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Harmonic Activation and Transport

by

Jacob S. Calvert

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

Statistics

in the

Graduate Division

of the

University of California, Berkeley

Committee in charge:

Professor Alan Hammond, Chair
Professor James Pitman
Professor Shirshendu Ganguly

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Harmonic Activation and Transport

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Abstract

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Professor Alan Hammond, Chair

Harmonic activation and transport (HAT) is a random process which rearranges a set, one element at a time. More precisely, for integers $n \geqslant 2$ and $d \geqslant 1$, and given an n-element subset U of $\mathbb{Z}^d$, HAT is a Markov chain with the following dynamics. HAT removes x from U according to the harmonic measure of x in U, and then adds y according to the probability that a simple random walk from x, conditioned to hit the remaining set, leaves from y when it first does so. This process is then repeated for the resulting set, and so on. We are primarily interested in the classification of HAT as recurrent or transient, as the dimension d and number of elements n in the initial set vary.

Chapter 2 concerns HAT in two dimensions. When $d = 2$, HAT exhibits a phenomenon we call *collapse*: Informally, the diameter shrinks to its logarithm over a number of steps which is comparable to this logarithm. Collapse implies the existence of the stationary distribution of HAT, where configurations are viewed up to translation, and the exponential tightness of diameter at stationarity. Additionally, collapse produces a renewal structure with which we establish that the center of mass process, properly rescaled, converges in distribution to two-dimensional Brownian motion.

To characterize the phenomenon of collapse, we address fundamental questions about the extremal behavior of harmonic measure and escape probabilities. Among n-element subsets of $\mathbb{Z}^2$, what is the least positive value of harmonic measure? What is the probability of escape from the set to a distance of, say, r? Concerning the former, examples abound for which the harmonic measure is exponentially small in n. We prove that it can be no smaller than exponential in $n \log n$. Regarding the latter, the escape probability is at most the reciprocal of $\log r$, up to a constant factor. We prove it is always at least this much, up to an n-dependent factor.

Chapter 3 concerns HAT in higher dimensions. When $d \geqslant 5$ and $n \geqslant 4$, HAT is transient. We prove that, remarkably, transience occurs in only one "way": The initial state fragments into clusters of two or three elements—but no other number—which then grow indefinitely separated. We call these clusters dimers and trimers. Underlying this characterization of transience is the fact that,

from any state, HAT reaches a state consisting exclusively of dimers and trimers, in a number of steps and with at least a probability which depend on d and n only.

Together, our results establish that HAT exhibits a phase transition in both d and n, in the sense that HAT is positive recurrent when $d \leqslant 2$ or $n \leqslant 3$, but transient when $d \geqslant 5$ and $n \geqslant 4$. Specifically, the phase boundary has a "corner": There are $d \geqslant 3$ and $n \geqslant 4$ for which HAT is transient, but HAT is positive recurrent for any smaller d or n.

To Nathan and those who miss him

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I also thank Shirshendu for his mentorship, as well as for the closeness of his collaboration and the generosity of his humor, which have made working with him a delight. Shirshendu's gift for mathematics is total, and he has the purposeful urgency of one so brimming with ideas that they can hardly be put to page fast enough. For this reason, I am doubly grateful for the patience and care he has shown in providing me detailed feedback on my work.

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Chapter 1

Introduction

Harmonic activation and transport (HAT) is a random process which rearranges a set, one element at a time. HAT exhibits a remarkable phase transition, wherein the addition of just one element can alter its long-term behavior from stationarity to transience. Although HAT is not a growth model, it is connected to the class of Laplacian growth models, which describe the motion of many physical interfaces. This chapter defines HAT, describes its phase transition, and details its connection to Laplacian growth.

1.1 Harmonic activation and transport

More precisely, HAT is a Markov chain $(U_0, U_1, \dots)$ with a distribution $\mathbf{P}$, the initial state of which can be any *configuration*—an n-element subset of $\mathbb{Z}^d$, for a number of elements $n \geqslant 2$ and a dimension $d \geqslant 1$. A generic step of the HAT dynamics is defined as

$$U_{t+1} = \big(U_t \setminus \{X\}\big) \cup \{Y\}, \tag{1.1}$$

where X is the first site in U_t that a simple random walk "from infinity" visits, and Y is the site that it steps from when it first visits $U_t \setminus \{X\}$. In fact, we condition this simple random walk to visit U_t and then $U_t \setminus \{X\}$, to counteract its transience in dimension $d \geqslant 3$. We say that HAT *activates* the element at X and then *transports* it to Y (Figure 1.1).

We define the distribution of X in terms of harmonic measure. To define harmonic measure, we denote by $\mathbb{P}_z$ the distribution of simple random walk $(S_0, S_1, \dots)$ from $z \in \mathbb{Z}^d$. For $A \subseteq \mathbb{Z}^d$, we use τ_A to denote the first time that simple random walk returns to A, i.e.,

$$\tau_A = \inf\{t \geqslant 1 : S_t \in A\}.$$

Definition 1.1.1 (Harmonic measure)**.** *Let A be a finite, nonempty subset of $\mathbb{Z}^d$. The harmonic measure (from infinity) of A is the function $\mathbb{H}_A : \mathbb{Z}^d \to [0,1]$ defined by*

$$\mathbb{H}_A(x) = \lim_{z \to \infty} \mathbb{P}_z(S_{\tau_A} = x \mid \tau_A < \infty). \tag{1.2}$$

This limit exists (see, e.g., [Law13, Chapter 2]), so $\mathbb{H}_A$ is well defined.

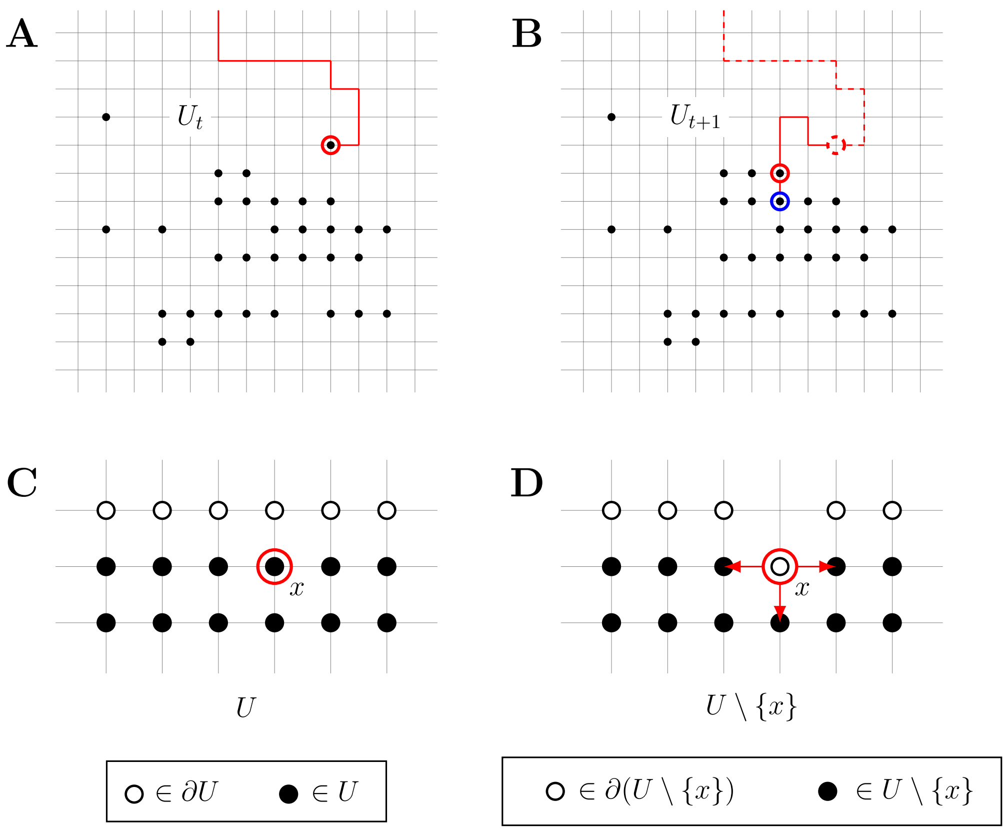

Figure 1.1: The HAT dynamics in $\mathbb{Z}^2$. (**A**) An element (indicated by a solid, red circle) in the configuration U_t is activated according to harmonic measure. (**B**) The activated element (following the solid, red path) hits another element (indicated by a solid, blue circle); it is then fixed at the site visited during the previous step (indicated by a solid, red circle), giving U_{t+1}. (**C**) An element of U (indicated by a red circle) is activated and (**D**) if it tries to move into $U \setminus \{x\}$, the element will be transported back to x.

Harmonic measure gets its name from the fact that, for fixed x, the conditional probability in (1.2) is a harmonic function, in the following sense. We will write $y \sim z$ if z and y are neighbors in $\mathbb{Z}^d$—that is, if $y, z \in \mathbb{Z}^d$ and if $\|y - z\| = 1$, where $\|\cdot\|$ is the Euclidean norm. The Laplacian of a function $f : \mathbb{Z}^d \to \mathbb{R}$ is defined by

$$\Delta f(z) = \frac{1}{2d} \sum_{y \sim z} f(y) - f(z).$$

The function f is said to be (discrete) harmonic on $B \subseteq \mathbb{Z}^d$ if $\Delta f(z) = 0$ for each $z \in B$. For fixed x, the conditional probability in (1.2) is a harmonic function of z, outside of the closure of A, defined as $\overline{A} = A \cup \partial A$, where $\partial A = \{y \notin A : y \sim x \text{ for some } x \in A\}$ denotes the boundary of A.

The distributions of X and Y, hence the activation and transport components of the HAT dynamics, are harmonic in essentially the same sense as the harmonic measure. For X, this is simply because the conditional distribution of X given U_t is $\mathbb{H}_{U_t}$. For Y, this is because the conditional distribution of Y given U_t and X is

$$\mathbb{P}_X\big(S_{\tau-1} = y \mid \tau < \infty\big),$$

where τ abbreviates $\tau_{U_t \setminus \{X\}}$. For fixed y, when $X = x$, this conditional probability is a harmonic function of x, outside of the closure of $\overline{U_t \setminus \{x\}}$.

We conclude this section by formally stating the transition probabilities of HAT. According to (1.1), given U_t, the probability that activation occurs at $x \in \mathbb{Z}^d$ and transport occurs to $y \in \mathbb{Z}^d$ at time t is

$$p_{U_t}(x,y) = \mathbb{H}_{U_t}(x)\,\mathbb{P}_x(S_{\tau-1} = y \mid \tau < \infty), \tag{1.3}$$

where τ again abbreviates $\tau_{U_t \setminus \{x\}}$. We define HAT in terms of p_{U_t} in the following way.

Definition 1.1.2 (HAT). *Given a configuration $U_0 \subset \mathbb{Z}^d$, HAT is the Markov chain $(U_0, U_1, \dots)$ with transition probabilities*

$$\mathbf{P}\big(U_{t+1} = (U_t \setminus \{x\}) \cup \{y\} \bigm| U_t\big) = \begin{cases} p_{U_t}(x,y) & \text{if } x \neq y, \text{ and} \\ \sum_{z \in \mathbb{Z}^d} p_{U_t}(z,z) & \text{if } x = y, \end{cases} \tag{1.4}$$

for $t \in \mathbb{Z}_{\geqslant 0}$ and $x, y \in \mathbb{Z}^d$.

Remark 1.1.3. *We use the random time $\tau - 1$ in* (1.3) *as opposed to, say, the first hitting time of the boundary of $U_t \setminus \{x\}$, for the following reason. For the scenario depicted in Figure 1.1C–D, wherein x neighbors elements of $U_t \setminus \{x\}$, this hitting time would be zero and therefore U_{t+1} would necessarily equal U_t. This possibility would complicate arguments in Section 2.7 and is therefore undesirable.*

1.2 Classification of HAT

Having stated the activation and transport dynamics and explained the sense in which they are harmonic, we turn to the focus of this thesis: the classification of HAT as recurrent or transient, for different values of d and n (Figure 1.2). We organize our study of HAT's classification in this way because d and n are fixed by U_0—if U_0 is an n-element subset of $\mathbb{Z}^d$, then U_t will be too. The work we discuss concerning HAT in $\mathbb{Z}^2$ comes from a paper with Shirshendu Ganguly and Alan Hammond [CGH21]; work concerning HAT in higher dimensions comes from [Cal21].

To discuss the classification of HAT, we must first identify the sets that HAT can reach (i.e., form as U_t at some time $t \geqslant 1$). According to (1.1), the element Y must have a neighbor in U_{t+1}. Moreover, it is easy to see that $\mathbb{H}_{U_{t+1}}(Y)$ must be positive. Hence, it cannot be that every element of U_{t+1} with positive harmonic measure is neighborless. This observation implies that HAT cannot reach sets for which every element with positive harmonic measure is neighborless. In fact, we will later prove that these are the only sets that HAT cannot reach, and we give them a name.

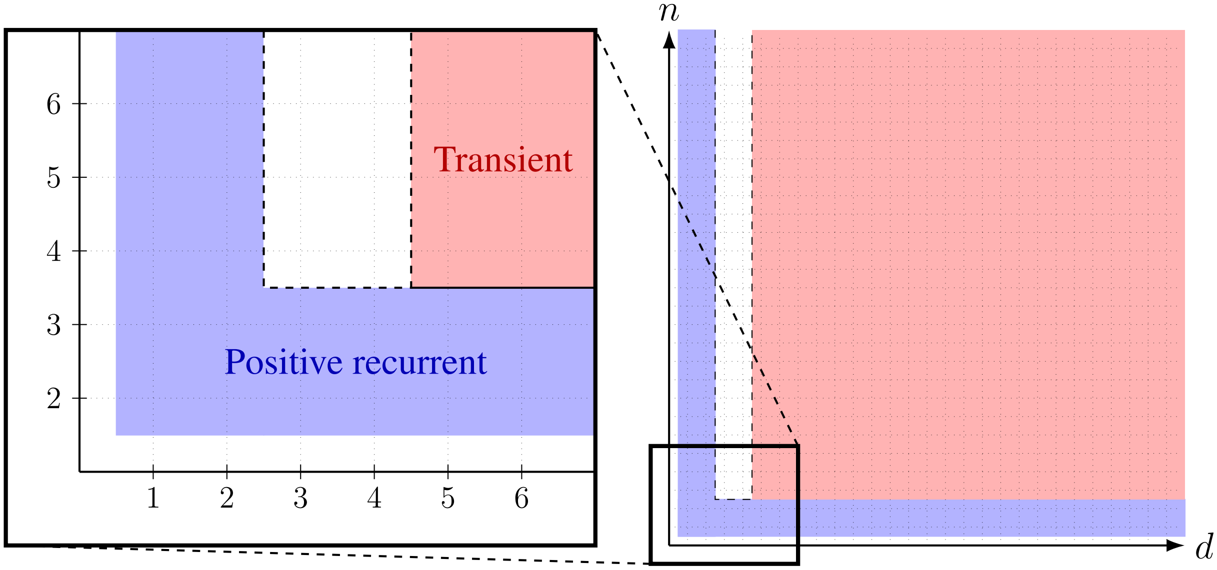

Figure 1.2: The phase diagram for HAT in the d–n grid. HAT is positive recurrent on $\widehat{\mathrm{NonIso}}_{d,n}$ in the blue-shaded region and transient in the red-shaded region. The classification of HAT has not been established in the unshaded region.

Definition 1.2.1 (Exposed elements, isolated and non-isolated sets)**.** *We say that an element x of finite $A \subset \mathbb{Z}^d$ is exposed in A if $\mathbb{H}_A(x) > 0$. We say that an n-element $A \subset \mathbb{Z}^d$ is isolated if every exposed x in A has no neighbor in A, and we denote the collection of such sets by $\mathrm{Iso}_{d,n}$. We denote the collection of all other n-element subsets of $\mathbb{Z}^d$ by $\mathrm{NonIso}_{d,n}$ and call its members non-isolated.*

We also observe that the HAT dynamics is invariant under the symmetries of $\mathbb{Z}^d$. If U_t is an n-element subset of $\mathbb{Z}^d$ and if $\mathcal{G}_d$ denotes the symmetry group of $\mathbb{Z}^d$, then the transition probabilities satisfy

$$\mathbf{P}(U_{t+1} = V \mid U_t) = \mathbf{P}(\, gU_{t+1} = gV \mid gU_t),$$

for every n-element $V \subset \mathbb{Z}^d$ and $g \in \mathcal{G}_d$. Accordingly, to each such V, we can associate the equivalence class

$$\widehat{V} = \left\{W \subset \mathbb{Z}^d : W = gV \text{ for some } g \in \mathcal{G}_d\right\}.$$

We denote the collection of equivalence classes of isolated and non-isolated sets by $\widehat{\mathrm{Iso}}_{d,n}$ and $\widehat{\mathrm{NonIso}}_{d,n}$. For brevity, we will often refer to an equivalence class of configurations as a configuration.

The fact that the HAT dynamics is irreducible on the collection of non-isolated sets and invariant under the symmetries of $\mathbb{Z}^d$ suggests that, if HAT did have a stationary distribution, it would naturally be supported on $\widehat{\mathrm{NonIso}}_{d,n}$. The main result of Chapter 2 states that HAT in $\mathbb{Z}^2$ has a unique stationary distribution, supported on $\widehat{\mathrm{NonIso}}_{2,n}$, to which the distribution of U_t converges, from any n-element configuration in $\mathbb{Z}^2$. In particular, HAT is positive recurrent on $\widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$. We will prove this result largely as a consequence of a phenomenon called *collapse*, which HAT exhibits in $\mathbb{Z}^2$.

Informally, collapse occurs when the diameter of a configuration is reduced to its logarithm over a number of steps proportional to this logarithm. Our characterization of collapse is essentially a quantitative version of a basic aspect of the HAT dynamics—the asymmetric behavior of diameter. Specifically, while the diameter of the HAT configuration can increase by at most one with each step, its diameter can decrease in one step by an amount which is nearly this diameter. For example, a configuration consisting of two elements separated by a large distance will have an equally large diameter that is reduced to one after the next step.

Chapter 2 features two other consequences of collapse. The first is a tail bound on the diameter of the HAT configuration under the stationary distribution, which is nearly exponential in the diameter. The second is the fact that, properly rescaled, the center of mass process converges to two-dimensional Brownian motion. Chapter 2 also proves a lower bound of harmonic measure, which is needed for our characterization of collapse and which may be of independent interest. Indeed, this bound has implications for a model of Laplacian growth, which we discuss in the next section.

In Chapter 3, our focus turns to HAT in higher dimensions. The main result of the chapter is that HAT is transient when $d \geqslant 5$ and $n \geqslant 4$. Remarkably, transience occurs in only one "way." The initial set fragments into clusters of two or three elements—but no other number—which then grow indefinitely separated. We call these clusters dimers and trimers. Underlying this characterization of transience is the fact that, from any set, HAT reaches a set consisting exclusively of dimers and trimers, in a number of steps and with at least a probability which depend on d and n only.

Taken together, the results of Chapters 2 and 3 establish that HAT exhibits a kind of phase transition in its long-term behavior, which is mediated not by a continuous parameter but by two discrete ones—d and n (Figure 1.2). Moreover, although we do not complete the phase diagram, these results suffice to show that the phase diagram exhibits a "corner": there are $d \geqslant 3$ and $n \geqslant 4$ for which HAT is transient, but HAT is positive recurrent on $\widehat{\mathrm{N}}\mathrm{onIso}_{d-1,n}$ and $\widehat{\mathrm{N}}\mathrm{onIso}_{d,n-1}$.

1.3 Connection to Laplacian growth

While HAT is not a growth model (indeed, it conserves the number of elements in a set), it is related by harmonic measure to a class of interfacial growth models, known as Laplacian growth models. In this context, "Laplacian" refers to the fact that each point of such an interface advances with a velocity or probability proportional to the gradient of a harmonic field, i.e., a field that satisfies the Laplace equation [BDLP01]. For example, Laplacian growth can describe the motion of an interface in response to gradients in pressure, temperature, or an electric field, which connects it to diverse natural phenomena such as finger formation between viscous fluids [ST58], dendritic crystal growth [LMK78], branched discharges in dielectric breakdown [NPW84], lung and vascular morphogenesis [LM95, FS99], the formation of river networks [PDSR13], and many others.

The connection between HAT and Laplacian growth arises from the fact that the harmonic measure of a finite set A is proportional to the gradient of a field which is harmonic outside of A, as we now explain. Let $d \geqslant 3$, $A \subseteq \mathbb{Z}^d$, and $f : \overline{A} \to \mathbb{R}$. The gradient of f at $x \in A$ is the function

$$\nabla f(x) = \frac{1}{2d} \sum_{y \sim x,\, y \in \partial A} \big(f(y) - f(x)\big).$$

We also define the escape probability $g_A(x) = \mathbb{P}_x(\sigma_A = \infty)$ for $x \in \mathbb{Z}^d$, where $\sigma_A = \inf\{t \geqslant 0 : S_t \in A\}$ denotes the first hitting time of A by simple random walk. Note that $g_A(x)$ is harmonic, in the sense that $\Delta g_A(x) = 0$, for x outside of A. In these terms, if A is finite, then

$$\mathbb{H}_A(x) = \frac{\nabla g_A(x)}{\sum_{y \in A} \nabla g_A(y)}$$

[Law13, Exercise 2.2.7]. For $\mathbb{Z}^2$, there is an analogous expression for harmonic measure, where g_A is defined differently, but remains harmonic outside of A [Law13, Proposition 2.3.2].

In this way, models of interfaces which grow in proportion to harmonic measure are Laplacian. Paradigmatic models of Laplacian growth include diffusion-limited aggregation (DLA) [WJS81, WS83], internal DLA [MD86, DF91], the Hastings-Levitov model [HL98], and the Abelian sandpile model [BTW87, Dha90, BLS91].

Definition 1.3.1 (DLA). *Let o denote the origin in $\mathbb{Z}^d$ and let $D_0 = \{o\}$. DLA is the Markov chain $(D_0, D_1, \dots)$, a generic step of which is defined by*

$$D_{t+1} = D_t \cup \{X\},$$

where the distribution of X is $\mathbb{H}_{\partial D_t}$.

Whereas HAT rearranges a set, DLA grows one. The simplicity of its definition belies the challenge DLA presents to rigorous analysis. Despite being introduced over 40 years ago, there are only *two* rigorous results about DLA in $\mathbb{Z}^2$, the original setting of its study [WJS81]. The potential value of rigorous contributions is underscored by the discrepancies between predictions about DLA from non-rigorous analytical approximations and simulation studies. In Chapter 2, we elaborate this point and prove a lower bound of harmonic measure which, although originally motivated by our analysis of HAT, effectively rules-out a prediction about DLA from the physics literature.

Chapter 2

HAT in two dimensions

This chapter is based on joint work with Shirshendu Ganguly and Alan Hammond [CGH21].

2.1 Main results

In this chapter, because we will exclusively work in $\mathbb{Z}^2$, we will repurpose d to denote a diameter instead of a dimension.

The variable connectivity of HAT configurations and concomitant opportunity for unchecked diameter growth would seem to jeopardize the positive recurrence of the HAT dynamics. Indeed, if the diameter were to grow unabatedly, the HAT dynamics could not return to a configuration or equivalence class thereof, and would therefore be doomed to transience. However, due to the asymmetric behavior of diameter under the HAT dynamics, this will not be the case. For an arbitrary initial configuration of $n \geqslant 2$ particles, we will prove—up to a factor depending on n—sharp bounds on the "collapse" time which, informally, is the first time the diameter is at most a certain function of n.

Definition 2.1.1. *For a positive real number R, we define the level-R collapse time to be $\mathcal{T}(R) = \inf\{t \geqslant 0 : \operatorname{diam}(U_t) \leqslant R\}$.*

For a real number $r \geqslant 0$, we define $\theta_m = \theta_m(r)$ through

$$\theta_0 = r \quad \text{and} \quad \theta_m = \theta_{m-1} + e^{\theta_{m-1}} \ \text{ for } m \geqslant 1. \tag{2.1}$$

In particular, $\theta_n(r)$ is approximately the n^{th} iterated exponential of r.

Theorem 2.1.2 (Collapse). *Let U be a finite subset of $\mathbb{Z}^2$ with $n \geqslant 2$ elements and denote the diameter of U by d. There exists a universal positive constant c such that, if d exceeds $\theta_{4n}(cn)$, then*

$$\mathbf{P}_U \left(\mathcal{T}(\theta_{4n}(cn)) \leqslant (\log d)^{1+o_n(1)} \right) \geqslant 1 - e^{-n}.$$

For the sake of concreteness, this is true with n^{-4} in the place of $o_n(1)$.

In words, for a given n, it typically takes $(\log d)^{1+o_n(1)}$ steps before the configuration of initial diameter d reaches a configuration with a diameter of no more than a large function of n.

Recall the definition of non-isolated configurations (Definition 1.2.1). As a consequence of Theorem 2.1.2 and the preceding discussion, it will follow that the HAT dynamics constitutes an aperiodic, irreducible, and positive recurrent Markov chain on $\widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$. In particular, this means that, from any configuration of $\widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$, the time it takes for the HAT dynamics to return to that configuration is finite in expectation. Aperiodicity, irreducibility, and positive recurrence imply the existence and uniqueness of the stationary distribution π_n, to which HAT converges from any n-element configuration. Moreover—again, due to Theorem 2.1.2—the stationary distribution is exponentially tight.

Theorem 2.1.3 (Existence of the stationary distribution)**.** *For every $n \geqslant 2$, from any n-element subset of $\mathbb{Z}^2$, HAT converges to a unique probability measure π_n supported on $\widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$. Moreover, π_n satisfies the following tightness estimate. There exists a universal positive constant c such that, for any $r \geqslant 2\theta_{4n}(cn)$,*

$$\pi_n\big(\mathrm{diam}(\widehat{U}) \geq r\big) \leq \exp\left(-\frac{r}{(\log r)^{1+o_n(1)}}\right).$$

In particular, this is true with $6n^{-4}$ in the place of $o_n(1)$.

As a further consequence of Theorem 2.1.2, we will find that the HAT dynamics exhibits a renewal structure which underlies the diffusive behavior of the corresponding center of mass process.

Definition 2.1.4. *For a sequence of configurations $(U_t)_{t\in\mathbb{N}}$, define the corresponding center of mass process $(\mathscr{M}_t)_{t\geqslant 0}$ by $\mathscr{M}_t = |U_t|^{-1}\sum_{x\in U_t} x$.*

For the following statement, denote by $\mathcal{C}([0,1])$ the continuous functions $f : [0,1] \to \mathbb{R}^2$ with $f(0) = (0,0)$, equipped with the topology induced by the supremum norm $\|f\|_\infty = \sup_{0\leqslant t\leqslant 1}\|f(t)\|$.

Theorem 2.1.5 (Convergence to two-dimensional Brownian motion)**.** *If $\mathscr{M}_t$ is linearly interpolated, then the law of the process $\big(t^{-1/2}\mathscr{M}_{st},\ s \in [0,1]\big)$, viewed as a measure on $\mathcal{C}([0,1])$, converges weakly as $t \to \infty$ to two-dimensional Brownian motion on $[0,1]$ with coordinate diffusivity $\chi^2 = \chi^2(n)$. Moreover, for a universal positive constant c, χ^2 satisfies:*

$$\theta_{5n}(cn)^{-1} \leq \chi^2 \leqslant \theta_{5n}(cn).$$

We have not tried to optimize the bounds on χ^2; indeed, they primarily serve to show that χ^2 is positive and finite.

As we elaborate in Section 2.2, the timescale of diameter collapse in Theorem 2.1.2 arises from novel estimates of harmonic measure and hitting probabilities, which control the activation and transport dynamics of HAT. Beyond their relevance to HAT, these results further the characterization of the extremal behavior of harmonic measure.

Estimates of harmonic measure often apply only to connected sets or depend on the diameter of the set. The discrete analogues of Beurling's projection theorem [Kes87]—which was used to prove the upper bound on the growth rate of DLA in [Kes90]—and Makarov's theorem [Law93] are notable examples. Furthermore, estimates of hitting probabilities often approximate sets by disks which contain them (for example, the estimates in Chapter 2 of [Law13]). Such approximations work well for connected sets, but not for sets which are "sparse" in the sense that they have large diameters relative to their cardinality; we provide examples to support this claim in Section 2.2.2. For the purpose of controlling the HAT dynamics, which adopts such sparse configurations, existing estimates of harmonic and hitting measures are either inapplicable or suboptimal.

To highlight the difference in the behavior of harmonic measure for general (i.e., potentially sparse) and connected sets, consider a finite subset A of $\mathbb{Z}^2$ with $n \geqslant 2$ elements. We ask: *What is the greatest value of* $\mathbb{H}_A(x)$*?* If we assume no more about A, then we can say no more than $\mathbb{H}_A(x) \leqslant \frac{1}{2}$ (see Section 2.5 of [Law13] for an example). However, if A is connected, then the discrete analogue of Beurling's projection theorem [Kes87] provides a finite constant c such that

$$\mathbb{H}_A(x) \leqslant cn^{-1/2}.$$

This upper bound is realized (up to a constant factor) when A is a line segment and x is one of its endpoints.

Our next result provides lower bounds of harmonic measure to complement the preceding upper bounds, addressing the question: *What is the least positive value of* $\mathbb{H}_A(x)$*?*

Theorem 2.1.6 (Lower bound of harmonic measure)**.** *There exists a universal positive constant* c *such that, if* A *is a subset of* $\mathbb{Z}^2$ *with* $n \geqslant 1$ *elements, then either* $\mathbb{H}_A(x) = 0$ *or*

$$\mathbb{H}_A(x) \geqslant e^{-cn \log n}. \tag{2.2}$$

If A *is connected, then* (2.2) *can be replaced by*

$$\mathbb{H}_A(x) \geqslant e^{-cn}. \tag{2.3}$$

The lower bound of (2.3) is optimal in terms of its dependence on n, as we can choose A to be a narrow, rectangular "tunnel" with a depth of order n, in which case the harmonic measure at the "bottom" of the tunnel is exponentially small in n; we will shortly discuss a related example in greater detail. We expect that the bound in (2.2) can be improved to an exponential decay with a rate of order n instead of $n \log n$.

If one could improve (2.2) as we anticipate, we believe that the resulting lower bound would be realized by the harmonic measure of the innermost element of a square spiral (Figure 2.1). The virtue of the square spiral is that, essentially, with each additional element, the shortest path to the innermost element lengthens by two steps. This heuristic suggests that the least positive value of harmonic measure should decay no faster than 4^{-2n}, as $n \to \infty$. Indeed, Example 2.1.8 suggests an asymptotic decay rate of $(2+\sqrt{3})^{-2n}$. We formalize this observation as a conjecture. To state it, denote the origin by $o = (0,0)$ and let $\mathscr{H}_n$ be the collection of n-element subsets A of $\mathbb{Z}^2$ such that $\mathbb{H}_A(o) > 0$.

Conjecture 2.1.7. *Asymptotically, the square spiral of Figure 2.1 realizes the least positive value of harmonic measure, in the sense that*

$$\lim_{n\to\infty} -\frac{1}{n} \log \inf_{A\in\mathscr{H}_n} \mathbb{H}_A(o) = 2\log(2+\sqrt{3}).$$

Example 2.1.8. *Figure 2.1 depicts the construction of an increasing sequence of sets* $(A_1, A_2, \dots)$ *such that, for all* $n \geqslant 1$, A_n *is an element of* $\mathscr{H}_n$, *and the shortest path* $\Gamma = (\Gamma_1, \Gamma_2, \dots, \Gamma_{|\Gamma|})$ *from the exterior boundary of* $A_n \cup \partial A_n$ *to the origin, which satisfies* $\Gamma_i \notin A_n$ *for* $1 \leqslant i \leqslant |\Gamma| - 1$, *has a length of* $2(1 - o_n(1))n$.

Since Γ_1 *separates the origin from infinity in* A_n^c, *we have*

$$\mathbb{H}_{A_n}(o) = \mathbb{H}_{A_n\cup\{\Gamma_1\}}(\Gamma_1) \cdot \mathbb{P}_{\Gamma_1}\left(S_{\tau_{A_n}} = o\right). \tag{2.4}$$

Concerning the first factor of (2.4), *one can show that there exist positive constants* $b, c < \infty$ *such that, for all sufficiently large* n,

$$cn^{-b} \leqslant \mathbb{H}_{A_n\cup\{\Gamma_1\}}(\Gamma_1) \leqslant 1.$$

To address the second factor of (2.4), *we observe that*

$$\mathbb{P}_{\Gamma_1}\left(S_{\tau_{A_n}} = o\right) = \mathbb{P}_{\Gamma_1}\left(S_1 = \Gamma_2 \,\middle|\, \tau_{A_n} < \tau_{\Gamma_1}\right) \cdot \mathbb{P}_{\Gamma_2}\left(S_{\sigma_{A_n}} = o \,\middle|\, \sigma_{A_n} < \sigma_{\Gamma_1}\right). \tag{2.5}$$

It is easy to see that the first factor of (2.5) *satisfies*

$$\frac{1}{2} \leqslant \mathbb{P}_{\Gamma_1}\left(S_1 = \Gamma_2 \,\middle|\, \tau_{A_n} < \tau_{\Gamma_1}\right) \leqslant 1.$$

The second factor of (2.5) *can be explicitly calculated using a system of difference equations. To this end, we define*

$$f(i) = \mathbb{P}_{\Gamma_i}\left(S_{\sigma_{A_n}} = o \,\middle|\, \sigma_{A_n} < \sigma_{\Gamma_1}\right) \quad \forall\, 1 \leqslant i \leqslant |\Gamma|,$$

which satisfies:

$$f(1) = 0, \quad f(|\Gamma|) = 1, \quad \textit{and} \quad f(i) = \frac{1}{4} f(i+1) + \frac{1}{4} f(i-1) \quad \forall\, 2 \leqslant i \leqslant |\Gamma| - 1.$$

The solution of this system yields

$$\mathbb{P}_{\Gamma_2}\left(S_{\sigma_{A_n}} = o \,\middle|\, \sigma_{A_n} < \sigma_{\Gamma_1}\right) = \frac{2\sqrt{3}}{(2+\sqrt{3})^{|\Gamma|-1} - (2-\sqrt{3})^{|\Gamma|-1}}. \tag{2.6}$$

Combining (2.4) *through* (2.6), *we find that, for all sufficiently large* n,

$$\frac{\frac{1}{2}cn^{-b}}{(2+\sqrt{3})^{|\Gamma|-1}} \leqslant \mathbb{H}_{A_n}(o) \leqslant \frac{1}{(2+\sqrt{3})^{|\Gamma|-2}}. \tag{2.7}$$

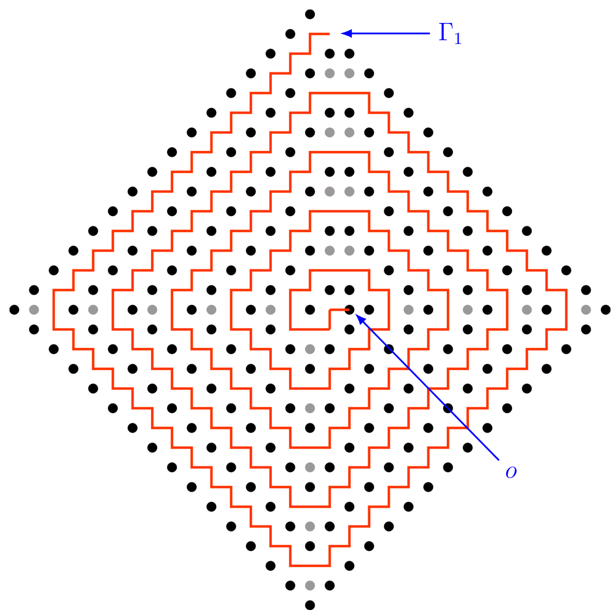

Figure 2.1: A square spiral. The shortest path Γ (red) from Γ_1 to the origin, which first hits A_n (black and gray dots) at the origin, has a length of approximately $2n$. Some elements (gray dots) of A_n could be used to continue the spiral pattern (indicated by the black dots), but are presently placed to facilitate a calculation in Example 2.1.8.

Substituting $|\Gamma| = 2(1 - o_n(1))n$ *into* (2.7) *and simplifying, we obtain*

$$(2 + \sqrt{3})^{-2(1+o_n(1))n} \leqslant \mathbb{H}_{A_n}(o) \leqslant (2 + \sqrt{3})^{-2(1-o_n(1))n},$$

which implies

$$\lim_{n\to\infty} -\frac{1}{n} \log \mathbb{H}_{A_n}(o) = 2 \log(2 + \sqrt{3}).$$

The problem of estimating $\mathbb{H}_A(x)$ is connected to an extensive literature on the supposed multifractal nature of DLA aggregates. In this context, the use of "multifractal" is an assertion that the harmonic measure on the boundaries of DLA configurations obeys power-law scaling with the linear size L of the aggregate [Mea87, HJK$^+$87] (see [Mak98] for a detailed mathematical formalism). To maintain the notation prevalent in this literature, we respectively denote by $p_{\max}(n)$ and $p_{\min}(n)$ the greatest and least positive values of harmonic measure on the boundary of a typical DLA configuration at time n. We note that, to make "typical" precise, we could write

$$p_{\max}(n) = \exp\left(\mathbf{E}\left[\log \max_x \mathbb{H}_{\partial U_n}(x)\right]\right), \tag{2.8}$$

where the expectation is with respect to the DLA dynamics and where the minimum is over x with positive $\mathbb{H}_{\partial U_n}(x)$; we could define $p_{\min}$ in an analogous fashion. Alternative definitions of $p_{\max}$ and $p_{\min}$ appear in the literature, but we will conflate them as the distinction is unimportant to the present discussion.

Suppose, for example, that there is a $\beta \in (0,1)$ such that linear size of a DLA configuration at time n satisfies $L = \Theta(n^\beta)$ as $n \to \infty$, a.s. Then the assumption of multifractality with respect to L implies the existence of finite exponents $\alpha_{\min} < \alpha_{\max}$ such that

$$p_{\max}(n) = \Theta\left(n^{-\beta\,\alpha_{\min}}\right) \quad \text{and} \quad p_{\min}(n) = \Theta\left(n^{-\beta\,\alpha_{\max}}\right). \tag{2.9}$$

To test the hypothesis that DLA configurations are multifractal, a number of studies have estimated the quantities in (2.9). Virtually every analytical approximation has concluded that β is $\frac{3}{5}$ [Mea83, Mut85, BBRT85, TS85, HL92], while almost every simulation study suggests a slightly smaller value of approximately $\frac{1}{1.71}$ [WS83, HMP86, TM89, Wol91, DHO$^+$99, DP00, DLP00]. Additionally, there appears to be a consensus that $\alpha_{\min}$ is approximately $\frac{2}{3}$ [TS85, HMP86, Wol91, HL92, JMP03]. Regarding mathematically rigorous constraints on β and $\alpha_{\min}$, Kesten's bound on the growth rate of DLA guarantees $\beta \leqslant \frac{2}{3}$ [Kes90] and the discrete analogue of Beurling's projection theorem [Kes87] concludes that the greatest value of the harmonic measure of *any* connected set of linear size L is $O(L^{-1/2})$. In this case, we must have $p_{\max}(n) = O(n^{-\beta/2})$ which, by (2.9), implies $\alpha_{\min} \geqslant \frac{1}{2}$. Thus, the conjectured values for β and $\alpha_{\min}$ are permitted by mathematically rigorous bounds.

In contrast with the apparent consensus on the value of $\alpha_{\min}$, the value of $\alpha_{\max}$ and the validity of the multifractal form (2.9) for $p_{\min}(n)$ are controversial. Concerning the latter, many studies have suggested that $p_{\min}(n)$ may decay more rapidly in n than a power law, ultimately corresponding to the failure of (2.9) [LS88, BA89, SLB$^+$90, ME90, EJM91, Wol91, Hen92, Man04a, Man04b]. For example, the authors of [LS88] found $-\log p_{\min}(n)$ to be $\Theta\left(n^{2\beta}\right)$, while the authors of [BA89] suggested $\Theta(n^a)$ for an unspecified $a > 0$. In fact, because DLA configurations are connected, Theorem 2.1.6 shows that a rate of exponential decay exceeding $O(n)$ is impossible. In particular, as β is accepted to be larger than $\frac{1}{2}$ (i.e., DLA is not a disk), Theorem 2.1.6 effectively rules-out the prediction of [LS88] concerning $p_{\min}(n)$, as well as the rate in [BA89] for $a > 1$.

While our results have consequences for $p_{\min}(n)$, they do not undermine the supposed multifractality of DLA. In fact, a pair of studies [JLMP02, ASSZ09] claimed to have definitively established a finite value of $\alpha_{\max}$ on the basis of an iterated conformal map approximation of DLA over roughly 10^4 steps and off-lattice DLA simulations over 10^6 steps, respectively. However, this conclusion may be influenced by insufficient simulation length and inherent biases in the corresponding simulation methods [Loh14, GB17]. Indeed, a state of the art, on-lattice simulation of DLA over 10^8 steps [GB17] produces a cluster which differs qualitatively from those of [JLMP02] and [ASSZ09], and which appears to exhibit fjords similar to those suggested in [SLB$^+$90] to lead to a violation of (2.9). If multifractality is ultimately violated, extremal harmonic measure estimates may play an important role in a refined understanding of the scaling of harmonic measure on DLA configurations.

We conclude the discussion of our main results by stating an estimate of hitting probabilities of the form $\mathbb{P}_x\left(\tau_{\partial A_d} < \tau_A\right)$, for $x \in A$ and where A_d is the set of all elements of $\mathbb{Z}^2$ within distance d of A; we will call these *escape probabilities* from A. Among n-element subsets A of $\mathbb{Z}^2$, when d is sufficiently large relative to the diameter of A, the greatest escape probability to a distance d from A is at most the reciprocal of $\log d$, up to a constant factor. We find that, in general, it is at least this much, up to an n-dependent factor.

Theorem 2.1.9 (Lower bound of escape probability)**.** *There exists a universal positive constant c such that, if A is a finite subset of $\mathbb{Z}^2$ with $n \geqslant 2$ elements and if $d \geqslant 2\,\mathrm{diam}(A)$, then, for any $x \in A$,*

$$\mathbb{P}_x(\tau_{\partial A_d} < \tau_A) \geqslant \frac{c\mathbb{H}_A(x)}{n \log d}. \tag{2.10}$$

In particular,

$$\max_{x \in A} \mathbb{P}_x\left(\tau_{\partial A_d} < \tau_A\right) \geqslant \frac{c}{n^2 \log d}. \tag{2.11}$$

In the context of the HAT dynamics, we will use (2.11) to control the transport step, ultimately producing the $\log d$ timescale appearing in Theorem 2.1.2. In the setting of its application, A and d will respectively represent a subset of a HAT configuration and the separation of A from the rest of the configuration. Reflecting the potential sparsity of HAT configurations, d may be arbitrarily large relative to n.

Organization

HAT motivates the development of new estimates of harmonic measure and escape probabilities. We attend to these estimates in Section 2.3, after we provide a conceptual overview of the proofs of Theorems 2.1.2 and 2.1.3 in Section 2.2. To analyze configurations of large diameter, we will decompose them into well separated "clusters," using a construction introduced in Section 2.5 and used throughout Section 2.6. The estimates of Section 2.3 control the activation and transport steps of the dynamics and serve as the critical inputs to Section 2.6, in which we analyze the "collapse" of HAT configurations. We then identify the class of configurations to which the HAT dynamics can return and prove the existence of a stationary distribution supported on this class; this is the primary focus of Section 2.7. The final section, Section 2.8, uses an exponential tail bound on the diameter of configurations under the stationary distribution—a result we obtain at the end of Section 2.7—to show that the center of mass process, properly rescaled, converges in distribution to two-dimensional Brownian motion.

2.2 Conceptual overview

2.2.1 Estimating the collapse time and proving the existence of the stationary distribution

Before providing precise details, we discuss some of the key steps in the proofs of Theorems 2.1.2 and 2.1.3. Since the initial configuration U of n particles is arbitrary, it will be advantageous to decompose any such configuration into clusters such that the separation between any two clusters is at least exponentially large relative to their diameters. For the purpose of illustration, let us start by assuming that U consists of just two clusters with separation d and hence the individual diameters of the clusters are no greater than $\log d$ (Figure 2.2).

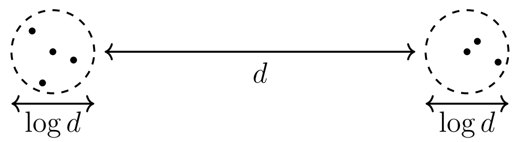

Figure 2.2: Exponentially separated clusters.

The first step in our analysis is to show that in time comparable to $\log d$, the diameter of U will shrink to $\log d$. This is the phenomenon we call *collapse*. Theorem 2.1.6 implies that every particle with positive harmonic measure has harmonic measure of at least $e^{-cn \log n}$. In particular, the particle in each cluster with the greatest escape probability from that cluster has at least this harmonic measure. Our choice of clustering will ensure that each cluster is separated by a distance which is at least twice its diameter and has positive harmonic measure. Accordingly, we will treat each cluster as the entire configuration and Theorem 2.1.9 will imply that the greatest escape probability from each cluster will be at least $(\log d)^{-1}$, up to a factor depending upon n.

Together, these results will imply that, in $O_n(\log d)$ steps, with a probability depending only upon n, all the particles from one of the clusters in Figure 2.2 will move to the other cluster. Moreover, since the diameter of a cluster grows at most linearly in time, the final configuration will have diameter which is no greater than the diameter of the surviving cluster plus $O_n(\log d)$. Essentially, we will iterate this estimate—by clustering anew the surviving cluster of Figure 2.2—each time obtaining a cluster with a diameter which is the logarithm of the original diameter, until d becomes smaller than a deterministic function θ_{4n}, which is approximately the $4n^{\text{th}}$ iterated exponential of cn, for a constant c.

Let us denote the corresponding stopping time by $\mathcal{T}(\text{below } \theta_{4n})$. In the setting of the application, there may be multiple clusters and we collapse them one by one, reasoning as above. If any such collapse step fails, we abandon the experiment and repeat it. Of course, with each failure, the set we attempt to collapse may have a diameter which is additively larger by $O_n(\log d)$. Ultimately, our estimates allow us to conclude that the attempt to collapse is successful within the first $(\log d)^{1+o_n(1)}$ tries with a high probability.

The preceding discussion roughly implies the following result, uniformly in the initial configuration U:

$$\mathbf{P}_U \left(\mathcal{T}(\text{below } \theta_{4n}) \leq (\log d)^{1+o_n(1)} \right) \geq 1 - e^{-n}.$$

At this stage, we prove that, given any configuration $\widehat{U}$ and any configuration $\widehat{V} \in \widehat{\mathrm{NonIso}}_{2,n}$, if K is sufficiently large in terms of n and the diameters of $\widehat{U}$ and $\widehat{V}$, then

$$\mathbf{P}_{\widehat{U}} \left(\mathcal{T}(\text{hits } \widehat{V}) \leqslant K^5 \right) \geqslant 1 - e^{-K},$$

where $\mathcal{T}(\text{hits } \widehat{V})$ is the first time the configuration is $\widehat{V}$. This estimate is obtained by observing that the particles of $\widehat{U}$ form a line segment of length n in K^3 steps with high probability, and then showing by induction on n that any other non-isolated configuration $\widehat{V}$ is reachable from the line segment in K^5 steps, with high probability. In addition to implying irreducibility of the HAT dynamics on $\widehat{\mathrm{NonIso}}_{2,n}$, we use this result to obtain a finite upper bound on the expected return time to any non-isolated configuration (i.e., it proves the positive recurrence of HAT on $\widehat{\mathrm{NonIso}}_{2,n}$). Irreducibility and positive recurrence on $\widehat{\mathrm{NonIso}}_{2,n}$ imply the existence and uniqueness of the stationary distribution.

2.2.2 Improved estimates of hitting probabilities for sparse sets

HAT configurations may include subsets with large diameters relative to the number of elements they contain, and in this sense they are sparse. Two such cases are depicted in Figure 2.3. A key component of the proofs of Theorems 2.1.6 and 2.1.9 is a method which improves two standard estimates of hitting probabilities when applied to sparse sets, as summarized by Table 2.1.

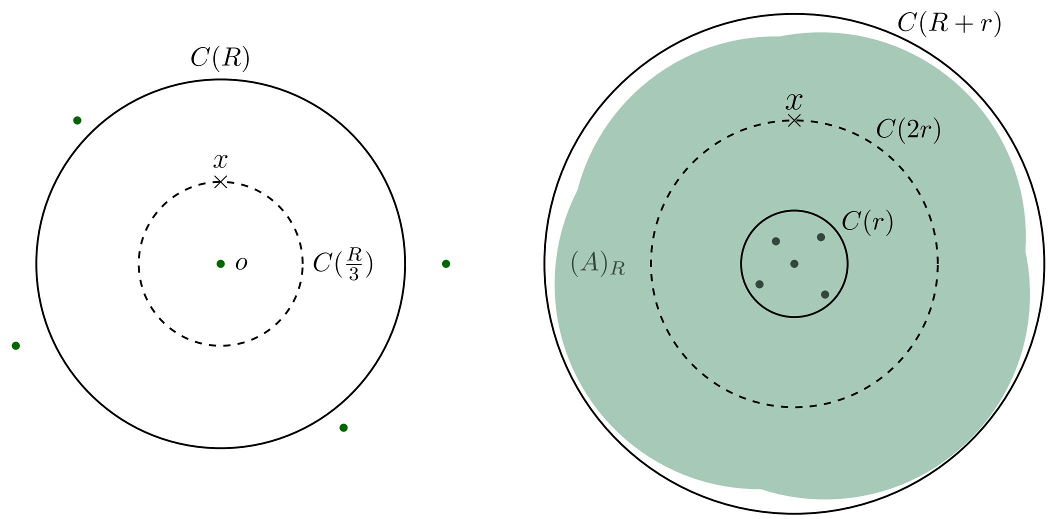

Figure 2.3: Sparse sets like ones which appear in the proofs of Theorems 2.1.6 (left) and 2.1.9 (right). The elements of A are represented by dark green dots. On the left, $A\backslash\{o\}$ is a subset of $D(R)^c$. On the right, A is a subset of $D(r)$ and the R-fattening of A (shaded green) is a subset of $D(R+r)$. The figure is not to scale, as $R \geqslant e^n$ on the left, while $R \geqslant e^r$ on the right.

Table 2.1: Summary of improvements to standard estimates in sparse settings. The origin is denoted by o and A_R denotes the set of all points in $\mathbb{Z}^d$ within a distance R of A.

Setting	Quantity	Standard estimate	New estimate
Fig. 2.3 (left), $R \geqslant e^n$	$\mathbb{P}_x(\tau_o < \tau_{A \cap D(R)^c})$	$\Omega\left(\frac{1}{\log R}\right)$	$\Omega\left(\frac{1}{n}\right)$
Fig. 2.3 (right), $R \geqslant e^r$	$\mathbb{P}_x(\tau_{\partial A_R} < \tau_A)$	$\Omega_n\left(\frac{1}{\log R}\right)$	$\Omega_n\left(\frac{\log r}{\log R}\right)$

For the scenario depicted in Figure 2.3 (left), we estimate the probability that a random walk from $x \in C(\frac{R}{3})$ hits the origin before any element of $A\backslash\{o\}$. Since $C(R)$ separates x from $A\backslash\{o\}$, this probability is at least $\mathbb{P}_x(\tau_o < \tau_{C(R)})$. We can calculate this lower bound by combining the fact that the potential kernel (defined in Section 2.3) is harmonic away from the origin with the optional stopping theorem (e.g., Proposition 1.6.7 of [Law13]):

$$\mathbb{P}_x\left(\tau_o < \tau_{C(R)}\right) = \frac{\log R - \log\|x\| + O(R^{-1})}{\log R + O(R^{-1})}.$$

This implies $\mathbb{P}_x(\tau_o < \tau_{A\cap D(R)^c}) = \Omega(\frac{1}{\log R})$, since $x \in C(\frac{R}{3})$ and $R \geqslant e^n$.

We can improve the lower bound to $\Omega(\frac{1}{n})$ by using the sparsity of A. We define the random variable $W = \sum_{y\in A\setminus\{o\}} \mathbf{1}\left(\tau_y < \tau_o\right)$ and write

$$\mathbb{P}_x\left(\tau_o < \tau_{A\setminus\{o\}}\right) = \mathbb{P}_x\left(W = 0\right) = 1 - \frac{\mathbb{E}_x W}{\mathbb{E}_x[W \bigm| W > 0]}.$$

We will show that $\mathbb{E}_x[W \bigm| W > 0] \geqslant \mathbb{E}_x W + \delta$ for some δ which is uniformly positive in A and n. We will be able to find such a δ because random walk from x hits a given element of $A\setminus\{o\}$ before o with a probability of at most $1/2$, so conditioning on $\{W > 0\}$ effectively increases W by $1/2$. Then

$$\mathbb{P}_x\left(\tau_o < \tau_{A\setminus\{o\}}\right) \geqslant 1 - \frac{\mathbb{E}_x W}{\mathbb{E}_x W + \delta} \geqslant 1 - \frac{n}{n+\delta} = \Omega(\tfrac{1}{n}).$$

The second inequality follows from the monotonicity of $\frac{\mathbb{E}_x W}{\mathbb{E}_x W + \delta}$ in $\mathbb{E}_x W$ and the fact that $|A| \leqslant n$, so $\mathbb{E}_x W \leqslant n$. This is a better lower bound than $\Omega(\frac{1}{\log R})$ when R is at least e^n.

A variation of this method also improves a standard estimate for the scenario depicted in Figure 2.3 (right). In this case, we estimate the probability that a random walk from $x \in C(2r)$ hits ∂A_R before A, where A is contained in $D(r)$ and A_R consists of all elements of $\mathbb{Z}^2$ within a distance $R \geqslant e^r$ of A. We can bound below this probability using the fact that

$$\mathbb{P}_x\left(\tau_{\partial A_R} < \tau_A\right) \geqslant \mathbb{P}_x(\tau_{C(R+r)} < \tau_{C(r)}).$$

A standard calculation using the potential kernel of random walk (e.g., Exercise 1.6.8 of [Law13]) shows that this lower bound is $\Omega_n(\frac{1}{\log R})$, since $R \geqslant e^r$ and $r = \Omega(n^{1/2})$.

We can improve the lower bound to $\Omega_n(\frac{\log r}{\log R})$ by using the sparsity of A. We define $W' = \sum_{y\in A} \mathbf{1}\left(\tau_y < \tau_{\partial A_R}\right)$ and write

$$\mathbb{P}_x\left(\tau_{\partial A_R} < \tau_A\right) = 1 - \frac{\mathbb{E}_x W'}{\mathbb{E}_x[W' \bigm| W' > 0]} \geqslant 1 - \frac{n\alpha}{1 + (n-1)\beta},$$

where α bounds above $\mathbb{P}_x\left(\tau_y < \tau_{\partial A_R}\right)$ and β bounds below $\mathbb{P}_z\left(\tau_y < \tau_{\partial A_R}\right)$, uniformly for $x \in C(2r)$ and distinct $y, z \in A$. We will show that $\alpha \leqslant \beta$ and $\beta \leqslant 1 - \frac{\log(2r)}{\log R}$. The former is plausible because $\|x - y\|$ is at least as great as $\|y - z\|$; the latter because $\operatorname{dist}(z, A) \geqslant R$ while $\|y - z\| \leqslant 2r$, and because of (2.14). We apply these facts to the preceding display to conclude

$$\mathbb{P}_x\left(\tau_{\partial A_R} < \tau_A\right) \geqslant n^{-1}(1-\beta) = \Omega_n(\tfrac{\log r}{\log R}).$$

This is a better lower bound than $\Omega_n(\frac{1}{\log R})$ because r can be as large as $\log R$.

In summary, by analyzing certain conditional expectations, we can better estimate hitting probabilities for sparse sets than we can by applying standard results. This approach may be useful in obtaining other sparse analogues of hitting probability estimates.

2.3 Harmonic measure estimates

The purpose of this section is to prove Theorem 2.1.6. We will describe the proof strategy in Section 2.3.1, before proving several estimates in Section 2.3.2 which will streamline the presentation of the proof in Section 2.3.3. The majority of our effort is devoted to the proof of (2.2); we will obtain (2.3) as a corollary of a geometric lemma in Section 2.3.2.

Consider a subset A of $\mathbb{Z}^2$ with $n \geqslant 2$ elements, which satisfies $\mathbb{H}_A(o) > 0$ (i.e., $A \in \mathscr{H}_n$). We frame the proof of Theorem 2.1.6—in particular, the proof of (2.2)—in terms of "advancing" a random walk from infinity to the origin in three or four stages, while avoiding all other elements of A. These stages are defined in terms of a sequence of annuli which partition $\mathbb{Z}^2$.

Denote the disk of radius r about x by $D_x(r) = \{y \in \mathbb{Z}^2 : \|x - y\| < r\}$, or $D(r)$ if $x = o$, and denote its boundary by $C_x(r) = \partial D_x(r)$, or $C(r)$ if $x = o$. Additionally, denote by $\mathcal{A}(r, R) = D(R) \backslash D(r)$ the annulus with inner radius r and outer radius R. We will frequently need to reference the subset of A which lies within or beyond a disk. We denote $A_{<r} = A \cap D(r)$ and $A_{\geqslant r} = A \cap D(r)^c$.

Define radii $R_1, R_2, \dots$ and annuli $\mathcal{A}_1, \mathcal{A}_2, \dots$ through $R_1 = 10^5$, and $R_\ell = R_1^\ell$ and $\mathcal{A}_\ell = \mathcal{A}(R_\ell, R_{\ell+1})$ for $\ell \geqslant 1$. We fix $\delta = 10^{-2}$ for use in intermediate scales, like $C(\delta R_{\ell+1}) \subset \mathcal{A}_\ell$. Additionally, we denote by n_0, n_ℓ, m_ℓ, and $n_{>J}$ the number of elements of A in $D(R_1)$, $\mathcal{A}_\ell$, $\mathcal{A}_\ell \cup \mathcal{A}_{\ell+1}$, and $D(R_{J+1})^c$, respectively.

We will split the proof of (2.2) into an easy case when $n_0 = n$ and a difficult case when $n_0 \neq n$. If $n_0 \neq n$, then $A_{\geqslant R_1}$ is nonempty and the following indices $I = I(A)$ and $J = J(A)$ are well defined:

$$I = \min\{\ell \geqslant 1 : \mathcal{A}_\ell \text{ contains an element of } A\backslash\{o\}\}, \text{ and}$$
$$J = \min\{\ell > I : \mathcal{A}_\ell \text{ contains no element of } A\backslash\{o\}\}.$$

We explain the roles of I and J in the following subsection.

2.3.1 Strategy for the proof of Theorem 2.1.6

This section outlines a proof of (2.2) by induction on n. The induction step is easy when $n_0 = n$; the following strategy concerns the difficult case when $n_0 \neq n$. The proof of (2.3) is a simple consequence of an input to the proof of (2.2), so we address it separately, in Section 2.3.2.

Stage 1: Advancing to $C(R_J)$. Assume $n_0 \neq n$ and $n \geqslant 3$. By the induction hypothesis, there is universal constant c_1 such that the harmonic measure at the origin is at least $e^{-c_1 k \log k}$, for any set in $\mathscr{H}_k$, $1 \leqslant k < n$. Denote the law of random walk from ∞ by $\mathbb{P}$ (without a subscript) and let $k = n_{>J} + 1$. Because a random walk from ∞ which hits the origin before $A_{\geqslant R_J}$ also hits $C(R_J)$ before A, the induction hypothesis applied to $A_{\geqslant R_J} \cup \{o\} \in \mathscr{H}_k$ implies that $\mathbb{P}(\tau_{C(R_J)} < \tau_A)$ is no smaller than exponential in $k \log k$. Note that $k < n$ because $A_{<R_{I+1}}$ has at least two elements by the definition of I.

The reason we advance the random walk to $C(R_J)$ instead of the boundary of a smaller disk is that an adversarial choice of A could produce a "choke point" which likely dooms the walk to be intercepted by $A \backslash \{o\}$ in the second stage of advancement (Figure 2.4). To avoid a choke point when advancing to the boundary of a disk D, it suffices for the conditional hitting distribution of ∂D given $\{\tau_{\partial D} < \tau_A\}$ to be comparable to the uniform hitting distribution on ∂D. To prove this comparison, the annular region immediately beyond D and extending to a radius at least twice that of D must be empty of A, hence the need for exponentially growing radii and for $\mathcal{A}_J$ to be empty of A.

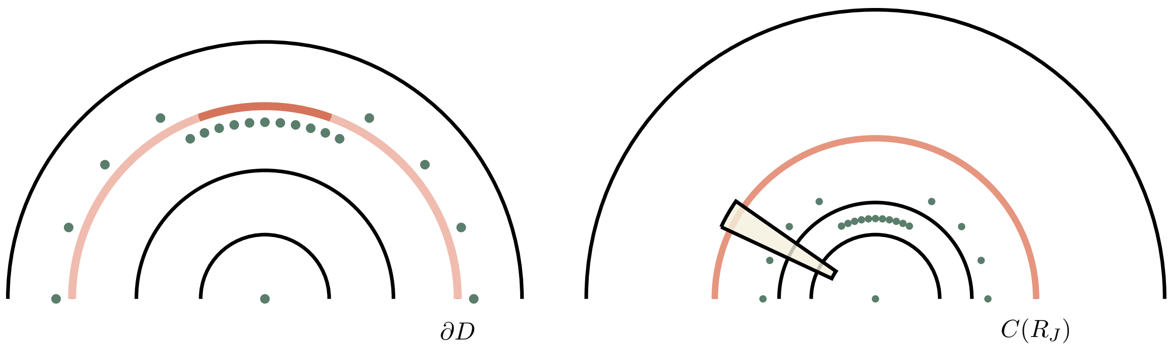

Figure 2.4: An example of a "choke point" (left) and a strategy for avoiding it (right). The hitting distribution of a random walk conditioned to reach ∂D before A (green dots) may favor the avoidance of $A \cap D^c$ in a way which localizes the walk (e.g., as indicated by the dark red arc of ∂D) prohibitively close to $A \cap D$. The hitting distribution on $C(R_J)$ will be approximately uniform if the radii grow exponentially. The random walk can then avoid the choke point by "tunneling" through it (e.g., by passing through the tan-shaded region).

Stage 2: Advancing into $\mathcal{A}_{I-1}$. For notational convenience, assume $I \geqslant 2$ so that $\mathcal{A}_{I-1}$ is defined; the argument is the same when $I = 1$. Each annulus $\mathcal{A}_\ell$, $\ell \in \{I, \dots, J-1\}$, contains one or more elements of A, which the random walk must avoid on its journey to $\mathcal{A}_{I-1}$. We build an overlapping sequence of rectangular and annular *tunnels*, through and between each annulus, which are empty of A and through which the walk can enter $\mathcal{A}_{I-1}$ (Figure 2.5). (In fact, depending on A, we may not be able to tunnel into $\mathcal{A}_{I-1}$, but this case will be easier; we address it at the end of this subsection.) Specifically, the walk reaches a particular subset Arc_{I-1} in $\mathcal{A}_{I-1}$ at the conclusion of the tunneling process. We will define Arc_{I-1} in Lemma 2.3.2 as an arc of a circle in $\mathcal{A}_{I-1}$.

By the pigeonhole principle applied to the radial coordinate, for each $\ell \geqslant I+1$, there is a sector of aspect ratio $m_\ell = n_\ell + n_{\ell-1}$, from the lower "$\delta^{\text{th}}$" of $\mathcal{A}_\ell$ to that of $\mathcal{A}_{\ell-1}$, which contains no element of A (Figure 2.5). To reach the entrance of the analogous tunnel between $\mathcal{A}_{\ell-1}$ and $\mathcal{A}_{\ell-2}$, the random walk may need to circle the lower δ^{th} of $\mathcal{A}_{\ell-1}$. We apply the pigeonhole principle to the angular coordinate to conclude that there is an annular region contained in the lower δ^{th} of $\mathcal{A}_{\ell-1}$, with an aspect ratio of $n_{\ell-1}$, which contains no element of A.

The probability that the random walk reaches the annular tunnel before exiting the rectangular tunnel from $\mathcal{A}_\ell$ to $\mathcal{A}_{\ell-1}$ is no smaller than exponential in m_ℓ. Similarly, the random walk reaches the rectangular tunnel from $\mathcal{A}_{\ell-1}$ to $\mathcal{A}_{\ell-2}$ before exiting the annular tunnel in $\mathcal{A}_{\ell-1}$ with a probability no smaller than exponential in $n_{\ell-1}$. Overall, we conclude that the random walk reaches Arc_{I-1} without leaving the union of tunnels—and therefore without hitting an element of A—with a probability no smaller than exponential in $\sum_{\ell=I}^{J-1} n_\ell$.

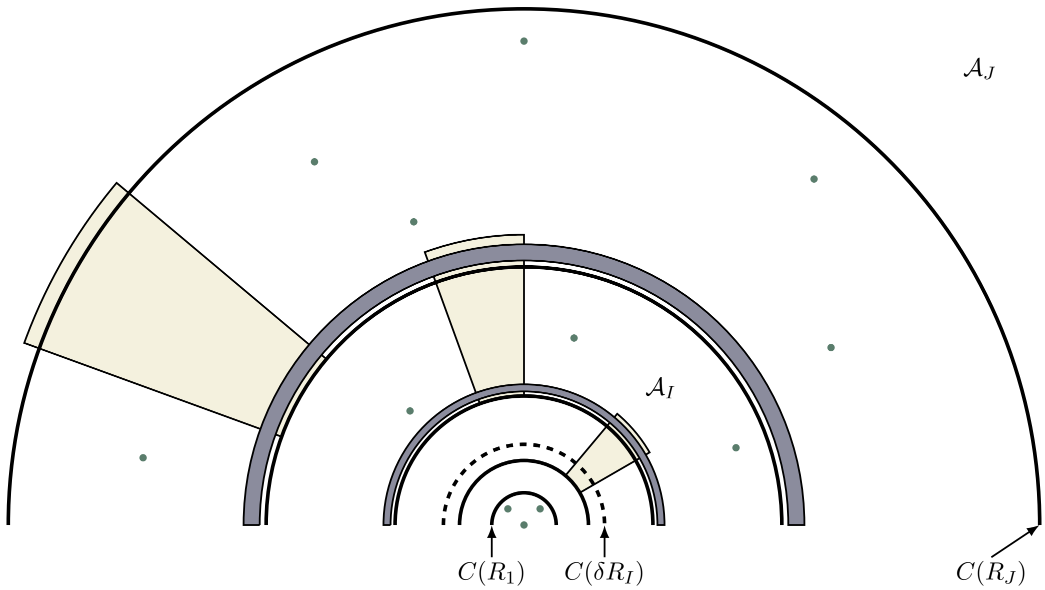

Figure 2.5: Tunneling through nonempty annuli. We construct a contiguous series of sectors (tan) and annuli (blue) which contain no elements of A (green dots) and through which the random walk may advance from $C(R_{J-1})$ to $C(\delta R_{I-1})$ (dashed).

Stage 3: Advancing to $C(R_1)$. Figure 2.3 (left) essentially depicts the setting of the random walk upon reaching $x \in \mathrm{Arc}_{I-1}$, except with $C(R_I)$ in the place of $C(R)$ and the circle containing Arc_{I-1} in the place of $C(\frac{R}{3})$, and except for the possibility that $D(R_1)$ contains other elements of A. Nevertheless, if the radius of Arc_{I-1} is at least e^n, then by pretending that $A_{<R_1} = \{o\}$, the method highlighted in Section 2.2.2 will show that $\mathbb{P}_x(\tau_{C(R_1)} < \tau_A) = \Omega(\frac{1}{n})$. A simple calculation will give the same lower bound (for a potentially smaller constant) in the case when the radius is less than e^n.

Stage 4: Advancing to the origin. Once the random walk reaches $C(R_1)$, we are in the setting of Lemma 2.3.12. There can be no more than $O(R_1^2)$ elements of $A_{<R_1}$, so there is a path of length $O(R_1^2)$ to the origin which avoids all other elements of A, and a corresponding probability of at least a constant that the random walk follows it.

Conclusion of Stages 1–4. The lower bounds from the four stages imply that there are universal constants c_1 through c_4 such that

$$\mathbb{H}_A(o) \geqslant e^{-c_1 k \log k - c_2 \sum_{\ell=I}^{J-1} n_\ell - \log(c_3 n) - \log c_4} \geqslant e^{-c_1 n \log n}.$$

It is easy to show that the second inequality holds if $c_1 \geqslant 8 \max\{1, c_2, \log c_3, \log c_4\}$, using the fact that $n - k = \sum_{\ell=I}^{J-1} n_\ell > 1$ and $\log n \geqslant 1$. We are free to adjust c_1 to satisfy this bound, because c_2 through c_4 do not depend on the induction hypothesis. This concludes the induction step.

A complication in Stage 2. If R_ℓ is not sufficiently large relative to m_ℓ, then we cannot tunnel the random walk through $\mathcal{A}_\ell$ into $\mathcal{A}_{\ell-1}$. We formalize this through the failure of the condition

$$\delta R_\ell > R_1(m_\ell + 1). \tag{2.12}$$

The problem is that, if (2.12) fails, then there are too many elements of A in $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell-1}$, and we cannot guarantee that there is a tunnel between the annuli which avoids A. We note that, while it may seem that this problem could be avoided by choosing R_1 in proportion to n, this choice would ultimately worsen (2.2) to e^{-cn^2}.

Accordingly, we will stop *Stage 2* tunneling once the random walk reaches a particular subset Arc_{K-1} of a circle in $\mathcal{A}_{K-1}$, where $\mathcal{A}_{K-1}$ is the outermost annulus which fails to satisfy (2.12). Specifically, we define K as:

$$K = \begin{cases} I, & \text{if (2.12) holds for } \ell \in \{I, \dots, J\}; \\ \min\{k \in \{I, \dots, J\} : \text{(2.12) holds for } \ell \in \{k, \dots, J\}\}, & \text{otherwise.} \end{cases} \tag{2.13}$$

The failure of (2.12) for $\ell = K - 1$ when $K \neq I$ will imply that there is a path of length $O(\sum_{\ell=I}^{K-1} n_\ell)$ from Arc_{K-1} to the origin which otherwise avoids A. In this case, *Stage 3* consists of random walk from Arc_{K-1} following this path to the origin with a probability no smaller than exponential in $\sum_{\ell=I}^{K-1} n_\ell$, and there is no *Stage 4*.

Overall, if $K \neq I$, *Stages 2,3* contribute a rate of $\sum_{\ell=I}^{J-1} n_\ell$. This rate is smaller than the one contributed by *Stages 2–4* when $K = I$, so the preceding conclusion holds.

2.3.2 Preparation for the proof of Theorem 2.1.6

First, we introduce some conventions, notation, and some objects associated with random walk.

All universal constants will be positive and finite. For subsets B and elements x of $\mathbb{Z}^2$, we will denote corresponding hitting times by $\sigma_B = \inf\{t \geqslant 0 : S_t \in B\}$ or σ_x. For $r > 0$, we will denote the r-fattening of B by $B_r = \{x \in \mathbb{Z}^2 : \mathrm{dist}(x, B) < r\}$. We will use $\mathrm{rad}(C)$ to denote the radius of a circle C (e.g., $\mathrm{rad}(C(r)) = r$). We will denote the minimum of random times τ_1 and τ_2 by $\tau_1 \wedge \tau_2$.

We will use the potential kernel associated with random walk on $\mathbb{Z}^2$. We denote the former by $\mathfrak{a}$. It has the form

$$\mathfrak{a}(x) = \frac{2}{\pi} \log \|x\| + \kappa + O\left(\|x\|^{-2}\right), \tag{2.14}$$

where $\kappa \in (1.02, 1.03)$ is an explicit constant. The potential kernel satisfies $\mathfrak{a}(o) = 0$ and is harmonic on $\mathbb{Z}^2 \setminus \{o\}$. As shown in [KS04], the constant hidden in the error term, which we call λ, is less than 0.06882. In some instances, we will want to apply $\mathfrak{a}$ to an element which belongs to $C(r)$. It will be convenient to denote, for $r > 0$,

$$\mathfrak{a}'(r) = \frac{2}{\pi} \log r + \kappa.$$

Input to Stage 1

Let $A \in \mathscr{H}_n$. Like in Section 2.3.1, we assume that $n_0 \neq n$ (i.e., $A_{\geqslant R_1} \neq \emptyset$) and defer the simpler complementary case to Section 2.3.3. The annulus $\mathcal{A}_J$ is important because of the following result. To state it, denote the uniform distribution on $C(R_J)$ by μ_J.

Lemma 2.3.1. *There is a constant c_1 such that, for every $z \in C(R_J)$,*

$$\mathbb{P}\big(S_{\tau_{C(R_J)}} = z \bigm| \tau_{C(R_J)} < \tau_A\big) \geqslant c_1 \mu_J(z). \tag{2.15}$$

Under the conditioning in (2.15), the random walk reaches $C(\delta R_{J+1})$ before hitting A, and typically proceeds to hit $C(R_J)$ before returning to $C(R_{J+1})$. The inequality (2.15) then follows from the fact that harmonic measure on $C(R_J)$ is comparable to μ_J.

Proof of Lemma 2.3.1. Under the conditioning, the random walk must reach $C(\delta R_{J+1})$ before $C(R_J)$. It therefore suffices to prove that there exists a positive constant c_1 such that, uniformly for all $x \in C(\delta R_{J+1})$ and $z \in C(R_J)$,

$$\mathbb{P}_x\big(S_\eta = z \bigm| \tau_{C(R_J)} < \tau_A\big) \geqslant c_1 \mu_{R_J}(z), \tag{2.16}$$

where $\eta = \tau_{C(R_J)} \wedge \tau_A$. Because $\partial \mathcal{A}_J$ separates x from A, the conditional probability in (2.16) is at least

$$\mathbb{P}_x\big(S_\eta = z \bigm| \tau_{C(R_J)} < \tau_{C(R_{J+1})},\, \tau_{C(R_J)} < \tau_A\big)\mathbb{P}_x\big(\tau_{C(R_J)} < \tau_{C(R_{J+1})}\big). \tag{2.17}$$

The first factor of (2.17) simplifies to

$$\mathbb{P}_x\big(S_{\tau_{C(R_J)}} = z \bigm| \tau_{C(R_J)} < \tau_{C(R_{J+1})}\big), \tag{2.18}$$

which we will bound below using Lemma 2.9.4.

We will verify the hypotheses of Lemma 2.9.4 with $\varepsilon = \delta$ and $R = R_{J+1}$. The first hypothesis is $R \geqslant 10\varepsilon^{-2}$, which is satisfied because $R_{J+1} \geqslant R_1 = 10\delta^{-2}$. The second hypothesis is (2.163) which, in our case, can be written as

$$\max_{x \in C(\delta R_{J+1})} \mathbb{P}_x\left(\tau_{C(R_{J+1})} < \tau_{C(R_J)}\right) < \tfrac{9}{10}. \tag{2.19}$$

Exercise 1.6.8 of [Law13] states that

$$\mathbb{P}_x\left(\tau_{C(R_{J+1})} < \tau_{C(R_J)}\right) = \frac{\log(\frac{\|x\|}{R_J}) + O(R_J^{-1})}{\log(\frac{R_{J+1}}{R_J}) + O(R_J^{-1} + R_{J+1}^{-1})}, \tag{2.20}$$

where the implicit constants are at most 2 (i.e., the $O(R_J^{-1})$ term is at most $2R_J^{-1}$). For the moment, ignore the error terms and assume $\|x\| = \delta R_{J+1}$, in which case (2.20) evaluates to $\frac{5\log 10 - \log 25}{5 \log 10} < 0.73$. Because $R_J \geqslant 10^5$, even after allowing $\|x\|$ up to $\delta R_{J+1} + 1$ and accounting for the error terms, (2.20) is less than $\frac{9}{10}$, which implies (2.19).

Applying Lemma 2.9.4 to (2.18), we obtain a constant c_2 such that

$$\mathbb{P}_x\big(S_{\tau_{C(R_J)}} = z \bigm| \tau_{C(R_J)} < \tau_{C(R_{J+1})}\big) \geqslant c_2 \mu_J(z). \tag{2.21}$$

By (2.19), the second factor of (2.17) is bounded below by $\frac{1}{10}$. We conclude the claim of (2.16) by combining this bound and (2.21) with (2.17), and by setting $c_1 = \frac{1}{10}c_2$. □

Inputs to Stage 2

We continue to assume that $n_0 \neq n$, so that I, J, and K are well defined; the $n_0 = n$ case is easy and we address it in Section 2.3.3. In this subsection, we will prove an estimate of the probability that a random walk passes through annuli $\mathcal{A}_{J-1}$ to $\mathcal{A}_K$ without hitting A. First, in Lemma 2.3.2, we will identify a sequence of "tunnels" through the nonempty annuli, which are empty of A. Second, in Lemma 2.3.3 and Lemma 2.3.4, we will show that random walk traverses these tunnels through a series of rectangles, with a probability which is no smaller than exponential in the number of elements in $\mathcal{A}_K, \dots, \mathcal{A}_{J-1}$. We will combine these estimates in Lemma 2.3.5.

Recall from Section 2.3.1 that $\mathcal{A}_K$ is the last annulus before the random walk encounters an annulus which fails to satisfy (2.12). We call the set of such ℓ by $\mathbb{I} = \{K, \dots, J\}$. For each $\ell \in \mathbb{I}$, we define the annulus $\mathcal{B}_\ell = \mathcal{A}(R_{\ell-1}, \delta R_{\ell+1})$. The inner radius of $\mathcal{B}_\ell$ is at least R_1 because

$$\ell \in \mathbb{I} \implies R_\ell > \delta^{-1} R_1 (m_\ell + 1) \geqslant 10^7 \implies \ell \geqslant 2.$$

The first implication is due to (2.12) and (2.13); the second is due to the fact that $R_\ell = 10^{5\ell}$.

The following lemma identifies subsets of $\mathcal{B}_\ell$ which are empty of A (Figure 2.6). Recall that $m_\ell = n_\ell + n_{\ell-1}$.

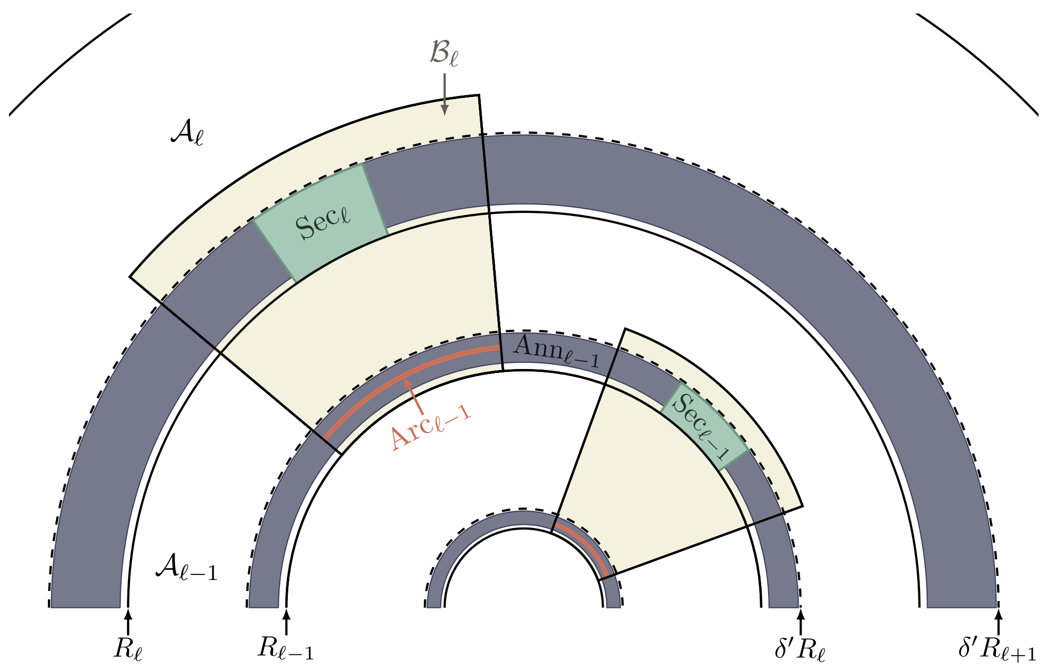

Figure 2.6: The regions identified in Lemma 2.3.2.

Lemma 2.3.2. *Let $\ell \in \mathbb{I}$. Denote $\varepsilon_\ell = (m_\ell + 1)^{-1}$ and $\delta' = \delta/10$. For every $\ell \in \mathbb{I}$, there is an angle $\vartheta_\ell \in [0, 2\pi)$ and a radius $a_{\ell-1} \in [10R_{\ell-1}, \delta' R_\ell)$ such that the following regions contain no element of A:*

- *the sector of* $\mathcal{B}_\ell$ *subtending the angular interval* $[\vartheta_\ell, \vartheta_\ell + 2\pi\varepsilon_\ell)$ *and, in particular, the "middle third" sub-sector*

$$\mathrm{Sec}_\ell = [R_\ell, \delta' R_{\ell+1}) \times \left[\vartheta_\ell + \tfrac{2\pi}{3}\varepsilon_\ell,\, \vartheta_\ell + \tfrac{4\pi}{3}\varepsilon_\ell\right);\ \ \textit{and}$$

- *the sub-annulus* $\mathrm{Ann}_{\ell-1} = \mathcal{A}(a_{\ell-1}, b_{\ell-1})$ *of* $\mathcal{B}_\ell$*, where we define*

$$b_{\ell-1} = a_{\ell-1} + \Delta_{\ell-1} \quad \textit{for} \quad \Delta_{\ell-1} = \delta' \varepsilon_\ell R_\ell \tag{2.22}$$

and, in particular, the circle $\mathrm{Circ}_{\ell-1} = C\left(\frac{a_{\ell-1}+b_{\ell-1}}{2}\right)$ *and the "arc"*

$$\mathrm{Arc}_{\ell-1} = \mathrm{Circ}_{\ell-1} \cap \left\{x \in \mathbb{Z}^2 : \arg x \in [\vartheta_\ell, \vartheta_\ell + 2\pi\varepsilon_\ell)\right\}.$$

We take a moment to explain the parameters and regions. Aside from $\mathcal{B}_\ell$, which overlaps $\mathcal{A}_\ell$ and $\mathcal{A}_{\ell-1}$, the subscripts of the regions indicate which annulus contains them (e.g., $\mathrm{Sec}_\ell \subset \mathcal{A}_\ell$ and $\mathrm{Ann}_{\ell-1} \subset \mathcal{A}_{\ell-1}$). The proof uses the pigeonhole principle to identify regions which contain none of the m_ℓ elements of A in $\mathcal{B}_\ell$ and $\mathrm{Ann}_{\ell-1}$; this motivates our choice of ε_ℓ. The key aspect of Sec_ℓ is that it is separated from $\partial\mathcal{B}_\ell$ by a distance of at least $R_{\ell-1}$. We also need the inner radius of $\mathrm{Ann}_{\ell-1}$ to be at least $R_{\ell-1}$ greater than that of $\mathcal{B}_\ell$, hence the lower bound on $a_{\ell-1}$. The other key aspect of $\mathrm{Ann}_{\ell-1}$ is its overlap with $\mathrm{Sec}_{\ell-1}$. The specific constants (e.g., $\frac{2\pi}{3}$, 10, and δ') are otherwise unimportant.

Proof of Lemma 2.3.2. Fix $\ell \in \mathbb{I}$. For $j \in \{0, \dots, m_\ell\}$, form the intervals

$$2\pi\varepsilon_\ell\,[j, j+1) \ \text{ and } \ 10R_{\ell-1} + \Delta_{\ell-1}[j, j+1).$$

$\mathcal{B}_\ell$ contains at most m_ℓ elements of A, so the pigeonhole principle implies that there are j_1 and j_2 in this range and such that, if $\vartheta_\ell = j_1 2\pi\varepsilon_\ell$ and if $a_{\ell-1} = 10R_{\ell-1} + j_2\Delta_{\ell-1}$, then

$$\mathcal{B}_\ell \cap \left\{x \in \mathbb{Z}^2 : \arg x \in \left[\vartheta_\ell, \vartheta_\ell + 2\pi\varepsilon_\ell\right)\right\} \cap A = \emptyset, \quad \text{and} \quad \mathcal{A}(a_{\ell-1}, a_{\ell-1} + \Delta_{\ell-1}) \cap A = \emptyset.$$

Because $\mathcal{B}_\ell \supseteq \mathrm{Sec}_\ell$ and $\mathrm{Ann}_{\ell-1} \supseteq \mathrm{Arc}_{\ell-1}$, for these choices of ϑ_ℓ and $a_{\ell-1}$, we also have $\mathrm{Sec}_\ell \cap A = \emptyset$ and $\mathrm{Arc}_{\ell-1} \cap A = \emptyset$. □

The next result bounds below the probability that the random walk tunnels "down" from Sec_ℓ to $\mathrm{Arc}_{\ell-1}$. We defer its proof to Section 2.9.5, as it is a simple consequence of the fact that random walk exits a rectangle through its far side with a probability which is no smaller than exponential in the aspect ratio of the rectangle (Lemma 2.9.5). In this case, the aspect ratio is $O(m_\ell)$.

Lemma 2.3.3. *There is a constant* c *such that, for any* $\ell \in \mathbb{I}$ *and every* $y \in \mathrm{Sec}_\ell$*,*

$$\mathbb{P}_y\left(\tau_{\mathrm{Arc}_{\ell-1}} < \tau_A\right) \geqslant c^{m_\ell}.$$

The following lemma bounds below the probability that the random walk tunnels "around" $\mathrm{Ann}_{\ell-1}$, from $\mathrm{Arc}_{\ell-1}$ to $\mathrm{Sec}_{\ell-1}$. Like Lemma 2.3.3, we defer its proof to Section 2.9.6 because it is a simple consequence of Lemma 2.9.5. Indeed, random walk from $\mathrm{Arc}_{\ell-1}$ can reach $\mathrm{Sec}_{\ell-1}$ without exiting $\mathrm{Ann}_{\ell-1}$ by appropriately exiting each rectangle in a sequence of $O(m_\ell)$ rectangles of aspect ratio $O(1)$. Applying Lemma 2.9.5 then implies (2.23).

Lemma 2.3.4. *There is a constant c such that, for any $\ell \in \mathbb{I}$ and every $z \in \mathrm{Arc}_{\ell-1}$,*

$$\mathbb{P}_z\big(\tau_{\mathrm{Sec}_{\ell-1}} < \tau_A\big) \geqslant c^{m_\ell}. \tag{2.23}$$

The next result combines Lemma 2.3.3 and Lemma 2.3.4 to tunnel from $\mathcal{A}_J$ into $\mathcal{A}_{K-1}$. Because the random walk tunnels from $\mathcal{A}_\ell$ to $\mathcal{A}_{\ell-1}$ with a probability no smaller than exponential in $m_\ell = n_\ell + n_{\ell-1}$, the bound in (2.24) is no smaller than exponential in $\sum_{\ell=K-1}^{J-1} n_\ell$ (recall that $n_J = 0$).

Lemma 2.3.5. *There is a constant c such that*

$$\mathbb{P}_{\mu_J}\left(\tau_{\mathrm{Arc}_{K-1}} < \tau_A\right) \geqslant c^{\sum_{\ell=K-1}^{J-1} n_\ell}. \tag{2.24}$$

Proof. Denote by G the event

$$\big\{\tau_{\mathrm{Arc}_{J-1}} < \tau_{\mathrm{Sec}_{J-1}} < \tau_{\mathrm{Arc}_{J-2}} < \cdots < \tau_{\mathrm{Arc}_K} < \tau_{\mathrm{Sec}_K} < \tau_{\mathrm{Arc}_{K-1}} < \tau_A\big\}.$$

Lemma 2.3.3 and Lemma 2.3.4 imply that there is a constant c_1 such that

$$\mathbb{P}_z(G) \geqslant c_1^{\sum_{\ell=K-1}^{J-1} n_\ell} \text{ for } z \in C(R_J) \cap \mathrm{Sec}_J. \tag{2.25}$$

The intersection of Sec_J and $C(R_J)$ subtends an angle of at least n_{J-1}^{-1}, so there is a constant c_2 such that

$$\mu_J(\mathrm{Sec}_J) \geqslant c_2 n_{J-1}^{-1}. \tag{2.26}$$

The inequality (2.24) follows from $G \subseteq \{\tau_{\mathrm{Arc}_{K-1}} < \tau_A\}$, and (2.25) and (2.26):

$$\mathbb{P}_{\mu_J}\left(\tau_{\mathrm{Arc}_{K-1}} < \tau_A\right) \geqslant \mathbb{P}_{\mu_J}(G) \geqslant c_2 n_{J-1}^{-1} \cdot c_1^{\sum_{\ell=K-1}^{J-1} n_\ell} \geqslant c_3^{\sum_{\ell=K-1}^{J-1} n_\ell}.$$

For the third inequality, we take $c_3 = (c_1 c_2)^2$. □

Inputs to Stage 3 when $K = I$

We continue to assume that $n_0 \neq n$, as the alternative case is addressed in Section 2.3.3. Additionally, we assume $K = I$. We briefly recall some important context. When $K = I$, at the end of *Stage 2*, the random walk has reached $\mathrm{Circ}_{I-1} \subseteq \mathcal{A}_{I-1}$, where Circ_{I-1} is a circle with a radius in $[R_{I-1}, \delta' R_I)$. Since $\mathcal{A}_I$ is the innermost annulus which contains an element of A, the random walk from Arc_{I-1} must simply reach the origin before hitting $A_{>R_I}$. In this subsection, we estimate this probability.

We will need the following standard hitting probability estimate (see, for example, Proposition 1.6.7 of [Law13]), which we state as a lemma because we will use it in other sections as well.

Lemma 2.3.6. *Let $y \in D_x(r)$ for $r \geqslant 2(\|x\|+1)$ and assume $y \neq o$. Then*

$$\mathbb{P}_y\left(\tau_o < \tau_{C_x(r)}\right) = \frac{\mathfrak{a}'(r) - \mathfrak{a}(y) + O\left(\frac{\|x\|+1}{r}\right)}{\mathfrak{a}'(r) + O\left(\frac{\|x\|+1}{r}\right)}. \tag{2.27}$$

The implicit constants in the error terms are less than one.

If $R_I < e^{4n}$, then no further machinery is needed to prove the *Stage 3* estimate.

Lemma 2.3.7. *There exists a constant c such that, if $R_I < e^{4n}$, then*

$$\mathbb{P}\left(\tau_{C(R_1)} < \tau_A \;\middle|\; \tau_{\mathrm{Circ}_{I-1}} < \tau_A\right) \geqslant \frac{c}{n}.$$

The bound holds because the random walk must exit $D(R_I)$ to hit $A_{\geqslant R_I}$. By a standard hitting estimate, the probability that the random walk hits the origin first is inversely proportional to $\log R_I$ which is $O(n)$ when $R_I < e^{4n}$.

Proof of Lemma 2.3.7. Uniformly for $y \in \mathrm{Circ}_{I-1}$, we have

$$\mathbb{P}_y\left(\tau_{C(R_1)} < \tau_A\right) \geqslant \mathbb{P}_y\left(\tau_o < \tau_{C(R_I)}\right) \geqslant \frac{\mathfrak{a}'(R_I) - \mathfrak{a}'(\delta R_{I-1}) - \frac{1}{R_I} - \frac{1}{\delta R_I}}{\mathfrak{a}'(R_I) + \frac{1}{R_I}} \geqslant \frac{1}{\mathfrak{a}'(R_I)}. \tag{2.28}$$

The first inequality follows from the observation that $C(R_1)$ and $C(R_I)$ separate y from o and A. The second inequality is due to Lemma 2.3.6, where we have replaced $\mathfrak{a}(y)$ by $\mathfrak{a}'(\delta R_I) + \frac{1}{\delta R_I}$ using (2.27) of Lemma 2.9.2 and the fact that $\|y\| \leqslant \delta R_I$. The third inequality follows from $\delta R_I \geqslant 10^3$. To conclude, we substitute $\mathfrak{a}'(R_I) = \frac{2}{\pi}\log R_I + \kappa$ into (2.28) and use assumption that $R_I < e^{4n}$. $\square$

We will use the rest of this subsection to prove the bound of Lemma 2.3.7, but under the complementary assumption $R_I \geqslant e^{4n}$. This is one of the two estimates we highlighted in Section 2.2.2.

Next is a standard result, which enables us to express certain hitting probabilities in terms of the potential kernel. We include a short proof for completeness.

Lemma 2.3.8. *For any pair of points $x, y \in \mathbb{Z}^2$, define*

$$M_{x,y}(z) = \frac{\mathfrak{a}(x-z) - \mathfrak{a}(y-z)}{2\mathfrak{a}(x-y)} + \frac{1}{2}.$$

Then $M_{x,y}(z) = \mathbb{P}_z(\sigma_y < \sigma_x)$.

Proof. Fix $x, y \in \mathbb{Z}^2$. Theorem 1.4.8 of [Law13] states that for any proper subset B of $\mathbb{Z}^2$ (including infinite B) and bounded function $F : \partial B \to \mathbb{R}$, the unique bounded function $f : B \cup \partial B \to \mathbb{R}$ which is harmonic in B and equals F on ∂B is $f(z) = \mathbb{E}_z[F(S_{\sigma_{\partial B}})]$. Setting $B = \mathbb{Z}^2 \backslash \{x, y\}$ and $F(z) = \mathbf{1}(z = y)$, we have $f(z) = \mathbb{P}_z(\sigma_y < \sigma_x)$. Since $M_{x,y}$ is bounded, harmonic on B, and agrees with f on ∂B, the uniqueness of f implies $M_{x,y}(z) = f(z)$. $\square$

The next two results partly implement the first estimate that we discussed in Section 2.2.2.

Lemma 2.3.9. *For any $z, z' \in \mathrm{Circ}_{I-1}$ and $y \in D(R_I)^c$,*

$$\mathbb{P}_z(\tau_y < \tau_o) \leq \frac{1}{2} \quad \textit{and} \quad |\mathbb{P}_z(\tau_y < \tau_o) - \mathbb{P}_{z'}(\tau_y < \tau_o)| \leqslant \frac{1}{\log R_I}. \tag{2.29}$$

The first inequality in (2.29) holds because z is appreciably closer to the origin than it is to y. The second inequality holds because a Taylor expansion of the numerator of $M_{z,y}(o) - M_{z',y}(o)$ shows that it is $O(1)$, while the denominator of $2\mathfrak{a}(y)$ is at least $\log R_I$.

Proof of Lemma 2.3.9. By Lemma 2.3.8,

$$\mathbb{P}_z(\tau_y < \tau_o) = \frac{1}{2} + \frac{\mathfrak{a}(z) - \mathfrak{a}(y-z)}{2\mathfrak{a}(y)}.$$

The first inequality of (2.29) holds because $\mathfrak{a}(y-z) \geqslant \mathfrak{a}(z)$. Indeed, Circ_{I-1} is a subset of $D(\delta R_I)$, so $\|z\| \leqslant \delta R_I + 1$ and $\|y - z\| \geqslant (1-\delta)R_I - 1$ by assumption. The latter is at least twice the former and $\|z\| \geqslant 2$, so by (1) of Lemma 2.9.1, $\mathfrak{a}(y-z) \geqslant \mathfrak{a}(z)$.

Using Lemma 2.3.8, the difference in (2.29) can be written as

$$|M_{z,y}(o) - M_{z',y}(o)| = \frac{|\mathfrak{a}(y-z') - \mathfrak{a}(y-z)|}{2\mathfrak{a}(y)}. \tag{2.30}$$

Concerning the denominator, $\|y\|$ is at least one, so $\mathfrak{a}(y)$ is at least $\frac{2}{\pi}\log\|y\| \geqslant \frac{2}{\pi}\log R_I$ by (2) of Lemma 2.9.1. We apply (3) of Lemma 2.9.1 with $R = R_I$ and $r = \mathrm{rad}(\mathrm{Circ}_{I-1}) \leqslant \delta R_I$ to bound the numerator by $\frac{4}{\pi}$. Substituting these bounds into (2.30) gives the second inequality in (2.29). □

Label the k elements in $A_{\geqslant R_1}$ by x_i for $1 \leqslant i \leqslant k$. Then let $Y_i = \mathbf{1}(\tau_{x_i} < \tau_o)$ and $W = \sum_{i=1}^k Y_i$. In words, W counts the number of elements of $A_{\geqslant R_1}$ which have been visited before the random walk returns to the origin.

Lemma 2.3.10. *If $R_I \geqslant e^{4n}$, then, for all $z \in \mathrm{Circ}_{I-1}$,*

$$\mathbb{E}_z[W \mid W > 0] \geq \mathbb{E}_z W + \frac{1}{4}. \tag{2.31}$$

The constant $\frac{1}{4}$ in (2.31) is unimportant, aside from being positive, independently of n. The inequality holds because random walk from Circ_{I-1} hits a given element of $A_{\geqslant R_1}$ before the origin with a probability of at most $\frac{1}{2}$. Consequently, given that some such element is hit, the conditional expectation of W is essentially larger than its unconditional one by a constant.

Proof of Lemma 2.3.10. Fix $z \in \mathrm{Circ}_{I-1}$. When $\{W > 0\}$ occurs, some labeled element, x_f, is hit first. After τ_{x_f}, the random walk may proceed to hit other x_i before returning to Circ_{I-1} at a time $\eta = \min\left\{t \geqslant \tau_{x_f} : S_t \in \mathrm{Circ}_{I-1}\right\}$. Let $\mathcal{V}$ be the collection of labeled elements that the walk visits before time η, $\{i : \tau_{x_i} < \eta\}$. In terms of $\mathcal{V}$ and η, the conditional expectation of W is

$$\mathbb{E}_z[W \mid W > 0] = \mathbb{E}_z\Big[|\mathcal{V}| + \mathbb{E}_{S_\eta} \sum_{i \notin \mathcal{V}} Y_i \,\Big|\, W > 0\Big]. \tag{2.32}$$

Let V be a nonempty subset of the labeled elements and let $z' \in \mathrm{Circ}_{I-1}$. We have

$$\Big| \mathbb{E}_z \sum_{i \notin V} Y_i - \mathbb{E}_{z'} \sum_{i \notin V} Y_i \Big| \leqslant \frac{n}{\log R_I} \leqslant \frac{1}{4}.$$

The first inequality is due to Lemma 2.3.9 and the fact that there are at most n labeled elements outside of V. The second inequality follows from the assumption that $R_I \geqslant e^{4n}$.

We use this bound to replace S_η in (2.32) with z:

$$\mathbb{E}_z[W \mid W > 0] \geqslant \mathbb{E}_z\Big[|\mathcal{V}| + \mathbb{E}_z \sum_{i \notin \mathcal{V}} Y_i \,\Big|\, W > 0\Big] - \frac{1}{4}. \tag{2.33}$$

By Lemma 2.3.9, $\mathbb{P}_z(\tau_{x_i} < \tau_o) \leqslant \frac{1}{2}$. Accordingly, for a nonempty subset V of labeled elements,

$$\mathbb{E}_z \sum_{i \notin V} Y_i \geqslant \mathbb{E}_z W - \frac{1}{2}|V|.$$

Substituting this into the inner expectation of (2.33), we find

$$\begin{aligned}\mathbb{E}_z[W \mid W > 0] &\geqslant \mathbb{E}_z\Big[|\mathcal{V}| + \mathbb{E}_z W - \frac{1}{2}|\mathcal{V}| \,\Big|\, W > 0\Big] - \frac{1}{4} \\ &\geqslant \mathbb{E}_z W + \mathbb{E}_z \left[\frac{1}{2}|\mathcal{V}| \,\Big|\, W > 0\right] - \frac{1}{4}.\end{aligned}$$

Since $\{W > 0\} = \{|\mathcal{V}| \geqslant 1\}$, this lower bound is at least $\mathbb{E}_z W + \frac{1}{4}$. ☐

We use the preceding lemma to prove the analogue of Lemma 2.3.7 when $R_I \geqslant e^{4n}$. The proof uses the method highlighted in Section 2.2.2 and Figure 2.3 (left).

Lemma 2.3.11. *There exists a constant* c *such that, if* $R_I \geqslant e^{4n}$, *then*

$$\mathbb{P}\left(\tau_{C(R_1)} < \tau_A \;\middle|\; \tau_{\mathrm{Circ}_{I-1}} < \tau_A\right) \geqslant \frac{c}{n}. \tag{2.34}$$

Proof. Conditionally on $\{\tau_{\mathrm{Circ}_{I-1}} < \tau_A\}$, let the random walk hit Circ_{I-1} at z. Denote the positions of the $k \leqslant n$ particles in $A_{\geqslant R_1}$ as x_i for $1 \leqslant i \leqslant k$. Let $Y_i = \mathbf{1}(\tau_{x_i} < \tau_o)$ and $W = \sum_{i=1}^k Y_i$, just as we did for Lemma 2.3.10. The claimed bound (2.34) follows from

$$\mathbb{P}_z(\tau_{C(R_1)} < \tau_A) \geqslant \mathbb{P}_z(W > 0) = \frac{\mathbb{E}_z W}{\mathbb{E}_z[W \mid W > 0]} \leqslant \frac{\mathbb{E}_z W}{\mathbb{E}_z W + 1/4} \leqslant \frac{n}{n + 1/4} \leqslant 1 - \frac{1}{5n}.$$

The first inequality follows from the fact that $C(R_1)$ separates z from the origin. The second inequality is due to Lemma 2.3.10, which applies because $R_I \geqslant e^{4n}$. Since the resulting expression increases with $\mathbb{E}_z W$, we obtain the third inequality by substituting n for $\mathbb{E}_z W$, as $\mathbb{E}_z W \leqslant n$. The fourth inequality follows from $n \geqslant 1$. ☐

Inputs to Stage 4 when $K = I$ and Stage 3 when $K \neq I$

The results in this subsection address the last stage of advancement in the two sub-cases of the case $n_0 \neq n$: $K = I$ and $K \neq I$. In the former sub-case, the random walk has reached $C(R_1)$; in the latter sub-case, it has reached Circ_{K-1}. Both sub-cases will be addressed by corollaries of the following geometric lemma.

Lemma 2.3.12. *Let $A \in \mathscr{H}_n$ and $r > 0$. From any $x \in C(r)\backslash A$, there is a path Γ in $(A\backslash\{o\})^c$ from $\Gamma_1 = x$ to $\Gamma_{|\Gamma|} = o$ with a length of at most $10\max\{r, n\}$. Moreover, if $A \subseteq D(r)$, then Γ lies in $D(r+2)$.*

We choose the constant factor of 10 for convenience; it has no special significance. We use a radius of $r+2$ in $D(r+2)$ to contain $D(r)$ and its $*$-visible boundary, defined as follows. Let m be a positive integer and denote by $\mathbb{Z}^{m*}$ the graph with vertex set $\mathbb{Z}^m$ and with an edge between distinct $x, y \in \mathbb{Z}^m$ when x and y differ by at most one in each coordinate. For $A \subseteq \mathbb{Z}^m$, we define the $*$-visible boundary $\partial_{\mathrm{vis}} A$ as

$$\partial_{\mathrm{vis}} A = \big\{ x \in \mathbb{Z}^m : x \text{ is adjacent in } \mathbb{Z}^{m*} \text{ to some } y \in A \\ \text{and there is a path from } \infty \text{ to } x \text{ disjoint from } A \big\}. \quad (2.35)$$

To prove Lemma 2.3.12, we need the following result, due to Kesten [Kes86] (alternatively, Theorem 4 of [Tim13]).

Lemma 2.3.13 (Lemma 2.23 of [Kes86])**.** *Let m be a positive integer. If A is a finite, $*$-connected subset of $\mathbb{Z}^m$, then $\partial_{\mathrm{vis}} A$ is connected in $\mathbb{Z}^m$.*

Proof of Lemma 2.3.12. Let $\{B_\ell\}_\ell$ be the collection of $*$-connected components of $A\backslash\{o\}$. By Lemma 2.3.13, because B_ℓ is finite and $*$-connected, $\partial^*_{\mathrm{ext}} B_\ell$ is connected.

Fix $r > 0$ and $x \in C(r)\backslash A$. Let Γ be the shortest path from x to the origin. If Γ is disjoint from $A\backslash\{o\}$, then we are done, as $|\Gamma|$ is no greater than $2r$. Otherwise, let ℓ_1 be the label of the first $*$-connected component intersected by Γ. Let i and j be the first and last indices such that Γ intersects $\partial^*_{\mathrm{ext}} B_{\ell_1}$, respectively. Because $\partial^*_{\mathrm{ext}} B_{\ell_1}$ is connected, there is a path Λ in $\partial^*_{\mathrm{ext}} B_{\ell_1}$ from Γ_i to Γ_j. We then edit Γ to form Γ' as

$$\Gamma' = \left(\Gamma_1, \dots, \Gamma_{i-1}, \Lambda_1, \dots, \Lambda_{|\Lambda|}, \Gamma_{j+1}, \dots, \Gamma_{|\Gamma|}\right).$$

If Γ' is disjoint from $A\backslash\{o\}$, then we are done, as Γ' is contained in the union of Γ and $\bigcup_\ell \partial^*_{\mathrm{ext}} B_\ell$. Since $\bigcup_\ell B_\ell$ has at most n elements, $\bigcup_\ell \partial^*_{\mathrm{ext}} B_\ell$ has at most $8n$ elements. Accordingly, the length of Γ' is at most $2r + 8n \leqslant 10\max\{r, n\}$. Otherwise, if Γ' intersects another $*$-connected component of $A\backslash\{o\}$, we can simply relabel the preceding argument to continue inductively and obtain the same bound.

Lastly, if $A \subseteq D(r)$, then $\bigcup_\ell \partial^*_{\mathrm{ext}} B_\ell$ is contained in $D(r+2)$. Since Γ is also contained in $D(r+2)$, this implies that Γ' is contained in $D(r+2)$. ☐

We now state three corollaries of Lemma 2.3.12. The first corollary addresses *Stage 4* when $K = I$. It follows from $|A_{<R_1}| = O(R_1^2)$ and $A_{\geqslant R_1} \subseteq D(R_1+2)^c$.

Corollary 2.3.14. *There is a constant c such that*

$$\mathbb{P}(\tau_o \leqslant \tau_A \mid \tau_{C(R_1)} < \tau_A) \geqslant c. \tag{2.36}$$

The second corollary addresses *Stage 3* when $K \neq I$.

Corollary 2.3.15. *Assume that $n_0 = 1$ and $K \neq I$. There is a constant c such that*

$$\mathbb{P}(\tau_o \leqslant \tau_A \mid \tau_{\mathrm{Circ}_{K-1}} < \tau_A) \geqslant c^{\sum_{\ell=I}^{K-1} n_\ell}. \tag{2.37}$$

The bound (2.37) follows from Lemma 2.3.12 because $K \neq I$ implies that the radius r of Circ_{K-1} is at most a constant factor times $|A_{<r}|$. Lemma 2.3.12 then implies that there is a path Γ from Circ_{K-1} to the origin with a length of $O(|A_{<r}|)$, which remains in $D(r+2)$ and otherwise avoids the elements of $A_{<r}$. In fact, because Circ_{K-1} is a subset of Ann_{K-1}, which contains no elements of A, by remaining in $D(r+2)$, Γ avoids $A_{\geqslant r}$ as well. This implies (2.37).

The third corollary implies (2.3) of Theorem 2.1.6 because any connected set belonging to $\mathscr{H}_n$ is contained in $D(n)$.

Corollary 2.3.16. *Let $n \geqslant 1$. There is a constant c such that, for any connected $A \in \mathscr{H}_n$,*

$$\mathbb{H}_A(o) \geqslant e^{-cn}.$$

2.3.3 Proof of Theorem 2.1.6

We only need to prove (2.2), because Corollary 2.3.16 establishes (2.3). The proof is by induction on n. Since (2.2) clearly holds for $n = 1$ and $n = 2$, we assume $n \geqslant 3$.

Let $A \in \mathscr{H}_n$. There are three cases: $n_0 = n$, $n_0 \neq n$ and $K = I$, and $n_0 \neq n$ and $K \neq I$. The first of these cases is easy: When $n_0 = n$, A is contained in $D(R_1)$, so Corollary 2.36 implies that $\mathbb{H}_A(o)$ is at least a universal constant. Accordingly, in what follows, we assume that $n_0 \neq n$ and address the two sub-cases $K = I$ and $K \neq I$.

First sub-case: $K = I$. If $K = I$, then we write

$$\mathbb{H}_A(o) = \mathbb{P}(\tau_o \leqslant \tau_A) \geqslant \mathbb{P}(\tau_{C(R_J)} < \tau_{\mathrm{Circ}_{I-1}} < \tau_{C(R_1)} < \tau_o \leqslant \tau_A).$$

Because $C(R_J)$, Circ_{I-1}, and $C(R_1)$ respectively separate Circ_{I-1}, $C(R_1)$, and the origin from ∞, we can express the lower bound as the following product:

$$\begin{aligned}\mathbb{H}_A(o) \geqslant \mathbb{P}(\tau_{C(R_J)} < \tau_A) \times \mathbb{P}\big(\tau_{\mathrm{Circ}_{I-1}} < \tau_A \,\big|\, \tau_{C(R_J)} < \tau_A\big) \\ \times \mathbb{P}\big(\tau_{C(R_1)} < \tau_A \,\big|\, \tau_{\mathrm{Circ}_{I-1}} < \tau_A\big) \times \mathbb{P}\big(\tau_o \leqslant \tau_A \,\big|\, \tau_{C(R_1)} < \tau_A\big).\end{aligned} \tag{2.38}$$

We address the four factors of (2.38) in turn. First, by the induction hypothesis, there is a constant c_1 such that

$$\mathbb{P}(\tau_{C(R_J)} < \tau_A) \geqslant e^{-c_1 k \log k},$$

where $k = n_{>J} + 1$. Second, by the strong Markov property applied to $\tau_{C(R_J)}$ and Lemma 2.3.1, and then by Lemma 2.3.5, there are constants c_2 and c_3 such that

$$\mathbb{P}\big(\tau_{\mathrm{Circ}_{I-1}} < \tau_A \;\big|\; \tau_{C(R_J)} < \tau_A\big) \geqslant c_2 \mathbb{P}_{\mu_J}\left(\tau_{\mathrm{Arc}_{I-1}} < \tau_A\right) \geqslant e^{-c_3 \sum_{\ell=I}^{J-1} n_\ell}. \tag{2.39}$$

Third and fourth, by Lemma 2.3.7 and Lemma 2.3.11, and by Corollary 2.3.14, there are constants c_4 and c_5 such that

$$\mathbb{P}\big(\tau_{C(R_1)} \leqslant \tau_A \;\big|\; \tau_{\mathrm{Circ}_{I-1}} < \tau_A\big) \geqslant (c_4 n)^{-1} \quad \text{and} \quad \mathbb{P}\big(\tau_o \leqslant \tau_A \;\big|\; \tau_{C(R_1)} \leqslant \tau_A\big) \geqslant c_5.$$

Substituting the preceding bounds into (2.38) completes the induction step for this sub-case:

$$\mathbb{H}_A(o) \geqslant e^{-c_1 k \log k - c_3 \sum_{\ell=I}^{J-1} n_\ell - \log(c_4 n) + \log c_5} \geqslant e^{-c_1 n \log n}.$$

The second inequality follows from $n - k = \sum_{\ell=I}^{J-1} n_\ell > 1$ and $\log n \geqslant 1$, and from potentially adjusting c_1 to satisfy $c_1 \geqslant 8 \max\{1, c_3, \log c_4, -\log c_5\}$. We are free to adjust c_1 in this way, since the other constants do not arise from the use of the induction hypothesis.

Second sub-case: $K \neq I$. If $K \neq I$, then we write $\mathbb{H}_A(o) \geqslant \mathbb{P}(\tau_{C(R_J)} < \tau_{\mathrm{Circ}_{K-1}} < \tau_o \leqslant \tau_A)$. Because $C(R_J)$ and Circ_{K-1} separate Circ_{K-1} and the origin from ∞, we can express the lower bound as:

$$\mathbb{H}_A(o) \geqslant \mathbb{P}(\tau_{C(R_J)} < \tau_A) \times \mathbb{P}\big(\tau_{\mathrm{Circ}_{K-1}} < \tau_A \;\big|\; \tau_{C(R_J)} < \tau_A\big) \times \mathbb{P}\big(\tau_o \leqslant \tau_A \;\big|\; \tau_{\mathrm{Circ}_{K-1}} < \tau_A\big). \tag{2.40}$$

As in the first sub-case, the first factor is addressed by the induction hypothesis and the lower bound (2.39) applies to the second factor of (2.40) with K in the place of I. Concerning the third factor, corollary 2.3.14 implies that there is a constant c_6 such that

$$\mathbb{P}\big(\tau_o \leqslant \tau_A \;\big|\; \tau_{\mathrm{Circ}_{K-1}} < \tau_A\big) \geqslant e^{-c_6 \sum_{\ell=I}^{K-1} n_\ell}.$$

Substituting the three bounds into (2.40) concludes the induction step in this sub-case:

$$\mathbb{H}_A(o) \geqslant e^{-c_1 k \log k - c_3 \sum_{\ell=K}^{J-1} n_\ell - c_6 \sum_{\ell=I}^{K-1} n_\ell} \geqslant e^{-c_1 n \log n}.$$

The second inequality follows from potentially adjusting c_1 to satisfy $c_1 \geqslant 8 \max\{1, c_3, c_6\}$.

This completes the induction and establishes (2.2). □

2.4 Escape probability estimates

The purpose of this section is to prove Theorem 2.1.9. It suffices to prove the escape probability lower bound (2.10), as (2.11) follows from (2.10) by the pigeonhole principle. Let A be an n-element subset of $\mathbb{Z}^2$ with at least two elements. We assume w.l.o.g. that $o \in A$. Denote $b = \mathrm{diam}(A)$, and suppose $d \geqslant 2b$. We aim to show that there is a constant c such that, if $d \geqslant 2b$, then, for every $x \in A$,

$$\mathbb{P}_x(\tau_{\partial A_d} < \tau_A) \geqslant \frac{c\mathbb{H}_A(x)}{n \log d}.$$

In fact, by adjusting c, we can reduce to the case when $d \geqslant kb$ for $k = 200$ and when b is at least a large universal constant, b'. We proceed to prove (2.10) when $d \geqslant 200b$, for sufficiently large b. Since $C(kb)$ separates A from ∂A_d, we can write the escape probability as the product of two factors:

$$\mathbb{P}_x(\tau_{\partial A_d} < \tau_A) = \mathbb{P}_x(\tau_{C(kb)} < \tau_A)\,\mathbb{P}_x\big(\tau_{\partial A_d} < \tau_A \;\big|\; \tau_{C(kb)} < \tau_A\big). \tag{2.41}$$

Concerning the first factor of (2.41), we have the following lemma.

Lemma 2.4.1. *Let $x \in A$. Then*

$$\mathbb{P}_x(\tau_{C(kb)} < \tau_A) \geqslant \frac{\mathbb{H}_A(x)}{4\log(kb)}. \tag{2.42}$$

The factor of $\log(kb)$ arises from evaluating the potential kernel at elements of $C(kb)$; the factor of 4 is unimportant. The proof is an application of the optional stopping theorem to the martingale $\mathfrak{a}(S_{j\wedge\tau_o})$.

Proof of Lemma 2.4.1. Let $x \in A$. By conditioning on the first step, we have

$$\mathbb{P}_x(\tau_{C(kb)} < \tau_A) = \frac{1}{4}\sum_{y\notin A, y\sim x} \mathbb{P}_y(\tau_{C(kb)} < \tau_A), \tag{2.43}$$

where $y \sim x$ means $\|x - y\| = 1$. We apply the optional stopping theorem to the martingale $\mathfrak{a}(S_{j\wedge\tau_o})$ with the stopping time $\tau_A \wedge \tau_{C(kb)}$ to find:

$$\frac{1}{4}\sum_{y\notin A, y\sim x} \mathbb{P}_y(\tau_{C(kb)} < \tau_A) = \frac{1}{4}\sum_{y\notin A, y\sim x} \frac{\mathfrak{a}(y) - \mathbb{E}_y\mathfrak{a}(S_{\tau_A})}{\mathbb{E}_y\big[\mathfrak{a}(S_{\tau_{C(kb)}}) - \mathfrak{a}(S_{\tau_A}) \;\big|\; \tau_{C(kb)} < \tau_A\big]}. \tag{2.44}$$

We need two facts. First, $\mathbb{H}_A(x)$ can be expressed as $\frac{1}{4}\sum_{y\notin A, y\sim x}\big(\mathfrak{a}(y) - \mathbb{E}_y\mathfrak{a}(S_{\tau_A})\big)$ [Pop21, Definition 3.15, Theorem 3.16]. Second, for any $z \in C(kb)$, $\mathfrak{a}(z) \leqslant 4\log(kb)$ by Lemma 2.9.1. Applying these facts to (2.44), and the result to (2.43), we find

$$\mathbb{P}_x(\tau_{C(kb)} < \tau_A) \geqslant \frac{1}{4\log(kb)} \cdot \frac{1}{4}\sum_{y\notin A, y\sim x}\big(\mathfrak{a}(y) - \mathbb{E}_y\mathfrak{a}(S_{\tau_A})\big) = \frac{\mathbb{H}_A(x)}{4\log(kb)}.$$

□

Concerning the second factor of (2.41), given that $\{\tau_{C(kb)} < \tau_A\}$ occurs, we are essentially in the setting depicted on the right side of Figure 2.3, with $x = S_{\tau_{C(kb)}}$, $r = b$, kb in the place of $2r$, and $R = d$. The argument highlighted in Section 2.2.2 suggests that the second factor of (2.41) is at least proportional to $\frac{\log b}{n \log d}$. We will prove this lower bound and combine it with (2.41) and (2.42) to obtain (2.10) of Theorem 2.1.9.

Lemma 2.4.2. *Let $y \in C(kb)$. If $d \geqslant kb$ and if b is sufficiently large, then*

$$\mathbb{P}_y(\tau_{\partial A_d} < \tau_A) \geqslant \frac{\log b}{2n\log d}. \tag{2.45}$$

Proof. Let $y \in C(kb)$. We will follow the argument of Section 2.2.2. Label the points of A as $x_1, x_2, \ldots, x_n$ and define

$$Y_i = \mathbf{1}\left(\tau_{x_i} < \tau_{\partial A_d}\right) \quad \text{and} \quad W = \sum_{i=1}^{n} Y_i.$$

From the definition of W, we see that $\{W = 0\} = \{\tau_{\partial A_d} < \tau_A\}$. Thus to obtain the lower bound in (2.45), it suffices to get a complementary upper bound on

$$\mathbb{P}_y(W > 0) = \frac{\mathbb{E}_y W}{\mathbb{E}_y[W \mid W > 0]}. \tag{2.46}$$

We will find α and β such that, uniformly for $y \in C(kb)$ and $x_i, x_j \in A$,

$$\mathbb{P}_y\left(\tau_{x_i} < \tau_{\partial A_d}\right) \leqslant \alpha \quad \text{and} \quad \mathbb{P}_{x_i}\left(\tau_{x_j} < \tau_{\partial A_d}\right) \geqslant \beta. \tag{2.47}$$

Moreover, α and β will satisfy

$$\alpha \leqslant \beta \quad \text{and} \quad 1 - \beta \geqslant \frac{\log b}{2 \log d}. \tag{2.48}$$

The requirement that $\alpha \leqslant \beta$ prevents us from choosing $\beta = 0$. Essentially, we will be able to satisfy (2.47) and the first condition of (2.48) because $\|x_i - x_j\|$ is smaller than $\|y - x_i\|$. We will be able to satisfy the second condition because $\operatorname{dist}(x_i, \partial A_d) \geqslant d$ while $\|x_i - x_j\| \leqslant b$, which implies that $\mathbb{P}_{x_i}(\tau_{x_j} < \tau_{\partial A_d})$ is roughly $1 - \frac{\log b}{\log d}$.

If α, β satisfy (2.47), then we can bound (2.46) as

$$\mathbb{P}_y(W > 0) \leqslant \frac{n\alpha}{1 + (n-1)\beta}. \tag{2.49}$$

Additionally, when α and β satisfy (2.48), (2.49) implies

$$\mathbb{P}_y(W = 0) \geqslant \frac{(1-\beta) + n(\beta - \alpha)}{(1-\beta) + n\beta} \geqslant \frac{1-\beta}{n} \geqslant \frac{\log b}{2n \log d},$$

which gives the claimed bound (2.45).

Identifying α. We now find the α promised in (2.47). Denote $F_i = C_{x_i}(d + b)$ (Figure 2.7). Since ∂A_d separates y from F_i, we have

$$\mathbb{P}_y\left(\tau_{x_i} < \tau_{\partial A_d}\right) \leqslant \mathbb{P}_y\left(\tau_{x_i} < \tau_{F_i}\right) = \mathbb{P}_{y - x_i}\left(\tau_o < \tau_{C(d+b)}\right). \tag{2.50}$$

The hypotheses of Lemma 2.3.6 are met because $y - x_i \neq o$ and $y - x_i \in D(d + b)$. Hence (2.27) applies as

$$\mathbb{P}_{y - x_i}\left(\tau_o < \tau_{C(d+b)}\right) = \frac{\mathfrak{a}'(d+b) - \mathfrak{a}(y - x_i) + O\left(\|y - x_i\|^{-1}\right)}{\mathfrak{a}'(d+b) + O\left(\|y - x_i\|^{-1}\right)}. \tag{2.51}$$

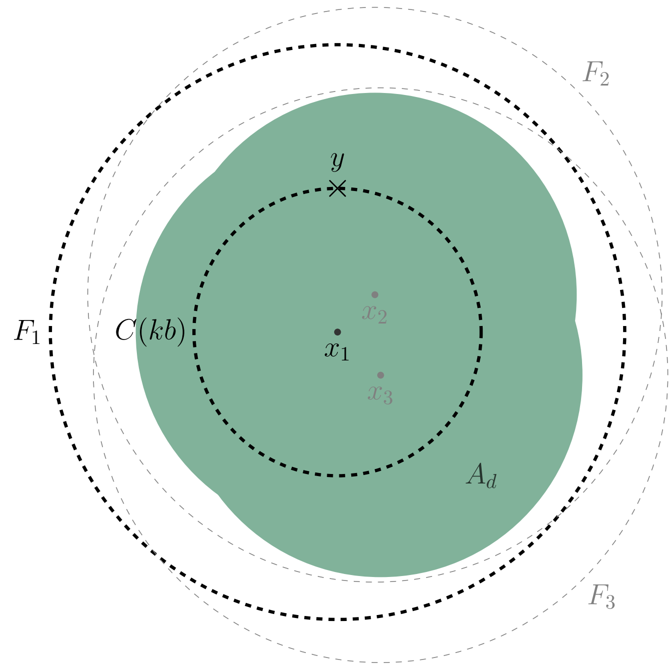

Figure 2.7: Escape to ∂A_d, for $n = 3$. Each F_i is a circle centered on $x_i \in A$, separating A_d from infinity. Lemma 2.3.6 bounds above the probability that the walk hits x_i before F_i, uniformly for $y \in C(kb)$.

Ignoring the error terms, the expression in (2.51) is at most $\frac{\log(d+b)-\log(kb)}{\log(d+b)}$. A more careful calculation gives

$$\mathbb{P}_{y-x_i}\left(\tau_o < \tau_{C(d+b)}\right) = \frac{\log(d+b)-\log(kb)}{\log(d+b)} + \delta_1 \leqslant \frac{(1+\varepsilon)\log d - \log(kb)}{\log d} + \delta_1 =: \alpha,$$

where $\delta_1 = (\frac{\pi\kappa}{2} + O(b^{-1}))(\log d)^{-1}$ and $\varepsilon = \frac{b}{d \log d}$. The inequality results from applying the inequality $\log(1+x) \leqslant x$, which holds for $x > -1$, to the $\log(d+b)$ term in the numerator, and reducing $\log(d+b)$ to $\log d$ in the denominator. By (2.50), α satisfies (2.47).

Identifying β. We now find a suitable β. Since $C_{x_i}(d)$ separates A from ∂A_d, we have

$$\mathbb{P}_{x_i}\left(\tau_{x_j} < \tau_{\partial A_d}\right) \geqslant \mathbb{P}_{x_i}\left(\tau_{x_j} < \tau_{C_{x_i}(d)}\right) = \mathbb{P}_{x_i - x_j}\left(\tau_o < \tau_{C(d)}\right). \tag{2.52}$$

The hypotheses of Lemma 2.3.6 are met because $x_i - x_j \neq o$ and $x_i - x_j \in D(d)$. Hence (2.27) applies as

$$\mathbb{P}_{x_i - x_j}\left(\tau_o < \tau_{C(d)}\right) = \frac{\mathfrak{a}'(d) - \mathfrak{a}(x_i - x_j) + O(\|x_i - x_j\|^{-1})}{\mathfrak{a}'(d) + O(\|x_i - x_j\|^{-1})}. \tag{2.53}$$

Ignoring the error terms, (2.53) is at least $\frac{\log d - \log b}{\log d + \kappa}$. A more careful calculation gives

$$\mathbb{P}_{x_i - x_j}\left(\tau_o < \tau_{C(d)}\right) = \frac{\log d - \log b}{\log d} - \delta_2 =: \beta,$$

where $\delta_2 = (\frac{\pi\kappa}{2} + O(b^{-1}))(\log d)^{-1}$. By (2.52), β satisfies (2.47).

Verifying (2.48). To verify the first condition of (2.48), we calculate

$$(\beta - \alpha)\log d = \log k - \tfrac{b}{d} - \pi\kappa + O(b^{-1}) \geqslant 1 + O(b^{-1}).$$

The inequality is due to $k = 200$, $\frac{b}{d} \leqslant 0.5$, and $\pi\kappa < 3.5$. If b is sufficiently large, then $1 + O(b^{-1})$ is nonnegative, which verifies (2.48).

Concerning the second condition of (2.48), if b is sufficiently large, then

$$1 - \beta = \frac{\log b + 1}{\log d} \leqslant \frac{\log b}{2\log d}.$$

We have identified α, β which satisfy (2.47) and (2.48) for sufficiently large b. By the preceding discussion, this proves (2.45). □

Proof of Theorem 2.1.9. By (2.41), Lemma 2.4.1, and Lemma 2.4.2, we have

$$\mathbb{P}_x(\tau_{\partial A_d} < \tau_A) \geqslant \frac{\mathbb{H}_A(x)}{4\log(kb)} \cdot \frac{\log b}{2n\log d} \geqslant \frac{\mathbb{H}_A(x)}{16n\log d}, \tag{2.54}$$

whenever $x \in A$ and $d \geqslant kb$, for sufficiently large b. The second inequality is due to the fact that $\log(kb) \leqslant 2\log b$ for sufficiently large b.

By the reductions discussed at the beginning of this section, (2.54) implies that there is a constant c such that (2.10) holds for $x \in A$ if A has at least two elements and if $d \geqslant 2\,\mathrm{diam}(A)$. (2.11) follows from (2.10) because, by the pigeonhole principle, some element of A has harmonic measure of at least n^{-1}. □

2.5 Clustering sets of relatively large diameter

When a HAT configuration has a large diameter relative to the number of particles, we can decompose the configuration into clusters of particles, which are well separated in a sense. This is the content of Lemma 2.5.2, which will be a key input to the results in Section 2.6.

Definition 2.5.1 (Exponential clustering). *For a finite $A \subset \mathbb{Z}^2$ with $|A| = n$, an exponential clustering of A with parameter $r \geqslant 0$, denoted $A \mapsto_r \{A^i, x_i, \theta^{(i)}\}_{i=1}^k$, is a partition of A into clusters $A^1, A^2, \dots, A^k$ with $1 \leqslant k \leqslant n$, such that each cluster arises as $A^i = A \cap D_{x_i}(\theta^{(i)})$ for $x_i \in \mathbb{Z}^2$, with $\theta^{(i)} \geqslant r$, and*

$$\mathrm{dist}(A^i, A^j) > \exp\left(\max\left\{\theta^{(i)}, \theta^{(j)}\right\}\right) \text{ for } i \neq j. \tag{2.55}$$

We will call x_i the center of cluster i. In some instances, the values of r, x_i, or $\theta^{(i)}$ will be irrelevant and we will omit them from our notation. For example, $A \mapsto \{A^i\}_{i=1}^k$.

An exponential clustering of A with parameter r always exists because, if $A^1 = A$, $x_1 \in A$, and $\theta^{(1)} \geqslant \max\{r, \operatorname{diam}(A)\}$, then $A \mapsto_r \{A^1, x_1, \theta^{(1)}\}$ is such a clustering. However, to ensure that there is an exponential clustering of A (with parameter r) with more than one cluster, we require that the diameter of A exceeds $2\theta_{n-1}(r)$. Recall that we defined $\theta_m(r)$ in (2.1) through $\theta_0(r) = 0$ and $\theta_m(r) = \theta_{m-1}(r) + e^{\theta_{m-1}(r)}$ for $m \geqslant 1$.

Lemma 2.5.2. *Let* $|A| = n$*. If* $\operatorname{diam}(A) > 2\theta_{n-1}(r)$*, then there exists an exponential clustering of* A *with parameter* r *into* $k > 1$ *clusters.*

To prove the lemma, we will identify disks with radii of at most $\theta_{n-1}(r)$, which cover A. Although it is not required of an exponential clustering, the disks will be centered at elements of A. These disks will give rise to at least two clusters, since $\operatorname{diam}(A)$ exceeds $2\theta_{n-1}(r)$. The disks will be surrounded by large annuli which are empty of A, which will imply that the clusters are exponentially separated.

Proof of Lemma 2.5.2. For each $x \in A$ and $m \geqslant 1$, consider the annulus $\mathcal{A}_x(\theta_m) = D_x(\theta_m) \backslash D_x(\theta_{m-1})$. For each x, identify the smallest m such that $\mathcal{A}_x(\theta_m) \cap A$ is empty and call it m_x. Note that since $|A| = n$, m_x can be no more than n and hence $\theta_{m_x} \leq \theta_n$. Call the corresponding annulus $\mathcal{A}_x^*$, and denote $D_x^* = D_x(\theta_{m_x - 1})$. For convenience, we label the elements of A as $x_1, x_2, \ldots, x_n$.

For $x_i \in A$, we collect those disks $D_{x_j}^*$ which contain it as

$$\mathcal{E}(x_i) = \left\{ D_{x_j}^* : x_i \in D_{x_j}^*,\ 1 \leqslant j \leqslant n \right\}.$$

We observe that $\mathcal{E}(x_i)$ is always nonempty, as it contains $D_{x_i}^*$. Now observe that, for any two distinct $D_{x_j}^*, D_{x_\ell}^* \in \mathcal{E}(x_i)$, it must be that

$$D_{x_j}^* \cap A \subseteq D_{x_\ell}^* \cap A \quad \text{or} \quad D_{x_\ell}^* \cap A \subseteq D_{x_j}^* \cap A. \tag{2.56}$$

To see why, assume for the purpose of deriving a contradiction that each disk contains an element of A which the other does not. Without loss of generality, suppose $\theta_{m_{x_j}} \geqslant \theta_{m_{x_\ell}}$ and let $y_\ell \in (D_{x_\ell}^* \backslash D_{x_j}^*) \cap A$. Because each disk must contain x_i, we have $\|y_\ell - x_i\| \leqslant 2\theta_{m_{x_\ell} - 1}$ and $\|x_i - x_j\| \leqslant \theta_{m_{x_j} - 1}$. The triangle inequality implies

$$\|y_\ell - x_j\| \leqslant \theta_{m_{x_j} - 1} + 2\theta_{m_{x_\ell} - 1} \leqslant \theta_{m_{x_j}} \implies y_\ell \in D_{x_j}(\theta_{m_{x_j}}) \cap A.$$

By assumption, y_ℓ is not in $D_{x_j}(\theta_{m_{x_j} - 1}) \cap A$, so y_ℓ must be an element of $\mathcal{A}_{x_j}(\theta_{m_{x_j}}) \cap A$, which contradicts the construction of m_{x_j}.

By (2.56), we may totally order the elements of $\mathcal{E}(x_i)$ by inclusion of intersection with A. For each x_i, we select the element of $\mathcal{E}(x_i)$ which is greatest in this ordering. If we have not already established it as a cluster, we do so. After we have identified a cluster for each x_i, we discard those $D_{x_j}^*$ which were not selected for any x_i. For the remainder of the proof, we only refer to those $D_{x_j}^*$ which were established as clusters, and we relabel the x_i so that the clusters can be expressed as the collection $\left\{ D_{x_j}^* \right\}_{j=1}^k$, for some $1 \leqslant k \leqslant n$. We will show that k is strictly greater than one.

The collection of clusters contains all elements of A, and is associated to the collection of annuli $\{\mathcal{A}^*_{x_j}\}_{j=1}^k$, which contain no elements of A. We observe that, for some distinct x_j and x_ℓ, it may be that $\mathcal{A}^*_{x_j} \cap D^*_{x_\ell} \neq \emptyset$. However, because the annuli contain no elements of A, it must be that

$$\begin{aligned}\operatorname{dist}(D^*_{x_j} \cap A, D^*_{x_\ell} \cap A) &> \max\left\{\theta_{m_{x_j}} - \theta_{m_{x_{j-1}}}, \theta_{m_{x_\ell}} - \theta_{m_{x_{\ell-1}}}\right\} \\ &= \max\left\{e^{\theta_{m_{x_j}-1}}, e^{\theta_{m_{x_\ell}-1}}\right\} \\ &= \exp\left(\max\left\{\operatorname{rad}(D^*_{x_j}), \operatorname{rad}(D^*_{x_\ell})\right\}\right),\end{aligned}$$

where we use rad to indicate the radius of a disk. As $D^*_{x_j} \cap A \subseteq D^*_{x_j}$ for any x_j in question, we conclude the desired separation of clusters by setting $A^i = D^*_{x_i} \cap A$ for each $1 \leqslant i \leqslant k$. Furthermore, since $m_{x_j} \leqslant n$ for all j, $\operatorname{rad}(D^*_{x_j}) \leq \theta_{n-1}$ for all j. Since A is contained in the union of the clusters, if $\operatorname{diam}(A) > 2\theta_{n-1}$, then there must be at least two clusters. Lastly, as $m_{x_j} \geqslant 0$ for all j, $\operatorname{rad}(D^*_{x_j}) \geqslant r$ for all j. □

2.6 Estimates of the time of collapse

We proceed to prove the main collapse result, Theorem 2.1.2. As the proof requires several steps, we begin by discussing the organization of the section and introducing some key definitions. We avoid discussing the proof strategy in detail before making necessary definitions; an in-depth proof strategy is covered in Section 2.6.2.

Briefly, to estimate the time until the diameter of the configuration falls below a given function of n, we will perform exponential clustering and consider the more manageable task of (i) estimating the time until some cluster loses all of its particles to the other clusters. By iterating this estimate, we can (ii) control the time it takes for the clusters to consolidate into a single cluster. We will find that the surviving cluster has a diameter which is approximately the logarithm of the original diameter. Then, by repeatedly applying this estimate, we can (iii) control the time it takes for the diameter of the configuration to collapse.

The purpose of Section 2.6.1 is to wield (ii) in the form of Proposition 2.6.3 and prove Theorem 2.1.2, thus completing (iii). The remaining subsections are dedicated to proving the proposition. An overview of our strategy will be detailed in Section 2.6.2. In particular, we describe how the key harmonic measure estimate of Theorem 2.1.6 and the key escape probability estimate of Theorem 2.1.9 contribute to addressing (i). We then develop basic properties of cluster separation and explore the geometric consequences of timely cluster collapse in Section 2.6.3. Lastly, in Section 2.6.4, we prove a series of propositions which collectively control the timing of individual cluster collapse, culminating in the proof of Proposition 2.6.3.

Implicit in this discussion is a notion of "cluster" which persists over several steps of the dynamics. We now make this precise in terms of an exponential clustering. Recall that an exponential clustering $U_0 \mapsto \{U_0^i, x_i, \theta^{(i)}\}_{i=1}^k$ of U_0 is defined such that: $\{U_0^i\}_{i=1}^k$ partitions U_0; each U_0^i equals $U_0 \cap D_{x_i}(\theta^{(i)})$; and every distinct pair of clusters U_0^i, U_0^j satisfies $\operatorname{dist}(U_0^i, U_0^j) > e^{\max\{\theta^{(i)},\theta^{(j)}\}}$.

Definition 2.6.1. *Let U_0 have an exponential clustering $U_0 \mapsto \{U_0^i, x_i, \theta^{(i)}\}_{i=1}^k$. For any time $t \geqslant 1$, if U_t is obtained from t steps of the HAT dynamics from initial configuration U_0, then we recursively define $\{U_t^i\}_{i=1}^k$ as*

$$U_t^i = U_t \cap \left(U_{t-1}^i \cup \partial U_{t-1}^i\right). \tag{2.57}$$

In principle, after many steps of the dynamics, clusters defined according to (2.57) may intersect one another. However, in our application, clusters will be disjoint.

Definition 2.6.2 (Cluster collapse times)**.** *Suppose U_0 has the exponential clustering $U_0 \mapsto \{U_0^i\}_{i=1}^k$. We define the ℓ-cluster collapse time as*

$$\mathcal{T}_\ell = \inf\left\{t \geqslant 0 : U_t^{j_1} = U_t^{j_2} = \cdots = U_t^{j_\ell} = \emptyset, \text{ for } 1 \leqslant j_1 < j_2 < \cdots < j_\ell \leqslant k\right\}.$$

We adopt the convention that $\mathcal{T}_0 \equiv 0$.

By (2.57), if for some time t the cluster U_t^i is empty, then $U_{t'}^i$ is empty for all times $t' \geqslant t$. Consequently, the collapse times are ordered: $\mathcal{T}_1 \leqslant \mathcal{T}_2 \leqslant \cdots \leqslant \mathcal{T}_\ell$.

2.6.1 Proving Theorem 2.1.2

We now state the proposition to which most of the effort in this section is devoted and, assuming it, prove Theorem 2.1.2. We will denote by

- n, the number of elements of U_0;
- $\Phi(r)$, the inverse function of $\theta_n(r)$ for all $r \geqslant 0$ ($\theta_n(r)$ is an increasing function of $r \geqslant 0$ for every n); and
- $\mathcal{F}_t$, the sigma algebra generated by the initial configuration U_0, the first t activation sites $X_0, X_1, \ldots, X_{t-1}$, and the first t random walks $S^0, S^1, \ldots, S^{t-1}$, which accomplish the transport component of the dynamics.

We note that Φ is defined so that, if $r = \Phi(\operatorname{diam}(U_0))$, then $\operatorname{diam}(U_0) > 2\theta_{n-1}(r)$ and, by Lemma 2.5.2, exponential clustering of U_0 with parameter r will produce at least two clusters.

Proposition 2.6.3. *There is a constant c such that, if the diameter d of U_0 exceeds $\theta_{4n}(cn)$, then for any number of clusters k resulting from exponential clustering of U_0 with parameter $r = \Phi(d)$ and with $\delta = (2n)^{-4}$, we have*

$$\mathbf{P}_{U_0}\left(\mathcal{T}_{k-1} \leqslant (\log d)^{1+7\delta}\right) \geqslant 1 - \exp\left(-2nr^\delta\right). \tag{2.58}$$

In words, if U_0 has a diameter of d, it takes no more than $(\log d)^{1+o_n(1)}$ steps to observe the collapse of all but one cluster, with high probability. Because no cluster begins with a diameter greater than $\log d$ (by exponential clustering) and, as the diameter of a cluster increases at most linearly in time, the remaining cluster at time $\mathcal{T}_{k-1}$ has a diameter of no more than $(\log d)^{1+o_n(1)}$.

We will obtain Theorem 2.1.2 by repeatedly applying Proposition 2.6.3. We prove the theorem here, assuming the proposition, and then prove the proposition in the following subsections.

Our argument takes the form of Algorithm 1 and an analysis of its outputs. We organize the proof in this way because it more compact and direct than the alternative. In the context of a configuration with k_ℓ clusters, we will set $E_\ell = \left\{\mathcal{T}_{k_\ell - 1} \leqslant (\log d_\ell)^{1+7\delta}\right\}$. The variable $\mathcal{T}_{k_\ell-1}$ is the time it takes for the k_ℓ clusters to collapse into one cluster. The algorithm takes as input an initial configuration U with number of elements n and diameter d. It defines variables V_ℓ, d_ℓ, and r_ℓ, which are the configuration, diameter, and clustering parameter after $\ell - 1$ collapses. We set V_1 equal to U; d_1 equal to d; r_1 to be $\Phi(d)$; two counting variables, ℓ and $\mathcal{T}$, equal to one and zero; and an indicator called flag to zero.

During the ℓ^{th} "loop," the algorithm performs exponential clustering with parameter r_ℓ on configuration V_ℓ to obtain k_ℓ clusters and checks the occurrence of E_ℓ^c. If E_ℓ^c occurs, the algorithm sets flag to one and "breaks" out of the current loop, upon which the algorithm terminates. If E_ℓ occurs, the algorithm assigns values for the configuration $V_{\ell+1}$, diameter $d_{\ell+1}$, and clustering parameter $r_{\ell+1}$, which will be used in the next loop (if another loop is entered). Additionally, the algorithm updates $\mathcal{T}$ to account for the $\mathcal{T}_{k_\ell-1}$ steps of the HAT dynamics and updates ℓ to $\ell+1$ so that the next loop uses the new configuration, diameter, and clustering parameter.

The algorithm terminates if, at the beginning of the ℓ^{th} loop, the current HAT configuration V_ℓ has a diameter d_ℓ less than or equal to $\theta_{4n}(cn)$ or if, at any time, flag $= 1$, indicating the occurrence of $E_{\ell-1}^c$. If the algorithm terminates with flag $= 0$, then it must have terminated because $d_\ell \leqslant \theta_{4n}(cn)$ and therefore the value of $\mathcal{T}$ returned by the algorithm is at least $\mathcal{T}(\theta_{4n}(cn))$. If the algorithm terminates with flag $= 1$, then we are unable to provide a bound on $\mathcal{T}(\theta_{4n}(cn))$ in terms of $\mathcal{T}$.

Proof of Theorem 2.1.2. In the context of the preceding discussion, it suffices to show that, with a probability of at least $1 - e^{-n}$, the algorithm terminates with flag $= 0$ and $\mathcal{T}$ which satisfies

$$\mathcal{T} \leqslant (\log d)^{1+o_n(1)}. \tag{2.59}$$

By Proposition 2.6.3, letting $\delta = (2n)^{-4}$, we have $\mathbf{P}_{V_\ell}(E_\ell^c) \leqslant e^{-2nr_\ell^\delta}$ for any ℓ. Consequently, if N is the number of loops (i.e., the number of times the `while` statement executes) before the algorithm terminates, then the procedure terminates with flag $= 0$ unless $\cup_{\ell=1}^N E_\ell^c$ occurs, which has a probability no greater than

$$\mathbf{P}_U\left(\cup_{\ell=1}^N E_\ell^c\right) \leqslant \sum_{\ell=1}^N e^{-2nr_\ell^\delta} = e^{-2nr_N^\delta} \sum_{\ell=1}^N e^{-2n(r_\ell^\delta - r_N^\delta)}. \tag{2.60}$$

For all $\ell < N$, the event E_ℓ occurs which implies (by some algebra) that $d_{\ell+1}$ is less than $(\log d_\ell)^{1+8\delta}$. Using this bound and the fact that d_ℓ is at least $\theta_{4n}(cn)$, some simple but cumbersome algebra shows

$$r_\ell^\delta - r_{\ell+1}^\delta = \Phi(d_\ell)^\delta - \Phi(d_{\ell+1})^\delta \geqslant 1.$$

Algorithm 1

Input : Configuration U, number of elements n, diameter $d = \operatorname{diam}(U)$
Output: Indicator of failed collapse time estimate flag, total collapse time $\mathcal{T}$

```
/* Assign initial values of parameters.                                    */
V_1 ← U,   d_1 ← d,   r_1 ← Φ(d),   ℓ ← 1,   T ← 0,   and   flag ← 0
/* While the diameter is large and preceding collapse time estimates
   have succeeded ...                                                      */
while d_ℓ > θ_{4n}(cn) and flag = 0 do
    /* Perform exponential clustering.                                     */
    V_ℓ ↦_{r_ℓ} {U_0^i}_{i=1}^{k_ℓ}
    /* Try to observe the collapse of a cluster.                           */
    if E_ℓ^c occurs then
        flag ← 1   // If collapse takes too long, indicate this with flag and
                      terminate.
        break
    else
        V_{ℓ+1} ← U_{T_{k_ℓ−1}},   d_{ℓ+1} ← diam(V_{ℓ+1}),   r_{ℓ+1} ← Φ(d_{ℓ+1})   // Else, prepare
                      the next loop.
    end
    T ← T + T_{k_ℓ−1},   ℓ ← ℓ + 1              // Restart the loop with the new
                                                    configuration.
end
return flag, T
```

Using (2.60), this implies

$$\mathbf{P}_U\left(\cup_{\ell=1}^N E_\ell^c\right) \leqslant e^{-2nr_N^\delta} \sum_{\ell=0}^{N-1} e^{-2n\ell} \leqslant 2e^{-2nr_N^\delta} \leqslant e^{-n}.$$

This establishes that the algorithm terminates with flag $= 0$ with a probability of at least $1 - e^{-n}$. It remains to establish (2.59) when $\cap_{\ell=1}^N E_\ell$ occurs.

Again, because $d_{\ell+1}$ is less than $(\log d_\ell)^{1+8\delta}$ and by the lower bound on d_ℓ, the ratio of $\log d_{\ell+1}$ to $\log d_\ell$ is at most $1/2$. In fact, it is much smaller, but this suffices to establish

$$\mathcal{T} = \sum_{\ell=1}^N \mathcal{T}_{k_\ell - 1} \leqslant \sum_{\ell=1}^N (\log d_\ell)^{1+7\delta} \leqslant (\log d_1)^{1+7\delta} \sum_{\ell=0}^{N-1} 2^{-\ell} \leqslant (\log d_1)^{1+8\delta}.$$

Because $d_1 = d$, we conclude (2.59) with $8\delta \leqslant n^{-4}$ in the place of $o_n(1)$. □

For applications in Section 2.7, we extend Theorem 2.1.2 to a more general tail bound of $\mathcal{T}(\theta_{4n})$.

Corollary 2.6.4 (Corollary of Theorem 2.1.2)**.** *Let U be an n-element subset of $\mathbb{Z}^2$ with a diameter of d. There exists a universal positive constant c such that*

$$\mathbf{P}_U\left(\mathcal{T}(\theta_{4n}(cn)) > t(\log \max\{t,d\})^{1+o_n(1)}\right) \leqslant e^{-t} \tag{2.61}$$

for all $t \geqslant 1$. For the sake of concreteness, this is true with $2n^{-4}$ in the place of $o_n(1)$.

In the proof of the corollary, it will be convenient to have notation for the timescale of collapse after j failed collapses, starting from a diameter of d. Because diameter increases at most linearly in time, if the initial configuration has a diameter of d and collapse does not occur in the next $(\log d)^{1+o_n(1)}$ steps, then the diameter after this period of time is at most $d + (\log d)^{1+o_n(1)}$. In our next attempt to observe collapse, we would wait at most $\left(\log(d+(\log d)^{1+o_n(1)})\right)^{1+o_n(1)}$ steps. This discussion motivates the definition of the functions $g_j = g_j(d,\varepsilon)$ by

$$g_0 = (\log d)^{1+\varepsilon} \quad \text{and} \quad g_j = \Big(\log\big(d + \sum_{i=0}^{j-1} g_i\big)\Big)^{1+\varepsilon} \quad \forall\, j \geqslant 1.$$

We will use $t_j = t_j(d,\varepsilon)$ to denote the cumulative time $\sum_{i=0}^{j} g_i$.

Proof of Corollary 2.6.4. Let $\varepsilon = n^{-4}$ and use this as the ε parameter for the collapse timescales g_j and cumulative times t_j. Additionally, denote $\theta = \theta_{4n}(cn)$ for the constant c from Theorem 2.1.2 (this will also be the constant in the statement of the corollary). The bound (2.61) clearly holds when d is at most θ, so we assume $d \geqslant \theta$.

Because the diameter of U is d and as diameter grows at most linearly in time, conditionally on $F_j = \{\mathcal{T}(\theta) > t_j\}$, the diameter of U_{t_j} is at most $d + t_j$. Consequently, by the Markov property applied to time t_j, and by Theorem 2.1.2 (the diameter is at least θ) and the fact that $n \geqslant 1$, the conditional probability $\mathbf{P}_U(F_{j+1}|F_j)$ satisfies

$$\mathbf{P}_U(F_{j+1}|F_j) = \mathbf{E}_U\left[\mathbf{P}_{U_{t_j}}(\mathcal{T}(\theta) > g_{j+1}) \frac{\mathbf{1}_{F_j}}{\mathbf{P}_U(F_j)}\right] \leqslant e^{-1} \text{ for any } j \geqslant 0. \tag{2.62}$$

In fact, Theorem 2.1.2 implies that the inequality holds with e^{-n} in the place of e^{-1}, but this will make no difference to us.

If the cumulative time t_J is at most t for an integer J, then there are at least J consecutive collapse attempts which must fail in order for $\mathcal{T}(\theta_{4n})$ to exceed t. Hence, for any such J, by (2.62),

$$\mathbf{P}_U(\mathcal{T}(\theta) > t) \leqslant \prod_{i=0}^{J-1} \mathbf{P}_U(F_{i+1}|F_i) \leqslant e^{-J}. \tag{2.63}$$

We now bound below J to obtain a further upper bound of (2.63) in terms of t. We have

$$J \geqslant \frac{t_J - g_J}{g_J} \geqslant \frac{t - 2g_J}{g_J} \geqslant \frac{t - 2\big(\log(d+t)\big)^{1+\varepsilon}}{\big(\log(d+t)\big)^{1+\varepsilon}}. \tag{2.64}$$

The first inequality holds because t_J is at most $(J+1)g_J$; the second because t_J is within g_J of t; and the third because, when the cumulative time t_J is at most t, the corresponding collapse timescale g_J is at most $\big(\log(d+t)\big)^{1+\varepsilon}$. Applying (2.64) to (2.63), we find

$$\mathbf{P}_U(\mathcal{T}(\theta) > t) \leqslant e^{-\frac{t-2(\log(d+t))^{1+\varepsilon}}{(\log(d+t))^{1+\varepsilon}}}.$$

We obtain (2.61) by replacing t with $t\big(\log \max\{t,d\}\big)^{1+2\varepsilon}$ in the preceding inequality and noting that, because $d \geqslant \theta$ and $\varepsilon < 1$, the resulting bound is at most e^{-t}. □

2.6.2 Proof strategy for Proposition 2.6.3

We turn our attention to the proof of Proposition 2.6.3, which finds a high-probability bound on the time it takes for all but one cluster to collapse. Heuristically, if there are only two clusters, separated by a distance ρ_1, then one of the clusters will lose all its particles to the other cluster in $\log \rho_1$ steps (up to factors depending on n), due to the harmonic measure and escape probability lower bounds of Theorems 2.1.6 and 2.1.9. This heuristic suggests that, among k clusters, we should observe the collapse of *some* cluster on a timescale which depends on the smallest separation between any two of the k clusters. Similarly, at the time the ℓ^{th} cluster collapses, if the least separation among the remaining clusters is $\rho_{\ell+1}$, then we expect to wait $\log \rho_{\ell+1}$ steps for the $(\ell+1)^{\text{st}}$ collapse.

If the timescale of collapse is small relative to the separation between clusters, the pairwise separation and diameters of clusters cannot appreciably change while collapse occurs. In particular, the separation between any two clusters will not significantly exceed the initial diameter d of the configuration, which suggests an overall bound of order $(\log d)^{1+o_n(1)}$ steps for all but one cluster to collapse, where the $o_n(1)$ factor accounts for various n-dependent factors. This is the upper bound we establish.

We now highlight some key aspects of the proof.

Expiry time

As described above, over the timescale typical of collapse, the diameters and separation of clusters will not change appreciably. Because these quantities determine the probability with which the least separated cluster loses a particle, we will be able to obtain estimates of this probability which hold uniformly from the time $\mathcal{T}_{\ell-1}$ of the $(\ell-1)^{\text{st}}$ cluster collapse and until the next time $\mathcal{T}_\ell$ that some cluster collapses, unless $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ is atypically large. Indeed, if $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ is as large as the separation ρ_ℓ of the least separated cluster at time $\mathcal{T}_{\ell-1}$, then two clusters may intersect. We avoid this by defining a $\mathcal{F}_{\mathcal{T}_{\ell-1}}$-measurable *expiry time* $\mathfrak{t}_\ell$ (which will effectively be $(\log \rho_\ell)^2$) and restricting our estimates to the interval from $\mathcal{T}_{\ell-1}$ to the minimum of $\mathcal{T}_{\ell-1} + \mathfrak{t}_\ell$ and $\mathcal{T}_\ell$. An expiry time of $(\log \rho_\ell)^2$ is short enough that the relative separation of clusters will not change significantly before it, but long enough so that some cluster will collapse before it with overwhelming probability.

Midway point

From time $\mathcal{T}_{\ell-1}$ to time $\mathcal{T}_\ell$ or until expiry, we will track activated particles which reach a circle of radius $\frac{1}{2}\rho_\ell$ surrounding one of the least separated clusters, which we call the *watched* cluster. We will use this circle, called the *midway point*, to organize our argument with the following three estimates, which will hold uniformly over this interval of time (Figure 2.8).

1. Activated particles which reach the midway point deposit at the watched cluster with a probability of at most 0.51.
2. With a probability of at least $(\log \rho_\ell)^{-1-o_n(1)}$, the activated particle reaches the midway point.
3. Conditionally on the activated particle reaching the midway point, the probability that it originated at the watched cluster is at least $(\log\log \rho_\ell)^{-1}$.

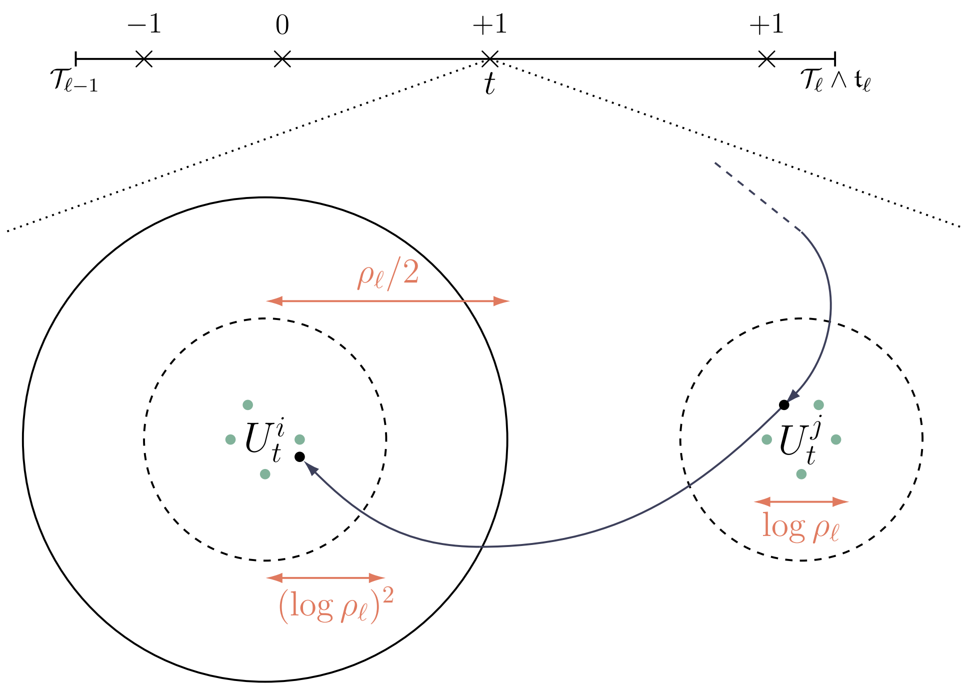

Figure 2.8: Setting of the proof of Proposition 2.6.3. Least separated clusters i and j (cluster i is the watched cluster), each with a diameter of approximately $\log \rho_\ell$, are separated by a distance ρ_ℓ at time $\mathcal{T}_{\ell-1}$. The diameters of the clusters grow at most linearly in time, so over approximately $(\log \rho_\ell)^2$ steps, the clusters remain within the dotted circles. Crosses on the timeline indicate times before collapse and expiry at which an activated particle reaches the midway point (solid circle). At these times, the number of particles in the watched cluster may remain the same or increase or decrease by one (indicated by $0, \pm 1$ above the crosses). At time t, the watched cluster gains a particle from cluster j.

To explain the third estimate, we make two observations. First, consider a cluster j separated from the watched cluster by a distance of ρ. In the relevant context, cluster j will essentially be exponentially separated, so its diameter will be at most $\log \rho$. Consequently, a particle activated at cluster j reaches the midway point with a probability of at most $\frac{\log\log\rho}{\log\rho}$. Because this probability is decreasing in ρ and because $\rho \geqslant \rho_\ell$, $\frac{\log\log\rho_\ell}{\log\rho_\ell}$ further bounds above it. Second, the probability that a particle activated at the watched cluster reaches the midway point is at least $(\log\rho_\ell)^{-1}$, up to a factor depending on n. Combining these two observations with Bayes's rule, a particle which reaches the midway point was activated at the watched cluster with a probability of at least $(\log\log\rho_\ell)^{-1}$, up to an n-dependent factor.

Coupling with random walk

Each time an activated particle reaches the midway point, there is a chance of at least $(\log\log\rho_\ell)^{-1}$ up to an n-dependent factor that the particle originated at the watched cluster and will ultimately deposit at another cluster. When this occurs, the watched cluster loses a particle. Alternatively, the activated particle may return to its cluster of origin—in which case the watched cluster retains its particles—or it deposits at the watched cluster, having originated at a different one—in which case the watched cluster gains a particle (Figure 2.8).

We will couple the number of elements in the watched cluster with a lazy, one-dimensional random walk, which will never exceed n and never hit zero before the size of the watched cluster does. It will take no more than $(\log\log\rho_\ell)^n$ instances of the activated particle reaching the midway point, for the random walk to make n consecutive down-steps. This is a coarse estimate; with more effort, we could improve the n-dependence of this term, but it would not qualitatively change the result. On a high probability event, ρ_ℓ will be sufficiently large to ensure that $(\log\log\rho_\ell)^n = (\log\rho_\ell)^{o_n(1)}$. Then, because it will typically take no more than $(\log\rho_\ell)^{1+o_n(1)}$ steps to observe a visit to the midway point, we will wait a number of steps on the same order to observe the collapse of a cluster.

2.6.3 Basic properties of clusters and collapse times

We will work in the following setting.

- For brevity, if we write θ_m with no parenthetical argument, we will mean $\theta_m(\gamma n)$ for the constant γ given by
$$\gamma = 18\max\{c_1, c_2^{-1}\} + 36, \tag{2.65}$$
where c_1 and c_2 are the constants in Theorems 2.1.6 and 2.1.9. Any constant larger than γ would also work in its place.

- U_0 has $n \geqslant 2$ elements and $\operatorname{diam}(U_0)$ is at least θ_{4n}.

- The clustering parameter r equals $\Phi(\operatorname{diam}(U_0))$, where we continue to denote by $\Phi(\cdot)$ the inverse function of $\theta_n(\cdot)$. In particular, r satisfies
$$r \geqslant \Phi(\theta_{4n}) = \theta_{3n} \geqslant e^n. \tag{2.66}$$

- We will assume that the initial configuration is exponentially clustered with parameter r as $U_0 \mapsto_r \{U_0^i, x_i, \theta^{(i)}\}_{i=1}^k$. In particular, we assume that clustering produces k clusters. We note that the choice of r guarantees $\operatorname{diam}(U_0) > 2\theta_{n-1}(r)$ which, by Lemma 2.5.2, guarantees that $k > 1$.
- We denote a generic element of $\{1, 2, \dots, k-1\}$ by ℓ.

Properties of cluster separation and diameter

We will use the following terms to describe the separation of clusters.

Definition 2.6.5. *We define pairwise cluster separation and the least separation by*

$$\operatorname{sep}(U_t^i) = \min_{j \neq i} \operatorname{dist}(U_t^i, U_t^j) \quad \textit{and} \quad \operatorname{sep}(U_t) = \min_i \operatorname{sep}(U_t^i).$$

(By convention, the distance to an empty set is ∞, so the separation of a cluster is ∞ at all times following its collapse.) If U_t^i satisfies $\operatorname{sep}(U_t^i) = \operatorname{sep}(U_t)$, then we say that U_t^i is least separated. Whenever there are at least two clusters, at least two clusters will be least separated. The least separation at a cluster collapse time will be an important quantity; we will denote it by

$$\rho_\ell = \operatorname{sep}(U_{\mathcal{T}_{\ell-1}}).$$

Next, we introduce the *expiry time* $\mathfrak{t}_\ell$ and the *truncated collapse time* $\mathcal{T}_\ell^-$. As discussed in Section 2.6.2, if at time $\mathcal{T}_{\ell-1}$ the least separation is ρ_ℓ, then we will obtain a lower bound on the probability that a least separated cluster loses a particle, which holds uniformly from time $\mathcal{T}_{\ell-1}$ to the first of $\mathcal{T}_{\ell-1} + \mathfrak{t}_\ell$ and $\mathcal{T}_\ell - 1$ (i.e., the time immediately preceding the ℓ^{th} collapse), which we call the truncated collapse time, $\mathcal{T}_\ell^-$. Here, $\mathfrak{t}_\ell$ is an $\mathcal{F}_{\mathcal{T}_{\ell-1}}$-measurable random variable which will effectively be $(\log \rho_\ell)^2$. It will be rare for $\mathcal{T}_\ell$ to exceed $\mathcal{T}_{\ell-1} + \mathfrak{t}_\ell$, so $\mathcal{T}_\ell^-$ can be thought of as $\mathcal{T}_\ell - 1$.

Definition 2.6.6. *Given the $\mathcal{F}_{\mathcal{T}_{\ell-1}}$ data (in particular ρ_ℓ and $\mathcal{T}_{\ell-1}$), we define the expiry time $\mathfrak{t}_\ell$ to be*

$$\mathfrak{t}_\ell = (\log \rho_\ell)^2 - 4 \log(\rho_\ell + \mathcal{T}_{\ell-1}) - \mathcal{T}_{\ell-1}.$$

We emphasize that $\mathfrak{t}_\ell$ should be thought of as $(\log \rho_\ell)^2$; the other terms will be much smaller and are included to simplify calculations which follow. Additionally, we define the truncated ℓ^{th} cluster collapse time to be

$$\mathcal{T}_\ell^- = (\mathcal{T}_{\ell-1} + \mathfrak{t}_\ell) \wedge (\mathcal{T}_\ell - 1).$$

Cluster diameter and separation have complementary behavior in the sense that diameter increases at most linearly in time but may decrease abruptly, while separation decreases at most linearly in time but may increase abruptly. We will not need a bound on decrease in diameter; we express the other properties in the following lemma.

Lemma 2.6.7. *Cluster diameter and separation obey the following properties.*

1. *Cluster diameter increases by at most one each step:*

$$\operatorname{diam}(U_t^i) \leqslant \operatorname{diam}(U_{t-1}^i) + 1. \tag{2.67}$$

2. *Cluster separation decreases by at most one each step:*

$$\operatorname{dist}(U_t^i, U_t^j) \geqslant \operatorname{dist}(U_{t-1}^i, U_{t-1}^j) - 1 \quad \textit{and} \quad \operatorname{sep}(U_t^i) \geqslant \operatorname{sep}(U_{t-1}^i) - 1. \tag{2.68}$$

3. *For any two times s and t satisfying $\mathcal{T}_{\ell-1} \leqslant s < t < \mathcal{T}_\ell$ and any two clusters i and j:*

$$\operatorname{dist}(U_t^i, U_t^j) \leqslant \operatorname{dist}(U_s^i, U_s^j) + \operatorname{diam}(U_s^i) + \operatorname{diam}(U_s^j) + (t-s).$$

Proof. The first two properties are obvious; we prove the third. Let i, j label two clusters which are nonempty at time $\mathcal{T}_{\ell-1}$ and let s, t satisfy the hypotheses. If there are m_i activations at the i^{th} cluster from time s to time t, then for any x' in U_t^i, there is an x in U_s^i such that $\|x - x'\| \leqslant m_i$. The same is true of any y' in the j^{th} cluster with m_j in the place of m_i. Since the sum of m_i and m_j is at most $t - s$, two uses of the triangle inequality give

$$\operatorname{dist}(U_t^i, U_t^j) \leqslant \max_{x' \in U_t^i,\, y' \in U_t^j} \|x' - y'\| \leqslant \max_{x \in U_s^i,\, y \in U_s^j} \|x - y\| + t - s.$$

This implies property (3) because, by two more uses of the triangle inequality,

$$\max_{x \in U_s^i,\, y \in U_s^j} \|x - y\| \leqslant \operatorname{dist}(U_s^i, U_s^j) + \operatorname{diam}(U_s^i) + \operatorname{diam}(U_s^j).$$

□

Consequences of timely collapse

If clusters collapse before their expiry times—i.e., if the event

$$\mathsf{Timely}(\ell) = \cap_{m=1}^{\ell} \{\mathcal{T}_m - \mathcal{T}_{m-1} \leqslant \mathfrak{t}_m\}$$

occurs—then we will be able to control the separation (Lemma 2.6.8) and diameters (Lemma 2.6.10) of the clusters by combining the initial exponential separation of the clusters with the properties of Lemma 2.6.7.

The next lemma states that, when cluster collapses are timely, cluster separation decreases little. To state it, we recall that $\operatorname{sep}(U_t^i)$ is the distance between U_t^i and the nearest other cluster, and that ρ_ℓ is the least of these distances among all pairs of distinct clusters at time $\mathcal{T}_{\ell-1}$. In particular, $\operatorname{sep}(U_{\mathcal{T}_{\ell-1}}^i) \geqslant \rho_\ell$ for each i.

Lemma 2.6.8. *For any cluster i, when $\mathsf{Timely}(\ell-1)$ occurs and when t is at most $\mathcal{T}_\ell^-$,*

$$\operatorname{sep}\left(U_t^i\right) \geqslant (1 - e^{-n}) \operatorname{sep}\left(U_{\mathcal{T}_{\ell-1}}^i\right). \tag{2.69}$$

Additionally, when $\mathsf{Timely}(\ell-1)$ occurs,

$$\rho_\ell \geqslant \tfrac{1}{2}\rho_1 \geqslant e^{\theta_{2n}}. \tag{2.70}$$

The factor of $1 - e^{-n}$ in (2.69) does not have special significance; other factors of $1 - o_n(1)$ would work, too. (2.69) and the first inequality in (2.70) are consequences of the fact (2.68) that separation decreases at most linearly in time and, when $\mathsf{Timely}(\ell - 1)$ occurs, $\mathcal{T}_{\ell-1}$ is small relative to the separation of the remaining clusters. The second inequality in (2.70) follows from our choice of r in (2.66).

Proof Lemma 2.6.8. We will prove (2.69) by induction, using the fact that separation decreases at most linearly in time (2.68) and that (by the definition of $\mathcal{T}_\ell^-$) at most $\mathfrak{t}_\ell$ steps elapse between $\mathcal{T}_{\ell-1}$ and $\mathcal{T}_\ell^-$.

For the base case, take $\ell = 1$. Suppose cluster i is nonempty at time $\mathcal{T}_{\ell-1}$. We must show that, when $t \leqslant \mathcal{T}_1^-$,

$$\operatorname{sep}\left(U_t^i\right) \geqslant (1 - e^{-n}) \operatorname{sep}\left(U_0^i\right).$$

Because separation decreases at most linearly in time (2.68) and because $t \leqslant \mathcal{T}_1^-$,

$$\operatorname{sep}(U_t^i) \geqslant \operatorname{sep}(U_0^i) - t \geqslant \operatorname{sep}(U_0^i) - \mathcal{T}_1^-.$$

This implies (2.69) for $\ell = 1$ because

$$\operatorname{sep}(U_0^i) - \mathcal{T}_1^- \geqslant \left(1 - \tfrac{(\log \rho_1)^2}{\rho_1}\right) \operatorname{sep}(U_0^i) \geqslant (1 - e^{2n} e^{-e^n}) \operatorname{sep}(U_0^i) \geqslant (1 - e^{-n}) \operatorname{sep}(U_0^i).$$

The first inequality is a consequence of the definitions of $\mathcal{T}_1^-$, $\mathfrak{t}_1$, and ρ_1, which imply $\mathcal{T}_1^- \leqslant \mathfrak{t}_1 \leqslant (\log \rho_1)^2$ and $\operatorname{sep}(U_0^i) \geqslant \rho_1$. Since the ratio of $(\log \rho_1)^2$ to ρ_1 decreases as ρ_1 increases, the second inequality follows from the bound $\rho_1 \geqslant e^{e^n}$, which is implied by the fact that U_0 satisfies the exponential separation property (2.55) with parameter $r \geqslant e^n$ (2.66). The third inequality is due to the fact that $e^n \geqslant 3n$ when $n \geqslant 2$.

The argument for $\ell > 1$ is similar. Assume (2.69) holds for $\ell - 1$. We have

$$\operatorname{sep}(U_t^i) \geqslant \operatorname{sep}(U_{\mathcal{T}_{\ell-1}}^i) - (t - \mathcal{T}_{\ell-1}) \geqslant \operatorname{sep}(U_{\mathcal{T}_{\ell-1}}^i) - \mathfrak{t}_\ell \geqslant \left(1 - \tfrac{(\log \rho_\ell)^2}{\rho_\ell}\right) \operatorname{sep}(U_{\mathcal{T}_{\ell-1}}^i). \tag{2.71}$$

The first inequality is implied by (2.68). The second inequality follows from the definitions of $\mathcal{T}_\ell^-$ and $\mathfrak{t}_\ell$, which imply $\mathcal{T}_\ell^- - \mathcal{T}_{\ell-1} \leqslant \mathfrak{t}_\ell \leqslant (\log \rho_\ell)^2$, and $t \leqslant \mathcal{T}_\ell^-$. The third inequality is due to the same upper bound on $\mathfrak{t}_\ell$ and the fact that $\operatorname{sep}(U_{\mathcal{T}_{\ell-1}}^i) \geqslant \rho_\ell$ by definition.

We will bound below ρ_ℓ to complete the induction step with (2.71), because the ratio of $(\log \rho_\ell)^2$ to ρ_ℓ decreases as ρ_ℓ increases. Specifically, we will prove (2.70). By definition, when $\mathsf{Timely}(\ell - 1)$ occurs, so too does $\mathsf{Timely}(\ell - 2)$. Accordingly, the induction hypothesis applies and we apply it $\ell - 1$ times:

$$\rho_{\ell-1} = \min_i \operatorname{sep}(U_{\mathcal{T}_{\ell-2}}^i) \geqslant (1 - e^{-n})^{\ell-1} \min_i \operatorname{sep}(U_0^i) = (1 - e^{-n})^{\ell-1} \rho_1.$$

The equalities follow from the definitions of $\rho_{\ell-1}$ and ρ_1. We also have

$$\rho_\ell \geqslant \rho_{\ell-1} - \mathfrak{t}_{\ell-1} \geqslant \left(1 - \tfrac{(\log \rho_{\ell-1})^2}{\rho_{\ell-1}}\right) \rho_{\ell-1} \geqslant (1 - e^{-n}) \rho_{\ell-1}.$$

The first inequality is due to (2.68) and the fact that at most $\mathfrak{t}_{\ell-1}$ steps elapse between $\mathcal{T}_{\ell-2}$ and $\mathcal{T}_{\ell-1}$ when $\mathsf{Timely}(\ell-1)$ occurs. The second inequality is due to $\mathfrak{t}_{\ell-1} \leqslant (\log \rho_{\ell-1})^2$ and the third is due to the fact that the ratio of $(\log \rho_{\ell-1})^2$ to $\rho_{\ell-1}$ decreases as $\rho_{\ell-1}$ increases.

Combining the two preceding displays and then using the fact that $\ell \leqslant n$ and $\rho_1 \geqslant e^{e^n}$, and the inequality $(1+x)^r \geqslant 1+rx$, which holds for $x > -1$ and $r > 1$, we find

$$\rho_\ell \geqslant (1-e^{-n})^\ell \rho_1 \geqslant (1-ne^{-n})\rho_1.$$

Because $ne^{-n} \leqslant \frac{1}{2}$ when $n \geqslant 2$, this proves $\rho_\ell \geqslant \frac{1}{2}\rho_1$, which is the first inequality of (2.70). To prove the second inequality in (2.70), we note that ρ_1 is at least θ_{3n} by (2.66).

We now apply $\rho_\ell \geqslant \frac{1}{2}\rho_1$ to the ratio in (2.71):

$$\frac{(\log \rho_\ell)^2}{\rho_\ell} \leqslant \frac{2(\log \rho_1)^2}{\rho_1} \leqslant e^{-n}.$$

The second inequality uses $\rho_1 \geqslant e^{e^n}$. We complete the induction step, proving (2.69), by substituting this bound into (2.71). □

When cluster collapses are timely, $\mathcal{T}_\ell^-$ is at most $(\log \rho_\ell)^2$, up to a factor depending on n.

Lemma 2.6.9. *When* $\mathsf{Timely}(\ell-1)$ *occurs,*

$$\mathcal{T}_\ell^- \leqslant 2n(\log \rho_\ell)^2. \tag{2.72}$$

The factor of 2 is for brevity; it could be replaced by $1+o_n(1)$. The lower bound on the least separation ρ_ℓ at time $\mathcal{T}_{\ell-1}$ in (2.70) indicates that, while ρ_ℓ may be much larger than ρ_1, it is at least half of ρ_1. Since the expiry time $\mathfrak{t}_\ell$ is approximately $(\log \rho_\ell)^2$, the truncated collapse time $\mathcal{T}_\ell^-$—which is at most the sum of the first ℓ expiry times—should be of the same order, up to a factor depending on ℓ (which we will replace with n since $\ell \leqslant n$).

Proof of Lemma 2.6.9. We write

$$\mathcal{T}_\ell^- = \mathcal{T}_\ell^- - \mathcal{T}_{\ell-1} + \sum_{m=1}^{\ell-1} (\mathcal{T}_m - \mathcal{T}_{m-1}) \leqslant \sum_{m=1}^{\ell} \mathfrak{t}_m \leqslant \sum_{m=1}^{\ell} (\log \rho_m)^2.$$

The first inequality follows from the fact that, when $\mathsf{Timely}(\ell-1)$ occurs, $\mathcal{T}_m - \mathcal{T}_{m-1} \leqslant \mathfrak{t}_m$ for $m \leqslant \ell-1$, and $\mathcal{T}_\ell^- - \mathcal{T}_{\ell-1} \leqslant \mathfrak{t}_\ell$. The second inequality holds because $\mathfrak{t}_m \leqslant (\log \rho_m)^2$ by definition.

Next, assume w.l.o.g. that cluster i is least separated at time $\mathcal{T}_{\ell-1}$, meaning $\rho_\ell = \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})$. Since $\mathsf{Timely}(\ell-1)$ occurs, Lemma 2.6.8 applies and with its repeated use we establish (2.72):

$$\sum_{m=1}^{\ell} (\log \rho_m)^2 \leqslant \sum_{m=1}^{\ell} \big(\log \mathrm{sep}(U^i_{\mathcal{T}_{m-1}})\big)^2 \leqslant \sum_{m=1}^{\ell} \Big(\log\big((1+\tfrac{e^{-n}}{1-e^{-n}})^{\ell-m}\rho_\ell\big)\Big)^2 \leqslant \ell(\log(2\rho_\ell))^2 \leqslant 2n(\log \rho_\ell)^2.$$

The first inequality is due to the definition of ρ_m as the least separation at time $\mathcal{T}_{m-1}$. This step is helpful because it replaces each summand with one concerning the i^{th} cluster. The second inequality holds because, by Lemma 2.6.8,

$$\rho_\ell = \text{sep}(U^i_{\mathcal{T}_{\ell-1}}) \geqslant (1-e^{-n})^{\ell-m}\text{sep}(U^i_{\mathcal{T}_{m-1}}) \implies \text{sep}(U^i_{\mathcal{T}_{m-1}}) \leqslant \big(1+\tfrac{e^{-n}}{1-e^{-n}}\big)^{\ell-m}\rho_\ell.$$

The third inequality follows from $\ell \leqslant n$ and $(1+\frac{e^{-n}}{1-e^{-n}})^n \leqslant 2$ when $n \geqslant 2$. The fourth inequality is due to $\ell \leqslant n$ and $\rho_\ell \geqslant e^{\theta_{2n}}$ from (2.70). (The factor of 2 could be replaced by $1+o_n(1)$.) Combining the displays proves (2.72). □

When cluster collapse is timely, we can bound cluster diameter at time $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$ from above, in terms of its separation at time $\mathcal{T}_{\ell-1}$ or at time t.

Lemma 2.6.10. *For any cluster i, when $\mathsf{Timely}(\ell-1)$ occurs and when t is at most $\mathcal{T}_\ell^-$,*

$$\text{diam}(U^i_t) \leqslant \big(\log \text{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2. \tag{2.73}$$

Additionally, if x_i is the center of the i^{th} cluster resulting from the exponential clustering of U_0, then when t is at most $\mathcal{T}_\ell^-$,

$$U^i_t \subseteq D_{x_i}\Big(\big(\log \text{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2\Big) \quad \textit{and} \quad U_t \backslash U^i_t \subseteq D_{x_i}\big(0.99\,\text{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^c. \tag{2.74}$$

Lastly, if i, j label any two clusters which are nonempty at time $\mathcal{T}_{\ell-1}$, then when t is at most $\mathcal{T}_\ell^-$,

$$\frac{\log \text{diam}(U^i_t)}{\log \text{dist}(U^i_t, U^j_t)} \leqslant \frac{2.1 \log\log \rho_\ell}{\log \rho_\ell}. \tag{2.75}$$

We use factors of 0.99 and 2.1 for concreteness; they could be replaced by $1-o_n(1)$ and $2+o_n(1)$. Lemma 2.6.10 implements the diameter and separation bounds we discussed in Section 2.6.2 (there, we used ρ in the place of $\text{sep}(U^i_{\mathcal{T}_{\ell-1}})$). Before proving the lemma, we discuss some heuristics which explain (2.73) through (2.75).

If a cluster is initially separated by a distance ρ, then it has a diameter of at most $2\log\rho$ by (2.55), which is negligible relative to an expiry time of order $(\log\rho)^2$. Diameter increases at most linearly in time by (2.67), so when cluster collapse is timely the diameter of U^i_t is at most $\big(\log \text{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2$. In fact, the definition of the expiry time subtracts the lower order terms, so the bound will be exactly this quantity. Moreover, since $(\log\rho)^2$ is negligible relative to the separation ρ, and as separation decreases at most linearly in time by (2.68), the separation of U^i_t should be at least $\text{sep}(U^i_{\mathcal{T}_{\ell-1}})$, up to a constant which is nearly one.

Combining these bounds on diameter and separation suggests that the ratio of the diameter of U^i_t to its separation from another cluster U^j_t should be roughly the ratio of $\big(\log \text{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2$ to $\text{sep}(U^i_{\mathcal{T}_{\ell-1}})$, up to a constant factor. Because this ratio is decreasing in the separation (for separation exceeding, say, e^2) and because the separation at time $\mathcal{T}_{\ell-1}$ is at least ρ_ℓ, the ratio $\frac{(\log\rho_\ell)^2}{\rho_\ell}$ should provide a further upper bound, again up to a constant factor. These three observations correspond to (2.73) through (2.75).

Proof of Lemma 2.6.10. We first address (2.73) and use it to prove (2.74). We then combine the results to prove (2.75). We bound $\mathrm{diam}(U_t^i)$ from above in terms of $\mathrm{diam}(U_0^i)$ as

$$\mathrm{diam}(U_t^i) \leqslant \mathrm{diam}(U_0^i) + \mathcal{T}_\ell^- \leqslant \mathrm{diam}(U_0^i) + \mathcal{T}_{\ell-1} + \mathfrak{t}_\ell. \tag{2.76}$$

The first inequality holds because diameter grows at most linearly in time (2.67) and because t is at most $\mathcal{T}_\ell^-$. The second inequality is due to the definition of $\mathcal{T}_\ell^-$. We then bound $\mathrm{diam}(U_0^i)$ from above in terms of $\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})$ as

$$\mathrm{diam}(U_0^i) \leqslant 2\log \mathrm{sep}(U_0^i) \leqslant 2\log\big(\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) + \mathcal{T}_{\ell-1}\big). \tag{2.77}$$

The exponential separation property (2.55) implies the first inequality and (2.68) implies the second.

Combining the two preceding displays, we find

$$\mathrm{diam}(U_t^i) \leqslant 2\log\big(\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) + \mathcal{T}_{\ell-1}\big) + \mathcal{T}_{\ell-1} + \mathfrak{t}_\ell.$$

Substituting the definition of $\mathfrak{t}_\ell$, the right-hand side becomes

$$2\log\big(\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) + \mathcal{T}_{\ell-1}\big) + (\log\rho_\ell)^2 - 4\log(\rho_\ell + \mathcal{T}_{\ell-1}).$$

By definition, ρ_ℓ is the least separation at time $\mathcal{T}_{\ell-1}$, so we can further bound $\mathrm{diam}(U_t^i)$ from above by substituting $\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})$ for ρ_ℓ:

$$\mathrm{diam}(U_t^i) \leqslant \big(\log \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2 - 2\log\big(\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) + \mathcal{T}_{\ell-1}\big). \tag{2.78}$$

Dropping the negative term gives (2.73).

We turn our attention to (2.74). To obtain the first inclusion of (2.74), we observe that U_t^i is contained in the disk $D_{x_i}\big(\mathrm{diam}(U_0^i) + \mathcal{T}_{\ell-1} + \mathfrak{t}_\ell\big)$, the radius of which is the quantity in (2.76) that we ultimately bounded above by $\big(\log \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2$.

Concerning the second inclusion of (2.74), we observe that for any y in $U_t \setminus U_t^i$, there is some y' in $U_{\mathcal{T}_{\ell-1}} \setminus U^i_{\mathcal{T}_{\ell-1}}$ such that $\|y - y'\|$ is at most $\mathfrak{t}_\ell$, because t is at most $\mathcal{T}_\ell^-$. By the triangle inequality and the bound on $\|y - y'\|$,

$$\|x_i - y\| \geqslant \|x_i - y'\| - \|y - y'\| \geqslant \|x_i - y'\| - \mathfrak{t}_\ell.$$

Next, we observe that the distance between x_i and y' is at least

$$\|x_i - y'\| \geqslant \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) - \mathrm{diam}(U_0^i).$$

The two preceding displays and (2.77) imply

$$\|x_i - y\| \geqslant \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) - 2\log\big(\mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}}) + \mathcal{T}_{\ell-1}\big) - \mathfrak{t}_\ell. \tag{2.79}$$

We continue (2.79) with

$$\|x_i - y\| \geqslant \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}) - \big(\log \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2 \geqslant \big(1 - \tfrac{(\log \rho_\ell)^2}{\rho_\ell}\big) \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}) \geqslant 0.99 \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}). \quad (2.80)$$

The first inequality follows from substituting the definition of $\mathfrak{t}_\ell$ into (2.79) and from $\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}) \geqslant \rho_\ell$. The second inequality holds because the ratio of $\big(\log \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2$ to $\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})$ decreases as $\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})$ increases and because $\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}) \geqslant \rho_\ell$. The fact (2.70) that ρ_ℓ is at least $e^{\theta_{2n}}$ when $\mathsf{Timely}(\ell-1)$ occurs implies that the ratio in (2.80) is at most 0.01, which justifies the third inequality. (2.80) proves the second inclusion of (2.74).

Lastly, to address (2.75), we observe that any element x in U^i_t is within a distance $\big(\log \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2$ of x_i by (2.78). So, by (2.80) and simplifying with $\rho_\ell \geqslant e^{\theta_{2n}}$, the distance between U^i_t and U^j_t is at least

$$\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}) - 2\big(\log \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2 \geqslant 0.99 \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}}).$$

Combining this with (2.73), and then using the fact that $\operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})$ is at least ρ_ℓ, gives

$$\frac{\log \operatorname{diam}(U^i_t)}{\log \operatorname{dist}(U^i_t, U^j_t)} \leqslant \frac{2 \log\log \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})}{\log\big(0.99 \operatorname{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)} \leqslant \frac{2.1 \log\log \rho_\ell}{\log \rho_\ell}.$$

□

The next lemma concerns two properties of the midway point introduced in Section 2.6.2. We recall that the midway point (for the period beginning at time $\mathcal{T}_{\ell-1}$ and continuing until $\mathcal{T}_\ell^-$) is a circle of radius $\frac{1}{2}\rho_\ell$, centered on the center x_i (given by the initial exponential clustering of U_0) of a cluster i which is least separated at time $\mathcal{T}_{\ell-1}$. The first property is the simple fact that, when collapse is timely, the midway point separates U^i_t from the rest of U_t until time $\mathcal{T}_\ell^-$. This is clear because the midway point is a distance of $\frac{1}{2}\rho_\ell$ from $U_{\mathcal{T}_{\ell-1}}$ and $\mathcal{T}_\ell^-$ is no more than $(\log \rho_\ell)^2$ steps away from $\mathcal{T}_{\ell-1}$ when collapse is timely. The second property is the fact that a random walk from anywhere in the midway point hits U^i_t before the rest of U_t (excluding the site of the activated particle) with a probability of at most 0.51, which is reasonable because the random walk begins effectively halfway between U^i_t and the rest of U_t. In terms of notation, when activation occurs at u, the bound applies to the probability of the event

$$\big\{\tau_{U^i_t \setminus \{u\}} < \tau_{U_t \setminus (U^i_t \cup \{u\})}\big\}.$$

We will stipulate that u belongs to a cluster in U_t which is not a singleton as, otherwise, its activation at time t necessitates $t = \mathcal{T}_\ell$.

Lemma 2.6.11. *Suppose cluster i is least separated at time $\mathcal{T}_{\ell-1}$ and recall that x_i denotes the center of the i^{th} cluster, determined by the exponential clustering of U_0. When* $\mathsf{Timely}(\ell-1)$ *occurs and when t is at most $\mathcal{T}_\ell^-$:*

1. *the midway point $C(i;\ell) = C_{x_i}\left(\frac{1}{2}\rho_\ell\right)$ separates U^i_t from $U_t \setminus U^i_t$, and*

2. *for any u in U_t which does not belong to a singleton cluster and any y in $C(i;\ell)$,*

$$\mathbb{P}_y\left(\tau_{U_t^i \setminus \{u\}} < \tau_{U_t \setminus (U_t^i \cup \{u\})}\right) \leqslant 0.51. \tag{2.81}$$

Proof. Property (1) is an immediate consequence of (2.74) of Lemma 2.6.10, since $\frac{1}{2}\rho_\ell$ is at least $(\log \rho_\ell)^2$ and less than $0.99\rho_\ell$.

Now let u and y satisfy the hypotheses, denote the center of the i^{th} cluster by x_i, and denote $C((\log \rho_\ell)^2)$ by B. To prove property (2), we will establish

$$\mathbb{P}_{y-x_i}\left(\tau_B < \tau_{z-x_i}\right) \leqslant 0.51, \tag{2.82}$$

for some $z \in U_t \setminus (U_t^i \cup \{u\})$. This bound implies (2.81) because, by (2.74), B separates U_t^i from the rest of U_t.

We can express the probability in (2.82) in terms of hitting probabilities involving only three points:

$$\begin{aligned}\mathbb{P}_{y-x_i}(\tau_B < \tau_{z-x_i}) &= \mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i}) + \mathbb{E}_{y-x_i}\left[\mathbb{P}_{S_{\tau_B}}(\tau_{z-x_i} < \tau_o)\mathbf{1}(\tau_B < \tau_{z-x_i})\right] \\ &\leqslant \mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i}) + \max_{v \in B} \mathbb{P}_v(\tau_{z-x_i} < \tau_o)\, \mathbb{P}_{y-x_i}(\tau_B < \tau_{z-x_i}).\end{aligned}$$

Rearranging, we find

$$\mathbb{P}_{y-x_i}(\tau_B < \tau_{z-x_i}) \leqslant \Big(1 - \max_{v \in B} \mathbb{P}_v(\tau_{z-x_i} < \tau_o)\Big)^{-1} \mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i}). \tag{2.83}$$

We will choose z so that the points $y - x_i$ and $z - x_i$ will be at comparable distances from the origin and, consequently, $\mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i})$ will be nearly $1/2$. In contrast, every element of B will be far nearer to the origin than to $z - x_i$, so $\mathbb{P}_v(\tau_{z-x_i} < \tau_o)$ will be nearly zero for every v in B. We will write these probabilities in terms of the potential kernel using Lemma 2.3.8. We will need bounds on the distances $\|z - x_i\|$ and $\|z - y\|$ to simplify the potential kernel terms; we take care of this now.

Suppose cluster j was nearest to cluster i at time $\mathcal{T}_{\ell-1}$. We then choose z to be the element of U_t^j nearest to U_t^i. Note that such an element exists because, when t is at most $\mathcal{T}_\ell^-$, every cluster surviving until time $\mathcal{T}_{\ell-1}$ survives until time t. By (2.74) of Lemma 2.6.10,

$$\|z - x_i\| \geqslant 0.99\rho_\ell.$$

Part (2) of Lemma 2.9.1 then gives the lower bound

$$\mathfrak{a}(z - x_i) \geqslant \frac{2}{\pi}\log(0.99\rho_\ell). \tag{2.84}$$

In the inter-collapse period before $\mathcal{T}_\ell^-$, the separation between z and y (initially $\frac{1}{2}\rho_\ell$) can grow by at most $\mathfrak{t}_\ell + \operatorname{diam}(U^j_{\mathcal{T}_{\ell-1}})$:

$$\|z - y\| \leqslant \tfrac{1}{2}\rho_\ell + \mathfrak{t}_\ell + \operatorname{diam}(U^j_{\mathcal{T}_{\ell-1}}).$$

By (2.73), the diameter of cluster j at time $\mathcal{T}_{\ell-1}$ is at most $(\log \rho_\ell)^2$; this upper bound applies to $\mathfrak{t}_\ell$ as well, so

$$\|z-y\| \leqslant \tfrac{1}{2}\rho_\ell + 2(\log \rho_\ell)^2 \leqslant 0.51\rho_\ell.$$

We obtained the second inequality using the fact (2.70) that, when $\mathsf{Timely}(\ell-1)$ occurs, ρ_ℓ is at least $e^{\theta_{2n}}$. (In what follows, we will use this fact without restating it.)

Accordingly, the difference between $\mathfrak{a}(z-y)$ and $\mathfrak{a}(y-x_i)$ satisfies

$$\mathfrak{a}(z-y) - \mathfrak{a}(y-x_i) \leqslant \frac{2}{\pi}\log(2\cdot 0.51) + 4\lambda\rho_\ell^{-2} \leqslant \frac{2}{\pi}. \tag{2.85}$$

By Lemma 2.3.8, the first term of (2.83) equals

$$\mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i}) = \frac{1}{2} + \frac{\mathfrak{a}(z-y)-\mathfrak{a}(y-x_i)}{2\mathfrak{a}(z-x_i)}. \tag{2.86}$$

Substituting (2.84) and (2.85) into (2.86), we find

$$\mathbb{P}_{y-x_i}(\tau_o < \tau_{z-x_i}) \leqslant \frac{1}{2} + \frac{1}{\log \rho_\ell} \leqslant 0.501. \tag{2.87}$$

We turn our attention to bounding above the maximum of $\mathbb{P}_v(\tau_{z-x_i} < \tau_o)$ over v in B. For any such v, Lemma 2.3.8 gives

$$\mathbb{P}_v(\tau_{z-x_i} < \tau_o) = \frac{1}{2} + \frac{\mathfrak{a}(v)-\mathfrak{a}(z-x_i-v)}{2\mathfrak{a}(z-x_i)}. \tag{2.88}$$

By Lemma 2.9.2, $\mathfrak{a}(v)$ is at most $\mathfrak{a}'((\log \rho_\ell)^2) + 2(\log \rho_\ell)^{-2}$. Then, since

$$\|z-x_i-v\| \geqslant 0.99\rho_\ell - (\log \rho_\ell)^2 \geqslant 0.98\rho_\ell,$$

we have

$$\mathfrak{a}(z-x_i-v) - \mathfrak{a}(v) \geqslant \frac{2}{\pi}\log(0.98\rho_\ell) - \frac{4}{\pi}\log\log\rho_\ell - 4(\log\rho_\ell)^{-2} \geqslant \frac{2\cdot 0.99}{\pi}\log(0.99\rho_\ell). \tag{2.89}$$

Substituting (2.84) and (2.89) into (2.88), we find

$$\mathbb{P}_v(\tau_{z-x_i} < \tau_o) \leqslant \frac{1}{2} - \frac{0.99}{2} \leqslant 0.005.$$

This bound holds uniformly over v in B. Applying it and (2.87) to (2.83), we find

$$\mathbb{P}_{y-x_i}(\tau_B < \tau_{z-x_i}) \leqslant (1-0.005)^{-1}0.501 \leqslant 0.51.$$

□

Combined with the separation lower bound (2.70) of Lemma 2.6.8, the inclusions (2.74) of Lemma 2.6.10 ensure that, when $\mathsf{Timely}(\ell-1)$ occurs, nonempty clusters at time $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$ are contained in well separated disks. A natural consequence is that, when $\mathsf{Timely}(\ell-1)$ occurs, every nonempty cluster has positive harmonic measure in U_t. Later, we will use this fact in conjunction with Theorem 2.1.6 to control the activation step of the HAT dynamics.

Lemma 2.6.12. *Let I_ℓ be the set of indices of nonempty clusters at time $\mathcal{T}_{\ell-1}$. When* $\mathsf{Timely}(\ell-1)$ *occurs and when t is at most $\mathcal{T}_\ell^-$, $\mathbb{H}_{U_t}(U_t^i) > 0$ for every $i \in I_\ell$.*

The proof is similar to that of Lemma 2.3.12. Recall the definition of the $*$-visible boundary (2.35) and define the disk D^i to be the one from (2.74)

$$D^i = D_{x_i}\big(\big(\log \mathrm{sep}(U^i_{\mathcal{T}_{\ell-1}})\big)^2\big) \text{ for each } i \in I_\ell. \tag{2.90}$$

For simplicity, assume $1 \in I_\ell$. Most of the proof is devoted to showing that there is a path Γ from $\partial^*_{\mathrm{ext}} D^1$ to a large circle C about U_t, which avoids $\cup_{i\in I_\ell} D^i$ and thus avoids U_t. To do so, we will specify a candidate path from $\partial^*_{\mathrm{ext}} D^1$ to C, and modify it as follows. If the path encounters a disk D^i, then we will reroute the path around $\partial^*_{\mathrm{ext}} D^i$ (which will be connected and will not intersect another disk). The modified path encounters one fewer disk. We will iterate this argument until the path avoids every disk and therefore never returns to U_t.

Proof of Lemma 2.6.12. Suppose $\mathsf{Timely}(\ell-1)$ occurs and $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, and assume w.l.o.g. that $1 \in I_\ell$. Let $y \in U_t^1$ satisfy $\mathbb{H}_{U_t^1}(y) > 0$. For each $i \in I_\ell$, let D^i be the disk defined in (2.90). As $\mathbb{H}_{U_t^1}(y)$ is positive, there is a path from y to $\partial^*_{\mathrm{ext}} D^1$ which does not return to U_t^1. In a moment, we will show that $\partial^*_{\mathrm{ext}} D^1$ is connected, so it will suffice to prove that there is a subsequent path from $\partial^*_{\mathrm{ext}} D^1$ to $C = C_{x_1}(2\,\mathrm{diam}(U_t))$ which does not return to $E = \cup_{i\in I_\ell} D^i$. This suffices when $\mathsf{Timely}(\ell-1)$ occurs because then, by Lemma 2.6.10, $U_t^i \subseteq D^i$ for each $i \in I_\ell$, so $U_t \subseteq E$.

We make two observations. First, because each D^i is finite and $*$-connected, Lemma 2.3.13 states that each $\partial^*_{\mathrm{ext}} D^i$ is connected. Second, $\partial^*_{\mathrm{ext}} D^i$ is disjoint from E when $\mathsf{Timely}(\ell-1)$ occurs; this is an easy consequence of (2.74) and the separation lower bound (2.70).

We now specify a candidate path from $\partial^*_{\mathrm{ext}} D^1$ to C and, if necessary, modify it to ensure that it does not return to E. Because $\mathbb{H}_{U_t^1}(y)$ is positive, there is a shortest path Γ from $\partial^*_{\mathrm{ext}} D^1$ to C, which does not return to U_t^1. Let L be the set of labels of disks encountered by Γ. If L is empty, then we are done. Otherwise, let i be the label of the first disk encountered by Γ, and let Γ_a and Γ_b be the first and last elements of Γ which intersect $\partial^*_{\mathrm{ext}} D^i$. By our first observation, $\partial^*_{\mathrm{ext}} D^i$ is connected, so there is a shortest path Λ in $\partial^*_{\mathrm{ext}} D^i$ from Γ_a to Γ_b. When edit Γ to form Γ' as

$$\Gamma' = \big(\Gamma_1, \ldots, \Gamma_{a-1}, \Lambda_1, \ldots, \Lambda_{|\Lambda|}, \Gamma_{b+1}, \ldots, \Gamma_{|\Gamma|}\big)\,.$$

Because Γ_b was the last element of Γ which intersected $\partial^*_{\mathrm{ext}} D^i$, Γ' avoids D^i. Additionally, by our second observation, Λ avoids E, so if L' is the set of labels of disks encountered by Γ', then $|L'| \leqslant |L| - 1$. If L' is empty, then we are done. Otherwise, we can relabel Γ to Γ' and L to L' in the preceding argument to continue inductively, obtaining Γ'' and $|L''| \leqslant |L| - 2$, and so on. Because $|L| \leqslant n$, we need to modify the path at most n times before the resulting path from y to C does not return to E. $\square$

The last result of this section bounds above escape probabilities; we will shortly specialize it for our setting. Note that ∂A_ρ denotes the exterior boundary of the ρ-fattening of A, not the ρ-fattening of ∂A.

Lemma 2.6.13. *If A is a subset of $\mathbb{Z}^2$ with at least two elements and if ρ is at least twice the diameter of A, then, for x in A,*

$$\mathbb{P}_x\left(\tau_{\partial(A\setminus\{x\})_\rho} < \tau_{A\setminus\{x\}}\right) \leqslant \frac{\log \operatorname{diam}(A) + 2}{\log \rho}. \tag{2.91}$$

The added 2 in (2.91) is unimportant. Note that, if A was a singleton set, then the probability in question would be proportional to $(\log \rho)^{-1}$. The $\log \operatorname{diam}(A)$ term arises from the fact that, if $|A| \geqslant 2$, then a random walk from x must avoid at least one element in $A\setminus\{x\}$, at a distance of at most $\operatorname{diam}(A)$ from x.

Proof of Lemma 2.6.13. We will replace the event in (2.91) with a more probable but simpler event and bound above its probability instead.

By hypothesis, A has at least two elements, so for any x in A, there is some y in $A\setminus\{x\}$ nearest to x. To escape to $\partial(A\setminus\{x\})_\rho$ without hitting $A\setminus\{x\}$ it is necessary to escape to $C_y(\rho)$ without hitting y. Accordingly, for a random walk from x, the following inclusion holds

$$\left\{\tau_{\partial(A\setminus\{x\})_\rho} < \tau_{A\setminus\{x\}}\right\} \subseteq \left\{\tau_{C_y(\rho)} < \tau_y\right\}. \tag{2.92}$$

To prove (2.91) it therefore suffices to obtain the same bound for the larger event.

The hypothesis $\rho \geqslant 2\operatorname{diam}(A)$ ensures that $x - y$ lies in $D(\rho)$, so we can apply the optional stopping theorem to the martingale $\mathfrak{a}(S_{j\wedge\tau_o})$ at the stopping time $\tau_{C(\rho)}$. Doing so, we find

$$\mathbb{P}_x(\tau_{C_y(\rho)} < \tau_y) = \mathbb{P}_{x-y}(\tau_{C(\rho)} < \tau_o) = \frac{\mathfrak{a}(x-y)}{\mathbb{E}_{x-y}[\mathfrak{a}(S_{\tau_{C(\rho)}}) \mid \tau_{C(\rho)} < \tau_o]}. \tag{2.93}$$

We apply Lemma 2.9.2 with $r = \rho$ and $x = o$ to find

$$\mathbb{E}_{x-y}[\mathfrak{a}(S_{\tau_{C(\rho)}}) \mid \tau_{C(\rho)} < \tau_o] \geqslant \mathfrak{a}'(\rho) - \rho^{-1} \geqslant \frac{2}{\pi}\log\rho. \tag{2.94}$$

By (2.14) and the facts that $1 \leqslant \|x-y\| \leqslant \operatorname{diam}(A)$ and $\kappa + \lambda \leqslant 1.1$, the numerator of (2.93), is at most

$$\mathfrak{a}(x-y) \leqslant \frac{2}{\pi}\log\|x-y\| + \kappa + \lambda\|x-y\|^{-2} \leqslant \frac{2}{\pi}\log\operatorname{diam}(A) + 1.1. \tag{2.95}$$

Substituting (2.94) and (2.95) into (2.93), and simplifying with $\frac{1.1\pi}{2} \leqslant 2$, we find

$$\mathbb{P}_x(\tau_{C_y(\rho)} < \tau_y) \leqslant \frac{\log\operatorname{diam}(A) + 2}{\log\rho}.$$

Due to the inclusion (2.92), this implies (2.91). □

2.6.4 Proof of Proposition 2.6.3

Recall that, for $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, the midway point is a circle which surrounds one of the clusters which is least separated at time $\mathcal{T}_{\ell-1}$. We call this cluster the watched cluster, to distinguish it from other clusters which are least separated at $\mathcal{T}_{\ell-1}$. The results of this section are phrased in these terms and through the following events.

Definition 2.6.14. *For any $x \in \mathbb{Z}^2$, time $t \geqslant 0$, and any $1 \leqslant i \leqslant k$, define the activation events*

$$\mathsf{Act}(x,t) = \{x \textit{ is activated at time } t\} \quad \textit{and} \quad \mathsf{Act}(i,t) = \bigcup_{x \in U_t^i} \mathsf{Act}(x,t).$$

Additionally, define the deposition event

$$\mathsf{Dep}(i,t) = \bigcup_{x \in U_t} \mathsf{Act}(x,t) \cap \left\{ \tau_{U_t^i \setminus \{x\}} < \tau_{U_t \setminus (U_t^i \cup \{x\})} \right\}.$$

In words, the deposition event requires that, at time t, the activated particle deposits at the i^{th} cluster.

When $\mathsf{Timely}(\ell-1)$ occurs, if the i^{th} cluster is the watched cluster at time $\mathcal{T}_{\ell-1}$, then for any time $t \in \left[\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-\right]$, define the "midway" event as

$$\mathsf{Mid}(i,t;\ell) = \bigcup_{x \in U_t} \mathsf{Act}(x,t) \cap \left\{ \tau_{C(i;\ell)} < \tau_{U_t \setminus \{x\}} \right\}.$$

In words, the midway event specifies that, at time t, the activated particle reaches $C(i;\ell)$ before deposition.

We will now use the results of the preceding subsection to bound below the probability that activation occurs at the watched cluster and that the activated particle subsequently reaches the midway point. Essentially, Theorem 2.1.6 addresses the former probability and Theorem 2.1.9 addresses the latter. However, it is necessary to first ensure that the watched cluster has positive harmonic measure, so that at least one of its particles can be activated and the lower bound (2.2) of Theorem 2.1.6 can apply. This is handled by Lemma 2.6.12, the hypotheses of which are satisfied whenever $\mathsf{Timely}(\ell-1)$ occurs and $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$. The hypotheses of Theorem 2.1.9 will be satisfied in this context so long as we estimate the probability of escape to a distance ρ which is at least twice the cluster diameter. The distance from the watched cluster to the midway point is roughly ρ_ℓ, while the cluster diameter is at most $(\log \rho_\ell)^2$ by (2.74) of Lemma 2.6.10, so this will be the case.

The lower bounds from Theorems 2.1.6 and 2.1.9 will imply that a particle with positive harmonic measure is activated and reaches the midway point with a probability of at least

$$\exp(-c_1 n \log n + \log(c_2 n^{-2})) \cdot (\log \rho_\ell)^{-1}$$

for constants c_1, c_2. From our choice of γ (2.65), the first factor in the preceding display is at least

$$\alpha_n = e^{\gamma n \log n}. \tag{2.96}$$

Proposition 2.6.15. *Let cluster i be least separated at time $\mathcal{T}_{\ell-1}$. When* $\mathsf{Timely}(\ell-1)$ *occurs and when $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, we have*

$$\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(i,t) \;\middle|\; \mathcal{F}_t\right) \geqslant (\alpha_n \log \rho_\ell)^{-1}. \tag{2.97}$$

Proof. Fix ℓ, suppose the i^{th} cluster is least separated at time $\mathcal{T}_{\ell-1}$ and $\mathsf{Timely}(\ell-1)$ occurs, and let $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$. For any $x \in U_t^i$, we have

$$\mathbf{P}(\mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(i,t) \mid \mathcal{F}_t) \geqslant \mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \;\middle|\; \mathsf{Act}(x,t),\, \mathcal{F}_t\right) \mathbf{P}(\mathsf{Act}(x,t) \mid \mathcal{F}_t). \tag{2.98}$$

Let B denote the set of all points within distance ρ_ℓ of U_t^i. We have the following inclusion when $\mathsf{Act}(x,t)$ occurs:

$$\left\{\tau_{\partial B} < \tau_{U_t^i}\right\} \subseteq \left\{\tau_{C(i;\ell)} < \tau_{U_t \setminus \{x\}}\right\} = \mathsf{Mid}(i,t;\ell). \tag{2.99}$$

From (2.99), we have

$$\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \;\middle|\; \mathsf{Act}(x,t),\, \mathcal{F}_t\right) \geqslant \mathbf{P}\left(\tau_{\partial B} < \tau_{U_t^i} \;\middle|\; \mathsf{Act}(x,t),\, \mathcal{F}_t\right) = \mathbb{P}_x\left(\tau_{\partial B} < \tau_{U_t^i}\right). \tag{2.100}$$

Now let x be an element of U_t^i which is exposed and which maximizes (2.100). Such an element must exist because, by Lemma 2.6.12, when $\mathsf{Timely}(\ell-1)$ occurs and when $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, $\mathbb{H}_{U_t}(U_t^i)$ is positive. We aim to apply Theorem 2.1.9 to bound below the probability in (2.100). The hypotheses of Theorem 2.1.9 require $|U_t^i| \geqslant 2$ and $\rho_\ell \geqslant 2\,\mathrm{diam}(U_t^i)$. First, the cluster U_t^i must contain at least two elements as, otherwise, activation at x would necessitate $t = \mathcal{T}_\ell$. Second, ρ_ℓ is indeed at least twice the diameter of U_t^i because, when $\mathsf{Timely}(\ell-1)$ occurs, U_t^i is contained in a disk of radius $(\log \rho_\ell)^2$ by (2.74). Theorem 2.1.9 therefore applies to (2.100), giving

$$\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \;\middle|\; \mathsf{Act}(x,t),\, \mathcal{F}_t\right) \geqslant c_2(n^2 \log \rho_\ell)^{-1}. \tag{2.101}$$

The harmonic measure lower bound (2.2) of Theorem 2.1.6 applies because x has positive harmonic measure. According to (2.2), the harmonic measure of x is at least

$$\mathbf{P}\left(\mathsf{Act}(x,t) \;\middle|\; \mathcal{F}_t\right) = \mathbb{H}_{U_t}(x) \geqslant e^{-c_1 n \log n}. \tag{2.102}$$

Combining (2.101) and (2.102), we find

$$\mathbf{P}(\mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(i,t) \mid \mathcal{F}_t) \geqslant c_2(n^2 \log \rho_\ell)^{-1} \cdot e^{-c_1 n \log n} \geqslant (\alpha_n \log \rho_\ell)^{-1}.$$

The second inequality is due to the definition of α_n (2.96). □

Next, we will bound below the conditional probability that activation occurs at the watched cluster, given that the activated particle reaches the midway point.

Proposition 2.6.16. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$. When* $\mathsf{Timely}(\ell-1)$ *occurs and when $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, we have*

$$\mathbf{P}\left(\mathsf{Act}(i,t) \;\middle|\; \mathsf{Mid}(i,t;\ell),\, \mathcal{F}_t\right) \geqslant (3\alpha_n \log\log \rho_\ell)^{-1}. \tag{2.103}$$

Proof. Suppose $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$ and $\mathsf{Timely}(\ell-1)$ occurs. If we obtain a lower bound p_1 on $\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(i,t) \;\middle|\; \mathcal{F}_t\right)$ and an upper bound p_2 on $\mathbf{P}\left(\bigcup_{j\neq i} \mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(j,t) \;\middle|\; \mathcal{F}_t\right)$, then

$$\mathbf{P}\left(\mathsf{Act}(i,t) \;\middle|\; \mathsf{Mid}(i,t;\ell),\, \mathcal{F}_t\right) \geqslant \frac{p_1}{p_1+p_2}. \tag{2.104}$$

First, the probability $\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \cap \mathsf{Act}(i,t) \;\middle|\; \mathcal{F}_t\right)$ is precisely the one we used to establish (2.98) in the proof of Proposition 2.6.15; p_1 is therefore at least $(\alpha_n \log \rho_\ell)^{-1}$.

Second, for any $j \neq i$, we use the trivial upper bound $\mathbf{P}(\mathsf{Act}(j,t) \mid \mathcal{F}_t) \leqslant 1$ and address the midway component by writing

$$\mathbf{P}\left(\mathsf{Mid}(i,t;\ell) \;\middle|\; \mathsf{Act}(j,t),\, \mathcal{F}_t\right) = \mathbf{E}\left[\mathbb{P}_X\left(\tau_{C(i;\ell)} < \tau_{U_t \setminus \{X\}}\right) \;\middle|\; \mathsf{Act}(j,t),\, \mathcal{F}_t\right]. \tag{2.105}$$

Use ρ to denote $\operatorname{dist}(U^i_{\mathcal{T}_{\ell-1}}, U^j_{\mathcal{T}_{\ell-1}})$ and B to denote the set of all points within a distance ρ of $U^j_t \setminus \{X\}$. (We use ρ instead of ρ_ℓ because j is not necessarily the cluster nearest cluster i.) We can use Lemma 2.6.13 to bound the probability in (2.105) because, for any random walk from X, the following inclusion holds:

$$\{\tau_{C(i;\ell)} < \tau_{U_t\setminus\{X\}}\} \subseteq \left\{\tau_B < \tau_{U^j_t \setminus \{X\}}\right\}.$$

Because the cluster U^j_t has at least two elements and because ρ is at least twice its diameter, an application of Lemma 2.6.13 with $A = U^j_t$ and ρ yields

$$\mathbb{P}_x\left(\tau_B < \tau_{U^j_t\setminus\{x\}}\right) \leqslant \frac{\log \operatorname{diam}(U^j_t) + 2}{\log \rho} \leqslant \frac{2.2 \log\log \rho_\ell}{\log \rho_\ell},$$

uniformly for x in U^j_t. The second inequality follows from (2.75), which bounds the ratio of $\log \operatorname{diam}(U^j_t)$ to $\log \rho$ by $\frac{2.1 \log\log\rho_\ell}{\log \rho_\ell}$.

Applying the preceding bound to (2.105), we find

$$\mathbf{P}(\mathsf{Mid}(i,t;\ell) \mid \mathsf{Act}(j,t),\, \mathcal{F}_t) \leqslant \frac{2.2\log\log\rho_\ell}{\log \rho_\ell} =: p_2.$$

Then, substituting p_1 and p_2 in (2.104), we conclude

$$\mathbf{P}\left(\mathsf{Act}(i,t) \;\middle|\; \mathsf{Mid}(i,t;\ell),\, \mathcal{F}_t\right) \geqslant (1 + 2.2\alpha_n \log\log\rho_\ell)^{-1} \geqslant (3\alpha_n \log\log \rho_\ell)^{-1}.$$

□

We now use Lemma 2.6.11 to establish that an activated particle, upon reaching the midway point, deposits at the watched cluster with a probability of no more than 0.51.

Proposition 2.6.17. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$. When* $\mathsf{Timely}(\ell-1)$ *occurs and when $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$, for x in U_t, we have*

$$\mathbf{P}\left(\mathsf{Dep}(i,t) \;\middle|\; \mathsf{Mid}(i,t;\ell),\, \mathsf{Act}(x,t),\, \mathcal{F}_t\right) \leqslant 0.51. \tag{2.106}$$

Proof. Using the definitions of $\mathsf{Dep}(i,t)$ and $\mathsf{Mid}(i,t;\ell)$, we write

$$\mathbf{P}\left(\mathsf{Dep}(i,t) \,\middle|\, \mathsf{Mid}(i,t;\ell),\, \mathsf{Act}(x,t),\, \mathcal{F}_t\right)$$
$$= \mathbf{P}\left(\bigcup_{y\in U_t} \mathsf{Act}(y,t) \cap \left\{\tau_{U_t^i\setminus\{y\}} < \tau_{U_t\setminus(U_t^i\cup\{y\})}\right\} \,\middle|\, \bigcup_{y\in U_t} \mathsf{Act}(y,t) \cap \left\{\tau_{C(i;\ell)} < \tau_{U_t\setminus\{y\}}\right\},\, \mathsf{Act}(x,t),\, \mathcal{F}_t\right).$$

Because $\mathsf{Act}(y,t)$ only occurs for one particle y in U_t^i at any given time t, the right-hand side simplifies to

$$\mathbf{E}\left[\mathbb{P}_x\left(\tau_{U_t^i\setminus\{x\}} < \tau_{U_t\setminus(U_t^i\cup\{x\})} \,\middle|\, \tau_{C(i;\ell)} < \tau_{U_t\setminus\{x\}}\right) \,\middle|\, \mathsf{Act}(x,t),\, \mathcal{F}_t\right].$$

We then apply the strong Markov property to $\tau_{C(i;\ell)}$ to find that the previous display equals

$$\mathbf{E}\left[\mathbb{P}_{S_{\tau_{C(i;\ell)}}}\left(\tau_{U_t^i\setminus\{x\}} < \tau_{U_t\setminus(U_t^i\cup\{x\})}\right) \,\middle|\, \mathsf{Act}(x,t),\, \mathcal{F}_t\right] \leqslant 0.51,$$

where the inequality follows from the estimate (2.81). □

The preceding three propositions realize the strategy of Section 2.6.2. We proceed to implement the strategy of Section 2.6.2. In brief, we will compare the number of particles in the watched cluster to a random walk and bound the collapse time using the hitting time of zero of the walk.

Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$ and denote by $(\eta_\ell(m))_{m\geqslant 0}$ the consecutive times at which the midway event $\mathsf{Mid}(i,\cdot;\ell)$ occurs. Set $\eta_\ell(0) \equiv \mathcal{T}_{\ell-1} - 1$ and for all $m \geqslant 1$ define

$$\eta_\ell(m) = \inf\{t > \eta_\ell(m-1) : \mathsf{Mid}(i,t;\ell) \text{ occurs}\}.$$

Additionally, we denote the number of midway event occurrences by time t as

$$N_\ell(t) = \sum_{m=1}^{\infty} \mathbf{1}\left(\eta_\ell(m) \leqslant t\right).$$

The number of elements in cluster i viewed at these times can be coupled to a lazy random walk $(W_m)_{m\geqslant 0}$ on $\{0,\ldots,n\}$ from $W_0 \equiv \left|U^i_{\mathcal{T}_{\ell-1}}\right|$, which takes down-steps with probability $q_W = (7\alpha_n \log\log\rho_\ell)^{-1}$ and up-steps with probability $1-q_W$, unless it attempts to take a down-step at $W_m = 0$ or an up-step at $W_m = n$, in which case it remains where it is.

When $\mathsf{Timely}(\ell-1)$ occurs, at each time $\eta_\ell(\cdot)$, the watched cluster has a chance of losing a particle of at least $0.49(3\alpha_n\log\log\rho_\ell)^{-1} \geqslant q_W$ (Propositions 2.6.15 and 2.6.16). The standard coupling of $|U^i_{\eta_\ell(m)+1}|$ and W_m will then guarantee $|U^i_{\eta_\ell(m)+1}| \leqslant W_m$. However, this inequality will only hold when $m \leqslant N_\ell(\mathcal{T}_\ell^-)$.

Lemma 2.6.18. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$. There is a coupling of $\left(\left|U^i_{\eta_\ell(m)+1}\right|\right)_{m\geqslant 0}$ and $(W_m)_{m\geqslant 0}$ such that, when $\mathsf{Timely}(\ell-1)$ and $\{N_\ell(\mathcal{T}_\ell^-) \geqslant M\}$ occur, $\left|U^i_{\eta_\ell(m)}\right| \leqslant W_m$ for all $m \leqslant M$.*

Proof. Define

$$q(m) = \mathbf{P}\left(\mathsf{Act}(i, \eta_\ell(m)) \cap \bigcup_{j \neq i} \mathsf{Dep}(j, \eta_\ell(m)) \,\middle|\, \mathcal{F}_{\eta_\ell(m)}\right).$$

In words, the event in the previous display is the occurrence of $\mathsf{Mid}(i, \eta_\ell(m); \ell)$, preceded by activation at cluster i and followed by deposition at cluster $j \neq i$; this is the probability that the watched cluster loses a particle.

Couple $\left(\left|U^i_{\eta_\ell(m)+1}\right|\right)_{m \geqslant 0}$ and $(W_m)_{m \geqslant 0}$ in the standard way. When $\mathsf{Timely}(\ell - 1)$ occurs, by Propositions 2.6.16 and 2.6.17, the estimates (2.103) and (2.106) hold for all $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$. In particular, these estimates hold at time $\eta_\ell(m)$ for any $m \leqslant M$ when $\{N_\ell(\mathcal{T}_\ell^-) \geqslant M\}$ occurs. Accordingly, for any such m, we have $q(m) \geqslant 0.49(3\alpha_n \log\log \rho_\ell)^{-1} \geqslant q_W$. □

Denote by τ_0^U and τ_0^W the first hitting times of zero for $\left(\left|U^i_{\eta_\ell(m)+1}\right|\right)_{m \geqslant 0}$ and $(W_m)_{m \geqslant 0}$. Under the coupling, τ_0^W cannot precede τ_0^U. So an upper bound on τ_0^W of m implies $\tau_0^U \leqslant m$ and therefore it takes no more than m occurrences of the midway event after $\mathcal{T}_{\ell-1}$ for the collapse of the watched cluster to occur. In other words, $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ is at most $\eta_\ell(m) + 1$.

Lemma 2.6.19. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$. When* $\mathsf{Timely}(\ell - 1)$ *and* $\{N_\ell(\mathcal{T}_\ell^-) \geqslant M\}$ *occur,*

$$\left\{\tau_0^W \leqslant M\right\} \subseteq \left\{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant \eta_\ell(M) + 1\right\}. \tag{2.107}$$

Proof. From Lemma 2.6.18, there is a coupling of $\left(\left|U^i_{\eta(m)+1}\right|\right)_{m \geqslant 0}$ and $(W_m)_{m \geqslant 0}$ such that, when $\mathsf{Timely}(\ell - 1)$ and $\{N_\ell(\mathcal{T}_\ell^-) \geqslant M\}$ occur, $\left|U^i_{\eta(m)+1}\right| \leqslant W_m$ for all $m \leqslant M$. In particular,

$$\{\tau_0^W \leqslant M\} \subseteq \{\tau_0^U \leqslant M\}.$$

If $\{\tau_0^U \leqslant M\}$ occurs, cluster i is empty after the time of the M^{th} occurrence of the midway event, $\eta_\ell(M) + 1$. That is, we have the inclusion

$$\{\tau_0^U \leqslant M\} \subseteq \{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant \eta_\ell(M) + 1\},$$

which implies (2.107). □

We now show that τ_0^W, the hitting time of zero for W_m, is not more than $\log(\log \rho_\ell)^n$, up to a factor depending on n, with high probability. With more effort, we could prove a much better bound (in terms of dependence on n), but this improvement would not affect the conclusion of Proposition 2.6.3. By Lemma 2.6.19, the bound on τ_0^W will imply a bound on $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ in terms of $\eta_\ell(\cdot)$. For brevity, denote $\beta_n = (8\alpha_n)^n$.

Lemma 2.6.20. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$ and let $K \geqslant 1$. If* $\mathsf{Timely}(\ell - 1)$ *and* $\{N_\ell(\mathcal{T}_\ell^-) \geqslant K \cdot \beta_n (\log\log \rho_\ell)^n\}$ *occur, then*

$$\mathbf{P}\left(\mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant \eta_\ell(K \cdot \beta_n (\log\log \rho_\ell)^n) + 1 \,\middle|\, \mathcal{F}_{\mathcal{T}_{\ell-1}}\right) \geqslant 1 - e^{-K}. \tag{2.108}$$

The factor $(\log\log\rho_\ell)^n$ appears because $(W_m)_{m\geqslant 0}$ takes down-steps with a probability which is the reciprocal of $O_n(\log\log\rho_\ell)$, and we will require it to take n consecutive down-steps. Note that the event involving K cannot occur if K is large enough, because $N_\ell(\mathcal{T}_\ell^-)$ cannot exceed $\mathcal{T}_\ell^- - \mathcal{T}_{\ell-1} \leqslant (\log\rho_\ell)^2$ (i.e., there can be no more occurrences of the midway event than there are HAT steps). The implicit bound on K is $(\log\rho_\ell)^{2-o_n(1)}$. We will apply the lemma with a K of approximately $(\log\rho_\ell)^\delta$ for a $\delta \in (0,1)$.

Proof of Lemma 2.6.20. Set $M = K \cdot \lfloor \beta_n (\log\log\rho_\ell)^n \rfloor$ and denote the distribution of $(W_m)_{m\geqslant 0}$ by $\mathbb{P}_W$. If $\mathsf{Timely}(\ell-1)$ and $\{N_\ell(\mathcal{T}_\ell^-) \geqslant M\}$ occur, then by Lemma 2.6.19, we have the inclusion (2.107):

$$\{\tau_0^W \leqslant M\} \subseteq \{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant \eta_\ell(M) + 1\}.$$

Since $(W_m)_{m\geqslant 0}$ is never greater than n, it never takes more than n down-steps for W_m to hit zero. Since $W_{m+1} = W_m - 1$ with a probability of q_W whenever $m \leqslant M - n$, we have

$$\mathbb{P}_W\left(\tau_0^W > m + n \;\middle|\; \tau_0^W > m\right) \leqslant 1 - q_W^n.$$

Applying this to all $m \leqslant M - n$, we find

$$\mathbb{P}_W\left(\tau_0^W > M\right) \leqslant (1 - q_W^n)^M \leqslant e^{-K}.$$

For the second inequality, we used the fact that $\lfloor \beta_n (\log\log\rho_\ell)^n \rfloor$ is at least q_W^{-n} and therefore M is at least $K \cdot q_W^{-n}$. Combining this with (2.107) gives (2.108). □

To conclude Proposition 2.6.3 from Lemma 2.6.20, we will show that if $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ exceeds, say, $(\log\rho_\ell)^{1+2\delta}$, then with high probability there are many—at least $(\log\rho_\ell)^\delta$—occurrences of the midway event (and therefore steps of the walk W_m), with high probability, for an appropriate choice of δ. Reflecting this aim, we define the event

$$\mathsf{Many}_\delta = \left\{\eta_\ell\big((\log\rho_\ell)^\delta\big) \leqslant (\log\rho_\ell)^{1+2\delta}\right\}.$$

When Many_δ occurs, we will find that $\mathcal{T}_\ell - \mathcal{T}_{\ell-1} > (\log\rho_\ell)^{1+2\delta}$ is unlikely, as the walk W_m will hit zero with high probability after $(\log\rho_\ell)^\delta$ steps.

For convenience, in what follows, we will treat terms of the form $(\log\rho_\ell)^\delta$ as integers, as the distinction will be unimportant.

Proposition 2.6.21. *Let cluster i be the watched cluster at time $\mathcal{T}_{\ell-1}$ and let $\delta = (2n)^{-4}$. If* $\mathsf{Timely}(\ell-1)$ *and $\{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} > (\log\rho_\ell)^{1+6\delta}\}$ occur, then*

$$\mathbf{P}\left(\mathsf{Many}_{3\delta} \;\middle|\; \mathcal{F}_{\mathcal{T}_{\ell-1}}\right) \geqslant 1 - e^{-5n(\log\rho_\ell)^{2\delta}}. \tag{2.109}$$

Proof. By Proposition 2.6.15, the estimate (2.97) holds for any $t \in [\mathcal{T}_{\ell-1}, \mathcal{T}_\ell^-]$. Accordingly, when $\left\{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} > (\log\rho_\ell)^{1+6\delta}\right\}$ occurs, (2.97) applies to every time t up to $(\log\rho_\ell)^{1+6\delta}$:

$$\mathbf{P}(\mathsf{Mid}(i,t;\ell) \mid \mathcal{F}_t) \geqslant (\alpha_n \log\rho_\ell)^{-1}. \tag{2.110}$$

Define the time

$$\mathfrak{s}_\ell(\delta) = 6n\alpha_n(\log\rho_\ell)^{1+2\delta}.$$

Suppose that the number of occurrences M of the midway event is such that the time $\eta_\ell(M)+\mathfrak{s}_\ell(\delta)$ is at most $(\log\rho_\ell)^{1+6\delta}$. We then define, for any $m \leqslant M$ the event that the m^{th} and $(m+1)^{\text{st}}$ occurrences of the midway event are "close" in time:

$$\mathsf{Close}_\delta(m) = \{\eta_\ell(m+1) - \eta_\ell(m) \leqslant \mathfrak{s}_\ell(\delta)\}\,.$$

In order for $\mathsf{Close}_\delta(m)$ to fail to occur, we must fail to observe the occurrence of $\mathsf{Mid}(i,t;\ell)$ in $\mathfrak{s}_\ell(\delta)$-many consecutive steps. Using the Markov property and the bound (2.110), we find that

$$\mathbf{P}\left(\mathsf{Close}_\delta(m)^c \;\middle|\; \mathcal{F}_{\eta_\ell(m)}\right) \leqslant \left(1 - \frac{1}{\alpha_n \log\rho_\ell}\right)^{\mathfrak{s}_\ell(\delta)} \leqslant e^{-6n(\log\rho_\ell)^{2\delta}}. \tag{2.111}$$

Denote $\mathsf{Close}_\delta = \bigcap_{m=0}^{(\log\rho_\ell)^{3\delta}-1} \mathsf{Close}_\delta(m)$. We claim that Close_δ is a subset of Many_δ and that

$$\mathbf{P}(\mathsf{Close}_\delta \mid \mathcal{F}_{\mathcal{T}_{\ell-1}}) \geqslant 1 - e^{-5n(\log\rho_\ell)^{2\delta}}, \tag{2.112}$$

which implies (2.109).

To prove the inclusion, we note that when Close_δ occurs, because ρ_ℓ is at least $e^{\theta_{2n}}$ when $\mathsf{Timely}(\ell-1)$ occurs (2.70), we have

$$\eta_\ell\left((\log\rho_\ell)^{3\delta}\right) \leqslant 6n\alpha_n(\log\rho_\ell)^{1+5\delta} \leqslant (\log\rho_\ell)^{1+6\delta}.$$

Specifically, the first bound holds due to the definitions of Close_δ and $\mathfrak{s}_\ell(\delta)$, and the second holds because $6n\alpha_n \leqslant (\log\rho_\ell)^\delta$ when $\rho_\ell \geqslant e^{\theta_{2n}}$ and $\delta = (2n)^{-4}$. This implies that Close_δ is a subset of Many_δ.

To prove (2.112) we use a union bound over the $(\log\rho_\ell)^{3\delta}$-many constituent events of Close_δ and (2.111), finding that

$$\mathbf{P}(\mathsf{Close}_\delta \mid \mathcal{F}_{\mathcal{T}_{\ell-1}}) \geqslant 1 - (\log\rho_\ell)^{3\delta} e^{-6n(\log\rho_\ell)^{2\delta}} \geqslant 1 - e^{-5n\log(\rho_\ell)^{2\delta}}.$$

□

We now have all the inputs required to complete the proof of Proposition 2.6.3.

Proof of Proposition 2.6.3. Let $\delta = (2n)^{-4}$. We will show that it is rare for $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ to exceed $(\log\rho_\ell)^{1+6\delta}$ by arguing that, if it does, then with high probability there are many occurrences of the midway event—and correspondingly many steps of the coupled random walk—over which the coupled random walk must avoid hitting zero.

In terms of notation, we will call this rare event F_δ:

$$F_\delta = \bigcap_{\ell=1}^{k-1} F_{\ell,\delta} \quad \text{where} \quad F_{\ell,\delta} = \left\{\mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant (\log\rho_\ell)^{1+6\delta}\right\}.$$

The event which bounds $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ in terms of the number of occurrences of the midway event is

$$G_{\ell,\delta} = \left\{ \mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant \eta_\ell((\log \rho_\ell)^{3\delta}) + 1 \right\}.$$

The event $G_{\ell,\delta}$ will be probable because, by Lemma 2.6.20, $\mathcal{T}_\ell - \mathcal{T}_{\ell-1}$ typically does not exceed the time it takes for the midway event to occur $\beta_n(\log\log\rho_\ell)^n = (\log \rho_\ell)^{o_n(1)}$ times. We will also use the probable event that there are approximately as many occurrences of the midway event as Proposition 2.6.15 suggests there should be:

$$\mathsf{Many}_{\ell,\delta} = \left\{ \eta_\ell\big((\log \rho_\ell)^{3\delta}\big) \leqslant (\log \rho_\ell)^{1+6\delta} \right\}.$$

We will be able to bound the probability of $F_{\ell,\delta}^c$ for each ℓ in terms of the probabilities of the rare events $G_{\ell,\delta}^c$ and $\mathsf{Many}_{\ell,\delta}^c$ because of the following inclusion:

$$F_{\ell,\delta} \subseteq G_{\ell,\delta} \cap \mathsf{Many}_{\ell,\delta}. \tag{2.113}$$

We will then apply Lemma 2.6.20 and Proposition 2.6.21 to bound the probabilities of $G_{\ell,\delta}^c$ and $\mathsf{Many}_{\ell,\delta}^c$. After bounding the probability of each event $F_{\ell,\delta}$, we will use a union bound to bound the probability of F_δ.

Consider $\ell = 1$. (Assumptions of the occurrence of $\mathsf{Timely}(\ell - 1)$ are satisfied automatically when $\ell = 1$.) Due to (2.113),

$$\mathbf{P}\left(F_{1,\delta}^c \,\middle|\, \mathcal{F}_0\right) \leqslant \mathbf{P}\left(F_{1,\delta}^c \cap G_{1,\delta}^c \cap \mathsf{Many}_{1,\delta} \,\middle|\, \mathcal{F}_0\right) + \mathbf{P}\left(F_{1,\delta}^c \cap \mathsf{Many}_{1,\delta}^c \,\middle|\, \mathcal{F}_0\right). \tag{2.114}$$

We apply Lemma 2.6.20 to bound the first term on the right-hand side of (2.114). It is easy to check that, because ρ_1 is at least $e^{\theta_{2n}}$,

$$(\log \rho_1)^\delta > 5n \cdot \beta_n (\log\log\rho_1)^n. \tag{2.115}$$

Here, $\beta_n = (8\alpha_n)^n$ is the same quantity which appears in the statement of Lemma 2.6.20. By (2.115), the quantity $(\log\rho_1)^{3\delta}$ which appears in the definition of $G_{1,\delta}$ satisfies

$$(\log \rho_1)^{3\delta} > 5n \cdot \beta_n (\log\log\rho_1)^n \cdot (\log\rho_1)^{2\delta}.$$

When $F_{1,\delta}^c \cap \mathsf{Many}_{1,\delta}$ occurs, there are at least $(\log\rho_1)^{3\delta}$ occurrences of the midway event. In the terminology of Lemma 2.6.20, $N_1(\mathcal{T}_1^-) \geqslant (\log\rho_1)^{3\delta}$ which, by (2.115), means we can take K as large as $5n(\log\rho_1)^{2\delta}$. We apply the bound of Lemma 2.6.20 with $K = 5n(\log\rho_1)^{2\delta}$, finding that

$$\mathbf{P}\left(F_{1,\delta}^c \cap G_{1,\delta}^c \cap \mathsf{Many}_{1,\delta} \,\middle|\, \mathcal{F}_0\right) \leqslant e^{-5n(\log\rho_1)^{2\delta}}.$$

Next, we can apply Proposition 2.6.21 directly to the second term on the right-hand side of (2.114):

$$\mathbf{P}\left(F_{1,\delta}^c \cap \mathsf{Many}_{1,\delta}^c \,\middle|\, \mathcal{F}_0\right) \leqslant e^{-5n(\log\rho_1)^{2\delta}}.$$

Substituting the bounds for the terms in (2.114), we find

$$\mathbf{P}\left(F_{1,\delta}^c \,\middle|\, \mathcal{F}_0\right) \leqslant 2e^{-5n(\log\rho_1)^{2\delta}}.$$

Continuing inductively, suppose $\bigcap_{i=1}^{\ell} F_{i,\delta}$ occurs. It is easy to show that

$$\bigcap_{i=1}^{\ell} F_{i,\delta} = \bigcap_{i=1}^{\ell} \left\{ \mathcal{T}_\ell - \mathcal{T}_{\ell-1} \leqslant (\log \rho_\ell)^{1+6\delta} \right\} \subseteq \mathsf{Timely}(\ell).$$

Accordingly, the hypotheses of Lemma 2.6.20 and Proposition 2.6.21 involving $\mathsf{Timely}(\ell-1)$ are satisfied.

By Lemma 2.6.8, $\rho_{\ell+1}$ is at least $e^{\theta_{2n}}$, so (2.115) holds analogously. Furthermore, by Lemma 2.6.8, for any ℓ' up to $\ell+1$, we have $\rho_{\ell'} \geqslant \rho_1/2$. An argument identical to the $\ell = 1$ case establishes

$$\mathbf{P}\left(F_{\ell+1,\delta}^c \mid \mathcal{F}_{\mathcal{T}_\ell}\right) \mathbf{1}\left(\cap_{i=1}^{\ell} F_{i,\delta} \right) \leqslant 2e^{-5n\log(\rho_{\ell+1})^{2\delta}} \leqslant 2e^{-5n\log(\rho_1/2)^{2\delta}} \leqslant e^{-3n(\log \rho_1)^{2\delta}}.$$

By a union bound and the preceding display,

$$\begin{aligned} \mathbf{P}\left(F_\delta^c \mid \mathcal{F}_0\right) &= \mathbf{P}\left(\cup_{i=1}^{k-1} \left\{ \cap_{j=1}^{i-1} F_{j,\delta} \cap F_{i,\delta}^c \right\} \,\Big|\, \mathcal{F}_0 \right) \\ &\leqslant \sum_{i=1}^{k-1} \mathbf{P}\left(\cap_{j=1}^{i-1} F_{j,\delta} \cap F_{i,\delta}^c \,\Big|\, \mathcal{F}_0 \right) \\ &\leqslant \sum_{i=1}^{k-1} \mathbf{E}\left[e^{-3n(\log\rho_1)^{2\delta}} \mathbf{1}\left(\cap_{j=1}^{i-1} F_{j,\delta} \right) \,\Big|\, \mathcal{F}_0 \right] \leqslant \sum_{i=1}^{k-1} e^{-3n(\log\rho_1)^{2\delta}} \leqslant e^{-2n(\log\rho_1)^{2\delta}}. \end{aligned} \tag{2.116}$$

It remains to bound the time $\mathcal{T}_{k-1}$ when F_δ occurs. One can show that, when F_δ occurs, ρ_ℓ is never more than twice the diameter d of the initial configuration U_0. We write

$$\mathcal{T}_{k-1} = \sum_{\ell=1}^{k-1} (\mathcal{T}_\ell - \mathcal{T}_{\ell-1}) \leqslant \sum_{\ell=1}^{k-1} (\log \rho_\ell)^{1+6\delta} \leqslant 2n(\log d)^{1+6\delta} \leqslant (\log d)^{1+7\delta}.$$

The preceding display and (2.116) establish (2.58). ☐

2.7 Existence of the stationary distribution

In this section, we will prove Theorem 2.1.3, which has two parts. The first part states the existence of a unique stationary distribution, π_n, supported on the equivalence classes of non-isolated configurations, $\widehat{\mathrm{NonIso}}_{2,n}$, to which the HAT dynamics converges from any n-element configuration. The second part provides a tail bound on the diameter of configurations under π_n. We will prove these parts separately, as the following two propositions.

Proposition 2.7.1. *For all $n \geqslant 1$, from any n-element subset U, HAT converges to a unique stationary distribution π_n on $\widehat{\mathrm{NonIso}}_{2,n}$, given by*

$$\pi_n(\widehat{U}) = \frac{1}{\mathbf{E}_{\widehat{U}} \mathcal{T}_{\widehat{U}}}, \quad \textit{for } \widehat{U} \in \widehat{\mathrm{NonIso}}_{2,n}, \tag{2.117}$$

in terms of the return time $\mathcal{T}_{\widehat{U}} = \inf\{t \geqslant 1 : \widehat{U}_t = \widehat{U}\}$.

Proposition 2.7.2. *For any $d \geqslant 2\theta_{4n}$,*

$$\pi_n\big(\mathrm{diam}(\widehat{U}) \geq d\big) \leq \exp\left(-\frac{d}{(\log d)^{1+o_n(1)}}\right). \tag{2.118}$$

For the sake of concreteness, this is true with $6n^{-4}$ in the place of $o_n(1)$.

Proof of Theorem 2.1.3. Combine Propositions 2.7.1 and 2.7.2. □

It will be relatively easy to establish Proposition 2.7.2 using the inputs to the proof of Proposition 2.7.1 and Corollary 2.6.4, so we focus on presenting the key components of the proof of Proposition 2.7.1.

By standard theory for countable state space Markov chains, to prove Proposition 2.7.1, we must prove that the HAT dynamics is positive recurrent, irreducible, and aperiodic. We address each of these in turn.

Proposition 2.7.3 (Positive recurrent)**.** *For any $U \in \mathrm{NonIso}_{2,n}$, $\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} < \infty$.*

To prove Proposition 2.7.3, we will estimate the return time to an arbitrary n-element configuration $\widehat{U}$ by separately estimating the time it takes to reach the line segment $\widehat{L}_n$ from $\widehat{U}$, where $L_n = \{y\, e_2 : y \in \{0, 1, \dots, n-1\}\}$, and the time it takes to hit $\widehat{U}$ from $\widehat{L}_n$. The first estimate is the content of the following result.

Proposition 2.7.4. *There is a constant c such that, if U is a configuration in $\mathrm{NonIso}_{2,n}$ with a diameter of R, then, for all $K \geqslant \max\{R, \theta_{4n}(cn)\}$,*

$$\mathbf{P}_U\left(\mathcal{T}_{\widehat{L}_n} \leqslant K^3\right) \geqslant 1 - e^{-K}. \tag{2.119}$$

The second estimate is provided by the next proposition.

Proposition 2.7.5. *There is a constant c such that, if U is a configuration in $\mathrm{NonIso}_{2,n}$ with a diameter of R, then, for all $K \geqslant \max\{e^{R^{2.1}}, \theta_{4n}(cn)\}$,*

$$\mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{U}} \leqslant K^5\right) \geqslant 1 - e^{-K}. \tag{2.120}$$

The proof of Proposition 2.7.3 applies (2.119) and (2.120) to the tail sum formula for $\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}}$.

Proof of Proposition 2.7.3. Let $U \in \mathrm{NonIso}_{2,n}$. We have

$$\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} = \sum_{t=0}^{\infty} \mathbf{P}_{\widehat{U}}\big(\mathcal{T}_{\widehat{U}} > t\big) \leqslant \sum_{t=0}^{\infty}\Big(\mathbf{P}_{\widehat{U}}\big(\mathcal{T}_{\widehat{L}_n} > \tfrac{t}{2}\big) + \mathbf{P}_{\widehat{L}_n}\big(\mathcal{T}_{\widehat{U}} > \tfrac{t}{2}\big)\Big). \tag{2.121}$$

Suppose U has a diameter of at most R and let $J = \max\{e^{R^{2.1}}, \theta_{4n}(cn)\}$, where c is the larger of the constants from Propositions 2.7.4 and 2.7.5. We group the sum (2.121) over t into blocks:

$$\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} \leqslant O(J^5) + \sum_{K=J}^{\infty}\sum_{t=2K^5}^{2(K+1)^5}\Big(\mathbf{P}_{\widehat{U}}\big(\mathcal{T}_{\widehat{L}_n} > \tfrac{t}{2}\big) + \mathbf{P}_{\widehat{L}_n}\big(\mathcal{T}_{\widehat{U}} > \tfrac{t}{2}\big)\Big).$$

By (2.119) and (2.120) of Propositions 2.7.4 and 2.7.5, each of the $O(K^4)$ summands in the K^{th} block is at most

$$\mathbf{P}_{\widehat{U}}\big(\mathcal{T}_{\widehat{L}_n} > K^5\big) + \mathbf{P}_{\widehat{L}_n}\big(\mathcal{T}_{\widehat{U}} > K^5\big) \leqslant 2e^{-K}. \tag{2.122}$$

Substituting (2.122) into (2.121), we find

$$\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} \leqslant O(J^5) + O(1)\sum_{K=J}^{\infty} K^4 e^{-K} < \infty.$$

□

Propositions 2.7.4 and 2.7.5 also imply irreducibility.

Proposition 2.7.6 (Irreducible)**.** *For any* $n \geqslant 1$, *HAT is irreducible on* $\widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$.

Proof. Let $\widehat{U}, \widehat{V} \in \widehat{\mathrm{N}}\mathrm{onIso}_{2,n}$. It suffices to show that HAT reaches $\widehat{V}$ from $\widehat{U}$ in a finite number of steps with positive probability. By Propositions 2.7.4 and 2.7.5, there is a finite number of steps $K = K(U, V)$ such that

$$\mathbf{P}_U\big(\mathcal{T}_{\widehat{L}_n} < K\big) > 0 \quad \text{and} \quad \mathbf{P}_{L_n}\big(\mathcal{T}_{\widehat{V}} < K\big) > 0.$$

By the Markov property applied to $\mathcal{T}_{\widehat{L}_n}$, the preceding bounds imply that $\mathbf{P}_{\widehat{U}}(\mathcal{T}_{\widehat{V}} < 2K) > 0$. □

Lastly, because aperiodicity is a class property, it follows from irreducibility and the simple fact that $\widehat{L}_n$ is aperiodic.

Proposition 2.7.7 (Aperiodic)**.** $\widehat{L}_n$ *is aperiodic.*

Proof. We claim that $\mathbf{P}_{L_n}(U_1 = L_n) \geqslant \frac{1}{4}$, which implies that $\mathbf{P}_{\widehat{L}_n}\big(\widehat{U}_1 = \widehat{L}_n\big) \geqslant \frac{1}{4} > 0$. Indeed, every element of L_n neighbors another, so, regardless of which one is activated, we can dictate one random walk step which results in transport to the site of activation and $U_1 = L_n$. □

The preceding results constitute a proof of Proposition 2.7.1.

Proof of Proposition 2.7.1. Combine Propositions 2.7.3, 2.7.6, and 2.7.7. □

The subsections are organized as follows. In Section 2.7.1, we prove some preliminary results, including a key lemma which states that it is possible to reach any configuration $U \in \mathrm{NonIso}_{2,n}$ from L_n, in a number of steps depending only on n and $\mathrm{diam}(U)$. These results support the proofs of Propositions 2.7.4 and 2.7.5 in Sections 2.7.2 and Sections 2.7.3, respectively. In Section 2.7.4, we prove Proposition 2.7.2.

2.7.1 Preliminaries of hitting estimates for configurations

The purpose of this section is to estimate the probability that HAT forms a given configuration $\widehat{V}$ from $\widehat{L}_n$. We accomplish this primarily through Lemma 2.7.9, which guarantees the existence of a sequence configurations from $\widehat{L}_n$ to $\widehat{V}$, which can be realized by HAT in a way which is amenable to estimates.

In this section, we will say that an element x of a configuration V is exposed if $\mathbb{H}_V(x) > 0$ and we will denote the exposed elements of a configuration V by $\partial_{\exp} V$. Additionally, we will continute to denote the radius of a set A by $\mathrm{rad}(A) = \sup\{\|x\| : x \in A\}$.

First, we have a consequence of Theorems 2.1.6 and 2.1.9.

Lemma 2.7.8. *There is a constant c such that, if V_0 is a subset of $\mathbb{Z}^2$ with $n \geqslant 2$ elements and a radius of at most $r > 1$, and if V_1 is such that $\mathbf{P}_{V_0}(U_1 = V_1) > 0$, then*

$$\mathbf{P}_{V_0}(U_1 = V_1) \geqslant e^{-cn \log n} (\log r)^{-1}. \tag{2.123}$$

Proof. We will prove (2.123) by factoring $\mathbf{P}_{V_0}(U_1 = V_1)$ into activation and transport components, and separately estimating the components with Theorems 2.1.6 and 2.1.9.

Let V_0 and V_1 satisfy the hypotheses. Because $\mathbf{P}_{V_0}(U_1 = V_1)$ is positive, there are exposed elements x of V_0 and y of $\partial(V_0 \backslash \{x\})$ such that $V_1 = V_0 \cup \{y\} \backslash \{x\}$. Denote $W = V_0 \backslash \{x\}$. We write

$$\mathbf{P}_{V_0}(U_1 = V_1) \geqslant \mathbb{H}_{V_0}(x)\, \mathbb{P}_x\big(S_{\tau_W - 1} = y\big) \geqslant e^{-c_1 n \log n} \mathbb{P}_x\big(S_{\tau_W - 1} = y\big). \tag{2.124}$$

Note that, for the first inequality to be an equality, we would need to sum the right-hand side over all x, y such that $V_1 = V_0 \cup \{y\} \backslash \{x\}$. The second inequality is implied by (2.2) of Theorem 2.1.6, because x is exposed in V_0, which has n elements.

In terms of a distance d (which we will specify shortly) and ∂W_d, the exterior boundary of the d-fattening of W, we address the second factor of (2.124) as

$$\mathbb{P}_x\big(S_{\tau_W - 1} = y\big) \geqslant \frac{1}{4} \mathbb{P}_x(\tau_{\partial W_d} < \tau_{\partial W})\, \mathbb{E}_x\left[\mathbb{P}_{S_{\tau_{\partial W_d}}}(S_{\tau_{\partial W}} = y) \;\middle|\; \tau_{\partial W_d} < \tau_{\partial W}\right]. \tag{2.125}$$

In words, the probability that a random walk from x first steps into W from y is at least the probability that it does so after first reaching ∂W_d. We choose this lower bound because the factors of (2.125) can be addressed by our escape probability and harmonic measure estimates. The factor of $\frac{1}{4}$ arises from forcing the walk to hit W in the next step, after reaching y at time $\tau_{\partial W}$.

To replace the hitting probability with harmonic measure, we recall a standard result. Theorem 2.1.3 of [Law13] states that there are constants c_2 and m such that, if A is a subset of $\mathbb{Z}^2$ contained in $D(r')$, if $z \in A$, and if $y \in D(mr')^c$, then

$$\mathbb{H}_A(z, y) \geqslant c_2 \mathbb{H}_A(y).$$

We apply this fact with $A = \partial W$ and $r' = r$, where $r > 1$ is an upper bound on the radius of V_0. Note that W and ∂W are contained in $D(r+1)$. Hence, if d is at least $(m+1)(r+1)$, then ∂W_d is contained in $D(m(r+1))^c$. This implies

$$\mathbb{P}_z(S_{\tau_{\partial W}} = y) \geqslant c_2 \mathbb{H}_{\partial W}(y) \geqslant e^{-c_3 n \log n} \text{ for every } z \in \partial W_d. \tag{2.126}$$

The second inequality is implied by (2.2) of Theorem 2.1.6, because y is exposed in a set of $|\partial W| \leqslant 4n$ elements.

We will now use (2.11) of Theorem 2.1.9 to bound the escape probability in (2.125). Recall that if A has at least two elements and if $d' \geqslant 2\,\mathrm{diam}(A)$, then (2.11) states

$$\mathbb{P}_x(\tau_{\partial A_{d'}} < \tau_A) \geqslant \frac{c_4 \mathbb{H}_A(x)}{n \log(d')} \text{ for every } x \in A.$$

We apply this fact with $A = \partial W$ and $d' = 4d$ to find

$$\mathbb{P}_x(\tau_{\partial W_d} < \tau_{\partial W}) \geqslant \mathbb{P}_x(\tau_{\partial A_{d'}} < \tau_A) \geqslant \frac{c_4 \mathbb{H}_{\partial W}(x)}{2n \log(4d)} \geqslant e^{-c_5 n \log n} (\log d)^{-1}. \tag{2.127}$$

The first inequality holds because A has a diameter of at most $2(r+1)$ and so, if $d \geqslant (m+1)(r+1)$, then $d + 2(r+1) \leqslant 4d$ and hence ∂W_d separates A from ∂A_{4d}. The second inequality is due to (2.11), which applies because $4d \geqslant 2\,\mathrm{diam}(A)$. The third inequality is due to (2.2), which applies because x is exposed in V_0, an n-element set.

Substituting (2.126) and (2.127) into (2.125), and replacing d with $(m+1)(r+1)$, we find

$$\mathbb{P}_x\big(S_{\tau_W - 1} = y\big) \geqslant e^{-c_6 n \log n} (\log r)^{-1}.$$

Lastly, applying this bound to (2.124), we find (2.123):

$$\mathbf{P}_{V_0}(U_1 = V_1) \geqslant e^{-c_7 n \log n} (\log r)^{-1}.$$

□

The preceding lemma will help us bound below the probability of realizing a given configuration V as U_t for some time t and from some initial configuration V_0. However, to apply the lemma, we need an upper bound on the number of HAT steps it takes to form V from V_0. Supplying such an upper bound is the purpose of the next result, which is a key input to the proof of Proposition 2.7.5.

Lemma 2.7.9. *For any number of elements $n \geqslant 2$ and configuration V in $\mathrm{NonIso}_{2,n}$, if the radius of V is at most an integer $r \geqslant 10n$, then there is a sequence of $k \leqslant 100nr$ activation sites $x_1, \dots, x_k$ and transport sites $y_1, \dots, y_k$ which can be "realized" by HAT from $V_0 = L_n$ to $V_k = V$ in the following sense: if we set $V_i = V_{i-1} \cup \{y_i\} \setminus \{x_i\}$ for each $i \in \{1, \dots, k\}$, then each transition probability $\mathbf{P}_{V_{i-1}}(U_i = V_i)$ is positive. Additionally, each V_i is contained in $D(r + 10n)$.*

The factors of 10 and 100 in the lemma statement are for convenience and have no further significance. We will prove Lemma 2.7.9 by induction on n. Informally, we will remove one element of L_n to facilitate the use of the induction hypothesis, forming most of V before returning the removed element. There is a complication in this step, as we cannot allow the induction hypothesis to "interact" with the removed element. We will resolve this problem by proving a slightly stronger claim than the lemma requires.

The proof will overcome two main challenges. First, removing an element from a configuration V in $\mathrm{NonIso}_{2,n}$ can produce a configuration in $\mathrm{Iso}_{2,n-1}$, in which case the induction hypothesis will not apply. Indeed, there are configurations of $\mathrm{NonIso}_{2,n}$ for which the removal of any exposed, non-isolated element produces a configuration of $\mathrm{Iso}_{2,n-1}$ (such a V is depicted in Figure 2.9). Second, if an isolated element is removed alone, it cannot be returned to form V by a single step of the HAT dynamics. To see how these difficulties interact, suppose $\partial_{\exp} V$ contains only one non-isolated element (say, at v), which is part of a two-element connected component of V. We cannot remove it and still apply the induction hypothesis, as $V \setminus \{v\}$ belongs to $\mathrm{Iso}_{2,n-1}$. We then have no choice but to remove an isolated element.

When we are forced to remove an isolated element, we will apply the induction hypothesis to form a configuration for which the removed element can be "treadmilled" to its proper location, chaperoned by a element which is non-isolated in the final configuration and so can be returned once the removed element reaches its destination.

We briefly explain what we mean by treadmilling a pair of elements. Consider elements v_1 and $v_1 + e_2$ of a configuration V. If $\mathbb{H}_V(v_1)$ is positive and if there is a path from v_1 to $v_1 + 2e_2$ which lies outside of $V \setminus \{v_1\}$, then we can activate at v_1 and transport to $v_1 + 2e_2$. The result is that the pair $\{v_1, v_1 + 2e_2\}$ has shifted by e_2. Call the new configuration V'. If $v_1 + e_2$ is exposed in V' and if there is a path from $v_1 + e_2$ to $v_1 + 3e_2$ in $V' \setminus \{v_1 + e_2\}$, we can analogously shift the pair $\{v_1 + e_2, v_1 + 2e_2\}$ by another e_2.

Proof of Lemma 2.7.9. The proof is by induction on $n \geqslant 2$. We will actually prove a stronger claim, because it facilitates the induction step. To state the claim, we denote by $W_i = V_{i-1} \setminus \{x_i\}$ the HAT configuration "in between" V_{i-1} and V_i and by E_i the event that, during the transition from V_{i-1} to V_i, the transport step takes place inside of $B_i = D(r + 10n) \setminus W_i$:

$$E_i = \big\{ \{S_0, \ldots, S_{\tau_{W_i}}\} \subseteq B_i \big\}.$$

We claim that Lemma 2.7.9 is true even if the conclusion $\mathbf{P}_{V_{i-1}}(U_i = V_i) > 0$ is replaced by $\mathbf{P}_{V_{i-1}}(U_i = V_i, E_i) > 0$.

To prove this claim, we will show that, for any V satisfying the hypotheses, there are sequences of at most $100nr$ activation sites $x_1, \ldots, x_k$, transport sites $y_1, \ldots, y_k$, and random walk paths $\Gamma^1, \ldots, \Gamma^k$ such that the activation and transport sites can be realized by HAT from $V_0 = L_n$ to $V_k = V$, and such that each Γ^i is a finite random walk path from x_i to y_i which lies in B_i. While it is possible to explicitly list these sequences of sites and paths in the proof which follows, the depictions in upcoming Figures 2.9 and 2.10 are easier to understand and so we omit some cumbersome details regarding them.

Concerning the base case of $n = 2$, note that $\mathrm{NonIso}_{2,2}$ has the same elements as the equivalence class $\widehat{L}_2$, so $x_1 = e_2$, $y_1 = e_2$, $\Gamma^1 = \emptyset$ works. Suppose the claim holds up to $n - 1$ for $n \geqslant 3$. There are two cases:

1. There is a non-isolated v in $\partial_{\exp} V$ such that $V \setminus \{v\}$ belongs to $\mathrm{NonIso}_{2,n-1}$.

2. For every non-isolated v in $\partial_{\exp} V$, $V \setminus \{v\}$ belongs to $\mathrm{Iso}_{2,n-1}$.

It will be easy to form V using the induction hypothesis in Case 1. In Case 2, we will need to use the induction hypothesis to form a set related to V, and subsequently form V from this related set. An instance of Case 2 is depicted in Figure 2.9.

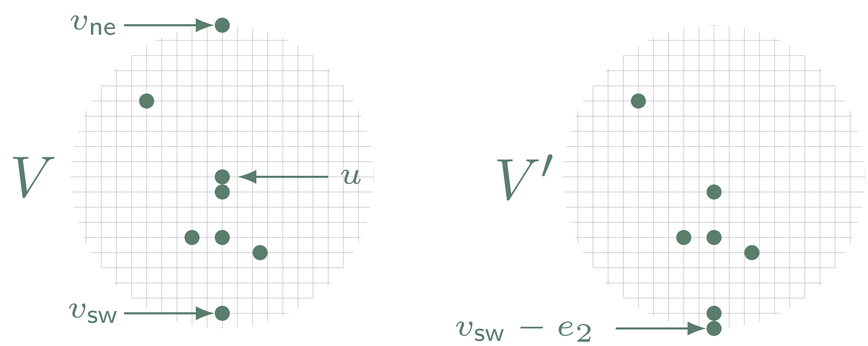

Figure 2.9: An instance of Case 2. If any non-isolated element of $\partial_{\mathrm{exp}} V$ is removed, the resulting set is isolated. We use the induction hypothesis to form $V' = (V \setminus \{v_{\mathsf{ne}}, u\}) \cup \{v_{\mathsf{sw}} - e_2\}$. The subsequent steps to obtain V from V' are depicted in Figure 2.10.

Case 1. Let r be an integer exceeding $10n$ and the radius of V and denote $R = r + 10(n-1)$. Recall that $V_0 = L_n$. Our strategy is to place one element of L_n outside of $D(R)$ and then apply the induction hypothesis to L_{n-1} to form most of V. This explains the role of the event E_i—it ensures that the element outside of the disk does not interfere with our use of the induction hypothesis.

To remove an element of L_n to $D(R)^c$, we treadmill (see the explanation following the lemma statement) the pair $\{(n-2)e_2, (n-1)e_2\}$ to $\{Re_2, (R+1)e_2\}$, after which we activate at Re_2 and transport to $(n-2)e_2$. This process requires $R - n + 2$ steps. It is clear that every transport step can occur via a finite random walk path which lies in $D(r + 10n)$. Call $a = (R+1)e_2$. The resulting configuration is $L_{n-1} \cup \{a\}$.

We will now apply induction hypothesis. Choose a non-isolated element v of $\partial_{\mathrm{exp}} V$ such that $V' = V \setminus \{v\}$ belongs to $\mathrm{NonIso}_{2,n-1}$. Such a v exists because we are in Case 1. By the induction hypothesis and because the radius of V' is at most r, there are sequences of at most $100(n-1)r$ activation and transport sites, which can be realized by HAT from $L_{n-1} \cup \{a\}$ to $V' \cup \{a\}$, and a corresponding sequence of finite random walk paths which lie in $D(R)$.

To complete this case, we activate at a and transport to v, which is possible because v was exposed and non-isolated in V. The existence of a random walk path from a to v which lies outside of V' is a consequence of Lemma 2.3.12. Recall that Lemma 2.3.12 applies only to sets in $\mathscr{H}_n$ (n-element sets which contain an exposed origin). If $A = V \cup \{a\}$, then $A - v$ belongs to $\mathscr{H}_n$. By Lemma 2.3.12, there is a finite random walk path from a to v which does not hit V' and which is contained in $D(R+3) \subseteq D(r + 10n)$.

In summary, there are sequences of at most $(R - n + 2) + 100(n-1)r + 1 \leqslant 100nr$ (the inequality follows from the assumption that $r \geqslant 10n$) activation and transport sites which can be realized by HAT from L_n to V, as well as corresponding finite random walk paths which remain within $D(r + 10n)$. This proves the claim in Case 1.

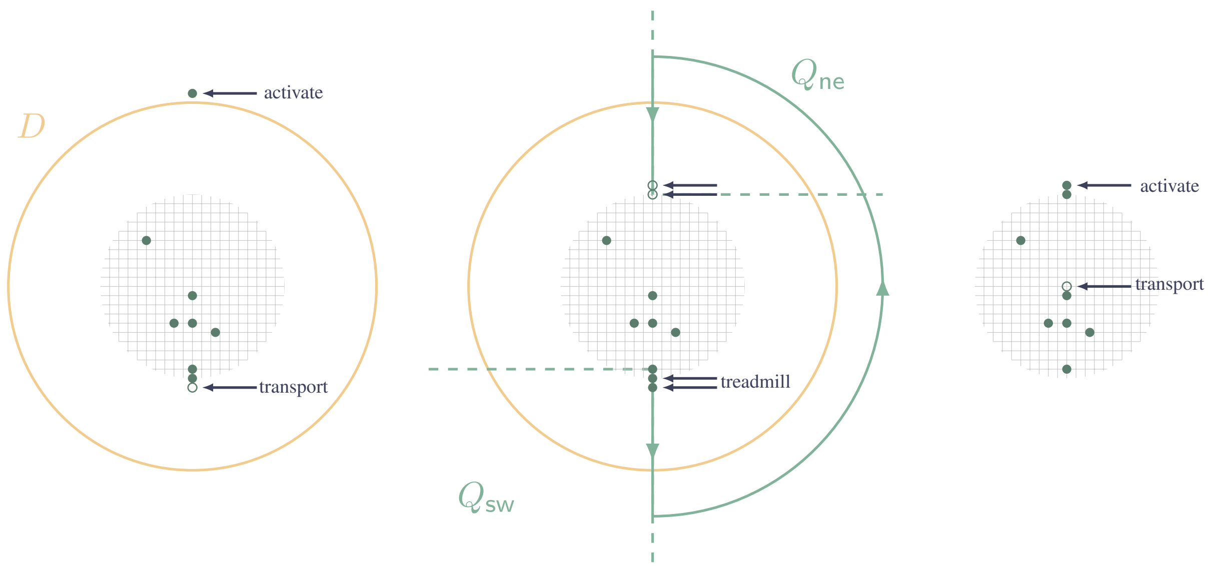

Figure 2.10: An instance of Case 2 (continued). On the left, we depict the configuration which results from the use of the induction hypothesis. The element outside of the disk D (the boundary of which is the orange circle) is transported to $v_{sw} - 2e_2$ (unfilled circle). In the middle, we depict the treadmilling of the pair $\{v_{sw} - e_2, v_{sw} - 2e_2\}$ through the quadrant Q_{sw}, around D^c, and through the quadrant Q_{ne}, until one of the treadmilled elements is at v_{ne}. The quadrants are depicted by dashed lines. On the right, the other element is returned to u (unfilled circle). The resulting configuration is V (see Figure 2.9).

Case 2. In this case, the removal of any non-isolated element v of $\partial_{\exp} V$ results in an isolated set $V \setminus \{v\}$, hence we cannot form such a set using the induction hypothesis. Instead, we will form a related, non-isolated set.

The first $R - n + 2$ steps, which produce $L_{n-1} \cup \{a\}$ from L_n, are identical to those of Case 1. We apply the induction hypothesis to form the set

$$V' = (V \setminus \{v_{ne}, u\}) \cup \{v_{sw} - e_2\},$$

which is depicted in Figure 2.9. Here, v_{ne} is the easternmost of the northernmost elements of V, v_{sw} is the westernmost of the southernmost elements of V, and u is any non-isolated element of $\partial_{\exp} V$ (e.g., $u = v_{ne}$ is allowed if v_{ne} is non-isolated).

The remaining steps are depicted in Figure 2.10. By the induction hypothesis and because the radius of V' is at most $r + 1$, there are sequences of at most $100(n - 1)(r + 1)$ activation and transport sites, which can be realized by HAT from $L_{n-1} \cup \{a\}$ to $V' \cup \{a\}$, and a corresponding sequence of finite random walk paths which lie in $D(R + 1)$.

Next, we activate at a and transport to $v_{\mathsf{sw}} - 2e_2$, which is possible because $v_{\mathsf{sw}} - 2e_2$ is exposed and non-isolated in V'. Like in Case 1, the existence of a finite random walk path from a to $v_{\mathsf{sw}} - 2e_2$ which lies in $D(R+3)\backslash V' \subseteq D(r+10n)$ is implied by Lemma 2.3.12. Denote the resulting configuration by V''.

The choice of v_{sw} ensures that $v_{\mathsf{sw}} - e_2$ and $v_{\mathsf{sw}} - 2e_2$ are the only elements of V'' which lie in the quadrant defined by

$$Q_{\mathsf{sw}} = (v_{\mathsf{sw}} - e_2) + \{v \in \mathbb{Z}^2 : v \cdot e_1 \leqslant 0,\ v \cdot e_2 \leqslant 0\}.$$

Additionally, the quadrant defined by

$$Q_{\mathsf{ne}} = v_{\mathsf{ne}} + \{v \in \mathbb{Z}^2 : v \cdot e_1 \geqslant 0,\ v \cdot e_2 \geqslant 0\}$$

contains no elements of V''. As depicted in Figure 2.10, this enables us to treadmill the pair $\{v_{\mathsf{sw}} - e_2, v_{\mathsf{sw}} - 2e_2\}$ from Q_{sw} to $D(R+3)^c$ and then to $\{v_{\mathsf{ne}}, v_{\mathsf{ne}} + e_2\}$ in Q_{ne}, without the pair encountering the remaining elements of V''. It is clear that this can be accomplished by fewer than $10(R+3)$ activation and transport sites, with corresponding finite random walk paths which lie in $D(R+6)$. The resulting configuration is $V''' = V \cup \{v_{\mathsf{ne}} + e_2\}\backslash\{u\}$.

Lastly, we activate at $v_{\mathsf{ne}} + e_2$ and transport to u, which is possible because the former is exposed in V''' and the latter is exposed and non-isolated in V. As before, the fact that there is a finite random walk path in $D(r+10n)$ which accomplishes the transport step is a consequence of Lemma 2.3.12. The resulting configuration is V.

In summary, there are sequences of fewer than $(R-n+2)+100(n-1)(r+1)+10(R+3)+2 \leqslant 100nr$ (the inequality follows from the assumption that $r \geqslant 10n$) activation and transport sites which can be realized by HAT from L_n to V, as well as corresponding finite random walk paths which remain in $D(r+10n)$. This proves the claim in Case 2. □

We can combine Lemma 2.7.8 and Lemma 2.7.9 to bound below the probability of forming a configuration from a line.

Lemma 2.7.10. *There is a constant c such that, if V is a configuration in* $\mathrm{NonIso}_{2,n}$ *with $n \geqslant 2$ and a diameter of at most $R \geqslant 10n$, then*

$$\mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{V}} \leqslant 200nR\right) \geqslant e^{-cn^3R^2}.$$

Proof. The hypotheses of Lemma 2.7.9 require an integer upper bound r on the radius of V of at least $10n$. We are free to assume that V contains the origin, in which case a choice of $r = \lfloor R \rfloor + 1$ works, due to the assumption $R \geqslant 10n$. We apply Lemma 2.7.9 with r to find that there is a sequence of configurations $V_0 = L_n, V_1, \dots, V_{k-1}, V_k = V$ such that $k \leqslant 100nr$, and such that $V_i \subseteq D(r+10n)$ and $\mathbf{P}_{V_{i-1}}(U_i = V_i) > 0$ for each i.

Because the transition probabilities are positive and because they concern sets V_{i-1} in the disk of radius $r + 10n$, Lemma 2.7.8 implies that each transition probability is at least

$$e^{-c_1 n \log n}(\log(r+10n))^{-1} \geqslant e^{-c_2 n^2 R}$$

for a constant c_1. The inequality follows from coarse bounds of $n \log n = O(n^2)$ and $\log r = O(R)$. We use this fact in the following string of inequalities:

$$\mathbf{P}_{L_n}(\mathcal{T}_{\widehat{V}} \leqslant 200nR) \geqslant \mathbf{P}_{L_n}(\mathcal{T}_{\widehat{V}} \leqslant k) \geqslant \mathbf{P}_{L_n}\big(\widehat{U}_k = \widehat{V}\big) \geqslant e^{-100nr\cdot c_2 n^2 R} \geqslant e^{-c_3 n^3 R^2}.$$

The first inequality holds because $k \leqslant 100nr \leqslant 200nR$; the second because $\{\widehat{U}_k = \widehat{V}\} \subseteq \{\mathcal{T}_{\widehat{V}} \leqslant k\}$; the third follows from the Markov property, $k \leqslant 100nr$, and the preceding bound from Lemma 2.7.8; the fourth from $100nr \leqslant 200nR$. □

2.7.2 Proof of Proposition 2.7.4

We now use Lemma 2.7.8 to obtain a tail bound on the time it takes for a given configuration to reach $\widehat{L}_n$. Our strategy is to repeatedly attempt to observe the formation of $\widehat{L}_n$ in n consecutive steps. If the attempt fails then, because the diameter of the resulting set may be larger—worsening the estimate (2.123)—we will wait until the diameter becomes smaller before the next attempt.

Proof of Proposition 2.7.4. To avoid confusion of U and U_t, we will use V_0 instead of U. We introduce a sequence of times, with consecutive times separated by at least n steps (which is enough time to attempt to form $\widehat{L}_n$) and at which the diameter of the configuration is at most $\theta_1 = \theta_{4n}(c_1 n)$ (where c_1 is the constant in Corollary 2.6.4). These will be the times at which we attempt to observe the formation of $\widehat{L}_n$. Define $\eta_0 = \inf\{t \geqslant 0 : \operatorname{diam}(U_t) \leqslant \theta_1\}$ and, for all $i \geqslant 1$, the times

$$\eta_i = \inf\{t \geqslant \eta_{i-1} + n : \operatorname{diam}(U_t) \leqslant \theta_1\}.$$

We use these times to define three events. Two of the events use a parameter K which we assume is at least the maximum of R and θ_2, where θ_2 equals $\theta_{4n}(cn)$ with $c = c_1 + 2c_2$ and c_2 is the constant guaranteed by Lemma 2.7.8. (The constant c is the one which appears in the statement of the proposition.) In particular, K is at least the maximum diameter $\theta_1 + n$ of a configuration at time $\eta_{i-1} + n$.

The first is the event that it takes an unusually long time for the diameter to fall below θ_1 for the first time:

$$E_1(K) = \left\{\eta_0 > 3K\big(\log(3K)\big)^{1+\varepsilon}\right\},$$

where $\varepsilon = 2n^{-4}$ is the $o_n(1)$ term from Corollary 2.6.4. The second is the event that an unusually long time elapses between $\eta_{i-1} + n$ and η_i for some $1 \leqslant i \leqslant m$:

$$E_2(m, K) = \bigcup_{i=1}^{m} \left\{\eta_i - (\eta_{i-1} + n) > 3K\big(\log(3K)\big)^{1+\varepsilon}\right\}.$$

The third is the event that we do not observe the formation of $\widehat{L}_n$ in $m \geqslant 1$ attempts:

$$E_3(m) = \bigcap_{i=1}^{m} \left\{\mathcal{T}_{\widehat{L}_n} > \eta_{i-1} + n\right\}.$$

Call $E(m,K) = E_1(K) \cup E_2(m,K) \cup E_3(m)$. When none of these events occur, we can bound $\mathcal{T}_{\widehat{L}_n}$:

$$\begin{aligned}\mathcal{T}_{\widehat{L}_n} \mathbf{1}_{E(m,K)^c} &\leqslant \left(\eta_0 + \sum_{i=1}^{m} (\eta_i - (\eta_{i-1} + n)) \right) \mathbf{1}_{E(m,K)^c} + n(m+1) \\ &\leqslant 3K\big(\log(3K)\big)^{1+\varepsilon} + 3mK\big(\log(3K)\big)^{1+\varepsilon} + n(m+1). \end{aligned} \tag{2.128}$$

We will show that if m is taken to be $3K(\log \theta_2)^n$, then $\mathbf{P}_{V_0}(E(m,K))$ is at most e^{-K}. Substituting this choice of m into (2.128) and using $(\log \theta_2)^{2n} \leqslant \theta_2 \leqslant K$ to simplify, we obtain a further upper bound of

$$\mathcal{T}_{\widehat{L}_n} \mathbf{1}_{E(m,K)^c} \leqslant K^3. \tag{2.129}$$

By (2.129), if we show $\mathbf{P}_{V_0}(E(m,K)) \leqslant e^{-K}$, then we are done. We start with a bound on $\mathbf{P}_{V_0}(E_1(K))$. Applying Corollary 2.6.4 with $3K$ in the place of t, r in the place of d, and $3K = \max\{3K, R\}$ in the place of $\max\{t,d\}$, gives

$$\mathbf{P}_{V_0}(E_1(K)) \leqslant e^{-3K}. \tag{2.130}$$

We will use Corollary 2.6.4 and a union bound to bound $\mathbf{P}_{V_0}(E_2(m,K))$. Because diameter grows at most linearly in time, the diameter of $U_{\eta_{i-1}+n} \in \mathcal{F}_{\eta_{i-1}}$ is at most $\theta_1 + n \leqslant 3K$. Consequently, Corollary 2.6.4 implies

$$\mathbf{P}_{V_0}\left(\eta_i - (\eta_{i-1}+n) > 3K\big(\log(3K)\big)^{1+\varepsilon} \,\Big|\, \mathcal{F}_{\eta_{i-1}+n} \right) \leqslant e^{-3K}. \tag{2.131}$$

A union bound over the constituent events of $E_2(m,K)$ and (2.131) give

$$\mathbf{P}_{V_0}(E_2(m,K)) \leqslant m e^{-3K}. \tag{2.132}$$

To bound the probability of $E_3(m)$, we will use Lemma 2.7.8. First, we need to identify a suitable sequence of HAT transitions. For any $0 \leqslant j \leqslant m-1$, given $\mathcal{F}_{\eta_j}$, set $V_0' = U_{\eta_j} \in \mathcal{F}_{\eta_j}$. There are pairs $\{(x_i, y_i) : 1 \leqslant i \leqslant n\}$ such that, setting $V_i' = V_{i-1}' \cup \{y_i\} \setminus \{x_i\}$ for $1 \leqslant i \leqslant n$, each transition probability $\mathbf{P}_{V_{i-1}'}(U_i = V_i')$ is positive and $V_n' \in \widehat{L}_n$. By Lemma 2.7.8, each transition probability is at least

$$\mathbf{P}_{V_{i-1}'}(U_i = V_i') \geqslant e^{-c_2 n \log n} \big(\log(\theta_1 + n)\big)^{-1} \geqslant (\log \theta_2)^{-1}. \tag{2.133}$$

For the first inequality we used the fact that the diameter of V_0' is at most θ_1, so after $i \leqslant n$ steps it is at most $\theta_1 + n$.

By the strong Markov property and (2.133),

$$\begin{aligned}\mathbf{P}_{V_0}\left(\mathcal{T}_{\widehat{L}_n} \leqslant \eta_j + n \,\Big|\, \mathcal{F}_{\eta_j} \right) &\geqslant \mathbf{P}_{V_0}\left(U_{\eta_j+1} = V_1', \ldots, U_{\eta_j+n} = V_n' \,\Big|\, \mathcal{F}_{\eta_j} \right) \\ &\geqslant \prod_{i=1}^{n} \mathbf{P}_{V_{i-1}'}(U_i = V_i') \geqslant (\log \theta_2)^{-n}. \end{aligned} \tag{2.134}$$

Because $E_3(j) \in \mathcal{F}_{\eta_j}$, (2.134) implies

$$\mathbf{P}_{V_0}\left(\mathcal{T}_{\widehat{L}_n} \leqslant \eta_j + n \,\middle|\, E_3(j)\right) \geqslant (\log \theta_2)^{-n}. \tag{2.135}$$

Using (2.135), we calculate

$$\mathbf{P}_{V_0}(E_3(m)) = \prod_{j=0}^{m-1} \mathbf{P}_{V_0}\left(\mathcal{T}_{\widehat{L}_n} > \eta_j + n \,\middle|\, E_3(j)\right) \leqslant \prod_{j=0}^{m-1} \left(1 - (\log \theta_2)^{-n}\right) \leqslant e^{-3K}. \tag{2.136}$$

Combining (2.130), (2.132), and (2.136), and simplifying using the fact that $K \geqslant \theta_2$, we find

$$\mathbf{P}_{V_0}(E(m,K)) \leqslant (m+2)e^{-3K} \leqslant e^{-K}.$$

□

2.7.3 Proof of Proposition 2.7.5

To prove this proposition, we will attempt to observe the formation of $\widehat{U}$ from $\widehat{L}_n$ and wait for the set to collapse if its diameter becomes too large, as we did in proving Proposition 2.7.4. However, there is an added complication: at the time that the set collapses, it does not necessarily form $\widehat{L}_n$, so we will need to use Proposition 2.7.4 to return to $\widehat{L}_n$ before another attempt at forming $\widehat{U}$. For convenience, we package these steps together in the following lemma.

Lemma 2.7.11. *There is a constant c such that, if V_0 is a configuration in $\mathrm{NonIso}_{2,n}$ with a diameter of R, then for any $K \geqslant \max\{R, \theta_{4n}(cn)\}$,*

$$\mathbf{P}_{V_0}\left(\mathcal{T}_{\widehat{L}_n} \leqslant 9K^3\right) \geqslant 1 - e^{-K}. \tag{2.137}$$

Proof. Call $\theta = \theta_{4n}(cn)$ where c is the constant guaranteed by Proposition 2.7.4. First, we wait until the diameter falls to θ. By Corollary 2.6.4,

$$\mathbf{P}_{V_0}\left(\mathcal{T}(\theta) \leqslant 2K\big(\log(2K)\big)^{1+\varepsilon}\right) \geqslant 1 - e^{-2K}, \tag{2.138}$$

where $\varepsilon = 2n^{-4}$ is the $o_n(1)$ term from Corollary 2.6.4. Second, from $U_{\mathcal{T}(\theta)}$, we wait until the configuration forms a line. By Proposition 2.7.4, for any $K \geqslant \theta$,

$$\mathbf{P}_{U_{\mathcal{T}(\theta)}}\left(\mathcal{T}_{\widehat{L}_n} \leqslant 8K^3\right) \geqslant 1 - e^{-2K}. \tag{2.139}$$

Simplifying with $K \geqslant \theta$, we have

$$2K\big(\log(2K)\big)^{1+\varepsilon} + 8K^3 \leqslant 9K^3.$$

Combining this bound with (2.138) and (2.139) gives (2.137). □

Proof of Proposition 2.7.5. We will use V to denote the target configuration instead of U, to avoid confusion with U_t. Recall that, for any configuration V in $\mathrm{NonIso}_{2,n}$ with a diameter upper bound of $r \geqslant 10n$, Lemma 2.7.10 gives a constant c_1 such that

$$\mathbf{P}_{\widehat{L}_n}(\mathcal{T}_{\widehat{V}} \leqslant 200nr) \geqslant e^{-c_1 n^3 r^2}.$$

Since $10nR \geqslant 10n$ is a diameter upper bound on V, we can apply the preceding inequality with $r = 10nR$:

$$\mathbf{P}_{\widehat{L}_n}\big(\mathcal{T}_{\widehat{V}} \leqslant 2000n^2 R\big) \geqslant e^{-c_1 n^4 R^2}. \tag{2.140}$$

With this result in mind, we denote $k = 2000n^2 R$ and define a sequence of times by

$$\zeta_0 \equiv 0 \quad \text{and} \quad \zeta_i = \inf\{t \geqslant \zeta_{i-1} + k : \widehat{U}_t = \widehat{L}_n\} \quad \text{for all } i \geqslant 1.$$

Here, the buffer of k steps is the period during which we attempt to observe the formation of V. After each failed attempt, because the diameter increases by at most one with each step, the diameter of $U_{\zeta_i + k}$ may be no larger than $k + n$.

We define two rare events in terms of these times and a parameter K, which we assume to be at least $\max\{e^{R^{2.1}}, \theta_{4n}(c_2 n)\}$, where c_2 is the greater of c_1 and the constant from Lemma 2.7.11. In particular, under this assumption, K is greater than $e^{4c_1 n^4 R^2}$ and $k+n$—a fact we will use later.

The first rare event is the event that an unusually long time elapses between $\zeta_{i-1} + k$ and ζ_i, for some $i \leqslant m$:

$$F_1(m, K) = \bigcup_{i=1}^{m} \left\{\zeta_i - (\zeta_{i-1} + k) > 72K^3\right\}.$$

The second is the event that we do not observe the formation of $\widehat{V}$ in $m \geqslant 1$ attempts:

$$F_2(m) = \bigcap_{i=1}^{m} \left\{\mathcal{T}_{\widehat{V}} > \zeta_{i-1} + k\right\}.$$

Call $F(m,K) = F_1(m,K) \cup F_2(m)$. When $F(m,K)^c$ occurs, we can bound $\mathcal{T}_{\widehat{V}}$ as

$$\mathcal{T}_{\widehat{V}} \mathbf{1}_{F(m,K)^c} = \sum_{i=0}^{m-1} (\zeta_i - (\zeta_{i-1} + k)) \mathbf{1}_{E(m,K)^c} + mk \leqslant 72mK^3 + mk. \tag{2.141}$$

We will show that if m is taken to be $2Ke^{c_1 n^4 R^2}$, then $\mathbf{P}_{\widehat{L}_n}(F(m,K))$ is at most e^{-K}. Substituting this value of m into (2.141) and simplifying with $K \geqslant k$ and then $K \geqslant e^{4c_1 n^4 R^2}$ gives

$$\mathcal{T}_{\widehat{V}} \mathbf{1}_{F(m,K)^c} \leqslant K^4 e^{2c_1 n^4 R^2} \leqslant K^5. \tag{2.142}$$

By (2.142), if we prove $\mathbf{P}_{\widehat{L}_n}(F(m,K)^c) \leqslant e^{-K}$, then we are done. We start with a bound on $\mathbf{P}_{\widehat{L}_n}(F_1(m,K))$. By the strong Markov property applied to the stopping time $\zeta_{i-1} + k$,

$$\mathbf{P}_{\widehat{L}_n}\left(\zeta_i - (\zeta_{i-1} + k) > 72K^3 \;\middle|\; \mathcal{F}_{\zeta_{i-1}+k}\right) = \mathbf{P}_{U_{\zeta_{i-1}+k}}\big(\zeta_1 > 72K^3\big) \leqslant e^{-2K}. \tag{2.143}$$

The inequality is due to Lemma 2.7.11, which applies to $U_{\zeta_{i-1}+k}$ and K because $U_{\zeta_{i-1}+k}$ is a non-isolated configuration with a diameter of at most $k+n$ and because $K \geqslant \max\{k+n, \theta_{4n}(c_2 n)\}$. From a union bound over the events which comprise $F_1(m,K)$ and (2.143), we find

$$\mathbf{P}_{\widehat{L}_n}(F_1(m,K)) \leqslant m e^{-2K}. \tag{2.144}$$

To bound $\mathbf{P}_{\widehat{L}_n}(F_2(m))$, we apply the strong Markov property to ζ_j and use (2.140):

$$\mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{V}} \leqslant \zeta_j + k \;\middle|\; \mathcal{F}_{\zeta_j}\right) \geqslant \mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{V}} \leqslant k\right) \geqslant 1 - e^{-c_1 n^4 R^2}. \tag{2.145}$$

Then, because $F_2(j) \in \mathcal{F}_{\zeta_j}$ and by (2.145),

$$\mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{V}} \leqslant \zeta_j + k \;\middle|\; F_2(j)\right) \geqslant 1 - e^{-c_1 n^4 R^2}. \tag{2.146}$$

We use (2.146) to calculate

$$\mathbf{P}_{\widehat{L}_n}(F_2(m)) = \prod_{j=0}^{m-1} \mathbf{P}_{\widehat{L}_n}\left(\mathcal{T}_{\widehat{V}} > \zeta_j + k \;\middle|\; F_2(j)\right) \leqslant \prod_{j=0}^{m-1}(1 - e^{-c_1 n^4 R^2}) \leqslant e^{-2K}. \tag{2.147}$$

The second inequality is due to the choice $m = 2K e^{c_1 n^4 R^2}$.

Recall that $F(m,K)$ is the union of $F_1(m,K)$ and $F_2(m)$. We have

$$\mathbf{P}_{\widehat{L}_n}(F(m,K)) \leqslant \mathbf{P}_{\widehat{L}_n}(F_1(m,K)) + \mathbf{P}_{\widehat{L}_n}(F_2(m)) \leqslant m e^{-2K} + e^{-2K} \leqslant e^{-K}.$$

The first inequality is a union bound; the second is due to (2.144) and (2.147); the third holds because $m+1 \leqslant e^K$. $\square$

2.7.4 Proof of Proposition 2.7.2

We now prove a tightness estimate for the stationary distribution—that is, an upper bound on $\pi_n\big(\operatorname{diam}(\widehat{U}) \geqslant d\big)$. By Proposition 2.7.1, the stationary probability $\pi_n(\widehat{U})$ of any non-isolated, n-element coniguration $\widehat{U}$ is the reciprocal of $\mathbf{E}_{\widehat{U}} \mathcal{T}_{\widehat{U}}$. When d is large (relative to θ_{4n}), this expected return time will be at least exponentially large in $\frac{d}{(\log d)^{1+o_n(1)}}$. This exponent arises from the consideration that, for a configuration with a diameter below θ_{4n} to increase its diameter to d, it must avoid collapse over the timescale for which it is typical (i.e., $(\log d)^{1+o_n(1)}$) approximately $\frac{d}{(\log d)^{1+o_n(1)}}$ times consecutively. Because the number of n-element configurations with a diameter of approximately d is negligible relative to their expected return times, the collective weight under π_n of such configurations will be exponentially small in $\frac{d}{(\log d)^{1+o_n(1)}}$.

We note that, while there are abstract results which relate hitting times to the stationary distribution (e.g., [GLPP17, Lemma 4]), we cannot directly apply results which require bounds on hitting times which hold uniformly for any initial configuration. This is because hitting times from $\widehat{V}$ depend on its diameter. We could apply such results after partitioning $\widehat{\mathrm{NonIso}}_{2,n}$ by diameter, but we would then save little effort from their use.

Proof of Proposition 2.7.2. Let d be at least $2\theta_{4n}$ and take $\varepsilon = 2n^{-4}$. We claim that, for any configuration $\widehat{U}$ with a diameter in $[2^j d, 2^{j+1} d)$ for an integer $j \geqslant 0$, the expected return time to $\widehat{U}$ satsfies

$$\mathbf{E}_{\widehat{U}} \mathcal{T}_{\widehat{U}} \geqslant \exp\left(\frac{2^j d}{(\log(2^j d))^{1+2\varepsilon}} \right). \tag{2.148}$$

We can use (2.148) to prove (2.118) in the following way. We write $\{\operatorname{diam}(\widehat{U}) \geqslant d\}$ as a disjoint union of events of the form $H_j = \{2^j \leqslant \operatorname{diam}(\widehat{U}) < 2^{j+1} d\}$ for $j \geqslant 0$. Because a disk with a diameter of at most $2^{j+1}d$ contains fewer than $\lfloor 4^{j+1} d^2 \rfloor$ elements of $\mathbb{Z}^2$, the number of non-isolated, n-element configurations with a diameter of at most $2^{j+1}d$ satisfies

$$\left| \left\{ \widehat{U} \text{ in } \widehat{\mathrm{N}}\mathrm{onIso}_{2,n} \text{ with } 2^j d \leqslant \operatorname{diam}(\widehat{U}) < 2^{j+1} d \right\} \right| \leqslant \binom{\lfloor 4^{j+1} d^2 \rfloor}{n} \leqslant (4^{j+1} d^2)^n. \tag{2.149}$$

We use (2.117) with (2.148) and (2.149) to estimate

$$\pi_n\big(\operatorname{diam}(\widehat{U}) \geqslant d\big) = \sum_{j=0}^{\infty} \pi_n(H_j) = \sum_{j=0}^{\infty} \sum_{\widehat{U} \in H_j} \pi_n(\widehat{U}) \leqslant \sum_{j=0}^{\infty} (4^{j+1} d^2)^n e^{-\frac{2^j d}{(\log(2^j d))^{1+2\varepsilon}}}. \tag{2.150}$$

Using the fact that $d \geqslant 2\theta_{4n}$, it is easy to check that the ratio of the $(j+1)^{\text{st}}$ summand to the j^{th} summand in (2.150) is at most e^{-j-1}, for all $j \geqslant 0$. Accordingly, we have

$$\pi_n\big(\operatorname{diam}(\widehat{U}) \geqslant d\big) \leqslant (4d^2)^n e^{-\frac{d}{(\log d)^{1+2\varepsilon}}} \sum_{j=0}^{\infty} e^{-j} \leqslant e^{-\frac{d}{(\log d)^{1+3\varepsilon}}},$$

where the second inequality is justified by the fact that $d \geqslant 2\theta_{4n}$. This proves (2.118) when the claimed bound (2.148) holds.

We will prove (2.148) by making a comparison with a geometric random variable on $\{0, 1, \dots\}$ with a "success" probability of $e^{-\frac{d}{(\log d)^{1+\varepsilon}}}$ (or with $2^j d$ in place of d). This geometric random variable will model the number of visits to configurations with diameters below θ_{4n} before reaching a diameter of d, and the success probability arises from the fact that, for a configuration to increase its diameter to d from θ_{4n}, it must avoid collapse over $d - \theta_{4n}$ steps. By Corollary 2.6.4, this happens with a probability which is exponentially small in $\frac{d}{(\log d)^{1+\varepsilon}}$.

Let $\widehat{U}$ be a non-isolated, n-element configuration with a diameter in $[2^j d, 2^{j+1} d)$. Additionally, let $\widehat{V}$ minimize $\mathbf{E}_{\widehat{V}} \mathcal{T}_{\widehat{U}}$ among $\widehat{\mathcal{V}}$, the configurations in $\mathrm{NonIso}_{2,n}$ with a diameter of at most θ_{4n}. Denoting by N the number of visits to configurations in $\widehat{\mathcal{V}}$ before $\mathcal{T}_{\widehat{U}}$, we claim

$$\mathbf{E}_{\widehat{U}} \mathcal{T}_{\widehat{U}} \geqslant (\log(2^{j+1} d))^{-2n}\, \mathbf{E}_{\widehat{V}} N. \tag{2.151}$$

By (2.133),

$$\mathbf{P}_{\widehat{U}}(\mathcal{T}_{\widehat{L}_n} < \mathcal{T}_{\widehat{U}}) \geqslant \big(\log(2^{j+1} d)\big)^{-2n}.$$

By this bound and the strong Markov property (applied to $\mathcal{T}_{\widehat{L}_n}$), and due to our choice of $\widehat{V}$,

$$\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} \geqslant \big(\log(2^{j+1}d)\big)^{-2n}\,\mathbf{E}_{\widehat{L}_n}\mathcal{T}_{\widehat{U}} \geqslant \big(\log(2^{j+1}d)\big)^{-2n}\mathbf{E}_{\widehat{V}}\mathcal{T}_{\widehat{U}}. \tag{2.152}$$

The time it takes to reach $\widehat{U}$ from $\widehat{V}$ is at least the number N of visits U_t makes to $\widehat{\mathcal{V}}$ before $\mathcal{T}_{\widehat{U}}$, so (2.152) implies (2.151).

The virtue of the lower bound (2.151) is that we can bound below $\mathbf{E}_{\widehat{V}}N$ as

$$\mathbf{E}_{\widehat{V}}N = \mathbf{P}_{\widehat{V}}(\mathcal{T}_{\widehat{\mathcal{V}}} < \mathcal{T}_{\widehat{U}})\left(1 + \mathbf{E}_{\widehat{V}}\left[N \mid \mathcal{T}_{\widehat{\mathcal{V}}} < \mathcal{T}_{\widehat{U}}\right]\right) \geqslant \mathbf{P}_{\widehat{V}}(\mathcal{T}_{\widehat{\mathcal{V}}} < \mathcal{T}_{\widehat{U}})\left(1 + \mathbf{E}_{\widehat{V}}N\right).$$

This bound implies that $\mathbf{E}_{\widehat{V}}N$ is at least the expected value of a geometric random variable on $\{0, 1, \dots\}$ with success parameter p of $\mathbf{P}_{\widehat{V}}(\mathcal{T}_{\widehat{U}} < \mathcal{T}_{\widehat{\mathcal{V}}})$:

$$\mathbf{E}_{\widehat{V}}N \geqslant (1-p)p^{-1}. \tag{2.153}$$

It remains to obtain an upper bound on p.

Because diameter increases at most linearly in time, $\mathcal{T}_{\widehat{U}}$ is at least $2^j d - \theta_{4n}$ under $\mathbf{P}_{\widehat{V}}$. Consequently,

$$\mathbf{P}_{\widehat{V}}(\mathcal{T}_{\widehat{U}} < \mathcal{T}_{\widehat{\mathcal{V}}}) \leqslant \mathbf{P}_{\widehat{V}}\big(\mathcal{T}(\theta_{4n}) > 2^j d - \theta_{4n}\big). \tag{2.154}$$

We apply Corollary 2.6.4 with t equal to $\frac{2^j d - \theta_{4n}}{(\log(2^j d))^{1+\varepsilon}}$, finding

$$\mathbf{P}_{\widehat{V}}(\mathcal{T}(\theta_{4n}) > 2^j d - \theta_{4n}) \leqslant \exp\left(-\frac{2^j d - \theta_{4n}}{(\log(2^j d))^{1+\varepsilon}}\right).$$

By (2.154), this is also an upper bound on $p < \frac{1}{2}$ and so, by (2.153), $\mathbf{E}_{\widehat{V}}N$ is at least $(2p)^{-1}$. Substituting these bounds into (2.151) and simplifying with the fact that $d \geqslant 2\theta_{4n}$, we find that the expected return time to $\widehat{U}$ satisfies (2.148):

$$\mathbf{E}_{\widehat{U}}\mathcal{T}_{\widehat{U}} \geqslant \tfrac{1}{2}\big(\log(2^{j+1}d)\big)^{-2n}\exp\left(\frac{2^j d - \theta_{4n}}{\big(\log(2^j d)\big)^{1+\varepsilon}}\right) \geqslant \exp\left(\frac{2^j d}{\big(\log(2^j d)\big)^{1+2\varepsilon}}\right).$$

□

2.8 Motion of the center of mass

As a consequence of the results of Section 2.7 and standard renewal theory, the center of mass process $(\mathscr{M}_t)_{t\geqslant 0}$, after linear interpolation and rescaling $(t^{-1/2}\mathscr{M}_{st})_{s\in[0,1]}$, and when viewed as a measure on $\mathcal{C}([0,1])$, converges weakly to two-dimensional Brownian motion as $t \to \infty$. This is the content of Theorem 2.1.5.

We will use the following lemma to bound the coordinate variances of the Brownian motion limit. To state it, we denote by $\tau_i = \inf\{t > \tau_{i-1} : \widehat{U}_t = \widehat{L}_n\}$ the i^{th} return time to $\widehat{L}_n$.

Lemma 2.8.1. *Let c be the constant from Proposition 2.7.5 and abbreviate $\theta_{4n}(cn)$ by θ. If, for some $i \geqslant 0$, X is one of the random variables*

$$\tau_{i+1} - \tau_i, \quad \|\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}\|, \quad \textit{or} \quad \|\mathscr{M}_t - \mathscr{M}_{\tau_i}\|\mathbf{1}(\tau_i \leqslant t \leqslant \tau_{i+1}),$$

then the distribution of X satisfies the following tail bound

$$\mathbf{P}_{\widehat{L}_n}\big(X > K^5\big) \leqslant e^{-K}, \quad K \geqslant \theta. \tag{2.155}$$

Consequently,

$$\mathbf{E}_{\widehat{L}_n} X \leqslant 2\theta^6 \quad \textit{and} \quad \operatorname{Var}_{\widehat{L}_n} X \leqslant 2\theta^{12}. \tag{2.156}$$

Proof. Because the diameter of $\widehat{L}_n$ is at most n, for any $K \geqslant \theta$, Proposition 2.7.5 implies

$$\mathbf{P}_{\widehat{L}_n}(\tau_1 > K^5) \leqslant e^{-K}.$$

Applying the strong Markov property to τ_i, we find (2.155) for $X = \tau_{i+1} - \tau_i$. Using (2.155) with the tail sum formulas for the first and second moments gives (2.156) for this X. The other cases of X then follow from

$$\big\|\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}\big\| \leqslant \tau_{i+1} - \tau_i.$$

□

Proof of Theorem 2.1.5. Standard arguments (e.g., Section 8 of [Bil99]) combined with the renewal theorem show that $\big(t^{-1/2}\mathscr{M}_{st}\big)_{t\geqslant 1}$ is a tight sequence of functions. We claim that the finite-dimensional distributions of the rescaled process converge as $t \to \infty$ to those of two-dimensional Brownian motion.

For any $m \geqslant 1$ and times $0 = s_0 \leqslant s_1 < s_2 < \cdots < s_m \leqslant 1$, form the random vector

$$t^{-1/2}\left(\mathscr{M}_{s_1 t}, \mathscr{M}_{s_2 t} - \mathscr{M}_{s_1 t}, \dots, \mathscr{M}_{s_m t} - \mathscr{M}_{s_{m-1} t}\right). \tag{2.157}$$

For s in $[0,1]$, we denote by $I(s)$ the number of returns to $\widehat{L}_n$ by time st. Lemma 2.8.1 and Markov's inequality imply that $\|\mathscr{M}_{s_i t} - \mathscr{M}_{\tau_{I(s_i)}}\| \to 0$ in probability as $t \to \infty$, hence, by Slutsky's theorem, the distributions of (2.157) and

$$t^{-1/2}\left(\mathscr{M}_{\tau_{I(s_1)}}, \mathscr{M}_{\tau_{I(s_2)}} - \mathscr{M}_{\tau_{I(s_1)+1}}, \dots, \mathscr{M}_{\tau_{I(s_m)}} - \mathscr{M}_{\tau_{I(s_{m-1})+1}}\right) \tag{2.158}$$

have the same $t \to \infty$ limit. By the renewal theorem, $I(s_1) < I(s_2) < \cdots < I(s_m)$ for all sufficiently large t, so the strong Markov property implies the independence of the entries in (2.158) for all such t.

A generic entry in (2.158) is a sum of independent increments of the form $\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}$. As noted in Section 2.1, the transition probabilities are unchanged when configurations are multiplied by elements of the symmetry group $\mathcal{G}$ of $\mathbb{Z}^2$. This implies

$$\mathbf{E}_{\widehat{L}_n}\left[\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}\right] = o \quad \text{and} \quad \Sigma = \nu^2 \mathbf{I},$$

where Σ is the variance-covariance matrix of $\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}$ and ν is a constant which, by Lemma 2.8.1, is finite. The renewal theorem implies that the scaled variance $t^{-1}\nu^2(I(s_i) - I(s_{i-1}))$ of the i^{th} entry converges almost surely to $(s_i - s_{i-1})\chi^2$ where $\chi^2 = \nu^2/\mathbf{E}_{\widehat{L}_n}[\tau_1]$, hence, by Slutsky's theorem, we can replace the scaled variance of each entry in (2.158) with its almost-sure limit, without affecting the limiting distribution of the vector.

By the central limit theorem,

$$\frac{1}{\chi\sqrt{t}}\left(\mathscr{M}_{\tau_{I(s_i)}} - \mathscr{M}_{\tau_{I(s_{i-1})+1}}\right) \xrightarrow{\mathrm{d}} \mathcal{N}\left(o, (s_i - s_{i-1})\mathbf{I}\right),$$

which, by the independence of the entries in (2.158) for all sufficiently large t, implies

$$\frac{1}{\chi\sqrt{t}}\left(\mathscr{M}_{s_1 t},\, \mathscr{M}_{s_2 t} - \mathscr{M}_{s_1 t},\, \dots,\, \mathscr{M}_{s_m t} - \mathscr{M}_{s_{m-1} t}\right) \xrightarrow{\mathrm{d}} \left(\mathbf{B}(s_1), \mathbf{B}(s_2 - s_1), \dots, \mathbf{B}(s_m - s_{m-1})\right), \quad (2.159)$$

as $t \to \infty$. Because m and the $\{s_i\}_{i=1}^m$ were arbitrary, the continuous mapping theorem and (2.159) imply the convergence of the finite-dimensional distributions of $\left(\frac{1}{\chi\sqrt{t}}\mathscr{M}_{st}, 0 \leqslant s \leqslant 1\right)$ to those of $(\mathbf{B}(s), 0 \leqslant s \leqslant 1)$. This proves the weak convergence component of Theorem 2.1.5.

It remains to bound χ^2, which we do by estimating $\mathbf{E}_{\widehat{L}_n}[\tau_1]$ and ν^2. $\mathbf{E}_{\widehat{L}_n}[\tau_1]$ is bounded above by $2\theta^5$, due to Lemma 2.8.1, and below by 1. Here, $\theta = \theta_{4n}(c_1 n)$ and c_1 is the constant from Proposition 2.7.5. To bound below ν^2, denote the e_2 component of $\mathscr{M}_{\tau_{i+1}} - \mathscr{M}_{\tau_i}$ by X and observe that $\mathbf{P}_{\widehat{L}_n}(X = n^{-1})$ is at least the probability that, from L_n, the element at o is activated and subsequently deposited at $(0, n)$ (recall that L_n is the segment from o to $(0, n-1)$), resulting in $\tau_1 = 1$ and $\mathscr{M}_{\tau_1} = \mathscr{M}_0 + n^{-1}e_2$. This probability is at least $e^{-c_2 n}$ for a constant c_2. Markov's inequality applied to X^2 then gives

$$\mathrm{Var}_{\widehat{L}_n} X \geqslant \mathbf{P}_{\widehat{L}_n}(X^2 \geqslant n^{-2}) \geqslant n^{-2}e^{-c_2 n} \geqslant e^{-c_3 n}.$$

By Lemma 2.8.1, ν^2 is at most $2\theta^{10}$. In summary,

$$1 \leqslant \mathbf{E}_{\widehat{L}_n}[\tau_1] \leqslant 2\theta^5 \quad \text{and} \quad e^{-c_3 n} \leqslant \nu^2 \leqslant 2\theta^{10},$$

which implies

$$\theta_{5n}(cn)^{-1} \leqslant e^{-c_3 n}(2\theta^5)^{-1} \leqslant \chi^2 \leqslant 2\theta^{10} \leqslant \theta_{5n}(cn),$$

with $c = \max\{c_1, c_3\}$. □

2.9 Proofs deferred from Section 2.3

In this section, we collect some results which support the proof of Theorem 2.1.6 in Section 2.3.

2.9.1 Potential kernel bounds

The following lemma states several facts about the potential kernel. As each fact is a simple consequence of (2.14), we omit its proof.

Lemma 2.9.1. *In what follows, x, y, z, z' are elements of $\mathbb{Z}^2$.*

1. *For $\mathfrak{a}(y)$ to be at least $\mathfrak{a}(x)$, it suffices to have*
$$\|y\| \geqslant \|x\|(1 + \pi\lambda\|x\|^{-2} + (\pi\lambda)^2\|x\|^{-4}).$$
In particular, if $\|x\| \geqslant 2$, then $\|y\| \geqslant 1.06\|x\|$ suffices.
2. *When $\|x\| \geqslant 1$, $\mathfrak{a}(x)$ is at least $\frac{2}{\pi}\log\|x\|$. When $\|x\| \geqslant 2$, $\mathfrak{a}(x)$ is at most $4\log\|x\|$.*
3. *If $z, z' \in C(r)$ and $y \in D(R)^c$ for $r \leqslant \frac{1}{100}R$ and $R \geqslant 100$, then*
$$|\mathfrak{a}(y-z) - \mathfrak{a}(y-z')| \leqslant \tfrac{4}{\pi}.$$
4. *If x and y satisfy $\|x\|, \|y\| \geqslant 1$ and $K^{-1} \leqslant \frac{\|y\|}{\|x\|} \leqslant K$ for some $K \geqslant 2$, then*
$$\mathfrak{a}(y) - \mathfrak{a}(x) \leqslant \log K.$$
5. *Let $x, y \in \mathbb{Z}^2$ with $\|x\| \geqslant 8\|y\|$ and $\|y\| \geqslant 10$. Then*
$$|\mathfrak{a}(x+y) - \mathfrak{a}(x)| \leqslant 0.7\frac{\|y\|}{\|x\|}.$$
6. *Let $R \geqslant 10r$ and $r \geqslant 10$. Then, uniformly for $x \in C(R)$ and $y \in C(r)$, we have*
$$0.56\log(R/r) \leqslant \mathfrak{a}(x) - \mathfrak{a}(y) \leqslant \log(R/r).$$

In the next section, we will need the following comparison of $\mathfrak{a}$ and $\mathfrak{a}'$.

Lemma 2.9.2. *Let μ be any probability measure on $C_x(r)$. Suppose $r \geqslant 2(\|x\|+1)$. Then*
$$\Big| \sum_{y \in C_x(r)} \mu(y)\mathfrak{a}(y) - \mathfrak{a}'(r) \Big| \leqslant \left(\frac{5}{2\pi} + 2\lambda\right)\left(\frac{\|x\|+1}{r}\right).$$

Proof. We recall that, for any $x \in \mathbb{Z}^2$, the potential kernel has the form specified in (2.14) where the error term conceals a constant of λ, which is no more than 0.07 [KS04]. That is,
$$\left|\mathfrak{a}(x) - \frac{2}{\pi}\log\|x\| - \kappa\right| \leqslant \lambda\|x\|^{-2}.$$

For $y \in C_x(r)$, we have $r - \|x\| - 1 \leqslant \|y\| \leqslant r + \|x\| + 1$. Accordingly,

$$\begin{aligned}\mathfrak{a}(y) &\leqslant \frac{2}{\pi}\log|r + \|x\| + 1| + \kappa + O(|r - \|x\| - 1|^{-2}) \\ &= \frac{2}{\pi}\log r + \kappa + \frac{2}{\pi}\log\left(1 + \frac{\|x\|+1}{r}\right) + O(|r - \|x\| - 1|^{-2}).\end{aligned}$$

Using the assumption $(\|x\| + 1)/r \in (0, 1/2)$ with Taylor's remainder theorem gives

$$\mathfrak{a}(y) \leqslant \mathfrak{a}'(r) + \frac{2}{\pi}\left(\frac{\|x\|+1}{r} + \frac{1}{2}\left(\frac{\|x\|+1}{r}\right)^2\right) + O(|r - \|x\| - 1|^{-2}).$$

Simplifying with $r \geqslant 2(\|x\| + 1)$ and $r \geqslant 2$ leads to

$$\mathfrak{a}(y) \leqslant \mathfrak{a}'(r) + \frac{2}{\pi}\left(\frac{5}{4} + \pi\lambda\right)\left(\frac{\|x\|+1}{r}\right) = \mathfrak{a}'(r) + \left(\frac{5}{2\pi} + 2\lambda\right)\left(\frac{\|x\|+1}{r}\right).$$

The lower bound is similar. Because this holds for any $y \in C_x(r)$, for any probability measure μ on $C_x(r)$, we have

$$\Big|\sum_{y \in C_x(r)} \mu(y)\mathfrak{a}(y) - \mathfrak{a}'(r)\Big| \leqslant \left(\frac{5}{2\pi} + 2\lambda\right)\left(\frac{\|x\|+1}{r}\right).$$

□

2.9.2 Comparison between harmonic measure and hitting probabilities

To prove Lemma 2.3.1, we require a comparison (Lemma 2.9.3) between certain values of harmonic measure and hitting probabilities. In fact, we need additional quantification of an error term which appears in standard versions of this result (e.g. [Law13, Theorem 2.1.3]). Effectively, this additional quantification comes from a bound on λ, the implicit constant in (2.14). The proof is similar to that of Theorem 3.17 in [Pop21].

Lemma 2.9.3. *Let $x \in D(R)^c$ for $R \geqslant 100r$ and $r \geqslant 10$. Then*

$$0.93\mathbb{H}_{C(r)}(y) \leqslant \mathbb{H}_{C(r)}(x, y) \leqslant 1.04\mathbb{H}_{C(r)}(y). \tag{2.160}$$

Proof. We have

$$\mathbb{H}_{C(r)}(x, y) - \mathbb{H}_{C(r)}(y) = -\mathfrak{a}(y - x) + \sum_{z \in C(r)} \mathbb{P}_y\left(S_{\tau_{C(r)}} = z\right)\mathfrak{a}(z - x). \tag{2.161}$$

Since $C(10r)$ separates x from $C(r)$, the optional stopping theorem applied to $\sigma_{C(10r)} \wedge \tau_{C(r)}$ and the martingale $\mathfrak{a}\left(S_{t\wedge\tau_x} - x\right)$ gives

$$\begin{aligned}\mathfrak{a}(y-x) = &\sum_{z \in C(r)} \mathbb{P}_y\left(S_{\tau_{C(r)}} = z\right)\mathfrak{a}(z-x) \\ &+ \mathbb{E}_y\left[\mathfrak{a}\left(S_{\sigma_{C(10r)}} - x\right) - \mathfrak{a}\left(S_{\tau_{C(r)}} - x\right) \;\middle|\; \sigma_{C(10r)} < \tau_{C(r)}\right]\mathbb{P}_y\left(\sigma_{C(10r)} < \tau_{C(r)}\right). \qquad (2.162)\end{aligned}$$

In the second term of (2.162), we analyze the difference in potentials by observing

$$S_{\sigma_{C(10r)}} - x - \left(S_{\tau_{C(r)}} - x\right) = S_{\sigma_{C(10r)}} - S_{\tau_{C(r)}}.$$

Accordingly, letting $u = S_{\tau_{C(r)}} - x$ and $v = S_{\sigma_{C(10r)}} - S_{\tau_{C(r)}}$,

$$\mathfrak{a}\left(S_{\sigma_{C(10r)}} - x\right) - \mathfrak{a}\left(S_{\tau_{C(r)}} - x\right) = \mathfrak{a}(u+v) - \mathfrak{a}(u).$$

We observe that $\|v\| \leqslant 11r + 2$ and $\|u\| \geqslant 99r - 2$, so $\|u\| \geqslant 8\|v\|$. Since we also have $\|v\| \geqslant 9r - 2 \geqslant 10$, (5) of Lemma 2.9.1 applies to give

$$\mathfrak{a}(u+v) - \mathfrak{a}(u) \leqslant 0.7\frac{\|v\|}{\|u\|} \leqslant \frac{2}{25}.$$

We analyze the other factor of (2.162) as

$$\begin{aligned}\mathbb{P}_y\left(\sigma_{C(10r)} < \tau_{C(r)}\right) &= \frac{1}{4}\sum_{z \notin C(r): z\sim y} \mathbb{P}_z\left(\sigma_{C(10r)} < \tau_{C(r)}\right) \\ &= \frac{1}{4}\sum_{z \notin C(r): z\sim y} \frac{\mathfrak{a}(z-z_0) - \mathbb{E}_z\mathfrak{a}\left(S_{\tau_{C(r)}} - z_0\right)}{\mathbb{E}_z\left[\mathfrak{a}\left(S_{\sigma_{C(10r)}}\right) - \mathfrak{a}\left(S_{\tau_{C(r)}}\right) \;\middle|\; \sigma_{C(10r)} < \sigma_{C(r)}\right]},\end{aligned}$$

where $z_0 \in A$. To obtain an upper bound on the potential difference in the denominator, we apply (6) of Lemma 2.9.1, which gives

$$\mathbb{P}_y\left(\sigma_{C(10r)} < \tau_{C(r)}\right) \leqslant \frac{1}{0.6\log 10}\mathbb{H}_{C(r)}(y).$$

Combining this with the other estimate for the second term of (2.162), we find

$$\mathfrak{a}(y-x) \leqslant \sum_{z \in C(r)} \mathbb{P}_y\left(S_{\tau_{C(r)}} = z\right)\mathfrak{a}(z-x) + \underbrace{\frac{2}{25}\cdot\frac{1}{0.56\log 10}}_{\leqslant 0.063}\mathbb{H}_{C(r)}(y).$$

Substituting this into (2.161), we have

$$\mathbb{H}_{C(r)}(x,y) - \mathbb{H}_{C(r)}(y) \geqslant -0.063\mathbb{H}_{C(r)}(y) \implies \mathbb{H}_{C(r)}(x,y) \geqslant 0.93\mathbb{H}_{C(r)}(y).$$

We again apply (5) and (6) of Lemma 2.9.1 to bound the factors in the second term of 2.162 as

$$\mathfrak{a}(u+v) - \mathfrak{a}(u) \geqslant -0.0875 \quad \text{and} \quad \mathbb{P}_y\left(\sigma_{C(10r)} < \tau_{C(r)}\right) \geqslant \frac{1}{\log 10}\mathbb{H}_{C(r)}(y).$$

Substituting these into (2.162), we find

$$\mathfrak{a}(y-x) \geqslant \sum_{z \in C(r)} \mathbb{P}_y\left(S_{\tau_{C(r)}} = z\right)\mathfrak{a}(z-x) - 0.0875 \cdot \frac{1}{\log 10}\mathbb{H}_{C(r)}(y).$$

Consequently, (2.161) becomes

$$\mathbb{H}_{C(r)}(x,y) - \mathbb{H}_{C(r)}(y) \leqslant \frac{0.0875}{\log 10}\mathbb{H}_{C(r)}(y) \leqslant \frac{1}{25}\mathbb{H}_{C(r)}(y).$$

Rearranging, we find

$$\mathbb{H}_{C(r)}(x,y) \leqslant 1.04\mathbb{H}_{C(r)}(y).$$

□

2.9.3 Uniform lower bound on a conditional entrance measure

We now use Lemma 2.9.3 to prove an inequality which is needed for the proof of Lemma 2.3.1. It is similar to Lemma 2.1 in [DPRZ06].

Lemma 2.9.4. *Let $\varepsilon > 0$, denote $\eta = \tau_{C(R)} \wedge \tau_{C(\varepsilon R)}$, and denote by μ the uniform measure on $C(\varepsilon R)$. There is a constant c such that, if $\varepsilon \leqslant \frac{1}{100}$ and $R \geqslant 10\varepsilon^{-2}$, and if*

$$\min_{x \in C(\varepsilon R)} \mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right) > \frac{1}{10}, \tag{2.163}$$

then, uniformly for $x \in C(\varepsilon R)$ and $y \in C(\varepsilon^2 R)$,

$$\mathbb{P}_x\left(S_\eta = y, \tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right) \geqslant c\mu(y)\,\mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right).$$

Proof. Fix ε and R which satisfy the hypotheses. Let $x \in C(\varepsilon R)$ and $y \in C(\varepsilon^2 R)$. We have

$$\mathbb{P}_x\left(S_{\tau_{C(\varepsilon^2 R)}} = y, \tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right) = \mathbb{H}_{C(\varepsilon^2 R)}(x,y) - \mathbb{P}_x\left(S_{\tau_{C(\varepsilon^2 R)}} = y, \tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right). \tag{2.164}$$

By the strong Markov property applied to $\tau_{C(R)}$,

$$\mathbb{P}_x\left(S_{\tau_{C(\varepsilon^2 R)}} = y, \tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right) = \mathbb{E}_x\left[\mathbb{H}_{C(\varepsilon^2 R)}\left(S_{\tau_{C(R)}}, y\right); \tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right]. \tag{2.165}$$

We will now use Lemma 2.9.3 to uniformly bound the terms of the form $\mathbb{H}_{C(\varepsilon^2 R)}(\cdot, y)$ appearing in (2.164) and (2.165).

For any $w \in C(R)$, the hypotheses of Lemma 2.9.3 are satisfied with $\varepsilon^2 R$ in the place of r and R as presently defined, because then $r \geqslant 10$ and $R \geqslant 100\varepsilon^2 R$. Therefore, by (2.160), uniformly for $w \in C(R)$,

$$\mathbb{H}_{C(\varepsilon^2 R)}(w, y) \leqslant 1.04\, \mathbb{H}_{C(\varepsilon^2 R)}(y). \tag{2.166}$$

Now, for any $x \in C(\varepsilon R)$, the hypotheses of Lemma 2.9.3 are again satisfied with the same r and with εR in the place of R, as $\varepsilon R \geqslant 100\varepsilon^2 R$ by assumption. We apply (2.160) to find

$$\mathbb{H}_{C(\varepsilon^2 R)}(x, y) \geqslant 0.93 \mathbb{H}_{C(\varepsilon^2 R)}(y). \tag{2.167}$$

Substituting (2.166) into (2.165), we find

$$\mathbb{P}_x\left(S_{\tau_{C(\varepsilon^2 R)}} = y, \tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right) \leqslant 1.04\, \mathbb{H}_{C(\varepsilon^2 R)}(y) \mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right).$$

Similarly, substituting (2.167) into (2.164) and using the previous display, we find

$$\begin{aligned}\mathbb{P}_x\left(S_{\tau_{C(\varepsilon^2 R)}} = y, \tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right) \geqslant\ & 0.93\, \mathbb{H}_{C(\varepsilon^2 R)}(y) \mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right) \\ & - (1.04 - 0.93)\ \mathbb{H}_{C(\varepsilon^2 R)}(y) \mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} > \tau_{C(R)}\right).\end{aligned}$$

Applying hypothesis (2.163), we find that the right-hand side is at least

$$c_1\, \mathbb{H}_{C(\varepsilon^2 R)}(y) \mathbb{P}_x\left(\tau_{C(\varepsilon^2 R)} < \tau_{C(R)}\right),$$

for a positive constant c_1. The result then follows the existence of a positive constant c_2 such that $\mathbb{H}_{C(\varepsilon^2 R)}(y) \geqslant c_2 \mu(y)$ for any $y \in C(\varepsilon^2 R)$. □

2.9.4 Estimate for the exit distribution of a rectangle

The purpose of this section is to prove an estimate which is needed in the proofs of Lemma 2.3.3 and Lemma 2.3.4. Informally, this estimate says that the probability that a random walk from one end of a rectangle (which may not be aligned with the coordinate axes) exits through the opposite end is no smaller than exponential in the aspect ratio of the rectangle. We believe this estimate is known but, as we are unable to find a reference for it, we prove it here. In brief, the proof uses an adaptive algorithm for constructing a sequence of squares which remain inside the rectangle and the sides of which are aligned with the axes. We then bound below the probability that the walk follows the path determined by the squares until exiting the opposite end of the rectangle.

Recall that $\mathrm{Rec}(\phi, w, \ell)$ denotes the rectangle of width w, centered along the line segment from $-e^{\mathbf{i}\phi} w$ to $e^{\mathbf{i}\phi}\ell$, intersected with $\mathbb{Z}^2$ (see Figure 2.11).

Lemma 2.9.5. *For any* $24 \leqslant w \leqslant \ell$ *and any* ϕ, *let* $\mathrm{Rec} = \mathrm{Rec}(\phi, w, \ell)$ *and* $\mathrm{Rec}^+ = \mathrm{Rec}(\phi, w, \ell + w)$. *Then,*

$$\mathbb{P}_o\left(\tau_{\partial \mathrm{Rec}} < \tau_{\partial \mathrm{Rec}^+}\right) \geqslant c^{\ell/w},$$

for a universal positive constant $c < 1$.

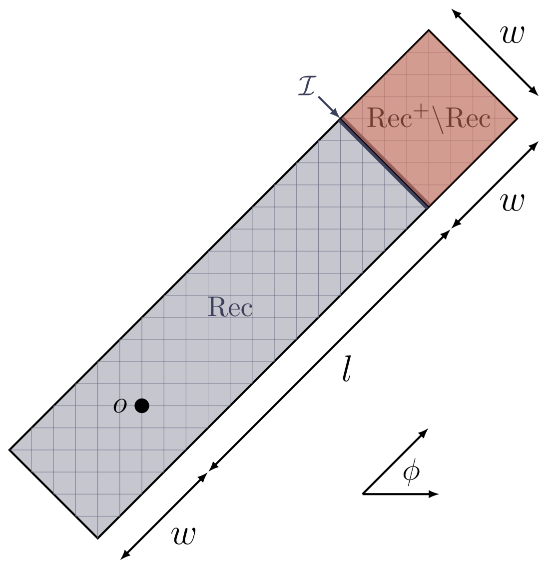

Figure 2.11: On the left, we depict the rectangles $\mathrm{Rec} = \mathrm{Rec}(\phi, w, l)$ (shaded blue) and $\mathrm{Rec}^+ = \mathrm{Rec}(\phi, w, l + w)$ (union of blue- and red-shaded regions) for $\phi = \pi/4$, $w = 4\sqrt{2}$, and $\ell = 11\sqrt{2}$. $\mathcal{I}$ denotes $\mathrm{Rec} \cap \partial(\mathrm{Rec}^+ \backslash \mathrm{Rec})$.

We use the hypothesis $w \geqslant 24$ to deal with the effects of discreteness; the constant 24 is otherwise unimportant, and many choices would work in its place.

Proof of Lemma 2.9.5. We will first define a square, centered at the origin and with each corner in $\mathbb{Z}^2$, which lies in Rec^+. We will then translate it to form a sequence of squares through which we will guide the walk to $\mathrm{Rec}^+ \backslash \mathrm{Rec}$ without leaving Rec^+ (see Figure 2.11). We split the proof into three steps: (1) constructing the squares; (2) proving that they lie in Rec^+; and (3) establishing a lower bound on the probability that the walk hits $\partial\mathrm{Rec}$ before hitting the interior boundary of Rec^+.

Step 1: Construction of the squares. Without loss of generality, assume $0 \leqslant \phi < \pi/2$. For $x \in \mathbb{Z}^2$, we will denote its first coordinate by x^1 and its second coordinate by x^2. We will use this convention only for this proof. Let $\mathfrak{l}$ be equal to $\lfloor \frac{w}{8} \rfloor$ if it is even and equal to $\lfloor \frac{w}{8} \rfloor - 1$ otherwise. With this choice, we define

$$Q = \{x \in \mathbb{Z}^2 : \max\{x^1, x^2\} \leqslant \mathfrak{l}\}.$$

Since $\mathfrak{l}$ is even, the translates of Q by integer multiples of $\frac{1}{2}\mathfrak{l}$ are also subsets of $\mathbb{Z}^2$.

We construct a sequence of squares Q_i in the following way, where we make reference to the line $L_\phi^\infty = e^{\mathbf{i}\phi}\mathbb{R}$. Let $y_1 = o$ and $Q_1 = y_1 + Q$. For $i \geqslant 1$, let

$$y_{i+1} = \begin{cases} y_i + \frac{1}{2}\mathfrak{l}\,(0,1) & \text{if } y_i \text{ lies on or below } L_\phi^\infty \\ y_i + \frac{1}{2}\mathfrak{l}\,(1,0) & \text{if } y_i \text{ lies above } L_\phi^\infty \end{cases} \quad \text{and} \quad Q_{i+1} = y_{i+1} + Q.$$

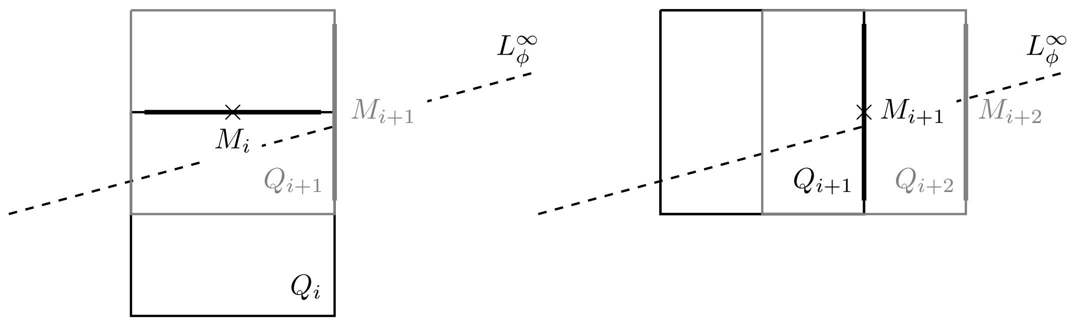

Figure 2.12: Two steps in the construction of squares. Respectively on the left and right, $y_{i+1} \in M_i$ and $y_{i+2} \in M_{i+1}$ (indicated by the $\times$ symbols) lie above L_ϕ^∞, so M_{i+1} and M_{i+2} are situated on the eastern sides of Q_{i+1} and Q_{i+2}. However, on the left, as Q_i was translated north to form Q_{i+1}, the relative orientation of M_i and M_{i+1} is perpendicular. In contrast, as Q_{i+1} is translated east to form Q_{i+2}, the right-hand side has parallel M_{i+1} and M_{i+2}.

In words, if the center of the present square lies on or below the line L_ϕ^∞, then we translate the center north by $\frac{1}{2}\mathfrak{l}$ to obtain the next square. Otherwise, we translate the center to the east by $\frac{1}{2}\mathfrak{l}$.

We further define, for $i \geqslant 1$,

$$M_i = \begin{cases} \left\{x \in Q_i : x^2 - y_i^2 = \frac{1}{2}\mathfrak{l} \text{ and } |y_i^1 - x^1| \leqslant \frac{1}{2}\mathfrak{l} - 1\right\} & \text{if } y_i \text{ lies on or below } L_\phi^\infty \\ \left\{x \in Q_i : x^1 - y_i^1 = \frac{1}{2}\mathfrak{l} \text{ and } |y_i^2 - x^2| \leqslant \frac{1}{2}\mathfrak{l} - 1\right\} & \text{if } y_i \text{ lies above } L_\phi^\infty. \end{cases} \tag{2.168}$$

In words, if y_i lies on or below the line L_ϕ^∞, we choose M_i to be the northernmost edge of Q_i, excluding the corners. Otherwise, we choose it to be the easternmost edge, excluding the corners. These possibilities are depicted in Figure 2.12. In fact, we leave the corners out of the M_i, as indicated, by the bounds of $\frac{1}{2}\mathfrak{l} - 1$ instead of $\frac{1}{2}\mathfrak{l}$ in (2.168). We must do so to ensure that $\mathbb{P}_\omega\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right)$ is harmonic for all $\omega \in M_i$; we will shortly need this to apply the Harnack inequality. Upcoming Figure 2.13 provides an illustration of M_i in this context.

We will guide the walk to ∂Rec without leaving Rec^+ by requiring that it exit each square Q_i through M_i for $1 \leqslant i \leqslant J$, where we define

$$J = \min\{i \geqslant 1 : M_i \subseteq \text{Rec}^c\}.$$

That is, J is the first index for which M_i is fully outside Rec. It is clear that J is finite.

Step 2: Proof that $\cup_{i=1}^J Q_i$ is a subset of Rec^+. Let v be the northeastern endpoint of L_ϕ, where L_ϕ is the segment of L_ϕ^∞ from o to $e^{\mathrm{i}\phi}(\ell + w/2)$ and define k to be the first index for which y_k satisfies

$$y_k^1 > v^1 \quad \text{or} \quad y_k^2 > v^2.$$

It will also be convenient to denote by $\mathcal{I}$ the interface between Rec and $\text{Rec}^+ \backslash \text{Rec}$ (the dashed line in Figure 2.11), given by

$$\mathcal{I} = \text{Rec} \cap \partial\left(\text{Rec}^+ \backslash \text{Rec}\right).$$

By construction, we have $\|y_k - y_{k-1}\| = \frac{1}{2}\mathfrak{l}$ and $\|y_k - v\| \leqslant \frac{1}{2}\mathfrak{l}$. By the triangle inequality, $\|y_{k-1} - v\| \leqslant \mathfrak{l}$. As $\|v\| = \ell + w/2$ and because $\mathrm{dist}(o, \mathcal{I}) \leqslant \ell + 1$, we must have—again by the triangle inequality—that $\mathrm{dist}(v, \mathcal{I}) \geqslant w/2 - 1$. From a third use of the triangle inequality and the hypothesized lower bound on w, we conclude

$$\mathrm{dist}(y_{k-1}, \mathcal{I}) \geqslant \frac{w}{2} - 1 - \mathfrak{l} \geqslant \frac{w}{2} - 1 - \frac{w}{8} \geqslant \frac{w}{3} > 2\mathfrak{l}. \tag{2.169}$$

To summarize in words, y_{k-1} is not in Rec and it is separated from Rec by a distance strictly greater than $2\mathfrak{l}$.

Because the sides of Q_{k-1} have length $\mathfrak{l}$, (2.169) implies $Q_{k-1} \subseteq \mathrm{Rec}^c$. Since M_{k-1} is a subset of Q_{k-1}, we must also have $M_{k-1} \subseteq \mathrm{Rec}^c$, which implies $J \leqslant k - 1$. As k was the first index for which $y_k^1 > v^1$ or $y_k^2 > v^2$, y_J satisfies $y_J^1 \leqslant v^1$ and $y_J^2 \leqslant v^2$. Then, by construction, for all $1 \leqslant i \leqslant J$, the centers satisfy

$$y^1 \leqslant y_i^1 \leqslant v^1 \quad \text{and} \quad y^2 \leqslant y_i^2 \leqslant v^2. \tag{2.170}$$

From (2.170) and the fact that $\mathrm{dist}(y_i, L_\phi^\infty) \leqslant \frac{1}{2}\mathfrak{l}$, we have

$$\mathrm{dist}(y_i, L_\phi) = \mathrm{dist}(y_i, L_\phi^\infty) \leqslant \frac{1}{2}\mathfrak{l} \quad \forall\ 1 \leqslant i \leqslant J.$$

As the diagonals of the Q_i have length $\sqrt{2}\mathfrak{l}$, (2.170) and the triangle inequality imply

$$\mathrm{dist}(x, L_\phi) \leqslant \mathrm{dist}(y_i, L_\phi) + \frac{1}{2}\sqrt{2}\mathfrak{l} = \frac{1}{2}(1 + \sqrt{2})\mathfrak{l} < \frac{w}{4} \qquad \forall\ x \in \bigcup_{i=1}^{J} Q_i.$$

To summarize, any element of Q_i for some $1 \leqslant i \leqslant J$ is within a distance $w/4$ of L_ϕ. As Rec^+ contains all points x within a distance $\frac{w}{2}$ of L_ϕ, we conclude

$$\bigcup_{i=1}^{J} Q_i \subseteq \mathrm{Rec}^+.$$

Step 3: Lower bound for $\mathbb{P}_o\left(\tau_{\partial \mathrm{Rec}} < \tau_{\partial \mathrm{Rec}^+}\right)$. From the previous step, to obtain a lower bound on the probability that the walk exits Rec before Rec^+, it suffices to obtain an upper bound J^* on J and a lower bound $c < 1$ on

$$\mathbb{P}_\omega\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\mathrm{int}} Q_{i+1}}\right),$$

uniformly for $\omega \in M_i$, for $0 \leqslant i \leqslant J - 1$. This way, if we denote $Y_0 \equiv y$ and $Y_i = S_{\tau_{\partial^{\mathrm{int}} Q_i}}$ for $1 \leqslant i \leqslant J - 1$, we can apply the strong Markov property to each τ_{M_i} and use the lower bound for each factor to obtain the lower bound

$$\mathbb{P}_o\left(\tau_{\partial \mathrm{Rec}} < \tau_{\partial \mathrm{Rec}^+}\right) \geqslant c^{J^*}. \tag{2.171}$$

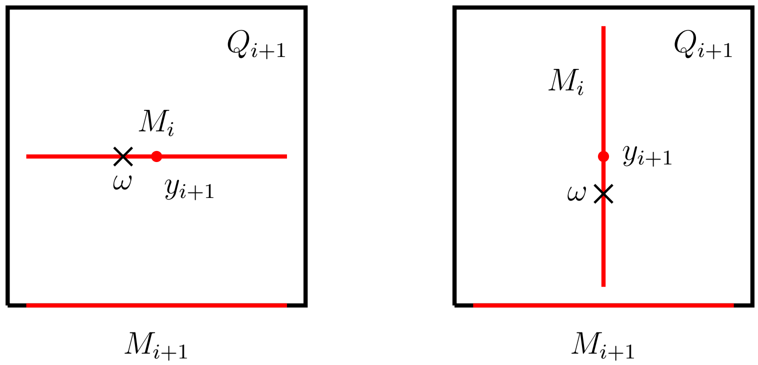

Figure 2.13: The two cases for lower-bounding M_{i+1} hitting probabilities.

To obtain an upper bound on J, we first recall that L_ϕ has a length of $\ell + w/2$, which satisfies

$$\ell + w/2 = \frac{\mathfrak{l}}{2}\left(\frac{2\ell}{\mathfrak{l}} + \frac{w}{\mathfrak{l}}\right) \leqslant \frac{\mathfrak{l}}{2}\left(\frac{2\ell}{w/8-1} + \frac{w}{w/8-1}\right) \leqslant \frac{\mathfrak{l}}{2}\left(48\frac{\ell}{w} + 24\right), \tag{2.172}$$

due to the fact that $\mathfrak{l} \geqslant \lfloor w/8 \rfloor - 1 \geqslant w/8 - 2$ and the hypothesis of $w \geqslant 24$. The number of steps to reach J is no more than twice the ratio $(\ell + w/2)/(\mathfrak{l}/2)$. Accordingly, using the bound in (2.172) and the hypothesis that $\ell/w \geqslant 1$, we have

$$J \leqslant 2\left(48\frac{\ell}{w} + 24\right) \leqslant 144\frac{\ell}{w} =: J^*. \tag{2.173}$$

We now turn to the hitting probability lower bounds.

From the construction, there are only two possible orientations of M_i relative to M_{i+1} (Figure 2.13). Either M_i and M_{i+1} have parallel orientation or they do not. Consider the former case. The hitting probability $\mathbb{P}_\omega\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right)$ is a harmonic function of ω for all ω in $Q_{i+1} \backslash \partial^{\text{int}} Q_{i+1}$ and M_{i+1} in particular. Therefore, by the Harnack inequality [Law13, Theorem 1.7.6], there is a constant a_1 such that

$$\mathbb{P}_\omega\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right) \geqslant a_1 \mathbb{P}_{y_{i+1}}\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right) \quad \forall\, \omega \in M_{i+1}. \tag{2.174}$$

The same argument applies to the case when M_i and M_{i+1} do not have parallel orientation and we find there is a constant a_2 such that (2.174) holds with a_2 in place of a_1. Setting $a = \min\{a_1, a_2\}$, we conclude that, for all $0 \leqslant i \leqslant J-1$ and any $\omega \in M_i$,

$$\mathbb{P}_\omega\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right) \geqslant a \mathbb{P}_{y_{i+1}}\left(\tau_{M_{i+1}} \leqslant \tau_{\partial^{\text{int}} Q_{i+1}}\right). \tag{2.175}$$

We have reduced the lower bound for any $\omega \in M_i$ and either of the two relative orientations of M_i and M_{i+1} to a lower bound on the hitting probability of one side of Q_{i+1} from the center. By symmetry, the walk hits M_{i+1} first with a probability of exactly $1/4$. We emphasize that the probability on the left-hand side of (2.175) is exactly $1/4$ as although M_{i+1} does not include the adjacent corners of Q_{i+1}, which are elements of $\partial^{\mathrm{int}} Q_{i+1}$, the corners are separated from y_{i+1} by the other elements of $\partial^{\mathrm{int}} Q_{i+1}$.

Calling $b = a/4$ and combining (2.173) and (2.175) with (2.171), we have

$$\mathbb{P}_o \left(\tau_{\partial \mathrm{Rec}} < \tau_{\partial \mathrm{Rec}^+}\right) \geqslant b^{J^*} = b^{144\ell/w} = c^{\ell/w}$$

for a positive constant $c < 1$. □

2.9.5 Proof of Lemma 2.3.3

The lower bound in Lemma 2.3.3 is a simple consequence of the fact that random walk exits a rectangle through its far side with a probability which is no smaller than exponential in the aspect ratio of the rectangle (Lemma 2.9.5). In this case, the aspect ratio is $O(m_\ell)$.

The proof has two steps. First, after specifying the rectangle, we will confirm that, if the random walk exits it appropriately, then $\{\tau_{\mathrm{Arc}_\ell} < \tau_A\}$ occurs. Second, we will apply Lemma 2.9.5 to estimate the probability with which the random walk appropriately exits the rectangle.

Proof of Lemma 2.3.3. Fix $\ell \in \mathbb{I}$ and $y \in \mathrm{Sec}_\ell$. Denote $\varepsilon_\ell = (m_\ell + 1)^{-1}$ and by $\mathrm{Rec}(\phi, w, l)$ the rectangle of width w, centered along the line segment from $-e^{\mathbf{i}\phi} w$ to $e^{\mathbf{i}\phi} l$, intersected with $\mathbb{Z}^2$ (see Figure 2.11). We will use the rectangles $\mathrm{Rec}_\ell = \mathrm{Rec}(\varphi_\ell, w_\ell, l_\ell)$ and $\mathrm{Rec}_\ell^+ = \mathrm{Rec}(\varphi_\ell, w_\ell, l_\ell + w_\ell)$ with

$$\varphi_\ell = \pi - \arg y, \quad w_\ell = \varepsilon_\ell R_{\ell-2}, \quad \text{and} \quad l_\ell = \mathrm{dist}(y, \mathrm{Arc}_\ell) + 4 w_\ell.$$

Additionally denote the "interface" between Rec_ℓ and Rec_ℓ^+ by $\mathcal{I}_\ell = \mathrm{Rec}_\ell \cap \partial(\mathrm{Rec}_\ell^+ \backslash \mathrm{Rec}_\ell)$.

We explain these choices as follows. The parameter φ_ℓ ensures that the rectangle is "pointing in the right direction." The factor of $R_{\ell-2}$ in w_ℓ reflects the need for the rectangle to remain within $\mathcal{B}_\ell$, the innermost radius of which is $R_{\ell-2}$. The factor of ε_ℓ arises from our earlier use of the pigeonhole principle. l_ℓ ensures that the random walk encounters Arc_ℓ if it exits Rec_ℓ through $\mathcal{I}_\ell$. Note that $l_\ell \geqslant w_\ell$ and $w_\ell \geqslant 100$ by (2.12), so the hypotheses of Lemma 2.9.5 are satisfied for Rec_ℓ.

We have

$$\mathbb{P}_o\big(\tau_{\partial \mathrm{Rec}_\ell} < \tau_{\partial \mathrm{Rec}_\ell^+}\big) = \mathbb{P}_y\big(\tau_{\mathcal{I}_\ell + y} < \tau_{\partial(\mathrm{Rec}_\ell^+ + y)}\big) \leqslant \mathbb{P}_y\big(\tau_{\mathrm{Arc}_\ell} < \tau_{\partial \mathcal{B}_\ell}\big) \leqslant \mathbb{P}_y\big(\tau_{\mathrm{Arc}_\ell} < \tau_A\big). \quad (2.176)$$

The equality follows from the translation invariance of the law of random walk and the definition of $\mathcal{I}_\ell$. The first inequality is due to the fact that $\mathcal{I}_\ell + y$ is a subset of $D\big(\frac{a_\ell + b_\ell}{2}\big)$ and $\mathrm{Rec}_\ell^+ + y$ is a subset of $\mathcal{B}_\ell$ (Figure 2.11). The second inequality follows from the fact that $\mathcal{B}_\ell$ is empty of A.

By (2.176) and Lemma 2.9.5, there are constants c_1 and c_2 such that

$$\mathbb{P}_y\big(\tau_{\mathrm{Arc}_\ell} < \tau_A\big) \geqslant \mathbb{P}_o\big(\tau_{\partial \mathrm{Rec}_\ell} < \tau_{\partial \mathrm{Rec}_\ell^+}\big) \geqslant c_1^{l_\ell / w_\ell} \geqslant c_1^{2R_\ell / \varepsilon_\ell R_{\ell-2}} = c_2^{m_\ell}.$$

The last inequality is due to the fact that, uniformly for $y \in \mathrm{Sec}_\ell$, $\mathrm{dist}(y, \mathrm{Arc}_\ell) \leqslant R_\ell$, so $l_\ell \leqslant 2R_\ell$. □

For convenience, we collect two facts about a_ℓ which we will use in the proof of Lemma 2.3.4.

Lemma 2.9.6. *Let $\ell \in \mathbb{I}$ and denote $\varepsilon_\ell = (m_\ell + 1)^{-1}$. Then*

$$10^3 \leqslant a_\ell \varepsilon_\ell \leqslant \Delta_\ell. \tag{2.177}$$

Additionally, if $x, y \in D(a_\ell)^c$ satisfy $\|x - y\| \leqslant 1$, then

$$|\arg x - \arg y| \leqslant 10^{-8} \varepsilon_{\ell-1}. \tag{2.178}$$

The specific constants 10^3 and 10^{-8} in (2.177) and (2.178) are unimportant. The factor of ε_ℓ in (2.178) reflects the angular width of $\mathrm{Sec}_{\ell-1}$ defined in Lemma 2.3.2. Both (2.177) and (2.178) are consequences of (2.12) and the definitions of regions in Lemma 2.3.2.

Proof of Lemma 2.9.6. By Lemma 2.3.2, a_ℓ is at least $10R_{\ell-2}$ which, by (2.12), is at least $10^3\varepsilon_\ell^{-1}$. Lemma 2.3.2 also states that a_ℓ is at most $\delta' R_{\ell-1}$, which equals $\varepsilon_\ell^{-1}\Delta_\ell$ (2.22). This proves (2.177).

Let x, y satisfy the hypotheses. Then $|\arg x - \arg y|$ is at most the reciprocal of a_ℓ and so at most the reciprocal of $10R_{\ell-2}$. (2.12) implies that $R_{\ell-2} \geqslant 10^7 \varepsilon_{\ell-1}^{-1}$, which implies $|\arg x - \arg y| \leqslant 10^{-8}\varepsilon_{\ell-1}$. □

2.9.6 Proof of Lemma 2.3.4

The reason we will use $O(m_\ell^{1/2})$ rectangles with aspect ratios of $O(m_\ell^{1/2})$, instead of, say, $O(1)$ rectangles with aspect ratios of $O(m_\ell)$, is that the rectangles must remain inside Ann_ℓ. (We could also use $O(m_\ell)$ rectangles of aspect ratio $O(1)$.) We briefly explain why this choice will work. As depicted in Figure 2.14, we will essentially center rectangles of width $\frac{1}{100}\Delta_\ell$ along chords of Circ_ℓ. The greatest distance between such a chord and Circ_ℓ is proportional to $r\omega^2$, where $r = \mathrm{rad}(\mathrm{Circ}_\ell)$ and ω is the angle subtended by the chord. Circ_ℓ is a distance $O(\Delta_\ell)$ from Ann_ℓ, so we must have $r\omega^2 = O(\Delta_\ell)$ as well. By (2.177), r is at most $m_\ell \Delta_\ell$, so ω must be $O(m_\ell^{-1/2})$. Consequently, we will need roughly $m_\ell^{1/2}$ rectangles to circle Ann_ℓ. This choice of ω results in rectangles with lengths of $O(r\omega) = O(m_\ell^{1/2}\Delta_\ell)$ and aspect ratios of $O(m_\ell^{1/2})$.

Proof of Lemma 2.3.4. Fix $\ell \in \mathbb{I}$. We will use $w = \frac{1}{100}\Delta_\ell$ as the width of the rectangles and take z to be an element of $(\mathrm{Circ}_\ell)_w$ (i.e., z is within w of Circ_ℓ). We choose z in this way, as opposed to fixing z in Circ_ℓ exclusively, because it will be useful later. As in Lemma 2.3.2, denote $\varepsilon_\ell = (m_\ell + 1)^{-1}$.

The lower bound in (2.177) implies that there is an element $v = v(z)$ of Circ_ℓ such that $\phi = \arg z - \arg v \in \varepsilon_\ell^{1/2}[\frac{\pi}{10}, \frac{\pi}{5}]$. We claim that, if $\psi = \arg(v - z)$, $l = \|v - z\|$, and $\mathrm{Rec}_v = \mathrm{Rec}(\psi, w, l)$, then $\mathrm{Rec}_v + z$ is contained in Ann_ℓ.

Establishing $\mathrm{Rec}_v + z \subseteq \mathrm{Ann}_\ell$. We note that $\mathrm{dist}(\mathrm{Circ}_\ell, \partial\mathrm{Ann}_\ell)$ is at least $0.49\Delta_\ell$ (the missing $0.01\Delta_\ell \geqslant 100$ accounts for discreteness). Accordingly, to show that $\mathrm{Rec}_v + z$ is contained in Ann_ℓ, it suffices to prove an upper bound of $0.49\Delta_\ell$ on the distance between an arbitrary $x \in \mathrm{Rec}_v + z$ and Circ_ℓ.

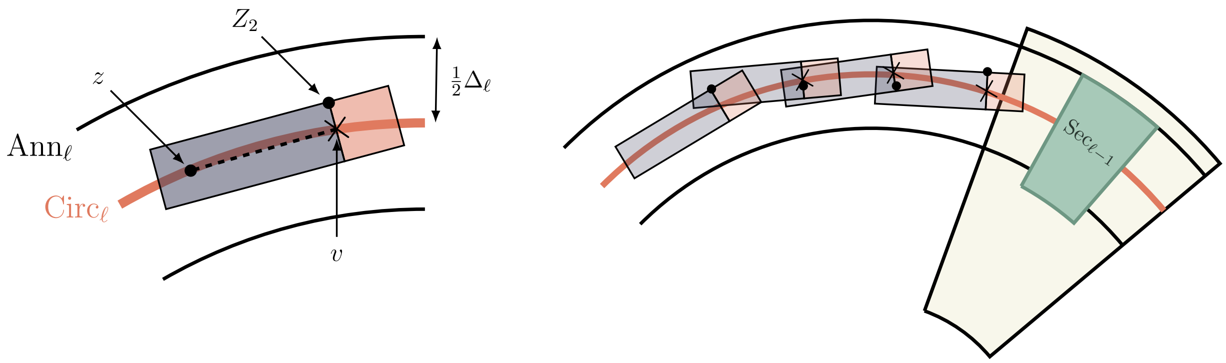

Figure 2.14: The rectangle $\mathrm{Rec}_v + z$ (left) and a sequence of rectangles approaching $\mathrm{Sec}_{\ell-1}$ in Ann_ℓ (right), which appear in the proof of Lemma 2.3.4. On the left, $\mathrm{Rec}_v + z$ is a rectangle centered along the dashed line segment from z (dot) to v (cross). A random walk from z exits the blue-shaded rectangle at Z_2 (dot). On the right, $\mathrm{Rec}_v + z$ and a sequence of three subsequent rectangles $\mathrm{Rec}_{V_2} + Z_2, \dots, \mathrm{Rec}_{V_4} + Z_4$. The rectangles are not to scale: their common width is $\frac{1}{100}\Delta_\ell$, while the distance from Circ_ℓ to Ann_ℓ is $\frac{1}{2}\Delta_\ell$.

Let z', v' be the elements of $\mathbb{R}^2$ nearest z, v which lie on the circle $C_{\mathbb{R}^2}$ in $\mathbb{R}^2$ with the same radius as Circ_ℓ, centered at the origin. Additionally, let $L_{\mathbb{R}^2} \subseteq \mathbb{R}^2$ be the chord connecting z' and v'. For $x \in \mathrm{Rec}_v + z$, we aim to apply the triangle inequality as:

$$\mathrm{dist}(x, \mathrm{Circ}_\ell) \leqslant \mathrm{dist}(x, L_{\mathbb{R}^2}) + \max_{q \in C_{\mathbb{R}^2}} \mathrm{dist}(q, L_{\mathbb{R}^2}) + \max_{y \in \mathrm{Circ}_\ell} \mathrm{dist}(y, C_{\mathbb{R}^2}). \tag{2.179}$$

Concerning the first term, the triangle inequality implies

$$\mathrm{Rec}_v + z \subseteq \big\{x \in \mathbb{Z}^2 : \mathrm{dist}(x, L_{\mathbb{R}^2}) < 2w + 2\big\}, \tag{2.180}$$

because the width of Rec_v is w, because $\mathrm{dist}(z, \mathrm{Circ}_\ell) \leqslant w$, and because

$$\max_{y \in \mathrm{Circ}_\ell} \mathrm{dist}(y, C_{\mathbb{R}^2}) \leqslant 1, \tag{2.181}$$

which also addresses the third term.

We now address the second term. The greatest distance between $L_{\mathbb{R}^2}$ and $C_{\mathbb{R}^2}$ is the sagitta of $L_{\mathbb{R}^2}$. An elementary formula expresses the length of the sagitta associated with a chord at a radius of r and subtending an angle 2ω as $r(1 - \cos(\omega))$. Accordingly, when $r = \frac{1}{2}(a_\ell + b_\ell)$ and $2\omega = \phi$, we have

$$\max_{q \in C_{\mathbb{R}^2}} \mathrm{dist}(q, L_{\mathbb{R}^2}) = r(1 - \cos(\phi/2)) \leqslant \tfrac{1}{8} r\phi^2 \leqslant \tfrac{\pi^2}{200} r\varepsilon_\ell \leqslant \tfrac{3\pi^2}{400}\Delta_\ell. \tag{2.182}$$

The first inequality follows from $\cos\psi \geqslant 1 - \frac{1}{2}\psi^2$, which holds for $\psi \in [0, \frac{\pi}{2}]$. The second inequality is the result of substituting the upper bound $\frac{\pi}{5}\varepsilon_\ell^{1/2}$ for ϕ. The third inequality follows from $r = a_\ell + \frac{1}{2}\Delta_\ell$ and (2.177).

By (2.179) through (2.182), and then the fact that $\Delta_\ell \geqslant 10^4$,

$$\operatorname{dist}(x, \operatorname{Circ}_\ell) \leqslant \tfrac{3\pi^2}{400}\Delta_\ell + \tfrac{1}{100}\Delta_\ell + 3 \leqslant 0.10\Delta_\ell. \tag{2.183}$$

By the preceding discussion, (2.183) implies $\operatorname{Rec}_v + z \subseteq \operatorname{Ann}_\ell$.

Checking the hypotheses of Lemma 2.9.5. We aim to apply Lemma 2.9.5 to Rec_v and $\operatorname{Rec}_v^+ = \operatorname{Rec}(\psi, w, l+w)$, so we verify that $24 \leqslant w \leqslant l$. The first inequality holds because (2.12) implies $\Delta_\ell \geqslant 10^4$ which, in turn, implies $w \geqslant 100$. To estimate $l = \|v - z\|$, we note that $\arg z' - \arg v' \geqslant \frac{\pi}{12}\varepsilon_\ell^{1/2}$ and recall the fact that a chord at a radius r and subtending an angle of 2ψ has a length of $2r\sin(\psi)$. Using the same r as before and taking $2\psi = \frac{\pi}{24}\varepsilon_\ell^{1/2}$, we find

$$\|v' - z'\| \geqslant 2r\sin\left(\tfrac{\pi}{24}\varepsilon_\ell^{1/2}\right) \geqslant \tfrac{1}{6} r \varepsilon_\ell^{1/2} \geqslant \tfrac{1}{60}\Delta_\ell. \tag{2.184}$$

The second equality is due to $\sin(\omega) \geqslant \frac{2}{\pi}\omega$, which holds for $\omega \in [0, \frac{\pi}{2}]$. The third inequality follows from $r \geqslant a_\ell$ and (2.177). By the triangle inequality, $\|v - z\| \geqslant \|v' - z'\| - 2$. By (2.184) and the lower bound on Δ_ℓ, we have $l = \|v - z\| \geqslant \frac{1}{100}\Delta_\ell = w$.

An upper bound on l. We also note an important upper bound on l. The difference $\arg z' - \arg v'$ is at most $\frac{\pi}{4}\varepsilon_\ell^{1/2}$. The chord length formula gives

$$\|v' - z'\| \leqslant 2r\sin\left(\tfrac{\pi}{4}\varepsilon_\ell^{1/2}\right) \leqslant \tfrac{\pi}{2} r \varepsilon_\ell^{1/2} \leqslant \tfrac{3\pi}{4}\varepsilon_\ell^{-1/2}\Delta_\ell. \tag{2.185}$$

The second inequality follows from $\sin(\omega) \leqslant \omega$, which holds for $\omega \in [0, \frac{\pi}{2}]$. The third inequality follows from $r \leqslant a_\ell + \frac{1}{2}\Delta_\ell$ and (2.177). By the triangle inequality, $\|v - z\| \leqslant \|v' - z'\| + 2$. By (2.185) and the lower bound on Δ_ℓ, we have $l \leqslant 3\varepsilon_\ell^{-1/2}\Delta_\ell$.

Applying Lemma 2.9.5. Call $E_1 = \{\tau_{\partial(\operatorname{Rec}_v + z)} < \tau_{\partial(\operatorname{Rec}_v^+ + z)}\}$. Lemma 2.9.5 and translation invariance imply that there is a constant c_1 such that

$$\mathbb{P}_z(E_1) \geqslant c_1^{l/w} \geqslant c_2^{m_\ell^{1/2}}. \tag{2.186}$$

The second inequality follows from $w = \frac{1}{100}\Delta_\ell$, the upper bound $3\varepsilon_\ell^{-1/2}\Delta_\ell$ on l, $\varepsilon_\ell^{-1/2} \leqslant 2m_\ell^{1/2}$, and $c_2 = c_1^{600}$.

Call $\eta_1 = \tau_{\partial(\operatorname{Rec}_v + z)} \wedge \tau_{\partial(\operatorname{Rec}_v^+ + z)}$. When E_1 occurs, $Z_2 = S_{\eta_1}$ belongs to $(\operatorname{Circ}_\ell)_w$, hence, given Z_2, we could equally well apply the preceding argument to Z_2 in the place of z. Given Z_2, define $V_2 = V_2(Z_2)$ in analogy with v and obtain $\operatorname{Rec}_{V_2}^+$, E_2, and η_2 by replacing v, z with V_2, Z_2 in Rec_v, E_1, and η_1. By the preceding argument, given Z_2, when E_1 occurs,

$$\mathbb{P}_{Z_2}(E_2) \geqslant c_2^{m_\ell^{1/2}}. \tag{2.187}$$

We can continue in this fashion, defining sequences of Z_k, V_k, E_k, η_k for $k \geqslant 3$ (Figure 2.14) and with bounds analogous to (2.187), except with Z_k, E_k in the place of Z_2, E_2.

A lower bound on $\mathbb{P}_z(\tau_{\operatorname{Sec}_{\ell-1}} < \tau_A)$. Let $N = \lceil 20\varepsilon_\ell^{-1/2} \rceil$ and denote $F_j = \cap_{i=1}^j E_i$. We claim that, when $F_N = \cap_{i=1}^N E_i$ occurs, there is some $j \in \{0, \dots, \eta_N\}$ such that $S_j \in \operatorname{Sec}_{\ell-1}$. Recall that

$\mathrm{Sec}_{\ell-1}$ overlaps Ann_ℓ in an angular interval of width $\frac{2\pi}{3}\varepsilon_{\ell-1}$. By (2.178), $|\arg S_{j+1} - \arg S_j|$ is less than $\frac{2\pi}{3}\varepsilon_{\ell-1}$ whenever S_j, S_{j+1} belong to $D(a_\ell)^c$. The preceding argument shows that, when F_N occurs, S_j belongs to $\mathrm{Ann}_\ell \subseteq D(a_\ell)^c$ for all $j \in \{0, \ldots, \eta_N\}$. So when F_N occurs, it suffices for $\sum_{k=0}^{N-1}(\arg Z_k - \arg Z_{k+1}) \geqslant 2\pi$ to hold, as this will imply that some $S_j \in \mathrm{Sec}_{\ell-1}$. This is the case, since $\arg Z_k - \arg Z_{k+1}$ is at least $\frac{\pi}{10}\varepsilon_\ell^{1/2}$ for each k:

$$\sum_{k=0}^{N-1}(\arg Z_k - \arg Z_{k+1}) \geqslant 20\varepsilon_\ell^{-1/2} \cdot \tfrac{\pi}{10}\varepsilon_\ell^{1/2} \geqslant 2\pi.$$

We have shown that, if z belongs to $(\mathrm{Circ}_\ell)_w$—in particular, if $z \in \mathrm{Arc}_\ell$—then

$$\mathbb{P}_z(\tau_{\mathrm{Sec}_{\ell-1}} < \tau_A) \geqslant \mathbb{P}_z(F_N). \tag{2.188}$$

Denote the σ-field generated by $(S_0, S_1, \ldots, S_t)$ by $\mathscr{F}_t$. We have

$$\begin{aligned}\mathbb{P}_z(F_N) &= \prod_{i=1}^{N} \mathbb{E}_z\Big[\mathbb{P}_z(E_i|\mathscr{F}_{\eta_{i-1}})\frac{\mathbf{1}_{F_{i-1}}}{\mathbb{P}_z(F_{i-1})}\Big] \\ &= \prod_{i=1}^{N} \mathbb{E}_z\Big[\mathbb{P}_{Z_i}(E_i)\frac{\mathbf{1}_{F_{i-1}}}{\mathbb{P}_z(F_{i-1})}\Big] \geqslant \prod_{i=1}^{N} \mathbb{E}_z\Big[c_2^{m_\ell^{1/2}}\frac{\mathbf{1}_{F_{i-1}}}{\mathbb{P}_z(F_{i-1})}\Big] = c_2^{Nm_\ell^{1/2}}. \end{aligned} \tag{2.189}$$

The first equality is due to $F_{i-1} \in \mathscr{F}_{\eta_{i-1}}$ and the second is due to the strong Markov property applied to η_{i-1}. The inequality is due to (2.186), (2.187), and the analogous bounds with $Z_3, Z_4, \ldots$ and $E_3, E_4, \ldots$ in the place of Z_2 and E_2.

Substituting the definition of N into (2.189) implies that there is a constant c_3 such that $\mathbb{P}_z(F_N) \geqslant c_3^{m_\ell}$. Then, by (2.188), $\mathbb{P}_z(\tau_{\mathrm{Sec}_{\ell-1}} < \tau_A) \geqslant c_3^{m_\ell}$. □

Chapter 3

HAT in higher dimensions

This chapter is based on [Cal21].

3.1 Main results

We revert to denoting the ambient dimension by d, as we did in Chapter 1.

Returning to the question of whether HAT is recurrent or transient, it is easy to see that HAT is positive recurrent on $\widehat{\mathrm{NonIso}}_{d,n}$, for any dimension d, when the number of elements n is two or three. Additionally, according to Theorem 2.1.3, HAT is positive recurrent on the class of non-isolated configurations, for every $n \geqslant 2$ when $d = 2$. In the context of Theorem 2.1.3, the first of our main results establishes that HAT exhibits a phase transition, in the sense that HAT is transient in any dimension $d \geqslant 5$, for every $n \geqslant 4$.

Theorem 3.1.1. *HAT is transient for every $d \geqslant 5$ and $n \geqslant 4$.*

Note that, because HAT is transient on $\widehat{\mathrm{Iso}}_{d,n}$ for any d and n, there is no need to qualify Theorem 3.1.1 further. At the end of this section, we briefly discuss a heuristic which explains why we assume $d \geqslant 5$ in Theorem 3.1.1, and which suggests that transience is plausible in four dimensions as well.

Figure 1.2 summarizes what is known about the phase diagram of HAT in the d–n grid. We highlight two of its features:

- Theorems 2.1.3 and 3.1.1 imply that the phase boundary has a "corner": There is a dimension $d' \geqslant 3$ and a number of elements $n' \geqslant 4$ such that HAT is transient for $(d,n) = (d',n')$, but positive recurrent on $\widehat{\mathrm{NonIso}}_{d,n}$ when $(d,n) = (d'-1,n')$ or $(d,n) = (d',n'-1)$.
- The classification of HAT is unknown when $d \in \{3,4\}$ and $n \geqslant 4$. However, a heuristic argument suggests that HAT is positive recurrent on $\widehat{\mathrm{NonIso}}_{3,n}$ when $n \in \{4,5\}$.

Our second main result, Theorem 3.1.5, is a detailed description of the "way" in which transience occurs when $d \geqslant 5$ and $n \geqslant 4$. To state the result, we need the notion of a clustering and some related definitions. Whereas in Chapter 2 we used D and C to denote a disk and its boundary, we will henceforth use them to denote clusterings.

Definition 3.1.2 (Clustering)**.** *A clustering C of a configuration U into k clusters is an ordered partition $(C^1, \dots, C^k)$ of U. In other words, C is a k-tuple of nonempty, disjoint subsets of U, the union of which is U.*

We need additional notation to state the next definition.

- First, observe that, if $C = (C^1, \dots, C^k)$ is a generic k-tuple of nonempty subsets of $\mathbb{Z}^d$ and if $x \in \cup_i C^i$, then x may belong to multiple subsets in C. However, if C is a clustering of a configuration U, then $x \in U$ belongs to exactly one cluster in C, which we call $\mathrm{clust}(C, x)$. For notational convenience, we define $\mathrm{clust}(C, \emptyset) = 1$.
- Second, in the context of a k-tuple $C = (C^1, \dots, C^k)$ of nonempty subsets of $\mathbb{Z}^d$, a label $i \in [\![k]\!]$ where $[\![k]\!]$ denotes $\{1, \dots, k\}$, and $A \subseteq \mathbb{Z}^d$, we will use the following notation:

$$C \cup^i A = (C^1, \dots, C^{i-1}, C^i \cup A, C^{i+1}, \dots, C^k)$$
$$C \setminus^i A = (C^1, \dots, C^{i-1}, C^i \setminus A, C^{i+1}, \dots, C^k).$$

We now introduce a way to associate a sequence of clusterings to a sequence of configurations. Given a clustering C_t of U_t and U_{t+1}, there is natural way to obtain a clustering of U_{t+1} from C_t: Treat every element of U_t like a element with a time-dependent location and a label fixed by C_t. The locations of the elements change according to the HAT dynamics, but their labels do not.

We implement this as follows. If $U_{t+1} = U_t$, then we set $C_{t+1} = C_t$. If $U_t \backslash U_{t+1}$ and $U_{t+1} \backslash U_t$ are singletons, then we call the corresponding elements X and Y, and set

$$C_{t+1} = (C_t \setminus^i \{X\}) \cup^i \{Y\} \tag{3.1}$$

where $i = \mathrm{clust}(C_t, X)$. Otherwise, we set C_{t+1} to be the clustering of U_{t+1} with one cluster.

Definition 3.1.3 (Natural clustering)**.** *Let $t \geqslant 0$ and let C be a clustering of U_t. The natural clustering of $(U_t, U_{t+1}, \dots)$ with C is the sequence of clusterings $(C_t, C_{t+1}, \dots)$ with $C_t = C$, and C_{s+1} determined by C_s, U_s, and U_{s+1} according to* (3.1) *for $s \geqslant t$.*

The last definition we need before stating Theorem 3.1.5 identifies a special kind of clustering, for which clusters have size two or three and satisfy bounds on separation, in terms both absolute and relative to diameter. Denote the set $C^{\neq i} = \cup_{j \neq i} C^i$. The separation of a clustering C is

$$\mathrm{sep}(C) = \min_i \mathrm{dist}(C^i, C^{\neq i}).$$

Definition 3.1.4 (Dimer-or-trimer clustering)**.** *For positive real numbers a and b, a clustering C is an (a, b) separated dimer-or-trimer (DOT) clustering if:*

(DOT.1) $|C^i| \in \{2, 3\}$ *for each i;*

(DOT.2) $\operatorname{sep}(C) \geqslant a$*; and*

(DOT.3) $\operatorname{diam}(C^i) \leqslant b \log \operatorname{dist}(C^i, C^{\neq i})$ *for each* i.

In this sense, a *constrains the absolute separation of the clustering, while* b *constrains cluster separation relative to diameter. We will refer to each cluster* C^i *with* $|C^i| = 2$ *as a dimer and each cluster* C^j *with* $|C^j| = 3$ *as a trimer*

For an n*-element configuration* $U \subset \mathbb{Z}^d$*, we will denote by* $\mathscr{C}(U, a, b)$ *the (possibly empty) set of* (a, b) *separated DOT clusterings of* U*. We will denote the set of* n*-element configurations in* $\mathbb{Z}^d$ *which have at least one* (a, b) *separated DOT clustering by* $\mathscr{U}_{d,n}(a, b)$*.*

Dimers and trimers have a special status relative to other clusters in dimension $d \geqslant 3$ for two reasons.

- Unlike a cluster consisting of a single element, a constituent element of a dimer or trimer can be activated without the cluster necessarily being "absorbed" by another; and
- Unlike clusters comprised of four or more elements, the distribution of the time it takes for a dimer or trimer to return to a given orientation has an exponential tail.

Regarding the second of these reasons, in dimension $d \geqslant 5$, an activated element would likely escape to infinity in the absence of the conditioning in the transport component of the HAT dynamics (1.3). Consequently, when clusters are well separated, the HAT dynamics favors a element activated at one cluster to be transported to the same cluster. As d increases, this effect becomes more pronounced. It is this fact about dimension $d \geqslant 5$ which allows dimers and trimers to persist for long periods of time without being absorbed by another cluster, despite comprising few elements.

Our discussion of the special status of dimers and trimers suggests that HAT configurations in dimension $d \geqslant 5$ with at least four elements may fragment until consisting exclusively of dimers and trimers, which then avoid one another indefinitely. The second of our main results shows that, remarkably, this is the fate of all HAT configurations.

Theorem 3.1.5. *Let* $d \in \mathbb{Z}_{\geqslant 5}$ *and* $n \in \mathbb{Z}_{\geqslant 4}$*, and let* U_0 *be an* n*-element configuration in* $\mathbb{Z}^d$*. There is a positive real number* $b = b(n)$*, and a* $\mathbf{P}_{U_0}$*-a.s. finite random time* θ *with the following property. There is a clustering* C *of* U_θ *for which the natural clustering* $(C_\theta, C_{\theta+1}, \dots)$ *of* $(U_\theta, U_{\theta+1}, \dots)$ *with* C *satisfies*

$$C_t \in \mathscr{C}(U_t, a_{t-\theta}, b) \text{ for } t \geqslant \theta, \tag{3.2}$$

where $a_s = s^{\frac{1}{2} - o_n(1)} + 100n$*. In particular,* (3.2) *implies*

$$U_t \in \mathscr{U}_{d,n}(a_{t-\theta}, b) \text{ for } t \geqslant \theta. \tag{3.3}$$

The term of $100n$ *in* a_s *is merely representative of a large multiple of* n*. For concreteness,* (3.2) *and* (3.3) *are true with* n^{-100} *in the place of* $o_n(1)$*.*

Theorem 3.1.5 identifies, for any initial configuration, an a.s. finite random time θ at which there is a clustering of U_θ into dimers or trimers and forever after which the same dimers or trimers become steadily, increasingly separated—in terms both absolute and relative to their diameters. In particular, there is no exchange of elements between clusters after θ. We emphasize that (3.2) is stronger than (3.3) because it precludes the exchange of elements between the clusters defined by C, while (3.3) does not.

Theorem 3.1.5 implies Theorem 3.1.1 because HAT is irreducible on non-isolated configurations.

Theorem 3.1.6. *HAT is irreducible on* $\widehat{\mathrm{NonIso}}_{d,n}$*, for every* $d \geqslant 5$ *and* $n \geqslant 4$.

Proof of Theorem 3.1.1. Let $d \geqslant 5$ and $n \geqslant 4$, and let U be the n-element segment $\{(j, 0, \ldots, 0) : j \in [\![n]\!]\} \subset \mathbb{Z}^d$. It suffices to show that $\widehat{U}$ is transient, since $\widehat{U} \in \widehat{\mathrm{NonIso}}_{d,n}$ and since HAT is irreducible on $\widehat{\mathrm{NonIso}}_{d,n}$ by Theorem 3.1.6. Theorem 3.1.5 implies that there is a $\mathbf{P}_U$-a.s. finite time θ such that $\mathrm{diam}(U_t) \geqslant 100n$ for $t \geqslant \theta$. Because $\mathrm{diam}(\widehat{U}) < 100n$, this implies that there are $\mathbf{P}_U$-a.s. finitely many returns to $\widehat{U}$, hence $\widehat{U}$ is transient. □

Key to the proof of Theorem 3.1.5 is the fact that, if a is a number which is sufficiently large in terms of d and n, then, in a number of steps $f = f(a, d, n)$ and with a probability of at least $g = g(a, d, n)$, HAT reaches a configuration in $\mathscr{U}_{d,n}(a, 1)$ from any configuration.

Theorem 3.1.7. *Let* $a \in \mathbb{Z}_{\geqslant 2}$, $d \in \mathbb{Z}_{\geqslant 5}$, *and* $n \in \mathbb{Z}_{\geqslant 4}$. *If* a *is sufficiently large in terms of* d *and* n, *then there are* $f \in \mathbb{Z}_{\geqslant 0}$ *and* $g > 0$ *such that, for any* n*-element configuration* $U \subset \mathbb{Z}^d$,

$$\mathbf{P}_U\big(U_f \in \mathscr{U}_{d,n}(a, 1)\big) \geqslant g. \tag{3.4}$$

The critical aspect of Theorem 3.1.7 is that f and g do not depend on the diameter of U. The proof takes the form of an analysis of three algorithms, which sequentially: (i) rearrange the configuration into well separated, connected clusters with at least two elements each; (ii) organize each cluster into a line segment; and (iii) split the segments into dimers and trimers. Collectively, the algorithms take as input an arbitrary configuration U and $a \in \mathbb{Z}_{\geqslant 2}$, and return a configuration in $\mathscr{U}_{d,n}(a, 1)$. It does not seem possible to appreciably simplify this process without introducing into g a dependence on the diameter of initial configuration.

Transience is plausible in four dimensions and provable in at least five dimensions

We preemptively address the question of "Why must $d \geqslant 5$ in Theorem 3.1.1?" with a discussion of some heuristics. Consider a pair of dimers. Until they begin to exchange elements, the difference of their centers of mass will behave like a d-dimensional random walk. If they never exchange elements, then, because random walk is transient in $d \geqslant 3$ dimensions, their separation will grow steadily and without bound as Theorem 3.1.5 predicts.

This basic picture is complicated by the fact that dimers exchange elements over a number of steps which depends on their separation. Specifically, if the dimers are separated by a distance a, then they will typically exchange elements over a^{d-2} steps. This timescale is related to the fact that a random walk from the origin in $d \geqslant 3$ dimensions escapes the origin to a distance a with a probability of roughly a^{2-d}. If the dimers do not exchange elements during a period of a^2 steps, then the separation of the dimers likely doubles over the same period, after which it takes 2^{d-2} times longer for them to exchange elements. Hence if $a^{d-2} \geqslant a^2$ and $2^{d-2} > 2$ (i.e., if $d \geqslant 4$), then it is plausible that dimer separation grows quickly enough that elements are never exchanged.

Now, consider replacing at least one of the dimers with a trimer, which can have a diameter as large as the logarithm of its separation (DOT.3). The larger the diameter of a trimer, the more likely it is for it to exchange a element with another cluster. Specifically, if an a separated trimer has a diameter of $\log(a)$, then it exchanges a element with another cluster over $a^{d-2}\log(a)^{-1}$ steps. This timescale is related to the fact that a random walk starting at a distance of $\log(a)$ from the origin in $d \geqslant 3$ dimensions escapes the origin to a distance a with a probability of roughly $a^{2-d}\log(a)$. For $a^{d-2}\log(a)^{-1}$ to be at least a^2, d must satisfy $d \geqslant 5$.

In summary, the greater the separation between DOTs, the longer it takes for them to exchange elements. This effect becomes more pronounced as d increases. Until DOTs exchange elements, the pairwise differences in their centers of mass behave like d-dimensional random walks, which inclines them to grow increasingly separated due to the transience of random walk in $d \geqslant 3$ dimensions. In $d \geqslant 5$ dimensions, we will be able to show that DOT separation grows rapidly enough in the absence of element exchange that it is typical for no element to be exchanged, leading to Theorem 3.1.5.

Organization

The machinery underlying the proof of Theorem 3.1.5 is an approximation of HAT by another Markov chain, called *intracluster* HAT (IHAT), which effectively applies the HAT dynamics to each cluster in a separate copy of $\mathbb{Z}^d$. Under IHAT, we will essentially treat the clusters' centers of mass as independent random walks. Section 3.4 explains how we will compare the transition probabilities of HAT and IHAT, and introduces a notion of harmonic measure adapted to clusterings, which is used to define IHAT. Section 3.5 proves estimates of harmonic measure, which are used to control the error arising from approximating HAT by IHAT in Section 3.6. Section 3.7 introduces a random walk model of the separation between a pair of clusters and Section 3.8 uses this random walk to obtain key estimates of separation growth. Beginning in Section 3.9, the focus shifts to the proof of Theorem 3.1.7. Section 3.10 presents some supporting results, which are applied in Section 3.11 to analyze the three algorithms around which the proof of Theorem 3.1.7 is organized in Section 3.11. The last section, Section 3.12, proves Theorem 3.1.6.

3.2 Proof of Theorem 3.1.5

Theorem 3.1.5 states that there is a random time θ from which the natural clustering of $(U_\theta, U_{\theta+1}, \dots)$ grows in separation according to (3.2), and which is $\mathbf{P}_U$-a.s. finite for every configuration U. Informally, we will define θ as the time of the first success in a sequence of trials, each of which attempts to observe the natural clustering with a sufficiently well separated DOT clustering satisfy (3.2). The fact that θ is a.s. finite will be a simple consequence of two results. First, Theorem 3.1.7 implies that, if the present trial fails, then we can conduct another, after waiting an a.s. finite number of steps for U_t to have a sufficiently well separated DOT clustering. Second, the following result states that each trial succeeds with a probability which is bounded away from zero, hence we need only conduct an a.s. finite number of trials before one succeeds.

Proposition 3.2.1. *Let $a, b \in \mathbb{R}_{>1}$, $W_0 \in \mathscr{U}_{d,n}(a,b)$ and $E_0 \in \mathscr{C}(W_0, a, b)$ for $d \in \mathbb{Z}_{\geqslant 5}$, and $n \in \mathbb{Z}_{\geqslant 4}$. Let $(C_0, C_1, \dots)$ be the natural clustering of $(U_0, U_1, \dots)$ with E_0 and let a_t denote the quantity $t^{\frac{1}{2}-n^{-100}} + 100n$ for $t \in \mathbb{Z}_{\geqslant 0}$. Define ξ to be the first time t that C_t is not an (a_t, b) separated DOT clustering of U_t:*

$$\xi = \inf\{t \geqslant 0 : C_t \notin \mathscr{C}(U_t, a_t, b)\}.$$

There is a number $\varepsilon > 0$ such that, if a, b are sufficiently large in terms of d, n, then

$$\mathbf{P}_{W_0}(\xi = \infty) \geqslant \varepsilon. \tag{3.5}$$

The proof of Proposition 3.2.1 will comprise several sections, and we dedicate the next section to a discussion of the proof strategy. For now, we assume it and use it to prove Theorem 3.1.5.

Proof of Theorem 3.1.5. Let $a \in \mathbb{Z}_{\geqslant 2}$ and $b \in \mathbb{R}_{>1}$ be sufficiently large in terms of d, n to satisfy the hypotheses of Theorem 3.1.7 and Proposition 3.2.1, and let ε be the number of the same name from Proposition 3.2.1. Additionally, let ψ be a function which, given a configuration $U \in \mathscr{U}_{d,n}(a,b)$, determines a clustering in $\mathscr{C}(U, a, b)$. (We use ψ to "pick" one of potentially multiple clusterings; it is otherwise unimportant.)

We define θ in terms of two sequences of random times, $(\tau_i)_{i\geqslant 1}$ and $(\xi_i)_{i\geqslant 0}$. Informally, τ_i is beginning of our i^{th} attempt to observe the natural clustering satisfy (3.2), and ξ_i is the time this attempt fails. More precisely, we define

$$\xi_0 = 0 \text{ and } \xi_i = \inf\{t \geqslant \tau_i : C_{i,t} \notin \mathscr{C}(U_t, a_{t-\tau_i}, b)\} \text{ for } i \geqslant 1,$$

where

- a_s is the quantity of the same name in the statement of Proposition 3.2.1;
- $\tau_i = \inf\{t \geqslant \xi_{i-1} : \mathscr{C}(U_t, a, b) \text{ is nonempty}\}$ for $i \geqslant 1$; and
- $(C_{i,\tau_i}, C_{i,\tau_i+1}, \dots)$ is the natural clustering of $(U_{\tau_i}, U_{\tau_i+1}, \dots)$ with $\psi(U_{\tau_i})$.

Lastly, define $\theta = \tau_I$ for $I = \inf\{i \geqslant 1 : \tau_i < \infty, \xi_i = \infty\}$. By the definition of θ, (3.2) is satisfied, which implies that (3.3) is, too.

We now show that θ is $\mathbf{P}_U$-a.s. finite. Let $J = \inf\{j : \xi_j = \infty\}$, in which case we have

$$\mathbf{P}_U(\theta < \infty) = \mathbf{P}_U(I < \infty) = \mathbf{P}_U(J < \infty). \tag{3.6}$$

The first equality follows from the definitions of I and θ. The second equality follows from the fact that $\mathbf{P}_W(\tau_1 < \infty) = 1$ for any n-element configuration $W \subset \mathbb{Z}^d$, which is a simple consequence of Theorem 3.1.7.

To bound the tail probabilities of J, we write

$$\mathbf{P}_U(J > j+1 \,|\, J > j) = \frac{\mathbf{E}_U\left[\mathbf{P}_{U_{\tau_j}}(\xi_1 < \infty); J > j\right]}{\mathbf{P}_U(J > j)} \leqslant 1 - \varepsilon. \tag{3.7}$$

The equality follows from the strong Markov property applied at time ξ_j and the fact that $\mathbf{P}_{U_{\xi_j}}(\tau_1 < \infty) = 1$. The inequality holds because Proposition 3.2.1 implies $\mathbf{P}_{U_{\tau_j}}(\xi_1 = \infty) \geqslant \varepsilon$.

The bound (3.7) implies that $\mathbf{P}_U(J > j)$ is summable, so the Borel-Cantelli lemma implies that J is $\mathbf{P}_U$-a.s. finite, which by (3.6) implies the same of θ. □

3.3 Strategy for the proof of Proposition 3.2.1

The proof of Proposition 3.2.1 has two main steps. First, we prove that, when DOTs are a separated in $d \geqslant 5$ dimensions, HAT approximates a related process, called *intracluster HAT* (IHAT), in which transport occurs only to the cluster at which activation occurred, over a^2 steps, up to an error of $O(a^{-1}\log(a))$. Specifically, we will prove that, if the natural clustering of $W_0, \ldots, W_t$ with a clustering C_0 satisfies separation conditions related to DOT.2 and DOT.3 (e.g., C_t is a separated), then the probability $\mathbf{P}_{W_0}(U_1 = W_1, \ldots, U_t = W_t)$ is within a factor of

$$\Big(1 - O\big(a^{2-d}\log(a)\big)\Big)^t$$

of the probability of the analogous event under IHAT. In our application, we will have $t = O(a^2)$ and $d \geqslant 5$, in which case this factor is $1 - O(a^{-1}\log(a))$.

Second, we show that over a^2 steps of IHAT, the separation between every pair of clusters effectively doubles, except with a probability of $O(a^{-1})$. We show this by considering the pairwise differences between clusters' centers of mass, viewed at consecutive times of return to given, "reference" orientations. Viewed in this way, the pairwise differences are d-dimensional random walks. We then apply the same argument with $2a$ in the place of a, then $4a$ in the place of $2a$, and so on. Each time the separation doubles, the approximation and exception errors halve, which implies that if a is sufficiently large, then the separation grows without bound, with positive probability.

The second step is possible because, under IHAT, we can treat each DOT as if it inhabited a separate copy of $\mathbb{Z}^d$, which simplifies our analysis of separation. We define IHAT by conditioning the transport component of the HAT dynamics on *intracluster transport*, i.e., transport can only occur to the boundary of the cluster at which activation occurred. Because intercluster transport over a^2 steps is atypical when clusters are a separated, IHAT is a good approximation of HAT over the period during which clusters typically double in separation.

The activation component of the IHAT dynamics is defined in terms of a generalization of harmonic measure to clusterings of subsets of $\mathbb{Z}^d$. This harmonic measure is proportional to the escape probability of each element, from the cluster to which the element belongs—not the entire configuration. In this way, the IHAT dynamics will treat each cluster in isolation, rather than collectively through the configuration. When clusters are a separated, the harmonic measure of a configuration and a clustering thereof will agree up to a factor of $1 - O(a^{2-d})$. In fact, the discrepancy between the transport components of HAT and IHAT will give rise to the dominant error factor; we discuss this in greater detail in the next section.

3.4 Intracluster HAT

The purpose of this section is to motivate the definition of intracluster HAT by examining the transition probabilities of HAT. We will observe that, when a HAT configuration consists of clusters which are well separated—in terms both absolute and relative to the diameters of the clusters—intercluster transport is rare. As a result, clusters evolve as if inhabiting separate copies of $\mathbb{Z}^d$, and the pairwise differences in their centers of mass can be modeled as random walks.

Although HAT makes no reference to clusters, we can lift HAT (in the sense of, e.g., [DHN00]) to a process on clusterings, and subsequently recover HAT through the map π which takes a tuple of sets C to their union:

$$\pi(C) = \bigcup_i C^i. \tag{3.8}$$

(We will no longer use π to denote the stationary distribution of HAT in two dimensions.) To define the lifted process, we give a name to the collection of pairs of elements of $\mathbb{Z}^d$ at which activation and to which transport can occur. We use the following notation for modifying a k-tuple of nonempty subsets of $\mathbb{Z}^d$, C:

$$C^{i,x,y} = (C \setminus^i \{x\}) \cup^i \{y\} \;\text{ for }\; i \in [\![k]\!],\;\; x, y \in \mathbb{Z}^d.$$

Definition 3.4.1 (A lifting of HAT)**.** *Let E_0 and E_1 be clusterings of two configurations in $\mathbb{Z}^d$. We define*

$$\mathsf{Pairs}(E_0, E_1) = \big\{(x,y) : x, y \in \mathbb{Z}^d,\;\; p_{\pi(E_0)}(x,y) > 0,\;\; \textit{and}\;\; E_1 = E_0^{i,x,y} \;\textit{for}\; i = \mathrm{clust}(E_0, x)\big\}.$$

Given a clustering C_0 of a configuration in $\mathbb{Z}^d$, we define the Markov chain $(C_0, C_1, \dots)$ with transition probabilities

$$\mathbf{P}(C_{t+1} = E \mid C_t) = \sum_{(x,y) \in \mathsf{Pairs}(C_t, E)} p_{\pi(C_t)}(x,y) \tag{3.9}$$

for $t \geqslant 0$. Note that, if we set $U_t = \pi(C_t)$ for each $t \geqslant 0$, then $(U_t)_{t\geqslant 0}$ has the same transition probabilities as HAT under $\mathbf{P}_{U_0}$. It this sense, $(C_t)_{t\geqslant 0}$ is a lifting of HAT. For simplicity, we will refer to both as HAT.

In the following subsections, we motivate IHAT by examining the transition probabilities of HAT (Section 3.4.1) and define IHAT (Section 3.4.2).

3.4.1 A calculation which motivates the definition of IHAT

Let C_0 be a clustering of a configuration U_0 in $\mathbb{Z}^d$. For each $t \geqslant 0$, we set $U_t = \pi(C_t)$, where $(C_t)_{t\geqslant 0}$ has the transition probabilities given by (3.9). We will examine the transition probabilities in the case when transport occurs to the boundary of the cluster at which activation occurred, i.e., $y \in \partial(C_t^i \backslash \{x\})$. We additionally assume that:

- $p_{U_t}(x, y)$ is positive; and
- C_t satisfies the absolute and relative separation conditions, DOT.2 and DOT.3, for some $a, b \in \mathbb{R}_{>1}$.

Call $\tau = \tau_{U_t \backslash \{x\}}$, $\tau' = \tau_{C_t^i \backslash \{x\}}$, and $\tau'' = \tau_{C_t^{\neq i}}$.

Under the assumptions, if $\{S_{\tau-1} = y\}$ occurs, then $\{\tau' < \tau''\}$ occurs too. Hence we can express the probability $p_{U_t}(x, y)$ (1.3) in terms of τ, τ', and τ'' as

$$\begin{aligned} p_{U_t}(x, y) &= \mathbb{H}_{U_t}(x)\, \mathbb{P}_x(S_{\tau-1} = y \mid \tau < \infty) \\ &= \mathbb{H}_{U_t}(x)\, \frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \tau'', \tau < \infty)}{\mathbb{P}_x(\tau < \infty)} \\ &= \mathbb{H}_{U_t}(x)\, \frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty) - \mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty)}{\mathbb{P}_x(\tau < \infty)}. \end{aligned}$$

Regrouping, we find

$$\begin{aligned} p_{U_t}(x, y) = \mathbb{H}_{U_t}(x)\, &\mathbb{P}_x(S_{\tau'-1} = y \mid \tau' < \infty) \\ &\times \frac{\mathbb{P}_x(\tau' < \infty)}{\mathbb{P}_x(\tau < \infty)} \left(1 - \frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty)}{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty)}\right). \end{aligned} \tag{3.10}$$

Concerning the factors in (3.10):

- The factor of
$$\mathbb{H}_{U_t}(x)\, \mathbb{P}_x(S_{\tau'-1} = y \mid \tau' < \infty) \tag{3.11}$$
can be thought of as an analogue of $p_{U_t}(x, y)$ for a variant of HAT in which only intracluster transport occurs. For the moment, denote this Markov chain by $(C_t')_{t\geqslant 0}$ and denote its law by $\mathbf{P}'$. We will modify the transition probabilities of this Markov chain in the next subsection to obtain a useful approximation of HAT.

- The factor of $\frac{\mathbb{P}_x(\tau'<\infty)}{\mathbb{P}_x(\tau<\infty)}$ is the probability that intracluster transport occurs. This probability is at least $1 - O(a^{2.1-d}b)$. The factor of $a^{2.1-d}$ arises from the asymptotic form of Green's function for random walk (3.18), the leading-order term of which is $O(a^{2-d})$, and a simple bound of $O(a^{0.1}b)$ on a reciprocal power of the diameters of the clusters, which satisfy DOT.3 with b by assumption.

- The ratio in parentheses in (3.10) is small when a is large because, for $\{\tau'' < \tau' < \infty\}$ to occur, a random walk from x in cluster i must escape to a different cluster, before returning. This probability is $O(a^{4-2d})$. In contrast, $\{S_{\tau'-1} = y, \tau' < \infty\}$ occurs with a probability which is at least a reciprocal power of the diameters of the clusters, which is $O(a^{0.1}b)$. Overall, the factor in parentheses will be $1 - O(a^{4.1-2d}b)$.

These observations suggest that, if E_0 is a clustering of a configuration in $\mathbb{Z}^d$ and if $(E_1, \dots, E_t)$ is a sequence of clusterings arising from transitions which satisfy the preceding assumptions, then

$$\mathbf{P}_{E_0}(C_1 = E_1, \dots, C_t = E_t) \geqslant \left(1 - O(a^{2.1-d}b) - O(a^{4.1-2d}b)\right)^t \mathbf{P}'_{E_0}(C'_1 = E_1, \dots, C'_t = E_t). \quad (3.12)$$

The relevant values of d and t for our purposes will be $d \geqslant 5$ and $t = O(a^2)$, in which case the first factor on the right-hand side of (3.12) is roughly $(1 - O(a^{-1}))$.

Even if we proved (3.12), it would not serve our strategy to model the clusters' centers of mass under $\mathbf{P}'$ as random walks in separate copies of $\mathbb{Z}^d$. The reason is that (3.11) still refers to clusters collectively, through U_t. We will resolve this in the next subsection, by replacing $\mathbb{H}_{U_t}(x)$ in (3.11) with a harmonic measure–like quantity that treaters clusters in isolation. The resulting transition probabilities will define IHAT.

3.4.2 The definition of IHAT

In $d \geqslant 3$ dimensions, there is an alternative definition of harmonic measure, which is equivalent to (1.2) (see, e.g., Chapter 2 of [Law13]). For finite $A \subset \mathbb{Z}^d$ and $x \in \mathbb{Z}^d$, we define the escape probability and capacity of A by

$$\mathrm{esc}_A(x) = \mathbb{P}_x(\tau_A = \infty)\, \mathbf{1}(x \in A) \quad \text{and} \quad \mathrm{cap}_A = \sum_{x \in A} \mathrm{esc}_A(x).$$

In these terms, the harmonic measure of A can be defined as

$$\mathbb{H}_A(x) = \frac{\mathrm{esc}_A(x)}{\mathrm{cap}_A} \quad \text{for } x \in A. \quad (3.13)$$

For a positive integer k, we generalize (3.13) to finite $A = (A^1, \dots, A^k) \subset (\mathbb{Z}^d)^k$ by defining $\mathrm{cap}_A = \sum_{i=1}^k \mathrm{cap}_{A^i}$ and

$$\mathbb{H}_A(i, x) = \frac{\mathrm{esc}_{A^i}(x)}{\mathrm{cap}_A} \quad \text{for } i \in [\![k]\!] \text{ and } x \in A^i. \quad (3.14)$$

We are ready to define IHAT, which we will denote by $(D_t)_{t\geqslant 0}$. Recalling (3.11), we replace the activation component $\mathbb{H}_{U_t}(x)$ with $\mathbb{H}_{D_t}(i,x)$ to obtain $q_{D_t}(i,x,y)$, the intracluster analogue of $p_{U_t}(x,y)$:

$$q_{D_t}(i,x,y) = \mathbb{H}_{D_t}(i,x)\,\mathbb{P}_x(S_{\tau-1} = y \mid \tau < \infty), \tag{3.15}$$

where τ abbreviates $\tau_{D_t \setminus^i \{x\}}$.

Definition 3.4.2 (Intracluster HAT). *Let E_0 and E_1 be clusterings of two configurations in $\mathbb{Z}^d$. We define*

$$\mathsf{Triples}(E_0,E_1) = \big\{(i,x,y) : i \in \mathbb{Z}_{\geqslant 1},\ x,y \in \mathbb{Z}^d,\ q_{E_0}(i,x,y) > 0,\ \textit{and}\ E_1 = E_0^{i,x,y}\big\}.$$

Given a clustering D_0 of a configuration in $\mathbb{Z}^d$, we define IHAT as the Markov chain $(D_0, D_1, \dots)$ with transition probabilities

$$\mathbf{Q}(D_{t+1} = E \mid D_t) = \sum_{(i,x,y)\in\mathsf{Triples}(D_t,E)} q_{D_t}(i,x,y) \tag{3.16}$$

for $t \geqslant 0$. Additionally, we set $V_t = \pi(D_t)$ for $t \geqslant 0$.

3.5 A lower bound of harmonic measure

The purpose of this section is to prove estimates of harmonic measure. We will use them in Section 3.6 to compare the activation components of HAT and IHAT and, ultimately, to establish an inequality of the form (3.12) for $\mathbf{Q}$ in the place of $\mathbf{P}'$.

Throughout this section, we assume $d \geqslant 3$. The first result concerns Green's function $G(x)$, which is defined for $x \in \mathbb{Z}^d$ by

$$G(x) = \sum_{t=0}^{\infty} \mathbb{P}_x(S_t = 0). \tag{3.17}$$

Green's function has the asymptotic form

$$G(x) = c\|x\|^{2-d} + O(\|x\|^{-d}) \tag{3.18}$$

for an explicit constant $c = c(d)$. In some instances, we will emphasize the dependence of Green's function on dimension by including a subscript: $G_d(x)$.

We need two monotonicity properties of Green's function. The first is phrased in terms of the following partial order on $\mathbb{Z}^d$. For $x,y \in \mathbb{Z}^d$, we write

$$x \succeq y \iff |x_i| \geqslant |y_i| \text{ for all } i \in [\![d]\!],$$

where x_i and y_i denote the i^{th} components of x and y.

Lemma 3.5.1 (Lemma 8 of [CC16]). *Let $x,y \in \mathbb{Z}^d$. If $x \succeq y$, then $G(x) \leqslant G(y)$.*

The second monotonicity property concerns Green's function at the origin $o \in \mathbb{Z}^d$.

Lemma 3.5.2. *$G_d(o)$ is a nonincreasing function of d.*

Proof. Montroll [Mon56, Eq. (2.10)] expresses $G_d(o)$ in terms of I_0, the modified Bessel function of the first kind, as

$$G_d(o) = \int_0^\infty e^{-t} I_0\Big(\frac{t}{d}\Big)^d dt.$$

Using the integral representation of I_0 [AS64, Eq. 9.6.19], we see that $I_0(\frac{t}{d})^d$ is the $L^{\frac{1}{d}}$ norm of $e^{t\cos\theta}$ with respect to the probability measure $\pi^{-1}d\theta$ on $[0,\pi]$:

$$I_0\Big(\frac{t}{d}\Big)^d = \bigg(\frac{1}{\pi}\int_0^\pi e^{\frac{t\cos\theta}{d}}\,d\theta\bigg)^d.$$

For any $d' \geqslant d$, the $L^{\frac{1}{d'}}$ norm is no larger than the $L^{\frac{1}{d}}$ norm, which implies

$$G_d(o) \geqslant \int_0^\infty e^{-t} I_0\Big(\frac{t}{d'}\Big)^{d'} dt = G_{d'}(o).$$

□

The next result is a consequence of Lemma 3.5.2.

Lemma 3.5.3. *If $d \in \mathbb{Z}_{\geqslant 5}$, then $G_d(o) \leqslant 1.2$.*

Proof. By Lemma 3.5.2, $G_d(o) \leqslant G_5(o)$ when $d \geqslant 5$. When $d = 5$, the probability p that random walk in $\mathbb{Z}^d$ returns to the origin is less than 0.14 [Mon56]. Since $G_5(o) = (1-p)^{-1}$, this implies $G_5(o) \leqslant 1.2$. □

We use the first monotonicity property to prove the next result.

Lemma 3.5.4. *Let U be an n-element configuration in $\mathbb{Z}^d$. There is a number $c = c(d)$ such that, if there is a clustering $\mathcal{C} = (C^1, C^2)$ of U with $|C^1| \leqslant 3$, then*

$$\mathrm{esc}_U(x) \geqslant \frac{3 - 2G(o)}{G(o)} - cn\,\mathrm{sep}(\mathcal{C})^{2-d} \quad \textit{for } x \in C^1. \tag{3.19}$$

The first term on the right-hand side of (3.19) is the escape probability of a three-element set, from an element with two neighbors. The term $n\,\mathrm{sep}(\mathcal{C})^{2-d}$ bounds the probability that random walk from C^1 hits C^2 before escaping; it arises from the leading-order term of Green's function (3.18) and the fact that $|C^2| \leqslant n$.

Proof of Lemma 3.5.4. Let U and $\mathcal{C}$ satisfy the hypotheses and let $x \in C^1$. Denote by N the number of returns made to U by random walk. We write $\mathrm{esc}_U(x)$ in terms of N as

$$\mathrm{esc}_U(x) = \mathbb{P}_x(N = 0) = 1 - \frac{\mathbb{E}_x[N]}{\mathbb{E}_x[N \mid N > 0]}. \tag{3.20}$$

We will bound $\mathrm{esc}_U(x)$ from below by substituting an upper bound for $\mathbb{E}_x[N]$ and a lower bound for $\mathbb{E}_x[N|N>0]$ into (3.20).

First, we find an upper bound for $\mathbb{E}_x[N]$. We separate the contributions to $\mathbb{E}_x[N]$ from C^1 and C^2:

$$\mathbb{E}_x[N] = \sum_{y \in C^1} G(x-y) + \sum_{y \in C^2} G(x-y) \; - 1. \tag{3.21}$$

Concerning the first sum in (3.21), since $x \in C^1$ and $|C^1| \leqslant 3$, the difference set $x - C^1$ contains o and at most two other elements. Hence there are $u_1, u_2 \in \mathbb{Z}^d$ such that $x - C^1 \subseteq \{o, u_1, u_2\}$ and which satisfy $u_1 \succeq e_j$ and $u_2 \succeq e_k$ for some $j,k \in [\![d]\!]$. Because Green's function is nonincreasing in $\succeq$ (Lemma 3.5.1) and because $G(e_j) = G(o) - 1$ for any $j \in [\![d]\!]$,

$$\sum_{y \in C^1} G(x-y) \leqslant 3G(o) - 2. \tag{3.22}$$

By (3.18), the second sum in (3.21) is at most $c_1 \mathrm{sep}(C)^{2-d}$ for a number $c_1 = c_1(d)$. Applying this bound and (3.22) to (3.21) gives

$$\mathbb{E}_x[N] \leqslant 3G(o) - 3 + c_1 n \mathrm{sep}(C)^{2-d}.$$

Second, we have a lower bound for the conditional expectation in (3.20):

$$\mathbb{E}_x[N \mid N > 0] = 1 + \mathbb{E}_x\big[\mathbb{E}_{S_{\tau_U}}[N] \;\big|\; \tau_U < \infty\big] \geqslant G(o).$$

Substituting the two preceding bounds into (3.20), we find

$$\mathrm{esc}_U(x) \geqslant 1 - \frac{3G(o) - 3 + c_1 n \mathrm{sep}(C)^{2-d}}{G(o)}.$$

Rearranging and setting $c = c_1/G(o)$ gives (3.19). □

For the next result, we need to assume $d \geqslant 5$ instead of $d \geqslant 3$, so that we can use the bound $G_d(o) \leqslant 1.2$ from Lemma 3.5.3.

Lemma 3.5.5. *Let $d \geqslant 5$ and let U be an n-element configuration in $\mathbb{Z}^d$. There is a real number $a = a(d,n)$ such that, if there is a clustering $C = (C^1, C^2)$ of U with $|C^1| \leqslant 3$ and $\mathrm{sep}(C) \geqslant a$, then*

$$\mathbb{H}_U(x) \geqslant \frac{1}{2n} \text{ for } x \in C^1. \tag{3.23}$$

Additionally, regardless of the separation of C,

$$\mathbb{H}_C(1,x) \geqslant \frac{1}{2n} \text{ for } x \in C^1. \tag{3.24}$$

The proof applies the escape probability lower bound (3.19) from Lemma 3.5.4 and the fact that $\mathrm{esc}_U(x) \leqslant G(o)^{-1}$ to get two-sided bounds on cap_U, hence harmonic measure by (3.13). Because this bound will be in terms of $G(o)$, to obtain (3.23)—which has no d-dependence—we will apply Lemma 3.5.3. Concerning the fact that (3.24) holds regardless of $\mathrm{sep}(C)$, recall that $\mathbb{H}_C(1,x)$ is proportional to the escape probability of C^1, which is agnostic to cluster separation.

Proof of Lemma 3.5.5. We prove (3.23) first. Let U and C satsify the hypotheses. The capacity satisfies $\mathrm{cap}_U \leqslant nG(o)^{-1}$ because U has n elements and because $\mathrm{esc}_U(x) \leqslant G(o)^{-1}$ for $x \in U$. Combining this with the escape probability lower bound from Lemma 3.5.4 to (3.13), we find

$$\mathbb{H}_U(x) = \frac{\mathrm{esc}_U(x)}{\mathrm{cap}_U} \geqslant \frac{3-2G(o)}{n} - c\,\mathrm{sep}(C)^{2-d} \text{ for } x \in C^1, \tag{3.25}$$

for a number $c = c(d)$. By assumption, $d \geqslant 5$, so Lemma 3.5.3 implies $G(o) \leqslant 1.2$. Substituting this bound into (3.25) gives

$$\mathbb{H}_U(x) \geqslant 0.6n^{-1} - c\,\mathrm{sep}(C)^{2-d}. \tag{3.26}$$

When $\mathrm{sep}(C)$ is at least $a = (10cn)^{\frac{1}{d-2}}$ the term $c\,\mathrm{sep}(C)^{2-d}$ is at most $0.1n^{-1}$, leading to the claimed bound (3.23). Note that, for every number a, the cluster C^1 considered in isolation is an a-separated clustering of C^1 with one cluster. Hence the same argument which gave (3.26), but with C^1 in place of U, implies (3.24). □

3.6 Approximation of HAT by intracluster HAT

Throughout this section, we will assume $d \in \mathbb{Z}_{\geqslant 5}$ to satisfy a hypothesis of Lemma 3.5.5, and $n \in \mathbb{Z}_{\geqslant 4}$ to ensure that DOT clusterings have at least two clusters. When we refer to a clustering with no further qualification, we will mean a clustering of an n-element configuration in $\mathbb{Z}^d$.

The main result of this section bounds below the probabilities of certain events under HAT in terms of their probabilities under IHAT. Specifically, this bound will apply to events that consist of sequences of clusterings that grow in separation sufficiently quickly, which will include the event in Proposition 3.2.1. To make "sufficiently quickly" precise, consider an initial separation $a \in \mathbb{R}_{\geqslant 1}$, a number $\delta > 0$, an increasing sequence of times $(t_k)_{k\geqslant 0} \subseteq \mathbb{Z}_{\geqslant 0}$ with $t_0 = 0$, and a sequence of separations $(\rho_k)_{k\geqslant 1} \subset \mathbb{R}_{\geqslant 1}$ which satisfies $\rho_k \geqslant \delta a$ for every $k \geqslant 1$. The bound will apply to events which consist of sequences of DOT clusterings $(E_1, E_2, \dots)$ such that $\mathrm{sep}(E_s) \geqslant \rho_k$ for $t_{k-1} < s \leqslant t_k$, where t_k and ρ_k satisfy

$$\sum_{k=1}^{\infty} t_k \rho_k^{2.1-d} \to 0 \text{ as } a \to \infty. \tag{3.27}$$

This condition will arise naturally in the proof, where the quantity $\exp(\sum_k t_k \rho_k^{2.1-d})$ will bound the multiplicative error that we incur from approximating HAT by IHAT. In our application, ρ_k will be roughly $2^k a$ and t_k will be roughly $(2^k a)^2$ (i.e., the expected number of steps for random walk to travel a distance of $2^k a$) for $k \geqslant 1$. Because $d \geqslant 5$, these choices of t_k and ρ_k will satisfy (3.27).

Proposition 3.6.1 (Main approximation result)**.** *Let $a, b \in \mathbb{R}_{>1}$ and let $\delta \in \mathbb{R}_{>0}$, let E_0 be an (a, b) separated DOT clustering, let $(t_k)_{k\geqslant 0} \subseteq \mathbb{Z}_{\geqslant 0}$ be an increasing sequence with $t_0 = 0$, and let $(\rho_k)_{k\geqslant 1} \subset \mathbb{R}_{\geqslant 1}$ satisfy $\rho_k \geqslant \delta a$ for every $k \geqslant 1$. Additionally, let $\ell \in \mathbb{Z}_{\geqslant 1}$ and let $\mathcal{D}$ be the event*

$$\mathcal{D} = \big\{(E_1, \dots, E_{t_\ell}) : \textit{for each } k \in [\![\ell]\!], \textit{for each } s \in \{t_{k-1}+1, \dots, t_k\}, \\ E_s \textit{ is a clustering which satisfies } \mathrm{sep}(E_s) \geqslant \rho_k\big\}. \tag{3.28}$$

If $(t_k)_{k\geqslant 1}$ *and* $(\rho_k)_{k\geqslant 1}$ *satisfy* (3.27)*, and if* $\mathcal{E} \subseteq \mathcal{D}$*, then*

$$\mathbf{P}_{E_0}(\mathcal{E}) \geqslant (1 - o_a(1))\mathbf{Q}_{E_0}(\mathcal{E}). \tag{3.29}$$

Proposition 3.6.1 is one of three inputs to the proof of Proposition 3.2.1. The other two inputs state that (i) the event which Proposition 3.2.1 concerns—that the natural clustering grows in separation as Theorem 3.1.5 predicts—is a subset of $\mathcal{D}$ and that (ii) its probability under $\mathbf{Q}_{E_0}$ is positive. We prove these other inputs in the next two sections. In the remainder of this section, we prove Proposition 3.6.1.

The key to the proof of Proposition 3.6.1 is a comparison of the transition probabilities of HAT and IHAT, which is the one-step analogue of (3.12), with $\mathbf{Q}$ in the place of $\mathbf{P}'$.

Proposition 3.6.2 (Approximation of transition probabilities)**.** *Let* $a, b \in \mathbb{R}_{>1}$*, and let* E_0, E_1 *be* (a, b) *separated DOT clusterings. There is a positive real number* $c = c(b, d, n)$ *such that, if* a *is sufficiently large in terms of* b *and* d*, then*

$$\mathbf{P}_{E_0}(C_1 = E_1) \geqslant \left(1 - ca^{2.1-d}\right)\mathbf{Q}_{E_0}(D_1 = E_1). \tag{3.30}$$

For the sake of concreteness, $a \geqslant e^{\max\{100b, 10d^2\}}$ *suffices.*

Proposition 3.6.1 is a consequence of a short calculation with Proposition 3.6.2.

Proof of Proposition 3.6.1. We express the probability of $\mathcal{E}$ in terms of transition probabilities and then apply Proposition 3.6.2:

$$\begin{aligned}
\mathbf{P}_{E_0}(\mathcal{E}) &= \sum_{(E_1,\dots,E_{t_\ell})\in\mathcal{E}} \prod_{k=1}^{\ell} \prod_{s=t_{k-1}+1}^{t_k} \mathbf{P}_{E_{s-1}}(C_1 = E_s) \\
&\geqslant \sum_{(E_1,\dots,E_{t_\ell})\in\mathcal{E}} \prod_{k=1}^{\ell} \prod_{s=t_{k-1}+1}^{t_k} \left(1 - c\rho_k^{2.1-d}\right)\mathbf{Q}_{E_{s-1}}(D_1 = E_s) \\
&= \mathbf{Q}_{E_0}(\mathcal{E}) \prod_{k=1}^{\ell} \left(1 - c\rho_k^{2.1-d}\right)^{t_k - t_{k-1}},
\end{aligned} \tag{3.31}$$

where $c = c(b, d, n)$ is the number of the same name from Proposition 3.6.2. The first equality holds by the Markov property of C_t. The inequality is due to Proposition 3.6.2, which applies in part because E_{s-1} and E_s are clusterings with separations of at least ρ_k by virtue of $\mathcal{E} \subseteq \mathcal{D}$. For the proposition to be applicable, E_{s-1} and E_s must also be DOT clusterings. We can assume that this is the case w.l.o.g., as sequences in $\mathcal{E}$ which include non-DOT clusterings have zero probability under $\mathbf{P}_{E_0}$, hence they do not contribute to the sum. Lastly, to use Proposition 3.6.2, ρ_k must be sufficiently large in terms of b and d. Since $\rho_k \geqslant \delta a$, we can arrange this by assuming that a is sufficiently large in terms of b, d, and δ. The second equality follows from the Markov property of D_t.

Concerning the product over k in (3.31), when a is sufficiently large to make $c\rho_k^{2.1-d} < 0.5$, we can apply the inequality $\log(1-x) \geqslant -2x$ (valid for $|x| < 0.5$) with $x = c\rho_k^{2.1-d}$ as

$$\sum_{k=1}^{\ell} (t_k - t_{k-1}) \log\left(1 - c\rho_k^{2.1-d}\right) \geqslant -2c \sum_{k=1}^{\ell} t_k \rho_k^{2.1-d} = -o_a(1).$$

The equality holds because $(t_k)_{k\geqslant 1}$ and $(\rho_k)_{k\geqslant 1}$ satisfy (3.27). By exponentiating both sides, we find that the product over k in (3.31) is at least $e^{-o_a(1)} = 1 - o_a(1)$. □

We will split the proof of Proposition 3.6.2 into three parts. Recall that $\mathbf{P}_{E_0}(C_1 = E_1)$ is a sum of $p_{\pi(E_0)}$ over $\mathsf{Pairs}(E_0, E_1)$, and $\mathbf{Q}_{E_0}(D_1 = E_1)$ is a sum of q_{E_0} over $\mathsf{Triples}(E_0, E_1)$. First, we will show that there is a correspondence between the elements of $\mathsf{Pairs}(E_0, E_1)$ and $\mathsf{Triples}(E_0, E_1)$ when E_0 and E_1 are sufficiently separated. Hence to show (3.30) it will suffice to prove the bound with $p_{\pi(E_0)}$ and q_{E_0} in the place of $\mathbf{P}_{E_0}(C_1 = E_1)$ and $\mathbf{Q}_{E_0}(D_1 = E_1)$. Second, for any element of $\mathsf{Pairs}(E_0, E_1)$, we will bound below the activation component of $p_{\pi(E_0)}$ by the activation component of q_{E_0}. Third, we will do the same for the transport components, and combine the bounds to give (3.30).

We state the correspondence between $\mathsf{Pairs}(E_0, E_1)$ and $\mathsf{Triples}(E_0, E_1)$ first.

Proposition 3.6.3 (Correspondence between Pairs and Triples)**.** *Let $a, b \in \mathbb{R}_{>1}$, and let E_0, E_1 be (a, b) separated DOT clusterings of n-element configurations in $\mathbb{Z}^d$. If $a \geqslant e^{\max\{100b, 10d^2\}}$, then*

$$\mathsf{Pairs}(E_0, E_1) = \left\{(x, y) : \left(\operatorname{clust}(E_0, x), x, y\right) \in \mathsf{Triples}(E_0, E_1)\right\} \tag{3.32}$$

and

$$\mathsf{Triples}(E_0, E_1) = \left\{\left(\operatorname{clust}(E_0, x), x, y\right) : (x, y) \in \mathsf{Pairs}(E_0, E_1)\right\}. \tag{3.33}$$

Next, we state the comparison of activation components.

Proposition 3.6.4 (Activation comparison)**.** *Let C be a clustering of an n-element configuration in $\mathbb{Z}^d$ which satisfies $|C^j| \leqslant 3$ for each j. There is a number $c = c(d)$ such that*

$$\mathbb{H}_{\pi(C)}(x) \geqslant \left(1 - cn\operatorname{sep}(C)^{2-d}\right)\mathbb{H}_C(i, x), \tag{3.34}$$

with $i = \operatorname{clust}(C, x)$ and for $x \in \mathbb{Z}^d$.

Lastly, we state the comparison of transport components.

Proposition 3.6.5 (Transport comparison)**.** *Let C be a clustering of an n-element configuration in $\mathbb{Z}^d$ and let $b \in \mathbb{R}_{>1}$. Suppose that C satisfies:*

- $|C^j| \leqslant 3$ *for each j;*
- $\operatorname{sep}(C)^{0.1} \geqslant \log(\operatorname{sep}(C))^{d-2}$*; and*
- $\operatorname{diam}(C^j) \leqslant b \log \operatorname{dist}(C^j, C^{\neq j})$ *for each j.*

Then there is a number $c = c(d)$ *such that, for any* $x \in C^i$ *and* $y \in \partial(C^i \setminus \{x\})$ *such that* $p_{\pi(C)}(x,y) > 0$*, we have*

$$\mathbb{P}_x(S_{\tau-1} = y \,|\, \tau < \infty) \geqslant \big(1 - cnb^{d-2}\mathrm{sep}(C)^{2.1-d}\big)\mathbb{P}_x(S_{\tau'-1} = y \,|\, \tau' < \infty), \tag{3.35}$$

where $\tau = \tau_{\pi(C)\setminus\{x\}}$ *and* $\tau' = \tau_{C^i\setminus\{x\}}$.

Concerning the hypotheses on C, the exponent of 0.1 in the second condition is merely representative of a number less than one. For concreteness, C satisfies this condition if $\mathrm{sep}(C) \geqslant e^{10d^2}$. Note that the conditions on x and y mean that intracluster transport can occur from x to y.

Let us prove Proposition 3.6.2, assuming the preceding three propositions.

Proof of Proposition 3.6.2. Let a, b, E_0, and E_1 satisfy the hypotheses of Proposition 3.6.3. Denote

$$\mathsf{Pairs}'(E_0, E_1) = \big\{(x,y) : \big(\mathrm{clust}(E_0, x), x, y\big) \in \mathsf{Triples}(E_0, E_1)\big\}.$$

By the definition of HAT and by Proposition 3.6.3, we have

$$\mathbf{P}_{E_0}(C_1 = E_1) = \sum_{(x,y)\in\mathsf{Pairs}'(E_0,E_1)} \mathbb{H}_{\pi(E_0)}(x)\,\mathbb{P}_x\big(S_{\tau-1} = y \;\big|\; \tau < \infty\big). \tag{3.36}$$

We will bound below each summand. Let $i = \mathrm{clust}(E_0, x)$, $\tau = \tau_{\pi(E_0)\setminus\{x\}}$ and $\tau' = \tau_{E_0^i\setminus\{x\}}$. We claim that

$$\begin{aligned}\mathbb{H}_{\pi(E_0)}(x)\,\mathbb{P}_x\big(S_{\tau-1} = y \;\big|\; \tau < \infty\big)&\\ \geqslant \big(1 - cnb^{d-2}a^{2.1-d}\big)&\mathbb{H}_{E_0}(i,x)\,\mathbb{P}_x\big(S_{\tau'-1} = y \;\big|\; \tau' < \infty\big),\end{aligned} \tag{3.37}$$

for a real number $c = c(d)$. Indeed, this follows from applying Proposition 3.6.4 and Proposition 3.6.5 with $C = E_0$, and the fact that E_0 is an (a,b) separated DOT clustering. Note that the propositions together require C to satisfy three conditions, but it is easy to see that these are met because E_0 is an (a,b) separated DOT clustering, for a which satisfies $a^{0.1} \geqslant \log(a)^{d-2}$ because $a \geqslant e^{10d^2}$. The use of Proposition 3.6.5 further requires $p_{\pi(E_0)}(x,y) > 0$ and $y \in \partial(E_0^i\setminus\{x\})$. The former holds because $(x,y) \in \mathsf{Pairs}(E_0, E_1)$. The latter must hold because, otherwise, y would belong to ∂E_0^j for a cluster $j \neq i$, in which case we would have $\mathrm{sep}(E_1) = 1$.

Returning to (3.36), and identifying the factor of $q_{E_0}(i,x,y)$ on the right hand side of (3.37), we find

$$\begin{aligned}\mathbf{P}_{E_0}(C_1 = E_1) &\geqslant \big(1 - cnb^{d-2}a^{2.1-d}\big) \sum_{(x,y)\in\mathsf{Pairs}'(E_0,E_1)} q_{E_0}(i,x,y)\\ &= \big(1 - cnb^{d-2}a^{2.1-d}\big) \sum_{(i,x,y)\in\mathsf{Triples}(E_0,E_1)} q_{E_0}(i,x,y)\\ &= \big(1 - cnb^{d-2}a^{2.1-d}\big)\mathbf{Q}_{E_0}(D_1 = E_1).\end{aligned}$$

The first equality holds by Proposition 3.6.3; the second holds by the definition of IHAT. $\square$

In the following two subsections, we prove Propositions 3.6.3–3.6.5.

3.6.1 Proof of Proposition 3.6.3

There are two potential barriers to a correspondence between $\mathsf{Pairs}(E_0, E_1)$ and $\mathsf{Triples}(E_0, E_1)$. The first is that one or more clusters could "surround" another cluster, disconnecting it from infinity. In this case, there would be an element $(i, x, y) \in \mathsf{Triples}(E_0, E_1)$ for which $(x, y) \notin \mathsf{Pairs}(E_0, E_1)$. The second barrier is the possibility that an element could move between clusters under HAT, whereas it could not under IHAT. In this case, there would be an element $(x, y) \in \mathsf{Pairs}(E_0, E_1)$, but no (i, x, y) in Triples. We resolve both of these barriers by requiring that E_0 and E_1 be sufficiently separated.

The following proposition handles a step in the proof of Proposition 3.6.3. In words, it states that, if a clustering C is sufficiently separated and if an element is exposed in C^i, then that element is also exposed in $\pi(C)$.

Proposition 3.6.6. *Let $a, b \in \mathbb{R}_{>1}$, and let C be an (a, b) separated DOT clustering of an n-element configuration in $\mathbb{Z}^d$. If $a \geqslant e^{\max\{100b, 10d^2\}}$, then*

$$\mathrm{esc}_{C^i}(x) > 0 \implies \mathrm{esc}_{\pi(C)}(x) > 0, \quad x \in \mathbb{Z}^d.$$

We will prove Proposition 3.6.6 at the end of this section. For now, we assume it and use it to prove Proposition 3.6.3.

Proof of Proposition 3.6.3. Let a, b, E_0, and E_1 satisfy the hypotheses. By the definitions of $\mathsf{Pairs}(E_0, E_1)$ and $\mathsf{Triples}(E_0, E_1)$, to establish (3.32) and (3.33), it suffices to show that

$$p_{\pi(E_0)}(x, y) > 0 \iff q_{E_0}(i, x, y) > 0 \tag{3.38}$$

for any $x, y \in \mathbb{Z}^d$ such that $E_1 = E_0^{i,x,y}$, where $i = \mathrm{clust}(E_0, x)$.

It is easy to see that $p_{\pi(E_0)}(x, y)$ is positive iff x is exposed in $\pi(E_0)$, y is exposed in $\pi(E_1)$, and y has a neighbor in $\pi(E_0)\setminus\{x\}$. In other words,

$$p_{\pi(E_0)}(x, y) > 0 \iff \begin{cases} \mathrm{esc}_{\pi(E_0)}(x) > 0, & \text{(3.39a)} \\ \mathrm{esc}_{\pi(E_1)}(y) > 0, \text{ and} & \text{(3.39b)} \\ y \in \partial\big(\pi(E_0)\setminus\{x\}\big). & \text{(3.39c)} \end{cases}$$

Analogously,

$$q_{E_0}(i, x, y) > 0 \iff \begin{cases} \mathrm{esc}_{E_0^i}(x) > 0, & \text{(3.40a)} \\ \mathrm{esc}_{E_1^i}(y) > 0, \text{ and} & \text{(3.40b)} \\ y \in \partial\big(E_0^i\setminus\{x\}\big). & \text{(3.40c)} \end{cases}$$

We claim that, if $a \geqslant e^{\max\{100b, 10d^2\}}$, then

$$\text{(3.39a)} \iff \text{(3.40a)}, \ \text{(3.39b)} \iff \text{(3.40b)}, \ \text{and (3.39c)} \iff \text{(3.40c)},$$

which together imply (3.38). We address the forward implications first.

Forward implications. Because $x \in E_0^i \subseteq \pi(E_0)$ and $y \in E_1^i \subseteq \pi(E_1)$,

$$\mathrm{esc}_{\pi(E_0)}(x) \leqslant \mathrm{esc}_{E_0^i}(x) \text{ and } \mathrm{esc}_{\pi(E_1)}(y) \leqslant \mathrm{esc}_{E_1^i}(y),$$

hence (3.39a) $\implies$ (3.40a) and (3.39b) $\implies$ (3.40b). Next, by the definition of π and because $\mathrm{sep}(E_0) > 1$, we have

$$\partial\big(\pi(E_0)\backslash\{x\}\big) = \partial\Big(\bigcup_i E_0^i\backslash\{x\}\Big) = \bigcup_i \partial\big(E_0^i\backslash\{x\}\big).$$

Consequently, (3.39c) implies that $y \in \partial\big(E_0^j\backslash\{x\}\big)$ for at least one choice of j. In fact, $j = i$ is the only choice which works as, otherwise, we would have $\mathrm{sep}(E_1) = 1$. We conclude that (3.39c) $\implies$ (3.40c).

Reverse implications. Because E_0 and E_1 are (a, b) separated for an a which is at least $e^{\max\{100b, 10d^2\}}$, Proposition 3.6.6 applies with E_0 or E_1 in the place of C. Using it, we conclude that (3.39a) $\impliedby$ (3.40a) and (3.39b) $\impliedby$ (3.40b). The argument we used for the last forward implication applies in reverse to show that (3.39c) $\impliedby$ (3.40c). □

In the remainder of this section, we prove Proposition 3.6.6 using a result of Kesten, which we stated as Lemma 2.3.13 in Chapter 2.

Proof of Proposition 3.6.6. Let $x \in \mathbb{Z}^d$ satisfy $\mathrm{esc}_{C^i}(x) > 0$ and let $U = \pi(C)$. To establish $\mathrm{esc}_U(x) > 0$, it suffices to show that there is a path from x to the boundary of

$$F = \{z \in \mathbb{Z}^d : \mathrm{dist}(z, U) \leqslant 2\,\mathrm{diam}(U)\},$$

which otherwise lies outside of U.

Because x is exposed in C^i, there is a path Γ from x to ∂F, which otherwise lies outside of C^i. We will modify Γ to obtain the path we desire. To this end, let B^j denote the fattening of the j^{th} cluster by its diameter, i.e., let

$$B^j = \big\{z \in \mathbb{Z}^d : \mathrm{dist}(z, C^j) \leqslant \mathrm{diam}(C^j)\big\}$$

for each j. We state two facts about the B^j.

Fact 1. Each B^j is finite and $*$-connected, so each $\partial_{\mathrm{vis}} B^j$ must be connected according to Lemma 2.3.13.

Fact 2. Each $\partial_{\mathrm{vis}} B^j$ is disjoint from $\cup_k B^k$. To see why, note that any element of $\partial_{\mathrm{vis}} B^j$ is within $\sqrt{d}$ of an element of B^j, while the distance between distinct B^j and B^k exceeds $\sqrt{d}$:

$$\begin{aligned}\mathrm{dist}(B^j, B^k) &\geqslant \mathrm{dist}(C^j, C^k) - \mathrm{diam}(C^j) - \mathrm{diam}(C^k) - 2\sqrt{d}\\ &\geqslant \mathrm{dist}(C^j, C^k)\Big(1 - \frac{2b\log\mathrm{dist}(C^j, C^k) + 2\sqrt{d}}{\mathrm{dist}(C^j, C^k)}\Big) > \sqrt{d}.\end{aligned}$$

The first inequality follows from the triangle inequality; the second from the fact that C satisfies DOT.3; the third from the fact that the ratio is decreasing in $\mathrm{dist}(C^j, C^k)$, which is at least $\mathrm{sep}(C)$, and the fact that $\mathrm{sep}(C) \geqslant e^{10d^2}$.

We will keep the part of Γ from x until it first encounters $\partial_{\text{vis}} B^i$, which otherwise avoids $\cup_k B^k$ by assumption. We denote by $\mathcal{L}$ the set of labels of the B^j subsequently hit by Γ. If $\mathcal{L}$ is empty, then we are done. Otherwise, let ℓ be the label of the first of the B^j that Γ hits, and let Γ_u and Γ_v be the first and last elements of Γ which intersect $\partial_{\text{vis}} B^\ell$. By Fact 1, $\partial_{\text{vis}} B^\ell$ is connected, so there is a shortest path Λ in $\partial_{\text{vis}} B^\ell$ from Γ_u to Γ_v. We then edit Γ to form Γ' as

$$\Gamma' = \big(\Gamma_1, \dots, \Gamma_{u-1}, \Lambda_1, \dots, \Lambda_{|\Lambda|}, \Gamma_{v+1}, \dots, \Gamma_{|\Gamma|}\big).$$

Because Γ_{v+1} was the last element of Γ which intersected $\partial_{\text{vis}} B^\ell$, Γ' avoids B^ℓ. By Fact 2, Λ avoids $\cup_k B^k$, so if $\mathcal{L}'$ is the set of labels of B^j encountered by Γ', then $|\mathcal{L}'| \leqslant |\mathcal{L}| - 1$.

If $\mathcal{L}'$ is empty, then we are done. Otherwise, we can relabel Γ to Γ' and $\mathcal{L}$ to $\mathcal{L}'$ in the preceding argument to continue inductively, obtaining Γ'' and $|\mathcal{L}''| \leqslant |\mathcal{L}| - 2$, and so on. Because $|\mathcal{L}| \leqslant n$, we need to modify the path at most n times before the resulting path to ∂F does not return to $\cup_k B^k$ after reaching $\partial_{\text{vis}} B^i$. In summary, we edited Γ to obtain a path from x to $\partial_{\text{vis}} B^i$ and then from $\partial_{\text{vis}} B^i$ to ∂F, which otherwise avoids U. By the preceding discussion, this proves $\text{esc}_U(x) > 0$. □

3.6.2 Proofs of Propositions 3.6.4 and 3.6.5

We will prove the comparison of the activation components of HAT and IHAT first.

Proof of Proposition 3.6.4. Let $U = \pi(C)$. We express the escape probability of U as

$$\text{esc}_U(x) = \text{esc}_{C^i}(x) - \mathbb{P}_x(\tau_{C^{\neq i}} < \infty, \tau_{C^i} = \infty), \quad x \in U. \tag{3.41}$$

By (3.18), there is a number $c_1 = c_1(d)$ such that, for any $y \in C^{\neq i}$,

$$\mathbb{P}_x(\tau_y < \infty) = \frac{G(x-y)}{G(o)} \leqslant c\text{sep}(C)^{2-d}.$$

By a union bound over the elements of $C^{\neq i}$ and by the preceding bound,

$$\mathbb{P}_x(\tau_{C^{\neq i}} < \infty, \tau_{C^i} = \infty) \leqslant n \max_{y \in C^{\neq i}} \mathbb{P}_x(\tau_y < \infty) \leqslant c_1 n\text{sep}(C)^{2-d}. \tag{3.42}$$

The right-hand side of (3.42) is small relative to the first term in (3.41) because $|C^i| \leqslant 3$ which, by Lemma 3.5.4, implies

$$\text{esc}_{C^i}(x) \geqslant \frac{3 - 2G(o)}{G(o)} \geqslant \frac{1}{2}. \tag{3.43}$$

The second inequality holds because the ratio decreases as $G(o)$ increases and $G(o) \leqslant 1.2$ by Lemma 3.5.3.

Substituting (3.42) and (3.43) into (3.41), we find

$$\text{esc}_{C^i}(x)\big(1 - cn\text{sep}(C)^{2-d}\big) \leqslant \text{esc}_U(x) \leqslant \text{esc}_{C^i}(x) \tag{3.44}$$

with $c = 2c_1$. In particular, summing the first inequality of (3.44) over x and i gives

$$\mathrm{cap}_C\big(1 - cn\mathrm{sep}(C)^{2-d}\big) \leqslant \mathrm{cap}_U. \tag{3.45}$$

Combining (3.44) and (3.45) gives (3.34):

$$\mathbb{H}_U(x) \geqslant \big(1 - cn\mathrm{sep}(C)^{2-d}\big)\frac{\mathrm{esc}_{C^i}(x)}{\mathrm{cap}_C} = \big(1 - cn\mathrm{sep}(C)^{2-d}\big)\mathbb{H}_C(i,x). \tag{3.46}$$

□

In addition to cluster sizes of at most three, the transport comparison requires that the clusters satisfy absolute and relative (to diameter) separation conditions, similar to DOT.1–DOT.3. The proof of (3.35) analyzes each factor of (3.10) in turn.

Proof of Proposition 3.6.5. According to (3.10),

$$\begin{aligned}\mathbb{P}_x(S_{\tau-1} = y \,|\, \tau < \infty) = \mathbb{P}_x&(S_{\tau'-1} = y \,|\, \tau' < \infty) \\ &\times \underbrace{\frac{\mathbb{P}_x(\tau' < \infty)}{\mathbb{P}_x(\tau < \infty)}}_{(3.47a)} \underbrace{\left(1 - \frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty)}{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty)}\right)}_{(3.47b)},\end{aligned} \tag{3.47}$$

where $\tau'' = \tau_{C^{\neq i}}$.

We will bound (3.47a) from below. We start with two observations. First, there must be an element $z \in C^i$ which is distinct from x because, otherwise, there can be no $y \in \partial(C^i \backslash \{x\})$. Second, for such a z, $\|x - z\|$ can be no larger than $b \log \mathrm{dist}(C^i, C^{\neq i})$ by the hypothesis on the diameter of C^i. Consequently, by (3.18), there is a positive number $c_1 = c_1(d)$ such that

$$\mathbb{P}_x(\tau' < \infty) \geqslant \mathbb{P}_x(\tau_z < \infty) = \frac{G(x-z)}{G(o)} \geqslant c_1(b \log \mathrm{dist}(C^i, C^{\neq i}))^{2-d}. \tag{3.48}$$

Concerning the denominator of (3.47a), because there are fewer than n elements in $C^{\neq i}$, a union bound over the elements of $C^{\neq i}$ and (3.18) imply that there is a number $c_2 = c_2(d)$ such that

$$\mathbb{P}_x(\tau'' < \infty) \leqslant c_2 n \mathrm{dist}(C^i, C^{\neq i})^{2-d}.$$

We use this fact with (3.48) in the following way:

$$\begin{aligned}\mathbb{P}_x(\tau < \infty) &\leqslant \left(1 + \frac{\mathbb{P}_x(\tau'' < \infty)}{\mathbb{P}_x(\tau' < \infty)}\right)\mathbb{P}_x(\tau' < \infty) \\ &\leqslant \left(1 + \frac{c_2 n \mathrm{dist}(C^i, C^{\neq i})^{2-d}}{c_1(b \log \mathrm{dist}(C^i, C^{\neq i}))^{2-d}}\right)\mathbb{P}_x(\tau' < \infty).\end{aligned} \tag{3.49}$$

This bound decreases as $\mathrm{dist}(C^i, C^{\neq i})$ increases, so (3.49) and the hypothesis on $\mathrm{sep}(C)$ imply

$$\mathbb{P}_x(\tau < \infty) \leqslant \big(1 + c_3 n b^{d-2} \mathrm{sep}(C)^{2.1-d}\big)\,\mathbb{P}_x(\tau' < \infty) \tag{3.50}$$

for $c_3 = c_2/c_1$.

The inequality (3.50) implies that (3.47a) satisfies

$$\frac{\mathbb{P}_x(\tau' < \infty)}{\mathbb{P}_x(\tau < \infty)} \geqslant 1 - c_3 n b^{d-2} \mathrm{sep}(C)^{2.1-d}. \tag{3.51}$$

We will now bound (3.47b) from below. For $\{\tau'' < \tau' < \infty\}$ to occur, a random walk from x must escape cluster i to a distance of at least $\mathrm{dist}(C^i, C^{\neq i})$, before returning to cluster i. Consequently, there is a positive number $c_4 = c_4(d)$ such that

$$\mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty) \leqslant c_4 n \mathrm{dist}(C^i, C^{\neq i})^{4-2d}. \tag{3.52}$$

Indeed, to obtain (3.52), we can write

$$\begin{aligned} \mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty) &= \mathbb{E}_x\left[\mathbb{P}_{S_{\tau''}}\left(S_{\tau'-1} = y\right); \tau'' < \tau'\right] \\ &\leqslant \max_{z \in C^{\neq i}} \mathbb{P}_z(\tau' < \infty)\, \mathbb{P}_x(\tau'' < \infty) \\ &\leqslant c_4 n \mathrm{dist}(C^i, C^{\neq i})^{4-2d}. \end{aligned}$$

Concerning the denominator of the ratio in (3.47b), because cluster i has at most three elements, the probability that $\{S_{\tau'-1} = y\}$ occurs is at least within a factor c_5 of hitting z (3.48):

$$\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty) \geqslant c_5 (b \log \mathrm{dist}(C^i, C^{\neq i}))^{2-d}. \tag{3.53}$$

Combining (3.52) and (3.53), we find that the ratio in (3.47b) satisfies

$$\frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty)}{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty)} \leqslant \frac{c_4 n \mathrm{dist}(C^i, C^{\neq i})^{4-2d}}{c_5 (b \log \mathrm{dist}(C^i, C^{\neq i}))^{2-d}}.$$

This bound increases as $\mathrm{dist}(C^i, C^{\neq i})$ decreases, so the hypothesis on $\mathrm{sep}(C)$ implies

$$\left(1 - \frac{\mathbb{P}_x(S_{\tau'-1} = y, \tau'' < \tau' < \infty)}{\mathbb{P}_x(S_{\tau'-1} = y, \tau' < \infty)}\right) \geqslant 1 - c_6 n b^{d-2} \mathrm{sep}(C)^{4.1-2d} \tag{3.54}$$

for $c_6 = c_4/c_5$.

Substituting the lower bounds (3.51) and (3.54) into (3.47), we find

$$\mathbb{P}_x(S_{\tau-1} = y \,|\, \tau < \infty) \geqslant \left(1 - c_7 n b^{d-2} \mathrm{sep}(C)^{2.1-d}\right) \mathbb{P}_x(S_{\tau'-1} = y \,|\, \tau' < \infty)$$

with $c_7 = 2\max\{c_3, c_6\}$. □

3.7 A random walk related to cluster separation

Proposition 3.6.1 will allow us to bound the probability in (3.5) of Proposition 3.2.1 with the corresponding probability under IHAT. The purpose of this section and Section 3.8 is to bound this IHAT probability away from zero. Our strategy is to argue that, for each pair of clusters, the difference in their centers of mass, viewed at certain renewal times, is a random walk in $\mathbb{Z}^d$. The purpose of this section is to define this random walk and prove some preliminary results about it. In Section 3.8, we will apply random walk estimates to complete the proof of Proposition 3.2.1.

3.7.1 Definitions

We will define several quantities associated with the natural clusterings C_t and D_t of U_t and V_t. To avoid repetition, we will use E_t and W_t as placeholders for C_t and U_t and D_t and V_t, and we will use E as a placeholder for $(C_t, t \geqslant 0)$ and $(D_t, t \geqslant 0)$. To ensure that there are at least two clusters in each clustering, we will assume that W_0 has $n \geqslant 4$ elements and that E_0 satisfies DOT.1.

Center of mass

For a finite, nonempty subset $A \subset \mathbb{Z}^d$, we define the (scaled) center of mass of A as

$$M(A) = \frac{6}{|A|} \sum_{x \in A} \|x\|. \tag{3.55}$$

The factor of six in (3.55) ensures that the center of mass is an element of $\mathbb{Z}^d$ when A has two or three elements.

Reference times

In words, the reference times for a pair of clusters are the consecutive times at which both clusters form line segments parallel to e_1, where $e_1 = (1, 0, \dots, 0) \in \mathbb{Z}^d$. Fix clusters $i < j$ in E_0. Their reference times are $\xi_0^{i,j}(E) = 0$ and

$$\xi_m^{i,j}(E) = \inf\{t > \xi_{m-1}^{i,j}(E) : E_t^i \in \mathrm{Ref}, E_t^j \in \mathrm{Ref}\}, \;\; m \in \mathbb{Z}_{\geqslant 1},$$

where Ref is the collection of *reference dimers and trimers*

$$\mathrm{Ref} = \{\{x, x + e_1\} : x \in \mathbb{Z}^d\} \cup \{\{x, x + e_1, x + 2e_1\} : x \in \mathbb{Z}^d\}.$$

A random walk

To make the notation easier to read, we will suppress the i, j superscript and the argument E, writing M_t^i instead of $M(E_t^i)$ and ξ_m instead of $\xi_m^{i,j}(E)$.

Consider the increments

$$\Delta_\ell^{i,j} = \left(M_{\xi_\ell}^i - M_{\xi_\ell}^j\right) - \left(M_{\xi_{\ell-1}}^i - M_{\xi_{\ell-1}}^j\right), \;\; \ell \geqslant 1,$$

which are i.i.d. and belong to $\mathbb{Z}^d$. In principle, the reference times can be arbitrarily large, so to ensure that the corresponding partial sums result in a finite-range random walk on $\mathbb{Z}^d$, we truncate them with a number $\kappa > 0$. We define $Z_0^{i,j;\kappa} = M_0^i - M_0^j$ and

$$Z_m^{i,j;\kappa} = Z_0^{i,j;\kappa} + \sum_{\ell=1}^{m} \Delta_\ell^{i,j} \mathbf{1}\left(\xi_\ell - \xi_{\ell-1} \leqslant \kappa\right), \;\; m \in \mathbb{Z}_{\geqslant 1}. \tag{3.56}$$

When $E_t = D_t$, for $x \in \mathbb{Z}^d$, we refer to the law of the process $(Z_m^{i,j;\kappa},\, m \in \mathbb{N})$ under $\mathbf{Q}$, conditioned on $Z_0^{i,j;\kappa} = x$, by $\mathbb{Q}_x^{i,j;\kappa}$.

For any κ, $\mathbb{Q}_x^{i,j;\kappa}$ is the law of a finite-range and symmetric random walk from x. Additionally, if $\kappa \geqslant 3$, then $\mathbb{Q}_x^{i,j;\kappa}$ is irreducible on $\mathbb{Z}^d$. This is because, for any pair of dimers or trimers, it is possible to return to Ref in three steps, in such a way that $\Delta_1^{i,j} = e_k$, for any $k \in [\![d]\!]$, where e_k denotes the k^{th} standard unit vector in $\mathbb{Z}^d$. Aside from these considerations, the truncation of reference times by κ is unimportant, because the distributions of reference times have exponentially small tails, as the next subsection shows.

3.7.2 Reference times have exponential tails under $\mathbf{Q}$

The distributions of reference times have exponentially small tails under $\mathbf{Q}$. This result requires no hypothesis about cluster separation. The key input to the proof of Lemma 3.7.1 is the lower bound on harmonic measure from Lemma 3.5.5.

Lemma 3.7.1. *Let D_0 be a clustering of an n-element configuration in $\mathbb{Z}^d$ with distinct clusters i and j. There is a number $c = c(d,n)$ such that, for all $t \in \mathbb{Z}_{\geqslant 0}$,*

$$\mathbf{Q}_{D_0}\big(\xi_1^{i,j} > t\big) \leqslant 2e^{-ct}. \tag{3.57}$$

Proof. We will prove (3.57) when both clusters are trimers; the resulting bound will hold for the other cases, which can be argued analogously.

We claim that D_{t+8}^i and D_{t+8}^j belong to Ref with a probability of at least p^8, where $p = (2n)^{-1}(2d)^{-4}$. Indeed, we can transition D_t^i to $D_{t+4}^i \in \mathrm{Ref}$ as follows:

1. If there is an isolated element, activate it (w.p. $\geqslant (2n)^{-1}$). Otherwise, "keep" the current cluster by transporting to wherever activation occurs (w.p. $\geqslant (2n \cdot 2d)^{-1}$). Repeat this step twice to ensure that the resulting cluster is connected.

2. Once the cluster is connected, if it does not belong to Ref, activate any element with the least e_1 component and transport it to $x + e_1$, where x is any element of the cluster with the greatest e_1 component (w.p. $\geqslant (2n)^{-1}(2d)^{-4}$). Otherwise, keep the current cluster (w.p. $\geqslant (2n \cdot 2d)^{-1}$). Repeat this step twice to ensure that the trimer belongs to Ref (i.e., equals $\{x, x+e_1, x+2e_1\}$ for some $x \in \mathbb{Z}^d$).

The factors of $(2n)^{-1}$ arise from the use of Lemma 3.5.5, which is justified because D_t^i is a trimer; factors of $(2d)^{-1}$ arise from dictating random walk steps during the transport component of the dynamics.

This process can be repeated for the j^{th} cluster to ensure $D_{t+8}^j \in \mathrm{Ref}$, while maintaining $D_{t+8}^i \in \mathrm{Ref}$. This implies

$$\mathbf{Q}_{D_0}\big(\xi_1^{i,j} > t+8 \bigm| \xi_1^{i,j} > t\big) \leqslant q$$

where $q = 1 - p^8$. Continuing inductively, we find

$$\mathbf{Q}_{D_0}\big(\xi_1^{i,j} > t\big) \leqslant q^{\lfloor \frac{t}{8} \rfloor} \leqslant q^{-1} e^{-ct}$$

for $c = -\frac{1}{8}\log q$. Since $q^{-1} \leqslant 2$, this implies (3.57). □

We mention that Lemma 3.7.1 will allow us to essentially ignore the truncation κ when we apply the following results. Proposition 2.4.5 of [LL10] gives a large deviations estimate for the number of steps it takes a random walk to exit a ball. We define the first hitting time of subset $A \subseteq \mathbb{Z}^d$ by

$$T_A^{i,j;\kappa} = \inf\{m \in \mathbb{Z}_{\geqslant 0} : Z_m^{i,j;\kappa} \in A\}.$$

Denote by $B(r)$ the ball of radius $r \in \mathbb{R}_{\geqslant 0}$ centered at the origin, i.e., $B(r) = \{z \in \mathbb{Z}^d : \|z\| < r\}$. There are numbers $\gamma_1 < \infty$ and $\gamma_2 > 0$ such that for any $\kappa \in [3, \infty)$, $r > 0$, and all $x \in \mathbb{Z}^d$ and $\alpha > 0$,

$$\mathbb{Q}_x^{i,j;\kappa}(T_{B(r)^{\mathsf{c}}}^{i,j;\kappa} > \alpha r^2) \leqslant \gamma_1 e^{-\gamma_2 \alpha}. \tag{3.58}$$

Additionally, Proposition 6.4.2 of [LL10] states that, if a number r is sufficiently large then, for any $\kappa \in [3, \infty)$ and $x \in B(r)^{\mathsf{c}}$, there is a constant $\gamma_3 \geqslant 1$ such that

$$\mathbb{Q}_x^{i,j;\kappa}\big(T_{B(r)}^{i,j;\kappa} < \infty\big) \leqslant \gamma_3 \Big(\frac{r}{\|x\|}\Big)^{d-2}. \tag{3.59}$$

Bounds (3.58) and (3.59) feature constants which could depend on κ through the increment distribution of the truncated random walk. In fact, due to Lemma 3.7.1, the coordinate variances of the increment $Z_1^{i,j;\kappa} - Z_0^{i,j;\kappa}$ are bounded above and below by finite, positive constants for any $\kappa \in [3, \infty)$. Consequently, we can (and do) assume w.l.o.g. that γ_1 through γ_3 do not depend on κ.

3.7.3 Results relating the separation of two clusters to the distance between their centers of mass

In Section 3.8, we will need to translate conditions involving the distance between two clusters to conditions involving the random walk Z_m^κ, and vice versa. In this subsection, we collect some simple facts which serve this purpose.

If the diameters of clusters i and j are small, then the triangle inequality implies that E_t^i and E_t^j are separated by a distance proportional to $\|M_t^i - M_t^j\|$.

Lemma 3.7.2. *If* $\max\{\mathrm{diam}(E_t^i), \mathrm{diam}(E_t^j)\} \leqslant r$ *and* $\|M_t^i - M_t^j\| \geqslant R$, *then*

$$\mathrm{dist}(E_t^i, E_t^j) \geqslant \frac{R - 8r}{6}. \tag{3.60}$$

Proof. We will prove (3.60) when cluster i is a dimer and cluster j is a trimer; the other cases are similar. Assign the elements of E_t^i labels $1, \dots, |E_t^i|$, and denote by $E_t^i(\ell)$ the element with label ℓ. Assume w.l.o.g. that $\mathrm{dist}(E_t^i, E_t^j) = \|E_t^i(1) - E_t^j(1)\|$. We write

$$M_t^i - M_t^j = 3E_t^i(1) + 3E_t^i(2) - 2E_t^j(1) - 2E_t^j(2) - 2E_t^j(3).$$

By the triangle inequality and the hypothesis on diameter, the right-hand side is at most $6\,\mathrm{dist}(E_t^i, E_t^j) + 8r$ in absolute value. Combining this observation with the hypothesis that $\|M_t^i - M_t^j\| \geqslant R$ gives (3.60). □

At reference times, clusters are connected and so have small diameters. Diameters increase at most linearly in time under IHAT. During each step the elements can move by a distance no larger than the diameter of their cluster. These two facts imply a quadratic bound on the difference $M_t^i - M_t^j$.

Lemma 3.7.3. *For any time $t \geqslant 0$, if $m \in \mathbb{N}$ is such that $\xi_m \leqslant t < \xi_{m+1}$, then*

$$\left\|\left(M_t^i - M_t^j\right) - \left(M_{\xi_m}^i - M_{\xi_m}^j\right)\right\| \leqslant 6\left(\xi_{m+1} - \xi_m\right)^2. \tag{3.61}$$

Proof. At time $t_0 = \xi_m$, both clusters i and j are connected and so have diameters of at most two. Their diameters can increase by at most one with each step, which implies

$$\max\left\{\operatorname{diam}(E_t^i), \operatorname{diam}(E_t^j)\right\} \leqslant (t - t_0) + 2. \tag{3.62}$$

We also know that no element can move in one step by a distance exceeding the present diameter of its cluster. Because at most one element moves in a given step, it must be that

$$\sum_{k\in\{i,j\}} \sum_{\ell=1}^{|E_t^k|} \left\|E_{t+1}^k(\ell) - E_t^k(\ell)\right\| \leqslant \max\left\{\operatorname{diam}(E_t^i), \operatorname{diam}(E_t^j)\right\}. \tag{3.63}$$

By the definition of M_t^i, the triangle inequality, and then (3.62) and (3.63),

$$\begin{aligned}
\left\|(M_t^i - M_t^j) - (M_{t_0}^i - M_{t_0}^j)\right\| &\leqslant 3 \sum_{k\in\{i,j\}} \sum_{\ell=1}^{|E_t^k|} \left\|E_t^k(\ell) - E_{t_0}^k(\ell)\right\| \\
&\leqslant 3 \sum_{s=t_0}^{t-1} \sum_{k\in\{i,j\}} \sum_{\ell=1}^{|E_s^k|} \left\|E_{s+1}^k(\ell) - E_s^k(\ell)\right\| \\
&\leqslant 3 \sum_{s=t_0}^{t-1} ((s - t_0) + 2) \leqslant 6(t - t_0)^2.
\end{aligned}$$

Because $t < \xi_{m+1}$, this implies (3.61). ☐

Lemma 3.7.4. *If t satsfies $\xi_m \leqslant t < \xi_{m+1}$, then*

$$\operatorname{dist}(E_t^i, E_t^j) \geqslant \frac{1}{6}\left\|M_{\xi_m}^i - M_{\xi_m}^j\right\| - 3\left(\xi_{m+1} - \xi_m\right)^2. \tag{3.64}$$

Proof. Let t satisfy the hypotheses. By Lemma 3.7.2,

$$6 \operatorname{dist}(E_t^i, E_t^j) \geqslant \|M_t^i - M_t^j\| - 8 \max\{\operatorname{diam}(E_t^i), \operatorname{diam}(E_t^j)\}.$$

By Lemma 3.7.3,

$$\|M_t^i - M_t^j\| \geqslant \left\|M_{\xi_m}^i - M_{\xi_m}^j\right\| - 6\left(\xi_{m+1} - \xi_m\right)^2,$$

and, by (3.62),

$$\max\{\operatorname{diam}(E_t^i), \operatorname{diam}(E_t^j)\} \leqslant t - \xi_m + 2 \leqslant \xi_{m+1} - \xi_m + 2.$$

Combining these inequalities and using the fact that $\xi_{m+1} - \xi_m \geqslant 1$, we find

$$\begin{aligned} 6\operatorname{dist}(E_t^i, E_t^j) &\geqslant \big\|M_{\xi_m}^i - M_{\xi_m}^j\big\| - 6\big(\xi_{m+1} - \xi_m\big)^2 - 8\big(\xi_{m+1} - \xi_m + 2\big) \\ &\geqslant \big\|M_{\xi_m}^i - M_{\xi_m}^j\big\| - 16\big(\xi_{m+1} - \xi_m\big)^2, \end{aligned}$$

which implies (3.64). □

The next lemma states that the distance between the centers of mass of two clusters can be bounded below in terms of the distance between two clusters and their diameters. We omit the proof, as it follows easily from the definition of M_t^i and the triangle inequality.

Lemma 3.7.5. *At every time $t \in \mathbb{Z}_{\geqslant 0}$,*

$$\big\|M_t^i - M_t^j\big\| \geqslant 6\operatorname{dist}(E_t^i, E_t^j) - 6\operatorname{diam}(E_t^i) - 6\operatorname{diam}(E_t^j).$$

3.8 Separation of clusters under HAT and intracluster HAT

The purpose of this section is to prove Proposition 3.2.1, which states that there is a positive probability that the natural clustering of HAT satisfies the separation condition (3.2) of Theorem 3.1.5, so long as the initial clustering is an (a, b) separated DOT clustering for sufficiently large numbers a and b. We do so by establishing the same result for IHAT and then invoking Proposition 3.6.1. We establish the result for IHAT by applying the standard estimates (3.58) and (3.59) to the random walk Z_m associated with the pair of clusters, and then translating these results into analogous conclusions about the separation of the clusters, using the results of Section 3.7.3.

3.8.1 Definitions of key quantities and events

In this subsection and those which follow it, we will assume $d \in \mathbb{Z}_{\geqslant 5}$ and $n \in \mathbb{Z}_{\geqslant 4}$. When we refer to a clustering, we will mean a clustering of an n-element configuration in $\mathbb{Z}^d$.

To state the main results of the section, we need to define several events, which formalize the following picture. Starting from a clustering E_0 with separation $a > 1$, we model the distance between two clusters, i and j, with the random walk Z_m. Accordingly, we aim to observe the distance between two clusters, i and j, double to $2a$ over roughly $(2a)^2$ steps of Z_m, without dropping below, say, $2a\delta$ for some $\delta \in (0, 1)$. We then aim to observe the separation double again, over $(4a)^2$ steps of Z_m, without dropping below $4a\delta$, and so on. In fact, we will budget slightly more time to observe the doubling, and δ will become smaller as we observe more doublings. Additionally, we will lessen the truncation of the reference times which define Z_m (3.56) (by increasing κ) as we observe more doublings.

We will use $\ell \in \mathbb{Z}_{\geqslant 1}$ to count the number of doublings. We introduce sequences of positive real numbers $(\delta_\ell)_{\ell \geqslant 0}$ and $(\kappa_\ell)_{\ell \geqslant 1}$, and a sequence of positive integer times $(t_\ell)_{\ell \geqslant 1}$. Each κ_ℓ and t_ℓ will depend on a. For positive real numbers c_1, c_2, and c_3, which will carry the same names throughout this section and which we will choose later, we define

$$\delta_{\ell-1} = \frac{c_1}{n\ell}, \; t_\ell(a) = [c_2 (n\ell)^4 \log(a)(2^\ell a)^2], \text{ and } \kappa_\ell(a) = c_3 \log(n\ell t_\ell(a))$$

for each $\ell \in \mathbb{Z}_{\geqslant 1}$, where $[r]$ denotes the integer part of a real number r. We think of $t_\ell(a)$ as $(2^\ell a)^2$; the other factors are convenient in later calculations.

Let E be a placeholder for $(C_t)_{t \geqslant 0}$ or $(D_t)_{t \geqslant 0}$. The events are defined in terms of the reference times $\xi_m^{i,j}(E)$, the number of reference times by time t,

$$N_t^{i,j}(E) = \sup\{m \in \mathbb{Z}_{\geqslant 1} : \xi_m^{i,j}(E) \leqslant t\},$$

the time

$$\mathcal{S}_\ell^{i,j}(E) = \inf\{s \geqslant t_{\ell-1} : \operatorname{dist}(E_s^i, E_s^j) \geqslant \delta_\ell^{-1} 2^\ell a\},$$

and the random walk $Z_m^{i,j;\kappa_\ell}(E)$. To reduce notational clutter—and because we intend to define the events for both HAT and IHAT—we will suppress the i, j superscripts and the argument E. For example, we will write ξ_1 instead of $\xi_1^{i,j}(E)$ and $Z_m^{\kappa_\ell}$ instead of $Z_m^{i,j;\kappa_\ell}(E)$.

We define four events for each $\ell \in \mathbb{Z}_{\geqslant 1}$, where ℓ counts the number of times the separation between clusters has doubled, starting from a separation of a. For each ℓ, we aim to observe:

1. the reference times after $t_{\ell-1}$ and up to t_ℓ differ by at most κ_ℓ (i.e., the truncation has no effect on the increments of $Z_m^{\kappa_\ell}$ between these times);
2. the clusters become separated by $\delta_\ell^{-1} 2^\ell a$, by time t_ℓ;
3. the separation remains above $\delta_\ell 2^{\ell-1} a$ during $\{t_{\ell-1}, \ldots, t_\ell\}$; and
4. the separation remains above $2^\ell a$ during $\{\mathcal{S}_\ell, \ldots, t_\ell\}$.

More precisely, we define the following events (Figure 3.1):

$$\begin{aligned}
\mathcal{G}_1^{i,j}(\ell) &= \left\{\xi_m - \xi_{m-1} \leqslant \kappa_\ell \text{ for } N_{t_{\ell-1}} < m \leqslant N_{t_\ell}\right\},\\
\mathcal{G}_2^{i,j}(\ell) &= \left\{\mathcal{S}_\ell \leqslant t_\ell\right\},\\
\mathcal{G}_3^{i,j}(\ell) &= \left\{\operatorname{dist}(E_s^i, E_s^j) \geqslant \delta_\ell 2^{\ell-1} a \text{ for } t_{\ell-1} \leqslant s \leqslant t_\ell\right\}, \text{ and}\\
\mathcal{G}_4^{i,j}(\ell) &= \left\{\operatorname{dist}(E_s^i, E_s^j) \geqslant 2^\ell a \text{ for } \mathcal{S}_\ell \leqslant s \leqslant t_\ell\right\}.
\end{aligned}$$

We also define $\mathcal{G}^{i,j}(\ell) = \bigcap_{k=1}^4 \mathcal{G}_k^{i,j}(\ell)$ and $\mathcal{G}(\ell) = \cap_{i<j} \mathcal{G}^{i,j}(\ell)$.

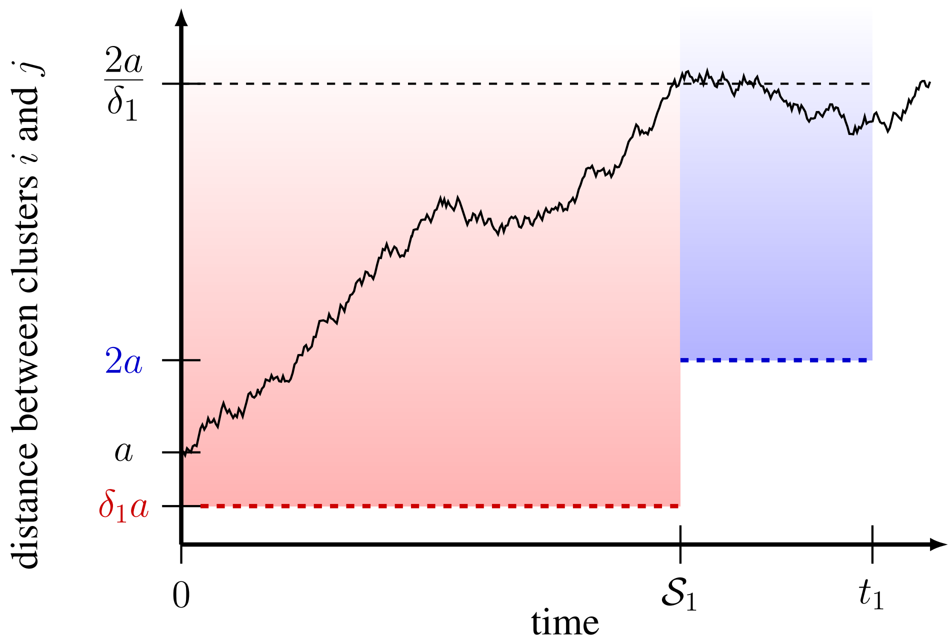

Figure 3.1: An occurrence of $\cap_{k=2}^4 \mathcal{G}_k(1)$.

3.8.2 Proof of Proposition 3.2.1

The event $\cap_{\ell \geqslant 1} \mathcal{G}(\ell)$ is significant because, when it occurs, the sequence of clusterings $(E_0, E_1, \dots)$ satisfies the separation condition which appears in Proposition 3.2.1 and Theorem 3.1.5. This is the content of the first main result of this section.

Proposition 3.8.1 (Separation when $\cap_{\ell \geqslant 1} \mathcal{G}(\ell)$ occurs)**.** *Let $a, b \in \mathbb{R}_{>1}$, let E_0 be an (a,b) separated DOT clustering, and let E_t denote either C_t or D_t for each $t \in \mathbb{Z}_{\geqslant 0}$. If b is sufficiently large in terms of n, and if a is sufficiently large in terms of b, d, and n, then, for any $\ell \in \mathbb{Z}_{\geqslant 1}$, when $\cap_{m=1}^{\ell} \mathcal{G}(m)$ occurs,*

$$\operatorname{diam}(E_s^i) \leqslant b \log \operatorname{dist}(E_s^i, E_s^{\neq i}) \text{ and } \operatorname{sep}(E_s) \geqslant s^{\frac{1}{2} - n^{-100}} + 100n, \tag{3.65}$$

for every cluster i and time $s \in [\![t_\ell]\!]$. In particular, denoting the separation lower bound in (3.65) *by a_s, if ξ is the first time t that E_t is not an (a_t, b) separated DOT clustering, i.e.,*

$$\xi = \inf\{t \in \mathbb{Z}_{\geqslant 0} : E_t \notin \mathscr{C}(\pi(E_t), a_t, b)\},$$

then

$$\cap_{m=1}^{\ell} \mathcal{G}(m) \subseteq \{\xi > t_\ell\}. \tag{3.66}$$

The second main result of this section is a bound on the probability under IHAT that $\mathcal{G}(\ell)$ occurs, for every $\ell \geqslant 1$.

Proposition 3.8.2 ($\cap_{\ell \geqslant 1} \mathcal{G}(\ell)$ is typical for IHAT when E_0 is well separated)**.** *Let $a, b \in \mathbb{R}_{>1}$ and let E_0 be an (a, b) separated DOT clustering. There is a number $\varepsilon > 0$ such that, if a is sufficiently large in terms of b, d, and n, then*

$$\mathbf{Q}_{E_0}\big(\cap_{\ell=1}^{\infty} \mathcal{G}(\ell) \big) > \varepsilon. \tag{3.67}$$

Together, the two preceding propositions and Proposition 3.6.1 imply Proposition 3.2.1.

Proof of Proposition 3.2.1. Let W_0, E_0 satisfy the hypotheses, let a_s, b, c, and ξ be the quantities with the same names in Proposition 3.8.1, and let $\mathcal{D}$ be the event defined in (3.28). Because the lifting of HAT with the law $\mathbf{P}_{E_0}$ has the same distribution as the natural clustering of $(U_0, U_1, \dots)$ with E_0 under $\mathbf{P}_{W_0}$, it suffices to show that there is an $\varepsilon > 0$ such that

$$\mathbf{P}_{E_0}(\xi = \infty) \geqslant \varepsilon.$$

We claim that, for sufficiently large a and b,

$$\mathbf{P}_{E_0}(\xi > t_\ell) \geqslant \mathbf{P}_{E_0}\big(\cap_{m=1}^{\ell} \mathcal{G}(m) \big) \geqslant \mathbf{Q}_{E_0}\big(\cap_{m=1}^{\ell} \mathcal{G}(m) \big)/2.$$

The first inequality holds by (3.66) of Proposition 3.8.1. For $k \geqslant 1$, let ρ_k denote the quantity $\delta_k 2^{k-1} a$, which bounds below $\operatorname{sep}(E_s)$ for $t_{k-1} \leqslant s \leqslant t_k$ when $\mathcal{G}(k)$ occurs. If $(t_k)_{k \geqslant 1}$ and $(\rho_k)_{k \geqslant 1}$ satisfy (3.27) and if $\rho_k \geqslant \delta a$ for some $\delta > 0$, then Proposition 3.6.1 will imply the second inequality because $\cap_{m=1}^{\ell} \mathcal{G}(m) \subseteq \mathcal{D}$. If these two conditions are met, then we can pass to the limit as $\ell \to \infty$ and apply Proposition 3.8.2 to conclude that there is an $\varepsilon > 0$ such that, when a is sufficiently large,

$$\mathbf{P}_{E_0}(\xi = \infty) \geqslant \mathbf{Q}_{E_0}\big(\cap_{m=1}^{\infty} \mathcal{G}(m) \big)/2 \geqslant \varepsilon.$$

It remains to verify the two conditions on $(t_k)_{k \geqslant 1}$ and $(\rho_k)_{k \geqslant 1}$. First, since $(a^2 \log(a))^{-1} t_k = O(2^{2.1k})$ and $a^{-1} \rho_k = \Omega(2^{0.9k})$, and since $d \geqslant 5$, the condition (3.27) is satisfied:

$$\sum_{k=1}^{\infty} t_k \rho_k^{2.1-d} \leqslant O(a^{4.1-d} \log(a)) \underbrace{\sum_{k=1}^{\infty} 2^{(4-0.9d)k}}_{<\infty} \to 0 \text{ as } a \to \infty.$$

Second, the fact that there is a $\delta > 0$ such that $\rho_k \geqslant \delta a$ for $k \geqslant 1$ is implied by $a^{-1} \rho_k = \Omega(2^{0.9k})$. $\square$

We turn our attention to the proofs of Propositions 3.8.1 and 3.8.2.

3.8.3 Proof of Proposition 3.8.1

Recall that the results of Section 3.7.3 relate cluster separation to the distance between their centers of mass and the differences between consecutive reference times. To prove Proposition 3.8.1, we will combine these results with the separation growth that the occurrence of $\cap_{m=1}^{\ell} \mathcal{G}(m)$ entails.

Proof of Proposition 3.8.1. The inclusion (3.66) follows directly from the definition of ξ and (3.65), the two bounds of which we prove in turn.

Fix an $\ell \in \mathbb{Z}_{\geqslant 1}$, a time $t_{\ell-1} < s \leqslant t_\ell$, and a cluster i of E_s. Consider the first bound of (3.65), which states that, if b is sufficiently large in terms of n, then

$$\operatorname{diam}(E_s^i) \leqslant b \log \operatorname{dist}(E_s^i, E_s^{\neq i}).$$

This is implied by the claim that, when $\cap_{m=1}^{\ell} \mathcal{G}(m)$ occurs,

$$\operatorname{diam}(E_s^i) \leqslant 2\kappa_\ell = O_n\big(\log(2^\ell a)\big), \text{ and}$$
$$\log \operatorname{dist}(E_s^i, E_s^{\neq i}) \geqslant \log\big(\delta_\ell 2^{\ell-1} a\big) = \Omega_n\big(\log(2^\ell a)\big).$$

We verify the claim as follows. Reusing (3.62), we see that the diameter of E_s^i is at most $\xi_{N_s+1} - \xi_{N_s} + 2$. When $\cap_{m=1}^{\ell} \mathcal{G}(m)$ occurs, $\xi_{N_s+1} - \xi_{N_s}$ is at most κ_ℓ. Since $\kappa_\ell \geqslant 2$, the diameter of E_s^i is at most $2\kappa_\ell$. Next, note that the occurrence of $\cap_{m=1}^{\ell} \mathcal{G}(m)$ implies $\operatorname{dist}(E_s^i, E_s^{\neq i}) \geqslant \delta_\ell 2^{\ell-1} a$. The equalities involving O_n and Ω_n follow from the definitions of κ_ℓ and δ_ℓ.

Next, consider the second bound of (3.65), which states that

$$\operatorname{sep}(E_s) \geqslant s^{\frac{1}{2} - n^{-100}} + 100n.$$

Because the occurrence of $\cap_{m=1}^{\ell} \mathcal{G}(m)$ implies that $\operatorname{sep}(E_s) \geqslant \delta_\ell 2^{\ell-1} a$, and because we can make this lower bound arbitrarily large relative to $100n$ by increasing a, it suffices to show that

$$\delta_\ell 2^{\ell-1} a \geqslant t_\ell^{\frac{1}{2} - n^{-100}} \tag{3.68}$$

for any ℓ (Figure 3.2).

We split into two cases in terms of $L = L(n)$, a positive integer which is sufficiently large to ensure $L^{-1} \log_2 L \leqslant n^{-100}$.

If $\ell \leqslant L$, then

$$\delta_\ell 2^{\ell-1} a = \Omega_n(a) \quad \text{and} \quad t_\ell^{\frac{1}{2} - n^{-100}} = O_n(a^{1-n^{-100}}).$$

Hence, to satisfy (3.68), we can simply take a sufficiently large in n. On the other hand, if $\ell > L$, then

$$\delta_\ell 2^{\ell-1} a = \Omega_n\big(2^\ell a\big) \quad \text{and} \quad t_\ell^{\frac{1}{2} - n^{-100}} = O_n\big(2^\ell a \cdot \ell^2 (2^\ell a)^{-2n^{-100}}\big).$$

Because $\ell > L$, we have $2^{-2n^{-100}\ell} \ell^2 \leqslant 1$ and, since the first quantity has an extra factor of $a^{2n^{-100}}$, we can take a sufficiently large to satisfy (3.68). ☐

3.8.4 Proof of Proposition 3.8.2

We will devote most of our effort in this subsection to a proof of the next result, from which Proposition 3.8.2 easily follows. To state it, we denote by $\mathscr{F}_t$ the σ-field generated by $D_0, \ldots, D_t$ for $t \geqslant 0$.

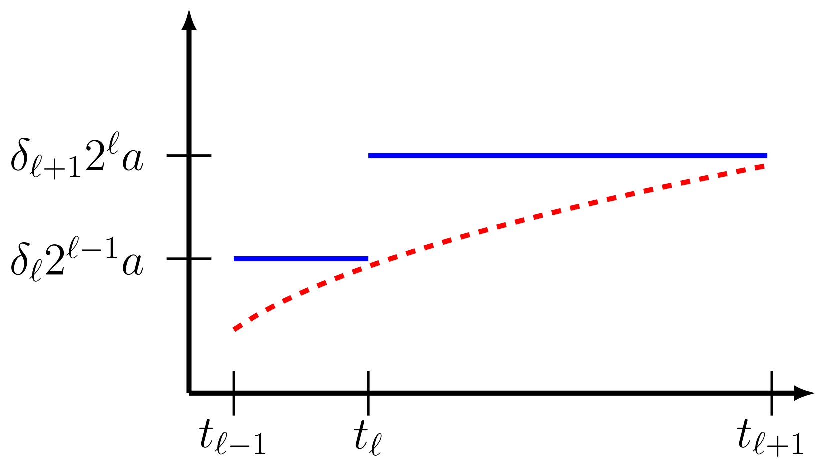

Figure 3.2: When $\cap_{m=1}^{\ell}\mathcal{G}(m)$ occurs, cluster separation lies above the blue lines. The red dashed line is a $t^{1/2-o_n(1)}$ lower bound on cluster separation.

Proposition 3.8.3. *Let $a, b \in \mathbb{R}_{>1}$, let E_0 be an (a, b) separated DOT clustering, and let $\ell \in \mathbb{Z}_{\geqslant 1}$. If a is sufficiently large in terms of b, d, and n, then*

$$\mathbf{Q}_{E_0}\big(\mathcal{G}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big) \leqslant \frac{1}{2\ell^2}. \tag{3.69}$$

Proof of Proposition 3.8.2. Let $b > 1$ and let a be sufficiently large to satisfy the hypotheses of Proposition 3.8.3, let E_0 be an (a, b) separated DOT clustering, and let $\ell \in \mathbb{Z}_{\geqslant 1}$. By conditioning on $\mathscr{F}_{t_{\ell-1}}$ and then applying Proposition 3.8.3, we find

$$\mathbf{Q}_{E_0}\Big(\mathcal{G}(\ell)^{\mathsf{c}} \cap_{m=1}^{\ell-1}\mathcal{G}(m)\Big) = \mathbf{E}_{E_0}\Big[\mathbf{Q}_{E_0}\big(\mathcal{G}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big)\Big] \leqslant \frac{1}{2\ell^2}.$$

Consequently,

$$\mathbf{Q}_{E_0}\Big(\big(\cap_{\ell=1}^{\infty}\mathcal{G}(\ell)\big)^{\mathsf{c}}\Big) = \sum_{\ell=1}^{\infty}\mathbf{Q}_{E_0}\Big(\mathcal{G}(\ell)^{\mathsf{c}} \cap_{m=1}^{\ell-1}\mathcal{G}(m)\Big) \leqslant \sum_{\ell=1}^{\infty}\frac{1}{2\ell^2} < 1.$$

□

To prove Proposition 3.8.3, we will prove (3.69) with a sequence of events $(\mathcal{H}(\ell))_{\ell\geqslant 1}$, which satisfy $\mathcal{H}(\ell) \subseteq \mathcal{G}(\ell)$, in the place of $(\mathcal{G}(\ell))_{\ell\geqslant 1}$. These events will refer to the random walk $Z_m^{\kappa_\ell}$, instead of the distance between the pairs of clusters, so that we can use random walk estimates (3.58) and (3.59) to estimate the probability that they occur.

For each $\ell \in \mathbb{Z}_{\geqslant 1}$, in terms of the time

$$\mathcal{T}_\ell = \inf\big\{m \geqslant N_{t_{\ell-1}} : Z_m^{\kappa_\ell} \notin B\big(6\delta_\ell^{-1}(2^\ell a) + 18\kappa_\ell^2\big)\big\},$$

we define

$$\begin{aligned}
&\mathcal{H}_1^{i,j}(\ell) = \mathcal{G}_1^{i,j}(\ell),\\
&\mathcal{H}_2^{i,j}(\ell) = \big\{\mathcal{T}_\ell \leqslant N_{t_\ell}\big\},\\
&\mathcal{H}_3^{i,j}(\ell) = \big\{Z_m^{\kappa_\ell} \notin B\big(6\delta_\ell(2^{\ell-1}a) + 18\kappa_\ell^2\big),\ N_{t_{\ell-1}} \leqslant m \leqslant N_{t_\ell}\big\},\ \text{and}\\
&\mathcal{H}_4^{i,j}(\ell) = \big\{Z_m^{\kappa_\ell} \notin B\big(6(2^\ell a) + 18\kappa_\ell^2\big),\ \mathcal{T}_\ell \leqslant m \leqslant N_{t_\ell}\big\}.
\end{aligned}$$

We also define $\mathcal{H}^{i,j}(\ell) = \bigcap_{k=1}^4 \mathcal{H}_k^{i,j}(\ell)$ and $\mathcal{H}(\ell) = \cap_{i<j}\mathcal{H}^{i,j}(\ell)$.

Proposition 3.8.4. *For each $\ell \in \mathbb{Z}_{\geqslant 1}$, $\mathcal{H}(\ell) \subseteq \mathcal{G}(\ell)$.*

Proof. Fix an $\ell \in \mathbb{Z}_{\geqslant 1}$ and a time $t_{\ell-1} \leqslant s \leqslant t_\ell$. When $\mathcal{G}(\ell)$ occurs,

$$Z_{N_s}^{\kappa_\ell} = M_{\xi_{N_s}}^i - M_{\xi_{N_s}}^j \quad \text{and} \quad \xi_{N_s+1} - \xi_{N_s} \leqslant \kappa_\ell.$$

Hence, by Lemma 3.7.4,

$$\operatorname{dist}(E_s^i, E_s^j) \geqslant \frac{1}{6}\big\|M_{\xi_{N_s}}^i - M_{\xi_{N_s}}^j\big\| - 3\big(\xi_{N_s+1} - \xi_{N_s}\big)^2 \geqslant \frac{1}{6}\big\|Z_{N_s}^{\kappa_\ell}\big\| - 3\kappa_\ell^2,$$

which implies the inclusions. □

Proposition 3.8.4 and the following four estimates will be the inputs to our proof of Proposition 3.8.3. The first estimate bounds above the probability that $\mathcal{H}_1^{i,j}(\ell)^{\mathsf{c}}$ occurs, i.e., some consecutive reference times between $t_{\ell-1}$ and t_ℓ differ by more than κ_ℓ. Note that we do not need a hypothesis on the separation of E_0 as the event $\mathcal{H}_1^{i,j}(\ell)$ only concerns reference times.

Proposition 3.8.5. *For $\ell \in \mathbb{Z}_{\geqslant 1}$,*

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_1^{i,j}(\ell)^{\mathsf{c}}\big) \leqslant \frac{1}{14(n\ell)^2}.$$

Proof. We calculate

$$\begin{aligned}
\mathbf{Q}_{E_0}\big(\mathcal{H}_1^{i,j}(\ell)^{\mathsf{c}}\big) &= \mathbf{Q}_{E_0}\left(\bigcup_{m=N_{t_{\ell-1}}+1}^{N_{t_\ell}} \big\{\xi_m - \xi_{m-1} > \kappa_\ell\big\}\right)\\
&\leqslant \mathbf{Q}_{E_0}\left(\bigcup_{m=N_{t_{\ell-1}}+1}^{N_{t_{\ell-1}}+t_\ell} \big\{\xi_m - \xi_{m-1} > \kappa_\ell\big\}\right)\\
&= \mathbf{E}_{E_0}\left[\mathbf{Q}_{D_{t_{\ell-1}}}\left(\bigcup_{m=1}^{t_\ell} \big\{\xi_m - \xi_{m-1} > \kappa_\ell\big\}\right)\right]\\
&\leqslant \mathbf{E}_{E_0}\left[\sum_{m=1}^{t_\ell} \mathbf{E}_{D_{t_{\ell-1}}}\Big[\mathbf{Q}_{D_{\xi_{m-1}}}\left(\xi_1 > \kappa_\ell\right)\Big]\right] \leqslant 2t_\ell e^{-c\kappa_\ell}.
\end{aligned}$$

The first equality is due to the definition of $\mathcal{H}_1^{i,j}(\ell)$. The first inequality holds because the number of reference times between $t_{\ell-1}$ and t_ℓ is never more than t_ℓ. The second equality is due to the Markov property applied to time $t_{\ell-1}$ and the fact that $(\xi_m - \xi_{m-1})_{m\in\mathbb{Z}_{\geqslant 1}}$ is an i.i.d. sequence. The second inequality follows from a union bound over m and the strong Markov property applied sequentially at the times $\xi_{m-1},\dots,\xi_{t_\ell-1}$. The third inequality is due to Lemma 3.7.1.

Recall that $t_\ell = [c_2(n\ell)^4\log(a)(2^\ell a)^2]$ and $\kappa_\ell = c_3\log(n\ell t_\ell)$, where c_2 and c_3 are yet unspecified positive numbers. By increasing c_2, we can assume w.l.o.g. that $t_\ell \geqslant 28$ and choose $c_3 = 2c^{-1}$, in which case

$$2t_\ell e^{-c\kappa_\ell} = \frac{2}{t_\ell(n\ell)^2} \leqslant \frac{1}{14(n\ell)^2}.$$

□

Proposition 3.8.6. *Let $\ell \in \mathbb{Z}_{\geqslant 1}$ and $\operatorname{sep}(E_0) \geqslant a$ for $a \in \mathbb{R}_{>1}$. If a is sufficiently large, then*

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_1^{i,j}(\ell)\cap\mathcal{H}_2^{i,j}(\ell)^{\mathsf{c}} \bigm| \mathscr{F}_{t_{\ell-1}}\big) \leqslant \frac{1}{14(n\ell)^2}.$$

Proof. Recall that, for $\mathcal{H}_2^{i,j}(\ell)^{\mathsf{c}}$ to occur, the random walk $Z_m^{\kappa_\ell}$ must fail to exit the ball of radius $r = 6\delta_\ell^{-1}(2^\ell a) + 18\kappa_\ell^2$ between steps $N_{t_{\ell-1}}$ and N_{t_ℓ}. By (3.58), the probability that this occurs is at most $\gamma_1 e^{-\gamma_2\alpha}$ when $N_{t_\ell} - N_{t_{\ell-1}}$ exceeds αr^2 for $\alpha > 0$. The occurrence of $\mathcal{H}_1^{i,j}(\ell)$ implies that

$$N_{t_\ell} - N_{t_{\ell-1}} \geqslant \big[\kappa_\ell^{-1}(t_\ell - t_{\ell-1} - 1)\big].$$

By taking a sufficiently large, we can ensure that the lower bound is at least $(4\kappa_\ell)^{-1}t_\ell$, hence to prove the proposition it suffices to prove that

$$(4\kappa_\ell)^{-1}t_\ell \geqslant \alpha r^2$$

where $\alpha = \gamma_2^{-1}\log\big(\gamma_1\cdot 14(n\ell)^2\big)$. Some algebra shows that $(r^2\kappa_\ell)^{-1}t_\ell = \Omega(\log(n\ell))$, which implies that c_2 can be taken sufficiently large to satisfy the preceding inequality. □

Proposition 3.8.7. *Let $\ell \in \mathbb{Z}_{\geqslant 1}$ and let $\operatorname{sep}(E_0) \geqslant a$ for $a \in \mathbb{R}_{>1}$. If a is sufficiently large, then*

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_3^{i,j}(\ell)^{\mathsf{c}} \bigm| \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big) \leqslant \frac{1}{14(n\ell)^2}.$$

Proof. Denote by $X = Z^{\kappa_{\ell-1}}_{N_{t_{\ell-1}}}$ the location of the random walk $Z_m^{\kappa_{\ell-1}}$ at time $t_{\ell-1}$. Recall that, for $\mathcal{H}_3^{i,j}(\ell)$ to occur, $Z_m^{\kappa_\ell}$ must avoid the ball of radius $r = 6\delta_\ell(2^\ell a) + 18\kappa_\ell^2$ between steps $N_{t_{\ell-1}}$ and N_{t_ℓ}. Clearly, it is enough for the random walk from X to escape $B(r)$. By the Markov property applied at time $t_{\ell-1}$ and by (3.59), this occurs with a probability of at least

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_3^{i,j}(\ell) \bigm| \mathscr{F}_{t_{\ell-1}}\big) = \mathbf{Q}_{D_{t_{\ell-1}}}\big(\mathcal{H}_3^{i,j}(\ell)\big) \geqslant 1 - \gamma_3\Big(\frac{r}{\|X\|}\Big)^{d-2}. \tag{3.70}$$

When $\cap_{m=1}^{\ell-1}\mathcal{G}(m)$ occurs, the separation of $D_{t_{\ell-1}}$ is at least $2^{\ell-1}a$ and the diameters of clusters i and j are at most $2\kappa_{\ell-1}$, which by Lemma 3.7.5 implies that

$$\|X\| = \left\|M^i_{t_{\ell-1}} - M^j_{t_{\ell-1}}\right\| \geqslant 2^{\ell-1}a - 24\kappa_{\ell-1}.$$

Call this lower bound R, in which case the preceding inequality and (3.70) imply

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_3^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big) \leqslant \gamma_3\Big(\frac{r}{R}\Big)^{d-2}.$$

Some algebra shows that $\frac{r}{R} = 12c_1(1+o_a(1))\cdot(n\ell)^{-1}$, so the preceding bound and the fact that $d \geqslant 5$ imply that

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_3^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big) \leqslant \frac{1}{14(n\ell)^2},$$

for an appropriate choice of c_1, for sufficiently large a. □

Proposition 3.8.8. *Let $\ell \in \mathbb{Z}_{\geqslant 1}$ and $\operatorname{sep}(E_0) \geqslant a$ for $a \in \mathbb{R}_{>1}$. If a is sufficiently large, then*

$$\mathbf{Q}_{E_0}\big(\mathcal{H}_2^{i,j}(\ell) \cap \mathcal{H}_4^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) \leqslant \frac{1}{14(n\ell)^2}.$$

Proof. When $\mathcal{H}_2^{i,j}(\ell)$ occurs, there is a number of steps $\mathcal{T}_\ell$ between $N_{t_{\ell-1}}$ and N_{t_ℓ} such that $Y = Z^{\kappa_\ell}_{\mathcal{T}_\ell}$ belongs to $B(R)^{\mathsf{c}}$, where $R = 6\delta_\ell^{-1}(2^\ell a) + 18\kappa_\ell^2$. For $\mathcal{H}_4^{i,j}(\ell)^{\mathsf{c}}$ to occur, $Z^{\kappa_\ell}_m$ must hit $B(r)$, where $r = 6(2^\ell a) + 18\kappa_\ell^2$, starting from Y. Hence by the strong Markov property applied at time $\xi_{\mathcal{T}_\ell}$ and by (3.59),

$$\begin{aligned}\mathbf{Q}_{E_0}\big(\mathcal{H}_2^{i,j}(\ell) \cap \mathcal{H}_4^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) &= \mathbf{E}_{D_{t_{\ell-1}}}\Big[\mathbf{Q}_{D_{\xi_{\mathcal{T}_\ell}}}\big(\mathcal{H}_4^{i,j}(\ell)^{\mathsf{c}}\big); \mathcal{H}_2^{i,j}(\ell)\Big]\\ &\leqslant \mathbf{E}_{D_{t_{\ell-1}}}\Big[\gamma_3\Big(\frac{r}{\|Y\|}\Big)^{d-2}\Big] \leqslant \gamma_3\Big(\frac{r}{R}\Big)^{d-2}.\end{aligned}$$

Some algebra shows that $\frac{r}{R} = c_1(1+o_a(1))\cdot(n\ell)^{-1}$, so the preceding bound is at most $(14(n\ell)^2)^{-1}$ when a is sufficiently large. □

We combine the preceding five propositions to prove Proposition 3.8.3.

Proof of Proposition 3.8.3. Let E_0 satisfy $\operatorname{sep}(E_0) \geqslant a$ for $a > 1$ and let $\ell \in \mathbb{Z}_{\geqslant 1}$. We aim to show that, if a is sufficiently large, then

$$\mathbf{Q}_{E_0}\big(\mathcal{G}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1}\mathcal{G}(m)\big) \leqslant \frac{1}{2\ell^2}. \tag{3.71}$$

By Proposition 3.8.4 and a union bound over distinct pairs of clusters, we have

$$\mathbf{Q}_{E_0}\big(\mathcal{G}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) \leqslant \mathbf{Q}_{E_0}\big(\mathcal{H}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) \leqslant \sum_{i<j}\mathbf{Q}_{E_0}\big(\mathcal{H}^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big). \tag{3.72}$$

Using the fact that, for events H_1 and H_2, H_2^{c} is contained in the disjoint union $(H_1 \cap H_2^{\mathsf{c}}) \cup H_1^{\mathsf{c}}$, we find

$$\mathbf{Q}_{E_0}\big(\mathcal{H}^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) \leqslant 3\mathbf{Q}_{E_0}\big(\mathcal{H}_1^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) + 2\mathbf{Q}_{E_0}\big(\mathcal{H}_1^{i,j}(\ell) \cap \mathcal{H}_2^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) + \mathbf{Q}_{E_0}\big(\mathcal{H}_3^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big) + \mathbf{Q}_{E_0}\big(\mathcal{H}_2^{i,j}(\ell) \cap \mathcal{H}_4^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big).$$

Applying Propositions 3.8.5 through 3.8.8 to bound the terms on the right hand side, we conclude that for sufficiently large a,

$$\mathbf{Q}_{E_0}\big(\mathcal{H}^{i,j}(\ell)^{\mathsf{c}} \mid \mathscr{F}_{t_{\ell-1}}\big)\mathbf{1}\big(\cap_{m=1}^{\ell-1} \mathcal{G}(m)\big) \leqslant \frac{1}{2(n\ell)^2}.$$

The bound (3.71) then follows from (3.72) and the fact that there are at most n^2 distinct pairs of clusters. □

3.9 Strategy for the proof of Theorem 3.1.7

Let us briefly summarize what the preceding sections have accomplished. In Section 3.2, we proved our main result, Theorem 3.1.5, assuming Proposition 3.2.1 and Theorem 3.1.7. In Sections 3.4 through 3.8, we proved Proposition 3.2.1, using an approximation of HAT by IHAT and a random walk model of cluster separation under IHAT. In this section, our focus shifts to proving Theorem 3.1.7.

We continue to assume that $d \in \mathbb{Z}_{\geqslant 5}$ and $n \in \mathbb{Z}_{\geqslant 4}$. Recall that Theorem 3.1.7 identifies a number of steps $f = f(a,d,n)$ and a positive probability $g = g(a,d,n)$ such that, if a is sufficiently large in terms of d and n, then the $\mathbf{P}_U$ probability that $U_f \in \mathscr{U}_{d,n}(a,1)$ is at least g, for any n-element configuration $U \subset \mathbb{Z}^d$.

If we permitted g to depend on U, it would be relatively straightforward to identify for any U a sequence of f' configurations which can be realized by HAT and which produce a configuration $U_{f'}$ belonging to $\mathscr{U}_{d,n}(a,1)$. Indeed, it would take only two "stages":

(1') First, we would rearrange U into a line segment emanating in the $-e_1$ direction from, say, the element of U which is least in the lexicographic ordering of $\mathbb{Z}^d$.

(2') Second, we would "treadmill" a pair of elements from the "tip" of the segment, in the $-e_1$ direction, until the pair was sufficiently far from the other elements. We would repeat this process, one pair at a time, until only two or three elements of the initial segment remained.

While stage (2') could be realized by HAT with at least a probability depending on d and n only, stage (1') could introduce a dependence on U into g. Indeed, it might require that we specify the transport of an activated element over a distance of roughly the diameter of U. We will avoid this by adding one preliminary stage; in the resulting, three-stage procedure, stages (1') and (2') are essentially stages (2) and (3).

To specify the stages, we need two definitions.

Definition 3.9.1 (Lined-up). *We say that a clustering C of a configuration U can be lined-up with separation r if $\mathrm{sep}(C) \geqslant 2r$, if each cluster of C has at least two elements, and if each cluster of C is connected. If such a clustering of U exists, we say that U can be lined-up with separation r.*

Definition 3.9.2 (Lex). *We say that an element x of a finite $A \subset \mathbb{Z}^d$ is lex in A, denoted $\mathrm{lex}(A) = x$, if it is least among the elements of A in the lexicographic order of $\mathbb{Z}^d$.*

Here is the three-stage procedure, which takes as input an initial, n-element configuration $U \subset \mathbb{Z}^d$ and $a \in \mathbb{Z}_{\geqslant 2}$, which we will later require to be sufficiently large in terms of d and n. Note that the output of each algorithm is a clustering—not a configuration.

1. First, we will use Algorithm $\mathcal{A}_2$ to construct a clustering $C = \mathcal{A}_2(U, a)$ of a configuration which can be lined-up with separation dn^2a. This algorithm is the most complicated of the three. In brief, the algorithm repeatedly attempts to create a non-isolated lex element of a cluster, so that it can be "treadmilled"—along with a neighboring element—to form a new dimer cluster.

2. In the second stage, we will apply Algorithm $\mathcal{A}_3$ to "line-up" the elements of each cluster C^i. Specifically, in terms of the line segment
$$L_k = \big\{ -je_1 : j \in \{0, 1, \ldots, k-1\} \big\},$$
we will rearrange the elements of C^i into the set $\mathrm{lex}(C^i) + L_{|C^i|}$. If C has m clusters, the resulting clustering will be
$$\mathcal{A}_3(C) = \big(\mathrm{lex}(C^i) + L_{|C^i|},\ i \in [\![m]\!]\big).$$

3. In the third stage, Algorithm $\mathcal{A}_4$ will iteratively treadmill pairs of elements from each segment in the $-e_1$ direction for multiples of a steps until only a dimer or a trimer of the original segment remains. The resulting clustering $\mathcal{A}(U, a) = \mathcal{A}_4(\mathcal{A}_3(C), a)$ will satisfy DOT.1–DOT.3 with a and $b = 1$, meaning that the configuration associated with the resulting clustering will belong to $\mathscr{U}_{d,n}(a, 1)$.

In the next section, we prove some results which will aid our analysis of the algorithms. In particular, we prove a harmonic measure lower bound for lex elements. After preparing these inputs, in Section 3.11, we will state and analyze the three algorithms to prove Theorem 3.1.7.

3.10 Some inputs to the proof of Theorem 3.1.7

3.10.1 A geometric lemma

To facilitate the use of the harmonic measure estimate in the next subsection, we need a geometric lemma and a consequence thereof. We state the following lemma with more generality than is needed for the immediate application; we will apply it again in a later section. The statement requires the notion of the $*$-visible boundary of a set, which we first defined in (2.35).

Lemma 3.10.1. *Let A be a finite subset of $\mathbb{Z}^d$ which contains the origin, and let x and y be distinct elements of $\partial_{\text{vis}} A$. There is a path Γ from x to y in A^c of length at most $\sqrt{d}\,\text{diam}(A) + 3^{d+1}|A|$. Moreover, $\Gamma \subseteq \{z \in \mathbb{Z}^d : \|z\| \leqslant \text{diam}(A) + \sqrt{d}\}$.*

Proof. Fix finite $A \subset \mathbb{Z}^d$ containing the origin, and fix two elements, x and y, in $\partial_{\text{vis}} A$. Let $\{B_i\}_i$ be the collection of $*$-connected components of A. Because A is finite, each B_i is finite and, as each B_i is also $*$-connected, each $\partial_{\text{vis}} B_i$ is connected in $\mathbb{Z}^d$ by Lemma 2.3.13.

Note that $\partial_{\text{vis}} A$ is contained in $\{z \in \mathbb{Z}^d : \text{dist}(z, A) < 3\}$, so $\text{diam}(\partial_{\text{vis}} A)$ is at most $\text{diam}(A) + 6$. If Γ is a path from x to y of least length, then, by the preceding observation and the Cauchy-Schwarz inequality, the length of Γ is at most $\sqrt{d}\,(\text{diam}(A) + 6)$. We will edit Γ to obtain a potentially longer path which does not intersect A.

If Γ does not intersect A, then we are done. Otherwise, let i_1 denote the label of the first $*$-connected component of A intersected by Γ. Additionally, denote by a and b the first and last indices of Γ which intersect $\partial_{\text{vis}} B_{i_1}$. Because $\partial_{\text{vis}} B_{i_1}$ is connected in $\mathbb{Z}^d$, there is a path Λ in $\partial_{\text{vis}} B_{i_1}$ from Γ_u to Γ_v. We may therefore edit Γ to form Γ':

$$\Gamma' = \big(\Gamma_1, \ldots, \Gamma_{u-1}, \Lambda_1, \ldots, \Lambda_{|\Lambda|}, \Gamma_{v+1}, \ldots, \Gamma_{|\Gamma|}\big).$$

If Γ' does not intersect A, then we are done, as Γ' is contained in the union of Γ and $\cup_i \partial_{\text{vis}} B_i$, and because $\cup_i \partial_{\text{vis}} B_i$ has at most $3^d|A|$ elements. Accordingly,

$$|\Gamma'| \leqslant \sqrt{d}(\text{diam}(A) + 6) + 3^d|A| \leqslant \sqrt{d}\,\text{diam}(A) + 3^{d+1}|A|. \tag{3.73}$$

Otherwise, if Γ' intersects another $*$-connected component B_{i_2} of A, we can argue in an analogous fashion to obtain a path Γ'' which neither intersects B_{i_1} nor B_{i_2}. Like Γ', Γ'' is contained in the union of Γ and $\cup_i \partial_{\text{vis}} B_i$ and so its length satisfies the same upper bound (3.73). Continuing inductively yields a path from x to y with a length of at most the right-hand side of (3.73).

The path is contained in the union of Γ and $\cup_i \partial_{\text{vis}} B_i$, which is contained in $\{z \in \mathbb{Z}^d : \|z\| \leqslant \text{diam}(A) + \sqrt{d}\}$ because A contains the origin by assumption. □

A consequence of this result is a simple comparison of harmonic measure at two points.

Lemma 3.10.2. *Let $d \geqslant 5$. There is a number $c = c(d, n)$ such that, if $A \cup B$ is an n-element subset of $\mathbb{Z}^d$ such that A is connected and $\text{dist}(A, B) \geqslant 4^d n$, then, for any distinct $x, y \in A$ which are exposed in $A \cup B$,*

$$\mathbb{H}_{A \cup B}(x) \geqslant c\mathbb{H}_{A \cup B}(y). \tag{3.74}$$

Proof. Let x, y be elements of A which are exposed in $A \cup B$. If u is any element of $\partial_{\text{vis}} A \cap \partial\{x\}$ and v is any element of $\partial_{\text{vis}} A \cap \partial\{y\}$, then, by Lemma 3.10.1, there is a path Γ_{uv} from u to v in A^c of length at most

$$\sqrt{d}\,\text{diam}(A) + 3^{d+1}|A| \leqslant 1.1 \cdot 3^{d+1} n \leqslant 4^d n.$$

The first inequality is due to the assumption that A is connected, which implies $\text{diam}(A) \leqslant n$, and the fact that $\sqrt{d} \leqslant 0.1 \cdot 3^{d+1}$ when $d \geqslant 5$. The second inequality holds because $d \geqslant 5$.

Because B is a distance of at least $4^d n$ from A, Γ_{uv} must also lie outside of B. This implies that there is a constant $c = c(d, n)$ such that

$$\mathrm{esc}_{A\cup B}(x) \geqslant c\,\mathrm{esc}_{A\cup B}(y).$$

Dividing by the capacity of $A \cup B$ gives (3.74). □

3.10.2 An estimate of harmonic measure for lex elements

We now prove a harmonic measure lower bound for lex elements.

Lemma 3.10.3. *Let $d \geqslant 5$. There are constants $r = r(n, d)$ and $c = c(d)$ such that, if $A \cup B$ is an n-element subset of $\mathbb{Z}^d$ satisfying $\mathrm{dist}(A, B) \geqslant r$ and if x is lex in A, then*

$$\mathrm{esc}_{A\cup B}(x) \geqslant cn^{-\frac{1}{d-2}-o_d(1)} \tag{3.75}$$

and, consequently,

$$\mathbb{H}_{A\cup B}(x) \geqslant cn^{-\frac{d-1}{d-2}-o_d(1)}. \tag{3.76}$$

For concreteness, the $o_d(1)$ quantity is never larger than 0.8 when $d \geqslant 5$, and the lower bound can be replaced with $cn^{-2.2}$.

Proof. Suppose x is lex in A and assume $\mathrm{dist}(A, B) \geqslant 2^{k+2}\ell$, for positive integers k and ℓ. With a probability of at least $(2d)^{-\ell}$, a random walk from x reaches $x_1 = x - \ell e_1$ before returning to $A \cup B$. The random walk can do so, for example, by following the ray $\{x - e_1, x - 2e_1, \dots\}$, which lies outside A because x is lex in A and B is a distance of $2^{k+2}\ell$ from x.

Center a cube Q_1 of side length 2ℓ at x_1; Q_1 does not intersect $A \cup B$. Denote the face of Q_1 in the $-e_1$ direction by F_1. By symmetry,

$$\mathbb{P}_{x_1}\big(S_{\tau_{Q_1}} \in F_1\big) = (2d)^{-1}.$$

Denote $S_{\tau_{F_1}}$ by X_2. Given X_2, we can center a cube Q_2 of side length 4ℓ at X_2; Q_2 does not intersect $A \cup B$. By analogously defining F_2, we have

$$\mathbb{P}_{X_2}\big(S_{\tau_{Q_2}} \in F_2\big) = (2d)^{-1}. \tag{3.77}$$

We can define X_j, Q_j, and F_j in this fashion, and (3.77) will hold with these variables in the place of X_2, Q_2, and F_2. The preceding bounds imply,

$$\mathbb{P}_x(\tau_{F_k} < \tau_{A\cup B}) \geqslant (2d)^{-(k+\ell)}. \tag{3.78}$$

Let $y = S_{\tau_{F_k}}$, in which case $\mathrm{dist}(y, A \cup B) \geqslant 2^k\ell$. Denote by N the number of returns made by random walk to $A \cup B$. By (3.18), if ℓ is sufficiently large (in a way which does not depend on n or $A \cup B$), then, for any $z \in A \cup B$,

$$G(y - z) \leqslant 2^{(2-d)k}.$$

Using the fact that $\mathbb{E}_y N = \sum_{z \in A\cup B} G(y-z)$, the preceding bound implies

$$\mathbb{E}_y N \leqslant n 2^{(2-d)k}.$$

By Markov's inequality, this implies

$$\mathbb{P}_y(N=0) \geqslant 1 - n2^{(2-d)k}.$$

Together with (3.78), we find

$$\mathrm{esc}_{A\cup B}(x) \geqslant c_1 (2d)^{-k}\big(1 - n2^{(2-d)k}\big), \tag{3.79}$$

for a constant $c_1 = c_1(d)$.

If n is at most 2^{d-3}, then choosing $k=1$ in (3.79) results in a lower bound $c_2 = c_2(d)$. Otherwise, if n is at least 2^{d-3}, then we can take k to be the integer part of $\log_2((2n)^{\frac{1}{d-2}})$, in which case (3.79) gives

$$\mathrm{esc}_{A\cup B}(x) \geqslant c_3 n^{-\frac{1+\log_2(d)}{d-2}}, \tag{3.80}$$

for another constant $c_3 = c_3(d)$. Because $\mathrm{cap}(A\cup B)$ is at most $nG(o)^{-1}$, (3.80) implies

$$\mathbb{H}_{A\cup B}(x) \geqslant c n^{-\frac{d-1+\log_2(d)}{d-2}}.$$

for $c = \min\{c_2, c_3\} G(o)$. We conclude the proof by setting $r = 2^{k+2}\ell$ and by replacing c_3 with c in (3.80). □

We apply the preceding lemma to prove the following conditional hitting estimate.

Lemma 3.10.4. *Let $d \geqslant 5$. There are constants $\rho = \rho(n,d)$ and $c = c(d)$ such that if x is lex in A, if $A \cup B$ is an n-element subset of $\mathbb{Z}^d$ such that $\mathrm{dist}(A,B) \geqslant \rho$, and if B can be written as a disjoint union $B^1 \cup B^2$ where $|B^1| \leqslant 3$ and $\mathrm{dist}(B^1, B^2) \geqslant \rho$, then*

$$\mathbb{P}_x\big(S_{\tau_{A\cup B}} \in B^1 \;\big|\; \tau_{A\cup B} < \infty\big) \geqslant c n^{-\frac{d}{d-2} - o_d(1)} \mathrm{diam}(A\cup B)^{2-d}. \tag{3.81}$$

For concreteness, the $o_d(1)$ quantity is smaller than 1.6 when $d \geqslant 5$.

Proof. Let A, B, B^1, and B^2 satisfy the hypotheses for the ρ in the statement of Lemma 3.10.3. Additionally, denote by F the set of points within a distance $r\mathrm{diam}(A\cup B)$ of $A \cup B$. Applying the strong Markov property to τ_{F^c}, we write

$$\begin{aligned}\mathbb{P}_x&\big(S_{\tau_{A\cup B}} \in B^1 \;\big|\; \tau_{A\cup B} < \infty\big)\\ &\geqslant \mathbb{E}_x\Big[\mathbb{P}_{S_{\tau_{F^c}}}\big(S_{\tau_{A\cup B}} \in B^1 \;\big|\; \tau_{A\cup B} < \infty\big)\mathbb{P}_{S_{\tau_{F^c}}}\big(\tau_{A\cup B} < \infty\big); \tau_{F^c} < \tau_{A\cup B}\Big].\end{aligned} \tag{3.82}$$

A standard result (e.g., [Law13, Theorem 2.1.3]) implies that for all sufficiently large r, if y belongs to F^c, then

$$\mathbb{P}_y\big(S_{\tau_{A\cup B}} \in B^1 \;\big|\; \tau_{A\cup B} < \infty\big) \geqslant \frac{1}{2}\mathbb{H}_{A\cup B}(B^1). \tag{3.83}$$

Because $B^1 \cup (A \cup B^2)$ satisfies the hypotheses of Lemma 3.10.3,

$$\mathbb{H}_{A\cup B}(B^1) \geqslant c_1 n^{-\frac{d-1}{d-2}-o_d(1)}. \tag{3.84}$$

for a constant c_1. By (3.18), for any such y, we have

$$\mathbb{P}_y\big(\tau_{A\cup B} < \infty\big) \geqslant c_2 \mathrm{dist}(y, A\cup B)^{2-d}, \tag{3.85}$$

for a constant $c_2 = c_2(d)$. Lastly, by Lemma 3.10.3, there is a constant c_3 such that

$$\mathbb{P}_x(\tau_{F^c} < \tau_{A\cup B}) \geqslant \mathrm{esc}_{A\cup B}(x) \geqslant c_3 n^{-\frac{1}{d-2}-o_d(1)}. \tag{3.86}$$

Applying (3.83) through (3.86) to (3.82), we find a constant $c_4 = c_4(d)$, such that

$$\mathbb{P}_x\big(S_{\tau_{A\cup B}} \in B^1 \bigm| \tau_{A\cup B} < \infty\big) \geqslant c_4 n^{-\frac{d}{d-2}-o_d(1)} \mathrm{diam}(A\cup B)^{2-d}.$$

Here, $o_d(1)$ can be taken to be 1.6 when $d \geqslant 5$. □

3.11 Proof of Theorem 3.1.7

In this section, we will analyze three algorithms which, when applied sequentially, dictate a sequence of HAT steps to form a configuration in $\mathscr{U}_{d,n}(a,1)$, from an arbitrary configuration and for any sufficiently large a. Each subsection will contain the statement of an algorithm and two results:

- informally, the first result will conclude that the algorithm does what it is intended to do; and
- the second will provide bounds on the number of steps and probability with which HAT realizes the steps dictated by the algorithm.

The final subsection will combine the bounds.

To prove that the configuration produced by the algorithms belongs to $\mathscr{U}_{d,n}(a,1)$, we must find an $(a,1)$ separated DOT clustering of the configuration. For this reason, it is convenient for the algorithms to return clusterings instead of configurations. Unlike the clusterings in the preceding sections, the clusterings will not be natural clusterings associated to HAT. Instead, the algorithms will actively assign and reassign elements to different clusters.

3.11.1 Algorithm 1

Before stating Algorithm $\mathcal{A}_2$, we give names to special elements that we reference in the algorithm. Let U be a configuration in $\mathbb{Z}^d$ containing an element $x \in \mathbb{Z}^d$, let C denote a clustering of a configuration, and denote $\tau = \tau_{U\setminus\{x\}}$. We define

- $\mu(U,x)$, an arbitrary maximizer y of $\mathbb{P}_x(S_{\tau-1} = y \mid \tau < \infty)$ over $U\setminus\{x\}$;

- $\nu(U, x)$, an arbitrary element $y \in \partial\{x\}$ which is exposed in U, assuming that $U \cap \partial\{x\}$ is nonempty; and
- $\mathrm{near}(C, y)$, the label of an arbitrary cluster which y neighbors or belongs to, i.e., an arbitrary element of $\{i : \mathrm{dist}(y, C^i) \leqslant 1\}$.

Additionally, we will refer to π, the map which takes a tuple of sets to their union (3.8).

To realize stage (1) of the strategy of Section 3.9, we must show that $\mathcal{A}_2$ produces a configuration which can be lined-up (Definition 3.9.1) and then show that HAT forms this configuration in a number of steps and with at least a probability which do not depend on the initial configuration. The following result addresses the former.

Proposition 3.11.1. *Given an n-element configuration $U \subset \mathbb{Z}^d$ and $a \in \mathbb{Z}_{\geqslant 2}$, $\mathcal{A}_2(U, a)$ can be lined-up with separation dn^2a.*

Proof. Consider the clustering C in line **27** of algorithm $\mathcal{A}_2$. It is easy to see that each cluster must have at least two elements and be connected. To prove that C can be lined-up with separation $r = dn^2a$, we must additionally show that C is $2r$ separated. Ignoring those elements which were assigned to clusters in lines **10** and **23**, due to line **19**, clusters $i < j$ are separated by at least

$$3d(n - i + 1)^3a - 3d(n - j + 1)^3a.$$

Because there are at most $[n/2]$ clusters, the preceding expression is at least

$$3d(n/2 + 1)^3a - 3d(n/2)^3a \geqslant 2dn^2a + n.$$

At most n elements are added to clusters by executing lines **10** and **23**. Because the clusters are connected, the preceding bound implies that the pairwise separation of clusters must be at least $2r = 2dn^2a$. □

We now verify that HAT realizes $\pi\big(\mathcal{A}_2(U, a)\big)$ in a number of steps and with at least a probability which do not depend on U.

Proposition 3.11.2. *Let U be an n-element configuration in $\mathbb{Z}^d$ and let $a \in \mathbb{Z}_{\geqslant 2}$. There are positive numbers $f_1 = f_1(a, d, n)$ and $g_1 = g_1(a, d, n)$ such that, if a is sufficiently large, then*

$$\mathbf{P}_U\Big(U_{f_1} = \pi\big(\mathcal{A}_2(U, a)\big)\Big) \geqslant g_1. \tag{3.87}$$

Proof. The proof takes the form of an analysis of Algorithm $\mathcal{A}_2$. Denote by u_k the configuration $U \cup \pi(C)$ after the k^{th} time $U \cup \pi(C)$ is changed (i.e., an element is moved) by the algorithm. Additionally, denote by N the number of times the configuration changes before the outer **while** loop terminates.

To establish (3.87), it suffices to show that there is a sequence of times $t_0 = 0 \leqslant t_1 < t_2 < \cdots < t_N \leqslant f_1$ such that $u_0 = U$, $u_N = \pi\big(\mathcal{A}_2(U, a)\big)$, and

$$\mathbf{P}_{u_0}\big(U_{t_1} = u_1, U_{t_2} = u_2, \ldots, U_{t_N} = u_N\big) \geqslant g_1. \tag{3.88}$$

Algorithm $\mathcal{A}_1$

Input : n-element configuration $U \subset \mathbb{Z}^d$ and $a \in \mathbb{Z}_{\geqslant 2}$
Output: Clustering C which can be lined-up with separation dn^2a
$C \leftarrow \emptyset, \quad i \leftarrow 1$ // Initialize variables.
while *U is nonempty* **do**
    $\ell \leftarrow \mathrm{lex}(U)$
    $R \leftarrow \mathrm{dist}\big(\ell, U\backslash\{\ell\}\big), \quad r \leftarrow 3d(n-i+1)^3 a$
    /* Form a non-isolated lex element if need be. */
    while $R > 1$ *and* $n > 1$ **do**
        $x \leftarrow \mu(U \cup \pi(C), \ell)$ // ℓ will be replaced by x.
        /* x either neighbors U ... */
        if $x \in \partial U$ **then**
            $U \leftarrow (U \cup \{x\})\backslash\{\ell\}$
        else
            /* ...or one of the existing clusters. */
            $U \leftarrow U\backslash\{\ell\}, \quad j \leftarrow \mathrm{near}(C, x)$
            $C \leftarrow C \cup^j \{x\}$
        end
        $\ell \leftarrow \mathrm{lex}(U)$ // The lex element of U may have changed.
        $R \leftarrow \mathrm{dist}(\ell, U\backslash\{\ell\})$
    end
    /* If the lex element is non-isolated, treadmill it. */
    if $R = 1$ **then**
        $y \leftarrow \nu(U \cup \pi(C), \ell)$ // y is an exposed neighbor of ℓ.
        $U \leftarrow U\backslash\{\ell, y\}$ // Remove the pair from U.
        $C \leftarrow C \cup^i \big\{\ell - re_1, \ell - (r-1)e_1\big\}$ // Treadmill the pair r steps.
        $i \leftarrow i + 1$ // Prepare to form the next cluster.
    else
        /* Otherwise, $U = \{\ell\}$; add it to an existing cluster. */
        $x \leftarrow \mu(U \cup \pi(C), \ell)$ // ℓ will be replaced by x.
        $U \leftarrow U\backslash\{\ell\}, \quad j \leftarrow \mathrm{near}(C, x)$
        $C \leftarrow C \cup^j \{x\}$
    end
end
return C

We will argue that $N \leqslant n(n+1)$, that we can take $t_N = n(n+1)r_1$ for $r_1 = 3dn^3 a$, and that

$$\mathbf{P}_{u_{k-1}}\big(U_{t_k} = u_k\big) \geqslant p, \tag{3.89}$$

for each $k \in [\![N]\!]$, for a positive number $p = p(a,d,n)$. The Markov property then implies that (3.88) holds with $f_1 = n(n+1)r_1$ and $g_1 = p^{n(n+1)}$.

Claim 1. We claim that $N \leqslant n(n+1)$. Observe that the outer and inner **while** loops starting on lines **2** and **5** each repeat at most n times. Indeed, U loses an element every time the outer loop repeats, which can happen no more than n times. Concerning the inner loop, no non-isolated element is made to be isolated, while, each time line **6** is executed, the isolated element ℓ is replaced by an element x which is non-isolated. This can happen at most n times consecutively. Accordingly, $U \cup \pi(C)$ changes at most $n+1$ times every time the outer loop repeats, hence $N \leqslant n(n+1)$.

Claim 2. We now claim that we can take $t_N = n(n+1)r_1$. It suffices to argue that, each time $U \cup \pi(C)$ changes, at most r_1 steps of HAT are required to realize the change. The configuration $U \cup \pi(C)$ changes due to the execution of lines **8**, **10** and **11**, **18** and **19**, or **23** and **24**. In all but one case—that of lines **18** and **19**—the transition requires only one HAT step. For lines **18** and **19**, at most r_1 steps are needed. Because there are at most $n(n+1)$ changes, t_N can be taken to be $n(n+1)r_1$.

Claim 3. We now verify (3.89) by considering each way $U \cup \pi(C)$ can change and by bounding below the probability that it is realized by HAT. Assume $U \cup \pi(C)$ has changed $k-1$ times so far.

- Lines **8**, **10** and **11**, or **23** and **24**: Activation at ℓ and transport to x. Assume a is sufficiently large in d and n to exceed the constant r in the statement of Lemma 3.10.3. Then, since ℓ is the lex element of U and since $\operatorname{dist}(U, \pi(C)) \geqslant a$, we can apply Lemma 3.10.3 with $A = U$ and $B = \pi(C)$ to find
$$\mathbb{H}_{U\cup\pi(C)}(\ell) \geqslant h,$$
for a positive number $h = h(d,n)$. By the definition of μ, x is the most likely destination of an element activated at ℓ. Transport from ℓ occurs to at most $2dn$ sites and so, by the pigeonhole principle, the element from ℓ is transported to x with a probability of at least $(2dn)^{-1}$. Together, these bounds imply
$$\mathbf{P}_{u_{k-1}}\big(U_{t_k} = u_k\big) \geqslant h(2dn)^{-1}. \tag{3.90}$$

- Lines **18** and **19**: Treadmilling of $\{\ell, y\}$. In the first step, we activate at y and transport to $\ell - e_1$. While y is not lex in U, by the definition of ν, it is an exposed neighbor of ℓ. Because $\operatorname{dist}(U, \pi(C)) \geqslant a$, if a is at least $4^d n$, we can apply Lemma 3.10.2 with $A = U$ and $B = \pi(C)$ to find $\mathbb{H}_{U\cup\pi(C)}(y) \geqslant c_1 h$, for a positive number $c_1 = c_1(d,n)$. Additionally, Lemma 3.10.1 implies that an element activated at y is transported to $\ell - e_1$ with a probability of at least $c_2 = c_2(d,n)$. Consequently, denoting
$$v_1 = \big(u_{k-1} \cup \{\ell - e_1, \ell\}\big) \setminus \{\ell, y\},$$
we have
$$\mathbf{P}_{u_{k-1}}(U_1 = v_1) \geqslant c_1 c_2 h. \tag{3.91}$$

Now, consider the configuration v_m resulting from starting at u_{k-1} and treadmilling $\{\ell, y\}$ a total of $m \geqslant 2$ steps in the $-e_1$ direction:

$$v_m = \big(u_{k-1} \cup \{\ell - me_1, \ell - (m-1)e_1\}\big) \setminus \{\ell, y\}.$$

To obtain v_{m+1}, we activate at $\ell - (m-1)e_1$ and transport to $\ell - (m+1)e_1$. By the same reasoning as before,

$$\mathbf{P}_{v_m}\big(U_1 = v_{m+1}\big) \geqslant c_1 c_2 h. \tag{3.92}$$

By (3.91), (3.92), and the Markov property,

$$\mathbf{P}_{u_{k-1}}\big(U_{t_k} = u_k\big) \geqslant \big(c_1 c_2 h\big)^{r_1}. \tag{3.93}$$

The bounds (3.90) and (3.93) show that, whenever $U \cup \pi(C)$ changes, the change can be realized by HAT (in one or more steps) with a probability of at least

$$p = \min \big\{ h(2dn)^{-1}, (c_1 c_2 h)^{r_1} \big\}.$$

This proves (3.89). We complete the proof by combining claims 1–3. □

3.11.2 Algorithm 2

To complete Stage 2 of the strategy of Section 3.9, we show that if C is a clustering which can be lined-up with separation dn^2a, then HAT forms the clusters into dn^2a separated line segments oriented parallel to e_1, in a number of steps and with at least a probability which depend on a, d, and n only (Proposition 3.11.4). The clustering to which we refer is

$$\mathcal{L}(C) = \big(\mathrm{lex}(C^i) + L_{|C^i|},\ i \in [\![m]\!]\big).$$

Proposition 3.11.3. *Let U be an n-element configuration in $\mathbb{Z}^d$ with a clustering C which can be lined-up with separation dn^2a, for $a \in \mathbb{Z}_{\geqslant 2}$. Then $\mathcal{A}_3(C) = \mathcal{L}(C)$.*

Proof. The only way the algorithm could fail to produce $\mathcal{L}(C)$ is if, for some outer **for** loop i and inner **for** loop j, the assignment in line **5** is impossible. This would mean that no element of $D = C^i \setminus \{\ell_i + L_j\}$ was exposed in $\pi(C)$. While there must be an element of D which is exposed in C^i, the elements of $C^{\neq i}$ could, in principle, separate D from ∞. In fact, as we argue now, this cannot occur because the clusters remain far enough apart while the algorithm runs.

Each C^i remains connected while the algorithm runs, so there is a ball B_i of radius n which contains C^i. The B_i are finite and $*$-connected, so each $*$-visible boundary $\partial_{\mathrm{vis}} B_i$ is connected by Lemma 2.3.13. Moreover, each $\partial_{\mathrm{vis}} B_i$ is disjoint from $\cup_j B_j$ because $\mathrm{dist}(B_i, B_j)$ exceeds $\sqrt{d}$. This lower bound holds because the clusters are initially $2dn^2a$ separated and the separation decreases by at most one with each of the n loops of the algorithm, hence

$$\mathrm{dist}(B^i, B^j) \geqslant \mathrm{dist}(C^i, C^j) - \mathrm{diam}(B_i) - \mathrm{diam}(B_j) - n \geqslant 2dn^2a - 5n > \sqrt{d}.$$

The rest of the argument, which constructs an infinite path from D which otherwise avoids C, is identical to the corresponding step in the proof of Proposition 3.6.6. We conclude that some element of D is exposed in $\pi(C)$, which completes the proof. □

Proposition 3.11.4. *Let U be an n-element configuration in $\mathbb{Z}^d$ with a clustering C which can be lined-up with separation dn^2a, for $a \in \mathbb{Z}_{\geqslant 2}$. There is a positive number $g_2 = g_2(d,n)$ such that, if a is sufficiently large, then*

$$\mathbf{P}_U\Big(U_n = \pi\big(\mathcal{A}_3(C)\big)\Big) \geqslant g_2. \tag{3.94}$$

Algorithm $\mathcal{A}_3$

Input : Clustering C of an n-element configuration $U \subset \mathbb{Z}^d$, which can be lined-up with separation dn^2a, for $a \in \mathbb{Z}_{\geqslant 2}$.

Output: $\mathcal{L}(C)$.

1 $m \leftarrow$ number of clusters in C

2 **for** $i \in [\![m]\!]$ **do**

3   **for** $j \in [\![|C^i| - 1]\!]$ **do**

4         $\ell_i \leftarrow \text{lex}(C^i)$ // The segment will grow from ℓ_i.

5         $x_j \leftarrow \text{lex}\big(\{z \in C^i \backslash \{\ell_i + L_j\} : \mathbb{H}_{\pi(C)}(z) > 0\}\big)$ // x_j is lex among exposed elements of C^i which have not yet been added to the growing segment.

6         $y_j \leftarrow \ell_i - je_1$ // y_j is the next addition to the segment.

7         $C \leftarrow (C \cup^i \{y_j\}) \backslash^i \{x_j\}$ // Update the i^{th} cluster.

8   **end**

9 **end**

10 **return** C

Proof of Proposition 3.11.4. Given a clustering C which satisfies the hypotheses, algorithm $\mathcal{A}_3$ specifies for each cluster i a sequence of $|C^i| - 1$ pairs (x_j, y_j)—where x_j is the site of activation and y_j is the site to which transport occurs—to rearrange C^i into $\text{lex}(C^i) + L_{|C^i|}$. We note that no pair will result in a decrease in cluster separation of more than one, or an increase in cluster diameter of more than one. Because the clusters are initially $2dn^2a$ separated, the clusters will remain $2dn^2a - n \geqslant a$ separated throughout.

Accordingly, if a is sufficiently large in terms of d and n, the combination of Lemma 3.10.2 and Lemma 3.10.3 imply that there is a constant $h = h(d,n)$ such that each x_j can be activated with a probability of at least h. Moreover, Lemma 3.10.1 implies that there is a positive number $c = c(d,n)$ such that an element from x_j can be transported to y_j with a probability of at least c. Consequently, denoting $C' = (C \cup^i \{y_j\}) \setminus^i \{x_j\}$, the transition in line **7** occurs with a probability of at least

$$\mathbf{P}_{\pi(C)}\big(U_1 = \pi(C')\big) \geqslant ch. \tag{3.95}$$

By (3.95) and the Markov property, and the fact that there are at most n pairs, we have

$$\mathbf{P}_U\Big(U_n = \pi\big(\mathcal{A}_3(C)\big)\Big) \geqslant (ch)^n.$$

Taking $g_2 = (ch)^n$ gives (3.94). □

3.11.3 Algorithm 3

At the beginning of Stage 3, the elements are neatly arranged into well separated line segments pointing in the $-e_1$ direction. In Stage 3, we iteratively treadmill pairs of elements in the $-e_1$ direction from each of the line segments, until only a dimer or trimer remains of the initial segment. We will label each treadmilled pair as a new cluster. To reflect this in our notation, when C has m clusters and when $A \subset \mathbb{Z}^d$, we will write $C \cup^{m+1} A$ to mean the addition of A to C as the $(m+1)^{\text{st}}$ cluster.

Algorithm $\mathcal{A}_4$

Input : An $n^2 a$ separated clustering C of an n-element configuration $U = \pi(\mathcal{L}(C)) \subset \mathbb{Z}^d$, for $a \in \mathbb{Z}_{\geqslant 2}$.
Output: A configuration in $\mathscr{U}_{d,n}(a, 2(\log a)^{-1})$.

1 $k \leftarrow 0, \quad m \leftarrow$ number of parts of C // Initialize variables.
2 **for** $i \in [\![m]\!]$ **do**
3   **for** $j \in [\![|C^i| \bmod 2 - 1]\!]$ **do**
4     $\ell \leftarrow \operatorname{lex}(C^i), \quad r \leftarrow 2(n-j)a$
5     $C \leftarrow \big(C \cup^{m+k} \{\ell - re_1, \ell - (r-1)e_1\}\big) \setminus^i \{\ell, \ell + e_1\}$ // Treadmill the pair r steps, labeling it as cluster $m+k$.
6     $k \leftarrow k+1$ // Account for the creation of a new cluster.
7   **end**
8 **end**
9 **return** C

Proposition 3.11.5. *If U is an n-element configuration in $\mathbb{Z}^d$ and if C is a clustering of U such that $U = \mathcal{L}(C)$ and C is $n^2 a$ separated for an integer $a > 1$, then $\pi\big(\mathcal{A}_4(C, a)\big)$ belongs to $\mathscr{U}_{d,n}(a, 2(\log a)^{-1})$.*

Proof. Denote by C_k the clustering C once it has been changed by the algorithm for the k^{th} time (i.e., the k^{th} time line **5** is executed). Denote by m the number of clusters of C_0 and N the number of times algorithm $\mathcal{A}_4$ changes C.

To prove that $\pi\big(\mathcal{A}_4(C_0, a)\big)$ belongs to $\mathscr{U}_{d,n}(a, 2\log(a)^{-1})$, we will verify DOT.1, DOT.2 with a, and observe that each cluster of C_N is connected; this will imply DOT.3 with $b = 2(\log a)^{-1}$.

Concerning DOT.1 and the claim that each cluster of C_N^i is connected, we note that line **5** creates connected clusters of size two and, because it is executed $|C^i| \bmod 2 - 1$ times for cluster i, when the inner **for** loop ends on line **7**, only two or three (connected) elements of the original cluster C^i remain. Accordingly, every cluster of C_N has two or three elements and is connected.

Concerning DOT.2, we observe that, for each $i \in [\![m]\!]$, the separation of cluster C_N^i is at least

$$\operatorname{dist}(C_N^i, C_N^{\neq i}) \geqslant \operatorname{dist}(C_0^i, C_0^{\neq i}) - 2(n-1)a \geqslant (n^2 - 2n + 2)a \geqslant a,$$

because no element is moved a distance exceeding $2(n-1)a$ by the algorithm. The same is true of $\operatorname{dist}(C_N^i, C_N^j)$ for each $i \in [\![m]\!]$ and every j, and for clusters i and j resulting from the treadmilling of different clusters of C_0. Concerning the pairwise separation of clusters $i \neq j$

formed by treadmilling pairs from the same cluster of C_0, by line **5**, we have

$$\operatorname{dist}(C_N^i, C_N^j) \geqslant 2(n-1)a - 2(n-2)a - 1 \geqslant a.$$

We conclude that every cluster i satisfies $\operatorname{dist}(C_N^i, C_N^{\neq i}) \geqslant a$, so C_N satisfies DOT.2 with a.

Because C_N satisfies DOT.1 and because each cluster is connected, the cluster diameters satisfy $\operatorname{diam}(C_N^i) \leqslant 2$. Then, because C_N satisfies DOT.2 with a, DOT.3 holds for $b = 2(\log a)^{-1}$. □

Proposition 3.11.6. *Let U be an n-element configuration in $\mathbb{Z}^d$ with an $n^2 a$ separated clustering C such that $U = \pi(\mathcal{L}(C))$, for $a \in \mathbb{Z}_{\geqslant 2}$. There are positive numbers $f_3 = f_3(a,d,n)$ and $g_3 = g_3(a,d,n)$ such that, if a is sufficiently large, then*

$$\mathbf{P}_U\Big(U_{f_3} = \pi\big(\mathcal{A}_4(C,a)\big)\Big) \geqslant g_3. \tag{3.96}$$

Proof. As in the proof of Proposition 3.11.5, denote by C_k the clustering C once it has been changed by the algorithm for the k^{th} time (i.e., the k^{th} time line **5** is executed). Denote by m the number of clusters of C_0 and N the number of times algorithm $\mathcal{A}_4$ changes C. Call $u_k = \pi(C_k)$.

To establish (3.96), it suffices to show that there is a sequence of times $t_0 = 0 \leqslant t_1 < t_2 < \cdots < t_N \leqslant f_3$ such that $u_0 = U$, $u_N = \pi\big(\mathcal{A}_4(C_0,a)\big)$, and

$$\mathbf{P}_{u_0}\big(U_{t_1} = u_1, U_{t_2} = u_2, \ldots, U_{t_N} = u_N\big) \geqslant g_3. \tag{3.97}$$

Consider outer **for** loop i, inner **for** loop j, and suppose that C has been changed a total of $k-1$ times thus far. Let $r = 2(n-j)a$. We will first bound below the probability that HAT realizes u_{k-1} as U_{r+1} from u_k (i.e., the transition reflected in line **5**). HAT can realize this transition by treadmilling the elements at $\ell = \operatorname{lex}(C_{k-1}^i)$ and $\ell + e_1$ to $\{\ell - re_1, \ell - (r-1)e_1\}$.

For example, in the first step, we activate at $\ell + e_1$ and transport to $\ell - e_1$. As observed in the proof of Proposition 3.11.5, C_k is a separated for every $0 \leqslant k \leqslant N$. Therefore, if a is sufficiently large in terms of d and n, then the hypotheses of Lemma 3.10.2 and Lemma 3.10.3 are satisfied with $A = \pi(C_k^i)$ and $B = \pi(C_k^{\neq i})$, and they together imply the existence of a positive lower bound $h = h(d,n)$ on $\mathbb{H}_{u_{k-1}}(\ell + e_1)$. It is clear that the element at $\ell + e_1$ can be transported to $\ell - e_1$ with a probability of at least $c = c(d)$, and so, denoting

$$v_s = \big(u_{k-1} \cup \{\ell - se_1, \ell - (s-1)e_1\}\big) \backslash \{\ell, \ell + e_1\},$$

we have

$$\mathbf{P}_{u_{k-1}}(U_s = v_1) \geqslant ch.$$

We can simply repeat this argument with ℓ and $\ell - e_1$ in the place of $\ell + e_1$ and ℓ, then $\ell - e_1$ and $\ell - 2e_1$, and so on. With the choice $u_k = v_r$, the Markov property implies

$$\mathbf{P}_{u_{k-1}}(U_r = u_k) \geqslant (ch)^r.$$

The same bound holds for any $k \in [\![N]\!]$, so, by another use of the Markov property and the fact that $N \leqslant n$, we find

$$\mathbf{P}_{u_0}\big(U_r = u_1, U_{2r} = u_2, \ldots, U_{Nr} = u_N\big) \geqslant (ch)^{rn}.$$

This proves (3.97) with $f_3 = nr$ and $g_3 = (ch)^{rn}$. □

3.11.4 Conclusion

We combine the results from the preceding subsections to prove the main result of this section.

Proof of Theorem 3.1.7. Let U be an n-element configuration in $\mathbb{Z}^d$. Take $a \in \mathbb{Z}_{\geqslant 2}$ to be sufficiently large in terms of d and n to satisfy the hypotheses of Propositions 3.11.2, 3.11.4, and 3.11.6. By Proposition 3.11.2, there are positive numbers $f_1 = f_1(a, d, n)$ and $g_1 = g_1(a, d, n)$ such that

$$\mathbf{P}_U\Big(U_{f_1} = \pi\big(\mathcal{A}_2(U, a)\big)\Big) \geqslant g_1. \tag{3.98}$$

By Proposition 3.11.1, $C_1 = \mathcal{A}_2(U, a)$ can be lined-up with separation dn^2a. Consequently, by Proposition 3.11.4, there is a positive number $g_2 = g_2(d, n)$ such that

$$\mathbf{P}_{\pi(C_1)}\Big(U_n = \pi\big(\mathcal{A}_3(C_1)\big)\Big) \geqslant g_2. \tag{3.99}$$

By Proposition 3.11.3, $C_2 = \mathcal{A}_3(C_1) = \mathcal{L}(C_1)$. Since C_2 is n^2a separated, by Proposition 3.11.6, there are positive numbers $f_3 = f_3(a, d, n)$ and $g_3 = g_3(a, d, n)$ such that

$$\mathbf{P}_{\pi(C_2)}\Big(U_{f_3} = \pi\big(\mathcal{A}_4(C_2, a)\big)\Big) \geqslant g_3. \tag{3.100}$$

Denote $C_3 = \mathcal{A}_4(C_2, a)$. By the Markov property and (3.98) through (3.100),

$$\mathbf{P}_U\big(U_{f_1+n+f_3} = \pi(C_3)\big) \geqslant g_1g_2g_3. \tag{3.101}$$

By Proposition 3.11.5, $\pi(C_3)$ belongs to $\mathscr{U}_{d,n}(a, 2(\log a)^{-1})$. Taking $a \geqslant e^2$, C_3 is an $(a, 1)$ separated DOT clustering. Setting $f = f_1 + n + f_3$ and $g = g_1g_2g_3$ concludes the proof. $\square$

3.12 Proof of Theorem 3.1.6

We continue to assume $d \in \mathbb{Z}_{\geqslant 5}$ and $n \in \mathbb{Z}_{\geqslant 4}$. To prove the irreducibility of HAT on $\widehat{\text{NonIso}}_{d,n}$, we show that HAT can form a line segment from any configuration and HAT can form any configuration from a line segment. This is the content of the next two propositions.

Proposition 3.12.1 (Set to line)**.** *Let $n \in \mathbb{Z}_{\geqslant 4}$ and let U be an n-element configuration in $\mathbb{Z}^d$. There are a finite number of steps $f_4 = f_4(d, n)$ and a positive number $g_4 = g_4(d, n, \operatorname{diam}(U))$ such that*

$$\mathbf{P}_U\big(\widehat{U}_{f_4} = \widehat{L}_n\big) \geqslant g_4. \tag{3.102}$$

To state the next result, define for finite $A \subset \mathbb{Z}^d$ its radius $\operatorname{rad}(A) = \sup\{\|x\| : x \in A\}$.

Proposition 3.12.2 (Line to set)**.** *Let $U \in \widehat{\text{NonIso}}_{d,n}$ have a radius of r. There are a finite number of steps $f_5 = f_5(d, n, r)$ and a positive probability $g_5 = g_5(d, n, r)$ such that*

$$\mathbf{P}_{L_n}\big(\widehat{U}_{f_5} = \widehat{U}\big) \geqslant g_5. \tag{3.103}$$

Theorem 3.1.6 is a simple consequence of the preceding propositions.

Proof of Theorem 3.1.6. Let $\widehat{U}$ and $\widehat{V}$ belong to $\widehat{\text{NonIso}}_{d,n}$. By Propositions 3.12.1 and 3.12.2, there are finite numbers of steps f and f', and positive probabilities g and g' such that

$$\mathbf{P}_U\big(\widehat{U}_f = \widehat{L}_n\big) \geqslant g \quad \text{and} \quad \mathbf{P}_{L_n}\big(\widehat{U}_{f'} = \widehat{V}\big) \geqslant g'.$$

Applying the Markov property at time f, the preceding bounds imply

$$\mathbf{P}_U\big(\widehat{U}_{f+f'} = \widehat{V}\big) \geqslant gg' > 0,$$

which implies that HAT is irreducible on $\widehat{\text{NonIso}}_{d,n}$. □

Next, we prove Proposition 3.12.1.

Proof of Proposition 3.12.1. Let $a = a(d,n)$ be an integer which is sufficiently large to satisfy the hypotheses of Theorem 3.1.7 and those of Lemma 3.10.3 and Lemma 3.10.4 in the place of r and ρ, respectively.

By Theorem 3.1.7, there is a positive integer $f = f(d,n)$ and a positive number $g = g(d,n)$, such that U_f belongs to $\mathscr{U}_{d,n}(a+n,1)$ with a probability of at least g. In particular, there is an $a+n$ separated clustering C of U_f.

Let ℓ be the lex element of U_f, which we assume w.l.o.g. belongs to C^1. Because C is $a+n$ separated, we can activate any lex element of any cluster with a probability of at least $h_1 = h_1(d,n)$ by Lemma 3.10.3. Then, by Lemma 3.10.4, we can transport to $\ell - e_1$ with a probability of at least $h_2 = h_2(d,n,\text{diam}(U))$. Reassigning the element at $\ell - e_1$ to cluster C^1, the resulting clusters are at least $a+n-1$ separated.

Because the resulting clusters are still a separated, we can simply repeat this process, transporting an element to $\ell - 2e_1$, and so on. Continuing in this fashion for a total of n steps results in $U_{f+n} = \ell + L_n$. The preceding discussion and the Markov property imply

$$\mathbf{P}_U\big(\widehat{U}_{f+n} = \widehat{L}_n\big) \geqslant g(h_1h_2)^n.$$

Setting $f_4 = f+n$ and $g_4 = g(h_1h_2)^n$ gives (3.102). □

We will prove Proposition 3.12.2 with an argument by induction. To facilitate the induction step, it is convenient to prove the following, more detailed claim, which assumes $n \in Z_{\geqslant 2}$ instead of $n \in \mathbb{Z}_{\geqslant 4}$.

Proposition 3.12.3. *Let $n \in \mathbb{Z}_{\geqslant 2}$ and let $U \in \widehat{\text{NonIso}}_{d,n}$. In terms of $r = \lceil \text{rad}(U) \rceil$, there are positive integers $f = 4^d n^2 r$ and $\ell = f/n$, and a sequence $((x_i,y_i), i \in [\![f]\!])$ of pairs in $\mathbb{Z}^d$ such that, setting*

$$W_0 = L_n \quad \textit{and} \quad W_j = (W_{j-1} \setminus \{x_j\}) \cup \{y_j\} \ \textit{for}\ j \in [\![f]\!],$$

the following conclusions hold:

(i) $W_f = U$.

(ii) *For each $j \in [\![f]\!]$, x_j is exposed in W_{j-1}.*

(iii) *For each $j \in [\![f]\!]$, there is a path Γ_j from x_j to y_j, which lies outside of $W_{j-1} \backslash \{x_j\}$ but inside of $B(r+dn)$, and which has a length of at most ℓ.*

Before proving the proposition, let us explain how Proposition 3.12.2 follows from it.

Proof of Proposition 3.12.2. By conclusion (iii) of Proposition 3.12.3, for each $j \in [\![f]\!]$, we have $W_{j-1} \subseteq B(r+dn)$. By this observation and conclusion (ii), the activation component $\mathbb{H}_{W_{j-1}}(x_j)$ of each transition is at least a positive number $h_1 = h_1(d,n,r)$. Again, by (iii), there is a path Γ_j, with a length of at most ℓ, which can realize the transport step from x_j to y_j. Consequently, in terms of $\tau = \tau_{W_{j-1}\backslash\{x_j\}}$, the transport component $\mathbb{P}_{x_j}(S_{\tau-1} = y_j \mid \tau < \infty)$ of each transition is at least $h_2 = (2d)^{-\ell-1}$. By the Markov property and conclusion (i), the probability in (3.103) is at least the product of these components, over f steps:

$$\mathbf{P}_{L_n}\big(\widehat{U}_f = \widehat{U}\big) \geqslant (h_1 h_2)^f.$$

□

Lastly, we prove Proposition 3.12.3.

Proof of Proposition 3.12.3. The proof is by induction on n. The base case of $n=2$ is trivial because $\widehat{\text{NonIso}}_{d,2}$ has the same elements as the equivalence class $\widehat{L}_2$. Now suppose the claim holds up to $n-1$ for $n \geqslant 3$.

There are two cases, which we phrase in terms of the "exposed" boundary of U:

$$\partial^*_{\text{ext}} U = \{x \in U : \mathbb{H}_U(x) > 0\}.$$

Either:

1. there is a non-isolated $x \in \partial^*_{\text{ext}} U$ such that $U \backslash \{x\} \in \text{NonIso}_{d,n-1}$; or
2. for every non-isolated $x \in \partial^*_{\text{ext}} U$, $U \backslash \{x\} \in \text{Iso}_{d,n-1}$.

Case 1. Peform the following steps. In what follows, denote $r = [\text{rad}(U)] + 1$.

Step 1: *"Treadmill" a pair of elements in the $-e_1$ direction for $f_1 = r + dn - 2$ steps.* Specifically, activate the element e_1 and transport it to $-e_1$, then activate the element at the origin and transport it to $-2e_1$, followed by activation at $-e_1$ and transport to $-3e_1$, and so on.

Step 2: *Isolate an element outside of $B(r + dn - 2)$.* At the end of Step 1, an element lies at $-f_1 e_1$ and another at $-(f_1 - 1)e_1$. Activate the latter and transport it to the e_1, then activate the element at $(n-1)e_1$ and transport it to the origin.

Step 3: *Use the induction hypothesis to form* $(U\backslash\{x\}) \cup \{-f_1 e_1\}$*, for a particular* x*.* By the end of Step 2, the configuration is $L_{n-1} \cup \{-f_1 e_1\}$. We use the induction hypothesis to form $U\backslash\{x\}$ from the L_{n-1} subset, where x is a non-isolated element of $\partial^*_{\text{ext}} U$ such that $U\backslash\{x\} \in \text{NonIso}_{d,n-1}$.

The use of the induction hypothesis guarantees that there is a sequence of $4^d(n-1)^2 r$ HAT steps from L_{n-1}, which: (i) result in $(U\backslash\{x\}) \cup \{-f_1 e_1\}$; (ii) have positive activation components; and (iii) have transport steps that are realized by random walk paths with lengths of at most $4^d(n-1)r$, which remain inside $B(r + d(n-1))$.

Step 4: *Transport the element at* $-f_1 e_1$ *to* x*.* We activate the element at $-f_1 e_1$ and transport it to x, which is possible because x is non-isolated and exposed in U. Because $U\backslash\{x\}$ lies in $B(r)$, Lemma 3.10.1 implies that there is a path from $-f_1 e_1$ to x which avoids $U\backslash\{x\}$, has a length of at most $\ell = 4^d nr$, and lies in $B(r + dn - 1)$.

Note that Steps 1 and 2 require $r + dn$ HAT steps, the activation components of which are positive and the transport components of which can be realized by paths of length at most $r + dn \leqslant \ell$. Steps 3 and 4 require $4^d(n-1)^2 r + 1$ HAT steps, again with positive activation components and transport components realized by paths of length at most ℓ. In total, at most $f = 4^d n^2 r$ HAT steps are needed and, since all paths lie in $B(r + dn - 1)$, conclusions (i) through (iii) hold.

Case 2. Because we cannot remove a non-isolated element of U without obtaining an isolated set—a set to which the induction hypothesis does not apply—we must instead use the induction hypothesis to form a set related to U. In fact, the first two steps are the same as in Case 1, so we begin with the configuration $L_{n-1} \cup \{-f_1 e_1\}$ and specify the third and subsequent steps.

Step 3': *Use the induction hypothesis.* Let w and y be the least and greatest elements of U in the lexicographic order, and let x be any non-isolated element of $\partial^*_{\text{ext}} U$. We use the induction hypothesis to form

$$U' = (U\backslash\{x, y\}) \cup \{w - e_1\},$$

which is possible because $U' \in \widehat{\text{N}}\text{onIso}_{d,n-1}$.

The result is a sequence of $4^d(n-1)^2 r$ HAT steps from L_{n-1} which form $U' \cup \{-f_1 e_1\}$ with positive activation components and transport components which are realized by random walk paths with the same properties as in Step 3.

Step 4': *Activate the element at* $-f_1 e_1$ *and transport it to* $w - 2e_1$*.*

Step 5': *Treadmill the pair* $\{w - e_1, w - 2e_1\}$*.* Since w is the least element of U in the lexicographic order, $w - e_1$ and $w - 2e_1$ are the only elements which lie in $O_1 = \{z \in \mathbb{Z}^d : z \cdot e_1 \leqslant w \cdot e_1\}$. Similarly, due to the choice of y, it is the only element which lies in $O_2 = \{z \in \mathbb{Z}^d : z \cdot e_d \geqslant y \cdot e_d\}$.

Consequently, it is possible to treadmill the pair $\{w - e_1, w - 2e_1\}$ to:

- $B(r + 3)^{\mathsf{c}}$ without leaving O_1; then

- O_2 without leaving $B(r+6)\backslash B(r+3)$; and
- $\{y, y+e_d\}$ without leaving O_2.

This requires at most $f_2 = 10r$ HAT steps, each of which has a positive activation component and a transport component realized by a random walk path of length five.

Step 6': *Activate at* $\{y+e_d\}$ *and transport to* x. The configuration at the end of Step 5' is $(U\backslash\{x\}) \cup \{y+e_d\}$, so activating the element at $\{y+e_d\}$ and transporting it to x (which is possible because x is an exposed, non-isolated element of $\partial^*_{\text{ext}}U$), forms U.

Recall that Steps 1 and 2 require $r + dn$ HAT steps, which can be realized by paths of length at most $r + dn \leqslant \ell$. Steps 3' and 4' require $4^d(n-1)^2r + 1$ HAT steps, realized by paths of length at most ℓ. Steps 5' and 6' require $10r + 1$ HAT steps, with paths satisfying the same length bound. All activation components are positive. At most f HAT steps are needed in total and, since all paths lie in $B(r + dn - 1)$, conclusions (i) through (iii) hold. $\square$

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