of fields has $(*)$. Thus, let $R = \prod_i k_i$, where k_i are fields. Using Proposition 3.1.5(iii), let I be an R -ideal, and $a = (a_i) \in (1 + I)^\sim$, such that $1 - ab \in I$ for some $b \in R$. Let J be the set of indices

j such that $a_j = 0$, and let e_J be the indicator vector of J , i.e. $e_J := (e_i) \in R$, where $e_i := \begin{cases} 1, & i \in J \\ 0, & i \notin J \end{cases}$. Then $e_J(1 - ab) \in I$, and satisfies $(e_J(1 - ab))_i = 0$ iff $i \notin J$ (note that if $i \in J$, then $a_i = 0$ and both $(1 - ab)_i = 1 - a_i b_i$ and $(e_J)_i$ equal 1, so their product is also $1 \neq 0$, whereas if $i \notin J$, then $(e_J)_i$ is already 0). Thus $(a + e_J(1 - ab))_i$ is nonzero for every i , hence $a + e_J(1 - ab) \in R^\times \implies a \in R^\times + I$.

ii) Via a different approach, we can also show that $(*)$ passes to finite products. Let $R = \prod_{i=1}^n R_i$, where R_i have $(*)$. Using Proposition 3.1.5(iv), let I be an R -ideal. Then $I = \prod_{i=1}^n I_i$ for R_i -ideals I_i . Let $a = (a_i) \in R$ be such that $1 - ab \in I$ for some $b = (b_i) \in R$. Then $1 - a_i b_i \in I_i$ for each i , so there exists $u_i \in R_i^\times$ with $1 - a_i u_i \in I_i$. Thus $u = (u_i) \in \prod_{i=1}^n R_i^\times = R^\times$, and $1 - au \in I$.

iii) In view of Example 3.2.6(i), as the diagonal map $\mathbb{Z} \hookrightarrow \prod_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$ is injective, we see that $(*)$ does not pass to subrings. On the other hand, it is easy to see that $(*)$ passes to quotient rings.

We briefly turn to the graded case. Let $R = \bigoplus_{i \geq 0} R_i$ be a $\mathbb{Z}_{\geq 0}$ -graded ring, $I = \bigoplus_{i \geq 0} I_i$ a graded R -ideal, and $p : R \twoheadrightarrow R/I$ the canonical surjection, a graded ring map of degree 0. Let $p_0 : R_0 \twoheadrightarrow (R/I)_0 = R_0/I_0$ be the induced ring map of degree 0 components. In general, the units of R need not be graded. However, with some primality assumptions we may reduce to the ungraded case, as follows:

Proposition 3.2.7. *Suppose I is prime. If p_0 has $(*)$, then p has $(*)$. The converse holds if R is a domain.*

Proof. If I is prime, then R/I is a positively graded domain, which has units only in degree 0, i.e. $(R/I)^\times \subseteq (R/I)_0$. Then $(R/I)^\times = ((R/I)_0)^\times = p_0^\times(R_0^\times) \subseteq p^\times(R^\times)$, and the first statement follows. Conversely, if R is a domain, then $R^\times \subseteq R_0$, so $R^\times = (R_0)^\times$ and $p(R^\times) = p_0(R_0^\times)$. \square

Corollary 3.2.8. *If $I \subseteq R_+ = \bigoplus_{i \geq 1} R_i$ is prime, then I has $(*)$.*

Proof. In this case, $I_0 = 0$, so $p_0 : R_0 \rightarrow R_0$ is the identity, hence p_0 has $(*)$. \square

To motivate the next section, we briefly summarize the results thus far: we have seen that property $(*)$ for a ring R depends only on the J -semisimple reduction $R/\text{Rad}(R)$. Since the J -semisimple reduction of a semilocal ring is a finite product of fields, this gives an alternate proof of Corollary 3.2.3. However, being semilocal is not a necessary condition for a ring to have $(*)$, as an infinite product of fields is never semilocal. Despite this, the examples given so far of rings with $(*)$ are quite similar - e.g. they all share the property that the J -semisimple reduction is 0-dimensional.

From a different angle, one can start with the observation that for any ring R , if $r \in R$ is a nonunit, then $R \twoheadrightarrow R/(r^2)$ is such that $1 + r$ goes to a unit in $R/(r^2)$, with inverse $1 - r$. In particular, if a ring R is to have $(*)$, then necessarily any element r must satisfy

$1+r \in R^\times + (r^2)$, i.e. for any $r \in R$, there exists $s \in R$ such that $1+r-sr^2 \in R^\times$. Recalling that $\text{Rad}(R) = \{r \in R \mid 1+(r) \subseteq R^\times\}$, this will certainly be satisfied if for every $r \in R$, there exists $s \in R$ with $r-sr^2 = r(1-sr) \in \text{Rad}(R)$. It is this last condition which we now examine in detail.

3.3 Semi-inverses

Returning to a general setting (laying aside for now the surjectivity question), let R be a ring, and $r \in R$. The failure of r to be a unit is encoded in the set of maximal ideals which contain r – namely, r is a unit iff r is not contained in any maximal ideal. Furthermore, when this occurs there is a unique element r^{-1} , with $1-r^{-1} \cdot r = 0 \in \mathfrak{m}$ for every maximal ideal \mathfrak{m} . Generalizing this basic fact gives an analogous notion for any $r \in R$:

Definition 3.3.1. Let R be a ring, $r \in R$. A subset $S \subseteq R$ is called a *semi-inverse set* for r if for every maximal ideal $\mathfrak{m} \in \text{mSpec}(R)$, either $r \in \mathfrak{m}$, or there exists $s \in S$ with $1-sr \in \mathfrak{m}$.

Notice that the two cases in the definition above are exhaustive and mutually exclusive: i.e. for any $r \in R$ and any $\mathfrak{m} \in \text{mSpec}(R)$, it is always the case that either $r \in \mathfrak{m}$ or there exists $s \in R$ with $1-sr \in \mathfrak{m}$, and both cases cannot occur simultaneously. Notice that existence of semi-inverse sets follows from the Axiom of Choice: for every maximal ideal \mathfrak{m} not containing r , the image $\bar{r} \in R/\mathfrak{m}$ is a unit, so there exists $\bar{s} \in R/\mathfrak{m}$ with $\bar{r} \cdot \bar{s} = \bar{1}$, i.e. $1-sr \in \mathfrak{m}$. This also shows that for any $r \in R$, the minimum size of a semi-inverse set for r is at most $|\text{mSpec}(R) \setminus V(r)|$, which leads to the following definition:

Definition 3.3.2. For a ring R , define a function $\rho : R \rightarrow \mathbb{N} \cup \{\infty\}$ by

$$\rho(r) := \begin{cases} \min\{|S| : S \text{ semi-inverse set for } r\}, & \text{if } r \text{ has a finite semi-inverse set} \\ \infty, & \text{if } r \text{ has no finite semi-inverse set} \end{cases}$$

The possible values that the function ρ can attain are rather limited:

Proposition 3.3.3. Let R be a ring, $r \in R$. Then $\rho(r) < \infty$ iff $\rho(r) \in \{0, 1\}$.

Proof. Suppose $\rho(r) \neq \infty$, and let $S = \{s_1, \dots, s_n\}$ be a finite semi-inverse set for r . Now $\prod_{i=1}^n (1-s_i r) = 1-sr$ for some $s \in R$ (since the product is finite). Thus $r(1-sr) = r \prod_{i=1}^n (1-s_i r) \in \text{Rad}(R)$, so $\{s\}$ is a semi-inverse set for r , and $\rho(r) \leq 1$. \square

Proposition 3.3.4. Let R be a ring, $r \in R$. Then $\rho(r) = 0$ iff $r \in \text{Rad}(R)$.

Proof. If $r \in \text{Rad}(R)$, then \emptyset is a semi-inverse set for r . Conversely, if $r \notin \mathfrak{m}$ for some $\mathfrak{m} \in \text{mSpec}(R)$, then if S is any semi-inverse set for r , there must exist $s \in S$ with $1-sr \in \mathfrak{m}$, so $|S| \geq 1$, hence $\rho(r) \geq 1$. \square

Proposition 3.3.5. *Let R be a ring. Then $R^\times \subseteq \rho^{-1}(\{1\})$, and equality holds iff $\text{Spec}(R/\text{Rad}(R))$ is connected.*

Proof. If $u \in R^\times$, then $\{u^{-1}\}$ is a semi-inverse set for u , so $\rho(u) = 1$ (as $u \notin \text{Rad}(R) \implies \rho(u) \neq 0$). For the second statement, suppose $R/\text{Rad}(R)$ has no idempotents, and pick $r \in R$, $\rho(r) = 1$. Let $\{s\}$ be a semi-inverse set for r , so $r(1 - sr) \in \text{Rad}(R)$. Then $\bar{r} = \bar{s} \cdot \bar{r}^2$ in $R/\text{Rad}(R)$, so $\bar{s} \cdot \bar{r}$ is idempotent in $R/\text{Rad}(R)$. By assumption $\bar{s} \cdot \bar{r} = \bar{0}$ or $\bar{1}$. If $\bar{s} \cdot \bar{r} = \bar{0}$, then $\bar{r} = (\bar{s} \cdot \bar{r})\bar{r} = \bar{0}$, i.e. $r \in \text{Rad}(R)$, but this cannot happen if $\rho(r) = 1$. Thus $\bar{s} \cdot \bar{r} = \bar{1}$, so r is a unit modulo $\text{Rad}(R)$, hence r is in fact a unit in R .

Conversely, suppose $\rho^{-1}(\{1\}) = R^\times$, and let $r \in R$ with $\bar{0} \neq \bar{r}$ idempotent in $R/\text{Rad}(R)$. Then $r - r^2 \in \text{Rad}(R)$, so $\{1\}$ is a semi-inverse set for r , i.e. $\rho(r) = 1$, so $r \in R^\times$. This implies $R/\text{Rad}(R)$ has only trivial idempotents, hence has connected spectrum. \square

Remark 3.3.6. i) If $\text{Spec}(R/\text{Rad}(R))$ is connected, then $\text{Spec}(R)$ is also connected: if $e \in R$ is idempotent, then $\bar{e} \in R/\text{Rad}(R)$ is also idempotent, so (replacing e by $1 - e$ if necessary) $\bar{0} = \bar{e} \implies e \in \text{Rad}(R) \implies 1 - e \in R^\times$, hence $e(1 - e) = 0 \implies e = 0$.

ii) If R is the coordinate ring of an (irreducible) affine variety (i.e. a finitely generated domain over a field), then $\text{Spec}(R/\text{Rad}(R))$ is connected.

Proposition 3.3.3 and Proposition 3.3.4 indicate that the only interesting behavior occurs for elements $r \in R$ with $\rho(r) = 1$, which motivates the following definition:

Definition 3.3.7. Let R be a ring, $r \in R$. If $\rho(r) = 1$, we say that r is a *semi-unit*. In this case, if $\{s\}$ is a semi-inverse set for r , we say that s is a *semi-inverse* of r . If every element of R is either a semi-unit or in the Jacobson radical (i.e. $\rho(R) \subseteq \{0, 1\}$), we say that R is a *semi-field*.

Remark 3.3.8. According to the definition, only semi-units can have semi-inverses, so although $\{1\}$ (or indeed any singleton set) is a semi-inverse set for 0, 1 is not treated as a semi-inverse of 0. Also, the relation of being a semi-inverse need not be symmetric: e.g. in $\mathbb{Z}/10\mathbb{Z}$, 3 is a semi-inverse of 2 (as $2 \equiv 3 \cdot 2^2 \pmod{10}$), but 2 is not a semi-inverse of 3 ($3 \not\equiv 2 \cdot 3^2 \pmod{10}$). However, notice that 2 and 8 are semi-inverses of each other.

The following proposition addresses uniqueness of semi-inverses:

Proposition 3.3.9. *Let R be a ring, $r \in R$ a semi-unit. If $s_1, s_2 \in R$ are semi-inverses of r , then $s_1 - s_2 \in \text{Rad}(R) :_R r$. Conversely, if s is a semi-inverse of r and $a \in \text{Rad}(R) :_R r$, then $s + a$ is a semi-inverse of r .*

Proof. If s_1, s_2 are semi-inverses of r , then $r(1 - s_1r), r(1 - s_2r) \in \text{Rad}(R)$, so $r(1 - s_1r) - r(1 - s_2r) = (s_2 - s_1)r^2 \in \text{Rad}(R)$, i.e. $s_2 - s_1 \in \text{Rad}(R) :_R r^2$. For the second statement, if s is a semi-inverse of r and $a \in \text{Rad}(R) :_R r^2$, then $r(1 - sr), ar^2 \in \text{Rad}(R)$, so $r(1 - (s + a)r) = r(1 - sr) - ar^2 \in \text{Rad}(R)$ also.

Finally, notice that $\text{Rad}(R) :_R r^2 = \text{Rad}(R) :_R r$, since if $ar^2 \in \text{Rad}(R)$, then $(ar)^2 = a(ar^2) \in \text{Rad}(R) \implies ar \in \text{Rad}(R)$, as $\text{Rad}(R)$ is a radical ideal. \square

Thus semi-inverses of r are unique precisely up to cosets of $\text{Rad}(R) : r$. In particular, semi-inverses of non-trivial semi-units are never unique:

Corollary 3.3.10. *Let R be a ring, $r \in R$ a semi-unit. Then r has a unique semi-inverse iff r is a unit and $\text{Rad}(R) = 0$.*

Proof. \Leftarrow : if r is a unit, then $\text{Rad}(R) : r = \text{Rad}(R) = 0$, so r^{-1} is the only semi-inverse of r .
 \Rightarrow : if r has a unique semi-inverse s , then $\text{Rad}(R) = 0$, and $r = sr^2$. But $0 = \text{Rad}(R) : r = 0 : r$, so r is a nonzerodivisor, hence $1 = sr$, i.e. $r \in R^\times$. \square

On the other hand, any semi-unit has a semi-inverse that is a unit. This follows from the following general decomposition theorem:

Theorem 3.3.11. *Let R be a ring, $r \in R$. Then r is a semi-unit iff $r = ue + t$ for some $t \in \text{Rad}(R)$, $u \in R^\times$, and $e \in R$ a semi-unit with 1 a semi-inverse of e ($\iff \bar{e}$ idempotent in $R/\text{Rad}(R)$). In particular, u^{-1} is a semi-inverse of r .*

Proof. Passing to $R/\text{Rad}(R)$, it suffices to show that \bar{r} is a product of a unit and an idempotent. Let s be a semi-inverse of r , so $\bar{r} = \overline{sr^2}$. Set $\bar{e} := \overline{rs}$. Then $\bar{e}^2 = \bar{e}$, so if e is any lift of \bar{e} , then e is a semi-unit in R with 1 as a semi-inverse. Notice also that $\bar{r} = \overline{re}$.

Next, set $\bar{u} := \overline{re} + (1 - \bar{e})$. Then $\overline{ue} = \overline{re}^2 + (1 - \bar{e})\bar{e} = \bar{r}$. Furthermore,

$$\begin{aligned} \bar{u} \cdot (\overline{se} + (1 - \bar{e})) &= (\overline{re} + (1 - \bar{e})) \cdot (\overline{se} + (1 - \bar{e})) \\ &= \overline{rse}^2 + (1 - \bar{e})^2 \\ &= \bar{e}^3 + (1 - \bar{e}) \\ &= 1 \end{aligned}$$

so \bar{u} is a unit. Lifting to R gives a unit $u \in R$, such that $t := r - ue \in \text{Rad}(R)$.

Finally, notice that $r(1 - u^{-1}r) = (ue + t)(1 - u^{-1}(ue + t)) = ue(1 - e) + t(1 - 2e - u^{-1}t) \in \text{Rad}(R)$, so u^{-1} is a semi-inverse of r . \square

3.4 Semi-fields

Having described the structure of semi-units, we now focus on the rings that have as many semi-units as possible, starting with the following criterion:

Proposition 3.4.1. *Let R be a ring. Then the following are equivalent:*

- i) R is a semi-field
- ii) $R/\text{Rad}(R)$ is von Neumann regular
- iii) $\dim R/\text{Rad}(R) = 0$.

Proof. R is a semi-field \iff for every $r \in R$, there exists $s \in R$ with $r(1 - sr) \in \text{Rad}(R)$ \iff for every $\bar{r} \in R/\text{Rad}(R)$, there exists $\bar{s} \in R/\text{Rad}(R)$ with $\bar{r} = \bar{s} \cdot \bar{r}^2$ \iff $R/\text{Rad}(R)$ is von Neumann regular. Since $R/\text{Rad}(R)$ is always reduced, this happens iff $\dim R/\text{Rad}(R) = 0$. \square

A geometric reformulation of the semi-field property is given by:

Proposition 3.4.2. *Let R be a ring. Then R is a semi-field iff $\text{mSpec } R$ is closed in $\text{Spec } R$.*

Proof. First, note that the closure of $\text{mSpec } R$ is equal to $V(\text{Rad } R)$: for any $\mathfrak{p} \in \text{Spec } R$, \mathfrak{p} is in $\overline{\text{mSpec } R} \iff$ for all $f \in R$ with $\mathfrak{p} \in D(f)$, there exists $\mathfrak{m} \in \text{mSpec } R$ with $\mathfrak{m} \in D(f) \iff R - \mathfrak{p} \subseteq \bigcup_{\mathfrak{m} \in \text{mSpec } R} (R - \mathfrak{m}) \iff \mathfrak{p} \supseteq \bigcap_{\mathfrak{m} \in \text{mSpec } R} \mathfrak{m} = \text{Rad } R$.

Thus, $\text{mSpec } R = \overline{\text{mSpec } R}$ iff $\text{mSpec } R = V(\text{Rad } R)$ iff $\dim R/\text{Rad}(R) = 0$, so the conclusion follows from Proposition 3.4.1. \square

Corollary 3.4.3. *The following are equivalent for a ring R :*

- i) R is semilocal*
- ii) $R/\text{Rad}(R)$ is Artinian*
- iii) R is a semi-field and $|\text{Min}(\text{Rad}(R))| < \infty$*
(here $\text{Min}(\cdot)$ denotes the set of minimal primes).

Proof. iii) \implies ii): If R is a semi-field with $\text{Min}(R/\text{Rad}(R)) = \text{Spec}(R/\text{Rad}(R))$ finite, then $R/\text{Rad}(R)$ is a von Neumann regular ring with finite spectrum, hence is Noetherian.

ii) \implies i): An Artinian ring is semilocal, and $R/\text{Rad}(R)$ semilocal $\implies R$ semilocal.

i) \implies iii): If R is semilocal, then by Chinese Remainder $R/\text{Rad}(R)$ is a finite direct product of fields. \square

We give two ways to produce new semi-fields:

Proposition 3.4.4. *The class of semi-fields is closed under quotients and products.*

Proof. Let R be a semi-field, and I an R -ideal. The surjection $p : R \rightarrow R/I$ sends $p(\text{Rad}(R)) \subseteq \text{Rad}(R/I)$, so $(R/I)/\text{Rad}(R/I)$ is a quotient of $R/(\text{Rad}(R) + I)$, which is itself a quotient of $R/\text{Rad}(R)$. Thus $\dim R/\text{Rad}(R) = 0$ implies $\dim(R/I)/\text{Rad}(R/I) = 0$.

If now R_i are semi-fields, then by Lemma 3.2.5

$$\left(\prod_i R_i\right)/\text{Rad}\left(\prod_i R_i\right) = \left(\prod_i R_i\right)/\left(\prod_i \text{Rad}(R_i)\right) = \prod_i R_i/\text{Rad}(R_i)$$

is a product of von Neumann regular rings, hence is von Neumann regular. \square

Remark 3.4.5. Geometrically, the first part of Proposition 3.4.4 says that the semi-field property passes to closed subschemes. However, the semi-field property does not pass to open subschemes – e.g. if R is any Noetherian ring, $x \in \text{Rad}(R)$ but x is not contained in any minimal prime of R , then $\dim R_x = \dim R - 1$, and if $\dim R < \infty$, then $\text{Rad}(R_x) = \text{nil}(R_x)$. Thus any Noetherian local domain (R, \mathfrak{m}) of dimension ≥ 2 and $0 \neq x \in \mathfrak{m}$ gives an example where R is a semi-field (being local), but R_x is not.

Even in light of Proposition 3.4.4, it is still reasonable to ask for nontrivial examples of semi-fields. One trivial reason for being a semi-field is that the set of closed points is finite, and Corollary 3.4.3 guarantees that this is the only possibility in the Noetherian case – thus, one must search among non-Noetherian rings for a nontrivial example.

Now one can easily form non-Noetherian rings by taking infinite products. However, products are an arguably trivial way to construct examples – for finite products, the geometric intuition is that the property of the closed points forming a closed set should pass to disjoint unions. This intuition fails for general von Neumann regular rings though, since not every von Neumann regular ring is a product of fields: e.g. if k is a finite field, then the subring of $\prod_{i \in \mathbb{N}} k$ consisting of eventually constant sequences is non-Noetherian and countable, whereas any product of fields is either Noetherian or uncountable. Despite this, von Neumann regular rings are trivially semi-fields for the same reason any zero-dimensional ring is: the set of closed points is certainly closed if every point is closed!

Nevertheless, there are indeed less trivial examples of semi-fields, which arise formally in a manner similar to Hilbert's basis theorem and (a general form of) the Nullstellensatz, which say that the Noetherian and Jacobson properties pass to rings of finite type. To emphasize the analogy, for a ring R , we say that a ring is of semi-finite type over R if it is of the form $R[[x_1, \dots, x_n]]/I$.

Proposition 3.4.6. *Let R be a semi-field. Then any ring of semi-finite type over R is a semi-field.*

Proof. By Proposition 3.4.4, it suffices to show R semi-field $\implies R[[x_1, \dots, x_n]]$ semi-field, and by induction it is enough to do the base case $n = 1$. This follows immediately from the fact that $x \in \text{Rad}(R[[x]])$, which in turn implies that every maximal ideal of $R[[x]]$ is of the form $\mathfrak{m}R[[x]] + (x)$ for a (uniquely determined) maximal ideal \mathfrak{m} of R , so $R/\text{Rad}(R) \cong R[[x]]/\text{Rad}(R[[x]])$. \square

3.5 Property (*) revisited

We finally return to the original surjectivity question. Proposition 3.4.1 shows that every example given earlier of a ring with (*) has been a semi-field. The following theorem gives the general phenomenon:

Theorem 3.5.1. *Let R be a semi-field. Then R has (*).*

Proof. By Proposition 3.2.4 we may pass to $R/\text{Rad}(R)$, so by Proposition 3.4.1 it suffices to show any von Neumann regular ring R has (*). For this we use Proposition 3.1.5(iv). Let $I \neq R$ be an ideal, and $a \in R$ such that $1 - ab \in I$ for some $b \in R$. As R is von Neumann regular, I is a radical ideal, so $I = \bigcap_i p_i$ for some primes $p_i \in \text{Spec } R$. Then $1 - ab \in p_i$ implies $a \notin p_i$, for all i . Now a is a semi-unit, so by Theorem 3.3.11, a has a semi-inverse which is a unit, i.e. there exists $u \in R^\times$ with $a = a^2u$. Then $a(1 - au) = 0 \in p_i$ for all i , so $1 - au \in p_i$ for all i , hence $1 - au \in I$. \square

Remark 3.5.2. Corollary 3.4.3, Proposition 3.4.4, and Theorem 3.5.1 give an alternate proof of Example 3.2.6, that an arbitrary product of semilocal rings has $(*)$. We do not know if the class of rings with $(*)$ is closed under arbitrary products.

Theorem 3.5.1 thus generalizes and gives a uniform proof of all the previous sufficient conditions for a ring to have $(*)$: Corollary 3.2.1, Corollary 3.2.3, and Example 3.2.6.

We conclude with an application and a generalization. Although the motivation in determining when an ideal or ring has $(*)$ has been mostly intrinsic, one possible application of these results is in constructing rings with trivial unit group.

Proposition 3.5.3. *Let $X \subseteq \mathbb{P}_{\mathbb{F}_2}^n$ be a reduced projective scheme. Then the homogeneous coordinate ring of X has trivial unit group.*

Proof. Let $S = \mathbb{F}_2[x_0, \dots, x_n] = \Gamma_*(\mathbb{P}_{\mathbb{F}_2}^n, \mathcal{O}_{\mathbb{P}_{\mathbb{F}_2}^n})$ and $R = \mathbb{F}_2[x_0, \dots, x_n]/I$, where I is a homogeneous radical ideal. Then $I = p_1 \cap \dots \cap p_m$, where p_i are homogeneous primes in S , so $R \hookrightarrow S/p_1 \times \dots \times S/p_m$. Thus $R^\times \subseteq \prod_{i=1}^m (S/p_i)^\times$, so it suffices to show $(S/p_i)^\times = \{1\}$ for each i . Now S is a polynomial ring over \mathbb{F}_2 , so $S^\times = (\mathbb{F}_2)^\times = \{1\}$, and each $p_i \subseteq S_+$, so by Corollary 3.2.8 there is a surjection $\{1\} = S^\times \twoheadrightarrow (S/p_i)^\times$. \square

In fact, the above reasoning holds in any number of variables. Thus, if $R = \mathbb{Z}[x_1, \dots]/I$ is any ring presented as a \mathbb{Z} -algebra, then homogenizing the defining ideal I with a new variable x_0 gives a standard graded ring $\tilde{R} := \mathbb{Z}[x_0, x_1, \dots]/\tilde{I}$, and then $(\tilde{R} \otimes_{\mathbb{Z}} \mathbb{F}_2)_{\text{red}} = \mathbb{F}_2[x_0, x_1, \dots]/\sqrt{\tilde{I}}$ has trivial unit group.

Conversely, every ring with trivial unit group has characteristic 2 (as $1 = -1$) and has trivial Jacobson radical (in particular, is reduced). Thus if $R^\times = \{1\}$, then R is the (affine) coordinate ring of a reduced scheme over \mathbb{F}_2 , and Proposition 3.5.3 realizes every (standard) graded ring with trivial unit group.

Finally, one possible generalization is to consider other functors from **CRing** to **Grp**. A natural choice which directly generalizes the group of units functor is $GL_n(_) : \mathbf{CRing} \rightarrow \mathbf{Grp}$, which for $n = 1$ coincides with $(_)^\times$. In order to treat the case of GL_n , it is necessary to consider noncommutative rings, and nonabelian groups.

It is also possible to define property $(*)$ for two-sided ideals in a noncommutative ring. It turns out that the key place where commutativity was used in Section 1 was to describe the preimage of the units of R/I as a saturation $(1 + I)^\sim$. To be precise, let us make the following definition:

Definition 3.5.4. Let R be an arbitrary (possibly noncommutative) ring, and $W \subseteq R$. Define the saturation of W as

$$W^\sim := \{x \in R \mid \exists y \in R : xy, yx \in W\}$$

When R is commutative, this reduces to the previous definition of \sim , and one can check that \sim is still a closure operation: e.g. $W \subseteq W^\sim$ follows from existence of a 1, and idempotence (i.e. $W^\sim = (W^\sim)^\sim$) follows from associativity of multiplication (note: if the condition “either $xy \in W$ or $yx \in W$ ” was used instead, then \sim would no longer be idempotent).

The definition of \sim above agrees well with units, since units are by definition two-sided. For example, $\{1\}^\sim = R^\times$, and the exact statement of Proposition 3.1.5 goes through without change.

However, this turns out to be unnecessary for GL_n , because of the fact that in the matrix ring, $AB = 1$ iff $BA = 1$. In other words, the definition for \sim above works well in a Dedekind-finite ring (i.e. $xy = 1 \iff yx = 1$ for all $x, y \in R$).

Proposition 3.5.5. *Let R be a ring, $I \subseteq \text{Rad}(R)$ an R -ideal, and $p : R \rightarrow R/I$ the canonical surjection. Then for any $n \in \mathbb{N}$, $\bar{p} : GL_n(R) \rightarrow GL_n(R/I)$ is surjective.*

Proof. Pick $B = (b_{ij}) \in GL_n(R/I)$, and let $A = (a_{ij}) \in M_n(R)$ be any (entrywise) lift of B to R , i.e. $p(a_{ij}) = b_{ij}$ for all i, j . Since $\det A$ is a polynomial in the entries of A , $p(\det A) = \det B$ is a unit in R/I . But $I \subseteq \text{Rad}(R)$, so $\det A$ is in fact a unit in R , i.e. $A \in GL_n(R)$. \square

Notice that the proof of Proposition 3.5.5 shows a stronger fact than preserving surjectivity; namely, any lift of a matrix in $GL_n(R/I)$ is already in $GL_n(R)$. In fact, the analogues of Corollary 3.2.1 and Proposition 3.2.4 hold for $GL_n(_)$ as well, and show that Proposition 3.5.5 holds for semilocal rings as well.

The most natural question remaining from the previous chapter is whether the converse of Theorem 3.5.1 holds: i.e. if every surjection out of a ring induces surjections on unit groups, must the ring be a semi-field? In arriving at the notion of semi-field, we broadened the notion of unit to semi-units. The present chapter [1] is concerned with an orthogonal generalization – rather than considering the complement of the union of all prime ideals, we consider (complements of) unions of infinitely many prime ideals. As finite unions of prime ideals are famously understood via the prime avoidance lemma, we seek to understand when the lemma extends to infinite unions; this will in turn lead to the resolution of the question above.

Chapter 4

Infinite prime avoidance

The classical prime avoidance lemma is one of the most ubiquitous results in commutative algebra. Prime avoidance, along with finiteness of associated primes, is one of the basic building blocks of the theory of Noetherian rings. For example, the two results can be jointly used to choose generic nonzerodivisors (such as in the converse of Krull's Altitude Theorem, cf. [7], Corollary 10.5), or to select a single annihilator for an ideal consisting of zerodivisors.

As a fundamental technical result, the prime avoidance lemma has found various extensions in the literature (cf. [14], [24]). Moreover, special cases of infinite prime avoidance have in the past been used to great effect, perhaps most famously as a crucial step in Nagata's example of an infinite-dimensional Noetherian ring. This indicates the potential utility of understanding and applying infinite prime avoidance methodically. The goal of this note is to make initial steps in this direction. To this end, we first make a definition. For a commutative ring R with $1 \neq 0$, $\text{Spec } R$ denotes the set (for now) of prime ideals of R .

Definition 4.0.1. Let R be a ring, $\Lambda \subseteq \text{Spec } R$. We say that Λ *satisfies prime avoidance* if $I \subseteq \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p} \implies I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Lambda$, for any R -ideal I .

Note that in the definition of prime avoidance, it is enough to check the condition for prime ideals I , since ideals which are maximal with respect to being contained in $\bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ (i.e. not meeting the multiplicative set $R \setminus \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$) exist by Zorn's lemma, and are prime.

Example 4.0.2. For any ring R , the set of maximal ideals $\text{mSpec } R$ satisfies prime avoidance: if $I \subseteq \bigcup_{\mathfrak{m} \in \text{mSpec } R} \mathfrak{m}$, then I consists of nonunits, hence is contained in a maximal ideal. This example, though basic, is actually representative of all examples in some sense: cf. Theorem 4.1.6(3), (6).

We now arrive at the classical prime avoidance lemma. For convenience we give a short direct proof (as opposed to one using induction):

Lemma (Prime Avoidance). *Let R be a ring, $\Lambda \subseteq \text{Spec } R$. If Λ is finite, then Λ satisfies prime avoidance.*

Proof. Write $\Lambda = \{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$. Suppose I is an R -ideal such that $I \not\subseteq \mathfrak{p}_i$ for any i , and choose $a_i \in I \setminus \mathfrak{p}_i$. Removing redundant primes for the union, we may choose $b_i \in \mathfrak{p}_i \setminus \bigcup_{j \neq i} \mathfrak{p}_j$ for each i . Set $c_i := a_i \prod_{j \neq i} b_j$. Then $c_i \in \mathfrak{p}_j$ iff $j \neq i$ by primeness of \mathfrak{p}_i , so $c_1 + \dots + c_n \in I \setminus \bigcup \mathfrak{p}_i$. \square

General though prime avoidance is, its single restriction is quite severe: the set Λ must be finite! The proof above offers no recourse to relaxing this constraint. But it is not without good reason that this is the case, as prime avoidance may simply fail when Λ is infinite, even for sets of minimal primes:

Example 4.0.3. Let k be a field, $R = k[x_0, x_1, \dots]/(x_{2i}x_{2i+1} \mid i \geq 0)$. Then the set of minimal primes $\text{Min}(R)$ has cardinality 2^{\aleph_0} : every minimal prime is of the form $(x_{a(i)} \mid i \geq 0)$ for a sequence $\{a(i)\}_{i \geq 0}$ with $a(i) \in \{2i, 2i+1\}$. Let $\mathfrak{p}_{\text{odd}} := (x_1, x_3, \dots)$ be the minimal prime of odd variables. Then $\mathfrak{p}_{\text{odd}} \subseteq \bigcup_{\mathfrak{p} \in \text{Min}(R) \setminus \{\mathfrak{p}_{\text{odd}}\}} \mathfrak{p}$.

To see this, pick $f \in \mathfrak{p}_{\text{odd}}$. Write f as an R -linear combination of finitely many generators of $\mathfrak{p}_{\text{odd}}$, say $x_1, x_3, \dots, x_{2j-1}$. Then e.g. $(x_1, x_3, \dots, x_{2j-1}, x_{2j}, x_{2j+2}, \dots)$ is a minimal prime of R containing f which is distinct from $\mathfrak{p}_{\text{odd}}$.

By similar reasoning, every minimal prime of R is contained in the union of the other minimal primes. We remark that in this ring, the set of all minimal primes does satisfy prime avoidance, but even this need not hold in general: there exist reduced rings of dimension > 0 where every nonzerodivisor is a unit.

Even in much tamer rings, infinite prime avoidance need not hold. For instance, Noetherian rings have only finitely many minimal primes, which prevents minimal primes from (mis)behaving as in Example 4.0.3. However, in this setting the principal ideal theorem can sometimes force infinite prime avoidance to fail:

Proposition 4.0.4. *Let R be a Noetherian ring.*

1. *For any $\mathfrak{q} \in \text{Spec } R$, $\mathfrak{q} \subseteq \bigcup_{\text{ht } \mathfrak{p} \leq 1} \mathfrak{p}$ (so prime avoidance fails if $\text{ht } \mathfrak{q} \geq 2$).*
2. *Suppose R is also Jacobson. Then for any $\mathfrak{m} \in \text{mSpec } R$ with $\text{ht } \mathfrak{m} \geq 2$, $\mathfrak{m} \subseteq \bigcup_{\mathfrak{n} \in \text{mSpec}(R) \setminus \{\mathfrak{m}\}} \mathfrak{n}$.*

Proof. (a): Pick $f \in \mathfrak{q}$, and take \mathfrak{p} a minimal prime of f . Then $f \in \mathfrak{p}$, and by Krull's Principal Ideal Theorem, $\text{ht } \mathfrak{p} \leq 1$.

(b): Pick $f \in \mathfrak{m}$, and let \mathfrak{p} be a minimal prime of f contained in \mathfrak{m} (i.e. the pullback to R of a minimal prime of $(R/(f))_{\mathfrak{m}}$). Now $\text{ht } \mathfrak{p} \leq 1 \implies \mathfrak{p} \neq \mathfrak{m} \implies \mathfrak{p}$ is not maximal; hence \mathfrak{p} is a (necessarily infinite) intersection of maximal ideals (as R is Jacobson). Thus there is a maximal ideal $\mathfrak{n} \neq \mathfrak{m}$ with $\mathfrak{p} \subseteq \mathfrak{n}$, so $f \in \mathfrak{n}$. \square

In spite of these examples, one can still ask for classes of infinite sets of primes which do satisfy prime avoidance. It turns out that this question does have some nice answers, which we will see in the next section.

4.1 Characterizations

Recall that $\text{Spec } R$ has the Zariski topology, with closed sets of the form $V(I) := \{\mathfrak{p} \mid I \subseteq \mathfrak{p}\}$ for an R -ideal I , and a ring map $\varphi : R \rightarrow S$ induces a continuous map $\varphi^* : \text{Spec } S \rightarrow \text{Spec } R$ via contraction.

Proposition 4.1.1. *Let $\varphi : R \rightarrow S$ be a ring map, which is either a surjection or a localization. If $\Lambda \subseteq \text{Spec } S$ satisfies prime avoidance, then so does $\varphi^*(\Lambda)$.*

Proof. Let $\mathfrak{p} \subseteq \bigcup_{\mathfrak{q} \in \Lambda} \varphi^{-1}(\mathfrak{q})$. Then for all $x \in \mathfrak{p}$, $x \in \varphi^{-1}(\mathfrak{q})$ for some $\mathfrak{q} \in \Lambda$, i.e. $\varphi(x) \in \mathfrak{q}$. Since φ is either a surjection or a localization, any element of $\mathfrak{p}S$ is of the form $s \cdot \varphi(x)$ for some $x \in \mathfrak{p}$, $s \in S$, so this shows that $\mathfrak{p}S \subseteq \bigcup_{\mathfrak{q} \in \Lambda} \mathfrak{q}$. By prime avoidance of Λ , $\mathfrak{p}S \subseteq \mathfrak{q}$ for some $\mathfrak{q} \in \Lambda$, hence $\mathfrak{p} \subseteq \varphi^{-1}(\mathfrak{p}S) \subseteq \varphi^{-1}(\mathfrak{q})$. \square

We use Proposition 4.1.1 to give examples of infinite sets satisfying prime avoidance. Hereafter when convenient, we view $\text{Spec}(U^{-1}R)$ inside $\text{Spec } R$ as $\{\mathfrak{p} \mid \mathfrak{p} \cap U = \emptyset\}$.

Corollary 4.1.2. *Let R be a ring, $U \subseteq R$ a multiplicative set, and I an R -ideal. Then $V(I) \cap \text{Spec}(U^{-1}R)$ satisfies prime avoidance.*

Proof. $\varphi : R \rightarrow U^{-1}(R/I)$ is a composite of localizations and surjections. Now apply Proposition 4.1.1 twice to $V(I) \cap \text{Spec}(U^{-1}R) = \varphi^*(\text{Spec}(U^{-1}(R/I)))$.

Notice: this shows that both $V(I)$ and $\text{Spec}(U^{-1}R)$ satisfy prime avoidance (by taking $U = \{1\}$ and $I = 0$, respectively). In addition, pulling back $\text{mSpec}(U^{-1}(R/I))$ above gives that $V(I) \cap \text{mSpec}(U^{-1}R) = \varphi^*(\text{mSpec}(U^{-1}(R/I)))$ satisfies prime avoidance. \square

Example 4.1.3. Proposition 4.1.1 may lead one to think that $\varphi^*(\text{Spec } S)$ satisfies prime avoidance for any ring epimorphism $\varphi : R \rightarrow S$, but this is not true. Let $k = \bar{k}$ be a field, $\tilde{R} = k[s, t, u]$, $S = k[x, y]$, and define $\tilde{\varphi} : \tilde{R} \rightarrow S$ by $s \mapsto x, t \mapsto xy, u \mapsto xy^2 - y$. Then $\tilde{\varphi}$ induces $\varphi : R := \tilde{R}/(su - t^2 + t) \rightarrow S$, which is a ring epimorphism. Since $R \cong k[x, xy, xy^2 - y] \subseteq S$, any nonunit in R is also a nonunit in S . Thus if $\Lambda := \varphi^*(\text{Spec } S)$, then $\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$ for any $\mathfrak{m} \in \text{mSpec } R$. However, $(s, t - 1, u)$ is a maximal ideal of R that is not in Λ : if $s \in \varphi^{-1}(x - a, y - b)$, then $a = 0$, and then $\varphi(t) = xy = x(y - b) + bx \in (x, y - b) \implies t \in \varphi^{-1}(x, y - b)$.

Example 4.1.4. It follows from Corollary 4.1.2 that basic Zariski-open sets (i.e. sets of the form $D(f) := (\text{Spec } R) \setminus V(f)$ for some $f \in R$) satisfy prime avoidance. However, arbitrary Zariski-open sets need not: if $R = k[x, y]$ for k a field, $\Lambda_1 := D(x)$, $\Lambda_2 := D(y)$, then $\Lambda_1 \cup \Lambda_2 = (\text{Spec } R) \setminus \{(x, y)\}$ does not satisfy prime avoidance, by Proposition 4.0.4(b). This example also shows that the class of sets satisfying prime avoidance is neither closed under union nor taking complements in $\text{Spec } R$.

Definition 4.1.5. Let R be a ring. For $\Lambda \subseteq \text{Spec } R$, define the following sets:

- $\Lambda_{max} := \{\mathfrak{p} \in \Lambda \mid \mathfrak{p} \not\subseteq \mathfrak{q}, \forall \mathfrak{q} \in \Lambda\}$, the subset of maximal elements of Λ . Notice: Λ_{max} may be empty, even if Λ is not!
- $\Lambda_{cl} := \{\mathfrak{q} \in \text{Spec } R \mid \exists \mathfrak{p} \in \Lambda, \mathfrak{q} \subseteq \mathfrak{p}\}$, the downward-closure of Λ in the poset $\text{Spec } R$. Notice: $(\cdot)_{cl}$ is a closure operation (i.e. monotonic, increasing, and idempotent). Indeed, $\Lambda_{cl} = \bigcup_{\mathfrak{p} \in \Lambda} (\{\mathfrak{p}\}_{cl}) = \bigcup_{\mathfrak{p} \in \Lambda} \text{Spec}(R_{\mathfrak{p}})$.

These definitions allow for various characterizations of prime avoidance. For a ring map $\varphi : R \rightarrow S$, we say that φ^* is surjective on closed points if $\text{mSpec } R \subseteq \varphi^*(\text{Spec } S)$ (or equivalently, $\text{mSpec } R \subseteq \varphi^*(\text{mSpec } S)$). In the following, keep in mind that although $W^{-1}I \subseteq W^{-1}J$ does not imply that $I \subseteq J$ in general, the implication does hold if J is prime (and does not meet W).

Theorem 4.1.6. *Let R be a ring, $\Lambda \subseteq \text{Spec } R$, $W := R \setminus \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$. Then the following are equivalent:*

- (1) Λ satisfies prime avoidance
- (2) $\text{mSpec}(W^{-1}R) \subseteq \Lambda_{max}$
- (3) $\text{mSpec}(W^{-1}R) = \Lambda_{max}$
- (4) $\text{mSpec}(W^{-1}R) \subseteq \Lambda_{cl}$
- (5) $\text{Spec}(W^{-1}R) = \Lambda_{cl}$
- (6) *There is a ring map $\varphi : R \rightarrow S$ such that*
 - (i) $\Lambda_{max} = \varphi^*(\text{mSpec } S)$ (so \exists induced map $W^{-1}R \rightarrow S$), and
 - (ii) $\text{Spec } S \rightarrow \text{Spec}(W^{-1}R)$ is surjective on closed points
- (7) Λ_{cl} satisfies prime avoidance
- (8) Λ_{max} satisfies prime avoidance and $\Lambda \subseteq (\Lambda_{max})_{cl}$.

Proof. (1) \iff (7): $\bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p} = \bigcup_{\mathfrak{p}' \in \Lambda_{cl}} \mathfrak{p}'$, and $I \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Lambda$ iff $I \subseteq \mathfrak{p}'$ for some $\mathfrak{p}' \in \Lambda_{cl}$.

(1) \implies (2): Take $\mathfrak{m} \in \text{mSpec}(W^{-1}R)$. Then $\mathfrak{m} = W^{-1}\mathfrak{q}$ where $\mathfrak{q} \in \text{Spec } R$ is maximal with respect to $\mathfrak{q} \cap W = \emptyset$. By prime avoidance, $\mathfrak{q} \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \Lambda$. But $\mathfrak{p} \cap W = \emptyset$, so $\mathfrak{q} = \mathfrak{p} \in \Lambda_{max}$ by maximality of \mathfrak{q} .

(2) \implies (3): Take $\mathfrak{p} \in \Lambda_{max}$. Then $W^{-1}\mathfrak{p}$ is a proper ideal in $W^{-1}R$, so $W^{-1}\mathfrak{p} \subseteq \mathfrak{m}$ for some maximal ideal $\mathfrak{m} \in \text{mSpec}(W^{-1}R)$. By assumption, $\mathfrak{m} = W^{-1}\mathfrak{q}$ for some $\mathfrak{q} \in \Lambda_{max}$. Localizing further at \mathfrak{q} gives $\mathfrak{p}R_{\mathfrak{q}} \subseteq \mathfrak{q}R_{\mathfrak{q}}$ which implies $\mathfrak{p} \subseteq \mathfrak{q}$, so by maximality of \mathfrak{p} in Λ , $\mathfrak{p} = \mathfrak{q}$, hence $W^{-1}\mathfrak{p} = \mathfrak{m} \in \text{mSpec}(W^{-1}R)$.

(3) \implies (4): Clear.

- (4) \implies (5): Follows from $\Lambda \subseteq \text{Spec}(W^{-1}R) = (\text{mSpec}(W^{-1}R))_{cl} \subseteq \Lambda_{cl}$.
(5) \implies (7): Follows from Corollary 4.1.2.
(3) \implies (6): Take $S := W^{-1}R$, $\varphi : R \rightarrow S$ the canonical map. Then (i) follows from (3), and (ii) is automatic.
(6) \implies (2): Clear.
(3) + (5) \implies (8): Clear.
(8) \implies (1): $\Lambda_{max} \subseteq \Lambda \subseteq (\Lambda_{max})_{cl} \implies (\Lambda_{max})_{cl} = \Lambda_{cl}$. Now apply (7). \square

Theorem 4.1.6(7) implies in particular that prime avoidance is determined by the downward-closed subsets of $\text{Spec } R$, and for downward-closed sets, prime avoidance behaves well with intersections:

Proposition 4.1.7. *Let R be a ring, $\{\Lambda_i\}$ a collection of downward-closed sets in $\text{Spec } R$ satisfying prime avoidance. Then $\Lambda := \bigcap \Lambda_i$ is also downward-closed and satisfies prime avoidance.*

Proof. It is clear that Λ is downward-closed. Let $\mathfrak{q} \in \text{Spec } R$, $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p} \subseteq \bigcap_i \bigcup_{\mathfrak{p} \in \Lambda_i} \mathfrak{p}$. By prime avoidance of Λ_i , there exist $\mathfrak{p}_i \in \Lambda_i$ such that \mathfrak{q} is contained in \mathfrak{p}_i , for every i . Then $\mathfrak{q} \in (\Lambda_i)_{cl} = \Lambda_i$ for every i , i.e. $\mathfrak{q} \in \Lambda$. \square

4.2 Dimension 1 and Arithmetic Rank

We can also give an analogue of Proposition 4.0.4(b) in (co)dimension 1 (whose proof we postpone until after Proposition 4.2.2):

Proposition 4.2.1. *Let R be a Noetherian normal ring of dimension 1.*

1. *For $\mathfrak{m} \in \text{mSpec } R$, $\text{mSpec}(R) \setminus \{\mathfrak{m}\}$ satisfies prime avoidance iff $[\mathfrak{m}]$ is torsion in $\text{Cl } R$ (the divisor class group of R).*
2. *Every $\Lambda \subseteq \text{Spec } R$ satisfies prime avoidance iff $\text{Cl } R$ is a torsion group.*

Proposition 4.2.1(b) naturally leads one to ask: what are the rings such that every set of primes satisfy prime avoidance? Such rings were introduced under the name of compactly-packed (C.P.) rings in [22], and have been fairly well-studied, e.g. in [25], [19]. The condition which replaces torsion in the class group turns out to be that of arithmetic rank 1. Recall that the arithmetic rank of an ideal I is defined as $\text{ara } I := \inf\{n \mid \exists x_1, \dots, x_n \in R, \sqrt{(x_1, \dots, x_n)} = \sqrt{I}\}$.

Proposition 4.2.2. *Let R be a ring. Then the following are equivalent:*

- (1) *For all $\Lambda \subseteq \text{Spec } R$, Λ satisfies prime avoidance*
- (2) *For all downward-closed $\Lambda \subseteq \text{Spec } R$, Λ satisfies prime avoidance*

- (3) For all Zariski-open sets $U \subseteq \text{Spec } R$, U satisfies prime avoidance
- (4) For all $\mathfrak{q} \in \text{Spec } R$, $(\text{Spec } R) \setminus V(\mathfrak{q})$ satisfies prime avoidance
- (5) For all $\mathfrak{q} \in \text{Spec } R$, $\text{ara } \mathfrak{q} \leq 1$.

Proof. (1) \implies (2) \implies (3) \implies (4): Clear.

(4) \implies (5): Let $\mathfrak{q} \in \text{Spec } R$. Then $\mathfrak{q} \not\subseteq \mathfrak{p}$ for all $\mathfrak{p} \notin V(\mathfrak{q})$, so by prime avoidance $\mathfrak{q} \not\subseteq \bigcup_{\mathfrak{p} \notin V(\mathfrak{q})} \mathfrak{p}$. Thus there is $x \in \mathfrak{q} \setminus \bigcup_{\mathfrak{p} \notin V(\mathfrak{q})} \mathfrak{p}$, and such an x has \mathfrak{q} as its only minimal prime (if $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \text{Spec } R$, then $\mathfrak{p} \in V(\mathfrak{q})$), i.e. $\sqrt{(x)} = \mathfrak{q}$.

(5) \implies (1): Let $\mathfrak{q} \in \text{Spec } R$, $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$. By hypothesis $\mathfrak{q} = \sqrt{(x)}$ for some $x \in R$. Then $x \in \mathfrak{p}$ for some $\mathfrak{p} \in \Lambda \implies \mathfrak{q} = \sqrt{(x)} \subseteq \mathfrak{p}$. \square

Proof of Proposition 4.2.1. If R is Dedekind and $\mathfrak{m} \in \text{mSpec } R$, then $\text{ara } \mathfrak{m} = 1$ iff $[\mathfrak{m}]$ is torsion in $\text{Cl } R$: by unique factorization of ideals, $\sqrt{(x)} = \mathfrak{m} \iff (x) = \mathfrak{m}^n$ for some $n \in \mathbb{N}$. If now R is any Noetherian normal ring of dimension 1, then R is a finite product of Dedekind domains and fields, so the above reasoning, along with (1) \iff (4) in Proposition 4.2.2, gives (a) and (b). \square

It is shown in [19] that if R is Noetherian, then (5) in Proposition 4.2.2 may be replaced with (5'): For all $\mathfrak{m} \in \text{mSpec } R$, $\text{ara } \mathfrak{m} = 1$ (which implies $\dim R \leq 1$, since $\text{ht } I \leq \text{ara } I$ in a Noetherian ring). In other words, under these assumptions the minimal primes also have arithmetic rank 1. In general though, it is possible for a minimal prime to be contained in a union of height 1 primes not containing it:

Example 4.2.3. Let k be a field, $R = k[x, y, z]/(xy, xz)$, and $\mathfrak{q} := (y, z)$, the non-principal minimal prime of R . If $\Lambda =$ all height 1 primes not containing \mathfrak{q} , then $\mathfrak{q} \subseteq \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$: to see this, take $0 \neq f \in \mathfrak{q}$, and let f_1 be an irreducible factor of \bar{f} in $R/(x) \cong k[y, z]$. Then (x, f_1) is a height 1 prime of R containing f , but not \mathfrak{q} . Together with the above reasoning, this shows that $\text{ara } \mathfrak{q} = 2$.

There is another interesting characterization of the C.P. property for domains via overrings: a Dedekind domain R is C.P. iff every overring of R (i.e. a ring S with $R \subseteq S \subseteq \text{Quot}(R)$) is a localization of R . Moreover, a Noetherian domain of dimension 1 is C.P. iff every sublocalization (i.e. an overring that is an intersection of localizations) is a localization. See [12], Corollaries 2.8 and 3.13 for more details.

It would also be remiss not to mention the geometric interpretation of prime avoidance, which is closer in spirit to the titular ‘‘avoidance’’. For an affine scheme $X = \text{Spec } R$, a (prime) cycle in X will mean an integral closed subscheme of X (i.e. a subscheme of the form $V(\mathfrak{p})$ for some $\mathfrak{p} \in \text{Spec } R$). A set of cycles $\{Z_i\}$ in X satisfies prime avoidance iff for any cycle Z not containing any Z_i , there is a hypersurface in X containing Z but not any Z_i . If the Z_i ’s consist of closed points, then this may be restated as: any cycle avoiding the Z_i can be extended to a hypersurface avoiding the Z_i . One can use this to see that a set Λ

of closed points in \mathbb{A}_k^2 with $|\Lambda| < |k|$ satisfies prime avoidance: if $p \notin \Lambda$, then there are $\geq |k|$ lines through p , and $\leq |\Lambda|$ of these can meet Λ . This includes e.g. any discrete (= has no limit points) set of points in $\mathbb{A}_{\mathbb{R}}^2$.

4.3 Applications and examples

We conclude with some applications of the ideas of prime avoidance. In general, prime avoidance is a constraint on a set of primes which can be used to justify one's intuition about the set (one interpretation of Theorem 4.1.6(3) is that prime avoidance means there are no "unexpected" closed points in the localization). In particular, prime avoidance can be used to construct rings essentially of finite type satisfying given conditions. Although the examples below are of independent interest, we use prime avoidance to verify certain properties of each:

Example 4.3.1. We give an example of a reduced, connected Noetherian affine scheme such that the closure of the closed points is a proper closed set of codimension 0. Algebraically, this is a Noetherian ring with no nilpotents or idempotents such that the Jacobson radical $\text{Rad } R$ is nonzero, but consists of zerodivisors. In other words, $\text{Rad } R$ lies strictly between the intersection and union of the minimal primes:

$$\bigcap_{\mathfrak{p} \in \text{Min } R} \mathfrak{p} \subsetneq \bigcap_{\mathfrak{m} \in \text{mSpec } R} \mathfrak{m} \subsetneq \bigcup_{\mathfrak{p} \in \text{Min } R} \mathfrak{p}$$

For the example: let $k = \bar{k}$ be a field, $T := k[x, y]/(xy)$, $\Lambda := V(x) \subseteq \text{Spec } T$, $W := T \setminus \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$, and $R := W^{-1}T$. By Corollary 4.1.2, Λ satisfies prime avoidance, so by Theorem 4.1.6, $\text{mSpec } R = \{W^{-1}(x, y - a) \mid a \in k\}$. Since k is infinite, $\text{Rad } R = \bigcap_{a \in k} W^{-1}(x, y - a) = W^{-1}(x) \neq 0$, and x is a zerodivisor in R .

Example 4.3.2. We give an example of a Jacobson ring R with the property $(*)$ (cf. Definition 3.1.4, Theorem 3.5.1). This resolves the question posed at the very beginning of this chapter, and shows that the converse of Theorem 3.5.1 is false. Let $T := \mathbb{C}[x]$, $\Lambda := \{(x - n) \mid n \in \mathbb{N}\}$, $W := T \setminus \bigcup_{\mathfrak{p} \in \Lambda} \mathfrak{p}$, and $R := W^{-1}T$. Since $\text{Cl } T = 0$, by Proposition 4.2.1 every subset of $\text{Spec } T$ satisfies prime avoidance, so by Theorem 4.1.6 $\text{mSpec}(R) = \{W^{-1}(x - n) \mid n \in \mathbb{N}\}$, hence R is a 1-dimensional Jacobson PID.

If $\varphi : R \rightarrow S$ is surjective, set $I := \ker \varphi$. Then $I = (f)R = W^{-1}(f)$ for some $f \in T$. Suppose $\exists g \in T$ with $\frac{g}{1} \notin R^\times$, but $\varphi\left(\frac{g}{1}\right) \in S^\times$. Since $S \cong R/I = W^{-1}T/W^{-1}(f) \cong T/(f)$, it suffices to show that $g + f_1 \in W$ for some $f_1 \in (f)$, i.e. $g + f_1$ has no roots in \mathbb{N} . Since f, g have no common roots, this is possible by taking $f_1 = cf^n$ where $c \in \mathbb{C}$, $n \in \mathbb{N}$ are such that $\deg f^n > \deg g$ and $|c| \gg 0$.

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