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NUMERICAL SOLUTION OF THE GEL'FAND-LEVITAN EQUATION

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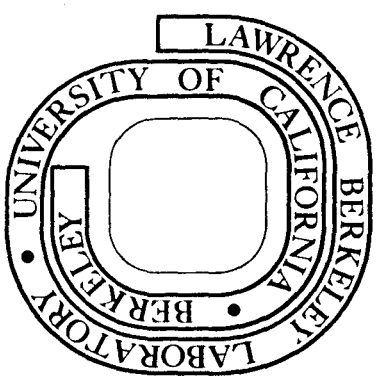

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NUMERICAL SOLUTION OF THE GEL'FAND-LEVITAN EQUATION

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ABSTRACT

The inverse Sturm-Liouville problem is solved by using the Gel'fand-Levitan equation. The equation is discretized by the trapezoidal rule and the problem reduced to solving a sequence of systems of linear equations. The convergence of the method is established. It is shown that the problem can be arbitrarily ill-conditioned. Finally the accuracy of the method is illustrated by two numerical examples.

Introduction

The Gel'fand-Levitan equation is the fundamental equation in inverse scattering theory. We shall consider the equation in connection with the inverse Sturm-Liouville problem. Roughly speaking, the question is this: How do you determine a coefficient in a differential equation from the eigenvalues and the integral of the square of the eigenfunctions? This problem can be reduced to solving a family of integral equations -- called the Gel'fand-Levitan equation (see [3]). In this paper we shall present a numerical technique for solving this equation and prove that the computed solution converges. It is generally believed that solving the Gel'fand-Levitan equation numerically is an ill-posed problem. Our numerical experiments do not support this belief. We shall show, however, that the Gel'fand-Levitan equation can be arbitrarily ill-conditioned.

The obvious discretization of the inverse Sturm-Liouville problem leads to the inverse eigenvalue problem for Jacobi matrices. In [2] de Boor and Golub have given a stable algorithm for this problem. It is extremely difficult to prove that the solution of the inverse problem for the tridiagonal matrix converges to the solution of the inverse problem for the differential equation. Hald [4] and Yen [9] have proved the convergence for a different kind of algorithm, based on the Rayleigh-Ritz method. The results are only local. The most powerful approach seems to be to analyze the differential equation and then discretize the resulting equations. Thus Hochstadt has reduced the inverse Sturm-Liouville problem to solving systems of ordinary differential equations (see [7,5]). In this paper we shall study a particular discretization of the Gel'fand-Levitan equation, and show how the numerical solutions can be organized.

1. The Gel'fand-Levitan equation

Consider the two point boundary value problem

$$-u'' + q(x)u = \lambda u \tag{1.1}$$

$$u'(0) - hu(0) = 0 \quad , \quad u'(\pi) + Hu(\pi) = 0$$

on the interval $0 \leqslant x \leqslant \pi$ with h and H finite. This problem has an infinite number of eigenvalues $\lambda_0, \lambda_1, \ldots$ and eigenfunctions $u_0(x), u_1(x), \ldots$. We assume that $u_j = 1$ at $x = 0$ and let ρ_j be the normalizing constants $\rho_j = \int_0^\pi u_j^2$. If $q \equiv 0$ and $h = H = 0$ then $\lambda_j = j^2$, $\rho_0 = \pi$ and $\rho_j = \pi/2$ for $j \geqslant 1$. Let

$$a(x) = \Sigma_{j=1}^{\infty} \left[\frac{\cos \sqrt{\lambda_j}\, x}{\rho_j} - \frac{\cos jx}{\pi/2} \right] + \frac{\cos \sqrt{\lambda_0}\, x}{\rho_0} - \frac{1}{\pi} \tag{1.2}$$

If $\lambda_j < 0$ then $\cos \sqrt{\lambda_j}\, x$ should be replaced by $\cosh \sqrt{|\lambda_j|}\, x$. The Gel'fand-Levitan equation is a one parameter family of integral equations and given by

$$f(x,y) + \int_0^x f(y,t)K(x,t)dt + K(x,y) = 0 \tag{1.3}$$

where $f(x,y) = [a(x+y) + a(x-y)]/2$ (see [3]). Here $y \leqslant x$ and x is a parameter. To find the potential q corresponding to the eigenvalues λ_j and the normalizing constants ρ_j we solve the Gel'fand-Levitan equation for each x in the interval $[0,\pi]$ and set $q(x) = 2 \frac{d}{dx} K(x,x)$. The constant h in the left boundary condition is $-f(0,0) = -a(0)$. To find the constant H in the boundary condition at the right we solve the

differential equation (1.1) with $\lambda = \lambda_0$ and initial conditions $u = 1$ and $u' = h$ at $x = 0$ and set $H = -u'(\pi)/u(\pi)$.

2. The matrix problem

To discretize Eq. (1.3) we approximate the integral by the trapezoidal rule. Let $h = \pi/N$ and let $(x_i,y_j) = (ih,jh)$. Then

$$f(x_i,y_j) + h\sum_{k=0}^{i} s_{ik}f(y_j,t_k)K_{ik} + K_{ij} = 0 \qquad (2.1)$$

with $j \leq i$, is a discrete version of the Gel'fand-Levitan equation. Here s_{ij} is 1/2 for $j = 0$ and $j = i$, and 1 otherwise. To evaluate K_{ii} we must solve Eq. (2.1) for $i = 1,2,\ldots,N$. For $i = 0$ we set $K_{00} = -f(0,0)$. If we solve the linear system of equations by Gaussian elimination, then the number of operations is of order $N^4/12$. We will reduce the cost to approximately $N^3/3$. Let $S_i = \mathrm{diag}(s_{i0}, \ldots, s_{ii})$ and let F_i and K_i be the principal submatrices of order $i+1$ of $F = (f(x_i,y_j))$ and $K = (K_{ij})$. Note that the upper triangular part of K is undefined. We can then rewrite Eq. (2.1) as $(I + hF_iS_i)k_i = -f_i$ where k_i^T and f_i^T are the last rows of K_i and F_i. By multiplying $I + hF_iS_i$ from the right by S_i^{-1} we get

$$A_iz_i = -f_i \qquad (2.2)$$

where $z_i = S_ik_i$ and A_i is symmetric. We will assume that

$A = \mathrm{diag}(2,1,\ldots,1) + hF$ is positive definite. This is reasonable as the corresponding fact holds for the Gel'fand-Levitan equation. Thus $A = LDL^T$ where L is unit triangular and D is diagonal (see [8]). Since A_i is the principal submatrix of order $i+1$ of $A + e_i e_i^T$ we see that $A_i = L_i D_i L_i^T$ where L_i and D_i are the principal submatrices of L and $D + e_i e_i^T$. Here $(e_i)_j = \delta_{ij}$. To calculate K_{ii} it is not necessary to solve Eq. (2.2) completely. Rather we solve $L_i w_i = -f_i$ and set

$$K_{ii} = 2(w_i)_i/(1 + (D)_{ii})$$

The approximate potential is then computed by centered differences: $q_i = (K_{i+1,i+1} - K_{i-1,i-1})/(2h)$ for $0 \leqslant i \leqslant N$. Here $K_{-1,-1}$ and $K_{N+1,N+1}$ are determined by polynomial extrapolation using four values of K_{ii} near the boundaries. This is equivalent to using unsymmetric differences in evaluating q_0 and q_N. To find the constant H in the boundary condition at the right we rewrite the differential equation (1.1) with $\lambda = \lambda_0$ as a first order system and solve it by the trapezoidal method with initial conditions $(u_0, u_0') = (1, -a(0))$. Thus

$$\begin{bmatrix} u_{j+1} \\ u'_{j+1} \end{bmatrix} = \frac{1}{1 - (h^2/4)Q_{j+1}} \begin{bmatrix} 1 + (h^2/4)Q_j & h \\ (h/2)(Q_j + Q_{j+1}) & 1 + (h^2/4)Q_{j+1} \end{bmatrix} \begin{bmatrix} u_j \\ u'_j \end{bmatrix} \tag{2.3}$$

for $0 \leqslant j < N$ where $Q_j = q_j - \lambda_0$, and we set $H = -u_N'/u_N$.

3. Convergence

We will assume that the function $a(x)$ is known exactly. In actual calculation we use only the first few eigenvalues. The remaining ones are equal to the squared integers. Thus the potential can be considered as a perturbation of the zero potential. The error in the computed potential comes from replacing the integral in the Gel'fand-Levitan equation by a finite sum and from the numerical differentiation. We will show that our method is second order accurate. The proof is based on asymptotic expansions and uses the stability of the Gel'fand-Levitan equation. By using the Euler-Maclaurin summation formula, we see that

$$f(x_i,y_j) + h\sum_{k=0}^{i} s_{ik} f(y_j,t_k)K(x_i,t_k) + K(x_i,y_j)$$

$$= \frac{h^2}{12}\left(f(y_j,t)K(x_i,t)\right)'\Big|_{t=0}^{x_i} - \frac{x_i}{720}h^4(fK)^{IV} \tag{3.1}$$

Thus $K(x_i,y_j)$ satisfies the discrete Gel'fand-Levitan equation except for terms of order $O(h^2)$. Here we assume that f and K are four times continuously differentiable. We will now show that there exists a function $E(x,y)$ such that $K + h^2E$ satisfies the discrete Gel'fand-Levitan equation except for terms of order $O(h^4)$. Let E be the solution of the one parameter family of integral equations

$$E(x,y) + \int_0^x f(y,t)E(x,t)dt = -\frac{1}{12}\left(f(y,t)K(x,t)\right)'\Big|_{t=0}^{x} .$$

Here $y \leqslant x$ and x is the parameter. This equation has the same structure as the Gel'fand-Levitan equation, and we assume that its solution is twice continuously differentiable. Note that $E(0,0) = 0$. By approximating the integral by the trapezoidal rule, we see that

$$E(x_i,y_j) + h\Sigma_{k=0}^{i} s_{ik} f(y_j,t_k)E(x_i,t_k)$$

$$= -\frac{1}{12}\left(f(y_j,t)K(x_i,t)\right)'\Big|_{t=0}^{x_i} + \frac{x_i}{12}h^2(fE)''$$

By multiplying both sides of this equation by h^2 and combining the result with Eq. (3.1), we conclude that $K+h^2E$ satisfies the discrete Gel'fand-Levitan equation, except for terms of order $O(h^4)$. Let $r_{ij} = K_{ij} - (K + h^2E)(x_i,y_j)$. Since Eq. (2.1) is linear, we find

$$(I + hF_iS_i)r_i = \frac{x_ih^4}{12}\left[-\frac{1}{60}(fK)^{IV} + (fE)''\right]$$

where r_i^T is the last row of the principal submatrix of order $i+1$ of (r_{ij}). If the inverse of $I + hF_iS_i$ is bounded by $1/\delta$ then the last component of r_i satisfies

$$|K_{ii} - (K+h^2E)(x_i,x_i)| \leqslant \frac{x_ih^4}{12\delta}\left[\frac{1}{60}\max_{t,y\leqslant x}|(fK)^{IV}| + \max_{t,y\leqslant x}|(fE)''|\right].$$

Thus we have found the first term in the asymptotic expansion of the diagonal elements of K. Note that $\|hF_iS_i\|_\infty \leq 1 - \delta$ if $|a| \leq (1-\delta)/\pi$. This happens if the eigenvalues and the normalizing constants are only slightly perturbed. The potential is computed by applying the centered difference formula to K_{ii}. By using Taylor's formula with remainder we get

$$\left|q_i - q(x_i) - h^2\left(\frac{1}{6}K''' + E'\right)\Big|_{x=x_i}\right|$$

$$\leq h^3\left[\frac{1}{24}\max|K^{IV}| + \frac{4}{3}\max|E''| + \frac{2\pi}{3\delta}\left(\frac{1}{60}\max|(fK)^{IV}| + \max|(fE)''|\right)\right].$$

This estimate holds for $0 \leq i \leq N$. The errors at the boundaries have been estimated by writing q_0 and q_N as unsymmetric differences of K_{ii}. It follows from the last equation that the second divided differences of q_i are bcunded and converge to q'' as h tends to zero. Finally we consider the evaluation of H. Since the trapezoidal method is second order accurate we conclude that the solution of Eq. (2.3) is second order accurate. The error in H is therefore of order $O(h^2)$.

4. Example of ill-conditioning

Our method works well if the Gel'fand-Levitan equation is well-conditioned. This is the case if the eigenvalues and the normalizing constants are only slightly perturbed. For example, if $|a(x)| \leq \delta/\pi$ then all A_i have a condition number less than $(2 + 2\delta)/(1 - 2\delta)$. We will now show that the Gel'fand-Levitan equation can be arbitrarily ill-

conditioned. This happens if two eigenvalues get very close or if a normalizing constant becomes either very large or very small. We will present the phenomenon for the discrete problem and will only perturb the lowest eigenvalue λ_0 and the corresponding normalizing constant ρ_0. We consider A_N with $N \geq 3$ as heuristic considerations indicate that this is the worst case.

Throughout this section we assume that $0 < \lambda_0 < 1$. This implies that the matrix F has rank 2 and that

$$hF = uu^T - vv^T$$

where the j'th elements of u and v are $\sqrt{h/\rho_0}\cos(\sqrt{\lambda_0}\,x_j)$ and $\sqrt{h/\pi}$. Let $U = (u,v)$, $V = (u,-v)$ and $D = S_N^{-1}$. Then $hF = UV^T$. To find the condition number for A_N we must estimate the eigenvalues of $D + UV^T$. Since $D = \mathrm{diag}(2,1,\ldots,1,2)$ we expect that some of the eigenvalues are close to 1 and 2. A direct calculation shows that 1 is an eigenvalue of A_N with multiplicity N-3. Here we have used that $\begin{pmatrix} u_0 v_0 \\ u_N v_N \end{pmatrix}$ is nonsingular and that the vectors $(u_1,\ldots,u_{N-1})$ and $(v_1,\ldots,v_{N-1})$ are linearly independent. The same technique shows that 2 cannot be an eigenvalue of A_N. To find the remaining 4 eigenvalues we observe that if $(D+UV^T)x = \lambda x$ and $\lambda \neq 1,2$ then $(I + V^T(D - \lambda)^{-1}U)V^Tx = 0$. The determinant of $I + V^T(D - \lambda)^{-1}U$ is

$$\begin{vmatrix} 1 + \dfrac{u_0^2 + u_N^2}{2-\lambda} + \dfrac{u_1^2 + \ldots + u_{N-1}^2}{1-\lambda} & \dfrac{u_0v_0 + u_Nv_N}{2-\lambda} + \dfrac{u_1v_1 + \ldots + u_{N-1}v_{N-1}}{1-\lambda} \\ -(1,2)\text{ element} & 1 - \dfrac{v_0^2 + v_N^2}{2-\lambda} - \dfrac{v_1^2 + \ldots + v_{N-1}^2}{1-\lambda} \end{vmatrix}$$

$$= \begin{vmatrix} 1 + \dfrac{a}{2-\lambda} + \dfrac{A}{1-\lambda} & \dfrac{c}{2-\lambda} + \dfrac{C}{1-\lambda} \\ -(1,2)\text{ element} & 1 - \dfrac{b}{2-\lambda} - \dfrac{B}{1-\lambda} \end{vmatrix} \qquad (4.1)$$

We can now prove that zero cannot be an eigenvalue of A_N. By using elementary trigonometric identities we see that

$$a/2 + A = \frac{\pi}{2\rho_0}\left[1 + E\left(2\pi\sqrt{\lambda_0}\right)\frac{\cos\left(h\sqrt{\lambda_0}\right)}{E\left(h\sqrt{\lambda_0}\right)}\right]$$

$$b/2 + B = 1$$

$$c/2 + C = \sqrt{\frac{\pi}{\rho_0}}\; E\left(\pi\sqrt{\lambda_0}\right)\frac{\cos\left(h/2\sqrt{\lambda_0}\right)}{E\left(h/2\sqrt{\lambda_0}\right)}$$

where $E(x) = \sin(x)/x$. Thus A_N is singular iff $c/2 + C$ vanishes. This cannot happen for $0 \leq \lambda_0 < 1$, but occurs for $\lambda_0 = 1$. By evaluating the determinant (4.1), multiplying the result by $(2 - \lambda)^2(1 - \lambda)^2$ and reordering the terms, we see that if $\lambda \neq 1,2$ then λ is an eigenvalue of A_N iff it is a root of the polynomial

$$(2-\lambda)^2\left[(1-\lambda)^2+(A-B)(1-\lambda)+C^2-AB\right]$$

$$+\left[(a-b)(2-\lambda)(1-\lambda)^2+(c^2-ab)(1-\lambda)^2+(2cC-aB-bA)(2-\lambda)(1-\lambda)\right].$$

Since A_N is real, the roots are real. We observe now that a, b and c tend to zero as h tends to zero, while the limits of A, B and C are finite. Since the roots depend continuously on the coefficients, at least two roots are close to 2. The remaining roots are determined approximately by the equation

$$\lambda^2 - (1 + A)\lambda + C^2 = 0 \qquad (4.2)$$

where A, 1 and C are the limits of A, B and C as h tends to zero. If we had attacked the Gel'fand-Levitan equation directly, we would have arrived at exactly the same equation. Eq. (4.2) has two real roots as $1 + A \geqslant 2C$. If C is small then the first root is small, whereas the second root is either of moderate size or large depending on the size of A. To cast this into quantitative terms we set $\lambda_0 = 1 - \varepsilon$ and find that the condition number of A_N is of order $8\rho_0/(\pi\varepsilon^2)$ in the first case, while it is of order $\pi/(\rho_0\varepsilon^2)$ in the second.

5. Calculation of f(x,y)

The evaluation of the function f(x,y) at the meshpoints can be reduced considerably by observing that the values $a(x_i + y_j)$ and $a(x_i - y_j)$ are included in the sequence a(ih) for $0 \leqslant i \leqslant 2N$. We will now discuss a case in which a finite number of eigenvalues and all

the normalizing constants are perturbed. This occurs if we insist that the potential is symmetric around the middle of the interval and perturb $\lambda_0,\ldots,\lambda_m$. It can be shown that if the potential is an even function around $\pi/2$ and $h = H$ then $q(x)$ and h are uniquely determined by $\lambda_0,\lambda_1,\ldots$ (see [5]). If the perturbations of the eigenvalues are not too large, say $|\lambda_j - j^2| < 1/3$ then the normalizing constants are determined by

$$\rho_j = \pi \prod_{\substack{i=0 \\ i\neq j}}^{m} \frac{\lambda_j - \lambda_i}{\lambda_j - i^2} \frac{\sqrt{\lambda_j}}{\sqrt{\lambda_j} + j} E\left((\sqrt{\lambda_j} - j)\pi\right), \quad 0 \leq j \leq m$$

$$= \frac{\pi}{2} \prod_{i=0}^{m} \frac{j^2 - \lambda_i}{j^2 - i^2}, \qquad m < j \qquad (5.1)$$

where $E(x) = \sin(x)/x$ (see [5]). For small x, $E(x)$ should be calculated using power series. This gives a stable computation of ρ_j. If large perturbations are considered then the normalizing constants should be derived directly from the definition $\rho_j = |\omega'(\lambda_j)|$ where

$$\omega(\lambda) = \prod_{i=0}^{m} \frac{\lambda - \lambda_i}{\lambda - i^2} \sqrt{\lambda} \sin(\pi\sqrt{\lambda}) .$$

We turn now to the calculation of $a(x)$ which we rewrite as

$$a(x) = \sum_{m+1}^{\infty} \left(\frac{1}{\rho_j} - \frac{1}{\pi/2}\right) \cos jx + \sum_{j=1}^{m} \frac{\cos\sqrt{\lambda_j}\, x}{\rho_j} - \frac{\cos jx}{\pi/2} + \frac{\cos\sqrt{\lambda_0}\, x}{\rho_0} - \frac{1}{\pi} . \qquad (5.2)$$

This series converges absolutely and uniformly, but slowly. To speed up the convergence we expand ρ_j^{-1} as a power series in j^{-2}. It follows from (5.1) that $\rho_j^{-1} = (2/\pi)(1 + A/j^2 + R_j)$ where $A = \Sigma_0^m(\lambda_i - i^2)$ and $R_j = O(j^{-4})$. By inserting this result in Eq. (5.2), reordering the terms and using that the Fourier series $\Sigma j^{-2}\cos jx$ can be summed explicitly, we obtain

$$a(x) = \frac{2}{\pi} A\left(x^2 - 2\pi x + \frac{2\pi^2}{3}\right)\frac{1}{4} + \Sigma_{j=1}^{m}\left[\frac{\cos\sqrt{\lambda_j}\,x}{\rho_j} - \frac{2}{\pi}\left(1 + \frac{A}{j^2}\right)\cos jx\right]$$

$$+ \frac{2}{\pi}\Sigma_{j=m+1}^{\infty} R_j \cos jx + \frac{\cos\sqrt{\lambda_j}\,x}{\rho_0} - \frac{1}{\pi} . \qquad (5.3)$$

The infinite sum converges quickly and the values of R_j can be computed from

$$R_j = \Pi_{i=0}^{m} \frac{j^2 - i^2}{j^2 - \lambda_i} - 1 - \frac{A}{j^2} .$$

One might argue that if only a finite number of eigenvalues are given then one should use the same number of normalizing constants. Numerical experiments show that the potentials computed in this manner oscillate around the previous potentials and are unsymmetric near the boundaries (see Figure 1).

6. Numerical examples

The algorithm presented in section 2 has been tested on a number of potentials. We have concentrated on those which are symmetric around the

middle of the interval. Thus the potential and the boundary conditions are uniquely determined by the eigenvalues. Our first problem is the reconstruction of the Mathieu equation with Neumann boundary conditions. The theory assumes the potential is smooth. This is the case if only a finite number of eigenvalues are perturbed. We have also considered the reconstruction of discontinuous potentials. The numerical results are encouraging, but we have no convergence proof for this kind of potential.

Our first test problem is $q = 2 \cos 2x$ and $h = H = 0$. The eigenvalues have been carefully tabulated (see [10]). Fig. 1 shows the potential computed from the first 6 eigenvalues with 80 points in $(0,\pi]$. The function $a(x)$ was evaluated by using (5.3). The exact potential corresponding to this $a(x)$ will be symmetric around $\pi/2$ and have the tabulated eigenvalues $\lambda_0,\ldots,\lambda_5$. For $j > 5$ the eigenvalues are $\lambda_j = j^2$. The potential converges toward $2 \cos 2x$ as we increase the number of eigenvalues. The obvious way of approximating $a(x)$ is to set $\lambda_j = j^2$ and $\rho_j = \pi/2$ for $j > 5$, thus truncating the series (1.2). Fig. 1 shows that this technique gives an unsymmetric potential. The phenomenon persists even if we increase the number of eigenvalues. To test the h^2 behavior of the method we computed q and H using the mesh length $h = \pi/N$ with $N = 20$, 40 and 80 and found $H = -.179$, -0.047 and -0.012.

In our second test problem the potential is -2 if $\pi/4 < x < 3\pi/4$ and 2 otherwise. The constants in the boundary conditions are $h = H = 0$. The eigenvalues are zeroes of a nonlinear equation, which only involve elementary functions. They were found by Newton's method. Fig. 2 shows the solution (called G-L) computed from $\lambda_0,\ldots,\lambda_7$ with 80 points in $(0,\pi]$. We have compared the Gel'fand-Levitan technique to Hochstadt's

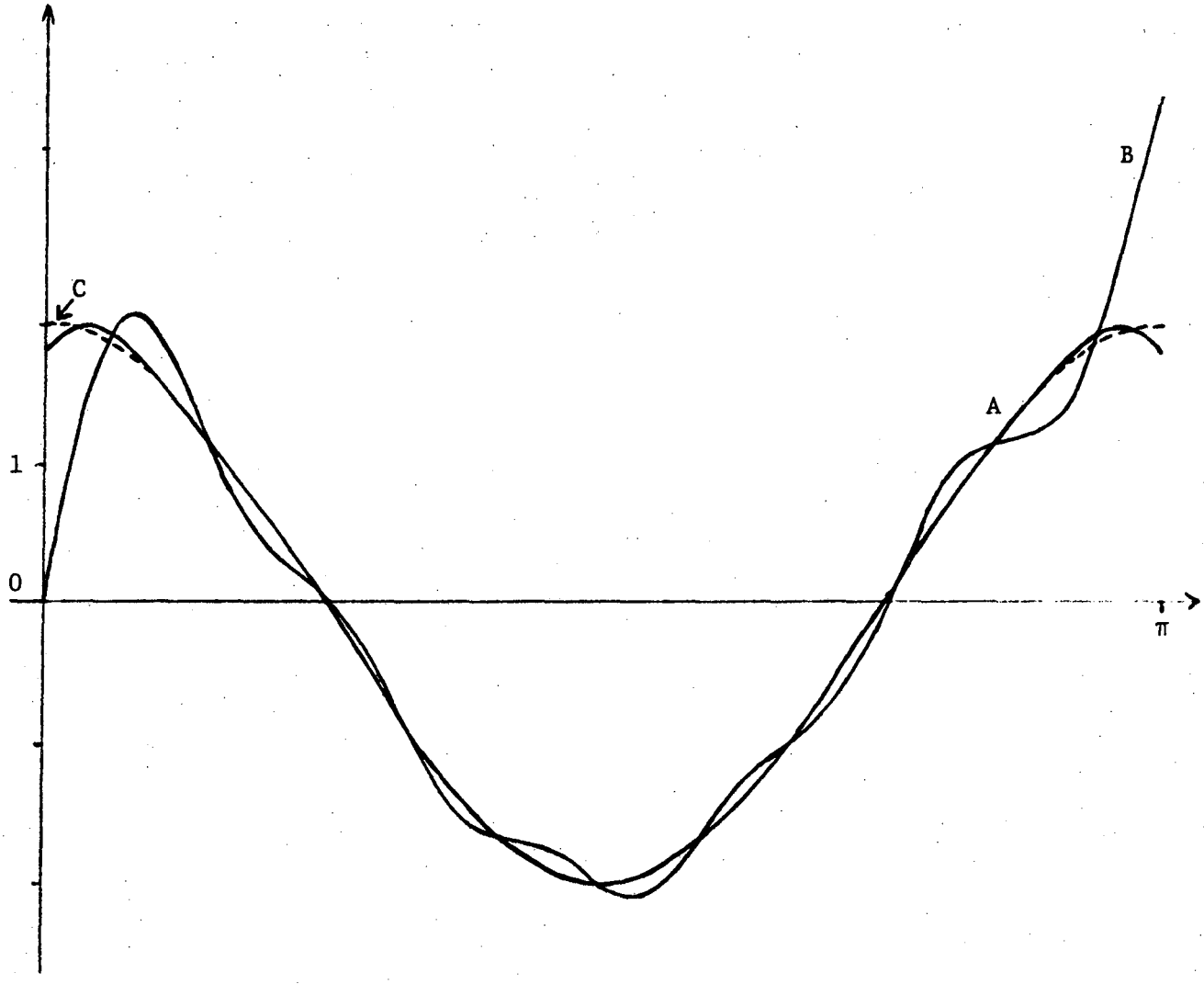

Fig. 1. Reconstruction of the Mathieu potential from 6 eigenvalues. a(x) computed from A: Eq. (5.3), B: Eq. (1.2) truncated. C = exact solution.

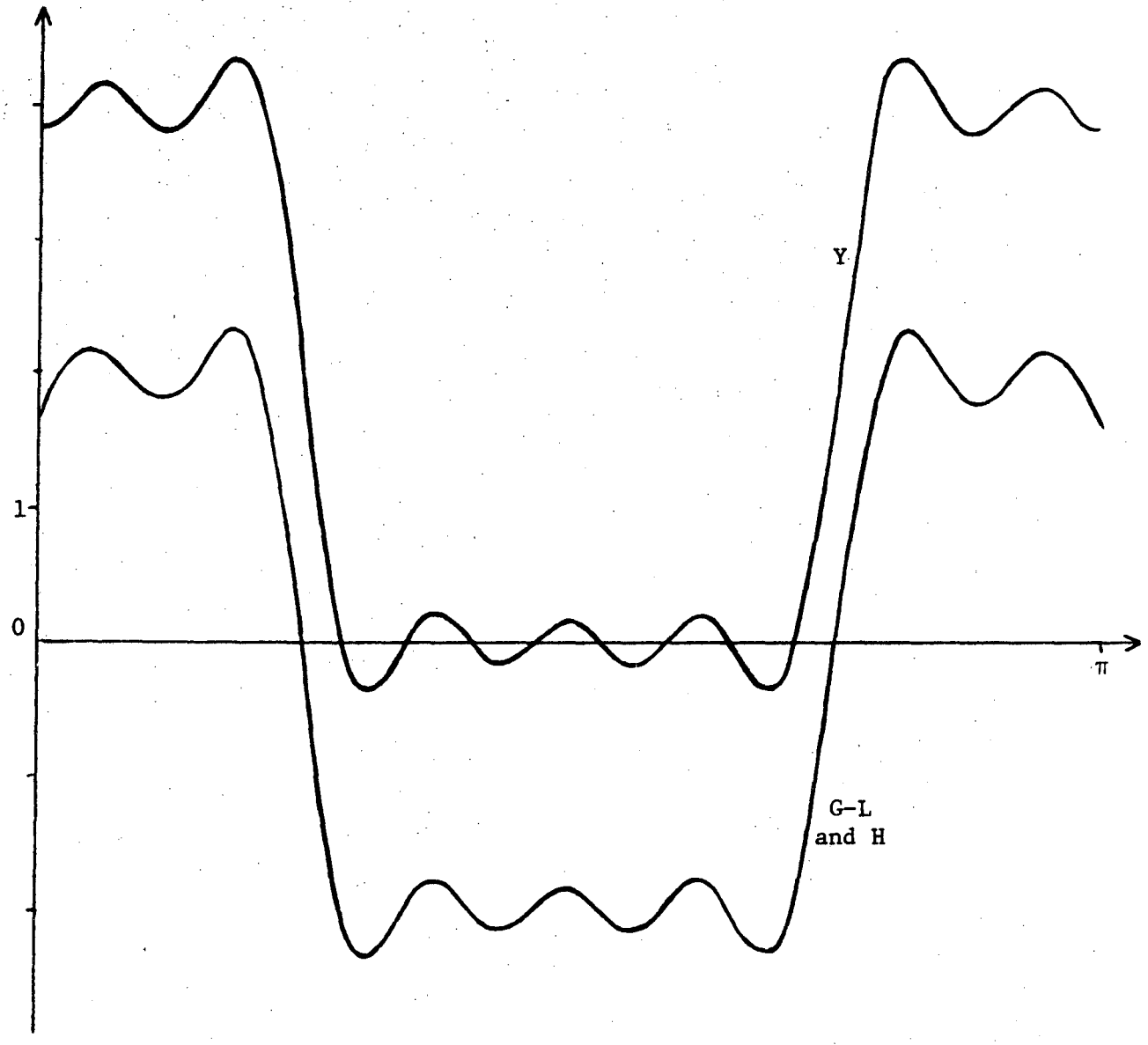

Fig. 2. Reconstruction of a discontinuous potential from 8 eigenvalues by three methods: G-L = Gel'fand-Levitan, H = Hochstadt, and Y = Yen (the latter shifted vertically by 2 for the sake of clarity).

method and to Yen's method. In Hochstadt's algorithm the inverse problem is reduced to solving two systems of ordinary differential equations, one linear and one nonlinear (see [7,5]). Yen [9] represents the potential by a Fourier series and finds the Fourier coefficients by solving a system of nonlinear equations. In Fig. 2 we display the solutions with a vertical shift of 2 for the sake of clarity. Hochstadt's solution differs from the solution of the Gel'fand-Levitan equation by at most 0.03. Yen's solution agrees with the other two in the interval $(\pi/4,3\pi/4)$. Note that Gibbs' phenomenon is present in all three cases. This shows the connection between the three methods which are completely different in theory and in implementation.

7. Open problems

In this paper we have concentrated on one particular inverse eigenvalue problem, namely finding the potential and the boundary conditions from the eigenvalues and the normalizing constants. This problem is discussed in the classical paper by Gel'fand and Levitan [3]. Here we have emphasized the reconstruction of symmetric potentials. There are a number of related inverse eigenvalue problems. For example: (a) If the boundary conditions are given and $h = H$ and the potential is symmetric, then it is uniquely determined by the reduced spectrum, i.e. all the eigenvalues except the first (see [1]). (b) The potential is uniquely determined by the eigenvalues and the normalizing constants if Eq. (1.1) has Dirichlet boundary conditions (see [3]). (c) If in addition the potential is symmetric then it is uniquely determined by all the eigenvalues (see [1]).

(d) An unsymmetric potential is uniquely determined by two interlacing spectra corresponding to two related sets of boundary conditions (for details see [7]). (e) If the boundary conditions are given, then the lowest eigenvalue in one of the spectra can be deleted (see [1]).

The numerical method presented in this paper is the simplest the author could think of and for which it is straightforward to give a convergence proof. However, many problems suggest themselves, such as establishing asymptotic expansions in general and studying the relative efficiency between Richardson's extrapolation and deferred correction. One can also consider replacing the trapezoidal rule by another quadrature formula or replacing the centered differences by higher order differences. Moreover, one need not solve the ordinary differential equation by the trapezoidal method, and different techniques can be used for interpolating the computed potentials. The question of stability of the method has not been studied sufficiently. It is straightforward to establish the stability of K_{ii}, provided the eigenvalues and the normalizing constants are only slightly perturbed, i.e. a is small. It it more difficult to obtain the general result. In practice only a finite amount of data is given. The effect of using finitely many eigenvalues when calculating symmetric potentials can be estimated, but the estimates are very pessimistic (see [5,9]). Finally, our method is unreasonably expensive if only a small number of eigenvalues and normalizing constants is perturbed. This problem should be solved directly by using the theory of degenerate integral kernels and trigonometric identities. It is expected that some of these questions will be answered in a forthcoming thesis by Jerome Coonen.

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