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Decoupling for the parabola and connections to efficient congruencing

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ABSTRACT OF THE DISSERTATION

Decoupling for the parabola and connections to efficient congruencing

by

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This thesis presents effective quantitative bounds for l^2 decoupling for the parabola. We first make effective the argument of Bourgain and Demeter in [BD17] for the case of the parabola. This allows us to improve upon the bound of $O_\varepsilon(\delta^{-\varepsilon})$ on the decoupling constant. Next, we give a new proof of l^2 decoupling for the parabola inspired from efficient congruencing. We also mention how efficient congruencing relates to decoupling for the cubic moment curve. This chapter contains the first known translation of an efficient congruencing argument into decoupling language. Finally, we discuss equivalences and monotonicity of various parabola decoupling constants and a "small ball" l^2 decoupling problem.

The dissertation of Zane Kun Li is approved.

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CHAPTER 1

Introduction

1.1 What is decoupling?

Consider a region $\Omega \subset \mathbb{R}^d$ and a partition $\{\theta\}$ of Ω. Let f_θ be defined on the Fourier side by $\widehat{f_\theta} = \widehat{f}1_\theta$. Then

$$f = \sum_\theta f_\theta.$$

Furthermore since the $\{\theta\}$ are a partition of Ω, Plancherel's theorem gives that

$$\|f\|_2 = (\sum_\theta \|f_\theta\|_2^2)^{1/2}$$

and hence to study $\|f\|_2$, it suffices to study $\|f_\theta\|_2$ for each θ. In this sense f has "decoupled" into the individual f_θ pieces.

We now ask instead of taking an L^2 norm, what happens in the case when we use instead an L^p norm. That is, let $D_p(\Omega = \bigcup \theta)$ be the best constant such that

$$\|f\|_p \leqslant D_p(\Omega = \bigcup \theta)(\sum_\theta \|f_\theta\|_p^2)^{1/2} \tag{1.1}$$

for all f with Fourier transform supported in Ω. What is the best estimate we can have for $D_p(\Omega = \bigcup \theta)$? From the triangle inequality and Cauchy-Schwarz, $D_p(\Omega = \bigcup \theta) \leqslant (\#\theta)^{1/2}$, however we seek the optimal bound of $D_p(\Omega = \bigcup \theta)$. In (1.1), we defined an $l^2 L^p$ decoupling for $\Omega = \bigcup \theta$, however we could have as well defined an $l^q L^p$ decoupling here where the l^2 sum is replaced by an l^q one. For brevity, we will often just use the phrase "l^2 decoupling" rather than "$l^2 L^p$ decoupling."

Decoupling-type inequalities were first studied by Wolff in [Wol00] who proved a sharp $l^p L^p$ decoupling theorem for the cone in $2+1$ dimensions for $p > 74$ and applied it to derive new local smoothing estimates. Wolff's work was further extended and generalized in [LW02, LP06, GS09, GS10]. Bourgain in [Bou13] was able to use induction on scales from [BG11] and multilinear restriction from [BCT06] to partially resolve l^2L^p decoupling for smooth compact hypersurfaces in $\mathbb{R}^n$ in the range $2 \leqslant p \leqslant \frac{2n}{n-1}$. Following the proof of l^2L^p decoupling for smooth compact hypersurfaces in $\mathbb{R}^n$ by Bourgain and Demeter in [BD15] for the full range $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$, decoupling inequalities for various curves and surfaces have found many applications to PDE ([Lee16, DGG17, DGL17, BBG18, BHS18, DGL18, FSW18, DZ19]), geometric measure theory ([DGO18, GIO18]), and analytic number theory ([BD16, BDG16, Bou17a, Bou17b, BDG17, Guo17, Hea17, BW18, GZ18a, GZ18b]). This list is by no means exhaustive, for a more complete list see [Pie19].

1.1.1 Decoupling for the paraboloid and moment curve

We now restriction attention to l^2 decoupling for the paraboloid [BD15] and moment curve [BDG16]. In the case of decoupling for the paraboloid, let

$$\Omega = \{(\underline{s}, |\underline{s}|^2 + t) : \underline{s} \in [0,1]^{n-1}, |t| \leqslant \delta^2\}$$

and we partition Ω into θ of the form

$$\{(\underline{s}, |\underline{s}|^2 + t) : \underline{s} \in Q, |t| \leqslant \delta^2\}$$

for frequency cube $Q \subset [0,1]^{n-1}$ of length δ. Then in [BD15], it was shown that $D_p(\Omega = \bigcup \theta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$. Note that having a δ^2 neighborhood is natural here since at this scale, the θ look like a $\delta \times \delta \times \cdots \times \delta \times \delta^2$ rectangular boxes.

For decoupling for the moment curve $t \mapsto (t, t^2, t^3, \ldots, t^n)$, let Ω be the δ^n-neighborhood of $\{(t, t^2, \ldots, t^n) : t \in [0,1]\}$ and the $\{\theta\}$ be the δ^n-neighborhood of $\{(t, t^2, \ldots, t^n) : t \in J\}$ where J runs through a partition of $[0,1]$ into intervals of length δ. Then in [BDG16], it was shown that $D_p(\Omega = \bigcup \theta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $2 \leqslant p \leqslant n(n+1)$. Similarly as the previous paragraph, a δ^n neighborhood is natural here since at this scale, the θ look like a $\delta \times \delta^2 \times \delta^3 \cdots \times \delta^n$ rectangular box. Applying this decoupling theorem to a particular f, then showed Vinogradov's mean value theorem.

We note that the ranges of $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$ and $2 \leqslant p \leqslant n(n+1)$ in decoupling for the paraboloid and moment curve, respectively, are sharp up to $\delta^{-\varepsilon}$-losses. That is, to have $D_p(\Omega = \bigcup \theta) \lesssim_\varepsilon \delta^{-\varepsilon}$ in the cases mentioned above, we need $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$ for the paraboloid and $2 \leqslant p \leqslant n(n+1)$ for the moment curve. To see the necessity of the upper bounds of $p \leqslant \frac{2(n+1)}{n-1}$ and $p \leqslant n(n+1)$, we can consider the example where $\widehat{f_\theta}(\xi)$ is a Schwartz function version of $\frac{1}{|\theta|}1_\theta(\xi)$. Finally to see the necessity of the lower bound $p \geqslant 2$ in both cases, we can consider the example where $\widehat{f_\theta}(\xi)$ is a Schwartz function version of $\frac{1}{|\theta|}1_\theta(\xi)e^{2\pi i c_\theta \cdot \xi}$ where $\{c_\theta\}$ are a collection of very far spaced points in $\mathbb{R}^n$.

1.1.2 The extension operator formulation

Instead of using the Fourier localized version of decoupling, we will instead use the extension operator formulation of decoupling. Both versions of decoupling are equivalent (see Sections 2.3 and 4.1 and Remark 5.2 of [BD15]) however the latter formulation makes it easier to see how decoupling estimates imply exponential sum estimates.

We define the extension operator formulation of decoupling for the paraboloid and moment curve. We note that we will use various different formulations in each of the chapters later, so the following two definitions are just for the reader to get a flavor of what definitions are ahead.

Let $P_\delta(Q)$ be the partition of $Q \subset \mathbb{R}^n$ into cubes of length δ. For a cube $B \subset \mathbb{R}^n$ centered at c_B of side length R, let

$$w_B(x) := (1 + \frac{|x - c_B|}{R})^{-100n}.$$

For the paraboloid, given an cube $Q \subset [0,1]^{n-1}$, let

$$(\mathcal{E}_Q g)(x) = \int_Q g(\xi)e(\xi \cdot \underline{x} + |\xi|^2 x_n)\, d\xi$$

where $e(z) := e^{2\pi i z}$ and $\underline{x} = (x_1, \dots, x_{n-1})$. Let $D_p^{parab}(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]^{n-1}} g\|_{L^p(B)} \leqslant D_p^{parab}(\delta)(\sum_{Q \in P_\delta([0,1]^{n-1})} \|\mathcal{E}_Q g\|_{L^p(w_B)}^2)^{1/2} \tag{1.2}$$

for all functions $g : [0,1]^{n-1} \to \mathbb{C}$ and cubes $B \subset \mathbb{R}^n$ of side length δ^{-2}. Then [BD15] showed that $D_p^{parab}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$.

Now we define the extension operator formulation of decoupling for the moment curve. For $J \subset [0,1]$, let

$$(\mathcal{E}_J g)(x) = \int_J g(\xi) e(\xi x_1 + \xi^2 x_2 + \cdots + \xi^n x_n)\, d\xi.$$

Let $D_p^{moment}(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]} g\|_{L^p(B)} \leqslant D_p^{moment}(\delta) (\sum_{J \in P_\delta([0,1])} \|\mathcal{E}_J g\|_{L^p(w_B)}^2)^{1/2} \tag{1.3}$$

for all functions $g : [0,1] \to \mathbb{C}$ and cubes $B \subset \mathbb{R}^n$ of side length δ^{-n}. Then [BDG16] showed that $D_p^{moment}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for $2 \leqslant p \leqslant n(n+1)$.

In all sections except Sections 3.7 and 4.4, we will be considering decoupling for the parabola. Note that the parabola is the moment curve in $\mathbb{R}^2$.

1.2 Vinogradov's mean value theorem

For integers $s, k \geqslant 1$, let $J_{s,k}(N)$ be the number of $2s$ tuples $(x_1, \ldots, x_s, y_1, \ldots, y_s) \in [1,N]^{2s}$ such that

$$\begin{aligned} x_1 + x_2 + \cdots + x_s &= y_1 + y_2 + \cdots + y_s \\ x_1^2 + x_2^2 + \cdots + x_s^2 &= y_1^2 + y_2^2 + \cdots + y_s^2 \\ &\vdots \\ x_1^k + x_2^k + \cdots + x_s^k &= y_1^k + y_2^k + \cdots + y_s^k. \end{aligned}$$

Since $1_{n=0} = \int_0^1 e(n\alpha)\, d\alpha$, we have

$$J_{s,k}(N) = \int_{[0,1]^k} |\sum_{n=1}^{N} e(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k)|^{2s}\, d\alpha. \tag{1.4}$$

If we set $x_i = y_i$ for $i = 1, 2, \ldots, s$, then $J_{s,k}(N) \geqslant N^s$. If we view the x_j and y_j as uniformly distributed in $[1,N]$, the ith power equation heuristically has a $1/N^i$ chance of being true and so this gives another $N^{2s}/\prod_{i=1}^k N^{-i} = N^{2s-k(k+1)/2}$ many solutions. This heuristic can be made rigorous as follows. Observe that for $1 \leqslant i \leqslant k$, since

$$|x_1^i + x_2^i + \cdots + x_s^i - y_1^i - y_2^i - \cdots - y_s^i| \leqslant 2sN^i.$$

Then

$$N^{2s} \lesssim \sum_{\substack{|h_1|\leqslant 2sX \\ \vdots \\ |h_k|\leqslant 2sX^k}} \int_{[0,1]^k} |\sum_{n=1}^{N} e(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k)|^{2s} e(-\alpha_1 h_1 - \alpha_2 h_2 - \cdots - \alpha_k h_k)\, d\alpha.$$

Applying the triangle inequality then shows that $J_{s,k}(N) \gtrsim_{s,k} N^{2s-\frac{k(k+1)}{2}}$. Thus we have obtained as a lower bound that

$$J_{s,k}(N) \gtrsim_{s,k} N^s + N^{2s-\frac{k(k+1)}{2}}.$$

In 1935, Vinogradov [Vin35] was motivated by applications to Waring's problem and the Riemann zeta function to study the mean value (1.4). The main conjecture in Vinogradov's mean value methods was that the lower bound on $J_{s,k}(N)$ is essentially an upper bound. That is,

$$J_{s,k}(N) \lesssim_{s,k,\varepsilon} N^{\varepsilon}(N^s + N^{2s-\frac{k(k+1)}{2}}) \tag{1.5}$$

or equivalently

$$\int_{[0,1]^k} |\sum_{n=1}^{N} e(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k)|^{2s}\, d\alpha \lesssim_{s,k,\varepsilon} N^{\varepsilon}(N^s + N^{2s-\frac{k(k+1)}{2}}). \tag{1.6}$$

From Hölder's inequality it suffices to just consider the critical case when $2s = k(k+1)$ in which case (1.6) reduces to showing

$$\int_{[0,1]^k} |\sum_{n=1}^{N} e(\alpha_1 n + \alpha_2 n^2 + \cdots + \alpha_k n^k)|^{k(k+1)}\, d\alpha \lesssim_{k,\varepsilon} N^{\frac{k(k+1)}{2}+\varepsilon}.$$

A change of variables and using periodicity shows that this is equivalent to showing that

$$\int_{[0,N^k]^k} |\sum_{n=1}^{N} e(\alpha_1 \frac{n}{N} + \alpha_2 (\frac{n}{N})^2 + \cdots + \alpha_k (\frac{n}{N})^k)|^{k(k+1)} \lesssim_{k,\varepsilon} N^{k^2+\frac{k(k+1)}{2}+\varepsilon}.$$

But this follows from l^2 decoupling for the moment curve (1.3) with the choice: $g(\xi) = \sum_{j=1}^{N} 1_{\xi=j/N}$, $p = k(k+1)$, and $\delta = 1/N$.

The critical case when $k = 2$ is classical. Wooley developed over a series of papers [Woo12, Woo13, Woo15, Woo17] the theory of efficient congruencing for Vinogradov's mean value theorem eventually proving in [Woo17] that (1.5) is true for all $1 \leqslant s \leqslant \frac{1}{2}k(k+1) - \frac{1}{3}k + o(k)$. Additionally in 2014 he was able to prove the critical $k = 3$ case ([Woo16], with a simplified approach by Heath-Brown in [Hea15]). In 2015, Bourgain-Demeter-Guth [BDG16] proved the sharp l^2 decoupling of the moment curve which then resolved Vinogradov's mean value conjecture for all $k \geqslant 2$. In 2017, Wooley [Woo19] then modified his efficient congruencing approach to also work for all $k \geqslant 2$. We refer the reader to [Pie19] for a more detailed summary of the history, background, and motivation of both efficient congruencing and decoupling methods.

Determining the dependence on ε for the implied constant in $J_{k(k+1)/2,k}(N) \lesssim_\varepsilon N^{k(k+1)/2+\varepsilon}$ is essential to applications of Vinogradov's mean value theorem to number theoretic results such as the growth of the zeta function in the critical strip, the zero free region, and zero density estimates [For02, Hea17]. See also [Hea17] and the MathOverflow question [Lew15] for applications of an effective Bourgain-Demeter-Guth result. One key point is that it is important to work out the dependence on the dimension n. The proof of decoupling for the moment curve in n dimensions relies on decoupling for the moment curve in $(n-1)$ dimensions. We then need to first study decoupling for $\xi \mapsto (\xi, \xi^2)$, in other words (2.4) with $n = 2$. This motivates why we study decoupling for the parabola in detail in this thesis.

Similarities between the efficient congruencing [Woo19] and decoupling [BDG16] methods such as the reliance on translation-dilation invariance for efficient congruencing and parabolic rescaling for decoupling have been observed (see Section 8.5 of [Pie19]). However, no precise dictionary between the two methods has been written down. Chapter 3 is the first to write down an efficient congruencing argument in decoupling language and makes precise how these two methods compare in the special case of a parabola. There is ongoing work joint with Shaoming Guo and Po-Lam Yung dealing with interpreting more complicated efficient congruencing arguments such as those found in [Hea15] and [Woo19].

1.3 Summary of the results

We now summarize all results in this thesis. We will let $D_p(\delta)$ be as in (1.2) with $n = 2$ (that is the decoupling constant for the parabola). Chapter 2 deals with obtaining explicit estimates in the decoupling constant for the parabola. By following the argument of [BD17], in Theorem 2.1.1, we show that

$$D_p(\delta) \lesssim \begin{cases} \exp(O((\log \frac{1}{\delta})^{1-c_p})) & \text{if } 2 \leqslant p < 6 \\ \exp(O(\frac{\log \frac{1}{\delta}}{\log\log \frac{1}{\delta}} \log\log\log \frac{1}{\delta})) & \text{if } p = 6 \end{cases}$$

where c_p is a small constant increasing to 1 as p increases to 6. We make all implied constants explicit and we carefully deal with various smoothed versions of 1_B that show up in the argument.

Chapter 3 was inspired from reading [Pie19, Section 4.3] and is the first concrete interpretation of an efficient congruencing proof into a decoupling language. The proof of l^2 decoupling for the parabola is boiled down the four basic steps: parabolic rescaling, bilinearization, ball inflation, and Hölder. Using our explicit estimates from Chapter 2, the argument we give in this chapter obtains that

$$D_6(\delta) \lesssim \exp(O(\frac{\log \frac{1}{\delta}}{\log\log \frac{1}{\delta}})).$$

This reproves

$$\| \sum_{|n|\leqslant N} a_n e^{2\pi i(nx+n^2t)} \|_{L^6_{x,t}(\mathbb{T}^2)} \lesssim \exp(O(\frac{\log N}{\log\log N}))(\sum_{|n|\leqslant N} |a_n|^2)^{1/2} \tag{1.7}$$

without using any number theory. Bourgain showed (1.7) in Proposition 2.36 of [Bou93] using the divisor bound. It is unknown whether the $\exp(O(\frac{\log N}{\log\log N}))$ can be improved. We also give three proofs of $D_6(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$, one that looks like an efficient congruencing proof (Section 3.2), a proof using language more familiar to decoupling (Sections 3.3 and 3.4) that includes a simplified ball inflation lemma, and finally a proof that looks more similar to that done by Bourgain-Demeter in [BD15, BD17] (Section 3.5). Finally, in Section 3.7, we outline work in progress with Shaoming Guo and Po-Lam Yung dealing with interpreting efficient congruencing as in [Hea15] into the decoupling language.

In our final chapter, we tie up some loose ends about the equivalence of various parabola decoupling constants (Section 4.1). Various equivalences of parabola decoupling constants were first dealt with in Section 2.3 to deal with issues arising from parabolic rescaling (Section 2.4). However all the decoupling constants in Section 2.3 were spatially localized (that is, have a $L^p(B)$ or $L^p(w_B)$) while in Section 4.1, we introduce some decoupling constants that are not spatially localized. This section complements the remark made in [BD15, Remark 5.2]. In Section 4.2, we give an immediate application of this equivalence and show that all eight parabola decoupling constants we define throughout this thesis (listed on Page 143) are equivalent and almost monotonic. Next we then given an elementary direct proof of l^2L^4 decoupling for the parabola in Section 4.3. Finally in Section 4.4, we discuss a "small ball" l^2 decoupling problem whose solution was first communicated to the author by Hong Wang.

CHAPTER 2

Effective l^2 decoupling for the parabola

2.1 Introduction

In [BD15] and later with a more streamlined proof [BD17], Bourgain and Demeter prove that the decoupling constant associated to the paraboloid $\{(\xi_1, \xi_2, \ldots, \xi_{n-1}, \xi_1^2 + \cdots + \xi_{n-1}^2) : \xi_i \in [0,1]\}$ is $O_{n,\varepsilon}(\delta^{-\varepsilon})$ for $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$. In [BDG16], Bourgain, Demeter, and Guth prove that the decoupling constant associated to the moment curve $\{(\xi, \xi^2, \ldots, \xi^n) : \xi \in [0,1]\}$ is $O_{n,\varepsilon}(\delta^{-\varepsilon})$ for $2 \leqslant p \leqslant n(n+1)$ which resolved Vinogradov's mean value conjecture. Both the moment curve and the paraboloid are the same when $n = 2$. It is this case we study and make effective.

For each interval $J \subset [0,1]$ and $g : [0,1] \to \mathbb{C}$, let

$$(\mathcal{E}_J g)(x) := \int_J g(\xi) e(\xi x_1 + \xi^2 x_2)\, d\xi$$

where here $e(z) = e^{2\pi i z}$. Note that $\mathcal{E}_{[0,1]} g$ is the extension operator for the parabola $\{(\xi, \xi^2) : \xi \in [0,1]\}$. For an integer $E \geqslant 1$ and a square $B = B(c_B, R) \subset \mathbb{R}^2$ centered at $c_B = (c_{B1}, c_{B2})$ of side length R, let

$$w_{B,E}(x) := (1 + \frac{|x - c_B|}{R})^{-E}.$$

If I is an interval in $[0,1]$ and $\delta \in (0,1)$, let $P_\delta(I)$ be the partition of I into $|I|/\delta$ many intervals of length δ. Note that when writing $P_\delta(I)$, we assume $|I|/\delta \in \mathbb{N}$. For $\delta \in \mathbb{N}^{-2}$, $2 \leqslant p < \infty$, and $E \geqslant 1$, let $D_{p,E}(\delta)$ be the smallest constant such that

$$\|\mathcal{E}_{[0,1]} g\|_{L^p(B)} \leqslant D_{p,E}(\delta) (\sum_{J \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2} \tag{2.1}$$

for all (axis-parallel) squares $B \subset \mathbb{R}^2$ of side length δ^{-1} and all functions $g : [0,1] \to \mathbb{C}$. Since $1_B \leqslant 2^E w_{B,E}$, the trivial bound for $D_{p,E}(\delta)$ is $2^{E/p}\delta^{-1/4}$ which follows from the triangle inequality and Cauchy-Schwarz. We will call $D_{p,E}(\delta)$ a (local) decoupling constant associated to the parabola $\{(\xi, \xi^2) : \xi \in [0,1]\}$. Note that $D_{p,E}(\delta)$ is essentially the same size as $\mathrm{Dec}_2(\delta, p, E)$ in [BD17] (a consequence of Proposition 2.2.11).

By making effective the arguments in [BD17], we have the following improvement over $D_{p,E}(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$.

Theorem 2.1.1. *Let* $E \geqslant 100$ *and* $0 < \delta < 2^{-64E^{15E}}$ *with* $\delta \in \mathbb{N}^{-2}$.

(i) *If* $2 \leqslant p \leqslant 4$, *then*

$$D_{p,E}(\delta) \leqslant \exp(E^{6E}(\log \frac{1}{\delta})^{2/3}).$$

(ii) *If* $4 < p < 6$, *then*

$$D_{p,E}(\delta) \leqslant \exp(E^{6E}(\log \frac{1}{\delta})^{\frac{2}{3}+\frac{1}{3}\log_2(\frac{p-2}{2})}).$$

(iii) *If* $p = 6$, *then*

$$D_{6,E}(\delta) \leqslant \exp(E^{6E}\frac{\log \frac{1}{\delta}}{\log\log \frac{1}{\delta}} \log\log\log \frac{1}{\delta}).$$

Using the trivial bound for $\delta > 2^{-64E^{15E}}$, one can obtain an upper bound on $D_{p,E}(\delta)$ that is valid for all $\delta \in \mathbb{N}^{-2}$.

In the proof of decoupling for the paraboloid or the moment curve in n dimensions, one crucial input is a decoupling in $(n-1)$ dimensions. This is most easily seen by the reliance on a Bourgain-Guth iteration to show the equivalence between linear and multilinear decoupling constants. In the case of the moment curve, this also makes an additional appearance in a step called lower dimensional decoupling (Lemma 8.2 of [BDG16]) since various sections of the moment curve look lower dimensional at certain scales. Thus ultimately we are reduced to first studying explicit decoupling in $n = 2$ dimensions. Because of this reduction of dimension argument, the arguments of [BD17, BDG16] should give an upper bound on the decoupling constant that is worse than those stated in Theorem 2.1.1.

While the argument in this chapter is similar to [BD17], we highlight some key features. One major feature is that we carefully work with the various weight functions that show up in the argument and obtain estimates with explicit constants. Section 2.2 develops all the estimates needed about the weight function $w_{B,E}$. The most crucial observation is that $w_{B(0,R),E} * w_{B(0,R'),E} \lesssim_E R'^2 w_{B(0,R),E}$ for $0 < R' \leqslant R$ (Lemma 2.2.1). The calculations in Section 2.2 can be easily generalized to n dimensions. A careful study of the weight $w_{B,E}$ reveals that the decoupling constant with weight $w_{B,E}$ does not behave too well under parabolic rescaling, see Lemma 2.2.18, Remark 2.2.19, and the proof of Proposition 2.4.1. Essentially this is because $w_{B,E}$ weights all directions evenly and so it is well-adapted for squares and circles but not rectangles and ellipses. To accommodate this, we introduce a second weight

$$\widetilde{w}_{B,E}(x) := w_{B,E}(x)(1 + \frac{|x_2 - c_{B2}|}{R})^{-E} \tag{2.2}$$

and let $\widetilde{D}_{p,E}(\delta)$ be defined similarly as in (2.1) but with $w_{B,E}$ replaced with $\widetilde{w}_{B,E}$. We will then need that $D_{p,E}(\delta) \sim_E \widetilde{D}_{p,E}(\delta)$ which is the topic of Section 2.3. Once we have this, we then recover almost multiplicativity of $D_{p,E}(\delta)$ in Section 2.4 and other applications of parabolic rescaling. This also introduces some slight changes compared to [BD17], namely our multilinear decoupling constant in Section 2.5 is defined with weight $\widetilde{w}_{B,E}$ rather than $w_{B,E}$ and in our iteration, A_p uses weight $\widetilde{w}_{B,E}$ rather than $w_{B,E}$. The ball inflation inequality of [BD17] is made effective in Section 2.6. We have chosen to keep track of the dependence on E since estimates for the decoupling constant in higher dimensions for a specific E may depend on an estimate for the decoupling constant at a lower dimension with a different E (see for example, Theorems 5.1 and 8.4 of [BD17]).

Another key feature is that we do not ignore integrality constraints about partitioning intervals into an integer number of smaller intervals. Tracing all the integrality constraints on the parameters in the argument, the iteration in Sections 2.7 and 2.8 gives a good upper bound for the linear decoupling constant along a lacunary sequence of scales (Section 2.9). Using almost multiplicativity of the linear decoupling constant (Proposition 2.4.1) and the trivial bound, we can upgrade this to be a good upper bound on all scales. This is done in Section 2.10. Finally optimizing in Section 2.11 completes the proof of Theorem 2.1.1.

2.2 Weight functions and consequences

2.2.1 The weights w_B and $\widetilde{w}_B$

As defined in Section 2.1, we recall that

$$w_B(x) := (1 + \frac{|x - c_B|}{R})^{-E}$$

and

$$\widetilde{w}_B(x) := w_B(x)(1 + \frac{|x_2 - c_{B2}|}{R})^{-E}.$$

If w is a weight function for B, let

$$\|f\|_{L^p_{\#}(w)} := (\frac{1}{|B|}\int_{\mathbb{R}^2} |f(x)|^p w(x)\, dx)^{1/p}.$$

We will make use of the following two inequalities that are immediate applications of Hölder's inequality: If $1/p = 1/q + 1/r$, then

$$\|fg\|_{L^p(w_{B,E})} \leqslant \|f\|_{L^q(w_{B,E})}\|g\|_{L^r(w_{B,E})}$$

and if $q > p$,

$$\|f\|_{L^p_{\#}(w_{B,E})} \leqslant \|f\|_{L^q_{\#}(w_{B,E})}. \tag{2.3}$$

The above two inequalities also hold with $w_{B,E}$ replaced with $\widetilde{w}_{B,E}$. When B is a square centered at the origin, w_B and $\widetilde{w}_B$ obey the following two important self-convolution estimates.

Lemma 2.2.1. *Let $E \geqslant 10$. For $0 < R' \leqslant R$,*

$$w_{B(0,R),E} * w_{B(0,R'),E} \leqslant 4^E R'^2 w_{B(0,R),E}. \tag{2.4}$$

We also have

$$R^2 w_{B(0,R),E} \leqslant 3^E 1_{B(0,R)} * w_{B(0,R),E}. \tag{2.5}$$

The same inequalities with the same constants hold true when $w_{B(0,R),E}$ is replaced with $\widetilde{w}_{B(0,R),E}$.

Proof. We first prove (2.4). We would like to give an upper bound for the expression

$$\frac{1}{R'^2}\int_{\mathbb{R}^2}(1+\frac{|x-y|}{R})^{-E}(1+\frac{|y|}{R'})^{-E}(1+\frac{|x|}{R})^{E}\,dy$$

depending only on E. A change of variables in y and rescaling x shows that it suffices to give an upper bound for

$$\int_{\mathbb{R}^2}(1+|x-\frac{R'}{R}y|)^{-E}(1+|y|)^{-E}(1+|x|)^{E}\,dy \tag{2.6}$$

depending only on E. If $|x|\leqslant 1$, then (2.6) is

$$\leqslant 2^E\int_{\mathbb{R}^2}(1+|y|)^{-E}\,dy\leqslant 2^E.$$

If $|x|>1$, then we split (2.6) into

$$(\int_{|x-\frac{R'}{R}y|\leqslant\frac{|x|}{2}}+\int_{|x-\frac{R'}{R}y|>\frac{|x|}{2}})(1+|x-\frac{R'}{R}y|)^{-E}(1+|y|)^{-E}(1+|x|)^{E}\,dy. \tag{2.7}$$

In the case of the first integral in (2.7), $(R'/R)|y|\geqslant|x|-|x-(R'/R)y|\geqslant|x|/2$ and hence

$$\begin{aligned}&\int_{|x-\frac{R'}{R}y|\leqslant\frac{|x|}{2}}(1+|x-\frac{R'}{R}y|)^{-E}(1+|y|)^{-E}(1+|x|)^{E}\,dy\\&\qquad\leqslant(\frac{(1+|x|)^E}{(1+(R/R')|x|/2)^E}\int_{\mathbb{R}^2}(1+|x-\frac{R'}{R}y|)^{-E}\,dy\leqslant(4R'/R)^E(R/R')^2\leqslant 4^E.\end{aligned}$$

In the case of the second integral in (2.7),

$$\begin{aligned}&\int_{|x-\frac{R'}{R}y|>\frac{|x|}{2}}(1+|x-\frac{R'}{R}y|)^{-E}(1+|y|)^{-E}(1+|x|)^{E}\,dy\\&\qquad\leqslant(\frac{1+|x|}{1+|x|/2})^E\int_{\mathbb{R}^2}(1+|y|)^{-E}\,dy\leqslant 2^E.\end{aligned}$$

This then proves (2.4).

To prove (2.5) it suffices to give a lower bound for

$$\frac{1}{R^2}\int_{B(0,R)}(1+\frac{|x-y|}{R})^{-E}(1+\frac{|x|}{R})^{E}\,dy$$

which depends only on E. As before, rescaling x and a change of variables in y gives that it suffices to give a lower bound independent of x for

$$\int_{B(0,1)}(\frac{1+|x|}{1+|x-y|})^E\,dy\geqslant(\frac{1+|x|}{2+|x|})^E\geqslant 2^{-E}.$$

Thus we have shown that $\frac{1}{R^2}(1_{B(0,R)} * w_{B(0,R),E}) \geqslant 2^{-E} w_{B(0,R),E}$ which shows (2.5).

We now prove the analogues for $\widetilde{w}_{B(0,R),E}$. We first prove the analogue of (2.4). We would like to give an upper bound for the expression

$$\frac{1}{R'^2}\int_{\mathbb{R}^2}(1+\frac{|x-y|}{R})^{-E}(1+\frac{|x_2-y_2|}{R})^{-E}(1+\frac{|y|}{R'})^{-E} \times (1+\frac{|y_2|}{R'})^{-E}(1+\frac{|x|}{R})^{E}(1+\frac{|x_2|}{R})^{E}\,dy.$$

A change of variables in y and rescaling x shows it suffices to bound

$$\int_{\mathbb{R}^2}(1+|x-\frac{R'}{R}y|)^{-E}(1+|y|)^{-E}(1+|x|)^{E} \times (1+|x_2-\frac{R'}{R}y_2|)^{-E}(1+|y_2|)^{-E}(1+|x_2|)^{E}\,dy. \tag{2.8}$$

By the triangle inequality,

$$(1+|x_2-\frac{R'}{R}y_2|)^{-E}(1+|y_2|)^{-E}(1+|x_2|)^{E} \leqslant \left(\frac{1+(R'/R)|y_2|}{1+|y_2|}\right)^{E} \leqslant 1.$$

The upper bound for (2.8) then reduces to finding an upper bound for (2.6).

To prove the analogue of (2.5) for $\widetilde{w}_{B(0,R),E}$, it suffices to give a lower bound for

$$\frac{1}{R^2}\int_{B(0,R)}(1+\frac{|x-y|}{R})^{-E}(1+\frac{|x_2-y_2|}{R})^{-E}(1+\frac{|x|}{R})^{E}(1+\frac{|x_2|}{R})^{E}\,dy$$

which depends only on E. Once again, a change of variables in y and a rescaling in x show that it suffices to give a lower bound for

$$\int_{B(0,1)}(1+|x-y|)^{-E}(1+|x|)^{E}(1+|x_2-y_2|)^{-E}(1+|x_2|)^{E}\,dy. \tag{2.9}$$

Since $y \in B(0,1)$, the triangle inequality gives

$$\frac{1+|x_2|}{1+|x_2-y_2|} \geqslant \frac{1+|x_2|}{3/2+|x_2|} \geqslant \frac{2}{3}.$$

Therefore (2.9) is bounded below by

$$(2/3)^{E}\int_{B(0,1)}(\frac{1+|x|}{1+|x-y|})^{E}\,dy \geqslant (2/3)^{E}(\frac{1+|x|}{2+|x|})^{E} \geqslant 3^{-E}.$$

This then proves the analogue of (2.5) for $\widetilde{w}_{B(0,R),E}$. This completes the proof of Lemma 2.2.1. □

Remark 2.2.2. As a corollary of Lemma 2.2.1 and the observation that $1_B \lesssim_E w_{B,E}$, we have $w_{B(0,R),E} * w_{B(0,R),E} \sim_E R^2 w_{B(0,R),E}$. This is also true for $\widetilde{w}_{B(0,R),E}$.

Remark 2.2.3. Let $I = [-R/2, R/2]$ and $I' = [-R'/2, R'/2]$ with $0 < R' \leqslant R$. For $x \in \mathbb{R}$, let $w_{I,E}(x) := (1 + \frac{|x|}{R})^{-E}$ and similarly define $w_{I',E}$. The same proof as (2.4) gives that

$$w_{I,E} * w_{I',E} \leqslant 4^E R' w_{I,E}.$$

This estimate will be used extensively in the proof of Lemma 2.3.17.

Lemma 2.2.1 has an immediate corollary which serves as the continuous analogue of the localization lemma given in Lemma 4.1 of [BD17]. This will allow us to upgrade from unweighted to weighted estimates, see later in Proposition 2.2.11. The inequality below is from the proof of Theorem 5.1 in [BD17].

Corollary 2.2.4. *For $1 \leqslant p < \infty$ and $E \geqslant 10$,*

$$\|f\|^p_{L^p(w_{B(0,R),E})} \leqslant 3^E \int_{\mathbb{R}^2} \|f\|^p_{L^p_{\#}(B(y,R))} w_{B(0,R),E}(y)\, dy.$$

This corollary is also true with $w_{B(0,R),E}$ replaced with $\widetilde{w}_{B(0,R),E}$.

Proof. Lemma 2.2.1 implies that

$$\begin{aligned} \int_{\mathbb{R}^2} \|f\|^p_{L^p_{\#}(B(y,R))} w_{B(0,R),E}(y)\, dy &= \int_{\mathbb{R}^2} |f(x)|^p (\frac{1}{R^2} 1_{B(0,R)} * w_{B(0,R),E})(x)\, dx \\ &\geqslant 3^{-E} \|f\|^p_{L^p(w_{B(0,R),E})} \end{aligned}$$

which completes the proof of Corollary 2.2.4. □

We close this section by proving two lemmas about the interaction between $\widetilde{w}_B$ and rotations which will be used in the proof of Theorem 2.6.1.

Lemma 2.2.5. *Let $c_J \in [\delta/2, 1 - \delta/2]$,*

$$R_J = \frac{1}{\sqrt{1 + 4c_J^2}} \begin{pmatrix} 1 & -2|c_J| \\ 2|c_J| & 1 \end{pmatrix},$$

and θ_J *be such that* $\cos\theta_J = 1/\sqrt{1+4c_J^2}$ *and* $\sin\theta_J = 2|c_J|/\sqrt{1+4c_J^2}$. *Suppose* $|a| \leqslant 2\delta^{-1}$, *then*

$$\widetilde{w}_{B(R_J(a,0)^T,\delta^{-1})}(s) \leqslant 16^E \widetilde{w}_{B(0,\delta^{-1})}(s).$$

Proof. We want to give an upper bound for

$$\left(\frac{\delta^{-1}+|s|}{\delta^{-1}+|s-(\cos\theta_J,\sin\theta_J)a|}\right)^E\left(\frac{\delta^{-1}+|s_2|}{\delta^{-1}+|s_2-(\sin\theta_J)a|}\right)^E \tag{2.10}$$

that only depends on E. We first consider the first expression in (2.10). If $|s| < 3\delta^{-1}$, then

$$\frac{\delta^{-1}+|s|}{\delta^{-1}+|s-(\cos\theta_J,\sin\theta_J)a|} \leqslant 4.$$

If $|s| \geqslant 3\delta^{-1}$, then

$$\frac{\delta^{-1}+|s|}{\delta^{-1}+|s-(\cos\theta_J,\sin\theta_J)a|} = \left(\frac{\delta^{-1}}{|s|}+1\right)\left(\frac{\delta^{-1}}{|s|}+\frac{|s-(\cos\theta_J,\sin\theta_J)a|}{|s|}\right)^{-1}. \tag{2.11}$$

Since $|s| \geqslant 3\delta^{-1}$ and $|a| \leqslant 2\delta^{-1}$,

$$\frac{|s-(\cos\theta_J,\sin\theta_J)a|}{|s|} \geqslant 1-\frac{|a|}{|s|} \geqslant \frac{1}{3}.$$

Therefore (2.11) is $\leqslant 4$ and so the first expression in (2.10) is $\leqslant 4^E$. We next consider the second expression in (2.10). The proof is almost exactly the same. If $|s_2| \leqslant 3\delta^{-1}$,

$$\frac{\delta^{-1}+|s_2|}{\delta^{-1}+|s_2-(\sin\theta_J)a|} \leqslant 4.$$

For $|s_2| > 3\delta^{-1}$,

$$\frac{\delta^{-1}+|s_2|}{\delta^{-1}+|s_2-(\sin\theta_J)a|} = \left(\frac{\delta^{-1}}{|s_2|}+1\right)\left(\frac{\delta^{-1}}{|s_2|}+\frac{|s_2-(\sin\theta_J)a|}{|s_2|}\right)^{-1}. \tag{2.12}$$

Since $|s_2| > 3\delta^{-1}$ and $|a| \leqslant 2\delta^{-1}$,

$$\frac{|s_2-(\sin\theta_J)a|}{|s_2|} \geqslant 1-\frac{|a|}{|s_2|} \geqslant \frac{1}{3}.$$

Therefore (2.12) is $\leqslant 4$ and so the second expression in (2.10) is $\leqslant 4^E$. This completes the proof of Lemma 2.2.5. □

Lemma 2.2.6. *Let R_J be as in Lemma 2.2.5. Then*

$$(1+\frac{|(R_J^{-1}x)_1|}{\delta^{-1}})^{-2E}(1+\frac{|(R_J^{-1}x)_2|}{\delta^{-1}})^{-2E} \leqslant \widetilde{w}_{B(0,\delta^{-1}),E}. \tag{2.13}$$

Proof. Since $(1+\delta|x|) \leqslant (1+\delta|x_1|)(1+\delta|x_2|)$, the left hand side of (2.13) is

$$\leqslant (1+\frac{|R_J^{-1}x|}{\delta^{-1}})^{-2E} = (1+\frac{|x|}{\delta^{-1}})^{-2E} \leqslant \widetilde{w}_{B(0,\delta^{-1}),E}$$

where the equality is because R_J is a rotation. This completes the proof of Lemma 2.2.6. □

2.2.2 Explicit Schwartz functions

In addition to our polynomial decaying weights w_B and $\widetilde{w}_B$, we will also need to construct an explicit Schwartz function weight. More specifically, in Corollary 2.2.9, we construct a nonnegative η in $\mathbb{R}^2$ such that $1_{B(0,1)}(x) \leqslant \eta(x)$ and $\text{supp}(\widehat{\eta}) \subset B(0,1)$. Such an η will be used in the proof of reverse Hölder (Lemma 2.2.20), l^2L^2 decoupling (Lemma 2.2.21), and will also allow us to reset the "E parameter" when we prove the equivalence of local decoupling constants in Section 2.3 (in particular, Lemma 2.3.8 and Proposition 2.3.11).

We also construct an explicit smoothed indicator function which is equal to 1 on $[-1,1]$ and vanishes outside $[-3,3]$. This will be used in the proof of ball inflation (Theorem 2.6.1) and the equivalence of local decoupling constants (Lemma 2.3.10).

Existence of such Schwartz functions is easy to justify, however our goal is to obtain explicit bounds and so not only will we need to construct such functions but also need to construct them in such a way as to make it easy to compute with. Both Schwartz functions rely on the following lemma which is a small modification of Theorem 1.3.5 of [Hor90].

Lemma 2.2.7. *Let $a_0 \geqslant a_1 \geqslant \cdots$ be a positive sequence such that $a := \sum_{i\geqslant 0} a_i < \infty$. For $i \geqslant 0$, let*

$$H_i(x) := \frac{1}{a_i}1_{[-a_i/2,a_i/2]}(x)$$

and let

$$u_k(x) := (H_0 * \cdots * H_k)(x).$$

Then for $k \geqslant 2$, $u_k \in C_c^{k-1}(\mathbb{R})$ is supported in $[-a/2, a/2]$ and converges (uniformly) to a function $u \in C_c^\infty(\mathbb{R})$ as $k \to \infty$ which is also supported in $[-a/2, a/2]$. Furthermore,

$$|u^{(j)}(x)| \leqslant \frac{2^j}{a_0 a_1 \cdots a_j}$$

for $j \geqslant 0$ and

$$\widehat{u}(\xi) = \prod_{i=0}^{\infty} \operatorname{sinc}(a_i \xi)$$

where $\operatorname{sinc}(x) = (\sin \pi x)/(\pi x)$.

Proof. The proof is the same as that in Theorem 1.3.5 of [Hor90] except in this case we have

$$u_k^{(j)} = [\prod_{i=0}^{j-1} \frac{1}{a_i}(\tau_{-a_i/2} - \tau_{a_i/2})](H_j * \cdots * H_k)$$

for $j \leqslant k-1$ where $(\tau_a f)(x) = f(x-a)$ and the product is a composition of operators.

For the claim about $\widehat{u}$, note that $\widehat{H_i}(\xi) = \operatorname{sinc}(a_i \xi)$ which implies $\widehat{u_k}(\xi) = \prod_{i=0}^{k} \operatorname{sinc}(a_i \xi)$. Since $u_k \to u$ uniformly as $k \to \infty$ and since u_k and u are both supported on $[-a/2, a/2]$, $\widehat{u_k} \to \widehat{u}$ uniformly as $k \to \infty$. This completes the proof of Lemma 2.2.7. □

We use Lemma 2.2.7 to construct a function ψ on $\mathbb{R}$ such that $\psi \geqslant 1_{[-1/2,1/2]}$ and $\operatorname{supp}(\widehat{\psi}) \subset [-1/2, 1/2]$.

Lemma 2.2.8. *For $x \in \mathbb{R}$, let*

$$\psi(x) := 4(\operatorname{sinc}(\frac{x}{6}) \prod_{i=1}^{\infty} \operatorname{sinc}(\frac{x}{6i^2}))^2.$$

Then $\psi \geqslant 1_{[-1/2,1/2]}$, $\operatorname{supp}(\widehat{\psi}) \subset [-1/2, 1/2]$, and for all $x \in \mathbb{R}$ and $E \geqslant 100$,

$$|\psi(x)| \leqslant \frac{E^{6E}}{(1+|x|)^{2E}}.$$

Proof. Let u be as in Lemma 2.2.7 with $a_0 = 1$ and $a_i = 1/i^2$. Then

$$\widehat{u}(x) = \operatorname{sinc}(x) \prod_{i=1}^{\infty} \operatorname{sinc}(x/i^2)$$

and u is supported in $[-3/2, 3/2]$.

Observe that $\psi(x) = F(x)^2$ with $F(x) = 2\widehat{u}(x/6)$. Since F is even, for $x \in [-1/2, 1/2]$, $F(x) \geqslant F(1/2) \geqslant 1$. As $\psi \geqslant 0$ for all $x \in \mathbb{R}$, $\psi \geqslant 1_{[-1/2,1/2]}$. From the support of u, the Fourier transform of F is supported in $[-1/4, 1/4]$. Since $\widehat{\psi} = \widehat{F} * \widehat{F}$, $\widehat{\psi}$ is supported in $[-1/2, 1/2]$.

By the construction of u,

$$|u^{(j)}(x)| \leqslant 2^j \prod_{k=0}^{j} a_k^{-1} = 2^j \prod_{k=1}^{j} k^2 \leqslant 2^j j^{2j}.$$

The support of u and integration by parts gives that for any $j \geqslant 0$ and $x \neq 0$,

$$|\widehat{u}(x)| \leqslant \frac{1}{(2\pi|x|)^j} \|u^{(j)}\|_{L^1(\mathbb{R})} \leqslant \frac{3j^{2j}}{\pi^j |x|^j}.$$

Applying the above bound to $j = E$ shows that for $x \neq 0$, $|\widehat{u}(x)| \leqslant E^{2E}|x|^{-E}$. Then for $|x| \geqslant 1$,

$$|\psi(x)| = 4|\widehat{u}(x/6)|^2 \leqslant E^{5E}|x|^{-2E}$$

Thus if $|x| \geqslant 1$, $(1 + |x|)^{2E}|\psi(x)| \leqslant E^{6E}$. If $|x| \leqslant 1$, then explicit computation gives that $(1 + |x|)^{2E}|\psi(x)| \leqslant 4^{E+1}$. This completes the proof of Lemma 2.2.8. □

Since $B(0, 1) = [-1/2, 1/2]^2$ and $(1+|x|)(1+|x_2|) \leqslant (1+|x_1|)(1+|x_2|)^2$, we immediately have the following corollary.

Corollary 2.2.9. *Let ψ be as in Lemma 2.2.8. For $x \in \mathbb{R}^2$, let*

$$\eta(x) = \psi(x_1)\psi(x_2).$$

Then $\eta \geqslant 1_{B(0,1)}$, $\operatorname{supp}(\widehat{\eta}) \subset B(0, 1)$, and for all $x \in \mathbb{R}^2$ and $E \geqslant 100$,

$$|\eta(x)| \leqslant \frac{E^{12E}}{(1 + |x_1|)^{2E}(1 + |x_2|)^{2E}}.$$

For $B = B(c_B, R)$, define

$$\eta_B(x) := \eta\left(\frac{x - c_B}{R}\right).$$

Then for all $x \in \mathbb{R}^2$ and arbitrary $E \geqslant 100$,

$$\eta_B(x) \leqslant E^{12E}\widetilde{w}_{B,E}(x) \leqslant E^{12E}w_{B,E}(x).$$

We now construct our smoothed indicator function and estimate the size of the Fourier transform of its moments.

Lemma 2.2.10. *Let u be as in Lemma 2.2.7 with $a_0 := 1/3$ and $a_i := 1/(3i^2)$. Then*

$$\Psi(x) := (u * 1_{[-2,2]})(x)$$

is a $C_c^\infty(\mathbb{R})$ function which is equal to 1 on $[-1,1]$ and vanishes outside $[-3,3]$. For $k \geqslant 0$, $x \in \mathbb{R}$, and $E \geqslant 100$ we have

$$|\int_{\mathbb{R}} t^k \Psi(t) e^{2\pi i t x}\, dt| \leqslant \frac{6^k E^{5E}}{(1+|x|)^{2E}}. \tag{2.14}$$

Proof. From Lemma 2.2.7, u is supported in $[-1,1]$. Since $u \geqslant 0$, $\|u\|_{L^1} = \widehat{u}(0) = 1$. Then

$$\Psi(x) = \int_{[x-2,x+2]\cap[-1,1]} u(s)\, ds = \begin{cases} 1 & \text{if } x \in [-1,1] \\ 0 & \text{if } x \notin [-3,3]. \end{cases}$$

To prove (2.14), we first prove that for $k \geqslant 0$,

$$|\partial^{2E}(x^k \Psi(x))| \leqslant 6^{2E+k} E^{4E} \tag{2.15}$$

where $\partial^E = d^E/dx^E$. From Lemma 2.2.7, for $j \geqslant 0$, $|u^{(j)}(x)| \leqslant 3(2^j)\prod_{i=1}^j 3i^2 = 3(6^j)(j!)^2$. Thus for $j \geqslant 0$,

$$|\Psi^{(j)}(x)| = |(u^{(j)} * 1_{[-2,2]})(x)| \leqslant 12(6^j)(j!)^2.$$

First suppose $2E \leqslant k$. Then since Ψ is supported on $[-3,3]$,

$$\begin{aligned} |\partial^{2E}(x^k \Psi(x))| &= |\sum_{j=0}^{2E} \binom{2E}{j} \partial^j(x^k) \Psi^{(2E-j)}(x)| \\ &\leqslant \sum_{j=0}^{2E} \binom{2E}{j} \frac{k!}{(k-j)!} 3^{k-j} 12(6^{2E-j})(2E-j)!^2 \\ &\leqslant 12(6^{2E}3^k)(2E!)^2 \sum_{j=0}^{2E} \binom{k}{j} \leqslant 12(6^{2E+k})(2E!)^2. \end{aligned}$$

Next suppose $k < 2E$. Then similarly,

$$|\partial^{2E}(x^k \Psi(x))| \leqslant \sum_{j=0}^{k} \binom{2E}{j} \frac{k!}{(k-j)!} 3^{k-j} 12(6^{2E-j})(2E-j)!^2 \leqslant 12(6^{2E+k})(2E!)^2.$$

Since $E \geqslant 100$, $12(2E!)^2 \leqslant E^{4E}$, and so when combined with the above implies

$$|\partial^{2E}(x^k\Psi(x))| \leqslant 6^{2E+k}E^{4E}$$

which proves (2.15).

We now prove (2.14). Integration by parts and (2.15) give that for $x \neq 0$,

$$|\int_{\mathbb{R}} t^k\Psi(t)e^{2\pi itx}\,dt| \leqslant \frac{6}{(2\pi|x|)^{2E}}\|\partial^{2E}(t^k\Psi(t))\|_{L^\infty} \leqslant \frac{6^kE^{4E}}{|x|^{2E}}.$$

Thus for $|x| \geqslant 1$,

$$(1+|x|)^{2E}|\int_{\mathbb{R}} t^k\Psi(t)e^{2\pi itx}\,dt| \leqslant 2^{2E}6^kE^{4E} \leqslant 6^kE^{5E}.$$

Observe that

$$\int_{\mathbb{R}}|t^k\Psi(t)|\,dt \leqslant 3^k\|\Psi\|_{L^1} = 4(3^k)$$

where the last equality we have used that $u \geqslant 0$ and $\|u\|_{L^1} = 1$. Then for $|x| < 1$,

$$(1+|x|)^{2E}|\int_{\mathbb{R}} t^k\Psi(t)e^{2\pi itx}\,dt| \leqslant 4^{E+1}3^k.$$

This completes the proof of Lemma 2.2.10. □

2.2.3 Immediate applications

Corollary 2.2.4 allows us to upgrade from estimates in $L^p(B)$ and $L^p(\eta_B)$ to estimates in $L^p(w_B)$ and $L^p(\widetilde{w}_B)$. We have the following proposition which contains all three different scenarios we will need to upgrade from an unweighted estimate to a weighted estimate.

Proposition 2.2.11. *Let $I \subset [0,1]$ and $\mathcal{P}$ be a disjoint partition of I.*

(a) Suppose for some $2 \leqslant p < \infty$, we have

$$\|\mathcal{E}_Ig\|_{L^p(B)} \leqslant C(\sum_{J\in\mathcal{P}}\|\mathcal{E}_Jg\|^2_{L^p(w_{B,E})})^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B of side length R. Then for each $E \geqslant 10$, we have

$$\|\mathcal{E}_Ig\|_{L^p(w_{B,E})} \leqslant 12^{E/p}C(\sum_{J\in\mathcal{P}}\|\mathcal{E}_Jg\|^2_{L^p(w_{B,E})})^{1/2} \tag{2.16}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B of side length R.

(*b*) *Suppose we have*

$$\|\mathcal{E}_I g\|_{L^2(B)} \leqslant C(\sum_{J\in\mathcal{P}} \|\mathcal{E}_J g\|^2_{L^2(\eta_B^2)})^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ *and all squares* B *of side length* R. *Then for each* $E \geqslant 100$, *we have*

$$\|\mathcal{E}_I g\|_{L^2(w_{B,E})} \leqslant 12^{E/2} E^{12E} C(\sum_{J\in\mathcal{P}} \|\mathcal{E}_J g\|^2_{L^2(w_{B,E})})^{1/2} \tag{2.17}$$

for all $g : [0,1] \to \mathbb{C}$ *and all squares* B *of side length* R.

(*c*) *Suppose for some* $1 \leqslant p < q < \infty$, *we have*

$$\|\mathcal{E}_I g\|_{L^q_\#(B)} \leqslant C\|\mathcal{E}_I g\|_{L^p_\#(\eta_B^p)}$$

for all $g : [0,1] \to \mathbb{C}$ *and all squares* B *of side length* R. *Then for each* $E \geqslant 100$, *we have*

$$\|\mathcal{E}_I g\|_{L^q_\#(w_{B,E})} \leqslant 12^{E/q} E^{12E} C\|\mathcal{E}_I g\|_{L^p_\#(w_{B,Ep/q})} \tag{2.18}$$

for all $g : [0,1] \to \mathbb{C}$ *and all squares* B *of side length* R.

The same results are also true with $w_{B,E}$ *replaced with* $\widetilde{w}_{B,E}$.

Proof. We first prove (a). Since for $a \in \mathbb{R}^2$, $(\mathcal{E}_J g)(x+a) = (\mathcal{E}_J h)(x)$ where $h(\xi) = g(\xi)e(a_1\xi + a_2\xi^2)$, a change of variables shows that it suffices to prove (2.16) in the case when B is centered at the origin. Corollary 2.2.4 implies that

$$\begin{aligned}
\|\mathcal{E}_I g\|^p_{L^p(w_{B,E})} &\leqslant 3^E \int_{\mathbb{R}^2} \|\mathcal{E}_I g\|^p_{L^p_\#(B(y,R))} w_{B,E}(y)\, dy \\
&\leqslant 3^E R^{-2} C^p \int_{\mathbb{R}^2} (\sum_{J\in\mathcal{P}} \|\mathcal{E}_J g\|^2_{L^p(w_{B(y,R),E})})^{p/2} w_{B,E}(y)\, dy \\
&= 3^E R^{-2} C^p \|\|\mathcal{E}_J g\|_{L^p(w_{B(y,R),E})}\|^p_{L^p_y(w_{B,E}) l^2_J}.
\end{aligned}$$

Since $p \geqslant 2$, we can interchange the $L^p_y(w_{B,E})$ and l^2_J norms and the above is

$$\begin{aligned}
&\leqslant 3^E R^{-2} C^p \|\|\mathcal{E}_J g\|_{L^p(w_{B(y,R),E})}\|^p_{l^2_J L^p_y(w_{B,E})} \\
&= 3^E R^{-2} C^p \bigg(\sum_{J\in\mathcal{P}} (\int_{\mathbb{R}^2} \|\mathcal{E}_J g\|^p_{L^p(w_{B(y,R),E})} w_{B,E}(y)\, dy)^{2/p} \bigg)^{p/2}.
\end{aligned} \tag{2.19}$$

Since B is assumed to be centered at the origin,

$$\int_{\mathbb{R}^2} \|\mathcal{E}_J g\|^p_{L^p(w_{B(y,R),E})} w_{B,E}(y)\,dy = \|\mathcal{E}_J g\|^p_{L^p(w_{B,E}*w_{B,E})} \leqslant 4^E R^2 \|\mathcal{E}_J g\|^p_{L^p(w_{B,E})}$$

where the inequality is an application of Lemma 2.2.1. Inserting this into (2.19) gives that

$$\|\mathcal{E}_I g\|^p_{L^p(w_{B,E})} \leqslant 12^E C^p (\sum_{J\in\mathcal{P}} \|\mathcal{E}_J g\|^2_{L^p(w_{B,E})})^{p/2}.$$

Taking $1/p$ powers of both sides completes the proof of (2.16).

We next prove (b). Once again it suffices to prove (2.17) in the case when B is centered at the origin. Corollary 2.2.4 implies that

$$\begin{aligned}\|\mathcal{E}_I g\|^2_{L^2(w_B)} &\leqslant 3^E \int_{\mathbb{R}^2} \|\mathcal{E}_I g\|^2_{L^2_{\#}(B(y,R))} w_B(y)\,dy \\ &= 3^E R^{-2} C^2 \sum_{J\in\mathcal{P}} \int_{\mathbb{R}^2} \|\mathcal{E}_J g\|^2_{L^2(\eta^2_{B(y,R)})} w_B(y)\,dy \\ &= 3^E R^{-2} C^2 \sum_{J\in\mathcal{P}} \|\mathcal{E}_J g\|^2_{L^2(\eta_B^2 * w_B)} \end{aligned} \tag{2.20}$$

By Corollary 2.2.9 and Lemma 2.2.1,

$$\eta_B^2 * w_B \leqslant E^{24E} w_{B,2E} * w_{B,E} \leqslant E^{24E} 4^E R^2 w_{B,E}$$

and hence (2.20) is

$$\leqslant E^{24E} 12^E C^2 \sum_{J'\in P_{1/R}(J)} \|\mathcal{E}_{J'} g\|^2_{L^2(w_B)}.$$

Taking $1/2$ powers of both sides completes the proof of (2.17).

We finally prove (c). Again it suffices to prove (2.18) in the case when B is centered at the origin. Corollary 2.2.4 implies that

$$\begin{aligned}\|\mathcal{E}_I g\|^q_{L^q(w_{B,E})} &\leqslant 3^E \int_{\mathbb{R}^2} \|\mathcal{E}_I g\|^q_{L^q_{\#}(B(y,R))} w_{B,E}(y)\,dy \\ &\leqslant 3^E C^q R^{-2q/p} \int_{\mathbb{R}^2} \|\mathcal{E}_I g\|^q_{L^p(\eta^p_{B(y,R)})} w_{B,E}(y)\,dy \\ &= 3^E C^q R^{-2q/p} \||\mathcal{E}_I g(s)|\eta_{B(y,R)}(s)\|^q_{L^q_y(w_{B,E})L^p_s}.\end{aligned}$$

Since $q > p$, we can interchange the norms and the above is

$$\begin{aligned} &\leqslant 3^E C^q R^{-2q/p} \| |\mathcal{E}_I g| \eta_{B(y,R)} \|^q_{L^p_s L^q_y(w_{B,E})} \\ &= 3^E C^q R^{-2q/p} \Big(\int_{\mathbb{R}^2} |\mathcal{E}_I g(s)|^p (\eta^q_B * w_{B,E})(s)^{p/q} \, ds \Big)^{q/p} \end{aligned} \tag{2.21}$$

Corollary 2.2.9 and Lemma 2.2.1 give that

$$\eta^q_B * w_{B,E} \leqslant E^{12Eq}(w_{B,Eq} * w_{B,E}) \leqslant E^{12Eq} 4^E R^2 w_{B,E}.$$

Inserting this into (2.21) shows that

$$\|\mathcal{E}_I g\|^q_{L^q(w_{B,E})} \leqslant 12^E E^{12Eq} C^q R^{2-2q/p} \|\mathcal{E}_I g\|^q_{L^p(w_{B,Ep/q})}$$

Changing L^q and L^p into $L^q_{\#}$ and $L^p_{\#}$, respectively, removes the factor of $R^{2-2q/p}$. Taking $1/q$ powers of both sides then completes the proof of (2.18).

Since the same estimates hold for $\widetilde{w}_{B,E}$ in Lemma 2.2.1, Corollary 2.2.4, and Corollary 2.2.9, the above proof also shows that the proposition also holds with every instance of $w_{B,E}$ replaced with $\widetilde{w}_{B,E}$. This completes the proof of Proposition 2.2.11. □

Remark 2.2.12. Note that a change of variables as in the beginning of the proof of Proposition 2.2.11 shows that knowing

$$\|\mathcal{E}_I g\|_{L^p(B(0,R))} \leqslant C\Big(\sum_{J \in \mathcal{P}} \|\mathcal{E}_J g\|^2_{L^p(w_{B(0,R),E})}\Big)^{1/2} \tag{2.22}$$

for all $g : [0,1] \to \mathbb{C}$ implies that

$$\|\mathcal{E}_I g\|_{L^p(B)} \leqslant C\Big(\sum_{J \in \mathcal{P}} \|\mathcal{E}_J g\|^2_{L^p(w_{B,E})}\Big)^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B of side length R. Therefore often to check the hypotheses of Proposition 2.2.11 we will just prove (2.22) instead.

Remark 2.2.13. Corollary 2.2.4 is not the only way to convert unweighted estimates to weighted estimates. Another approach is to prove an unweighted estimate where B is replaced by $2^n B$ for all $n \geqslant 0$ and then use that $w_{B,E} \sim \sum_{n \geqslant 0} 2^{-nE} 1_{2^n B}$ to conclude the weighted estimate.

Proposition 2.2.14. *Let B be a square of side length R and let $\mathcal{B}$ be a disjoint partition of B into squares Δ with side length $R' < R$. Then for $E \geqslant 10$,*

$$\sum_{\Delta\in\mathcal{B}} w_{\Delta,E} \leqslant 48^E w_{B,E}. \tag{2.23}$$

This inequality remains true with $w_{\Delta,E}$ and $w_{B,E}$ replaced with $\widetilde{w}_{\Delta,E}$ and $\widetilde{w}_{B,E}$.

Proof. It suffices to prove the case when B is centered at the origin. Since $\mathcal{B}$ is a disjoint partition of B,

$$\sum_{\Delta\in\mathcal{B}} 1_\Delta \leqslant 1_B.$$

Therefore

$$\sum_{\Delta\in\mathcal{B}} 1_\Delta * w_{B(0,R'),E} \leqslant 1_B * w_{B(0,R'),E}.$$

Lemma 2.2.1 gives that

$$3^{-E}R'^2 \sum_{\Delta\in\mathcal{B}} w_{\Delta,E} \leqslant \sum_{\Delta\in\mathcal{B}} 1_\Delta * w_{B(0,R'),E}$$

and

$$1_B * w_{B(0,R'),E} \leqslant 8^E R'^2 w_{B,E}$$

where here we have also used $1_B \leqslant 2^E w_{B,E}$. Rearranging then proves (2.23). Since $1_B \leqslant 4^E\widetilde{w}_{B,E}$, the same proof then proves (2.23) with $w_{\Delta,E}$ and $w_{B,E}$ replaced with $\widetilde{w}_{\Delta,E}$ and $\widetilde{w}_{B,E}$, respectively. This completes the proof of Proposition 2.2.14. □

Remark 2.2.15. The only property we really need in Proposition 2.2.14 is that $\sum_{\Delta\in\mathcal{B}} 1_\Delta \leqslant C1_B$ for some absolute constant C. In particular, the same proof will work with finitely overlapping covers and when $R/R' \notin \mathbb{N}$.

We illustrate two lemmas regarding how the weights w_B and $\widetilde{w}_B$ and shear matrices interact. Both lemmas are similar to Proposition 2.2.14 except now there is an additional shear matrix. Lemma 2.2.16 is used in the proof of Lemma 2.3.10. This lemma is a warmup to the proof of Lemma 2.2.18. Lemma 2.2.18 is the key lemma for the application of parabolic rescaling in Propositions 2.4.1 and 2.5.2 and is why we have two separate weights w_B and $\widetilde{w}_B$.

Lemma 2.2.16. *Let* $E \geqslant 10$ *and* $S = \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right)$ *where* $|a| \leqslant 2$. *Then*

$$w_{B(0,R),E}(Sx) \leqslant 90^E w_{B(0,R),E}(x).$$

Proof. Since our weights are centered at the origin, rescaling x, it suffices to prove the case when $R = 1$. Since $|a| \leqslant 2$, $S^{-1}B(0,1) \subset B(0,3)$ and so $1_{B(0,1)}(Sx) \leqslant 1_{B(0,3)}(x)$ for all $x \in \mathbb{R}^2$. Therefore

$$1_{B(0,1)}(x) \leqslant 1_{B(0,3)}(S^{-1}x)$$

for all $x \in \mathbb{R}^2$. Convolving both sides by $w_{B(0,1),E}$ and applying Lemma 2.2.1 gives that

$$3^{-E} w_{B(0,1),E} \leqslant (1_{B(0,3)} \circ S^{-1}) * w_{B(0,1),E}.$$

Thus it remains to prove that

$$(1_{B(0,3)} \circ S^{-1}) * w_{B(0,1),E} \leqslant 30^E w_{B(0,1),E} \circ S^{-1}.$$

This is the same as showing that

$$\int_{\mathbb{R}^2} 1_{B(0,3)}(S^{-1}y)(1 + |x-y|)^{-E}\, dy \leqslant 30^E (1 + |S^{-1}x|)^{-E}. \tag{2.24}$$

If $x \in 2^5 S(B(0,1))$, then $|S^{-1}x| \leqslant 2^4\sqrt{2}$ and so

$$\int_{\mathbb{R}^2} 1_{B(0,3)}(S^{-1}y)(1 + |x-y|)^{-E}\, dy \leqslant 1 \leqslant 24^E (1 + |S^{-1}x|)^{-E}$$

which proves (2.24) in this case. Next let $x \in 2^{n+1}S(B(0,1)) \backslash 2^n S(B(0,1))$ for some $n \geqslant 5$. Then

$$(1 + |S^{-1}x|)^{-E} \geqslant (1 + \sqrt{2} \cdot 2^n)^{-E} \geqslant (2 \cdot 2^n)^{-E}.$$

Thus in this case, to prove (2.24) it suffices to show that

$$\int_{\mathbb{R}^2} 1_{B(0,3)}(S^{-1}y)(1 + |x-y|)^{-E}\, dy \leqslant 15^E 2^{-nE}. \tag{2.25}$$

We have

$$\begin{aligned} \int_{\mathbb{R}^2} 1_{B(0,3)}(S^{-1}y)(1 + |x-y|)^{-E}\, dy &= \int_{S(B(0,3))} \frac{1}{(1 + |x-y|)^E}\, dy \\ &= \int_{x - S(B(0,3))} \frac{1}{(1 + |y|)^E}\, dy. \end{aligned} \tag{2.26}$$

For $y \in x - S(B(0,3))$, write $y = Sa - Sb$ where $a \in B(0, 2^{n+1}) \backslash B(0, 2^n)$ and $b \in B(0,3)$. Since $\|S^{-1}\| \leqslant 2\|S^{-1}\|_{\max} \leqslant 4$,

$$|y| = |S(a-b)| \geqslant \|S^{-1}\|^{-1}|a-b| \geqslant \frac{1}{4}(2^{n-1} - \frac{3}{2}\sqrt{2}) \geqslant \frac{1}{10}2^n.$$

Therefore the right hand side of (2.26) is bounded above by $9(10^E)2^{-nE}$ which proves (2.25) and hence (2.24). This completes the proof of Lemma 2.2.16. □

Remark 2.2.17. The same proof also shows that $w_{B(0,R),E}(S^t x) \leqslant 90^E w_{B(0,R),E}$ since the only two properties of S we used were $S^{-1}B(0,1) \subset B(0,3)$ and $\|S^{-1}\| \leqslant 4$. These properties are satisfied if we replace S with S^t.

Lemma 2.2.18. *For $0 < \delta \leqslant \sigma < 1$ with $\sigma^{-1/2} \in \mathbb{N}$, let*

$$T = \begin{pmatrix} \sigma^{1/2} & 2a\sigma^{1/2} \\ 0 & \sigma \end{pmatrix}$$

with $0 \leqslant a \leqslant 1 - \sigma^{1/2}$ and $B = B(0, \delta^{-1})$. Then $T(B)$ is contained in a $3\sigma^{1/2}\delta^{-1} \times \sigma\delta^{-1}$ rectangle centered at the origin. Let $\mathcal{B}$ denote the partition of this rectangle into $3\sigma^{-1/2}$ many squares with side length $\sigma\delta^{-1}$. Then for $E \geqslant 100$,

$$\sum_{\Delta \in \mathcal{B}} \widetilde{w}_{\Delta,E} \leqslant 720^E w_{B,E} \circ T^{-1}. \tag{2.27}$$

Proof. The proof is similar to what we did in Proposition 2.2.14 and Lemma 2.2.16. Since B is axis-parallel and centered at the origin, $T(B)$ is a parallelogram centered at the origin with a base parallel to the x-axis and height $\sigma\delta^{-1}$. The corners of B are given by $(\pm\delta^{-1}/2, \pm\delta^{-1}/2)$ and hence the corners of $T(B)$ are given by

$$(\frac{1}{2}\sigma^{1/2}(1+2a)\delta^{-1}, \frac{1}{2}\sigma\delta^{-1}) \qquad (-\frac{1}{2}\sigma^{1/2}(1+2a)\delta^{-1}, -\frac{1}{2}\sigma\delta^{-1})$$
$$(\frac{1}{2}\sigma^{1/2}(1-2a)\delta^{-1}, -\frac{1}{2}\sigma\delta^{-1}) \qquad (-\frac{1}{2}\sigma^{1/2}(1-2a)\delta^{-1}, \frac{1}{2}\sigma\delta^{-1}).$$

Then $T(B)$ is contained in a $3\sigma^{1/2}\delta^{-1} \times \sigma\delta^{-1}$ rectangle centered at the origin.

Note that $T(B) \subset \bigcup_{\Delta \in \mathcal{B}} \Delta \subset 10T(B)$ (we actually have $\bigcup_{\Delta \in \mathcal{B}} \Delta \subset (3+2a)T(B)$, but this is not needed) and so

$$\sum_{\Delta \in \mathcal{B}} 1_{B(c_\Delta, \sigma\delta^{-1})} \leqslant 1_{B(0,10\delta^{-1})} \circ T^{-1}.$$

Convolution with $\widetilde{w}_{B(0,\sigma\delta^{-1}),E}$ gives that

$$(\sigma\delta^{-1})^2 \sum_{\Delta \in \mathcal{B}} \widetilde{w}_{\Delta,E} \leqslant 3^E (1_{B(0,10\delta^{-1})} \circ T^{-1}) * \widetilde{w}_{B(0,\sigma\delta^{-1}),E}.$$

Thus it suffices to show that

$$(\sigma\delta^{-1})^{-2}(1_{B(0,10\delta^{-1})} \circ T^{-1}) * \widetilde{w}_{B(0,\sigma\delta^{-1}),E} \leqslant 240^E w_{B(0,\delta^{-1}),E} \circ T^{-1}.$$

That is,

$$(\sigma\delta^{-1})^{-2} \int_{\mathbb{R}^2} 1_{B(0,10\delta^{-1})}(T^{-1}y)(1 + \frac{|x-y|}{\sigma\delta^{-1}})^{-E}(1 + \frac{|x_2 - y_2|}{\sigma\delta^{-1}})^{-E}\, dy$$
$$\leqslant 240^E (1 + \frac{|T^{-1}x|}{\delta^{-1}})^{-E}.$$

Rescaling x and y (by setting $X = x/(\sigma\delta^{-1})$ and $Y = y/(\sigma\delta^{-1})$) shows it suffices to prove that

$$\int_{\mathbb{R}^2} 1_{B(0,10)}(S^{-1}y)(1 + |x-y|)^{-E}(1 + |x_2 - y_2|)^{-E}\, dy \leqslant 240^E (1 + |S^{-1}x|)^{-E} \qquad (2.28)$$

for all $x \in \mathbb{R}^2$ where $S = \sigma^{-1}T = \left(\begin{smallmatrix} \sigma^{-1/2} & 2a\sigma^{-1/2} \\ 0 & 1 \end{smallmatrix}\right)$. Suppose $x \in 2^6 S(B)$. Then $|S^{-1}x| \leqslant 32\sqrt{2}$ and so

$$\int_{\mathbb{R}^2} 1_{B(0,10)}(S^{-1}y)(1 + |x-y|)^{-E}(1 + |x_2 - y_2|)^{-E}\, dy \leqslant 1 \leqslant 50^E (1 + |S^{-1}x|)^{-E}.$$

It then remains to prove (2.28) for $x \in 2^{n+1}S(B)\backslash 2^n S(B)$ for all $n \geqslant 6$.

Fix an $n \geqslant 6$. For $x \in 2^{n+1}S(B)\backslash 2^n S(B)$, $|S^{-1}x| \leqslant 2^{n+1/2}$ and so $(2^{n+1})^{-E} \leqslant (1 + |S^{-1}x|)^{-E}$. Therefore to prove (2.28) it is enough to prove

$$\int_{10S(B(0,1))} \frac{1}{|x-y|^E (1 + |x_2 - y_2|)^E}\, dy \leqslant 120^E 2^{-nE}$$

for all $x \in 2^{n+1}S(B)\backslash 2^n S(B)$. A change of variables shows that we need to prove

$$\int_{x - 10S(B(0,1))} \frac{1}{|y|^E (1 + |y_2|)^E}\, dy \leqslant 120^E 2^{-nE} \qquad (2.29)$$

for all $x \in 2^{n+1}S(B(0,1))\backslash 2^n S(B(0,1))$.

Fix an $x \in 2^{n+1}S(B(0,1))\backslash 2^n S(B(0,1))$. First suppose $|x_2| \geqslant 2^{2n/E}$. If $y \in x - 10S(B(0,1))$, then $y = Sa - Sb$ for some $a \in B(0,2^{n+1})\backslash B(0,2^n)$ and $b \in B(0,10)$. Since $\|S^{-1}\| \leqslant 2\|S^{-1}\|_{\max} \leqslant 4$, we first have

$$|y| = |S(a-b)| \geqslant \|S^{-1}\|^{-1}|a-b| \geqslant \frac{1}{4}|a-b| \geqslant \frac{1}{4}(2^{n-1} - 5\sqrt{2}) \geqslant \frac{1}{20}2^n.$$

Next, $y_2 = x_2 - (Sb)_2 = x_2 - b_2$ and $b_2 \in [-5,5]$ and so

$$\frac{1+|x_2|}{1+|y_2|} = \frac{1+|x_2|}{1+|x_2-b_2|} \leqslant 1 + |b_2| \leqslant 6.$$

Therefore

$$\int_{x-10S(B(0,1))} \frac{1}{|y|^E(1+|y_2|)^E}\,dy \leqslant \Big(\frac{6}{1+|x_2|}\Big)^E \int_{|y| \geqslant 2^n/20} \frac{1}{|y|^E}\,dy \leqslant 120^E \frac{2^{2n}}{(1+|x_2|)^E}2^{-nE}$$

and since $|x_2| \geqslant 2^{2n/E}$, we have proven (2.29) in this case.

Next, suppose $|x_2| < 2^{2n/E}$. In this case, we claim that $y \in x - 10S(B(0,1))$ satisfies $|y| \gtrsim 2^n \sigma^{-1/2}$ and so we can bound the integral trivially. By assumption, $|(S^{-1}x)_2| = |x_2| < 2^{2n/E}$. Since $S^{-1}x \in 2^{n+1}B(0,1)\backslash 2^n B(0,1)$, $|S^{-1}x| \geqslant 2^{n-1}$. Thus

$$|(S^{-1}x)_1| \geqslant 2^{n-1} - 2^{2n/E}.$$

Since $(S^{-1}x)_1 = \sigma^{1/2}x_1 - 2ax_2$, it follows that

$$|x_1| \geqslant \sigma^{-1/2}(2^{n-1} - 3 \cdot 2^{2n/E}).$$

As in the previous paragraph, write $y = x - Sb$ for some $b \in B(0,10)$. Then

$$|y| \geqslant |y_1| = |x_1| - \sigma^{-1/2}|b_1 + 2ab_2| \geqslant \sigma^{-1/2}(2^{n-1} - 3 \cdot 2^{2n/E} - 15) \geqslant \frac{1}{5}\sigma^{-1/2}2^n$$

where the last inequality we have used that $n \geqslant 6$ and $E \geqslant 100$. Thus in the case when $|x_2| < 2^{2n/E}$,

$$\int_{x-10S(B(0,1))} \frac{1}{|y|^E(1+|y_2|)^E}\,dy \leqslant (100\sigma^{-1/2})5^E\sigma^{E/2}2^{-nE} \leqslant 6^E 2^{-nE}$$

which proves (2.29) in this case. This completes the proof of Lemma 2.2.18. □

Remark 2.2.19. The $\widetilde{w}_{\Delta,E}$ on the left hand side of (2.27) was needed to make sure the E on both sides stays the same which is needed when we iterate later (for example in Lemma 2.5.2). If the $\widetilde{w}_{\Delta,E}$ is replaced with $w_{\Delta,E}$, then by the same method as the proof above, one can obtain $\sum_{\Delta\in\mathcal{B}} w_{\Delta,E} \lesssim_E w_{B,E-2}\circ T^{-1}$. In this case, some loss in E must occur since we can consider the analogue of (2.28) and (2.29) and let $a = 0$ and $x = (0, 2^{n-1})$.

2.2.4 Bernstein's inequality

Another immediate application of Proposition 2.2.11 is Bernstein's inequality (also called reverse Hölder in [BD17]). This should be compared with (2.3) at the beginning of Section 2.2. Our proof of Lemma 2.2.20 is the same as that of Corollary 4.3 of [BD17] except we make effective all the implicit constants.

Lemma 2.2.20. *Let $1 \leqslant p < q \leqslant \infty$, $E \geqslant 100$, $J \subset [0,1]$ with $\ell(J) = 1/R$ and $B \subset \mathbb{R}^2$ a square with side length $R \geqslant 1$. If $q < \infty$, then*

$$\|\mathcal{E}_J g\|_{L^q_\#(\widetilde{w}_{B,E})} \leqslant E^{23E}\|\mathcal{E}_J g\|_{L^p_\#(\widetilde{w}_{B,Ep/q})}. \tag{2.30}$$

If $q = \infty$, then

$$\sup_{x\in B}|(\mathcal{E}_J g)(x)| \leqslant E^{23E}\|\mathcal{E}_J g\|_{L^p_\#(\widetilde{w}_{B,E})}. \tag{2.31}$$

Proof. Let η be as in Corollary 2.2.9. Since $\eta_B \geqslant 1_B$,

$$\|\mathcal{E}_J g\|_{L^q(B)} \leqslant \|\eta_B\mathcal{E}_J g\|_{L^q(\mathbb{R}^2)}.$$

Let $\theta(x) = \Psi(2x_1)\Psi(2x_2)$ where Ψ is defined as in Lemma 2.2.10. Then $\theta = 1$ on $B(0,1)$ and vanishes outside $B(0,3)$. Since $\widehat{\eta_B}$ is supported on $B(0,1/R)$, the Fourier transform of $\eta_B\mathcal{E}_J g$ is supported in some square S with side length $10/R$. Then we have the following self-replicating formula

$$\eta_B\mathcal{E}_J g = (\eta_B\mathcal{E}_J g) * \check{\theta}_S.$$

Young's inequality then gives

$$\|\eta_B\mathcal{E}_J g\|_{L^q(\mathbb{R}^2)} \leqslant \|\eta_B\mathcal{E}_J g\|_{L^p(\mathbb{R}^2)}\|\check{\theta}_S\|_{L^r(\mathbb{R}^2)} = \|\check{\theta}_S\|_{L^r(\mathbb{R}^2)}\|\mathcal{E}_J g\|_{L^p(\eta_B^p)}$$

where $1/q = 1/p + 1/r - 1$ (since $q > p$, we have $r > 1$ and $\check{\theta}_S \in L^r$). Since $\check{\theta}(\xi) = (1/4)\check{\Psi}(\xi_1/2)\check{\Psi}(\xi_2/2)$, $\|\check{\theta}\|_{L^r(\mathbb{R}^2)} = 4^{1/r-1}\|\check{\Psi}\|^2_{L^r(\mathbb{R})}$, applying Lemma 2.2.10 gives that

$$\|\check{\theta}_S\|_{L^r(\mathbb{R}^2)} = (10/R)^{2-2/r}\|\check{\theta}\|_{L^r(\mathbb{R}^2)} = 25^{1/r'}R^{-2/r'}\|\check{\Psi}\|^2_{L^r(\mathbb{R})} \leqslant 25^{1/r'}R^{-2/r'}E^{10E}.$$

Therefore

$$\|\mathcal{E}_J g\|_{L^q(B)} \leqslant 25^{1/r'}E^{10E}R^{-2/r'}\|\mathcal{E}_J g\|_{L^p(\eta_B^p)} \tag{2.32}$$

for all squares $B \subset \mathbb{R}^2$ with side length R. If $q < \infty$, applying Proposition 2.2.11 and then using that $q > p \geqslant 1$ and $E \geqslant 100$ proves (2.30).

If $q = \infty$, then (2.32) and Corollary 2.2.9 implies that

$$\sup_{x \in B}|(\mathcal{E}_J g)(x)| \leqslant 25^{1/p}E^{22E}R^{-2/p}\|\mathcal{E}_J g\|_{L^p(\tilde{w}_{B,E})}.$$

Since $E \geqslant 100$, (2.31) then follows. This completes the proof of Lemma 2.2.20. □

2.2.5 l^2L^2 decoupling

We now prove l^2L^2 decoupling which will follow from almost orthogonality. This proof is the same as that of Proposition 6.1 of [BD17] except we once again make explicit all implicit constants.

Lemma 2.2.21. *Let $J \subset [0,1]$ be an interval of length $\geqslant 1/R$ such that $|J|R \in \mathbb{N}$. Then for $E \geqslant 100$ and each square $B \subset \mathbb{R}^2$ with side length R,*

$$\|\mathcal{E}_J g\|^2_{L^2(\tilde{w}_{B,E})} \leqslant E^{13E}\sum_{J' \in P_{1/R}(J)}\|\mathcal{E}_{J'} g\|^2_{L^2(\tilde{w}_{B,E})}.$$

Proof. Let η be as in Corollary 2.2.9. Since $\eta_B^2 \geqslant 1_B$,

$$\|\mathcal{E}_J g\|^2_{L^2(B)} \leqslant \|\mathcal{E}_J g\|^2_{L^2(\eta_B^2)} = \|\eta_B\mathcal{E}_J g\|^2_{L^2(\mathbb{R}^2)} = \|\sum_{J' \in P_{1/R}(J)}\eta_B\mathcal{E}_{J'} g\|^2_{L^2(\mathbb{R}^2)}.$$

Note that the Fourier transform of $\eta_B\mathcal{E}_{J'}g$ is supported in the $1/R$-neighborhood of the piece of parabola above J'. Therefore $\eta_B\mathcal{E}_{J'}g$ and $\eta_B\mathcal{E}_{J''}g$ have disjoint Fourier support if J' and

J'' are separated by $\geqslant 2$ intervals. Applying this and Plancherel gives

$$\begin{aligned}
&\|\sum_{J' \in P_{1/R}(J)} \eta_B \mathcal{E}_{J'} g\|_{L^2(\mathbb{R}^2)}^2 \\
&\qquad \leqslant \sum_{J_1' \in P_{1/R}(J)} \sum_{\substack{J_2' \in P_{1/R}(J) \\ d(c_{J_1'}, c_{J_2'}) \leqslant 2/R}} \|\eta_B \mathcal{E}_{J_1'} g\|_{L^2} \|\eta_B \mathcal{E}_{J_2'} g\|_{L^2} \\
&\qquad \leqslant (\sum_{J_1' \in P_{1/R}(J)} \|\eta_B \mathcal{E}_{J_1'} g\|_{L^2}^2)^{1/2} (\sum_{J_1' \in P_{1/R}(J)} (\sum_{\substack{J_2' \in P_{1/R}(J) \\ d(c_{J_1'}, c_{J_2'}) \leqslant 2/R}} \|\eta_B \mathcal{E}_{J_2'} g\|_{L^2})^2)^{1/2} \\
&\qquad \leqslant \sqrt{5} (\sum_{J_1' \in P_{1/R}(J)} \|\eta_B \mathcal{E}_{J_1'} g\|_{L^2}^2)^{1/2} (\sum_{J_1' \in P_{1/R}(J)} \sum_{\substack{J_2' \in P_{1/R}(J) \\ d(c_{J_1'}, c_{J_2'}) \leqslant 2/R}} \|\eta_B \mathcal{E}_{J_2'} g\|_{L^2}^2)^{1/2} \\
&\qquad \leqslant 5 \sum_{J' \in P_{1/R}(J)} \|\mathcal{E}_{J'} g\|_{L^2(\eta_B^2)}^2.
\end{aligned}$$

Thus we have shown that

$$\|\mathcal{E}_J g\|_{L^2(B)} \leqslant \sqrt{5} (\sum_{J' \in P_{1/R}(J)} \|\mathcal{E}_{J'} g\|_{L^2(\eta_B^2)}^2)^{1/2}$$

for all squares $B \subset \mathbb{R}^2$ with side length R. Applying Proposition 2.2.11 then completes the proof of Lemma 2.2.21. □

Remark 2.2.22. To modify the weights w_B and $\widetilde{w}_B$, the main properties the weights need to satisfy are Lemma 2.2.1 and Lemma 2.2.18. The other lemmas such as Lemmas 2.2.5, 2.2.6, and 2.2.16 are also desired, but these should be easy to satisfy.

2.3 Equivalence of local decoupling constants

Recall that $\widetilde{D}_{p,E}(\delta)$ is defined similarly as $D_{p,E}(\delta)$ except instead of $w_{B,E}$ we use $\widetilde{w}_{B,E}$. The main goal of this section is to prove that

$$D_{p,E}(\delta) \sim_E \widetilde{D}_{p,E}(\delta) \tag{2.33}$$

for $2 \leqslant p \leqslant 6$, $E \geqslant 100$, and $\delta \in \mathbb{N}^{-2}$. This is proven in Proposition 2.3.11. This equivalence is a consequence of a larger equivalence of a collection of local decoupling constants. This section is similar to Remark 5.2 of [BD15] and may be of independent interest since it shows that an array of slightly different local decoupling constants are essentially the same size. The restriction $p \leqslant 6$ is very mild and can be removed with a bit more care (at the cost of introducing an implied constant that depends on p). However since $2 \leqslant p \leqslant 6$ is precisely the range we need, we restrict to this range. The appearance of the weight $\widetilde{w}_B$ in parabolic rescaling (arising from Lemma 2.2.18) means that (2.33) will play an essential part of the argument (for example in Proposition 2.4.1, Lemma 2.5.2, and Lemma 2.8.11).

Let f_R denote the Fourier restriction of f to R. For each $J = [n_J\delta^{1/2}, (n_J+1)\delta^{1/2}] \in P_{\delta^{1/2}}([0,1])$, let

$$\theta_J := \{(s, L_J(s)+t) : n_J\delta^{1/2} \leqslant s \leqslant (n_J+1)\delta^{1/2}, -5\delta \leqslant t \leqslant 5\delta\}$$

where

$$L_J(s) := (2n_J+1)\delta^{1/2}s - n_J(n_J+1)\delta$$

and $0 \leqslant n_J \leqslant \delta^{-1/2}-1$. Here θ_J is a parallelogram that has height 10δ and has base parallel to the straight line connecting $(n_J\delta^{1/2}, n_J^2\delta)$ and $((n_J+1)\delta^{1/2}, (n_J+1)^2\delta)$. We note that for $\xi \in \theta_J$,

$$|\xi_2 - L_J(\xi_1)| \leqslant 5\delta \tag{2.34}$$

and

$$|L_J(\xi_1) - \xi_1^2| \leqslant \delta/4. \tag{2.35}$$

Boundedness of the Hilbert transform implies that Fourier restriction to θ_J is a bounded operator from $L^p \to L^p$ with operator norm bounded independent of J, we make this explicit with the following lemma.

Lemma 2.3.1. *For each $J \in P_{\delta^{1/2}}([0,1])$ and $2 \leqslant p < \infty$, $\|f_{\theta_J}\|_p \leqslant C_p\|f\|_p$ with $C_p := (\frac{1}{2}+\frac{1}{2}\cot(\frac{\pi}{2p}))^4$.*

Proof. Fix $J \in P_{\delta^{1/2}}([0,1])$. Let R denote the operator defined by $\widehat{Rf} = \widehat{f}1_{\theta_J}$. Let S denote the operator defined by $\widehat{Sf} = \widehat{f}1_{[0,\infty)}$. Each θ_J is the intersection of four half planes in $\mathbb{R}^2$.

Since multiplier norms are unchanged after rotation and translation,

$$\|R\|_{p\to p} \leqslant \|S\|_{p\to p}^4. \tag{2.36}$$

Note that here we have also used that the operator norm of Fourier restriction to a half plane is bounded above by $\|S\|_{p\to p}$ which follows from Fubini's Theorem. If H denotes the Hilbert transform, observe that $\widehat{f}(\xi) + i\widehat{Hf}(\xi) = 2\widehat{f}(\xi)1_{[0,\infty)}(\xi)$ almost everywhere. Since $2 \leqslant p < \infty$, $\|H\|_{p\to p} \leqslant \cot(\frac{\pi}{2p})$. Therefore

$$\|S\|_{p\to p} \leqslant \frac{1}{2} + \frac{1}{2}\cot(\frac{\pi}{2p}).$$

Inserting this into (2.36) then completes the proof of Lemma 2.3.1. □

Remark 2.3.2. One can think of θ_J as a polygonal approximation of the set $\{(s, s^2 + t) : s \in J, |t| \leqslant \delta\}$. The reason why we use θ_J instead is because Fourier restriction to the aforementioned set is not bounded in L^p for $p \neq 2$.

To prove (2.33), we introduce two more local decoupling constants and show that all four decoupling constants are equivalent.

Definition 2.3.3. *Let $\delta \in \mathbb{N}^{-2}$, $2 \leqslant p < \infty$ and $E \geqslant 1$. Let η be as in Corollary 2.2.9. Let $\overline{D}_p(\delta)$ be the smallest constant such that*

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leqslant \overline{D}_p(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_J g\|_{L^p(\eta_B)}^2)^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B with side length δ^{-1}. Let $\widehat{D}_{p,E}(\delta)$ be the smallest constant such that

$$\|f\|_{L^p(B)} \leqslant \widehat{D}_{p,E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])} \|f_{\theta_J}\|_{L^p(w_{B,E})}^2)^{1/2}$$

for all f Fourier supported in $\Theta = \bigcup_{J\in P_{\delta^{1/2}}([0,1])} \theta_J$ and all squares B with side length δ^{-1}.

From our definitions of w_B, $\widetilde{w}_B$, and η_B, observe that

$$1_B \leqslant 2^E w_{B,E}, \quad 1_B \leqslant 4^E \widetilde{w}_{B,E}, \quad 1_B \leqslant \eta_B.$$

Furthermore, note that by the triangle inequality followed by Cauchy-Schwarz, all four local decoupling constants we have defined are $\lesssim_{E,p} \delta^{-1/4}$. Taking a specific $g : [0,1] \to \mathbb{C}$ or a specific f with Fourier support in Θ and using Proposition 2.2.11 shows that $D_{p,E}(\delta)$, $\widetilde{D}_{p,E}(\delta)$, and $\widehat{D}_{p,E}(\delta)$ are $\gtrsim_{E,p} 1$. We make this precise with $\widehat{D}_{p,E}$ which is the only decoupling constant we need an explicit lower bound.

Remark 2.3.4. Another consequence of the equivalence of the four local decoupling constants is that $\overline{D}_p(\delta) \gtrsim_{E,p} 1$ but this is not immediate from the definition.

Lemma 2.3.5. *For $p \geqslant 2$ and $E \geqslant 10$, $\widehat{D}_{p,E}(\delta) \geqslant 12^{-E/p}$.*

Proof. Let $\widehat{D}'_{p,E}(\delta)$ be the smallest constant such that

$$\|f\|_{L^p(w_{B,E})} \leqslant \widehat{D}'_{p,E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])} \|f_{\theta_J}\|^2_{L^p(w_{B,E})})^{1/2}$$

for all f Fourier supported in Θ and all squares B with side length δ^{-1}. Proposition 2.2.11 implies that $\widehat{D}'_{p,E}(\delta) \leqslant 12^{E/p}\widehat{D}_{p,E}(\delta)$. From the definition,

$$\widehat{D}'_{p,E}(\delta) = \sup_{f,B} \frac{\|f\|_{L^p(w_{B,E})}}{(\sum_{J\in P_{\delta^{1/2}}([0,1])} \|f_{\theta_J}\|^2_{L^p(w_{B,E})})^{1/2}} \tag{2.37}$$

where the sup is taken over the f and B as mentioned at the beginning of this proof. Taking an f with Fourier support on $\theta_{[0,\delta^{1/2}]}$ shows that $\widehat{D}'_{p,E}(\delta) \geqslant 1$. Here note that we needed the numerator of the right hand side of (2.37) to be $L^p(w_{B,E})$ rather than $L^p(B)$. Therefore $\widehat{D}_{p,E}(\delta) \geqslant 12^{-E/p}$ which completes the proof of Lemma 2.3.5. □

Remark 2.3.6. The decoupling constants $D_{p,E}(\delta)$ and $\widetilde{D}_{p,E}(\delta)$ are useful because $w_B * w_B \sim_E R^2 w_B$ and similarly for $\widetilde{w}_B$. This allows us to use Proposition 2.2.11 to upgrade from unweighted to weighted estimates which is an important part of the argument. The same cannot be said with the Schwartz weight decoupling constant $\overline{D}_p(\delta)$ since we do not necessarily have $\eta_B * \eta_B \sim R^2\eta_B$. This useful convolution property of the w_B and $\widetilde{w}_B$ makes $D_{p,E}(\delta)$ and $\widetilde{D}_{p,E}(\delta)$ ideal for iterative parts of the argument.

On the other hand, the decoupling constants $\overline{D}_p(\delta)$ and $\widehat{D}_{p,E}(\delta)$ are more useful for Fourier type arguments since the Fourier transform of w_B and $\widetilde{w}_B$ are of sinc type and so do not work well with Fourier arguments. One corollary of the results proven in this section is that all four local decoupling constants are essentially equivalent so we can easily swap between them.

To prove (2.33) we will prove the chain of inequalities

$$D_{p,E}(\delta) \leqslant \widetilde{D}_{p,E}(\delta) \lesssim_E \overline{D}_p(\delta) \lesssim_E \widehat{D}_{p,G}(\delta) \lesssim_E D_{p,E}(\delta) \tag{2.38}$$

for $2 \leqslant p \leqslant 6$ and some $G < E$ we will make explicit in our proof.

The first two inequalities follow from that $\eta_B \lesssim_E w_B \lesssim \widetilde{w}_B$. The third inequality follows from boundedness of the Hilbert transform (Lemma 2.3.1) and the last inequality will follow from adapting the proof of Theorem 5.1 in [BD17] to our case and is the most technical.

Lemma 2.3.7. *For $E \geqslant 100$ and $2 \leqslant p < \infty$,*

$$D_{p,E}(\delta) \leqslant \widetilde{D}_{p,E}(\delta) \leqslant E^{12E/p}\overline{D}_p(\delta).$$

Proof. The first inequality follows from the observation that $\widetilde{w}_B \leqslant w_B$. The second inequality follows from Corollary 2.2.9, in particular, $\eta_B \leqslant E^{12E}\widetilde{w}_{B,E}$. This completes the proof of Lemma 2.3.7. □

As mentioned above, the third inequality in (2.38) comes from boundedness of the Hilbert transform. In particular, we need the following lemma. Because $\overline{D}_p$ does not depend on E, this lemma allows us to "reset" the E parameter in $D_{p,E}$. This is useful because going up in the E parameter of $D_{p,E}$ is easy but going down is much harder.

Lemma 2.3.8. *For $\delta \in \mathbb{N}^{-2}$, $E \geqslant 1$, and $2 \leqslant p < \infty$, we have*

$$\overline{D}_p(\delta) \leqslant (3C_p + 5 \cdot 12^{E/p})\widehat{D}_{p,E}(\delta)$$

where C_p is as defined in Lemma 2.3.1.

Proof. We first assume that $\delta \in \mathbb{N}^{-2}$ and $\delta \leqslant 1/36$. Fix arbitrary $g : [0,1] \to \mathbb{C}$ and square B with side length δ^{-1}. We can write

$$g = g1_{[0,\delta^{1/2})\cup(1-\delta^{1/2},1]} + g1_{[\delta^{1/2},1-\delta^{1/2}]} := g_1 + g_2.$$

Then

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leqslant \|\mathcal{E}_{[0,1]}g_1\|_{L^p(B)} + \|\mathcal{E}_{[0,1]}g_2\|_{L^p(B)}.$$

Using the support of g_1, the triangle inequality, $1_B \leqslant \eta_B$, and Lemma 2.3.5, we have

$$\begin{aligned}\|\mathcal{E}_{[0,1]}g_1\|_{L^p(B)} &\leqslant \|\mathcal{E}_{[0,\delta^{1/2}]}g\|_{L^p(B)} + \|\mathcal{E}_{[1-\delta^{1/2},1]}g\|_{L^p(B)}\\ &\leqslant 2\cdot 12^{E/p}\widehat{D}_{p,E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])}\|\mathcal{E}_J g\|_{L^p(\eta_B)}^2)^{1/2}. \end{aligned} \tag{2.39}$$

Since g_2 is supported in $[\delta^{1/2}, 1-\delta^{1/2}]$, the Fourier transform of $\eta_B\mathcal{E}_{[0,1]}g_2 = \eta_B\mathcal{E}_{[\delta^{1/2},1-\delta^{1/2}]}g$ is supported in a δ-neighborhood of this interval which is contained in Θ. Therefore

$$\|\eta_B\mathcal{E}_{[0,1]}g_2\|_{L^p(B)} \leqslant \widehat{D}_{p,E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])}\|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_J}\|_{L^p(w_{B,E})}^2)^{1/2}. \tag{2.40}$$

Note that since $g_2 = g1_{[\delta^{1/2},1-\delta^{1/2}]}$,

$$\begin{aligned}(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_J} &= (\eta_B\mathcal{E}_{[\delta^{1/2},1-\delta^{1/2}]}g)_{\theta_J}\\ &= \begin{cases}(\eta_B\mathcal{E}_{J_r}g)_{\theta_J} & \text{if } J=[0,\delta^{1/2}]\\ (\eta_B\mathcal{E}_J g+\eta_B\mathcal{E}_{J_r}g)_{\theta_J} & \text{if } J=[\delta^{1/2},2\delta^{1/2}]\\ (\eta_B\mathcal{E}_{J_\ell}g+\eta_B\mathcal{E}_J g+\eta_B\mathcal{E}_{J_r}g)_{\theta_J} & \text{if } J\in P_{\delta^{1/2}}([2\delta^{1/2},1-2\delta^{1/2}])\\ (\eta_B\mathcal{E}_{J_\ell}g+\eta_B\mathcal{E}_J g)_{\theta_J} & \text{if } J=[1-2\delta^{1/2},1-\delta^{1/2}]\\ (\eta_B\mathcal{E}_{J_\ell}g)_{\theta_J} & \text{if } J=[1-\delta^{1/2},1].\end{cases}\end{aligned}$$

where J_ℓ and J_r denote the intervals to the left and right of J. Lemma 2.3.1 gives that for $J\in P_{\delta^{1/2}}([2\delta^{1/2},1-2\delta^{1/2}])$,

$$\|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_J}\|_{L^p(w_{B,E})} \leqslant \sum_{J'\in\{J_\ell,J,J_r\}}\|(\eta_B\mathcal{E}_{J'}g)_{\theta_J}\|_p \leqslant C_p\sum_{J'\in\{J_\ell,J,J_r\}}\|\mathcal{E}_{J'}g\|_{L^p(\eta_B)}.$$

Similarly we have

$$\begin{aligned}\|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,E})} &\leqslant C_p\|\mathcal{E}_{[\delta^{1/2},2\delta^{1/2}]}g\|_{L^p(\eta_B)}\\ \|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_{[1-\delta^{1/2},1]}}\|_{L^p(w_{B,E})} &\leqslant C_p\|\mathcal{E}_{[1-2\delta^{1/2},1-\delta^{1/2}]}g\|_{L^p(\eta_B)}\\ \|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_{[\delta^{1/2},2\delta^{1/2}]}}\|_{L^p(w_{B,E})} &\leqslant C_p(\|\mathcal{E}_{[\delta^{1/2},2\delta^{1/2}]}g\|_{L^p(\eta_B)}+\|\mathcal{E}_{[2\delta^{1/2},3\delta^{1/2}]}g\|_{L^p(\eta_B)})\end{aligned}$$

and

$$\|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_{[1-2\delta^{1/2},1-\delta^{1/2}]}}\|_{L^p(w_{B,E})}$$
$$\leqslant C_p(\|\mathcal{E}_{[1-3\delta^{1/2},1-2\delta^{1/2}]}g\|_{L^p(\eta_B)} + \|\mathcal{E}_{[1-2\delta^{1/2},1-\delta^{1/2}]}g\|_{L^p(\eta_B)})$$

where here we have used that $\delta \leqslant 1/36$. Applying Cauchy-Schwarz and using the above four inequalities gives that

$$\sum_{J\in P_{\delta^{1/2}}([0,1])}\|(\eta_B\mathcal{E}_{[0,1]}g_2)_{\theta_J}\|^2_{L^p(w_{B,E})} \leqslant 9C_p^2\sum_{J\in P_{\delta^{1/2}}([0,1])}\|\mathcal{E}_Jg\|^2_{L^p(\eta_B)}$$

Combining this with (2.40) and $1_B \leqslant \eta_B$ gives

$$\|\mathcal{E}_{[0,1]}g_2\|_{L^p(B)} \leqslant 3C_p\widehat{D}_{p,E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])}\|\mathcal{E}_Jg\|^2_{L^p(\eta_B)})^{1/2}. \tag{2.41}$$

Combining (2.39) and (2.41) proves that

$$\overline{D}_p(\delta) \leqslant (3C_p + 2\cdot 12^{E/p})\widehat{D}_{p,E}(\delta) \tag{2.42}$$

for all $\delta \in \mathbb{N}^{-2}$ and $\delta \leqslant 1/36$.

For $\delta = 1, 1/4, 1/9, 1/16$, and $1/25$, we resort to the trivial bound. Proceeding as in the proof of (2.39) shows that for each such $\delta = 1/i^2$, $i = 1, 2, \ldots, 5$, we have

$$\overline{D}_p(\delta) \leqslant 5\cdot 12^{E/p}\widehat{D}_{p,E}(\delta).$$

Combining this with (2.42) then completes the proof of Lemma 2.3.8. □

Remark 2.3.9. The reason why we split g up into g_1 and g_2 in proof above is because $\eta_B\mathcal{E}_{[0,1]}g$ is Fourier supported in a set that is slightly bigger than Θ.

The last inequality in (2.38) is the most technical of the four inequalities. The proof is similar to that of Theorem 5.1 in [BD17] however our proof is more complicated since our definition of $\widehat{D}_{p,E}(\delta)$ uses Fourier restriction to the parallelogram θ_J (to take advantage of L^p boundedness) rather than Fourier restriction to a δ-tube of a piece of parabola. We also want explicit constants and so we will need to spend some time to extract explicit constants from taking a large number of derivatives. We state our lemma below but due to the length of its proof, we defer the proof to the end of this section.

To simplify some constants, we also restrict to the range when $2 \leqslant p \leqslant 6$ since this is the range we care about. The restriction that $p \leqslant 6$ is only used once in the proof of Lemma 2.3.10 (in particular at the end of the proof of Lemma 2.3.16) and is a very mild assumption which can be removed with a bit more care.

Lemma 2.3.10. *For $E \geqslant 10$ and $2 \leqslant p \leqslant 6$,*

$$\widehat{D}_{p,E}(\delta) \leqslant E^{60E} D_{p,2E+7}(\delta).$$

Since $w_{B,E_2} \leqslant w_{B,E_1}$ for $E_1 \leqslant E_2$, $D_{p,E_1}(\delta) \leqslant D_{p,E_2}(\delta)$ and so we can increase the E parameter at no cost. Combining Lemmas 2.3.7-2.3.10 proves the following result which shows (2.38) and hence (2.33).

Proposition 2.3.11. *For $\delta \in \mathbb{N}^{-2}$, $E \geqslant 100$, and $2 \leqslant p \leqslant 6$, we have*

$$D_{p,E}(\delta) \leqslant \widetilde{D}_{p,E}(\delta) \leqslant E^{6E}\overline{D}_p(\delta) \leqslant E^{7E}\widehat{D}_{p,G}(\delta) \leqslant E^{70E}D_{p,E}(\delta)$$

where $G = \lfloor (E-7)/2 \rfloor$.

Proof. Fix arbitrary integer $E \geqslant 100$. Using Lemma 2.3.7 and that $2 \leqslant p \leqslant 6$, we have

$$D_{p,E}(\delta) \leqslant \widetilde{D}_{p,E}(\delta) \leqslant E^{6E}\overline{D}_p(\delta).$$

Now we use Lemma 2.3.8 to reset our E. Since $E \geqslant 100$, $G > 10$. From Lemmas 2.3.8 and 2.3.10,

$$E^{6E}\overline{D}_p(\delta) \leqslant E^{7E}\widehat{D}_{p,G}(\delta) \leqslant E^{7E}G^{60G}D_{p,2G+7}(\delta)$$

where in the first inequality we have used that $C_p \leqslant 32$ for $2 \leqslant p \leqslant 6$. Increasing $2G+7$ to E bounds the above by $E^{70E}D_{p,E}(\delta)$. This completes the proof of Proposition 2.3.11. $\square$

2.3.1 Proof of Lemma 2.3.10

This proof is similar to the proof of Theorem 5.1 in [BD17]. Our goal is to show that if f is Fourier supported on $\Theta = \bigcup_{J \in P_{\delta^{1/2}}([0,1])} \theta_J$, then

$$\|f\|_{L^p(B)} \lesssim_E D_{p,2E+7}(\delta)\Big(\sum_{J \in P_{\delta^{1/2}}([0,1])} \|f_{\theta_J}\|_{L^p(w_{B,E})}^2\Big)^{1/2}$$

for all squares B with side length δ^{-1} and some implied constant that will be made explicit in our proof. It suffices to show that this is true in the case when B is centered at the origin.

Since f is Fourier supported on Θ, for $x \in B$,

$$\begin{aligned} f(x) &= \sum_{J \in P_{\delta^{1/2}}([0,1])} \int_{\theta_J} \widehat{f}(\xi) e(\xi \cdot x)\, d\xi \\ &= \sum_{J \in P_{\delta^{1/2}}([0,1])} \int_{J \times [-5\delta, 5\delta]} \widehat{f}(s, L_J(s) + t) e(sx_1 + s^2 x_2) e((L_J(s) - s^2)x_2) e(tx_2)\, ds\, dt. \end{aligned}$$

Note that here both t and $L_J(s) - s^2$ are of size $O(\delta)$ and x_2 is of size $O(\delta^{-1})$, so the contribution from $e((L_J(s) - s^2)x_2)$ and $e(tx_2)$ should be negligible. We make this rigorous. Since

$$e(tx_2) = \sum_{j \geqslant 0} \frac{(2\pi)^j}{j!} (\frac{2ix_2}{\delta^{-1}})^j (\frac{\delta^{-1} t}{2})^j$$

and

$$e((L_J(s) - s^2)x_2) = \sum_{k \geqslant 0} \frac{(2\pi)^k}{k!} (\frac{2ix_2}{\delta^{-1}})^k (\frac{\delta^{-1}(L_J(s) - s^2)}{2})^k,$$

it follows that for $x \in B$,

$$|f(x)| \leqslant \sum_{j,k \geqslant 0} \frac{(2\pi)^k (2\pi)^j}{k! j!} | \sum_{J \in P_{\delta^{1/2}}([0,1])} (\mathcal{E}_J g_{j,k})(x)|$$

where $g_{j,k} : [0,1] \to \mathbb{C}$ is defined pointwise almost everywhere piecewise on each $J \in P_{\delta^{1/2}}([0,1])$ by

$$g_{j,k}(s) = (\frac{\delta^{-1}(L_J(s) - s^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(s, L_J(s) + t) (\frac{\delta^{-1} t}{2})^j\, dt$$

for $s \in J$. Let $F := 2E + 7$. We then have

$$\|f\|_{L^p(B)} \leqslant D_{p,F}(\delta) \sum_{j,k \geqslant 0} \frac{(2\pi)^k (2\pi)^j}{k! j!} (\sum_{J \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_J g_{j,k}\|_{L^p(w_{B,F})}^2)^{1/2}. \tag{2.43}$$

It then remains to prove that

$$\|\mathcal{E}_J g_{j,k}\|_{L^p(w_{B,F})} \lesssim_E \exp(O(j) + O(k)) \|f_{\theta_J}\|_{L^p(w_{B,E})} \tag{2.44}$$

for some implied constants that will be made explicit in our proof. We first claim it suffices to only prove (2.44) when $J = [0, \delta^{1/2}]$.

Lemma 2.3.12. *Suppose we knew that*

$$\|\mathcal{E}_{[0,\delta^{1/2}]}(\frac{\delta^{-1}(\delta^{1/2}s-s^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(s,\delta^{1/2}s+t)(\frac{\delta^{-1}t}{2})^j\,dt\|_{L^p(w_{B,F})} \leqslant C\|f_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,E})} \tag{2.45}$$

for some constant C. Then

$$\|\mathcal{E}_{[n_J\delta^{1/2},(n_J+1)\delta^{1/2}]}(\frac{\delta^{-1}(L_J(s)-s^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(s,L_J(s)+t)(\frac{\delta^{-1}t}{2})^j\,dt\|_{L^p(w_{B,F})} \leqslant 90^{(E+F)/p}C\|f_{\theta_{[n_J\delta^{1/2},(n_J+1)\delta^{1/2}]}}\|_{L^p(w_{B,E})}. \tag{2.46}$$

Remark 2.3.13. Here s is a dummy variable, so $\mathcal{E}_J g(s)$ means the extension operator applied to the function $g(s)$ creating the function $(\mathcal{E}_J g)(x)$.

Proof. This proof is essentially a change of variables. The idea is to translate $\theta_{[n_J,(n_J+1)\delta^{1/2}]}$ to the origin and apply a shear matrix to turn it into $\theta_{[0,\delta^{1/2}]}$. Then apply (2.45) and finally undo the shear transformation. The weights w_B are preserved from (2.45) because of Lemma 2.2.16.

We have

$$\left(\mathcal{E}_{[n_J\delta^{1/2},(n_J+1)\delta^{1/2}]}(\frac{\delta^{-1}(L_J(s)-s^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(s,L_J(s)+t)(\frac{\delta^{-1}t}{2})^j\,dt\right)(x)$$
$$= \int_{[n_J\delta^{1/2},(n_J+1)\delta^{1/2}]} (\frac{\delta^{-1}(L_J(s)-s^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(s,L_J(s)+t)(\frac{\delta^{-1}t}{2})^j\,dt\,e(sx_1+s^2x_2)\,ds.$$

The change of variables $u = s - n_J\delta^{1/2}$ and the observation that

$$L_J(u+n_J\delta^{1/2}) - (u+n_J\delta^{1/2})^2 = \delta^{1/2}u - u^2$$

gives that the above is equal in absolute value to

$$\int_{[0,\delta^{1/2}]} (\frac{\delta^{-1}(\delta^{1/2}u-u^2)}{2})^k \int_{-5\delta}^{5\delta} \widehat{f}(u+n_J\delta^{1/2},L_J(u+n_J\delta^{1/2})+t) \times (\frac{\delta^{-1}t}{2})^j e(u(x_1+2n_J\delta^{1/2}x_2)+u^2x_2)\,du.$$

Since $|2n_J\delta^{1/2}| \leqslant 2$, after a change of variables and an application of Lemma 2.2.16, the right hand side of (2.46) is bounded above by

$$90^{F/p}\|\mathcal{E}_{[0,\delta^{1/2}]}(\frac{\delta^{-1}(\delta^{1/2}s-s^2)}{2})^k \times \int_{-5\delta}^{5\delta} \widehat{f}(s+n_J\delta^{1/2},L_J(s+n_J\delta^{1/2})+t)(\frac{\delta^{-1}t}{2})^j\,dt\|_{L^p(w_{B,F})} \tag{2.47}$$

Observe that

$$L_J(s+n_J\delta^{1/2}) = n_J^2\delta + (2n_J+1)\delta^{1/2}s.$$

Let

$$g_J(x) := f(x)e^{-2\pi i x\cdot(n_J\delta^{1/2}, n_J^2\delta)}.$$

Then

$$\widehat{f}(s+n_J\delta^{1/2}, L_J(s+n_J\delta^{1/2})+t) = \widehat{g_J}(s, (2n_J+1)\delta^{1/2}s+t).$$

This implies that

$$\begin{aligned}&\mathcal{E}_{[0,\delta^{1/2}]}\Big(\frac{\delta^{-1}(\delta^{1/2}s-s^2)}{2}\Big)^k\int_{-5\delta}^{5\delta}\widehat{f}(s+n_J\delta^{1/2}, L_J(s+n_J\delta^{1/2})+t)\Big(\frac{\delta^{-1}t}{2}\Big)^j\,dt\\ &= \int_0^{\delta^{1/2}}\int_{-5\delta}^{5\delta}\Big(\frac{\delta^{-1}(\delta^{1/2}s-s^2)}{2}\Big)^k\widehat{g_J}(s,(2n_J+1)\delta^{1/2}s+t)\Big(\frac{\delta^{-1}t}{2}\Big)^j e(sx_1+s^2x_2)\,dt\,ds\end{aligned}$$

which is equal to

$$\int_{\theta_J-(n_J\delta^{1/2},n_J^2\delta)}\Big(\frac{\delta^{-1}(\delta^{1/2}\xi_1-\xi_1^2)}{2}\Big)^k\widehat{g_J}(\xi)\Big(\frac{\delta^{-1}(\xi_2-(2n_J+1)\delta^{1/2}\xi_1)}{2}\Big)^j e(\xi_1x_1+\xi_1^2x_2)\,d\xi. \tag{2.48}$$

Let

$$T_J = \begin{pmatrix} 1 & 0\\ -2n_J\delta^{1/2} & 1\end{pmatrix}.$$

Notice that T_J sends $\theta_J-(n_J\delta^{1/2}, n_J^2\delta)$ to $\theta_{[0,\delta^{1/2}]}$. Letting $\mu = T_J\xi$ gives that (2.48) is equal to

$$\begin{aligned}&\int_{\theta_{[0,\delta^{1/2}]}}\Big(\frac{\delta^{-1}(\delta^{1/2}\mu_1-\mu_1^2)}{2}\Big)^k\widehat{g_J}(T_J^{-1}\mu)\Big(\frac{\delta^{-1}(\mu_2-\delta^{1/2}\mu_1)}{2}\Big)^j e(\mu_1x_1+\mu_1^2x_2)\,d\mu\\ &\quad= \int_{\theta_{[0,\delta^{1/2}]}}\Big(\frac{\delta^{-1}(\delta^{1/2}\mu_1-\mu_1^2)}{2}\Big)^k\widehat{g_J\circ T_J^t}(\mu)\Big(\frac{\delta^{-1}(\mu_2-\delta^{1/2}\mu_1)}{2}\Big)^j e(\mu_1x_1+\mu_1^2x_2)\,d\mu\\ &\quad= \int_0^{\delta^{1/2}}\int_{-5\delta}^{5\delta}\Big(\frac{\delta^{-1}(\delta^{1/2}s-s^2)}{2}\Big)^k\widehat{g_J\circ T_J^t}(s,\delta^{1/2}s+t)\Big(\frac{\delta^{-1}t}{2}\Big)^j\,dt\,e(sx_1+s^2x_2)\,ds.\end{aligned}$$

Inserting the above into (2.47) and applying (2.45) shows that the left hand side of (2.46) is bounded by

$$90^{F/p}C\|(g_J\circ T_J^t)_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,E})}. \tag{2.49}$$

By Lemma 2.2.16 and the definitions of T_J and g_J, we have

$$\begin{aligned}
&\|(g_J\circ T_J^t)_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,E})}^p \\
&\quad= \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\widehat{g}_J(T_J^{-1}\xi)1_{\theta_{[0,\delta^{1/2}]}}(\xi)e^{2\pi i x\cdot\xi}\,d\xi\Big|^p w_{B,E}(x)\,dx\\
&\quad= \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\widehat{g}_J(\mu)1_{\theta_{[0,\delta^{1/2}]}}(T_J\mu)e^{2\pi i x\cdot\mu}\,d\mu\Big|^p w_{B,E}(T_J^{-t}x)\,dx\\
&\quad= \int_{\mathbb{R}^2}\Big|\int_{\mathbb{R}^2}\widehat{f}(\mu+(n_J\delta^{1/2},n_J^2\delta))1_{\theta_J}(\mu+(n_J\delta^{1/2},n_J^2\delta))e^{2\pi i x\cdot\mu}\,d\mu\Big|^p w_{B,E}(T_J^{-t}x)\,dx\\
&\quad\leqslant 90^E\|f_{\theta_J}\|_{L^p(w_{B,E})}^p.
\end{aligned}$$

Inserting this into (2.49) completes the proof of Lemma 2.3.12. □

We now prove (2.44) when $J=[0,\delta^{1/2}]$, in other words we will prove (2.45). Corollary 2.2.4 implies that it is enough to show that

$$\int_{\mathbb{R}^2}\|\mathcal{E}_{[0,\delta^{1/2}]}g_{j,k}\|_{L^p_\#(B(y,\delta^{-1}))}^p w_{B,F}(y)\,dy \lesssim_E \exp(p(O(j)+O(k)))\|f_{\theta_{[0,\delta^{1/2}]}}\|_{L^p(w_{B,E})}^p. \tag{2.50}$$

We have

$$\begin{aligned}
&(\mathcal{E}_{[0,\delta^{1/2}]})g_{j,k}(x)\\
&\qquad= \int_{\theta_{[0,\delta^{1/2}]}}\widehat{f}(\xi)\Big(\frac{\delta^{-1}(\delta^{1/2}\xi_1-\xi_1^2)}{2}\Big)^k\Big(\frac{\delta^{-1}(\xi_2-\delta^{1/2}\xi_1)}{2}\Big)^j e((\xi_1^2-\xi_2)x_2)e(\xi\cdot x)\,d\xi.
\end{aligned}$$

For $x\in B(y,\delta^{-1})$, since

$$e((\xi_1^2-\xi_2)x_2) = e((\xi_1^2-\xi_2)y_2)e((\xi_1^2-\xi_2)(x_2-y_2)),$$

a Taylor expansion of $e((\xi_1^2-\xi_2)(x_2-y_2))$ gives that for $x\in B(y,\delta^{-1})$,

$$|(\mathcal{E}_{[0,\delta^{1/2}]}g_{j,k})(x)| \leqslant \sum_{\ell\geqslant 0}\frac{(2\pi)^\ell}{\ell!}\Big|\int_{\theta_{[0,\delta^{1/2}]}}\widehat{f}(\xi)C_{j,k,\ell}(\xi)e((\xi_1^2-\xi_2)y_2)e(\xi\cdot x)\,d\xi\Big| \tag{2.51}$$

where

$$C_{j,k,\ell}(\xi) := \Big(\frac{\delta^{-1}(\delta^{1/2}\xi_1-\xi_1^2)}{2}\Big)^k\Big(\frac{\delta^{-1}(\xi_2-\delta^{1/2}\xi_1)}{2}\Big)^j\Big(\frac{\delta^{-1}(\xi_1^2-\xi_2)}{2}\Big)^\ell.$$

Let Ψ be as in Lemma 2.2.10 and so $\Psi\in C_c^\infty(\mathbb{R})$, $\Psi=1$ on $[-1,1]$ and vanishes outside $[-3,3]$. For positive integer k and $\lambda>0$, let

$$M_{k,\lambda}(x) := x^k\Psi(x/\lambda).$$

Because the integral on the right hand side of (2.51) is restricted to $\theta_{[0,\delta^{1/2}]}$, we can insert some Schwartz cutoffs into $C_{j,k,\ell}$. From (2.34) and (2.35), for $\xi \in \theta_{[0,\delta^{1/2}]}$,

$$\frac{\delta^{-1}}{2}|\delta^{1/2}\xi_1 - \xi_1^2| \leqslant \frac{1}{8}, \quad \frac{\delta^{-1}}{2}|\xi_2 - \delta^{1/2}\xi_1| \leqslant \frac{5}{2}, \quad \frac{\delta^{-1}}{2}|\xi_1^2 - \xi_2| \leqslant \frac{21}{8}.$$

Furthermore, for $\xi \in \theta_{[0,\delta^{1/2}]}$, $|\xi_1| \leqslant \delta^{1/2}$ and $|\xi_2| \leqslant 6\delta$. Let

$$\begin{aligned} F(\xi) &:= \Psi(\delta^{-1/2}\xi_1)\Psi(\frac{\delta^{-1}\xi_2}{6}), \\ M_1(\xi_1) &:= M_{k,1/8}(\frac{\delta^{-1}(\delta^{1/2}\xi_1 - \xi_1^2)}{2}), \\ M_2(\xi) &:= M_{j,5/2}(\frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2}), \\ M_3(\xi) &:= M_{\ell,21/8}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2}), \end{aligned} \tag{2.52}$$

and

$$\widetilde{C}_{j,k,\ell}(\xi) := F(\xi)M_1(\xi_1)M_2(\xi)M_3(\xi).$$

Thus we can replace the $C_{j,k,\ell}$ on the right hand side of (2.51) with $\widetilde{C}_{j,k,\ell}$. It then remains to prove that

$$\begin{aligned} \int_{\mathbb{R}^2} \Big\| \int_{\theta_{[0,\delta^{1/2}]}} \widehat{f}(\xi)\widetilde{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x)\, d\xi \Big\|^p_{L^p_{\#}(B(y,\delta^{-1}))} w_{B,F}(y)\, dy \\ \lesssim_E \exp(p(O(j) + O(k) + O(\ell)))\|f_{\theta_{[0,\delta^{1/2}]}}\|^p_{L^p(w_{B,E})}. \end{aligned} \tag{2.53}$$

For each fixed j, k, ℓ, y, let

$$m(\xi) := e(\xi_1^2 y_2)\widetilde{C}_{j,k,\ell}(\xi) = e(\xi_1^2 y_2)M_1(\xi_1)M_2(\xi)M_3(\xi)F(\xi). \tag{2.54}$$

Fix arbitrary $y \in \mathbb{R}^2$. Therefore

$$\begin{aligned} &\int_{\theta_{[0,\delta^{1/2}]}} \widehat{f}(\xi)\widetilde{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x)\, d\xi \\ &\qquad = \int_{\mathbb{R}^2} \widehat{f_{\theta_{[0,\delta^{1/2}]}}}(\xi)m(\xi)e(\xi_1 x_1 + \xi_2(x_2 - y_2))\, d\xi \\ &\qquad = (f_{\theta_{[0,\delta^{1/2}]}} * \check{m})(x_1, x_2 - y_2). \end{aligned}$$

This implies

$$\begin{aligned} &\Big\| \int_{\theta_{[0,\delta^{1/2}]}} \widehat{f}(\xi)\widetilde{C}_{j,k,\ell}(\xi)e((\xi_1^2 - \xi_2)y_2)e(\xi \cdot x)\, d\xi \Big\|^p_{L^p_{\#}(B(y,\delta^{-1}))} \\ &\qquad = \delta^2 \int_{\mathbb{R}^2} |f_{\theta_{[0,\delta^{1/2}]}} * \check{m}|^p(x)1_B(x_1 - y_1, x_2)\, dx. \end{aligned}$$

Hölder's inequality implies that

$$|f_{\theta_{[0,\delta^{1/2}]}} * \check{m}|^p \leqslant (|f_{\theta_{[0,\delta^{1/2}]}}|^p * |\check{m}|)\|\check{m}\|_{L^1}^{p-1}.$$

Note that the L^1 norm on the right hand side depends on y since $\check{m}$ depends on y. To show (2.53), it is enough to show that for all $z \in \mathbb{R}^2$,

$$\delta^2 \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} |\check{m}|(x-z)1_B(x_1-y_1,x_2)\|\check{m}\|_{L^1}^{p-1} w_{B,F}(y)\,dx\,dy \lesssim_E \exp(p(O(j)+O(k)+O(\ell)))w_{B,E}(z). \tag{2.55}$$

We claim that for integers $a, b \geqslant 0$,

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b m\|_{L^\infty} \leqslant C(a,b)(\delta^{-1/2}+\delta^{1/2}|y_2|)^a\delta^{-b} \tag{2.56}$$

where

$$C(a,b) = 12^5 40^a 3^b 15^j 3^k 16^\ell a^{7a} b^{2b}(a+b)!^4(a+1)^5(b+1)^3.$$

The proof of (2.56) is deferred to the end of this section. The calculation is straightforward but rather tedious. With (2.56), integration by parts gives the following lemma.

Lemma 2.3.14. *For $a, b \geqslant 0$, we have*

$$|\check{m}(x)| \leqslant 2^a 2^b 216 C(a,b)(\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(1+\frac{|x_2|}{\delta^{-1}})^{-b}).$$

Proof. Note that for $|x| \leqslant 1$, $1 \leqslant 2/(1+|x|)$ and for $|x| \geqslant 1$, $1/|x| \leqslant 2/(1+|x|)$. There are four regions to consider.

First consider the case when $|x_1| > \delta^{-1/2}+\delta^{1/2}|y_2|$ and $|x_2| > \delta^{-1}$. Since m is supported in a $6\delta^{1/2} \times 36\delta$ rectangle centered at the origin, integration by parts gives that

$$\begin{aligned}\left|\int_{\mathbb{R}^2} m(\xi)e^{2\pi i(x_1\xi_1+x_2\xi_2)}\,d\xi\right| &= \left|\int_{\mathbb{R}^2} m(\xi)\frac{1}{(2\pi i x_1)^a(2\pi i x_2)^b}\partial_{\xi_1}^a\partial_{\xi_2}^b e^{2\pi i(x_1\xi_1+x_2\xi_2)}\,d\xi\right| \\ &\leqslant \frac{216}{(2\pi|x_1|)^a(2\pi|x_2|)^b}C(a,b)(\delta^{-1/2}+\delta^{1/2}|y_2|)^a\delta^{-b}\delta^{3/2} \\ &\leqslant \frac{216C(a,b)}{(2\pi)^a(2\pi)^b}(\delta^{1/2}(\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(\frac{|x_2|}{\delta^{-1}})^{-b}) \\ &\leqslant \frac{216C(a,b)}{\pi^a\pi^b}(\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(1+\frac{|x_2|}{\delta^{-1}})^{-b}).\end{aligned}$$

Next consider the case when $|x_1| \leqslant \delta^{-1/2} + \delta^{1/2}|y_2|$ and $|x_2| \leqslant \delta^{-1}$. Then we just use the trivial bound in this case. We have

$$\Big|\int_{\mathbb{R}^2} m(\xi)e^{2\pi i(x_1\xi_1+x_2\xi_2)}\,d\xi\Big| \leqslant 216C(0,0)\delta^{3/2}$$
$$\leqslant 2^a 2^b 216C(0,0)(\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(1+\frac{|x_2|}{\delta^{-1}})^{-b}).$$

For the case when $|x_1| \leqslant \delta^{-1/2} + \delta^{1/2}|y_2|$ and $|x_2| > \delta^{-1}$ we integrate by parts in ξ_2 but use trivial bounds in ξ_1. Thus

$$\Big|\int_{\mathbb{R}^2} m(\xi)e^{2\pi i(x_1\xi_1+x_2\xi_2)}\,d\xi\Big| \leqslant \frac{216}{(2\pi|x_2|)^b}C(0,b)\delta^{-b}\delta^{3/2}$$
$$\leqslant \frac{2^a 216C(0,b)}{\pi^b}(\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(1+\frac{|x_2|}{\delta^{-1}})^{-b}).$$

Similarly, when $|x_1| > \delta^{-1/2} + \delta^{1/2}|y_2|$ and $|x_2| \leqslant \delta^{-1}$ we obtain

$$\Big|\int_{\mathbb{R}^2} m(\xi)e^{2\pi i(x_1\xi_1+x_2\xi_2)}\,d\xi\Big| \leqslant \frac{2^b 216C(a,0)}{\pi^a}(\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-a})(\delta(1+\frac{|x_2|}{\delta^{-1}})^{-b}).$$

Combining the estimates in the above four cases completes the proof of Lemma 2.3.14. □

In particular, taking $a, b = E \geqslant 10$ in Lemma 2.3.14 gives the following corollary.

Corollary 2.3.15. *For $E \geqslant 10$, let*

$$\phi_1(x_1) := \delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-E}, \quad \phi_2(x_2) := \delta(1+\frac{|x_2|}{\delta^{-1}})^{-E}.$$

Then

$$|\check{m}(x)| \leqslant 15^j 3^k 16^\ell E^{30E}\phi_1(x_1)\phi_2(x_2).$$

We now prove (2.55). The following lemma is the only place where $p \leqslant 6$ is used.

Lemma 2.3.16. *For $2 \leqslant p \leqslant 6$,*

$$\|\check{m}\|_{L^1}^{p-1} \leqslant 15^{j(p-1)}3^{k(p-1)}16^{\ell(p-1)}E^{30E(p-1)}(1+\delta|y_2|)^5.$$

Proof. From Corollary 2.3.15,

$$\|\check{m}\|_{L^1} \leqslant 15^j 3^k 16^\ell E^{30E} \int_{\mathbb{R}} \phi_1(x_1)\, dx_1 \int_{\mathbb{R}} \phi_2(x_2)\, dx_2.$$

A change of variables gives that

$$\int_{\mathbb{R}} \phi_1(x_1)\, dx_1 = \delta^{1/2}(\delta^{-1/2} + \delta^{1/2}|y_2|) \int_{\mathbb{R}} (1+|x_1|)^{-E}\, dx_1 \leqslant 1 + \delta|y_2|$$

and

$$\int_{\mathbb{R}} \phi_2(x_2)\, dx_2 = \int_{\mathbb{R}} (1+|x_2|)^{-E}\, dx_2 \leqslant 1.$$

Therefore

$$\|\check{m}\|_{L^1} \leqslant 15^j 3^k 16^\ell E^{30E}(1+\delta|y_2|).$$

Raising both sides to the $(p-1)$-power and then using that $p \leqslant 6$ completes the proof of the lemma. $\square$

A change of variables gives

$$\delta^2 \int_{\mathbb{R}^2} |\check{m}|(x-z) 1_B(x_1 - y_1, x_2)\, dx = (|\check{m}| * \delta^2 1_B)(y_1 - z_1, -z_2)$$

and so combining this with Lemma 2.3.16 shows that the left hand side of (2.55) is bounded above by

$$15^{j(p-1)} 3^{k(p-1)} 16^{\ell(p-1)} E^{30E(p-1)} \int_{\mathbb{R}^2} (|\check{m}| * \delta^2 1_B)(y_1 - z_1, -z_2)(1+\delta|y_2|)^5 w_{B,F}(y)\, dy. \quad (2.57)$$

Corollary 2.3.15 gives that

$$(|\check{m}| * \delta^2 1_B)(x) \leqslant 15^j 3^k 16^\ell E^{30E} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(x_1)(\phi_2 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(x_2).$$

Since $1_{[-\delta^{-1}/2, \delta^{-1}/2]} \leqslant 2^E w_{[-\delta^{-1}/2, \delta^{-1}/2], E}$, Remark 2.2.3 shows

$$(\phi_2 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(x_2) \leqslant 8^E \delta (1 + |x_2|/\delta^{-1})^{-E}.$$

Therefore

$$(|\check{m}| * \delta^2 1_B)(y_1 - z_1, -z_2) \leqslant 15^j 16^\ell E^{30E} 8^E \delta (1 + \frac{|z_2|}{\delta^{-1}})^{-E} (\phi_1 * \delta 1_{[-\delta^{-1}/2, \delta^{-1}/2]})(y_1 - z_1).$$

Thus (2.57) is bounded above by

$$15^{jp}3^{kp}16^{\ell p}E^{30Ep}8^{E}\times \\ \delta(1+\frac{|z_2|}{\delta^{-1}})^{-E}\int_{\mathbb{R}^2}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy. \tag{2.58}$$

The following lemma will complete the proof of (2.55).

Lemma 2.3.17. *Let $E \geqslant 10$ and $F = 2E+7$, then*

$$\int_{\mathbb{R}^2}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy \leqslant 9\cdot 128^E\delta^{-1}(1+\frac{|z_1|}{\delta^{-1}})^{-E}. \tag{2.59}$$

Proof. We break the left hand side of (2.59) into the sum of integrals over the regions (recall that $\delta \in \mathbb{N}^{-2}$)

$$I := \{y : |y_2| \leqslant \delta^{-1}\}$$
$$II := \bigcup_{1\leqslant k<\delta^{-1/2}} \{y : k\delta^{-1} < |y_2| \leqslant (k+1)\delta^{-1}\}$$
$$III := \bigcup_{k\geqslant 0}\{y : 2^k\delta^{-3/2} < |y_2| \leqslant 2^{k+1}\delta^{-3/2}\}.$$

We also note that for $a \geqslant 1$,

$$(1+\frac{|x|}{a})^{-E} \leqslant a^E(1+|x|)^{-E}. \tag{2.60}$$

We first consider the integral over region I. When $|y_2| \leqslant \delta^{-1}$,

$$\phi_1(x_1) = \delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-E} \leqslant \delta^{1/2}(1+\frac{|x_1|}{2\delta^{-1/2}})^{-E} \leqslant 2^E\delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}})^{-E}.$$

Therefore by Remark 2.2.3,

$$(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1) \leqslant 16^E\delta(1+\frac{|y_1-z_1|}{\delta^{-1}})^{-E}$$

and so

$$\int_I(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy \\ \leqslant 16^E\delta\int_{\mathbb{R}^2}(1+\frac{|y_1-z_1|}{\delta^{-1}})^{-E}(1+\frac{|y_2|}{\delta^{-1}})^5(1+\frac{|y_1|}{\delta^{-1}})^{-E}(1+\frac{|y_2|}{\delta^{-1}})^{-E-7}\,dy.$$

Applying Remark 2.2.3 in the y_1 variable bounds this by

$$64^E(1+\frac{|z_1|}{\delta^{-1}})^{-E}\int_{\mathbb{R}}(1+\frac{|y_2|}{\delta^{-1}})^{-E-2}\,dy_2 \leqslant 64^E\delta^{-1}(1+\frac{|z_1|}{\delta^{-1}})^{-E}. \tag{2.61}$$

We next consider the integral over region II. For each $1 \leqslant k < \delta^{-1/2}$ and y such that $k\delta^{-1} < |y_2| \leqslant (k+1)\delta^{-1}$, we have

$$\phi_1(x_1) = \delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-E} \leqslant \delta^{1/2}(1+\frac{|x_1|}{3k\delta^{-1/2}})^{-E} \leqslant 3^E\delta^{1/2}(1+\frac{|x_1|}{k\delta^{-1/2}})^{-E}.$$

Therefore by Remark 2.2.3,

$$(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1) \leqslant 24^E k\delta(1+\frac{|y_1-z_1|}{\delta^{-1}})^{-E}$$

and so

$$\begin{aligned}
&\int_{II}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy \\
&\quad= \sum_{1\leqslant k<\delta^{-1/2}}\int_{k\delta^{-1}<|y_2|\leqslant(k+1)\delta^{-1}}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy \\
&\quad\leqslant 96^E\sum_{1\leqslant k<\delta^{-1/2}}k(1+\frac{|z_1|}{\delta^{-1}})^{-E}\int_{k\delta^{-1}<|y_2|\leqslant(k+1)\delta^{-1}}(1+\frac{|y_2|}{\delta^{-1}})^{-E-2}\,dy_2 \\
&\quad\leqslant 96^E\sum_{1\leqslant k<\delta^{-1/2}}k(1+\frac{|z_1|}{\delta^{-1}})^{-E}2\delta^{-1}k^{-E-2} \leqslant 4\cdot 96^E\delta^{-1}(1+\frac{|z_1|}{\delta^{-1}})^{-E}
\end{aligned} \tag{2.62}$$

where in the last inequality we have used that $E \geqslant 10$.

Finally we consider the integral over region III. For each $k \geqslant 0$ and y such that $2^k\delta^{-3/2} < |y_2| \leqslant 2^{k+1}\delta^{-3/2}$, we have

$$\phi_1(x_1) = \delta^{1/2}(1+\frac{|x_1|}{\delta^{-1/2}+\delta^{1/2}|y_2|})^{-E} \leqslant \delta^{1/2}(1+\frac{|x_1|}{4\cdot 2^k\delta^{-1}})^{-E} \leqslant 4^E\delta^{1/2}(1+\frac{|x_1|}{2^k\delta^{-1}})^{-E}.$$

Therefore by Remark 2.2.3,

$$(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1) \leqslant 32^E\delta^{1/2}(1+\frac{|y_1-z_1|}{2^k\delta^{-1}})^{-E}$$

and so

$$\begin{aligned}
&\int_{III}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy\\
&=\sum_{k\geqslant 0}\int_{2^k\delta^{-3/2}<|y_2|\leqslant 2^{k+1}\delta^{-3/2}}(\phi_1 * \delta 1_{[-\delta^{-1}/2,\delta^{-1}/2]})(y_1-z_1)(1+\frac{|y_2|}{\delta^{-1}})^5 w_{B,F}(y)\,dy\\
&\leqslant 32^E\delta^{1/2}\times\\
&\quad\sum_{k\geqslant 0}\int_{\mathbb{R}}(1+\frac{|y_1-z_1|}{2^k\delta^{-1}})^{-E}(1+\frac{|y_1|}{\delta^{-1}})^{-E}\,dy_1\int_{2^k\delta^{-3/2}<|y_2|\leqslant 2^{k+1}\delta^{-3/2}}(1+\frac{|y_2|}{\delta^{-1}})^{-E-2}\,dy_2\\
&\leqslant 128^E\sum_{k\geqslant 0}\delta^{-1/2}(1+\frac{|z_1|}{2^k\delta^{-1}})^{-E}2^{k+1}\delta^{-3/2}(2^k\delta^{-1/2})^{-E-2}\\
&=128^E\sum_{k\geqslant 0}\delta^{-2+(E+2)/2}2^{k+1-k(E+2)}(1+\frac{|z_1|}{2^k\delta^{-1}})^{-E}\leqslant 4\cdot 128^E\delta(1+\frac{|z_1|}{\delta^{-1}})^{-E}
\end{aligned}$$

where in the third inequality we have used (2.60). Summing this with (2.61) and (2.62) shows that the left hand side of (2.59) is bounded above by $9\cdot 128^E\delta(1+|z_1|/\delta^{-1})^{-E}$ which completes the proof of Lemma 2.3.17. □

Thus Lemma 2.3.17 shows that (2.58) is bounded above by

$$9\cdot 15^{jp}3^{kp}16^{\ell p}E^{30Ep}2^{10E}(1+\frac{|z_1|}{\delta^{-1}})^{-E}(1+\frac{|z_2|}{\delta^{-1}})^{-E}\leqslant 15^{jp}3^{kp}16^{\ell p}E^{40Ep}w_{B,E}(z). \tag{2.63}$$

We now trace back all the implied constants to finish the proof of Lemma 2.3.10. From (2.63), the implied constants in (2.55) and (2.53) are both $15^{jp}3^{kp}16^{\ell p}E^{40Ep}$. By (2.51) and (2.53), the left hand side of (2.50) is

$$\begin{aligned}
&\Big\|\|\mathcal{E}_{[0,\delta^{1/2}]}g_{j,k}\|_{L^p_\#(B(y,\delta^{-1}))}\Big\|^p_{L^p_y(w_{B,F})}\\
&\leqslant\Big\|\sum_{\ell\geqslant 0}\frac{(2\pi)^\ell}{\ell!}\|\int_{\theta_{[0,\delta^{1/2}]}}\widehat{f}(\xi)\widetilde{C}_{j,k,\ell}(\xi)e((\xi_1^2-\xi_2)y_2)e(\xi\cdot x)\,d\xi\|_{L^p_\#(B(y,\delta^{-1}))}\Big\|^p_{L^p_y(w_{B,F})}\\
&\leqslant\Big(\sum_{\ell\geqslant 0}\frac{(2\pi)^\ell}{\ell!}\Big\|\|\int_{\theta_{[0,\delta^{1/2}]}}\widehat{f}(\xi)\widetilde{C}_{j,k,\ell}(\xi)e((\xi_1^2-\xi_2)y_2)e(\xi\cdot x)\,d\xi\|_{L^p_\#(B(y,\delta^{-1}))}\Big\|_{L^p_y(w_{B,F})}\Big)^p\\
&\leqslant 15^{jp}3^{kp}e^{32\pi p}E^{40Ep}\|f_{\theta_{[0,\delta^{1/2}]}}\|^p_{L^p(w_{B,E})}
\end{aligned}$$

which gives the implied constant in (2.50). Using this, Lemma 2.3.12, and Lemma 2.2.4, we have

$$\|\mathcal{E}_J g_{j,k}\|_{L^p(w_{B,F})} \leqslant \|f_{\theta_J}\|_{L^p(w_{B,E})} \times \begin{cases} 3^{E/p}15^j3^kE^{40E}e^{32\pi} & \text{if } J = [0, \delta^{1/2}] \\ 90^{(E+F)/p}3^{E/p}15^j3^kE^{40E}e^{32\pi} & \text{if } J \neq [0, \delta^{1/2}] \end{cases}$$
$$\leqslant 15^j3^kE^{54E}\|f_{\theta_J}\|_{L^p(w_{B,E})}$$

where in the last inequality we have used that $E \geqslant 10$, $2 \leqslant p \leqslant 6$, and $F = 2E+7$. Inserting this estimate into (2.43) gives that

$$\|f\|_{L^p(B)} \leqslant \mathrm{Dec}'(\delta, p, F)E^{54E}e^{32\pi}(\sum_{J \in P_{\delta^{1/2}}([0,1])} \|f_{\theta_J}\|^2_{L^p(w_{B,E})})^{1/2}.$$

Since $E \geqslant 10$, $e^{36\pi} \leqslant 10^{50} \leqslant E^{5E}$ and this completes the proof of Lemma 2.3.10.

2.3.2 Proof of (2.56)

Let F, M_1, M_2, M_3, and m be as in (2.52) and (2.54). We will prove (2.56).

Lemma 2.3.18. *Let $\lambda > 0$ and let*

$$M_{k,\lambda}(x) := x^k\Psi(x/\lambda)$$

where Ψ is as defined in Lemma 2.2.10. Then for integer $a \geqslant 0$,

$$\|\partial^a M_{k,\lambda}\|_{L^\infty} \leqslant 12 \cdot 6^a3^k(1+\lambda)^k(a!)^2. \tag{2.64}$$

If $\lambda \geqslant 1$, this bound can be replaced with $12(6^{a+k}\lambda^k)(a!)^2$.

Proof. This proof is essentially the same as that of the beginning of the proof of Lemma 2.2.10. From the proof of Lemma 2.2.10, we have that $|\Psi^{(j)}(x)| \leqslant 12(6^j)(j!)^2$ for all $j \geqslant 0$. Since Ψ is supported in $[-3, 3]$, $\Psi(x/\lambda)$ is supported in $[-3\lambda, 3\lambda]$.

If $a = 0$, then $\|M_{k,\lambda}\|_{L^\infty} \leqslant 12(3\lambda)^k$ which proves (2.64) in this case. Now consider when

$a \geqslant 1$. First suppose that $a \leqslant k$, then

$$\begin{aligned}|\partial^a(M_{k,\lambda}(x))| &= |\sum_{j=0}^{a}\binom{a}{j}\partial^j(x^k)\Psi^{(a-j)}(x)| \\ &\leqslant \sum_{j=0}^{a}\binom{a}{j}\frac{k!}{(k-j)!}(3\lambda)^{k-j}12(6^{a-j})(a-j)!^2 \\ &\leqslant 12(6^a 3^k)(a!)^2\sum_{j=0}^{a}\binom{k}{j}\lambda^{k-j} \leqslant 12\cdot 6^a 3^k(1+\lambda)^k(a!)^2.\end{aligned}$$

Next suppose that $k < a$, then

$$|\partial^a(M_{k,\lambda}(x))| \leqslant \sum_{j=0}^{k}\binom{a}{j}\frac{k!}{(k-j)!}(3\lambda)^{k-j}12(6^{a-j})(a-j)!^2 \leqslant 12\cdot 6^a 3^k(1+\lambda)^k(a!)^2.$$

This completes the proof of Lemma 2.3.18. $\square$

Our goal is to obtain an estimate on $\|\partial_{\xi_1}^a\partial_{\xi_2}^b m\|_{L^\infty}$ depending only on a, b, δ and y_2 and where m is as defined in (2.54) and (2.52). Since we want exact constants, we will need to differentiate exactly each of the five functions that make up $m(\xi)$. Note that since Ψ is supported in $[-3,3]$, m is supported in a $6\delta^{1/2}\times 36\delta$ rectangle centered at the origin. In particular, for all $\xi \in \operatorname{supp}(m)$,

$$-3\delta^{1/2} \leqslant \xi_1 \leqslant 3\delta^{1/2}. \tag{2.65}$$

The bounds in Lemmas 2.3.20 and 2.3.21 are valid when we take no derivatives (either $a=0$ or $b=0$) provided we use the convention that $0^0 = 1$.

To compute $\partial_{\xi_1}^a\partial_{\xi_2}^b m$, we will need to take arbitrarily many derivatives of a composition of functions. We will use the Faa di Bruno formula. We briefly recall all needed formulas (see [Joh02] for a reference, note that Johnson defined $B_{m,0} = 0$ for $m > 0$ since the sum conditions would be vacuous). For $m, k \geqslant 1$, define the Bell polynomials

$$B_{m,k}(x_1, x_2, \ldots, x_{m-k+1}) = \frac{1}{k!}\sum_{\substack{j_1+\cdots+j_k=m \\ j_i\geqslant 1}}\binom{m}{j_1,\ldots,j_k}x_{j_1}\cdots x_{j_k}.$$

Let

$$Y_m(x_1,\ldots,x_m) := \sum_{k=1}^{m} B_{m,k}(x_1,\ldots,x_{m-k+1}). \tag{2.66}$$

The Faa di Bruno formula states that

$$\frac{d^m}{dt^m} g(f(t)) = \sum_{k=1}^{m} g^{(k)}(f(t)) B_{m,k}(f'(t), f''(t), \ldots, f^{(m-k+1)}(t)).$$

Finally we will abuse notation slightly by writing $Y_m(x, y, 0, \ldots, 0)$ as $Y_m(x, y)$.

Lemma 2.3.19. *Let $m \geqslant 1$ and $x, y \neq 0$ such that $|x| \leqslant C|y|^{1/2}$ with $C \geqslant 1$. Then*

$$|Y_m(x, y)| \leqslant C^m m^m |y|^{m/2}.$$

Proof. From [Joh02, p. 220], $Y_m(x, y)$ is equal to the determinant of the $m \times m$ matrix

$$\begin{pmatrix} x & (m-1)y & 0 & \cdots & 0 & 0 \\ -1 & x & (m-2)y & \cdots & 0 & 0 \\ 0 & -1 & x & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & x & y \\ 0 & 0 & 0 & \cdots & -1 & x \end{pmatrix}.$$

Cofactor expansion gives that $Y_m(x, y)$ obeys the recurrence $Y_m = xY_{m-1} + (m-1)yY_{m-2}(1, 1)$ with $Y_1 = x$, $Y_2 = x^2 + y$. Therefore $Y_m(1, 1)$ obeys the recurrence $Y_m(1, 1) = Y_{m-1}(1, 1) + (m-1)Y_{m-2}$ and so $Y_m(1, 1) \leqslant m! \leqslant m^m$. Each

$$Y_m(x, y) = x^m + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j x^{m-2j} y^j = y^{m/2}\left(\frac{x^m}{y^{m/2}} + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j \frac{x^{m-2j}}{y^{m/2-j}}\right) \tag{2.67}$$

and $Y_m(1, 1) = 1 + \sum_j c_j \leqslant m^m$. Thus $Y_m(x, 0) = x^m$ and

$$|Y_m(x, y)| \leqslant |y|^{m/2}\left(C^m + \sum_{j=1}^{\lfloor m/2 \rfloor} c_j C^{m-2j}\right) \leqslant C^m m^m |y|^{m/2}.$$

This completes the proof of Lemma 2.3.19. ☐

Lemma 2.3.20. *For $a \geqslant 0$ and $\xi \in \operatorname{supp}(m)$,*

$$\|\partial_{\xi_1}^a e^{2\pi i y_2 \xi_1^2}\|_{L^\infty} \leqslant (12\pi)^a a^a \times \begin{cases} \delta^{-a/2} & \text{if } |y_2| \leqslant \delta^{-1} \\ \delta^{a/2}|y_2|^a & \text{if } |y_2| > \delta^{-1}. \end{cases}$$

In particular,

$$\|\partial_{\xi_1}^a e^{2\pi i y_2 \xi_1^2}\|_{L^\infty} \leqslant (12\pi)^a a^a (\delta^{-1/2} + \delta^{1/2}|y_2|)^a.$$

Proof. If $a = 0$, then L^∞ norm is equal to 1 and the above formula still holds true. Now suppose $a \geqslant 1$. From Faa di Bruno's formula,

$$\partial_{\xi_1}^a e^{2\pi i y_2 \xi_1^2} = \sum_{k=1}^{a} (2\pi i)^k e^{2\pi i y_2 \xi_1^2} B_{a,k}(2\xi_1 y_2, 2y_2, 0, \ldots, 0)$$

and so,

$$\|\partial_{\xi_1}^a e^{2\pi i y_2 \xi_1^2}\|_{L^\infty} \leqslant (2\pi)^a Y_a(2|\xi_1||y_2|, 2|y_2|). \tag{2.68}$$

Suppose $|y_2| \leqslant \delta^{-1}$, then $\delta^{1/2}|y_2| \leqslant |y_2|^{1/2}$ and so from (2.65),

$$2|\xi_1||y_2| \leqslant 6|y_2|^{1/2}.$$

Therefore Lemma 2.3.19 gives that

$$Y_a(2|\xi_1||y_2|, 2|y_2|) \leqslant 6^a a^a |y_2|^{a/2} \leqslant 6^a a^a \delta^{-a/2}.$$

Inserting this into (2.68) then finishes this case.

If $|y_2| > \delta^{-1}$, then from (2.67),

$$Y_a(2|\xi_1||y_2|, 2|y_2|) \leqslant Y_a(6\delta^{1/2}|y_2|, 2|y_2|) = 6^a \delta^{a/2}|y_2|^a (1 + \sum_{j=1}^{\lfloor a/2 \rfloor} 18^{-j} c_j (\delta |y_2|)^{-j}).$$

Since $\delta|y_2| > 1$ and $1 + \sum_j c_j \leqslant a^a$, the above is bounded by $6^a a^a \delta^{a/2} |y_2|^a$ which completes the proof of Lemma 2.3.20. □

Lemma 2.3.21. *For integers $a, b \geqslant 0$ and $\xi \in \operatorname{supp}(m)$,*

$$\|\partial_{\xi_1}^a M_1\|_{L^\infty} \leqslant 12(21^a a^{3a} 3^k)\delta^{-a/2} \tag{2.69}$$

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b M_2\|_{L^\infty} \leqslant 12(6^a 3^b 15^j)(a+b)!^2 \delta^{-b-a/2} \tag{2.70}$$

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b M_3\|_{L^\infty} \leqslant 12(18^a 3^b 16^\ell) a^a (a+b)!^2 \delta^{-b-a/2} \tag{2.71}$$

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b F\|_{L^\infty} \leqslant 12^2 6^a (a!)^2 (b!)^2 \delta^{-b-a/2}. \tag{2.72}$$

Proof. We first prove (2.69). If $a = 0$, then from Lemma 2.3.18,

$$\|M_1\|_{L^\infty} = \|M_{k,1/8}\|_{L^\infty} \leqslant 12 \cdot 3^k$$

which proves (2.69) in this case. Next suppose $a \geqslant 1$. We compute that

$$\partial_{\xi_1}^a M_1 = \sum_{s=1}^{a} M_{k,1/8}^{(s)}(\frac{\delta^{-1}(\delta^{1/2}\xi_1 - \xi_1^2)}{2})B_{a,s}(\frac{1}{2}\delta^{-1/2} - \delta^{-1}\xi_1, -\delta^{-1}, 0, \ldots, 0)$$

and so applying Lemma 2.3.18 and (2.66) gives that

$$\|\partial_{\xi_1}^a M_1\|_{L^\infty} \leqslant 12(3^k 6^a)(a!)^2 Y_a(\delta^{-1/2}|\frac{1}{2} - \delta^{-1/2}\xi_1|, \delta^{-1}). \tag{2.73}$$

Since

$$\delta^{-1/2}|\frac{1}{2} - \delta^{-1/2}\xi_1| \leqslant \frac{7}{2}(\delta^{-1})^{1/2},$$

Lemma 2.3.19 implies that

$$Y_a(\delta^{-1/2}|\frac{1}{2} - \delta^{-1/2}\xi_1|, \delta^{-1}) \leqslant (7/2)^a a^a \delta^{-a/2}.$$

Inserting this into (2.73) completes the proof of (2.69).

We now prove (2.70). We compute

$$\begin{aligned}\partial_{\xi_1}^a \partial_{\xi_2}^b M_2 &= (\frac{\delta^{-1}}{2})^b \partial_{\xi_1}^a M_{j,5/2}^{(b)}(\frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2}) \\ &= (\frac{\delta^{-1}}{2})^b (-\frac{\delta^{-1/2}}{2})^a M_{j,5/2}^{(a+b)}(\frac{\delta^{-1}(\xi_2 - \delta^{1/2}\xi_1)}{2}).\end{aligned}$$

Applying Lemma 2.3.18 gives

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b M_2\|_{L^\infty} \leqslant 12(6^a 3^b 15^j)(a+b)!^2 \delta^{-b-a/2}$$

which proves (2.70).

Next we prove (2.71). If $a = 0$, then

$$\partial_{\xi_2}^b M_3 = (-\frac{\delta^{-1}}{2})^b M_{\ell,21/8}^{(b)}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2})$$

and so

$$\|\partial_{\xi_2}^b M_3\|_{L^\infty} \leqslant 12(3^b 16^\ell)(b!)^2 \delta^{-b}$$

which proves (2.71) in this case. Now suppose $a \geqslant 1$. Faa di Bruno's formula gives that

$$\partial_{\xi_1}^a \partial_{\xi_2}^b M_3 = (-\frac{\delta^{-1}}{2})^b \sum_{s=1}^{a} M_{\ell,21/8}^{(s+b)}(\frac{\delta^{-1}(\xi_1^2 - \xi_2)}{2})B_{a,s}(\delta^{-1}\xi_1, \delta^{-1}, 0, \ldots, 0).$$

Applying Lemma 2.3.18 and (2.66) gives that

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b M_3\|_{L^\infty} \leqslant 12(6^a 3^b 16^\ell)(a+b)!^2 \delta^{-b} Y_a(\delta^{-1}|\xi_1|, \delta^{-1}) \tag{2.74}$$

Since $\delta^{-1}|\xi_1| \leqslant 3(\delta^{-1})^{1/2}$, it follows that

$$Y_a(\delta^{-1}|\xi_1|, \delta^{-1}) \leqslant 3^a a^a \delta^{-a/2}.$$

Inserting this into (2.74) completes the proof of (2.71).

Finally we prove (2.72). We compute

$$\partial_{\xi_1}^a \partial_{\xi_2}^b F = \delta^{-a/2} (\frac{\delta^{-1}}{6})^b \Psi^{(a)}(\delta^{-1/2}\xi_1) \Psi^{(b)}(\frac{\delta^{-1}\xi_2}{6}).$$

Lemma 2.2.10 then implies that

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b F\|_{L^\infty} \leqslant 12^2 6^a (a!)^2 (b!)^2 \delta^{-b-a/2}$$

which proves (2.72). This completes the proof of Lemma 2.3.21. □

We are now ready to prove (2.56).

Lemma 2.3.22. *For $a, b \geqslant 0$,*

$$\|\partial_{\xi_1}^a \partial_{\xi_2}^b m\|_{L^\infty} \leqslant 12^5 40^a 3^b 15^j 3^k 16^\ell a^{7a} b^{2b} (a+b)!^4 (a+1)^5 (b+1)^3 (\delta^{-1/2} + \delta^{1/2}|y_2|)^a \delta^{-b}.$$

Proof. We compute

$$\partial_{\xi_1}^a \partial_{\xi_2}^b m = \sum_{\substack{s_1+s_2+s_3=b \\ t_1+t_2+t_3+t_4+t_5=a \\ s_i, t_i \geqslant 0}} \left(\frac{b!}{s_1! s_2! s_3!}\right) \left(\frac{a!}{t_1! t_2! t_3! t_4! t_5!}\right) \times$$
$$(\partial_{\xi_1}^{t_1} e(\xi_1^2 y_2))(\partial_{\xi_1}^{t_2} M_1)(\partial_{\xi_1}^{t_3} \partial_{\xi_2}^{s_1} M_2)(\partial_{\xi_1}^{t_4} \partial_{\xi_2}^{s_2} M_3)(\partial_{\xi_1}^{t_5} \partial_{\xi_2}^{s_3} F).$$

Applying crude bounds and Lemmas 2.3.20 and 2.3.21 gives that

$$\begin{aligned}\|\partial_{\xi_1}^a \partial_{\xi_2}^b m\|_{L^\infty} &\leqslant 12^5 40^a 3^b 15^j 3^k 16^\ell a! b! \delta^{-b-a/2} (1 + \delta|y_2|)^a \times \\ &\qquad \sum_{\substack{s_1+s_2+s_3=b \\ t_1+t_2+t_3+t_4+t_5=a \\ s_i, t_i \geqslant 0}} t_1^{t_1} t_2^{3t_2} t_4^{t_4} (t_3+s_1)!^2 (t_4+s_2)!^2 t_5! s_3! \\ &\leqslant 12^5 40^a 3^b 15^j 3^k 16^\ell a^{7a} b^{2b} (a+b)!^4 (a+1)^5 (b+1)^3 \delta^{-b-a/2} (1+\delta|y_2|)^a\end{aligned}$$

where in the first inequality we have used that

$$(\delta^{-1/2}+\delta^{1/2}|y_2|)^{t_1}=\delta^{-t_1/2}(1+\delta|y_2|)^{t_1}\leqslant \delta^{-t_1/2}(1+\delta|y_2|)^{a}$$

and we have removed a $t_5!$ and $s_3!$ using the multinomial coefficient. This completes the proof of Lemma 2.3.22 and the proof of (2.56). □

2.4 Parabolic rescaling: an application

As an application of Lemma 2.2.18 and Proposition 2.3.11, we will prove that the decoupling constant is essentially multiplicative. This will play an important role in Section 2.10 when we upgrade knowledge about decoupling at a lacunary sequence of scales to knowledge about decoupling on all possible scales in $\mathbb{N}^{-2}$. The restriction that $p \leqslant 6$ is once again an artifact that only arises from our application of Proposition 2.3.11.

Proposition 2.4.1. *Let $E \geqslant 100$ and $2 \leqslant p \leqslant 6$. For $0 < \delta < \sigma < 1$ with $\delta, \sigma, \delta/\sigma \in \mathbb{N}^{-2}$, we have*

$$D_{p,E}(\delta) \leqslant E^{100E} D_{p,E}(\sigma) D_{p,E}(\delta/\sigma).$$

Proof. Fix an arbitrary $E \geqslant 100$ and $2 \leqslant p \leqslant 6$. We need to show that for all $g : [0,1] \to \mathbb{C}$ and all squares B of side length δ^{-1}, we have

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leqslant E^{100E} D_{p,E}(\sigma) D_{p,E}(\delta/\sigma)(\sum_{J\in P_{\delta^{1/2}}([0,1])}\|\mathcal{E}_J g\|^2_{L^p(w_{B,E})})^{1/2}.$$

It suffices to assume that B is centered at the origin.

Since $\delta/\sigma \in \mathbb{N}^{-2}$, we can partition B into a collection of squares $\{\Sigma\}$ of side length σ^{-1}. Then

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(\Sigma)} \leqslant D_{p,E}(\sigma)(\sum_{J\in P_{\sigma^{1/2}}([0,1])}\|\mathcal{E}_J g\|^2_{L^p(w_{\Sigma,E})})^{1/2}.$$

Raising both sides to the pth power and summing over all Σ, then using Minkowski's inequality (since $p \geqslant 2$), and finally applying Proposition 2.2.14 gives that

$$\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} \leqslant 48^{E/p} D_{p,E}(\sigma)(\sum_{J\in P_{\sigma^{1/2}}([0,1])}\|\mathcal{E}_J g\|^2_{L^p(w_{B,E})})^{1/2}. \tag{2.75}$$

For each $J = [a, a+\sigma^{1/2}]$, we will first show that

$$\|\mathcal{E}_J g\|_{L^p(B)} \lesssim_E D_{p,E}(\delta/\sigma)(\sum_{J'\in P_{\delta^{1/2}}(J)} \|\mathcal{E}_{J'}g\|_{L^p(w_{B,E})}^2)^{1/2}. \tag{2.76}$$

Afterwards we will apply Proposition 2.2.11 to (2.76) and then insert the result into (2.75) to finish.

Let T be as in Lemma 2.2.18, $L(\xi) = (\xi - a)/\sigma^{1/2}$, $g_L = g \circ L^{-1}$. Then a change of variables gives that

$$\|\mathcal{E}_J g\|_{L^p(B)} = \sigma^{\frac{1}{2}-\frac{3}{2p}}\|\mathcal{E}_{[0,1]}g_L\|_{L^p(T(B))}.$$

Let $\mathcal{B}$ be as in Lemma 2.2.18. Thus we cover $T(B)$ by a collection of squares $\mathcal{B} = \{\Delta\}$ of side length σ/δ, use decoupling constant $\widetilde{D}_{p,E}$ at scale δ/σ and undo change of variables. This gives

$$\begin{aligned}
\sigma^{\frac{p}{2}-\frac{3}{2}}\|\mathcal{E}_{[0,1]}g_L\|_{L^p(T(B))}^p &\leqslant \sigma^{\frac{p}{2}-\frac{3}{2}}\sum_{\Delta\in\mathcal{B}}\|\mathcal{E}_{[0,1]}g_L\|_{L^p(\Delta)}^p \\
&\leqslant \widetilde{D}_{p,E}(\delta/\sigma)^p\sigma^{\frac{p}{2}-\frac{3}{2}}\sum_{\Delta\in\mathcal{B}}(\sum_{J''\in P_{(\delta/\sigma)^{1/2}}([0,1])}\|\mathcal{E}_{J''}g_L\|_{L^p(\widetilde{w}_{\Delta,E})}^2)^{p/2} \\
&\leqslant \widetilde{D}_{p,E}(\delta/\sigma)^p(\sum_{J'\in P_{\delta^{1/2}}(J)}\|\mathcal{E}_{J'}g\|_{L^p(\sum_\Delta \widetilde{w}_{\Delta,E}\circ T)}^2)^{p/2} \\
&\leqslant \widetilde{D}_{p,E}(\delta/\sigma)^p 720^E(\sum_{J'\in P_{\delta^{1/2}}(J)}\|\mathcal{E}_{J'}g\|_{L^p(w_{B,E})}^2)^{p/2}
\end{aligned}$$

where the third inequality we have used Minkowski's inequality and $p \geqslant 2$ and the last inequality we have used Lemma 2.2.18. Combining this with Proposition 2.3.11 gives that

$$\|\mathcal{E}_J g\|_{L^p(B)} \leqslant E^{70E}720^{E/p}D_{p,E}(\delta/\sigma)(\sum_{J'\in P_{\delta^{1/2}}(J)}\|\mathcal{E}_{J''}g\|_{L^p(w_{B,E})}^2)^{1/2}.$$

Applying Proposition 2.2.11 gives that

$$\|\mathcal{E}_J g\|_{L^p(w_{B,E})} \leqslant E^{80E}D_{p,E}(\delta/\sigma)(\sum_{J'\in P_{\delta^{1/2}}(J)}\|\mathcal{E}_{J''}g\|_{L^p(w_{B,E})}^2)^{1/2}.$$

Inserting this into (2.75) then completes the proof of Proposition 2.4.1. □

Remark 2.4.2. Combining Propositions 2.3.11 and 2.4.1, we see that all four decoupling constants $D_{p,E}$, $\widetilde{D}_{p,E}$, $\overline{D}_p$, and $\widehat{D}_{p,E}$ obey a similar multiplicative property.

2.5 Bilinear equivalence

We now define the bilinear decoupling constant and show that it is essentially the same size as the linear decoupling constant. In [BD17], Bourgain and Demeter use a Bourgain-Guth type argument to do this. However in two dimensions, there is a simpler proof using Hölder's inequality and parabolic rescaling by Tao in [Tao15]. It is this version we follow.

For each $m \in \mathbb{N}$, $E \geqslant 100$, let

$$\nu := 2^{-16 \cdot 2^m E^{10E}}.$$

For $\delta \in (0,1)$ such that $\nu\delta^{-1/2} \in \mathbb{N}$, let $D_{p,E}(\delta, m)$ be the best constant such that

$$\| \operatorname{geom} |\mathcal{E}_{I_i} g| \|_{L^p(B)} \leqslant D_{p,E}(\delta, m) \operatorname{geom}(\sum_{J \in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|^2_{L^p(\widetilde{w}_{B,E})})^{1/2}$$

for all pairs of intervals $I_1, I_2 \in P_\nu([0,1])$ which are at least ν-separated, functions $g : [0,1] \to \mathbb{C}$, and squares B of side length δ^{-1}. Note that the right hand side uses the weight function $\widetilde{w}_{B,E}$ rather than $w_{B,E}$.

We first give the trivial bound for the bilinear decoupling constant which is a useful bound at large scales.

Lemma 2.5.1. *Let m, E, ν be defined as above. If $\nu\delta^{-1/2} \in \mathbb{N}$, then $D_{p,E}(\delta, m) \leqslant 4^E \nu^{1/2} \delta^{-1/4}$.*

Proof. Hölder's inequality gives that

$$\| \operatorname{geom} |\mathcal{E}_{I_i} g| \|_{L^p(B)} \leqslant \operatorname{geom} \|\mathcal{E}_{I_i} g\|_{L^p(B)}.$$

The triangle inequality, Cauchy-Schwarz, and that $1_B \leqslant 4^E \widetilde{w}_{B,E}$ gives

$$\|\mathcal{E}_{I_i} g\|_{L^p(B)} = \| \sum_{J \in P_{\delta^{1/2}}(I_i)} \mathcal{E}_J g\|_{L^p(B)} \leqslant 4^E \nu^{1/2} \delta^{-1/4} (\sum_{J \in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|^2_{L^p(\widetilde{w}_{B,E})})^{1/2}$$

which completes the proof of Lemma 2.5.1. □

Lemma 2.5.2. *Let $E \geqslant 100$ and $2 \leqslant p \leqslant 6$. If $\delta^{1/2} \in 2^{-\mathbb{N}}$ and $\delta^{1/2}\nu^{-1} \in 2^{-\mathbb{N}}$, then*

$$D_{p,E}(\delta) \leqslant E^{100E} (D_{p,E}(\frac{\delta}{\nu^2}) + \frac{1}{\nu} D_{p,E}(\delta, m)).$$

Proof. This proof is essentially an application of parabolic rescaling. The restriction $2 \leqslant p \leqslant 6$ comes only from the application of Proposition 2.3.11. Fix an arbitrary square B of side length δ^{-1} and function $g : [0,1] \to \mathbb{C}$. It suffices to assume B is centered at the origin. Partition $[0,1]$ into $1/\nu$ many intervals $I_1, \ldots, I_{1/\nu}$ of length ν (here we have used that $\nu \in 2^{-\mathbb{N}}$). Then

$$\begin{aligned}\|\mathcal{E}_{[0,1]}g\|_{L^p(B)} &= \|\sum_{1 \leqslant i \leqslant 1/\nu} \mathcal{E}_{I_i}g\|_{L^p(B)} \leqslant \|\sum_{1 \leqslant i,j \leqslant 1/\nu} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)}^{1/2} \\ &\leqslant \Big(\|\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)} + \|\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)}\Big)^{1/2} \\ &\leqslant \sqrt{2}\Big(\|\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)}^{1/2} + \|\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)}^{1/2}\Big).\end{aligned}$$

We first consider the off-diagonal terms. This will be controlled by the bilinear decoupling constant. Hölder's inequality gives that

$$(\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|)^{p/2} \leqslant \nu^{-(p-2)} \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} (|\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|)^{p/2}$$

and hence

$$\int_B (\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|)^{p/2}\, dx \leqslant \nu^{-(p-2)} \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} \int_B (|\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|)^{p/2}\, dx.$$

By bilinear decoupling, the above is bounded above by

$$\nu^{-(p-2)} D_{p,E}(\delta, m)^p \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| > 1}} (\sum_{J \in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|_{L^p(\widetilde{w}_{B,E})}^2)^{p/4} (\sum_{J \in P_{\delta^{1/2}}(I_j)} \|\mathcal{E}_J g\|_{L^p(\widetilde{w}_{B,E})}^2)^{p/4}.$$

Note that here we have used that $\nu/\delta^{1/2} \in 2^{\mathbb{N}}$. Since $\delta^{1/2}$ is dyadic and I_i and I_j are dyadic intervals, this is bounded above by

$$\nu^{-p} D_{p,E}(\delta, m)^p (\sum_{J \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_J g\|_{L^p(\widetilde{w}_{B,E})}^2)^{p/2}.$$

Now we consider the diagonal contribution. The triangle inequality followed by Cauchy-Schwarz gives that

$$\|\sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^{p/2}(B)} \leqslant \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} \|\mathcal{E}_{I_i}g\|_{L^p(B)} \|\mathcal{E}_{I_j}g\|_{L^p(B)} \tag{2.77}$$

Let $I = a + [0,\nu]$ be an interval of length ν. Let $L(\xi) = (\xi - a)/\nu$, $g_L := g \circ L^{-1}$, and $T = \left(\begin{smallmatrix} \nu & 2a\nu \\ 0 & \nu^2 \end{smallmatrix}\right)$. A change of variables then gives that $|(\mathcal{E}_I g)(x)| = \nu|(\mathcal{E}_{[0,1]} g_L)(Tx)|$ and therefore

$$\|\mathcal{E}_I g\|_{L^p(B)} = \nu^{1-3/p}\|\mathcal{E}_{[0,1]} g_L\|_{L^p(T(B))}. \tag{2.78}$$

Note that $T(B)$ is a parallelogram contained in a $3\nu\delta^{-1} \times \nu^2\delta^{-1}$ rectangle. Covering $T(B)$ by squares $\mathcal{B} = \{\Delta\}$ of side length $\nu^2\delta^{-1}$ gives that

$$\nu^{1-3/p}\|\mathcal{E}_{[0,1]} g_L\|_{L^p(T(B))} \leqslant \nu^{1-3/p}(\sum_{\Delta \in \mathcal{B}} \|\mathcal{E}_{[0,1]} g_L\|_{L^p(\Delta)}^p)^{1/p}. \tag{2.79}$$

Applying the definition of the decoupling constant (and using that $\nu\delta^{-1/2} \in 2^{\mathbb{N}}$), gives that for each square Δ,

$$\|\mathcal{E}_{[0,1]} g_L\|_{L^p(\Delta)}^p \leqslant \widetilde{D}_{p,E}(\delta/\nu^2)^p (\sum_{J \in P_{\delta^{1/2}/\nu}([0,1])} \|\mathcal{E}_J g_L\|_{L^p(\widetilde{w}_{\Delta,E})}^2)^{p/2}.$$

Inserting this into (2.79) bounds the left hand side of (2.79) by

$$\widetilde{D}_{p,E}(\delta/\nu^2)(\sum_{\Delta \in \mathcal{B}}(\sum_{J \in P_{\delta^{1/2}/\nu}([0,1])} (\nu^{1-3/p}\|\mathcal{E}_J g_L\|_{L^p(\widetilde{w}_{\Delta,E})})^2)^{p/2})^{1/p}.$$

Applying the same change of variables as in (2.78) followed by Minkowski's inequality (using that $p \geqslant 2$) gives that the above is bounded by

$$\begin{aligned} &\widetilde{D}_{p,E}(\delta/\nu^2)(\sum_{J \in P_{\delta^{1/2}}(I)} \|\mathcal{E}_J g\|_{L^p(\sum_{\Delta \in \mathcal{B}} \widetilde{w}_{\Delta,E} \circ T)}^2)^{1/2} \\ &\qquad\qquad \leqslant 720^{E/p}\widetilde{D}_{p,E}(\delta/\nu^2)(\sum_{J \in P_{\delta^{1/2}}(I)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}. \end{aligned}$$

By Proposition 2.3.11, $\widetilde{D}_{p,E}(\delta) \leqslant E^{70E} D_{p,E}(\delta)$ and so the above gives that

$$\|\mathcal{E}_I g\|_{L^p(B)} \leqslant E^{75E} D_{p,E}(\delta/\nu^2)(\sum_{J \in P_{\delta^{1/2}}(I)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2}$$

for each interval I of length ν.

Using this for each interval that shows up on the right hand side of (2.77) gives an upper bound of

$$E^{150E} D_{p,E}(\delta/\nu^2)^2 \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} (\sum_{J \in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|_{L^p(w_{B,E})}^2)^{1/2} (\sum_{J' \in P_{\delta^{1/2}}(I_j)} \|\mathcal{E}_{J'} g\|_{L^p(w_{B,E})}^2)^{1/2}.$$

Using that $2ab \leqslant a^2 + b^2$, the above is bounded by

$$E^{150E} D_{p,E}(\delta/\nu^2)^2 \cdot \frac{1}{2} \sum_{\substack{1 \leqslant i,j \leqslant 1/\nu \\ |i-j| \leqslant 1}} \left(\sum_{J \in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|^2_{L^p(w_{B,E})} + \sum_{J' \in P_{\delta^{1/2}}(I_j)} \|\mathcal{E}_{J'} g\|^2_{L^p(w_{B,E})} \right)$$
$$\leqslant 2 \cdot E^{150E} D_{p,E}(\delta/\nu^2)^2 \sum_{J \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_J g\|^2_{L^p(w_{B,E})}.$$

Therefore if $\delta^{1/2} \in 2^{-\mathbb{N}}$ and $\delta^{1/2}\nu^{-1} \in 2^{-\mathbb{N}}$, we have

$$D_{p,E}(\delta) \leqslant 2 \cdot E^{75E} D_{p,E}(\frac{\delta}{\nu^2}) + \frac{\sqrt{2}}{\nu} D_{p,E}(\delta, m)$$

which completes the proof of Lemma 2.5.2. □

Proposition 2.5.3. *Let $E \geqslant 100$ and $2 \leqslant p \leqslant 6$. Fix an arbitrary integer $m \geqslant 1$. Let $\delta^{1/2} \in 2^{-\mathbb{N}}$ and K be the largest positive integer such that $\delta^{1/2}\nu^{-K} \in 2^{-\mathbb{N}}$. Then*

$$D_{p,E}(\delta) \leqslant \delta^{100E \log_\nu E} \nu^{-1} \max(1, \max_{i=0,1,\dots,K-1} D_{p,E}(\delta\nu^{-2i}, m)).$$

Proof. Note that $\delta^{1/2} \in 2^{-\mathbb{N}}$ and $\delta^{1/2}\nu^{-K} \in 2^{-\mathbb{N}}$ imply that for $i = 0, 1, \dots, K$, $\delta^{1/2}\nu^{-i} \in 2^{-\mathbb{N}}$. In particular for each $i = 1, 2, \dots, K$, both $\delta^{1/2}\nu^{-i+1}$ and $\delta^{1/2}\nu^{-i}$ are in $2^{-\mathbb{N}}$ and hence

$$D_{p,E}(\delta\nu^{-2i+2}) \leqslant E^{100E}(D_{p,E}(\delta\nu^{-2i}) + \frac{1}{\nu} D_{p,E}(\delta\nu^{-2i+2}, m)).$$

Combining these K inequalities then gives that

$$D_{p,E}(\delta) \leqslant E^{100EK}(D_{p,E}(\delta\nu^{-2K}) + 2\nu^{-1} \max_{i=0,1,\dots,K-1} D_{p,E}(\delta\nu^{-2i}, m)). \tag{2.80}$$

To control $D_p(\delta\nu^{-2K})$, we use the definition of K. In particular, since $\delta^{1/2} \in 2^{-\mathbb{N}}$, $\delta^{1/2}\nu^{-(K+1)}$ is dyadic but $\geqslant 1$. Therefore $\delta^{1/2}\nu^{-K-1} \geqslant 1$ and so $\delta^{1/2}\nu^{-K} \geqslant \nu$. The trivial bound then gives that

$$D_{p,E}(\delta\nu^{-2K}) \leqslant 2^{E/p}(\delta\nu^{-2K})^{-1/4} \leqslant 2^E \nu^{-1/2}.$$

Since $\delta^{1/2}\nu^{-K} \leqslant 1$, $K \leqslant \log_{\nu^{-1}} \delta^{-1/2}$ and hence

$$E^{100EK} \leqslant \delta^{50E \log_\nu E}.$$

Inserting the above two centered equations into (2.80) then completes the proof of Proposition 2.5.3. □

2.6 Ball inflation

We first discuss some basic geometry. Let $\mathbb{P} := \{(\xi, \xi^2) : \xi \in [0,1]\}$ and $\pi : \mathbb{P} \to [0,1]$ be the projection map which sends $(\xi, \xi^2) \mapsto \xi$. Since I_1, I_2 are d-separated, for any $P \in I_1, Q \in I_2$, we have $|P - Q| \geqslant d$. Observe that

$$n(\pi^{-1}(P)) = \frac{(-2P, 1)}{\sqrt{1 + 4P^2}}$$

and similarly for Q (where here $n(\pi^{-1}(P))$ refers to the normal vector to the parabola at the point $\pi^{-1}(P)$). Let θ be the angle between $n(\pi^{-1}(P))$ and $n(\pi^{-1}(Q))$. Then since $|P - Q| \geqslant d$,

$$\sin\theta = \frac{2|P-Q|}{\sqrt{(1+4P^2)(1+4Q^2)}} \geqslant \frac{2}{5}d.$$

In the terminology of [BD17], I_1 and I_2 are $2d/5$-transverse.

We will now prove the following effective ball inflation inequality.

Theorem 2.6.1. *Let $p \geqslant 4$, $0 < \delta < 1/10$, $E \geqslant 100$, and $0 < d < 1/2$. Let $I_1, I_2 \subset [0,1]$ be two d-separated intervals of length $\geqslant \delta$ such that $|I_i|/\delta \in \mathbb{N}$. Let B be an arbitrary square in $\mathbb{R}^2$ with side length δ^{-2} and let $\mathcal{B}$ be the unique partition of B into squares Δ of side length δ^{-1}. Then for all $g : [0,1] \to \mathbb{C}$, we have*

$$\frac{1}{|\mathcal{B}|}\sum_{\Delta \in \mathcal{B}} \text{geom}\Big(\sum_{J \in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_\#(\tilde{w}_{\Delta,E})}\Big)^{p/2} \leqslant E^{50Ep} d^{-1} (\log\frac{1}{\delta})^{p/2}\, \text{geom}\Big(\sum_{J \in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_\#(\tilde{w}_{B,E})}\Big)^{p/2}. \tag{2.81}$$

Furthermore, for $p = 4$, the estimate is true without the logarithm.

This inequality allows us to keep the frequency scale the same while increasing (inflating) the spatial scale and is a key step in the iteration. We will first prove a version of Theorem 2.6.1 where we additionally assume that all the $\|\mathcal{E}_J g\|$ are of comparable size (for each I_i). Then we remove this assumption by dyadic pigeonholing to obtain (2.81).

Lemma 2.6.2. *Let $p > 4$ and everything else be as defined in Theorem 2.6.1. Furthermore, let $\mathcal{F}_1$ be a collection of intervals in $P_\delta(I_1)$ such that for each pair of intervals $J, J' \in \mathcal{F}_1$, we* *have*

$$\frac{1}{2} < \frac{\|\mathcal{E}_J g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}}{\|\mathcal{E}_{J'} g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}} \leqslant 2. \tag{2.82}$$

Similarly define $\mathcal{F}_2$. Then for all $g : [0,1] \to \mathbb{C}$ we have

$$\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} \operatorname{geom}(\sum_{J\in\mathcal{F}_i} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^{p/2} \leqslant E^{30Ep} d^{-1} \operatorname{geom}(\sum_{J\in\mathcal{F}_i} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}. \tag{2.83}$$

Proof. For each $J \in P_\delta(I_i)$ centered at c_J, cover B by a set $\mathcal{T}_J$ of mutually parallel nonoverlapping boxes P_J with dimension $\delta^{-1} \times \delta^{-2}$ with longer side pointing in the direction of the normal vector to $\mathbb{P}$ at $\pi^{-1}(c_J)$. Note that any $\delta^{-1} \times \delta^{-2}$ box outside $4B$ cannot cover B itself. Thus we may assume that all the boxes in $\mathcal{T}_J$ are contained in $4B$. Finally, let $P_J(x)$ denote the box in $\mathcal{T}_J$ containing x and let $2P_J$ be the $2\delta^{-1} \times 2\delta^{-2}$ box having the same center and orientation as P_J.

Since $p > 4$, Hölder's inequality yields that

$$(\sum_{J\in\mathcal{F}_i} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^{p/2} \leqslant (\sum_{J\in\mathcal{F}_i} \|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^2 |\mathcal{F}_i|^{p/2-2}.$$

Thus the left hand side of (2.83) is bounded above by

$$(\prod_{i=1}^{2} |\mathcal{F}_i|^{p/4-1}) \frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} \prod_{i=1}^{2} (\sum_{J\in\mathcal{F}_i} \|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})}). \tag{2.84}$$

For $x \in 4B$, define

$$H_J(x) := \begin{cases} \sup_{y\in 2P_J(x)} \|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B(y,\delta^{-1}),E})} & \text{if } x \in \bigcup_{P_J\in\mathcal{T}_J} P_J \\ 0 & \text{if } x \in 4B \backslash \bigcup_{P_J\in\mathcal{T}_J} P_J. \end{cases} \tag{2.85}$$

For each $x \in \Delta$, observe that $\Delta \subset 2P_J(x)$. Therefore for each $x \in \Delta$, $c_\Delta \in 2P_J(x)$ and hence

$$\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})} \leqslant H_J(x) \tag{2.86}$$

for all $x \in \Delta$. Thus

$$\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\tilde{w}_{\Delta,E})})$$

$$= \sum_{\substack{J_1\in\mathcal{F}_1\\ J_2\in\mathcal{F}_2}}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\|\mathcal{E}_{J_1}g\|^{p/2}_{L^{p/2}_{\#}(\tilde{w}_{\Delta,E})}\|\mathcal{E}_{J_2}g\|^{p/2}_{L^{p/2}_{\#}(\tilde{w}_{\Delta,E})}\frac{1}{|\Delta|}\int_{\Delta}dx$$

$$\leqslant \sum_{\substack{J_1\in\mathcal{F}_1\\ J_2\in\mathcal{F}_2}}\frac{1}{|B|}\int_B H_{J_1}(x)H_{J_2}(x)\,dx \tag{2.87}$$

where the last inequality we have used (2.86). By how H_J is defined, H_J is constant on each $P_J \in \mathcal{T}_J$. That is, for each $x \in \bigcup_{P_J\in\mathcal{T}_J} P_J$,

$$H_J(x) = \sum_{P_J\in\mathcal{T}_J} c_{P_J}1_{P_J}(x)$$

for some constants $c_{P_J} \geqslant 0$. Then

$$\frac{1}{|B|}\int_B H_{J_1}(x)H_{J_2}(x)\,dx = \frac{1}{|B|}\sum_{\substack{P_{J_1}\in\mathcal{T}_{J_1}\\ P_{J_2}\in\mathcal{T}_{J_2}}} c_{P_{J_1}}c_{P_{J_2}}|(P_{J_1}\cap P_{J_2})\cap B|$$

$$\leqslant \frac{1}{|B|}\sum_{\substack{P_{J_1}\in\mathcal{T}_{J_1}\\ P_{J_2}\in\mathcal{T}_{J_2}}} c_{P_{J_1}}c_{P_{J_2}}|P_{J_1}\cap P_{J_2}|$$

where the last inequality is because $c_{P_J} \geqslant 0$ for all P_J. Since $|P_J| = \delta^{-3}$ we also have

$$\frac{1}{|B|}\int_{4B} H_J(x)\,dx = \frac{1}{|B|}\int_{\bigcup_{P_J\in\mathcal{T}_J}P_J}\sum_{P_J\in\mathcal{T}_J}c_{P_J}1_{P_J}(x)\,dx = \delta\sum_{P_J\in\mathcal{T}_J}c_{P_J}.$$

Recall that $J_1 \in \mathcal{F}_1 \subset P_\delta(I_1)$ and $J_2 \in \mathcal{F}_2 \subset P_\delta(I_2)$. Since I_1 and I_2 are d-separated, so are J_1 and J_2. Let $\angle_{J_1,J_2}$ be the angle between the directions of J_1 and J_2. By geometry discussion at the beginning of this section, $\sin(\angle_{J_1,J_2}) \geqslant 2d/5$. Therefore

$$|P_{J_1}\cap P_{J_2}| \leqslant \frac{\delta^{-2}}{\sin(\angle_{J_1,J_2})} \leqslant \frac{\delta^{-2}}{2d/5}.$$

Applying this gives

$$\frac{1}{|B|}\sum_{\substack{P_{J_1}\in\mathcal{T}_{J_1}\\ P_{J_2}\in\mathcal{T}_{J_2}}} c_{P_{J_1}}c_{P_{J_2}}|P_{J_1}\cap P_{J_2}|$$

$$\leqslant \frac{3\delta^{-2}d^{-1}}{|B|}\prod_{i=1}^{2}(\frac{\delta^{-1}}{|B|}\int_{4B}H_{J_i}(x)\,dx) = \frac{3d^{-1}}{|B|^2}\prod_{i=1}^{2}\int_{4B}H_{J_i}(x)\,dx.$$

Therefore (2.87) is bounded above by

$$3d^{-1}\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\frac{1}{|B|}\int_{4B}H_J(x)\,dx) = 768d^{-1}\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\frac{1}{|4B|}\int_{4B}H_J(x)\,dx). \tag{2.88}$$

We now apply Lemma 2.6.3, proven later, to (2.88). This gives that an upper bound of

$$E^{20Ep}d^{-1}\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})$$

where here we have also used that $E \geqslant 100$ and $p \geqslant 2$. Thus (2.84) is bounded above by

$$E^{20Ep}d^{-1}(\prod_{i=1}^{2}|\mathcal{F}_i|^{p/4-1})\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}). \tag{2.89}$$

To obtain the right hand side of (2.83) we now use that intervals in $\mathcal{F}_i$ satisfy (2.82). We have

$$\begin{aligned}(\prod_{i=1}^{2}|\mathcal{F}_i|^{p/4-1})\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}) &\leqslant \prod_{i=1}^{2}|\mathcal{F}_i|^{p/4-1}\prod_{i=1}^{2}(|\mathcal{F}_i|\max_{J'\in\mathcal{F}_i}\|\mathcal{E}_{J'} g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})\\
&= \left(\prod_{i=1}^{2}(|\mathcal{F}_i|\max_{J'\in\mathcal{F}_i}\|\mathcal{E}_{J'} g\|^{2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{1/2}\right)^{p/2}\\
&\leqslant \left(\prod_{i=1}^{2}(\sum_{J\in\mathcal{F}_i}4\|\mathcal{E}_J g\|^{2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{1/2}\right)^{p/2}\\
&= 2^p\operatorname{geom}(\sum_{J\in\mathcal{F}_i}\|\mathcal{E}_J g\|^{2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}\end{aligned}$$

where the second inequality is due to (2.82). Inserting this into (2.89) then completes the proof of Lemma 2.6.2. □

Lemma 2.6.3. *Let* H_J *be as defined in* (2.85)*. Then*

$$\frac{1}{|4B|}\int_{4B}H_J(x)\,dx \leqslant E^{8Ep}\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}.$$

Proof. This is the inequality proven in (29) of [BD17] without explicit constants. We follow their proof, this time paying attention to the implied constants.

Fix arbitrary $J \subset [0,1]$ of length δ and center c_J. For $x \in \bigcup_{P_J\in\mathcal{T}_J}P_J = \operatorname{supp} H_J \subset 4B$, fix arbitrary $y \in 2P_J(x)$. Note that $2P_J(x)$ points is a rectangle of dimension $2\delta^{-1}\times 2\delta^{-2}$ with the longer side pointing in the direction of $(-2c_J, 1)$.

Let R_J and θ_J be as in Lemma 2.2.5. Since $c_J \in [\delta/2, 1-\delta/2]$, both $\cos\theta_J$ and $\sin\theta_J$ are nonzero. Note that R_J is the rotation matrix such that R_J^{-1} applied to $2P_J(x)$ gives an axis parallel rectangle of dimension $2\delta^{-1} \times 2\delta^{-2}$ with the longer side pointing in the vertical direction. Since $y \in 2P_J(x)$, we can write

$$R_J^{-1}y = R_J^{-1}x + \overline{y}$$

where $|\overline{y}_1| \leqslant 2\delta^{-1}$ and $|\overline{y}_2| \leqslant 2\delta^{-2}$. We then have

$$\|\mathcal{E}_J g\|_{L^{p/2}(\widetilde{w}_{B(y,\delta^{-1}),E})}^{p/2} = \int_{\mathbb{R}^2} |(\mathcal{E}_J g)(s)|^{p/2} \widetilde{w}_{B(x+R_J\overline{y},\delta^{-1}),E}(s)\, ds$$

Writing $\overline{y} = (\overline{y}_1, 0)^T + (0, \overline{y}_2)^T$ and a change of variables gives that the above is equal to

$$\int_{\mathbb{R}^2} |(\mathcal{E}_J g)(s + x + R_J(0,\overline{y}_2)^T)|^{p/2} \widetilde{w}_{B(R_J(\overline{y}_1,0)^T,\delta^{-1}),E}(s)\, ds. \tag{2.90}$$

Inserting Lemma 2.2.5 into (2.90) gives that

$$\|\mathcal{E}_J g\|_{L^{p/2}(\widetilde{w}_{B(y,\delta^{-1}),E})}^{p/2} \leqslant 16^E \int_{\mathbb{R}^2} |(\mathcal{E}_J g)(s + x + R_J(0,\overline{y}_2)^T)|^{p/2} \widetilde{w}_{B(0,\delta^{-1}),E}(s)\, ds. \tag{2.91}$$

Observe that

$$|(\mathcal{E}_J g)(s + x + R_J(0,\overline{y}_2)^T)| = |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(\lambda) e(\lambda \cdot (s+x)) e(\lambda \cdot R_J(0,\overline{y}_2)^T)\, d\lambda|.$$

Since R_J is a rotation matrix, a change of variables gives that the above is equal to

$$|\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J\lambda) e(\lambda \cdot R_J^{-1}(s+x)) e(\lambda \cdot (0,\overline{y}_2)^T)\, d\lambda| \tag{2.92}$$

Writing

$$e(\lambda \cdot (0,\overline{y}_2)^T) = e((\lambda_2 - c_J^2)\overline{y}_2) e(c_J^2 \overline{y}_2) = e(c_J^2\overline{y}_2) \sum_{k=0}^{\infty} \frac{(2\pi i)^k \overline{y}_2^k}{k!} (\lambda_2 - c_J^2)^k$$

and using that $|\overline{y}_2| \leqslant 2\delta^{-2}$ shows that (2.92) is

$$\leqslant \sum_{k=0}^{\infty} \frac{(4\pi)^k}{k!} |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J\lambda) e(\lambda \cdot R_J^{-1}(s+x)) (\frac{\lambda_2 - c_J^2}{\delta^2})^k\, d\lambda|$$

Applying the change of variables $\eta = \lambda - \pi^{-1}(c_J)$ gives that the above is

$$\leqslant \sum_{k=0}^{\infty} \frac{30^k}{k!} |\int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J(\eta + \pi^{-1}(c_J))) e(\eta \cdot R_J^{-1}(s+x)) (\frac{\eta_2}{2\delta^2})^k\, d\eta|. \tag{2.93}$$

Note that $\widehat{\mathcal{E}_J g}(R_J(\eta+\pi^{-1}(c_J)))$ is supported in a $4\delta \times 4\delta^2$ box centered at the origin pointing in the horizontal direction. Thus we may insert the cutoff Ψ from Lemma 2.2.10 in (2.93). Then (2.93) becomes

$$\sum_{k=0}^{\infty} \frac{30^k}{k!} \Big| \int_{\mathbb{R}^2} \widehat{\mathcal{E}_J g}(R_J(\eta + \pi^{-1}(c_J))) e(\eta \cdot R_J^{-1}(s+x)) (\frac{\eta_2}{2\delta})^k \Psi(\frac{\eta_1}{2\delta}) \Psi(\frac{\eta_2}{2\delta}) \, d\eta \Big|.$$

Note that we are a bit wasteful since $\Psi(\eta_1/(2\delta))\Psi(\eta_2/(2\delta))$ is equal to 1 on $[-2\delta, 2\delta]^2$ rather than $[-2\delta, 2\delta] \times [-2\delta^2, 2\delta^2]$, but this will turn out to not matter.

Let $\Phi_k(t) := t^k \Psi(t)$ and let

$$(M_k f)(x) = \int_{\mathbb{R}^2} \widehat{f}(R_J(\eta + \pi^{-1}(c_J))) e(\eta \cdot x) \Psi(\frac{\eta_1}{2\delta}) \Phi_k(\frac{\eta_2}{2\delta}) \, d\eta.$$

Thus we have shown that

$$|(\mathcal{E}_J g)(s + x + R_J(0, \overline{y}_2)^T)| \leqslant \sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_J^{-1}(s+x))|$$

and combining this with (2.91) gives that for $x \in \bigcup_{P_J \in \mathcal{T}_J} P_J$ and $y \in 2P_J(x)$,

$$\|\mathcal{E}_J g\|^{p/2}_{L^{p/2}_{\#}(\widetilde{w}_{B(y,\delta^{-1}),E})} \leqslant 16^E \delta^2 \int_{\mathbb{R}^2} (\sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_J^{-1}(s+x))|)^{p/2} \widetilde{w}_{B(0,\delta^{-1}),E}(s) \, ds.$$

Thus

$$\begin{aligned} &\frac{1}{|4B|} \int_{4B} H_J(x) \, dx \\ &\quad \leqslant 16^{E-1} \delta^6 \int_{4B} \int_{\mathbb{R}^2} (\sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(R_J^{-1}(s+x))|)^{p/2} \widetilde{w}_{B(0,\delta^{-1}),E}(s) \, ds \, dx \\ &\quad = 16^{E-1} \delta^6 \int_{\mathbb{R}^2} (\sum_{k=0}^{\infty} \frac{30^k}{k!} |(M_k \mathcal{E}_J g)(u)|)^{p/2} (\int_{4B} \widetilde{w}_{B(x,\delta^{-1}),E}(R_J u) \, dx) \, du. \end{aligned} \tag{2.94}$$

As $1_{4B} \leqslant 4^E \widetilde{w}_{4B,E} \leqslant 64^E \widetilde{w}_{B,E}$ and since B is centered at the origin,

$$\begin{aligned} \int_{4B} \widetilde{w}_{B(x,\delta^{-1}),E}(R_J u) \, dx &= (1_{4B} * \widetilde{w}_{B(0,\delta^{-1}),E})(R_J u) \\ &\leqslant 64^E (\widetilde{w}_{B,E} * \widetilde{w}_{B(0,\delta^{-1}),E})(R_J u) \leqslant 256^E \delta^{-2} \widetilde{w}_{B,E}(R_J u). \end{aligned}$$

Thus it follows that (2.94) is bounded by

$$2^{12E} \delta^4 (\sum_{k=0}^{\infty} \frac{30^k}{k!} \|M_k \mathcal{E}_J g \circ R_J^{-1}\|_{L^{p/2}(\widetilde{w}_{B,E})})^{p/2}. \tag{2.95}$$

Inserting an extra $e(R_J\pi^{-1}(c_J)\cdot s)$ and applying a change of variables gives

$$\begin{aligned}|(M_k\mathcal{E}_Jg)(R_J^{-1}s)| &= |\int_{\mathbb{R}^2}\widehat{\mathcal{E}_Jg}(R_J(\eta+\pi^{-1}(c_J)))e(R_J\eta\cdot s)\Psi(\frac{\eta_1}{2\delta})\Phi_k(\frac{\eta_2}{2\delta})\,d\eta| \\ &= |\int_{\mathbb{R}^2}\widehat{\mathcal{E}_Jg}(\gamma)e(\gamma\cdot s)\widehat{m_k}(\gamma)\,d\gamma|\end{aligned}$$

where

$$\widehat{m_k}(\gamma) = \Psi(\frac{\gamma_1\cos\theta_J+\gamma_2\sin\theta_J-c_J}{2\delta})\Phi_k(\frac{\gamma_2\cos\theta_J-\gamma_1\sin\theta_J-c_J^2}{2\delta}).$$

Then $|M_k\mathcal{E}_Jg\circ R_J^{-1}| = |\mathcal{E}_Jg * m_k| \leqslant |\mathcal{E}_Jg| * |m_k|$ and Hölder's inequality implies

$$(|\mathcal{E}_Jg| * |m_k|)^{p/2} \leqslant (|\mathcal{E}_Jg|^{p/2} * |m_k|)\|m_k\|_{L^1}^{p/2-1}.$$

Therefore

$$\|M_k\mathcal{E}_Jg\circ R_J^{-1}\|_{L^{p/2}(\widetilde{w}_{B,E})} \leqslant \|m_k\|_{L^1(\mathbb{R}^2)}^{1-2/p}\|\mathcal{E}_Jg\|_{L^{p/2}(\widetilde{w}_{B,E}*|m_k|(-\cdot))} \tag{2.96}$$

where here $|m_k|(-\cdot)$ is the function $|m_k|(-x)$. Since Φ and Ψ are both Schwartz functions, our goal will be to use the rapid decay to show that $|m_k| \lesssim_E \widetilde{w}_{B,E}$. A change of variables gives

$$\begin{aligned}|m_k(x)| &= |\int_{\mathbb{R}^2}\widehat{m_k}(\gamma)e^{2\pi i x\cdot\gamma}\,d\gamma| \\ &= 4\delta^2|\int_{\mathbb{R}}\Psi(w_1)e^{2\pi i(R_J^{-1}x)_1(2\delta w_1)}\,dw_1\int_{\mathbb{R}}\Phi_k(w_2)e^{2\pi i(R_J^{-1}x)_2(2\delta w_2)}\,dw_2|.\end{aligned}$$

Since $\Psi = \Phi_0$, by Lemma 2.2.10,

$$|\int_{\mathbb{R}}\Psi(w_1)e^{2\pi i(R_J^{-1}x)_1(2\delta w_1)}\,dw_1| \leqslant \frac{E^{5E}}{(1+2\delta|(R_J^{-1}x)_1|)^{2E}}$$

and

$$|\int_{\mathbb{R}}\Phi_k(w_2)e^{2\pi i(R_J^{-1}x)_2(2\delta w_2)}\,dw_2| \leqslant \frac{6^kE^{5E}}{(1+2\delta|(R_J^{-1}x)_2|)^{2E}}.$$

Therefore

$$|m_k(x)| \leqslant 4\delta^2 6^k E^{10E}(1+\frac{|(R_J^{-1}x)_1|}{\delta^{-1}})^{-2E}(1+\frac{|(R_J^{-1}x)_2|}{\delta^{-1}})^{-2E}. \tag{2.97}$$

Thus we have

$$\|m_k\|_{L^1(\mathbb{R}^2)}^{1-2/p} \leqslant (6^k E^{11E})^{1-2/p}. \tag{2.98}$$

Applying Lemma 2.2.6 to (2.97) shows

$$|m_k(x)| \leqslant 4(6^k E^{10E})\delta^2 \widetilde{w}_{B(0,\delta^{-1}),E}(x).$$

Note that this inequality does not change if we replace x with $-x$ on the left hand side since the right hand side is radial. Lemma 2.2.1 then implies that

$$\widetilde{w}_{B,E} * |m_k|(-\cdot) \leqslant 6^k E^{11E} \widetilde{w}_{B,E}$$

and hence

$$\|\mathcal{E}_J g\|_{L^{p/2}(\widetilde{w}_{B,E}*|m_k|(-\cdot))} \leqslant (6^k E^{11E})^{2/p} \|\mathcal{E}_J g\|_{L^{p/2}(\widetilde{w}_{B,E})}.$$

Combining this with (2.95), (2.96), and (2.98) shows that

$$\frac{1}{|4B|}\int_{4B} H_J(x)\,dx \leqslant 2^{12E} E^{11Ep/2} \delta^4 (\sum_{k=0}^{\infty} \frac{180^k}{k!} \|\mathcal{E}_J g\|_{L^{p/2}(\widetilde{w}_{B,E})})^{p/2} \leqslant E^{8Ep} \|\mathcal{E}_J g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}^{p/2}$$

where in the last inequality we have used that $E \geqslant 100$ and $p \geqslant 2$. This completes the proof of Lemma 2.6.3. □

Proof of Theorem 2.6.1. If $p = 4$, the proof of Lemma 2.6.2 (in particular (2.89)) implies that we can just take $\mathcal{F}_i = P_\delta(I_i)$ and discard the requirement in (2.82) since the only reason we dyadically decomposed and restricted to $p > 4$ was to match the $L^{p/2}_{\#}$ with the ℓ^2 sum over $\sum_{J \in \mathcal{F}_i}$ in (2.83).

From now on we assume $p > 4$. For $i = 1, 2$, let

$$M_i := \max_{J \in P_\delta(I_i)} \|\mathcal{E}_J g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})}.$$

For each $i = 1, 2$, let $\mathcal{F}_{i,0}$ denote the set of intervals $J' \in P_\delta(I_i)$ such that

$$\|\mathcal{E}_{J'} g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})} \leqslant \delta^3 M_i$$

and partition the remaining intervals in $P_\delta(I_i)$ into $\lceil \log_2(\delta^{-3})\rceil$ many classes $\mathcal{F}_{i,k}$ (with $k = 1, 2, \ldots, \lceil \log_2(\delta^{-3})\rceil$) such that

$$2^{k-1}\delta^3 M_i < \|\mathcal{E}_{J'}g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})} \leqslant 2^k\delta^3 M_i$$

for all $J' \in \mathcal{F}_{i,k}$. Note that $\mathcal{F}_{i,k}$ satisfies the hypothesis (2.82) given in Lemma 2.6.2. For $1 \leqslant k, l \leqslant \lceil \log_2(\delta^{-3})\rceil$, let

$$F_\Delta(k,l) := (\sum_{J\in\mathcal{F}_{1,k}} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^{p/4}(\sum_{J\in\mathcal{F}_{2,l}} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^{p/4}.$$

Note that $F_\Delta(a,b) = F_\Delta(b,a)$.

The left hand side of (2.81) is equal to

$$\begin{aligned}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}(&\sum_{0\leqslant k,l\leqslant\lceil\log_2(\delta^{-3})\rceil}\sum_{\substack{J\in\mathcal{F}_{1,k}\\ J'\in\mathcal{F}_{2,l}}} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})}\|\mathcal{E}_{J'} g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})})^{p/4}\\ &\leqslant (\lceil \log_2(\delta^{-3})\rceil + 1)^{\frac{p}{2}-2}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\sum_{0\leqslant k,l\leqslant\lceil\log_2(\delta^{-3})\rceil} F_\Delta(k,l). \end{aligned} \tag{2.99}$$

We then have

$$\begin{aligned}&\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}}\sum_{k,l=0}^{\lceil\log_2(\delta^{-3})\rceil} F_\Delta(k,l)\\ &\quad= \frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(0,0) + 2\sum_{k=1}^{\lceil\log_2(\delta^{-3})\rceil}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(0,k) + \sum_{k,l=1}^{\lceil\log_2(\delta^{-3})\rceil}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(k,l).\end{aligned} \tag{2.100}$$

We first consider the third sum on the right hand side of (2.100). In this case, both families of intervals satisfy (2.82) in Lemma 2.6.2. Thus applying Lemma 2.6.2 gives that

$$\sum_{k,l=1}^{\lceil\log_2(\delta^{-3})\rceil}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(k,l) \leqslant \lceil \log_2(\delta^{-3})\rceil^2 E^{30Ep} d^{-1}\operatorname{geom}(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}. \tag{2.101}$$

The first two sums on the right hand side of (2.100) are taken care of by trivial estimates. We consider the first sum. From Proposition 2.2.14, $\widetilde{w}_{\Delta,E} \leqslant 48^E \widetilde{w}_{B,E}$ (we can obtain a better constant using Lemma 2.2.1 and $1_\Delta \leqslant 1_B$ but this is not needed). Therefore for $J' \in \mathcal{F}_{i,0}$,

$$\max_{\Delta\in\mathcal{B}} \|\mathcal{E}_{J'}g\|_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})} \leqslant \delta^{-4/p}48^{2E/p}\|\mathcal{E}_{J'}g\|_{L^{p/2}_{\#}(\widetilde{w}_{B,E})} \leqslant \delta^{3-4/p}48^{2E/p}M_i. \tag{2.102}$$

Since $|\mathcal{F}_{i,0}| \leqslant |P_\delta(I_i)| \leqslant \delta^{-1}$,

$$\begin{aligned} \frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(0,0) &\leqslant (|\mathcal{F}_{1,0}||\mathcal{F}_{2,0}|\delta^{12-16/p}48^{8E/p}M_1^2M_2^2)^{p/4} \\ &\leqslant \delta^{5p/2-4}48^{2E}\operatorname{geom}(M_i^2)^{p/2}. \end{aligned} \tag{2.103}$$

Since $p > 4$, $5p/2 - 4 > 6$ and so the union bound implies that (2.103) is bounded by

$$48^{2E}\operatorname{geom}(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}. \tag{2.104}$$

Finally we consider the second sum on the right hand side of (2.100). From the same proof as (2.102), for $J' \in \mathcal{F}_{2,k}$ with $k \neq 0$ we have

$$\max_{\Delta\in\mathcal{B}} \|\mathcal{E}_{J'}g\|_{L^{p/2}_{\#}(\widetilde{w}_{\Delta,E})} \leqslant \delta^{-4/p}48^{2E/p}M_2.$$

Therefore by the same reasoning as in the previous paragraph we have

$$\begin{aligned} \frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(0,k) &\leqslant (|\mathcal{F}_{1,0}||\mathcal{F}_{2,k}|(\delta^{3-4/p}48^{2E/p}M_1)^2(\delta^{-4/p}48^{2E/p}M_2)^2)^{p/4} \\ &\leqslant \delta^{p-4}48^{2E}\operatorname{geom}(M_i^2)^{p/2}. \end{aligned}$$

Since $p > 4$, we can discard the power of δ and hence

$$2\sum_{k=1}^{\lceil \log_2(\delta^{-3})\rceil}\frac{1}{|\mathcal{B}|}\sum_{\Delta\in\mathcal{B}} F_\Delta(0,k) \leqslant 2\lceil \log_2(\delta^{-3})\rceil 48^{2E}\operatorname{geom}(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}.$$

Combining this with (2.100), (2.101), and (2.104) shows that (2.99) (and hence the left hand side of (2.81)) is bounded above by

$$(\cdots)\operatorname{geom}(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^{p/2}_{\#}(\widetilde{w}_{B,E})})^{p/2}$$

where $(\cdots)$ is equal to

$$(\lceil \log_2(\delta^{-3})\rceil + 1)^{\frac{p}{2}-2}\Big(\lceil \log_2(\delta^{-3})\rceil^2 E^{30Ep}d^{-1} + 2\lceil \log_2(\delta^{-3})\rceil^2 48^{2E} + 48^{2E}\Big).$$

Since $\delta < 1/10$ and $E \geqslant 100$, this is bounded above by $E^{50Ep}d^{-1}(\log\frac{1}{\delta})^{p/2}$ which completes the proof of Theorem 2.6.1. $\square$

2.7 The iteration: preliminaries

We now setup the iteration scheme as in [BD17] except this time we pay attention to various integrality constraints from previous sections. Let $g : [0,1] \to \mathbb{C}$, $t \geqslant 1$, $q \leqslant r$, and I_1, I_2 two intervals in $[0,1]$. Let B^r be a square in $\mathbb{R}^2$ with side length δ^{-r}. Define

$$G_t(q,r) := \operatorname{geom}\Big(\sum_{J \in P_{\delta^q}(I_i)} \|\mathcal{E}_J g\|^2_{L^t_{\#}(\widetilde{w}_{B^r,E})}\Big)^{1/2}$$

and

$$A_p(q,r) = (\operatorname*{Avg}_{B^q \in P_{\delta^{-q}}(B^r)} G_2(q,q)^p)^{1/p} := \left(\frac{1}{|P_{\delta^{-q}}(B^r)|}\sum_{B^q \in P_{\delta^{-q}}(B^r)} G_2(q,q)^p\right)^{1/p}.$$

Strictly speaking we should be writing $G_t(q, B^r)$ instead of $G_t(q,r)$ since this expression is different for different B^r, however all that matters is keeping track of what our frequency and spatial scales are so for simplicity we will write r instead of B^r.

Remark 2.7.1. Note that for $G_t(q,r)$ and $A_p(q,r)$ to be defined, we need $|I_i|\delta^{-q} \in \mathbb{N}$ and $\delta^{-r+q} \in \mathbb{N}$.

For a square B^q, note that $A_p(q,q) = G_2(q,q)$ for all p. In $A_p(q,r)$, increasing q represents smaller frequency scales and increasing r represents larger spatial scales.

We note that G_t and A_p here are essentially the same as D_p and A_p, respectively in [BD17]. The only difference is that here we use the weight $\widetilde{w}_B$ instead of w_B. This is because our bilinear decoupling constant is defined with weight $\widetilde{w}_B$ rather than w_B.

Observe that G_t and A_p obey the following two basic properties. First the t parameter in G_t obeys Hölder's inequality.

Lemma 2.7.2 (Hölder's inequality for G_t)**.** *For each square $B^r \subset \mathbb{R}^2$, if $(1-\alpha)/p_1 + \alpha/p_2 = 1/t$, then*

$$G_t(q,r) \leqslant G_{p_1}(q,r)^{1-\alpha} G_{p_2}(q,r)^{\alpha}.$$

Proof. The factor $1/|B^r|$ in the definition of G_t balances out by how α is defined and hence we may replace $L^t_{\#}$, $L^{p_1}_{\#}$, and $L^{p_2}_{\#}$ with L^t, L^{p_1}, and L^{p_2}, respectively. Next, it suffices to prove that

$$\sum_{J\in P_{\delta^q}(I_i)} \|\mathcal{E}_J g\|_{L^t(\widetilde{w}_{B^r})}^2 \leqslant \Big(\sum_{J\in P_{\delta^q}(I_i)} \|\mathcal{E}_J g\|_{L^{p_1}(\widetilde{w}_{B^r})}^2\Big)^{1-\alpha}\Big(\sum_{J\in P_{\delta^q}(I_i)} \|\mathcal{E}_J g\|_{L^{p_2}(\widetilde{w}_{B^r})}^2\Big)^{\alpha}.$$

Applying Hölder's inequality gives that

$$\|\mathcal{E}_J g\|_{l_J^2 L^t}^2 \leqslant \Big\| \|\mathcal{E}_J g\|_{L^{p_1}}^{1-\alpha} \|\mathcal{E}_J g\|_{L^{p_2}}^{\alpha}\Big\|_{l_J^2}^2 = \Big\| \|\mathcal{E}_J g\|_{L^{p_1}}^{2(1-\alpha)} \|\mathcal{E}_J g\|_{L^{p_2}}^{2\alpha}\Big\|_{l_J^1} \leqslant \|\mathcal{E}_J g\|_{l_J^2 L^{p_1}}^{2(1-\alpha)} \|\mathcal{E}_J g\|_{l_J^2 L^{p_2}}^{2\alpha}$$

where here by L^p we mean $L^p(\widetilde{w}_{B^r})$. This completes the proof Lemma 2.7.2. □

Second, the averaging in the r parameter in A_p allows us to increase it.

Lemma 2.7.3. *Fix arbitrary positive integers $r \leqslant s \leqslant t$ and suppose δ is such that $|I_i|\delta^{-r} \in \mathbb{N}$, $\delta^{-s+r} \in \mathbb{N}$, and $\delta^{-t+s} \in \mathbb{N}$. Then for each square $B^t \subset \mathbb{R}^2$,*

$$\operatorname*{Avg}_{B^s \in P_{\delta^{-s}}(B^t)} A_p(r,s)^p = A_p(r,t)^p.$$

Proof. Fix arbitrary square $B^t \subset \mathbb{R}^2$. Expanding the left hand side, we have

$$\operatorname*{Avg}_{B^s\in P_{\delta^{-s}}(B^t)} A_p(r,s)^p = \operatorname*{Avg}_{B^s\in P_{\delta^{-s}}(B^t)} \operatorname*{Avg}_{B^r\in P_{\delta^{-r}}(B^s)} G_2(r,r)^p = \operatorname*{Avg}_{B^r\in P_{\delta^{-r}}(B^t)} G_2(r,r)^p = A_p(r,t)^p.$$

This completes the proof of Lemma 2.7.3. □

Finally, we end this section with an outline of our strategy. As in Section 2.5, let $m \geqslant 1$, $E \geqslant 100$, $2 \leqslant p \leqslant 6$, and $\nu := 2^{-16\cdot 2^m E^{10E}}$. Let I_1, I_2 be two arbitrary intervals in $P_\nu([0,1])$ which are at least ν-separated.

Lemma 2.7.4. *Suppose δ was such that $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$. Then for each square B^1 of side length δ^{-1}, we have*

$$\|\operatorname{geom}|\mathcal{E}_{I_i} g|\|_{L^p_{\#}(B^1)} \leqslant E^{100E}\nu^{1/2}\delta^{-1/2^{m+1}} A_p(\frac{1}{2^m}, 1).$$

Proof. Note that since $\delta^{-1/2^m} \in 2^{\mathbb{N}}$, $\delta^{-1+1/2^m} \in \mathbb{N}$ since $m \geqslant 1$. This proof is just an application of Hölder, Minkowski, and Bernstein inequalities. We have

$$\begin{aligned}
&\| \operatorname{geom} |\mathcal{E}_{I_i} g|\|_{L^p_{\#}(B^1)}^p = \frac{1}{|B^1|}\int_{B^1} \operatorname{geom} |\mathcal{E}_{I_i} g|^p = \frac{1}{|B^1|}\int_{B^1} \operatorname{geom} |\sum_{J \in P_{\delta^{1/2^m}}(I_i)} \mathcal{E}_J g|^p \\
&\leqslant (\nu^{1/2}\delta^{-1/2^{m+1}})^p \frac{1}{|B^1|}\int_{B^1} \operatorname{geom}(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} |\mathcal{E}_J g|^2)^{p/2} \\
&= (\nu^{1/2}\delta^{-1/2^{m+1}})^p \operatorname*{Avg}_{B^{1/2^m} \in P_{\delta^{-1/2^m}}(B^1)} \| \operatorname{geom}(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} |\mathcal{E}_J g|^2)^{1/2}\|_{L^p_{\#}(B^{1/2^m})}^p.
\end{aligned}$$

Note that

$$\| \operatorname{geom}(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} |\mathcal{E}_J g|^2)^{1/2}\|_{L^p(B^{1/2^m})}^p \leqslant \operatorname{geom} \|(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} |\mathcal{E}_J g|^2)^{1/2}\|_{L^p(B^{1/2^m})}^p.$$

Since $p \geqslant 2$,

$$\|(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} |\mathcal{E}_J g|^2)^{1/2}\|_{L^p(B^{1/2^m})}^p \leqslant (\sum_{J \in P_{\delta^{1/2^m}}(I_i)} \|\mathcal{E}_J g\|_{L^p(B^{1/2^m})}^2)^{p/2}.$$

Combining the above three centered equations gives that

$$\begin{aligned}
&\| \operatorname{geom}|\mathcal{E}_{I_i} g|\|_{L^p_{\#}(B^1)} \\
&\qquad \leqslant \nu^{1/2}\delta^{-1/2^{m+1}} (\operatorname*{Avg}_{B^{1/2^m} \in P_{\delta^{-1/2^m}}(B^1)} \operatorname{geom}(\sum_{J \in P_{\delta^{1/2^m}}(I_i)} \|\mathcal{E}_J g\|_{L^p_{\#}(B^{1/2^m})}^2)^{p/2})^{1/p}.
\end{aligned}$$

Bernstein's inequality (Lemma 2.2.20) and that $p \leqslant 6$, $E \geqslant 100$ gives that

$$\|\mathcal{E}_J g\|_{L^p_{\#}(B^{1/2^m})} \leqslant 4^{pE/2}(pE/2)^{23pE/2}\|\mathcal{E}_J g\|_{L^2_{\#}(\tilde{w}_{B^{1/2^m},E})} \leqslant E^{100E}\|\mathcal{E}_J g\|_{L^2_{\#}(\tilde{w}_{B^{1/2^m},E})}.$$

Inserting this above gives that

$$\| \operatorname{geom} |\mathcal{E}_{I_i} g|\|_{L^p_{\#}(B^1)} \leqslant E^{100E}\nu^{1/2}\delta^{-1/2^{m+1}} A_p(\frac{1}{2^m}, 1)$$

which completes the proof of Lemma 2.7.4. □

Our target will be to prove an estimate of the form

$$A_p(2^{-m}, 1) \lesssim_{\delta,\nu,E,m} G_p(\frac{1}{2}, 1) \tag{2.105}$$

because then combining this with Lemma 2.7.4 gives an upper bound on the bilinear decoupling constant. Proposition 2.5.3 then allows us to control the linear decoupling constant. To prove (2.105), we will use ball inflation, l^2L^2 decoupling to prove an estimate of the form $A_p(2^{-\ell}, 2^{-\ell+1}) \lesssim_{\nu,E} A_p(2^{-\ell+1}, 2^{-\ell+1})$ for each $\ell = 2, 3, \ldots, m$. Then Lemma 2.7.3 allows us to patch all the estimates together.

The iteration is easier in the $2 \leqslant p \leqslant 4$ regime and so we will first do that case, then we will move on to the case when $4 < p < 6$. Finally, to control the decoupling constant at $p = 6$, we will apply Bernstein's inequality and use the decoupling constant at p' for some p' suitably close to 6.

2.8 Control of the bilinear decoupling constant

We now iterate to control the bilinear decoupling constant. We have two separate but similar cases. Our goal is to prove the following result.

Proposition 2.8.1. *Fix integers $m \geqslant 3$ and $E \geqslant 100$. Let $\nu := 2^{-16\cdot 2^m \cdot E^{10E}}$ and suppose δ is such that $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$.*

(a) If $2 \leqslant p \leqslant 4$, then

$$D_{p,E}(\delta, m) \leqslant \nu^{1/2}(E^{300E}\nu^{-1/4})^m \delta^{-\frac{1}{2^{m+1}}}.$$

(b) If $4 < p < 6$, let $a = \frac{p-4}{p-2}$, then

$$D_{p,E}(\delta, m) \leqslant \nu^{1/2}(E^{300E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2})^m \delta^{-\frac{1}{2^{m+1}}} D_{p,E}(\delta)^{1-(1-a)^{m-1}}.$$

2.8.1 Case $2 \leqslant p \leqslant 4$

Lemma 2.8.2. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^{2/2^\ell} \subset \mathbb{R}^2$, we have*

$$A_4(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant E^{100E}\nu^{-1/4} A_4(\frac{2}{2^\ell}, \frac{2}{2^\ell}).$$

Proof. Fix an arbitrary square $B^{2/2^\ell}$ of side length $\delta^{-2/2^\ell}$. Note that our restrictions on δ and ν also imply that $\nu\delta^{-2/2^\ell} \in \mathbb{N}$. We have

$$A_4(\frac{1}{2^\ell}, \frac{2}{2^\ell})^4 = \operatorname*{Avg}_{B^{1/2^\ell} \in P_{\delta^{-1/2^\ell}}(B^{2/2^\ell})} G_2(\frac{1}{2^\ell}, \frac{1}{2^\ell})^4 \leqslant E^{200E}\nu^{-1}G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell})^4 \tag{2.106}$$

where the inequality is by an application of Theorem 2.6.1. By l^2L^2 decoupling (Lemma 2.2.21), for each interval $J \in P_{\delta^{1/2^\ell}}(I_i)$, we have

$$\|\mathcal{E}_J g\|^2_{L^2_\#(\tilde{w}_{B^{2/2^\ell},E})} \leqslant E^{13E} \sum_{J' \in P_{\delta^{2/2^\ell}}(J)} \|\mathcal{E}_{J'} g\|^2_{L^2_\#(\tilde{w}_{B^{2/2^\ell},E})}.$$

Therefore

$$\sum_{J \in P_{\delta^{1/2^\ell}}(I_i)} \|\mathcal{E}_J g\|^2_{L^2_\#(\tilde{w}_{B^{2/2^\ell},E})} \leqslant E^{13E} \sum_{J \in P_{\delta^{1/2^\ell}}(I_i)} \sum_{J' \in P_{\delta^{2/2^\ell}}(J)} \|\mathcal{E}_{J'} g\|^2_{L^2_\#(\tilde{w}_{B^{2/2^\ell},E})}.$$

Since I_i, J and J' are all dyadic intervals, the above is equal to

$$E^{13E} \sum_{J' \in P_{\delta^{2/2^\ell}}(I_i)} \|\mathcal{E}_{J'} g\|^2_{L^2_\#(\tilde{w}_{B^{2/2^\ell},E})}.$$

Therefore

$$G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant E^{13E/2} G_2(\frac{2}{2^\ell}, \frac{2}{2^\ell}) = E^{13E/2} A_4(\frac{2}{2^\ell}, \frac{2}{2^\ell}).$$

Combining this with (2.106) completes the proof of Lemma 2.8.2. □

Hölder's inequality allows us to change from A_4 to A_p for $2 \leqslant p \leqslant 4$ at no cost.

Corollary 2.8.3. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^{2/2^\ell} \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant E^{100E}\nu^{-1/4} A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell}).$$

Proof. Applying Hölder's inequality to the definition of A_p shows that for $2 \leqslant p \leqslant 4$, $A_p(q, r) \leqslant A_4(q, r)$. Lemma 2.8.2 and that

$$A_4(\frac{2}{2^\ell}, \frac{2}{2^\ell}) = G_2(\frac{2}{2^\ell}, \frac{2}{2^\ell}) = A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})$$

then completes the proof of Corollary 2.8.3. □

Now for each square B^1 with side length δ^{-1}, we partition into squares of side length $\delta^{-2/2^\ell}$ and sum the previous corollary over all such squares. This yields the following result.

Lemma 2.8.4. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^\ell}, 1) \leqslant E^{100E}\nu^{-1/4}A_p(\frac{1}{2^{\ell-1}}, 1).$$

Proof. Fix an arbitrary square B^1 of side length δ^{-1}. Since $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$, we can dyadically partition B^1 into squares of side length $\delta^{-1/2^\ell}$. Lemma 2.7.3 and Corollary 2.8.3 then give that

$$\begin{aligned} A_p(\frac{1}{2^\ell}, 1)^p &= \operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^p \\ &\leqslant E^{100Ep}\nu^{-p/4} \operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})^p = E^{100Ep}\nu^{-p/4}A_p(\frac{2}{2^\ell}, 1)^p. \end{aligned}$$

This completes the proof of Lemma 2.8.4. □

Now we combine the $m-1$ inequalities together to obtain the following result.

Lemma 2.8.5. *Suppose $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^m}, 1) \leqslant (E^{100E}\nu^{-1/4})^{m-1}A_p(\frac{1}{2}, 1).$$

Proof. Since $\delta^{-1/2^m} \in 2^{\mathbb{N}}$, $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ for $\ell = 1, 2, \ldots, m$. Since $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, it follows that $\nu\delta^{-1/2^{m-1}} \in \mathbb{N}$. Since $\delta^{-1/2^{m-1}} \in 2^{\mathbb{N}}$, we have that $\nu\delta^{-1/2^{m-2}} \in \mathbb{N}$. Continuing this shows that $\nu\delta^{-1/2^\ell} \in \mathbb{N}$ for $\ell = 1, 2, \ldots, m$. Iterating Lemma 2.8.4 a total of $m-1$ times then completes the proof of Lemma 2.8.4. □

We now finally relate $A_p(1/2, 1)$ to $G_p(1/2, 1)$ which will prove (2.105) in the case when $2 \leqslant p \leqslant 4$.

Lemma 2.8.6. *If $\delta^{-1/2}, \nu\delta^{-1/2} \in \mathbb{N}$, then*

$$A_p(\frac{1}{2}, 1) \leqslant 48^{E/p}G_p(\frac{1}{2}, 1).$$

Proof. Hölder's inequality (2.3) implies that

$$G_2(\frac{1}{2},\frac{1}{2}) \leqslant \text{geom}(\sum_{J\in P_{\delta^{1/2}}(I_i)} \|\mathcal{E}_J g\|^2_{L^p_{\#}(\tilde{w}_{B^{1/2},E})})^{1/2}.$$

Since $\|\text{ geom } f_i\|_p \leqslant \text{geom }\|f_i\|_p$ and so

$$A_p(\frac{1}{2},1) \leqslant \frac{1}{|P_{\delta^{-1/2}}(B^1)|^{1/p}}\text{geom}(\sum_{B^{1/2}\in P_{\delta^{-1/2}}(B^1)}(\sum_{J\in P_{\delta^{1/2}}(I_i)}\|\mathcal{E}_J g\|^2_{L^p_{\#}(\tilde{w}_{B^{1/2},E})})^{p/2})^{1/p}.$$

Changing the $L^p_{\#}$ to L^p, interchanging the l^2 and l^p norms, and then applying Proposition 2.2.14 shows that this is $\leqslant 48^{E/p}G_p(1/2,1)$ which completes the proof of Lemma 2.8.6. □

Combining Lemmas 2.8.4 and 2.8.6 then proves (2.105) in the case when $2 \leqslant p \leqslant 4$.

Lemma 2.8.7. *Suppose $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^m},1) \leqslant (E^{200E}\nu^{-1/4})^{m-1}G_p(\frac{1}{2},1).$$

Combining Lemma 2.8.7 with Lemma 2.7.4 and applying the definition of the bilinear decoupling constant gives Proposition 2.8.1 in the case when $2 \leqslant p \leqslant 4$.

2.8.2 Case $4 < p < 6$

We now implement the iteration in the case when $4 < p < 6$. This case is similar to the case when $2 \leqslant p \leqslant 4$. For $4 < p < 6$, $a = \frac{p-4}{p-2}$ satisfies

$$\frac{1}{p/2} = \frac{a}{p} + \frac{1-a}{2}.$$

Note that $2(1-a)$ decreases monotonically to 1 as p increase to 6. The analogue of Lemma 2.8.2 and Corollary 2.8.3 is as follows.

Lemma 2.8.8. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^{2/2^\ell} \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^\ell},\frac{2}{2^\ell}) \leqslant E^{60E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2}A_p(\frac{2}{2^\ell},\frac{2}{2^\ell})^{1-a}G_p(\frac{1}{2^\ell},\frac{2}{2^\ell})^a.$$

Proof. The proof is similar to that of Lemma 2.8.2. Since $p \geqslant 4$, in the definition of A_p, we can increase the $L^2_{\#}(\widetilde{w}_{B^{1/2^\ell},E})$ to $L^{p/2}_{\#}(\widetilde{w}_{B^{1/2^\ell},E})$ using Hölder's inequality. Combining this with Theorem 2.6.1 gives that

$$A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant E^{50E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2}G_{p/2}(\frac{1}{2^\ell}, \frac{2}{2^\ell}).$$

Hölder's inequality for G_t (Lemma 2.7.2) then shows that

$$G_{p/2}(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^a G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell})^{1-a}.$$

Proceeding as at the end of the proof of Lemma 2.8.2 gives that

$$G_2(\frac{1}{2^\ell}, \frac{2}{2^\ell}) \leqslant E^{13E/2}A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})$$

Putting the above three centered equations together then completes the proof of Lemma 2.8.8. □

The analogue of Lemma 2.8.4 is as follows. The strategy of proof is essentially the same as that in Lemma 2.8.4 except this time we also need to deal with the $G_p(2^{-\ell}, 2^{-\ell+1})^a$ term from Lemma 2.8.8.

Lemma 2.8.9. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^\ell}, 1) \leqslant E^{100E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2}A_p(\frac{1}{2^{\ell-1}}, 1)^{1-a}G_p(\frac{1}{2^\ell}, 1)^a.$$

Proof. Fix an arbitrary square B^1 of side length δ^{-1}. Since $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$, we can dyadically partition B^1 into squares of side length $\delta^{-1/2^\ell}$. Lemmas 2.7.3 and 2.8.8 and Hölder's inequality gives that

$$\begin{aligned} A_p(\frac{1}{2^\ell}, 1)^p &= \operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} A_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^p \\ &\leqslant E^{60Ep}\nu^{-\frac{p}{4}}(\log\frac{1}{\delta})^{p/2}\left(\operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} A_p(\frac{2}{2^\ell}, \frac{2}{2^\ell})^p\right)^{1-a}\left(\operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^p\right)^a. \end{aligned}$$

Lemma 2.7.3 gives that the first parenthetical term is equal to $A_p(\frac{2}{2^\ell}, 1)^{p(1-a)}$. Thus the lemma is complete if we can show that

$$\operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^p \leqslant E^{40Ep} G_p(\frac{1}{2^\ell}, 1)^p. \tag{2.107}$$

Expanding definitions and interchanging geometric mean and the sum over $B^{2/2^\ell}$ gives that

$$\operatorname*{Avg}_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} G_p(\frac{1}{2^\ell}, \frac{2}{2^\ell})^p$$
$$\leqslant \frac{1}{|B^1|} \operatorname{geom}(\sum_{B^{2/2^\ell} \in P_{\delta^{-2/2^\ell}}(B^1)} (\sum_{J \in P_{\delta^{1/2^\ell}}(I_i)} \|\mathcal{E}_J g\|^2_{L^p(\widetilde{w}_{B^{2/2^\ell}, E})})^{p/2}).$$

Since $p \geqslant 2$, we can switch the l^2 and l^p norms inside the geometric mean. Finally, apply Proposition 2.2.14 then proves that the above is $\leqslant 48^E G_p(\frac{1}{2^\ell}, 1)^p$ which proves (2.107). This completes the proof of Lemma 2.8.9. □

Combining the above $m - 1$ inequalities in Lemma 2.8.9 gives the following result.

Lemma 2.8.10. *Suppose $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^m}, 1) \leqslant (E^{100E} \nu^{-1/4} (\log \frac{1}{\delta})^{1/2})^{m-1} A_p(\frac{1}{2}, 1)^{(1-a)^{m-1}} \prod_{\ell=2}^{m} G_p(\frac{1}{2^\ell}, 1)^{a(1-a)^{m-\ell}}.$$

Proof. The proof is the same as that of Lemma 2.8.5. □

To control $A_p(\frac{1}{2}, 1)$, we use Lemma 2.8.6. However, now we also need to control $G_p(\frac{1}{2^\ell}, 1)$ which we achieve by the following trivial bound.

Lemma 2.8.11. *Fix an integer $2 \leqslant \ell \leqslant m$. Suppose $\delta^{-1/2^\ell} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^\ell} \in \mathbb{N}$. Then*

$$G_p(\frac{1}{2^\ell}, 1) \leqslant E^{100E} D_{p,E}(\delta) G_p(\frac{1}{2}, 1).$$

Proof. For each $J \in P_{\delta^{1/2^\ell}}(I_i)$, we have

$$\begin{aligned} \|\mathcal{E}_J g\|_{L^p(B^1)} &= \|\mathcal{E}_{[0,1]}(g 1_J)\|_{L^p(B^1)} \\ &\leqslant \widetilde{D}_{p,E}(\delta) (\sum_{J' \in P_{\delta^{1/2}}([0,1])} \|\mathcal{E}_{J'}(g 1_J)\|^2_{L^p(\widetilde{w}_{B^1, E})})^{1/2} \\ &= \widetilde{D}_{p,E}(\delta) (\sum_{J' \in P_{\delta^{1/2}}(J)} \|\mathcal{E}_{J'} g\|^2_{L^p(\widetilde{w}_{B^1, E})})^{1/2} \end{aligned}$$

where the last equality is because both $\delta^{1/2^{\ell}}$ and $\delta^{1/2}$ are dyadic. Applying Propositions 2.2.11 and 2.3.11 then shows that

$$\|\mathcal{E}_J g\|_{L^p(\widetilde{w}_{B^1,E})} \leqslant 12^{E/p} E^{70E} D_{p,E}(\delta) (\sum_{J' \in P_{\delta^{1/2}}(J)} \|\mathcal{E}_{J'} g\|^2_{L^p(\widetilde{w}_{B^1,E})})^{1/2}.$$

Combining this with the definition of $G_p(1/2^{\ell}, 1)$ then completes the proof of Lemma 2.8.11.

□

Combining Lemmas 2.8.6, 2.8.10, and 2.8.11 gives the following result.

Lemma 2.8.12. *Suppose $\delta^{-1/2^m} \in 2^{\mathbb{N}}$ and $\nu\delta^{-1/2^m} \in \mathbb{N}$, then for each square $B^1 \subset \mathbb{R}^2$, we have*

$$A_p(\frac{1}{2^m}, 1) \leqslant (E^{100E} \nu^{-1/4} (\log \frac{1}{\delta})^{1/2})^m D_{p,E}(\delta)^{1-(1-a)^{m-1}} G_p(\frac{1}{2}, 1)$$

This with Lemma 2.7.4 then proves Proposition 2.8.1 when $4 < p < 6$. Note that in this case we obtain a small improvement over the trivial bound of $D_{p,E}(\delta, m) \lesssim_{p,E} D_{p,E}(\delta)$ which is the key to obtaining control of the linear decoupling constant when $4 < p < 6$.

2.9 Decoupling at lacunary scales

Using Propositions 2.5.3 and 2.8.1 we bound the linear decoupling constant at a sequence of lacunary scales. The lacunary scales are because of the integrality conditions in Proposition 2.8.1. Our goal will be to prove the following result.

Proposition 2.9.1. *Let $E \geqslant 100$, $m \geqslant 3$, $\nu := 2^{-16 \cdot 2^m E^{10E}}$, and $\delta \in \{\nu^{2^m n}\}_{n=1}^{\infty}$.*

(a) If $2 \leqslant p \leqslant 4$, then

$$D_{p,E}(\delta) \leqslant 2^{m^2} E^{400Em} \nu^{-2^m} \delta^{-\frac{1}{2^m}}.$$

(b) If $4 < p < 6$, then

$$D_{p,E}(\delta) \leqslant (2^{m^2} E^{400Em} \nu^{-2^m})^{\frac{1}{[2/(p-2)]^{m-1}}} \delta^{-\frac{1}{2[4/(p-2)]^{m-1}}}.$$

(c) *If $p = 6$, then for $p' \in (4,6)$, we have*

$$D_{6,E}(\delta) \leqslant E^{50E}(2^{m^2}E^{400Em}\nu^{-2^m})^{\frac{1}{(2/(p'-2))^{m-1}}}\delta^{-\frac{1}{2[4/(p'-2)]^{m-1}}-2(\frac{1}{p'}-\frac{1}{6})}.$$

The proof of Proposition 2.9.1 actually shows that $D_{p,E}(\delta) \leqslant E^{400Em}\nu^{-2^m}\delta^{-1/2^m}$ for $2 \leqslant p \leqslant 4$, but the extra 2^{m^2} is harmless and will allow us to treat all three cases in essentially the same manner. Note that in Propositions 2.8.1 and 2.9.1, the bound when $2 \leqslant p \leqslant 4$ is same as the bound for $4 < p < 6$ except with $p = 4$ (and so $a = 0$) and no $(\log \frac{1}{\delta})^{1/2}$. When we prove Proposition 2.9.1, we will only consider the more complicated case when $4 < p < 6$ and $p = 6$.

2.9.1 Case $4 < p < 6$

We first prove the following lemma.

Lemma 2.9.2. *Let $\nu = 2^{-16\cdot 2^m\cdot E^{10E}}$, $\delta^{1/2} \in 2^{-\mathbb{N}}$, and $a = \frac{p-4}{p-2}$. Let K be the largest integer such that $\delta^{1/2}\nu^{-K} \in 2^{-\mathbb{N}}$. Suppose $(\delta\nu^{-2i})^{-1/2^m} \in 2^{\mathbb{N}}$ for all $i = 0, 1, \ldots, K-1$. Then*

$$D_{p,E}(\delta) \leqslant 2^{m^2}E^{400Em}\nu^{-2^m}\delta^{-\frac{1}{2^m}} \max_{i=0,1,\ldots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2i})^{1-(1-a)^{m-1}}.$$

Proof. Observe that

$$\nu(\delta\nu^{-2i})^{-1/2^m} = (\delta\nu^{-2(i+2^{m-1})})^{-1/2^m}$$

and so for $i = 0, 1, \ldots, K - 2^{m-1} - 1$, we have that $\nu(\delta\nu^{-2i})^{-1/2^m} \in \mathbb{N}$.

For $i = 0, 1, \ldots, K - 2^{m-1} - 1$, we may apply Proposition 2.8.1 which gives that for such i,

$$D_{p,E}(\delta\nu^{-2i}, m) \leqslant (E^{300E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2})^m\delta^{-\frac{1}{2^{m+1}}}D_{p,E}(\delta\nu^{-2i})^{1-(1-a)^{m-1}}.$$

For $i = K - 2^{m-1}, \ldots, K-1$, the trivial bound (Lemma 2.5.1) gives that

$$D_{p,E}(\delta\nu^{-2i}, m) \leqslant 4^E\nu^{1/2}(\delta\nu^{-2i})^{-1/4} \leqslant 4^E(\delta^{-1/2}\nu^K)^{1/2}\nu^{-\frac{1}{2}(2^{m-1}-1)}. \tag{2.108}$$

By how K is defined, $\delta^{1/2}\nu^{-K-1} \notin 2^{-\mathbb{N}}$. Since $\delta^{1/2}$ and ν are dyadic numbers, we must then have $\delta^{1/2}\nu^{-K-1} \in 2^{\mathbb{Z}}$ and hence $\delta^{1/2}\nu^{-K-1} \geqslant 1$ which implies that $\delta^{-1/2}\nu^K \leqslant \nu^{-1}$. Inserting this into (2.108) gives that for such i,

$$D_{p,E}(\delta\nu^{-2i}, m) \leqslant 4^E \nu^{-2^m/4}.$$

Therefore Proposition 2.5.3 gives that

$$\begin{aligned}
&D_{p,E}(\delta) \\
&\quad \leqslant \delta^{100E\log_\nu E}\nu^{-1}\max(1, 4^E\nu^{-2^m/4}, \max_{i=0,1,\dots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2i}, m)) \\
&\quad \leqslant \delta^{100E\log_\nu E}\nu^{-1}\max\Big(4^E\nu^{-2^m/4}, \\
&\qquad\qquad (E^{300E}\nu^{-1/4}(\log\frac{1}{\delta})^{1/2})^m \delta^{-\frac{1}{2^{m+1}}} \max_{i=0,1,\dots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2i})^{1-(1-a)^{m-1}}\Big) \\
&\quad \leqslant E^{300Em}\nu^{-2^m}(\log\frac{1}{\delta})^{m/2}\delta^{-\frac{1}{2^{m+1}}+100E\log_\nu E} \max_{i=0,1,\dots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2i})^{1-(1-a)^{m-1}}
\end{aligned}$$

where in the last inequality we have used that $D_{p,E}(\delta) \geqslant 12^{-E/p}$ for all δ which follows from the same proof as Lemma 2.3.5. Observe that $\log\frac{1}{\delta} \leqslant \frac{1}{ae}\delta^{-a}$ for $a > 0$, and hence

$$(\log\frac{1}{\delta})^{m/2} \leqslant 2^{m^2}E^{4Em}\delta^{-\frac{5}{2^m \cdot E^{8E}}}.$$

Furthermore, from our definition of ν, $\delta^{100E\log_\nu E} \leqslant \delta^{-\frac{10}{2^m E^{8E}}}$. Inserting this into the above completes the proof of Lemma 2.9.2. □

Because of the generality of the statement of the previous lemma, we can upgrade the above result so that the same maximum appears on both left and right hand sides.

Lemma 2.9.3. *Suppose ν, δ, K, and a are as in Lemma 2.9.2. The left hand side of the inequality in Lemma 2.9.2 can be replaced with* $\max_{i=0,1,\dots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2i})$.

Proof. Fix a $j = 0, 1, \dots, K - 2^{m-1} - 1$. Let $K(j) := K - j$. Since K is the largest integer such that $\delta^{1/2}\nu^{-K} \in 2^{-\mathbb{N}}$, it follows that $K(j)$ is the largest integer such that

$$(\delta\nu^{-2j})^{1/2}\nu^{-K(j)} = \delta^{1/2}\nu^{-(K(j)+j)} \in 2^{-\mathbb{N}}.$$

We similarly also have $(\delta\nu^{-2(i+j)})^{-1/2^m} \in 2^{\mathbb{N}}$ for $i = 0, 1, \dots, K(j) - 1$. Therefore Lemma 2.9.2 gives that

$$\begin{aligned}
D_{p,E}(\delta\nu^{-2j}) &\leqslant 2^{m^2}E^{400Em}\nu^{-2^m}\delta^{-\frac{1}{2^m}} \max_{\ell=0,1,\dots,K-2^{m-1}-1-j} D_{p,E}(\delta\nu^{-2(j+\ell)})^{1-(1-a)^{m-1}} \\
&\leqslant 2^{m^2}E^{400Em}\nu^{-2^m}\delta^{-\frac{1}{2^m}} \max_{\ell=0,1,\dots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2\ell})^{1-(1-a)^{m-1}}.
\end{aligned}$$

Since j on the left hand side of the above inequality is arbitrary and the right hand side is independent of j, the above inequality is still true if we take the maximum over all j on the left hand side. This completes the proof of Lemma 2.9.3. □

This gives the following corollary.

Corollary 2.9.4. *Suppose ν, δ, K, and a are as in Lemma 2.9.2. Then*

$$\max_{\ell=0,1,\ldots,K-2^{m-1}-1} D_{p,E}(\delta\nu^{-2\ell}) \leqslant (2^{m^2}E^{400Em}\nu^{-2^m}\delta^{-\frac{1}{2^m}})^{\frac{1}{(1-a)^{m-1}}}$$

Taking $\ell = 0$ in Corollary 2.9.4 and observing that the choice of $\delta \in \{\nu^{2^m n}\}_{n=1}^{\infty}$ satisfies the hypothesis of Lemma 2.9.2 completes the proof of Proposition 2.9.1 when $4 < p < 6$. Indeed, with this choice of δ, $K = 2^{m-1}n - 1$ and so observe that

$$(\delta\nu^{-2i})^{-1/2^m} = (\nu^{-1})^{n-2i/2^m}$$

and for $i = 0, 1, \ldots, K-1$, we have $n - 2i/2^m \geqslant 0$.

2.9.2 Case $p = 6$

At $p = 6$ the argument no longer gives a better than trivial estimate since here $2(1-a) = 1$. The advantage we have however is that we know a good bound on $D_{p',E}(\delta)$ for all p' arbitrary close to 6. This combined with reverse Hölder and Hölder is enough to give a better than trivial bound at $p = 6$.

Let $4 < p' < 6$ to be chosen later. The proof of Lemma 2.2.20 along with Corollary 2.2.9 and Proposition 2.2.11 imply that

$$\begin{aligned}\|\mathcal{E}_{[0,1]}g\|_{L^6(B)} &\leqslant 25^{(1/p'-1/6)}E^{22E}\|\mathcal{E}_{[0,1]}g\|_{L^{p'}(w_{B,E})}\\ &\leqslant E^{23E}D_{p',E}(\delta)(\sum_{J\in P_{\delta^{1/2}}([0,1])}\|\mathcal{E}_J g\|^2_{L^{p'}(w_{B,E})})^{1/2}.\end{aligned}$$

Hölder's inequality to increase $L^{p'}$ to L^6 then implies that

$$D_{6,E}(\delta) \leqslant E^{50E}(\delta^{-2})^{1/p'-1/6}D_{p',E}(\delta).$$

Combining this with Proposition 2.9.1 for $4 < p' < 6$ shows that under the hypothesis of Proposition 2.9.1 and arbitrary $4 < p' < 6$, we have

$$D_{6,E}(\delta) \leqslant E^{50E}(2^{m^2}E^{400Em}\nu^{-2^m})^{\frac{1}{(2/(p'-2))^{m-1}}}\delta^{-\frac{1}{2[4/(p'-2)]^{m-1}}-2(\frac{1}{p'}-\frac{1}{6})}.$$

Thus if we choose p' so that $1/p' - 1/6$ is sufficiently small and then choose m sufficiently large, we once again can do better than the trivial bound of $O_{E,p}(\delta^{-1/4})$. This completes the proof of Proposition 2.9.1 when $p = 6$.

2.10 Decoupling at all scales

While Proposition 2.9.1 is for a lacunary sequence of scales, recall that the decoupling constant defined in (2.1) is for $\delta \in \mathbb{N}^{-2}$. To upgrade Proposition 2.9.1 to all scales $\delta \in \mathbb{N}^{-2}$ we use lacunarity and Proposition 2.4.1.

Lemma 2.10.1. *Suppose $\delta \in [\delta_1, \delta_2] \cap \mathbb{N}^{-2}$ and $\delta_2/\delta_1 = c$. Then*

$$D_{p,E}(\delta) \leqslant E^{100E}2^{E/p}c^{1/4}D_{p,E}(\delta_2).$$

Proof. Using Proposition 2.4.1 and the trivial bound on decoupling we have

$$\begin{aligned} D_{p,E}(\delta) &\leqslant E^{100E}D_{p,E}(\delta_2)D_{p,E}(\frac{\delta}{\delta_2}) \\ &\leqslant E^{100E}2^{E/p}(\frac{\delta_2}{\delta})^{1/4}D_{p,E}(\delta_2) \leqslant E^{100E}2^{E/p}c^{1/4}D_{p,E}(\delta_2) \end{aligned}$$

which completes the proof of Lemma 2.10.1. □

Combining this lemma with Proposition 2.9.1 gives the following result.

Proposition 2.10.2. *Let $E \geqslant 100$, $m \geqslant 3$, and suppose $\delta \in \mathbb{N}^{-2}$.*

(*a*) *If $2 \leqslant p \leqslant 4$, then*

$$D_{p,E}(\delta) \leqslant 2^{4^m E^{15E}}\delta^{-\frac{1}{2^m}}.$$

(*b*) *If $4 < p < 6$, then*

$$D_{p,E}(\delta) \leqslant (2^{4^m E^{15E}}\delta^{-\frac{1}{2^m}})^{\frac{1}{[2/(p-2)]^{m-1}}}.$$

(c) *If $p = 6$, then for $p' \in (4,6)$ we have*

$$D_{p,E}(\delta) \leqslant (2^{4^m E^{15E}} \delta^{-\frac{1}{2^m}})^{\frac{1}{[2/(p'-2)]^{m-1}}} \delta^{-2(\frac{1}{p'}-\frac{1}{6})}.$$

Proof. Recall that $\nu = 2^{-16 \cdot 2^m E^{10E}}$. The proof of all three parts is essentially the same, so we only concentrate on the $2 \leqslant p \leqslant 4$ case. If $\delta \in [\nu^{2^m}, 1] \cap \mathbb{N}^{-2}$, the trivial bound gives that

$$D_{p,E}(\delta) \leqslant 2^{E/p} \nu^{-2^m/4} = 2^{E/p + 4 \cdot 4^m E^{10E}}. \tag{2.109}$$

From Lemma 2.10.1, if $\delta \in [\nu^{2^m(n+1)}, \nu^{2^m n}] \cap \mathbb{N}^{-2}$ for some $n \geqslant 1$, then

$$D_{p,E}(\delta) \leqslant E^{100E} 2^{E/p} \nu^{-2^m/4} D_{p,E}(\nu^{2^m n}).$$

Inserting the bound from Proposition 2.9.1 gives that the above is bounded by

$$E^{100E} 2^{E/p} \nu^{-2^m/4} 2^{m^2} E^{400Em} \nu^{-2^m} \delta^{-\frac{1}{2^m}} \leqslant 2^{m^2} E^{500Em} \nu^{-\frac{5}{4} 2^m} \delta^{-\frac{1}{2^m}}.$$

Using that $E \geqslant 100$ and the definition of ν, we have

$$2^{m^2} E^{500Em} \nu^{-\frac{5}{4} 2^m} \leqslant 2^{100 \cdot 4^m \cdot E^{10E}} \leqslant 2^{4^m E^{15E}}.$$

This then shows

$$D_{p,E}(\delta) \leqslant 2^{4^m E^{15E}} \delta^{-\frac{1}{2^m}}$$

for all $\delta \in [\nu^{2^m(n+1)}, \nu^{2^m n}]$, $n \geqslant 1$. Combining with (2.109) completes the proof of Proposition 2.10.2 when $2 \leqslant p \leqslant 4$. When $4 < p < 6$, $\frac{1}{2/(p-2)} > 1$ and so we can repeat the same proof as above in the remaining two cases of the proposition. This completes the proof of Proposition 2.10.2. □

2.11 Proof of Theorem 2.1.1

Since Proposition 2.10.2 is true for all $m \geqslant 3$ and $\delta \in \mathbb{N}^{-2}$, we now optimize the bound on $D_{p,E}(\delta)$ in m. This will give the proof of Theorem 2.1.1.

Proof of Theorem 2.1.1. We combine the cases of $2 \leqslant p \leqslant 4$ and $4 < p < 6$. Fix arbitrary $\delta \in \mathbb{N}^{-2}$ and $E \geqslant 100$. Let m be the largest integer such that

$$2^{-m} \leqslant E^{5E} (\log_2 \delta^{-1})^{-1/3} < 2^{-m+1}. \tag{2.110}$$

Since $\delta < 2^{-64E^{15E}}$, $m \geqslant 3$. Then

$$2^{4^m E^{15E}} \delta^{-\frac{1}{2^m}} \leqslant \exp(5(\log 2)^{1/3} E^{5E} (\log \frac{1}{\delta})^{2/3}) \leqslant \exp(5 \cdot E^{5E} (\log \frac{1}{\delta})^{2/3}) \tag{2.111}$$

which finishes the case of Theorem 2.1.1 when $2 \leqslant p \leqslant 4$. For $4 < p < 6$, observe that

$$(\frac{2}{p-2})^{-(m-1)} = \exp(-(m-1)\log \frac{2}{p-2}) \leqslant 2(\log \frac{1}{\delta})^{-\frac{1}{3}\log_2(\frac{2}{p-2})}. \tag{2.112}$$

Combining (2.111) and (2.112) then proves Theorem 2.1.1 in the case when $4 < p < 6$.

For the case when $p = 6$, choose m as in (2.110). Then for $4 < p' < 6$,

$$\begin{aligned} D_{6,E}(\delta) &\leqslant \exp(10 \cdot E^{5E} (\log \frac{1}{\delta})^{\frac{2}{3} - \frac{1}{3}\log_2(\frac{2}{p'-2})}) \delta^{-2(\frac{1}{p'} - \frac{1}{6})} \\ &\leqslant \exp(E^{6E} (\log \frac{1}{\delta})[(\log \frac{1}{\delta})^{-\frac{1}{3}\log_2(\frac{4}{p'-2})} + (\frac{1}{p'} - \frac{1}{6})]). \end{aligned} \tag{2.113}$$

It thus remains to optimize

$$(\log \frac{1}{\delta})^{-\frac{1}{3}\log_2(\frac{4}{p'-2})} + (\frac{1}{p'} - \frac{1}{6})$$

for $4 < p' < 6$.

Let $\lambda := \frac{1}{p'} - \frac{1}{6}$ and suppose we choose p' sufficiently close to 6 such that $\lambda < 1/4$. Then $\frac{4}{p'-2} = \frac{1+6\lambda}{1-3\lambda}$ and

$$\log \frac{4}{p'-2} \geqslant 8\lambda.$$

Thus

$$(\log \frac{1}{\delta})^{-\frac{1}{3}\log_2 \frac{4}{p'-2}} + (\frac{1}{p'} - \frac{1}{6}) \leqslant (\log \frac{1}{\delta})^{-3\lambda} + \lambda.$$

Setting

$$\lambda = \frac{\log(3 \log\log \frac{1}{\delta})}{3 \log\log \frac{1}{\delta}}$$

gives that

$$(\log \frac{1}{\delta})^{-3\lambda} + \lambda = \frac{1 + \log 3 + \log\log\log \frac{1}{\delta}}{3 \log\log \frac{1}{\delta}} \leqslant \frac{\log\log\log \frac{1}{\delta}}{\log\log \frac{1}{\delta}} \tag{2.114}$$

where we have used that $1 + \log 3 \leqslant \log\log\log \frac{1}{\delta}$ for our range of δ. Note that for our range of δ, $\lambda < 1/4$ since this is equivalent to $3 \log\log \frac{1}{\delta} < (\log \frac{1}{\delta})^{3/4}$ which is certainly satisfied if $\delta^{-1} > 10^8$. Inserting (2.114) into (2.113) then completes the proof of Theorem 2.1.1. $\square$

CHAPTER 3

An l^2 decoupling interpretation of efficient congruencing in 2D

3.1 Introduction

Since we will once again be studying l^2 decoupling for the parabola, we adopt essentially the same notation as in Chapter 2 with a few small differences (namely δ in Chapter 2 is δ^2 in this chapter and we just set $E = 100$). For an interval $J \subset [0,1]$ and $g : [0,1] \to \mathbb{C}$, we define

$$(\mathcal{E}_J g)(x) := \int_J g(\xi) e(\xi x_1 + \xi^2 x_2)\, d\xi$$

where $e(a) := e^{2\pi i a}$. For an interval I, let $P_\ell(I)$ be the partition of I into intervals of length ℓ. By writing $P_\ell(I)$, we are assuming that $|I|/\ell \in \mathbb{N}$. We will also similarly define $P_\ell(B)$ for squares B in $\mathbb{R}^2$. Next if $B = B(c, R)$ is a square in $\mathbb{R}^2$ centered at c of side length R, let

$$w_B(x) := (1 + \frac{|x-c|}{R})^{-100}.$$

We will always assume that our squares have sides parallel to the x and y-axis. We observe that $1_B \leqslant 2^{100} w_B$. For a function w, we define

$$\|f\|_{L^p(w)} := (\int_{\mathbb{R}^2} |f(x)|^p w(x)\, dx)^{1/p}.$$

For $\delta \in \mathbb{N}^{-1}$, let $D(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]} g\|_{L^6(B)} \leqslant D(\delta)(\sum_{J \in P_\delta([0,1])} \|\mathcal{E}_J g\|^2_{L^6(w_B)})^{1/2} \tag{3.1}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B in $\mathbb{R}^2$ of side length δ^{-2}. Let $D_p(\delta)$ be the decoupling constant where the L^6 in (3.1) is replaced with L^p. Since $1_B \lesssim w_B$, the triangle inequality combined with Cauchy-Schwarz shows that $D_p(\delta) \lesssim_p \delta^{-1/2}$. The l^2 decoupling theorem for the paraboloid proven by Bourgain and Demeter in [BD15] implies that for the parabola we have $D_p(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for $2 \leqslant p \leqslant 6$ and this range of p is sharp.

This chapter attempts to probe the connections between efficient congruencing and l^2 decoupling in the simplest case of the parabola. Our proof of l^2 decoupling for the parabola is inspired by the exposition of efficient congruencing in Pierce's Bourbaki seminar exposition [Pie19]. This proof will give the following result.

Theorem 3.1.1. *For $\delta \in \mathbb{N}^{-1}$ such that $0 < \delta < e^{-200^{200}}$, we have*

$$D(\delta) \leqslant \exp(30\frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}}).$$

This improves upon a previous result of Theorem 2.1.1 in Chapter 2. In the context of discrete Fourier restriction, Theorem 3.1.1 implies that for all N sufficiently large and arbitrary sequence $\{a_n\} \subset l^2$, we have

$$\|\sum_{|n|\leqslant N} a_n e^{2\pi i(nx+n^2t)}\|_{L^6(\mathbb{T}^2)} \lesssim \exp(O(\frac{\log N}{\log\log N}))(\sum_{|n|\leqslant N}|a_n|^2)^{1/2}$$

which rederives (up to constants) the upper bound obtained by Bourgain in [Bou93, Proposition 2.36] but without resorting to using a divisor bound. It is an open problem whether the $\exp(O(\frac{\log N}{\log\log N}))$ can be improved.

3.1.1 More notation

Once again we will let η be a Schwartz function such that $\eta \geqslant 1_{B(0,1)}$ and $\mathrm{supp}(\widehat{\eta}) \subset B(0,1)$. For $B = B(c,R)$ we also define $\eta_B(x) := \eta(\frac{x-c}{R})$. Since we care about explicit constants in Section 3.2, we will use the explicit η constructed in Corollary 2.2.9. In particular, for this η, $\eta_B \leqslant 10^{2400} w_B$. For the remaining sections in this chapter, we will ignore this constant. We refer the reader to [BD17, Section 4] and Chapter 2, Section 2.2 for some useful properties of the weight w_B and η_B.

Finally we define

$$\|f\|_{L^p_\#(B)} := (\frac{1}{|B|}\int_B |f(x)|^p\, dx)^{1/p}$$

and given a collection $\mathcal{C}$ of squares, we let

$$\operatorname*{Avg}_{\Delta\in\mathcal{C}} f(\Delta) := \frac{1}{|\mathcal{C}|}\sum_{\Delta\in\mathcal{C}} f(\Delta).$$

3.1.2 Outline of proof of Theorem 3.1.1

Our argument is inspired by the discussion of efficient congruencing in [Pie19, Section 4] which in turn is based off Heath-Brown's simplification [Hea15] of Wooley's proof of the cubic case of Vinogradov's mean value theorem [Woo16].

Our first step, much like the first step in both 2D efficient congruencing and decoupling, is to bilinearize the problem. Throughout we will assume $\delta^{-1}\in\mathbb{N}$ and $\nu\in\mathbb{N}^{-1}\cap(0,1/100)$.

Fix arbitrary integers $a,b\geqslant 1$. Suppose δ and ν were such that $\nu^a\delta^{-1},\nu^b\delta^{-1}\in\mathbb{N}$. This implies that $\delta\leqslant\min(\nu^a,\nu^b)$ and the requirement that $\nu^{\max(a,b)}\delta^{-1}\in\mathbb{N}$ is equivalent to having $\nu^a\delta^{-1},\nu^b\delta^{-1}\in\mathbb{N}$. For this δ and ν, let $M_{a,b}(\delta,\nu)$ be the best constant such that

$$\int_B |\mathcal{E}_I g|^2|\mathcal{E}_{I'}g|^4 \leqslant M_{a,b}(\delta,\nu)^6(\sum_{J\in P_\delta(I)}\|\mathcal{E}_J g\|^2_{L^6(w_B)})(\sum_{J'\in P_\delta(I')}\|\mathcal{E}_{J'}g\|^2_{L^6(w_B)})^2 \tag{3.2}$$

for all squares B of side length δ^{-2}, $g:[0,1]\to\mathbb{C}$, and all intervals $I\in P_{\nu^a}([0,1])$, $I'\in P_{\nu^b}([0,1])$ with $d(I,I')\geqslant 3\nu$. We will say that such I and I' are 3ν-separated. Applying Hölder followed by the triangle inequality and Cauchy-Schwarz shows that $M_{a,b}(\delta,\nu)$ is finite. This is not the only bilinear decoupling constant we can use (see (3.27) and (3.31) in Sections 3.4 and 3.5, respectively), but in this outline we will use (3.2) because it is closest to the one used in [Pie19] and the one we will use in Section 3.2.

Our proof of Theorem 3.1.1 is broken into the following four lemmas. We state them below ignoring explicit constants for now.

Lemma 3.1.2 (Parabolic rescaling). *Let $0<\delta<\sigma<1$ be such that $\sigma,\delta,\delta/\sigma\in\mathbb{N}^{-1}$. Let I be an arbitrary interval in $[0,1]$ of length σ. Then*

$$\|\mathcal{E}_I g\|_{L^6(B)} \lesssim D(\frac{\delta}{\sigma})(\sum_{J\in P_\delta(I)}\|\mathcal{E}_J g\|^2_{L^6(w_B)})^{1/2}$$

for every $g : [0,1] \to \mathbb{C}$ *and every square* B *of side length* δ^{-2}.

Lemma 3.1.3 (Bilinear reduction)**.** *Suppose* δ *and* ν *were such that* $\nu\delta^{-1} \in \mathbb{N}$. *Then*

$$D(\delta) \lesssim D(\frac{\delta}{\nu}) + \nu^{-1} M_{1,1}(\delta, \nu).$$

Lemma 3.1.4. *Let* a *and* b *be integers such that* $1 \leqslant a \leqslant 2b$. *Suppose* δ *and* ν *were such that* $\nu^{2b}\delta^{-1} \in \mathbb{N}$. *Then*

$$M_{a,b}(\delta, \nu) \lesssim \nu^{-1/6} M_{2b,b}(\delta, \nu).$$

Lemma 3.1.5. *Suppose* b *is an integer and* δ *and* ν *were such that* $\nu^{2b}\delta^{-1} \in \mathbb{N}$. *Then*

$$M_{2b,b}(\delta, \nu) \lesssim M_{b,2b}(\delta, \nu)^{1/2} D(\frac{\delta}{\nu^b})^{1/2}.$$

Applying Lemma 3.1.4, we can move from $M_{1,1}$ to $M_{2,1}$ and then Lemma 3.1.5 allows us to move from $M_{2,1}$ to $M_{1,2}$ at the cost of a square root of $D(\delta/\nu)$. Applying Lemma 3.1.4 again moves us to $M_{2,4}$. Repeating this we can eventually reach $M_{2^{N-1},2^N}$ paying some $O(1)$ power of ν^{-1} and the value of the linear decoupling constants at various scales. This combined with Lemma 3.1.3 and the choice of $\nu = \delta^{1/2^N}$ leads to the following result.

Lemma 3.1.6. *Let* $N \in \mathbb{N}$ *and suppose* δ *was such that* $\delta^{-1/2^N} \in \mathbb{N}$ *and* $0 < \delta < 100^{-2^N}$. *Then*

$$D(\delta) \lesssim D(\delta^{1-\frac{1}{2^N}}) + \delta^{-\frac{4}{3\cdot 2^N}} D(\delta^{1/2})^{\frac{1}{3\cdot 2^N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}.$$

This then gives a recursion which shows that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ (see Section 3.2.3 for more details).

The proof of Lemma 3.1.2 is essentially a change of variables and applying the definition of the linear decoupling constant (some technical issues arise because of the weight w_B, see Chapter 2, Section 2.4). The idea is that a cap on the paraboloid can be stretched to the whole paraboloid without changing any geometric properties. The bilinear reduction Lemma 3.1.3 follows from Hölder's inequality. The argument we use is from Tao's exposition on the Bourgain-Demeter-Guth proof of Vinogradov's mean value theorem [Tao15]. In general dimension, the multilinear reduction follows from a Bourgain-Guth argument (see [BG11] and [BD17, Section 8]). We note that if a and b are so large that $\nu^a, \nu^b \approx \delta$ then $M_{a,b} \approx O(1)$ and so the goal of the iteration is to efficiently move from small a and b to very large a and b.

Lemma 3.1.4 is the most technical of the four lemmas and is where we use a Fefferman-Cordoba argument in Section 3.2. It turns out we can still close the iteration with Lemma 3.1.4 replaced by $M_{a,b} \lesssim M_{b,b}$ for $1 \leqslant a < b$ and $M_{b,b} \lesssim \nu^{-1/6} M_{2b,b}$. Both these estimates come from the same proof of Lemma 3.1.4 and is how we approach the iteration in Sections 3 and 4 (see Lemmas 3.3.3 and 3.3.5 and their rigorous counterparts Lemmas 3.4.7 and 3.4.8). The proof of these lemmas is a consequence of $l^2 L^2$ decoupling and bilinear Kakeya.

As a and b get larger and larger the estimate in Lemma 3.1.4 generally gets better and better than the trivial bound of $M_{a,b} \lesssim \nu^{-(2b-a)/6} M_{2b,b}$. The $\nu^{-1/6}$ comes from the ν-transversality of I_1 and I_2 in the definition of $M_{a,b}$. In particular, should be viewed as $(\nu^{-(2-1)})^{1/6}$ where the $1/6$ comes from that we are working in L^6 and the $\nu^{-(2-1)}$ comes from $\nu^{-(d-1)}$ with $d = 2$ which is the power of ν arising from multilinear Kakeya. Finally, Lemma 3.1.5 is an application of Hölder and parabolic rescaling.

3.1.3 Comparison with 2D efficient congruencing as in [Pie19, Section 4]

The main object of iteration in [Pie19, Section 4] is the following bilinear object

$$\begin{aligned} &I_1(X; a, b) \\ &= \max_{\xi \neq \eta \pmod{p}} \int_{(0,1]^k} \Big| \sum_{\substack{1 \leqslant x \leqslant X \\ x \equiv \xi \pmod{p^a}}} e(\alpha_1 x + \alpha_2 x^2) \Big|^2 \Big| \sum_{\substack{1 \leqslant y \leqslant X \\ y \equiv \eta \pmod{p^a}}} e(\alpha_1 y + \alpha_2 y^2) \Big|^4 \, d\alpha. \end{aligned}$$

Lemma 3.1.2-3.1.5 correspond directly to Lemmas 4.2-4.5 of [Pie19, Section 4]. The observation that Lemmas 4.2 and 4.3 of [Pie19] correspond to parabolic rescaling and bilinear reduction, respectively was already observed by Pierce in [Pie19, Section 8].

We can think of p as ν^{-1}, $J(X)/X^3$ as $D(\delta)$, and $p^{a+2b} I_1(X; a, b)/X^3$ as $M_{a,b}(\delta, \nu)^6$. In the definition of I_1, the $\max_{\xi \neq \eta \pmod{p}}$ condition can be thought of as corresponding to the transversality condition that I_1 and I_2 are ν-transverse (or since we are in 2D, ν-separated) intervals of length ν. The integral over $(0, 1]^2$ corresponds to an integral over B. Finally the expression

$$|\sum_{\substack{1\leqslant x\leqslant X\\ x\equiv\xi \pmod{p^a}}} e(\alpha_1 x+\alpha_2 x^2)|,$$

can be thought of as corresponding to $|\mathcal{E}_I g|$ for I an interval of length ν^a and so the whole of $I_1(X;a,b)$ can be thought of as $\int_B |\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4$ where $\ell(I_1)=\nu^a$ and $\ell(I_2)=\nu^b$ with I_1 and I_2 are $O(\nu)$-separated. This will be our interpretation in Section 3.2.

Interpreting the proof of Lemma 3.1.4 using the uncertainty principle, we reinterpret $I_1(X;a,b)$ as (ignoring weight functions)

$$\operatorname*{Avg}_{\Delta\in P_{\nu^{-\max(a,b)}}(B)}\|\mathcal{E}_I g\|^2_{L^2_\#(\Delta)}\|\mathcal{E}_{I'}g\|^4_{L^4_\#(\Delta)} \tag{3.3}$$

where I and I' are length ν^a and ν^b, respectively and are ν-separated. The uncertainty principle says that (3.3) is essentially equal to $\frac{1}{|B|}\int_B |\mathcal{E}_I g|^2|\mathcal{E}_{I'}g|^4$.

Finally in Section 3.5 we replace (3.3) with

$$\operatorname*{Avg}_{\Delta\in P_{\nu^{-b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I)}\|\mathcal{E}_J g\|^2_{L^2_\#(\Delta)}\Big)\Big(\sum_{J'\in P_{\nu^b}(I')}\|\mathcal{E}_{J'}g\|^2_{L^2_\#(\Delta)}\Big)^2$$

where I and I' are length ν and ν-separated. Note that when $b=1$ this then is exactly equal to $\frac{1}{|B|}\int_B |\mathcal{E}_I g|^2|\mathcal{E}_{I'}g|^4$. The interpretation given above is now similar to the A_p object studied by Bourgain-Demeter in [BD17].

3.1.4 Comparison with 2D l^2 decoupling as in [BD17]

Let $M^{(2,4)}_{a,b}(\delta,\nu)$ be the bilinear constant defined in (3.2). Let $M^{(3,3)}_{1,1}(\delta,\nu)$ be the bilinear constant defined as in (3.2) with $a=b=1$ except instead we use the true geometric mean. This latter bilinear decoupling constant is the one used by Bourgain and Demeter in [BD17].

The largest difference between our proof and the Bourgain-Demeter proof is how we iterate. Both proofs obtain that

$$D(\delta)\lesssim D(\frac{\delta}{\nu})+\nu^{-1}M^{(s,6-s)}_{1,1}(\delta,\nu) \tag{3.4}$$

where $s=3$ corresponds to the Bourgain-Demeter proof while $s=2$ corresponds to our proof. However we proceed to analyze the iteration slightly differently. Bourgain-Demeter applies (3.4) to $D(\delta/\nu)$ and $D(\delta/\nu^2)$ to obtain

$$\begin{aligned} D(\delta) &\lesssim D(\frac{\delta}{\nu^2}) + \nu^{-1}(M_{1,1}^{(3,3)}(\frac{\delta}{\nu},\nu) + M_{1,1}^{(3,3)}(\delta,\nu)) \\ &\lesssim D(\frac{\delta}{\nu^3}) + \nu^{-1}(M_{1,1}^{(3,3)}(\frac{\delta}{\nu^2},\nu) + M_{1,1}^{(3,3)}(\frac{\delta}{\nu},\nu) + M_{1,1}^{(3,3)}(\delta,\nu)) \end{aligned}$$

and we continue to iterate until δ/ν^{2^n} is of size 1. It now remains to analyze $M_{1,1}^{(3,3)}(\delta,\nu)$ for various scales δ which is done by the A_p expressions that are used in [BD17]. For our proof, in two steps (of applying Lemmas 3.1.4 and 3.1.5) we obtain

$$\begin{aligned} D(\delta) &\lesssim D(\frac{\delta}{\nu}) + \nu^{-7/6}M_{1,2}^{(2,4)}(\delta,\nu)^{1/2}D(\frac{\delta}{\nu})^{1/2} \\ &\lesssim D(\frac{\delta}{\nu}) + \nu^{-5/4}M_{2,4}^{(2,4)}(\delta,\nu)^{1/4}D(\frac{\delta}{\nu^2})^{1/4}D(\frac{\delta}{\nu})^{1/2} \end{aligned}$$

and we continue to iterate δ/ν^{2^n} is of size 1. Note that while the iteration here is able to tackle the endpoint L^6 estimate directly and as written [BD17] could not do so, the iteration in [BD17] can be slightly modified so it can handle the endpoint estimate directly (thanks to Pavel Zorin-Kranich for pointing this out).

3.1.5 Comparison of the iteration in Section 3.2 and 3.4

The way we iterate in Section 3.2 will be slightly different than how we iterate in Section 3.4. In Section 3.2, we first apply the trivial bound $M_{1,1} \lesssim \nu^{-1/6}M_{1,2}$. Then Lemmas 3.1.4 and 3.1.5 imply that for integer $b \geqslant 2$,

$$M_{b/2,b}(\delta,\nu) \lesssim \nu^{-1/6}M_{b,2b}(\delta,\nu)^{1/2}D(\frac{\delta}{\nu^b})^{1/2}.$$

Thus from this we can access $M_{2^{N-1},2^N}$ for arbitrary large N but lose only $\nu^{-O(1)}$. In contrast, for Section 3.4, we use that $M_{a,b} \lesssim M_{b,b}$ for $1 \leqslant a < b$ (from l^2L^2 decoupling) and $M_{b,b} \lesssim \nu^{-1/6}M_{2b,b}$ (from bilinear Kakeya). Combining these two inequalities with Lemma 3.1.5 gives that for integer $b \geqslant 1$,

$$M_{b,b}(\delta,\nu) \lesssim \nu^{-1/6}M_{2b,2b}(\delta,\nu)^{1/2}D(\frac{\delta}{\nu^b})^{1/2}.$$

Now we can access the constant $M_{2^N,2^N}$ for arbitrary large N but lose only $\nu^{-O(1)}$. Both iterations give similar quantitative estimates.

3.1.6 Overview of chapter

Theorem 3.1.1 will be proved in Section 3.2 via a Fefferman-Cordoba argument. This argument does not generalize to proving that $D_p(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ except for $p = 4, 6$. However in Section 3.3, by the uncertainty principle we reinterpret a key lemma from Section 2 (Lemma 3.2.8) which allows us to generalize the argument in Section 3.2 so that it can work for all $2 \leqslant p \leqslant 6$. We make this completely rigorous in Section 3.4 by defining a slightly different (but morally equivalent) bilinear decoupling constant. This will make use of l^2L^2 decoupling, Bernstein's inequality, and bilinear Kakeya. A basic version of the ball inflation inequality similar to that used in [BD17, Theorem 9.2] and [BDG16, Theorem 6.6] makes an appearance. Finally in Section 3.5, we reinterpret the argument made in Section 3.4 and write an argument that is more like that given in [BD17]. We create a 1-parameter family of bilinear constants which in some sense "interpolate" between the Bourgain-Demeter argument and our argument here.

The three arguments in Sections 3.2-3.5 are similar but will use slightly different bilinear decoupling constants. We will only mention explicit constants in Section 3.2. In Sections 3.4 and 3.5, for simplicity, we will only prove that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. The estimates in those sections can be made explicit by using explicit constants obtained from Chapter 2. Because the structure of the iteration in Sections 3.4 and 3.5 is the same as that in Section 3.2, we obtain essentially the same quantitative bounds as in Theorem 3.1.1 when making explicit the bounds in Sections 3.4 and 3.5.

In Section 3.6 we modify the argument in the previous sections to illustrate how to tackle l^2L^p decoupling for the parabola for $2 < p < 6$, taking $p = 4$ as an example. Finally in Section 3.7, we address ongoing work with Shaoming Guo and Po-Lam Yung about efficient congruencing in [Hea15] and sketch how we give a new (bilinear) proof of sharp l^4L^{12} decoupling for the moment curve $t \mapsto (t, t^2, t^3)$.

3.2 Proof of Theorem 3.1.1

We recall the definition of the bilinear decoupling constant $M_{a,b}$ as in (3.2). The arguments in this section will rely strongly on that the exponents in the definition of $M_{a,b}$ are 2 and 4, though we will only essentially use this in Lemma 3.2.8.

Given two expressions x_1 and x_2, let

$$\operatorname{geom}_{2,4} x_i := x_1^{2/6} x_2^{4/6}.$$

Hölder gives $\| \operatorname{geom}_{2,4} x_i\|_p \leqslant \operatorname{geom}_{2,4} \|x_i\|_p$.

3.2.1 Parabolic rescaling and consequences

The linear decoupling constant $D(\delta)$ obeys the following important property.

Lemma 3.2.1 (Parabolic rescaling)**.** *Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$. Let I be an arbitrary interval in $[0, 1]$ of length σ. Then*

$$\|\mathcal{E}_I g\|_{L^6(B)} \leqslant 10^{20000} D(\frac{\delta}{\sigma})(\sum_{J \in P_\delta(I)} \|\mathcal{E}_J g\|^2_{L^6(w_B)})^{1/2}$$

for every $g : [0, 1] \to \mathbb{C}$ and every square B of side length δ^{-2}.

Proof. See [BD17, Proposition 7.1] for the proof without explicit constants and Section 2.4 with $E = 100$ for a proof with explicit constants (and a clarification of parabolic rescaling with weight w_B). □

As an immediate application of parabolic rescaling we have almost multiplicativity of the decoupling constant.

Lemma 3.2.2 (Almost multiplicativity)**.** *Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$, then*

$$D(\delta) \leqslant 10^{20000} D(\sigma) D(\delta/\sigma).$$

Proof. See Proposition 2.4.1 with $E = 100$. □

The trivial bound of $O(\nu^{(a+2b)/6}\delta^{-1/2})$ for $M_{a,b}(\delta,\nu)$ is too weak for applications. We instead give another trivial bound that follows from parabolic rescaling.

Lemma 3.2.3. *If δ and ν were such that $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$, then*

$$M_{a,b}(\delta,\nu) \leqslant 10^{20000} D(\frac{\delta}{\nu^a})^{1/3} D(\frac{\delta}{\nu^b})^{2/3}.$$

Proof. Fix arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ which are 3ν-separated. Hölder's inequality gives that

$$\| \operatorname{geom}_{2,4} |\mathcal{E}_{I_i} g|\|_{L^6(B)}^6 \leqslant \|\mathcal{E}_{I_1} g\|_{L^6(B)}^2 \|\mathcal{E}_{I_2} g\|_{L^6(B)}^4.$$

Parabolic rescaling bounds this by

$$10^{120000} D(\frac{\delta}{\nu^a})^2 D(\frac{\delta}{\nu^b})^4 (\sum_{J \in P_\delta(I_1)} \|\mathcal{E}_J g\|_{L^6(w_B)}^2)(\sum_{J' \in P_\delta(I_2)} \|\mathcal{E}_{J'} g\|_{L^6(w_B)}^2)^2.$$

Taking sixth roots then completes the proof of Lemma 3.2.3. □

Hölder and parabolic rescaling allows us to interchange the a and b in $M_{a,b}$.

Lemma 3.2.4. *Suppose $b \geqslant 1$ and δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then*

$$M_{2b,b}(\delta,\nu) \leqslant 10^{10000} M_{b,2b}(\delta,\nu)^{1/2} D(\delta/\nu^b)^{1/2}.$$

Proof. Fix arbitrary I_1 and I_2 intervals of length ν^{2b} and ν^b, respectively which are ν-separated. Hölder's inequality then gives

$$\||\mathcal{E}_{I_1} g|^{1/3} |\mathcal{E}_{I_2} g|^{2/3}\|_{L^6(B)}^6 \leqslant (\int_B |\mathcal{E}_{I_1} g|^4 |\mathcal{E}_{I_2} g|^2)^{1/2} (\int_B |\mathcal{E}_{I_2} g|^6)^{1/2}.$$

Applying the definition of $M_{b,2b}$ and parabolic rescaling bounds the above by

$$(10^{20000})^3 M_{b,2b}(\delta,\nu)^3 D(\frac{\delta}{\nu^b})^3 (\sum_{J \in P_\delta(I_1)} \|\mathcal{E}_J g\|_{L^6(w_B)}^2)(\sum_{J' \in P_\delta(I_2)} \|\mathcal{E}_{J'} g\|_{L^6(w_B)}^2)^2$$

which completes the proof of Lemma 3.2.4. □

Lemma 3.2.5 (Bilinear reduction)**.** *Suppose δ and ν were such that $\nu\delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \leqslant 10^{30000} (D(\frac{\delta}{\nu}) + \nu^{-1} M_{1,1}(\delta,\nu)).$$

Proof. Let $\{I_i\}_{i=1}^{\nu^{-1}} = P_\nu([0,1])$. We have

$$\|\mathcal{E}_{[0,1]}g\|_{L^6(B)} = \|\sum_{1\leqslant i\leqslant \nu^{-1}} \mathcal{E}_{I_i}g\|_{L^6(B)} \leqslant \|\sum_{1\leqslant i,j\leqslant \nu^{-1}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2}$$
$$\leqslant \sqrt{2}\Big(\|\sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|\leqslant 3}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2} + \|\sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|> 3}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2}\Big). \tag{3.5}$$

We first consider the diagonal terms. The triangle inequality followed by Cauchy-Schwarz gives that

$$\|\sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|\leqslant 3}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)} \leqslant \sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|\leqslant 3}} \|\mathcal{E}_{I_i}g\|_{L^6(B)}\|\mathcal{E}_{I_j}g\|_{L^6(B)}.$$

Parabolic rescaling bounds this by

$$10^{40000}D(\frac{\delta}{\nu})^2 \sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|\leqslant 3}} (\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2)^{1/2}(\sum_{J\in P_\delta(I_j)} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2)^{1/2}$$
$$\leqslant \frac{10^{40000}}{2}D(\frac{\delta}{\nu})^2 \sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|\leqslant 3}} \Big(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2 + \sum_{J\in P_\delta(I_j)} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2\Big)$$
$$\leqslant 10^{40010}D(\frac{\delta}{\nu})^2 \sum_{J\in P_\delta([0,1])} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2.$$

Therefore the first term in (3.5) is bounded above by

$$10^{30000}D(\frac{\delta}{\nu})(\sum_{J\in P_\delta([0,1])} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2)^{1/2}. \tag{3.6}$$

Next we consider the off-diagonal terms. We have

$$\|\sum_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|> 3}} |\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2} \leqslant \nu^{-1} \max_{\substack{1\leqslant i,j\leqslant \nu^{-1}\\ |i-j|> 3}} \||\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2}$$

Hölder's inequality gives that

$$\||\mathcal{E}_{I_i}g||\mathcal{E}_{I_j}g|\|_{L^3(B)}^{1/2} \leqslant \||\mathcal{E}_{I_i}g|^{1/3}|\mathcal{E}_{I_j}g|^{2/3}\|_{L^6(B)}^{1/2}\||\mathcal{E}_{I_i}g|^{2/3}|\mathcal{E}_{I_j}g|^{1/3}\|_{L^6(B)}^{1/2} \tag{3.7}$$

and therefore from (3.2) (and using that $\nu\delta^{-1} \in \mathbb{N}$), the second term in (3.5) is bounded by

$$\sqrt{2}\nu^{-1}M_{1,1}(\delta,\nu)(\sum_{J\in P_\delta([0,1])} \|\mathcal{E}_Jg\|_{L^6(w_B)}^2)^{1/2}.$$

Combining this with (3.6) and applying the definition of $D(\delta)$ then completes the proof of Lemma 3.2.5. ∎

3.2.2 A Fefferman-Cordoba argument

In the proof of Lemma 3.2.8 we need a version of $M_{a,b}$ with both sides being $L^6(w_B)$. The following lemma shows that these two constants are equivalent.

Lemma 3.2.6. *Suppose δ and ν were such that $\nu^a\delta^{-1}$, $\nu^b\delta^{-1} \in \mathbb{N}$. Let $M'_{a,b}(\delta,\nu)$ be the best constant such that*

$$\int |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^4 w_B \leqslant M'_{a,b}(\delta,\nu)^6 (\sum_{J\in P_\delta(I)} \|\mathcal{E}_J g\|^2_{L^6(w_B)})(\sum_{J'\in P_\delta(I')} \|\mathcal{E}_{J'} g\|^2_{L^6(w_B)})^2$$

for all squares B of side length δ^{-2}, $g : [0,1] \to \mathbb{C}$, and all 3ν-separated intervals $I \in P_{\nu^a}([0,1])$ and $I' \in P_{\nu^b}([0,1])$. Then

$$M'_{a,b}(\delta,\nu) \leqslant 12^{100/6} M_{a,b}(\delta,\nu).$$

Remark 3.2.7. Since $1_B \lesssim w_B$, $M_{a,b}(\delta,\nu) \lesssim M'_{a,b}(\delta,\nu)$ and hence Lemma 3.2.6 implies $M_{a,b} \sim M'_{a,b}$.

Proof. Fix arbitrary 3ν-separated intervals $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$. It suffices to assume that B is centered at the origin.

Corollary 2.2.4 gives

$$\| \operatorname{geom}_{2,4} |\mathcal{E}_{I_i} g|\|^6_{L^6(w_B)} \leqslant 3^{100} \int_{\mathbb{R}^2} \| \operatorname{geom}_{2,4} |\mathcal{E}_{I_i} g|\|^6_{L^6_\#(B(y,\delta^{-2}))} w_B(y)\, dy.$$

Applying the definition of $M_{a,b}$ gives that the above is

$$\begin{aligned}
&\leqslant 3^{100}\delta^4 M_{a,b}(\delta,\nu)^6 \int_{\mathbb{R}^2} \operatorname{geom}_{2,4}(\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^6(w_{B(y,\delta^{-2})})})^3 w_B(y)\, dy\\
&\leqslant 3^{100}\delta^4 M_{a,b}(\delta,\nu)^6 \operatorname{geom}_{2,4} \int_{\mathbb{R}^2} (\sum_{J\in P_\delta(I_i)} \|\mathcal{E}_J g\|^2_{L^6(w_{B(y,\delta^{-2})})})^{\frac{1}{2}\cdot 6} w_B(y)\, dy\\
&\leqslant 3^{100}\delta^4 M_{a,b}(\delta,\nu)^6 \operatorname{geom}_{2,4}(\sum_{J\in P_\delta(I_i)} (\int_{\mathbb{R}^2} \|\mathcal{E}_J g\|^6_{L^6(w_{B(y,\delta^{-2})})} w_B(y)\, dy)^{1/3})^3
\end{aligned}$$

where the second inequality is by Hölder and the third inequality is by Minkowski. Since B is centered at the origin, $w_B * w_B \leqslant 4^{100}\delta^{-4} w_B$ (Lemma 2.2.1) and hence

$$\delta^4 \int_{\mathbb{R}^2} \|\mathcal{E}_J g\|^6_{L^6(w_{B(y,\delta^{-2})})} w_B(y)\, dy \leqslant 4^{100} \|\mathcal{E}_J g\|^6_{L^6(w_B)}.$$

This then immediately implies that $M'_{a,b}(\delta,\nu) \leqslant 12^{100/6} M_{a,b}(\delta,\nu)$ which completes the proof of Lemma 3.2.6. $\square$

We have the following key technical lemma of this paper. We encourage the reader to compare the argument with that of [Pie19, Lemma 4.4]. This lemma is a large improvement over the trivial bound of $M_{a,b} \lesssim \nu^{-(2b-a)/6} M_{2b,b}$ especially at very small scales (large a, b).

Lemma 3.2.8. *Let a and b be integers such that $1 \leqslant a \leqslant 2b$. Suppose δ and ν was such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then*

$$M_{a,b}(\delta,\nu) \leqslant 10^{1000} \nu^{-1/6} M_{2b,b}(\delta,\nu).$$

Proof. It suffices to assume that B is centered at the origin with side length δ^{-2}. The integrality conditions on δ and ν imply that $\delta \leqslant \nu^{2b}$ and $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$. Fix arbitrary intervals $I_1 = [\alpha, \alpha+\nu^a] \in P_{\nu^a}([0,1])$ and $I_2 = [\beta, \beta+\nu^b] \in P_{\nu^b}([0,1])$ which are 3ν-separated.

Let $g_\beta(x) := g(x+\beta)$, $T_\beta = \left(\begin{smallmatrix} 1 & 2\beta \\ 0 & 1 \end{smallmatrix}\right)$, and $d := \alpha - \beta$. Shifting I_2 to $[0,\nu^b]$ gives that

$$\begin{aligned}\int_B |(\mathcal{E}_{I_1} g)(x)|^2 |(\mathcal{E}_{I_2} g)(x)|^4\, dx &= \int_B |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(T_\beta x)|^2 |(\mathcal{E}_{[0,\nu^b]} g_\beta)(T_\beta x)|^4\, dx \\ &= \int_{T_\beta(B)} |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(x)|^2 |(\mathcal{E}_{[0,\nu^b]} g_\beta)(x)|^4\, dx. \end{aligned} \tag{3.8}$$

Note that d can be negative, however since $g : [0,1] \to \mathbb{C}$ and $d = \alpha - \beta$, $\mathcal{E}_{[d,d+\nu^a]} g_\beta$ is defined. Since $|\beta| \leqslant 1$, $T_\beta(B) \subset 100B$. Combining this with $1_{100B} \leqslant \eta_{100B}$ gives that (3.8) is

$$\begin{aligned} &\leqslant \int_{\mathbb{R}^2} |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(x)|^2 |(\mathcal{E}_{[0,\nu^b]} g_\beta)(x)|^4 \eta_{100B}(x)\, dx \\ &= \sum_{J_1, J_2 \in P_{\nu^{2b}}([d,d+\nu^a])} \int_{\mathbb{R}^2} (\mathcal{E}_{J_1} g_\beta)(x) \overline{(\mathcal{E}_{J_2} g_\beta)(x)} |(\mathcal{E}_{[0,\nu^b]} g_\beta)(x)|^4 \eta_{100B}(x)\, dx. \end{aligned} \tag{3.9}$$

We claim that if $d(J_1, J_2) > 10\nu^{2b-1}$, the integral in (3.9) is equal to 0.

Suppose $J_1, J_2 \in P_{\nu^{2b}}([d, d+\nu^a])$ such that $d(J_1, J_2) > 10\nu^{2b-1}$. Expanding the integral in (3.9) for this pair of J_1, J_2 gives that it is equal to

$$\int_{\mathbb{R}^2} \left(\int_{J_1 \times [0,\nu^b]^2 \times J_2 \times [0,\nu^b]^2} \prod_{i=1}^{3} g_\beta(\xi_i) \overline{g_\beta(\xi_{i+3})} e(\cdots) \prod_{i=1}^{6} d\xi_i \right) \eta_{100B}(x)\, dx \tag{3.10}$$

where the expression inside the $e(\cdots)$ is

$$((\xi_1 - \xi_4)x_1 + (\xi_1^2 - \xi_4^2)x_2) + ((\xi_2 + \xi_3 - \xi_5 - \xi_6)x_1 + (\xi_2^2 + \xi_3^2 - \xi_5^2 - \xi_6^2)x_2).$$

Interchanging the integrals in ξ and x shows that the integral in x is equal to the Fourier inverse of η_{100B} evaluated at

$$(\sum_{i=1}^{3}(\xi_i - \xi_{i+3}), \sum_{i=1}^{3}(\xi_i^2 - \xi_{i+3}^2)).$$

Since the Fourier inverse of η_{100B} is supported in $B(0, \delta^2/100)$, (3.10) is equal to 0 unless

$$|\sum_{i=1}^{3}(\xi_i - \xi_{i+3})| \leqslant \delta^2/200$$
$$|\sum_{i=1}^{3}(\xi_i^2 - \xi_{i+3}^2)| \leqslant \delta^2/200. \tag{3.11}$$

Since $\delta \leqslant \nu^{2b}$ and $\xi_i \in [0, \nu^b]$ for $i = 2, 3, 5, 6$, (3.11) implies

$$|\xi_1 - \xi_4||\xi_1 + \xi_4| = |\xi_1^2 - \xi_4^2| \leqslant 5\nu^{2b}. \tag{3.12}$$

Since I_1, I_2 are 3ν-separated, $|d| \geqslant 3\nu$. Recall that $\xi_1 \in J_1$, $\xi_4 \in J_2$ and J_1, J_2 are subsets of $[d, d + \nu^a]$. Write $\xi_1 = d + r$ and $\xi_4 = d + s$ with $r, s \in [0, \nu^a]$. Then

$$|\xi_1 + \xi_4| = |2d + (r + s)| \geqslant 6\nu - |r + s| \geqslant 6\nu - 2\nu^a \geqslant 4\nu. \tag{3.13}$$

Since $d(J_1, J_2) > 10\nu^{2b-1}$, $|\xi_1 - \xi_4| > 10\nu^{2b-1}$. Therefore the left hand side of (3.12) is $> 40\nu^{2b}$, a contradiction. Thus the integral in (3.9) is equal to 0 when $d(J_1, J_2) > 10\nu^{2b-1}$.

The above analysis implies that (3.9) is

$$\leqslant \sum_{\substack{J_1, J_2 \in P_{\nu^{2b}}([d, d+\nu^a]) \\ d(J_1, J_2) \leqslant 10\nu^{2b-1}}} \int_{\mathbb{R}^2} |(\mathcal{E}_{J_1} g_\beta)(x)||(\mathcal{E}_{J_2} g_\beta)(x)||(\mathcal{E}_{[0,\nu^b]} g_\beta)(x)|^4 \eta_{100B}(x)\, dx.$$

Undoing the change of variables as in (3.8) gives that the above is equal to

$$\sum_{\substack{J_1, J_2 \in P_{\nu^{2b}}(I_1) \\ d(J_1, J_2) \leqslant 10\nu^{2b-1}}} \int_{\mathbb{R}^2} |(\mathcal{E}_{J_1} g)(x)||(\mathcal{E}_{J_2} g)(x)||(\mathcal{E}_{I_2} g)(x)|^4 \eta_{100B}(T_\beta x)\, dx. \tag{3.14}$$

Observe that

$$\eta_{100B}(T_\beta x) \leqslant 10^{2400} w_{100B}(T_\beta x) \leqslant 10^{2600} w_{100B}(x) \leqslant 10^{2800} w_B(x)$$

where the second inequality is an application of Lemma 2.2.16 and the last inequality is because $w_B(x)^{-1}w_{100B}(x) \leqslant 10^{200}$. An application of Cauchy-Schwarz shows that (3.14) is

$$\leqslant 10^{2800} \sum_{\substack{J_1,J_2\in P_{\nu^{2b}}(I_1)\\ d(J_1,J_2)\leqslant 10\nu^{2b-1}}} \Big(\int_{\mathbb{R}^2}|\mathcal{E}_{J_1}g|^2|\mathcal{E}_{I_2}g|^4 w_B\Big)^{1/2}\Big(\int_{\mathbb{R}^2}|\mathcal{E}_{J_2}g|^2|\mathcal{E}_{I_2}g|^4 w_B\Big)^{1/2}.$$

Note that for each $J_1 \in P_{\nu^{2b}}(I_1)$, there are $\leqslant 10000\nu^{-1}$ intervals $J_2 \in P_{\nu^{2b}}(I_1)$ such that $d(J_1, J_2) \leqslant 10\nu^{2b-1}$. Thus two applications of Cauchy-Schwarz bounds the above by

$$10^{2802}\nu^{-1/2}\Big(\sum_{J_1\in P_{\nu^{2b}}(I_1)}\int_{\mathbb{R}^2}|\mathcal{E}_{J_1}g|^2|\mathcal{E}_{I_2}g|^4 w_B\Big)^{1/2}\times$$
$$\Big(\sum_{J_1\in P_{\nu^{2b}}(I_1)}\sum_{\substack{J_2\in P_{\nu^{2b}}(I_2)\\ d(J_1,J_2)\leqslant 10\nu^{2b-1}}}\int_{\mathbb{R}^2}|\mathcal{E}_{J_1}g|^2|\mathcal{E}_{I_2}g|^4 w_B\Big)^{1/2}.$$

Since there are $\leqslant 10000\nu^{-1}$ relevant J_2 for each J_1, the above is

$$\leqslant 10^{3000}\nu^{-1}\sum_{J\in P_{\nu^{2b}}(I_1)}\int_{\mathbb{R}^2}|\mathcal{E}_J g|^2|\mathcal{E}_{I_2}g|^4 w_B$$
$$\leqslant 10^{3000}12^{100}M_{2b,b}(\delta,\nu)^6\Big(\sum_{J\in P_\delta(I_1)}\|\mathcal{E}_J g\|_{L^6(w_B)}^2\Big)\Big(\sum_{J'\in P_\delta(I_2)}\|\mathcal{E}_{J'} g\|_{L^6(w_B)}^2\Big)^2$$

where the last inequality is an application of Lemma 3.2.6. This completes the proof of Lemma 3.2.8. □

Iterating Lemmas 3.2.4 and 3.2.8 repeatedly gives the following estimate.

Lemma 3.2.9. *Let $N \in \mathbb{N}$ and suppose δ and ν were such that $\nu^{2^N}\delta^{-1} \in \mathbb{N}$. Then*

$$M_{1,1}(\delta,\nu) \leqslant 10^{60000}\nu^{-1/3}D\Big(\frac{\delta}{\nu^{2^{N-1}}}\Big)^{\frac{1}{3\cdot 2^N}}D\Big(\frac{\delta}{\nu^{2^N}}\Big)^{\frac{2}{3\cdot 2^N}}\prod_{j=0}^{N-1}D\Big(\frac{\delta}{\nu^{2^j}}\Big)^{1/2^{j+1}}.$$

Proof. Lemmas 3.2.4 and 3.2.8 imply that if $1 \leqslant a \leqslant 2b$ and δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$, then

$$M_{a,b}(\delta,\nu) \leqslant 10^{20000}\nu^{-1/6}M_{b,2b}(\delta,\nu)^{1/2}D\Big(\frac{\delta}{\nu^b}\Big)^{1/2}. \tag{3.15}$$

Since $\nu^{2^N}\delta^{-1} \in \mathbb{N}$, $\nu^i\delta^{-1} \in \mathbb{N}$ for $i = 0, 1, 2, \ldots, 2^N$. Applying (3.15) repeatedly gives

$$M_{1,1}(\delta,\nu) \leqslant 10^{40000}\nu^{-1/3}M_{2^{N-1},2^N}(\delta,\nu)^{\frac{1}{2^N}}\prod_{j=0}^{N-1}D\Big(\frac{\delta}{\nu^{2^j}}\Big)^{1/2^{j+1}}.$$

Bounding $M_{2^{N-1},2^N}$ using Lemma 3.2.3 then completes the proof of Lemma 3.2.9. □

Remark 3.2.10. A similar analysis as in (3.11)-(3.13) shows that if $1 \leqslant a < b$ and δ and ν were such that $\nu^b\delta^{-1} \in \mathbb{N}$, then $M_{a,b}(\delta,\nu) \lesssim M_{b,b}(\delta,\nu)$. Though we do not iterate this way in this section, it is enough to close the iteration with $M_{a,b} \lesssim M_{b,b}$ for $1 \leqslant a < b$, and $M_{b,b} \lesssim \nu^{-1/6}M_{2b,b}$, and Lemma 3.2.4. This gives $M_{b,b} \lesssim \nu^{-1/6}M_{2b,2b}^{1/2}D(\delta/\nu^b)^{1/2}$ which is much better than the trivial bound. We interpret the iteration and in particular Lemma 3.2.8 this way in Sections 3.3-3.5.

3.2.3 The $O_\varepsilon(\delta^{-\varepsilon})$ bound

Combining Lemma 3.2.9 with Lemma 3.2.5 gives the following.

Corollary 3.2.11. *Let $N \in \mathbb{N}$ and suppose δ and ν were such that $\nu^{2^N}\delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \leqslant 10^{10^5}\left(D(\frac{\delta}{\nu}) + \nu^{-4/3}D(\frac{\delta}{\nu^{2^{N-1}}})^{\frac{1}{3\cdot 2^N}}D(\frac{\delta}{\nu^{2^N}})^{\frac{2}{3\cdot 2^N}}\prod_{j=0}^{N-1}D(\frac{\delta}{\nu^{2^j}})^{1/2^{j+1}}\right)$$

Choosing $\nu = \delta^{1/2^N}$ in Corollary 3.2.11 and requiring that $\nu = \delta^{1/2^N} \in \mathbb{N}^{-1} \cap (0, 1/100)$ gives the following result.

Corollary 3.2.12. *Let $N \in \mathbb{N}$ and suppose δ was such that $\delta^{-1/2^N} \in \mathbb{N}$ and $\delta < 100^{-2^N}$. Then*

$$D(\delta) \leqslant 10^{10^5}\left(D(\delta^{1-\frac{1}{2^N}}) + \delta^{-\frac{4}{3\cdot 2^N}}D(\delta^{1/2})^{\frac{1}{3\cdot 2^N}}\prod_{j=0}^{N-1}D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}\right).$$

Corollary 3.2.12 allows us to conclude that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. To see this, the trivial bounds for $D(\delta)$ are $1 \lesssim D(\delta) \lesssim \delta^{-1/2}$ for all $\delta \in \mathbb{N}^{-1}$. Let λ be the smallest real number such that $D(\delta) \lesssim_\varepsilon \delta^{-\lambda-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$. From the trivial bounds, $\lambda \in [0, 1/2]$. We claim that $\lambda = 0$. Suppose $\lambda > 0$.

Choose N to be an integer such that

$$\frac{5}{6} + \frac{N}{2} - \frac{4}{3\lambda} \geqslant 1. \tag{3.16}$$

Then by Corollary 3.2.12, for $\delta^{-1/2^N} \in \mathbb{N}$ with $\delta < 100^{-2^N}$,

$$\begin{aligned} D(\delta) &\lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon} + \delta^{-\frac{4}{3\cdot 2^N}-\frac{\lambda}{6\cdot 2^N}-\sum_{j=0}^{N-1}(1-\frac{1}{2^{N-j}})\frac{\lambda}{2^{j+1}}-\varepsilon} \\ &\lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon} + \delta^{-\lambda(1-(\frac{5}{6}+\frac{N}{2}-\frac{4}{3\lambda})\frac{1}{2^N})-\varepsilon} \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon} \end{aligned}$$

where in the last inequality we have used (3.16). Applying almost multiplicativity of the linear decoupling constant (similar to Section 2.10 or the proof of Lemma 3.2.14 later) then shows that for all $\delta \in \mathbb{N}^{-1}$,

$$D(\delta) \lesssim_{N,\varepsilon} \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}.$$

This then contradicts minimality of λ. Therefore $\lambda = 0$ and thus we have shown that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$.

3.2.4 An explicit bound

Having shown that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$, we now make this dependence on ε explicit. Fix arbitrary $0 < \varepsilon < 1/100$. Then $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$.

Lemma 3.2.13. *Fix arbitrary $0 < \varepsilon < 1/100$ and suppose $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$. Let integer $N \geqslant 1$ be such that*

$$\frac{5}{6} + \frac{N}{2} - \frac{4}{3\varepsilon} > 0.$$

Then for δ such that $\delta^{-1/2^N} \in \mathbb{N}$ and $\delta < 100^{-2^N}$, we have

$$D(\delta) \leqslant 2 \cdot 10^{10^5} C_\varepsilon^{1-\frac{\varepsilon}{2^N}} \delta^{-\varepsilon}.$$

Proof. Inserting $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ into Corollary 3.2.12 gives that for all integers $N \geqslant 1$ and δ such that $\delta^{-1/2^N} \in \mathbb{N}$, $\delta < 100^{-2^N}$, we have

$$D(\delta) \leqslant 10^{10^5} (C_\varepsilon \delta^{\frac{\varepsilon}{2^N}} + C_\varepsilon^{1-\frac{2}{3\cdot 2^N}} \delta^{\frac{\varepsilon}{2^N}(\frac{5}{6}+\frac{N}{2}-\frac{4}{3\varepsilon})}) \delta^{-\varepsilon}.$$

Thus by our choice of N,

$$D(\delta) \leqslant 10^{10^5} (C_\varepsilon \delta^{\frac{\varepsilon}{2^N}} + C_\varepsilon^{1-\frac{2}{3\cdot 2^N}}) \delta^{-\varepsilon}. \tag{3.17}$$

There are two possibilities. If $\delta < C_\varepsilon^{-1}$, then since $0 < \varepsilon < 1/100$, (3.17) becomes

$$D(\delta) \leqslant 10^{10^5} (C_\varepsilon^{1-\frac{\varepsilon}{2^N}} + C_\varepsilon^{1-\frac{2}{3\cdot 2^N}}) \delta^{-\varepsilon} \leqslant 2 \cdot 10^{10^5} C_\varepsilon^{1-\frac{\varepsilon}{2^N}} \delta^{-\varepsilon}. \tag{3.18}$$

On the other hand if $\delta \geqslant C_\varepsilon^{-1}$, the trivial bound gives

$$D(\delta) \leqslant 2^{100/6} \delta^{-1/2} \leqslant 2^{100/6} C_\varepsilon^{1/2}$$

which is bounded above by the right hand side of (3.18). This completes the proof of Lemma 3.2.13. □

Note that Lemma 3.2.13 is only true for δ satisfying $\delta^{-1/2^N} \in \mathbb{N}$ and $\delta < 100^{-2^N}$. We now use almost multiplicativity to upgrade the result of Lemma 3.2.13 to all $\delta \in \mathbb{N}^{-1}$.

Lemma 3.2.14. *Fix arbitrary $0 < \varepsilon < 1/100$ and suppose $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$. Then*

$$D(\delta) \leqslant 10^{10^6} 2^{4 \cdot 8^{1/\varepsilon}} C_\varepsilon^{1-\frac{\varepsilon}{8^{1/\varepsilon}}} \delta^{-\varepsilon}$$

for all $\delta \in \mathbb{N}^{-1}$.

Proof. Choose

$$N := \lceil \frac{8}{3\varepsilon} - \frac{5}{3} \rceil \tag{3.19}$$

and $\delta \in \{2^{-2^N n}\}_{n=7}^{\infty} = \{\delta_n\}_{n=7}^{\infty}$. Then for these δ, $\delta^{-1/2^N} \in \mathbb{N}$ and $\delta < 100^{-2^N}$. If $\delta \in (\delta_7, 1] \cap \mathbb{N}^{-1}$, then

$$D(\delta) \leqslant 2^{100/6} \delta^{-1/2} \leqslant 2^{100/6} 2^{2^{N-1} \cdot 7}.$$

If $\delta \in (\delta_{n+1}, \delta_n]$ for some $n \geqslant 7$, then almost multiplicativity and Lemma 3.2.13 gives that

$$\begin{aligned} D(\delta) &\leqslant 10^{20000} D(\delta_n) D(\frac{\delta}{\delta_n}) \\ &\leqslant 10^{20000} (2 \cdot 10^{10^5} C_\varepsilon^{1-\frac{\varepsilon}{2^N}} \delta_n^{-\varepsilon})(2^{100/6} (\frac{\delta_n}{\delta})^{1/2}) \\ &\leqslant 10^{10^6} 2^{2^{N-1}} C_\varepsilon^{1-\frac{\varepsilon}{2^N}} \delta^{-\varepsilon} \end{aligned}$$

where N is as in (3.19) and the second inequality we have used the trivial bound for $D(\delta/\delta_n)$.

Combining both cases above then shows that if N is chosen as in (3.19), then

$$D(\delta) \leqslant 10^{10^6} 2^{7 \cdot 2^{N-1}} C_\varepsilon^{1-\frac{\varepsilon}{2^N}} \delta^{-\varepsilon}$$

for all $\delta \in \mathbb{N}^{-1}$. Since we are no longer constrained by having $N \in \mathbb{N}$, we can increase N to be $3/\varepsilon$ and so we have that

$$D(\delta) \leqslant 10^{10^6} 2^{4 \cdot 8^{1/\varepsilon}} C_\varepsilon^{1-\frac{\varepsilon}{8^{1/\varepsilon}}} \delta^{-\varepsilon}$$

for all $\delta \in \mathbb{N}^{-1}$. This completes the proof of Lemma 3.2.14. □

Lemma 3.2.15. *For all $0 < \varepsilon < 1/100$ and all $\delta \in \mathbb{N}^{-1}$, we have*

$$D(\delta) \leqslant 2^{200^{1/\varepsilon}} \delta^{-\varepsilon}.$$

Proof. Let $P(C, \lambda)$ be the statement that $D(\delta) \leqslant C\delta^{-\lambda}$ for all $\delta \in \mathbb{N}^{-1}$. Lemma 3.2.14 implies that for $\varepsilon \in (0, 1/100)$,

$$P(C_\varepsilon, \varepsilon) \implies P(10^{10^6} 2^{4 \cdot 8^{1/\varepsilon}} C_\varepsilon^{1-\frac{\varepsilon}{8^{1/\varepsilon}}}, \varepsilon).$$

Iterating this M times gives that

$$P(C_\varepsilon, \varepsilon) \implies P([10^{10^6} 2^{4 \cdot 8^{1/\varepsilon}}]^{\sum_{j=0}^{M-1} (1-\frac{\varepsilon}{8^{1/\varepsilon}})^j} C_\varepsilon^{(1-\frac{\varepsilon}{8^{1/\varepsilon}})^M}, \varepsilon).$$

Letting $M \to \infty$ thus gives that for all $0 < \varepsilon < 1/100$,

$$D(\delta) \leqslant (10^{10^6} 2^{4 \cdot 8^{1/\varepsilon}})^{8^{1/\varepsilon}/\varepsilon} \delta^{-\varepsilon} \leqslant 2^{100^{1/\varepsilon}/\varepsilon} \delta^{-\varepsilon} \leqslant 2^{200^{1/\varepsilon}} \delta^{-\varepsilon}$$

for all $\delta \in \mathbb{N}^{-1}$. This completes the proof of Lemma 3.2.15. □

Optimizing in ε then gives the proof of our main result.

Proof of Theorem 3.1.1. Note that if $\eta = \log A - \log\log A$, then $\eta \exp(\eta) = A(1 - \frac{\log\log A}{\log A}) \leqslant A$. Choose ε such that $A = (\log_2 200)(\log \frac{1}{\delta})$, $\eta = \frac{1}{\varepsilon} \log 200$, and $\eta = \log A - \log\log A$. Then

$$200^{1/\varepsilon} \log 2 \leqslant \varepsilon \log \frac{1}{\delta}$$

and hence

$$2^{200^{1/\varepsilon}} \delta^{-\varepsilon} \leqslant \exp(2\varepsilon \log \frac{1}{\delta}). \tag{3.20}$$

Since $\eta = \log A - \log\log A$, we need to ensure that our choice of ε is such that $0 < \varepsilon < 1/100$. Thus we need

$$\varepsilon = \frac{\log 200}{\log((\log_2 200)(\log \frac{1}{\delta})) - \log\log((\log_2 200)(\log \frac{1}{\delta}))} < \frac{1}{100}.$$

Note that for all $x > 0$, $\log\log x < (\log x)^{1/2}$ and hence for all $0 < \delta < e^{-\frac{4}{\log_2 200}}$,

$$\begin{aligned}\log((\log_2 200)(\log\frac{1}{\delta})) &- \log\log((\log_2 200)(\log\frac{1}{\delta}))\\ &\geqslant \log((\log_2 200)(\log\frac{1}{\delta})) - [\log((\log_2 200)(\log\frac{1}{\delta}))]^{1/2}\\ &\geqslant \frac{1}{2}\log((\log_2 200)(\log\frac{1}{\delta})) \geqslant \frac{1}{2}\log\log\frac{1}{\delta}. \qquad (3.21)\end{aligned}$$

Thus we need $0 < \delta < e^{-\frac{4}{\log_2 200}}$ to also be such that

$$\frac{2\log 200}{\log\log\frac{1}{\delta}} < \frac{1}{100}$$

and hence $\delta < e^{-200^{200}}$. Therefore using (3.20) and (3.21), we have that for $\delta \in (0, e^{-200^{200}}) \cap \mathbb{N}^{-1}$,

$$D(\delta) \leqslant \exp(30\frac{\log\frac{1}{\delta}}{\log\log\frac{1}{\delta}}).$$

This completes the proof of Theorem 3.1.1. □

3.3 An uncertainty principle interpretation of Lemma 3.2.8

The main point was of Lemma 3.2.8 was to show that if $1 \leqslant a \leqslant 2b$, δ and ν such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$, then

$$\int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \nu^{-1} \sum_{J \in P_{\nu^{2b}}(I_1)} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4 \qquad (3.22)$$

for arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$. From Lemma 3.2.9, we only need (3.22) to be true for $1 \leqslant a \leqslant b$. Our goal of this section is to prove (heuristically under the uncertainty principle) the following two statements:

(I) For $1 \leqslant a < b$, $M_{a,b}(\delta, \nu) \lesssim M_{b,b}(\delta, \nu)$; in other words

$$\int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \sum_{J \in P_{\nu^b}(I_1)} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4 \qquad (3.23)$$

for arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$.

(II) $M_{b,b}(\delta,\nu) \lesssim \nu^{-1/6} M_{2b,b}(\delta,\nu)$; in other words

$$\int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \nu^{-1} \sum_{J \in P_{\nu^{2b}}(I_1)} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4 \tag{3.24}$$

for arbitrary $I_1, I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$.

Replacing 4 with $p-2$ then allows us to generalize to $2 \leqslant p < 6$ (in Section 3.6 we illustrate this in the case of $p = 4$). Note that all results in this section are only heuristically true. In this section we will pretend all weight functions are just indicator functions and will make these heuristics rigorous in the next section.

The particular instance of the uncertainty principle we will use is the following. Let I be an interval of length $1/R$ with center c. Fix an arbitrary $R \times R^2$ rectangle T oriented in the direction $(-2c, 1)$. Heuristically for $x \in T$, $(\mathcal{E}_I g)(x)$ behaves like $a_{T,I} e^{2\pi i \omega_{T,I} \cdot x} 1_T(x)$. Here the amplitude a_T depends on g, T, and I and the phase ω_T depends on T and I. In particular, $|(\mathcal{E}_I g)(x)|$ is essentially constant on every $R \times R^2$ rectangle oriented in the direction $(-2c, 1)$. This also implies that if Δ is a square of side length R, then $|(\mathcal{E}_I g)(x)|$ is essentially constant on Δ (with constant depending on Δ) and $\|\mathcal{E}_I g\|_{L^p_\#(\Delta)}$ is essentially constant with the same constant independent of p.

We introduce two standard tools from [BD17, BDG16].

Lemma 3.3.1 (Bernstein's inequality)**.** *Let I be an interval of length $1/R$ and Δ a square of side length R. If $1 \leqslant p \leqslant q < \infty$, then*

$$\|\mathcal{E}_I g\|_{L^q_\#(\Delta)} \lesssim \|\mathcal{E}_I g\|_{L^p_\#(\Delta)}.$$

We also have

$$\|\mathcal{E}_I g\|_{L^\infty(\Delta)} \lesssim \|\mathcal{E}_I g\|_{L^p_\#(\Delta)}.$$

Proof. See [BD17, Corollary 4.3] or Lemma 2.2.20 for a rigorous proof. □

The reverse inequality in the above lemma is just an application of Hölder.

Lemma 3.3.2 (l^2L^2 decoupling)**.** *Let I be an interval of length $\geqslant 1/R$ such that $R|I| \in \mathbb{N}$ and Δ a square of side length R. Then*

$$\|\mathcal{E}_I g\|_{L^2(\Delta)} \lesssim \Big(\sum_{J \in P_{1/R}(I)} \|\mathcal{E}_J g\|_{L^2(\Delta)}^2\Big)^{1/2}.$$

Proof. See [BD17, Proposition 6.1] or Lemma 2.2.21 for a rigorous proof. □

The first inequality (3.23) is an immediate application of the uncertainty principle and l^2L^2 decoupling.

Lemma 3.3.3. *Suppose $1 \leqslant a < b$ and δ and ν were such that $\nu^b\delta^{-1} \in \mathbb{N}$. Then*

$$\int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \sum_{J \in P_{\nu^b}(I_1)} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4$$

for arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$. In other words, $M_{a,b}(\delta,\nu) \lesssim M_{b,b}(\delta,\nu)$.

Proof. It suffices to show that for each $\Delta' \in P_{\nu^{-b}}(B)$, we have

$$\int_{\Delta'} |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \sum_{J \in P_{\nu^b}(I_1)} \int_{\Delta'} |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4.$$

Since I_2 is an interval of length ν^b, $|\mathcal{E}_{I_2} g|$ is essentially constant on Δ'. Therefore the above reduces to showing

$$\int_{\Delta'} |\mathcal{E}_{I_1} g|^2 \lesssim \sum_{J \in P_{\nu^b}(I_1)} \int_{\Delta'} |\mathcal{E}_J g|^2$$

which since $a < b$ and I_1 is of length ν^a is just an application of l^2L^2 decoupling. This completes the proof of Lemma 3.3.3. □

Inequality (3.24) is a consequence of the following ball inflation lemma which is reminiscent of the ball inflation in the Bourgain-Demeter-Guth proof of Vinogradov's mean value theorem. The main point of this lemma is to increase the spatial scale so we can apply l^2L^2 decoupling while keep the frequency scales constant.

Lemma 3.3.4 (Ball inflation). *Let $b \geqslant 1$ be a positive integer. Suppose I_1 and I_2 are ν-separated intervals of length ν^b. Then for any square Δ' of side length ν^{-2b}, we have*

$$\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(\Delta)} \lesssim \nu^{-1} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(\Delta')} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(\Delta')}.$$

Proof. The uncertainty principle implies that $|\mathcal{E}_{I_1} g|$ and $|\mathcal{E}_{I_2} g|$ are essentially constant on Δ. Therefore we essentially have

$$\begin{aligned}\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(\Delta)} &\sim \frac{1}{|P_{\nu^{-b}}(\Delta')|} \sum_{\Delta \in P_{\nu^{-b}}(\Delta')} \frac{1}{|\Delta|} \int_\Delta |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \\ &= \frac{1}{|\Delta'|} \int_{\Delta'} |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4.\end{aligned}$$

On Δ', note that $|\mathcal{E}_{I_1} g| \sim \sum_{T_1} |c_{T_1}| 1_{T_1}$ and similarly for I_2 where $\{T_i\}$ are the $\nu^{-b} \times \nu^{-2b}$ rectangles covering Δ' and pointing in the normal direction of the cap on the parabola living above I_i. Since I_1 and I_2 are ν-separated, for any two tubes T_1, T_2 corresponding to I_1, I_2, we have $|T_1 \cap T_2| \lesssim \nu^{-1-2b}$. Therefore

$$\frac{1}{|\Delta'|} \int_{\Delta'} |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \sim \nu^{-1} \frac{\nu^{-2b}}{|\Delta'|} \sum_{T_1, T_2} |c_{T_1}|^2 |c_{T_2}|^4.$$

Since

$$\|\mathcal{E}_{I_1} g\|^2_{L^2_\#(\Delta')} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(\Delta')} \sim \frac{\nu^{-6b}}{|\Delta'|^2} \sum_{T_1, T_2} |c_{T_1}|^2 |c_{T_2}|^4$$

and $|\Delta'| = \nu^{-4b}$, this completes the proof of Lemma 3.3.4. □

We now prove inequality (3.24).

Lemma 3.3.5. *Suppose δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then*

$$\int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^4 \lesssim \nu^{-1} \sum_{J \in P_{\nu^{2b}}(I_1)} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{I_2} g|^4$$

for arbitrary $I_1 \in P_{\nu^b}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ such that $d(I_1, I_2) \gtrsim \nu$. In other words, $M_{b,b}(\delta, \nu) \lesssim \nu^{-1/6} M_{2b,b}(\delta, \nu)$.

Proof. This is an application of ball inflation, l^2L^2 decoupling, Bernstein, and the uncertainty principle. Since $\nu^{2b}\delta^{-1} \in \mathbb{N}$, $\nu^b\delta^{-1} \in \mathbb{N}$ and $\delta \leqslant \nu^{2b}$. Fix arbitrary $I_1, I_2 \in P_{\nu^b}([0,1])$. We have

$$\begin{aligned}\frac{1}{|B|}\int_B |\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4 &= \frac{1}{|B|}\sum_{\Delta\in P_{\nu^{-b}}(B)}\int_\Delta |\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4\\ &\leqslant \frac{1}{|B|}\sum_{\Delta\in P_{\nu^{-b}}(B)}(\int_\Delta |\mathcal{E}_{I_1}g|^2)\|\mathcal{E}_{I_2}g\|^4_{L^\infty(\Delta)}\\ &\lesssim \frac{1}{|P_{\nu^{-b}}(B)|}\sum_{\Delta\in P_{\nu^{-b}}(B)}(\frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_{I_1}g|^2)\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta)}\\ &= \operatorname*{Avg}_{\Delta\in P_{\nu^{-b}}(B)}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta)} \end{aligned}\tag{3.25}$$

where the second inequality is because of Bernstein. From ball inflation we know that for each $\Delta' \in P_{\nu^{-2b}}(B)$,

$$\operatorname*{Avg}_{\Delta\in P_{\nu^{-2b}}(\Delta')}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta)} \lesssim \nu^{-1}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta')}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta')}.$$

Averaging the above over all $\Delta' \in P_{\nu^{-2b}}(B)$ shows that (3.25) is

$$\lesssim \nu^{-1}\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta')}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta')}.$$

Since I_1 is of length ν^b, l^2L^2 decoupling gives that the above is

$$\begin{aligned}&\lesssim \nu^{-1}\sum_{J\in P_{\nu^{2b}}(I_1)}\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\|\mathcal{E}_Jg\|^2_{L^2_\#(\Delta')}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta')}\\ &= \nu^{-1}\frac{1}{|B|}\sum_{J\in P_{\nu^{2b}}(I_1)}\sum_{\Delta'\in P_{\nu^{-2b}}(B)}\|\mathcal{E}_{I_2}g\|^4_{L^4(\Delta')}\|\mathcal{E}_Jg\|^2_{L^2_\#(\Delta')}\\ &= \nu^{-1}\frac{1}{|B|}\sum_{J\in P_{\nu^{2b}}(I_1)}\sum_{\Delta'\in P_{\nu^{-2b}}(B)}(\int_{\Delta'}|\mathcal{E}_{I_2}g|^4)\|\mathcal{E}_Jg\|^2_{L^2_\#(\Delta')}.\end{aligned}$$

Since $|\mathcal{E}_Jg|$ is essentially constant on Δ', the uncertainty principle gives that essentially we have

$$(\int_{\Delta'}|\mathcal{E}_{I_2}g|^4)\|\mathcal{E}_Jg\|^2_{L^2_\#(\Delta')} \sim \int_{\Delta'}|\mathcal{E}_Jg|^2|\mathcal{E}_{I_2}g|^4.$$

Combining the above two centered equations then completes the proof of Lemma 3.3.5. □

Remark 3.3.6. The proof of Lemma 3.3.5 is reminiscent of our proof of Lemma 3.2.8. The $\|\mathcal{E}_{I_2}g\|_{L^\infty(\Delta)}$ can be thought as using the trivial bound for ξ_i, $i = 2,3,5,6$ to obtain (3.12). Then we apply some data about separation, much like in ball inflation here to get large amounts of cancelation.

3.4 An alternate proof of $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$

The ball inflation lemma and our proof of Lemma 3.3.5 inspire us to define a new bilinear decoupling constant that can make our uncertainty principle heuristics from the previous section rigorous.

The left hand side of the definition of $D(\delta)$ is unweighted, however recall that Proposition 2.2.11 implies that

$$\|\mathcal{E}_{[0,1]}g\|_{L^6(w_B)} \lesssim D(\delta)(\sum_{J\in P_\delta([0,1])}\|\mathcal{E}_J g\|^2_{L^6(w_B)})^{1/2}. \tag{3.26}$$

for all $g : [0,1] \to \mathbb{C}$ and squares B of side length δ^{-2}.

We will assume that $\delta^{-1} \in \mathbb{N}$ and $\nu \in \mathbb{N}^{-1} \cap (0, 1/100)$. Let $\mathcal{M}_{a,b}(\delta,\nu)$ be the best constant such that

$$\begin{aligned} &\operatorname*{Avg}_{\Delta\in P_{\nu^{-\max(a,b)}}(B)}\|\mathcal{E}_I g\|^2_{L^2_\#(w_\Delta)}\|\mathcal{E}_{I'}g\|^4_{L^4_\#(w_\Delta)} \\ &\qquad\leqslant \mathcal{M}_{a,b}(\delta,\nu)^6(\sum_{J\in P_\delta(I)}\|\mathcal{E}_J g\|^2_{L^6_\#(w_B)})(\sum_{J\in P_\delta(I')}\|\mathcal{E}_{J'}g\|^2_{L^6_\#(w_B)})^2 \end{aligned} \tag{3.27}$$

for all squares B of side length δ^{-2}, $g : [0,1] \to \mathbb{C}$ and all intervals $I \in P_{\nu^a}([0,1])$, $I' \in P_{\nu^b}([0,1])$ with $d(I,I') \geqslant \nu$.

Suppose $a > b$ (the proof when $a \leqslant b$ is similar). The uncertainty principle implies that

$$\begin{aligned} \operatorname*{Avg}_{\Delta\in P_{\nu^{-a}}(B)}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(\Delta)} &\sim \frac{1}{|P_{\nu^{-a}}(B)|}\sum_{\Delta\in P_{\nu^{-a}}(B)}(\frac{1}{|\Delta|}\int_\Delta|\mathcal{E}_{I_2}g|^4)\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(\Delta)} \\ &\sim \frac{1}{|B|}\int_B|\mathcal{E}_{I_1}g|^2|\mathcal{E}_{I_2}g|^4 \end{aligned}$$

where the last $\sim$ is because $|\mathcal{E}_{I_1}g|$ is essentially constant on Δ. Therefore our bilinear constant $\mathcal{M}_{a,b}$ is essentially the same as the bilinear constant $M_{a,b}$ we defined in (3.2).

3.4.1 Some basic properties

Lemma 3.4.1 (Bernstein)**.** *Let I be an interval of length $1/R$ and Δ a square of side length R. Then*

$$\|\mathcal{E}_I g\|_{L^\infty(\Delta)} \lesssim \|\mathcal{E}_I g\|_{L^p_\#(w_\Delta)}.$$

Proof. See [BD17, Corollary 4.3] for a proof without explicit constants or Lemma 2.2.20 for a version with explicit constants. □

Lemma 3.4.2 (l^2L^2 decoupling)**.** *Let I be an interval of length $\geqslant 1/R$ such that $R|I| \in \mathbb{N}$ and Δ a square of side length R. Then*

$$\|\mathcal{E}_I g\|_{L^2(w_\Delta)} \lesssim (\sum_{J \in P_{1/R}(I)} \|\mathcal{E}_J g\|^2_{L^2(w_\Delta)})^{1/2}.$$

Proof. See [BD17, Proposition 6.1] for a proof without explicit constants or Lemma 2.2.21 for a version with explicit constants. □

We now run through the substitutes of Lemmas 3.2.3-3.2.5.

Lemma 3.4.3. *Suppose δ and ν were such that $\nu^a\delta^{-1}$, $\nu^b\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{a,b}(\delta,\nu) \lesssim D(\frac{\delta}{\nu^a})^{1/3} D(\frac{\delta}{\nu^b})^{2/3}.$$

Proof. Let $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$. Hölder's inequality gives that

$$\begin{aligned}
&\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(w_\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(w_\Delta)} \\
&\qquad \leqslant \operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|^2_{L^6_\#(w_\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^6_\#(w_\Delta)} \\
&\qquad \leqslant (\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|^6_{L^6_\#(w_\Delta)})^{1/3} (\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_2} g\|^6_{L^6_\#(w_\Delta)})^{2/3} \\
&\qquad \lesssim \|\mathcal{E}_{I_1} g\|^2_{L^6_\#(w_B)} \|\mathcal{E}_{I_2} g\|^4_{L^6_\#(w_B)}
\end{aligned}$$

where the last inequality we have used that $\sum_\Delta w_\Delta \lesssim_n w_B$ (see Proposition 2.2.14). Finally applying (3.26) with parabolic rescaling then completes the proof of Lemma 3.4.3. □

Lemma 3.4.4. *Suppose $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{a,b}(\delta, \nu) \lesssim \mathcal{M}_{b,a}(\delta, \nu)^{1/2} D(\frac{\delta}{\nu^b})^{1/2}.$$

Proof. Let $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$. We have

$$\begin{aligned}
&\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_\Delta)}^2 \|\mathcal{E}_{I_2} g\|_{L^4_\#(w_\Delta)}^4 \\
&\quad \leqslant \operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_\Delta)}^2 \|\mathcal{E}_{I_2} g\|_{L^2_\#(w_\Delta)} \|\mathcal{E}_{I_2} g\|_{L^6_\#(w_\Delta)}^3 \\
&\quad \leqslant \Big(\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_\Delta)}^4 \|\mathcal{E}_{I_2} g\|_{L^2_\#(w_\Delta)}^2\Big)^{1/2} \Big(\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_2} g\|_{L^6_\#(w_\Delta)}^6\Big)^{1/2} \\
&\quad \lesssim \Big(\operatorname*{Avg}_{\Delta \in P_{\nu^{-\max(a,b)}}(B)} \|\mathcal{E}_{I_1} g\|_{L^4_\#(w_\Delta)}^4 \|\mathcal{E}_{I_2} g\|_{L^2_\#(w_\Delta)}^2\Big)^{1/2} \|\mathcal{E}_{I_2} g\|_{L^6_\#(w_B)}^3
\end{aligned}$$

where the first and second inequalities are because of Hölder and the third inequality is an application of Hölder and the estimate $\sum_\Delta w_\Delta \lesssim w_B$. Applying parabolic rescaling and the definition of $\mathcal{M}_{b,a}$ then completes the proof of Lemma 3.4.4. □

Lemma 3.4.5 (Bilinear reduction)**.** *Suppose δ and ν were such that $\nu\delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \lesssim_n D(\frac{\delta}{\nu}) + \nu^{-1}\mathcal{M}_{1,1}(\delta, \nu).$$

Proof. The proof is essentially the same as that of Lemma 3.2.5 except when analyzing (3.7) in the off-diagonal terms we use

$$\begin{aligned}
\||\mathcal{E}_{I_i} g|^{1/3} |\mathcal{E}_{I_j} g|^{2/3}\|_{L^6_\#(B)}^6 &= \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \frac{1}{|\Delta|} \int_\Delta |\mathcal{E}_{I_i} g|^2 |\mathcal{E}_{I_j} g|^4 \\
&\leqslant \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \|\mathcal{E}_{I_i} g\|_{L^2_\#(\Delta)}^2 \|\mathcal{E}_{I_j} g\|_{L^\infty(\Delta)}^4 \\
&\lesssim \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \|\mathcal{E}_{I_i} g\|_{L^2_\#(w_\Delta)}^2 \|\mathcal{E}_{I_j} g\|_{L^4_\#(w_\Delta)}^4
\end{aligned}$$

where the second inequality we have used Bernstein. □

3.4.2 Ball inflation

We now prove rigorously the ball inflation lemma we mentioned in the previous section.

Lemma 3.4.6 (Ball inflation)**.** *Let $b \geqslant 1$ be a positive integer. Suppose I_1 and I_2 are ν-separated intervals of length ν^b. Then for any square Δ' of side length ν^{-2b}, we have*

$$\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(w_\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(w_\Delta)} \lesssim \nu^{-1} \|\mathcal{E}_{I_1} g\|^2_{L^2_\#(w_{\Delta'})} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(w_{\Delta'})}. \tag{3.28}$$

Proof. Without loss of generality we may assume that Δ' is centered at the origin. Fix intervals I_1 and I_2 intervals of length ν^b which are ν-separated with centers c_1 and c_2, respectively.

Cover Δ' by a set $\mathcal{T}_1$ of mutually parallel nonoverlapping rectangles T_1 of dimensions $\nu^{-b} \times \nu^{-2b}$ with longer side pointing in the direction of $(-2c_1, 1)$ (the normal direction of the piece of parabola above I_1). Note that any $\nu^{-b} \times \nu^{-2b}$ rectangle outside $4\Delta'$ cannot cover Δ' itself. Thus we may assume that all rectangles in $\mathcal{T}_1$ are contained in $4\Delta'$. Finally let $T_1(x)$ be the rectangle in $\mathcal{T}_1$ containing x. Similarly define $\mathcal{T}_2$ except this time we use I_2.

For $x \in 4\Delta'$, define

$$F_1(x) := \begin{cases} \sup_{y \in 2T_1(x)} \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_{B(y,\nu^{-b})})} & \text{if } x \in \bigcup_{T_1 \in \mathcal{T}_1} T_1 \\ 0 & \text{if } x \in 4\Delta' \setminus \bigcup_{T_1 \in \mathcal{T}_1} T_1 \end{cases}$$

and

$$F_2(x) := \begin{cases} \sup_{y \in 2T_2(x)} \|\mathcal{E}_{I_2} g\|_{L^4_\#(w_{B(y,\nu^{-b})})} & \text{if } x \in \bigcup_{T_2 \in \mathcal{T}_2} T_2 \\ 0 & \text{if } x \in 4\Delta' \setminus \bigcup_{T_2 \in \mathcal{T}_2} T_2. \end{cases}$$

Given a $\Delta \in P_{\nu^{-b}}(\Delta')$, if $x \in \Delta$, then $\Delta \subset 2T_i(x)$. This implies that the center of Δ, $c_\Delta \in 2T_i(x)$ for $x \in \Delta$ and hence for all $x \in \Delta$,

$$\|\mathcal{E}_{I_1} g\|_{L^2_\#(w_\Delta)} \leqslant F_1(x)$$

and

$$\|\mathcal{E}_{I_2} g\|_{L^4_\#(w_\Delta)} \leqslant F_2(x).$$

Therefore

$$\|\mathcal{E}_{I_1} g\|^2_{L^2_\#(w_\Delta)} \|\mathcal{E}_{I_2} g\|^4_{L^4_\#(w_\Delta)} \leqslant \frac{1}{|\Delta|} \int_\Delta F_1(x)^2 F_2(x)^4 \, dx. \tag{3.29}$$

By how F_i is defined, F_i is constant on each $T_i \in \mathcal{T}_i$. That is, for each $x \in \bigcup_{T_i \in \mathcal{T}_i} T_i$,

$$F_i(x) = \sum_{T_i \in \mathcal{T}_i} c_{T_i} 1_{T_i}(x)$$

for some constants $c_{T_i} \geqslant 0$.

Thus using (3.29) and that the T_i are disjoint, the left hand side of (3.28) is bounded above by

$$\frac{1}{|\Delta'|}\int_{\Delta'} F_1(x)^2 F_2(x)^4\, dx = \frac{1}{|\Delta'|}\sum_{T_1,T_2} c_{T_1}^2 c_{T_2}^4 |T_1 \cap T_2| \lesssim \nu^{-1}\frac{\nu^{-2b}}{|\Delta'|}\sum_{T_1,T_2} c_{T_1}^2 c_{T_2}^4 \tag{3.30}$$

where the last inequality we have used that since I_1 and I_2 are ν-separated, sine of the angle between T_1 and T_2 is $\gtrsim \nu$ and hence $|T_1 \cap T_2| \lesssim \nu^{-1-2b}$. Note that

$$\|F_1\|_{L^2_\#(4\Delta')}^2 = \frac{\nu^{-3b}}{|4\Delta'|}\sum_{T_1} c_{T_1}^2$$

and

$$\|F_2\|_{L^4_\#(4\Delta')}^4 = \frac{\nu^{-3b}}{|4\Delta'|}\sum_{T_2} c_{T_2}^4.$$

Therefore (3.30) is

$$\lesssim \nu^{-1}\|F_1\|_{L^2_\#(4\Delta')}^2 \|F_2\|_{L^4_\#(4\Delta')}^4.$$

Thus we are done if we can prove that

$$\|F_1\|_{L^2_\#(4\Delta')}^2 \lesssim \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_{\Delta'})}^2$$

and

$$\|F_2\|_{L^4_\#(4\Delta')}^4 \lesssim \|\mathcal{E}_{I_2} g\|_{L^4_\#(w_{\Delta'})}^4$$

but this was exactly what was shown in [BD17, Eq. (29)] (and Lemma 2.6.3 for the same inequality but with explicit constants). □

Our choice of bilinear constant (3.27) makes the rigorous proofs of Lemmas 3.3.3 and 3.3.5 immediate consequences of ball inflation and l^2L^2 decoupling.

Lemma 3.4.7. *Suppose $1 \leqslant a < b$ and δ and ν were such that $\nu^b\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{a,b}(\delta,\nu) \lesssim \mathcal{M}_{b,b}(\delta,\nu).$$

Proof. For arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ which are ν-separated, it suffices to show that

$$\operatorname*{Avg}_{\Delta\in P_{\nu^{-b}}(B)}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(w_\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(w_\Delta)} \lesssim \sum_{J\in P_{\nu^b}(I_1)}\operatorname*{Avg}_{\Delta\in P_{\nu^{-b}}(B)}\|\mathcal{E}_{J}g\|^2_{L^2_\#(w_\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(w_\Delta)}.$$

But this is immediate from l^2L^2 decoupling which completes the proof of Lemma 3.4.7. □

Lemma 3.4.8. *Let $b \geqslant 1$ and suppose δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{b,b}(\delta,\nu) \lesssim \nu^{-1/6}\mathcal{M}_{2b,b}(\delta,\nu).$$

Proof. For arbitrary $I_1 \in P_{\nu^a}([0,1])$ and $I_2 \in P_{\nu^b}([0,1])$ which are ν-separated, it suffices to prove that

$$\operatorname*{Avg}_{\Delta\in P_{\nu^{-b}}(B)}\|\mathcal{E}_{I_1}g\|^2_{L^2_\#(w_\Delta)}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(w_\Delta)} \lesssim \nu^{-1}\sum_{J\in P_{\nu^{2b}}(I_1)}\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\|\mathcal{E}_{J}g\|^2_{L^2_\#(w_{\Delta'})}\|\mathcal{E}_{I_2}g\|^4_{L^4_\#(w_{\Delta'})}.$$

But this is immediate from ball inflation followed by l^2L^2 decoupling which completes the proof of Lemma 3.4.8. □

Combining Lemmas 3.4.4, 3.4.7, and 3.4.8 gives the following corollary.

Corollary 3.4.9. *Suppose δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{b,b}(\delta,\nu) \lesssim \nu^{-1/6}\mathcal{M}_{2b,2b}(\delta,\nu)^{1/2}D(\frac{\delta}{\nu^b})^{1/2}.$$

This corollary should be compared to the trivial estimate obtained from Lemma 3.4.3 which implies $\mathcal{M}_{b,b}(\delta,\nu) \lesssim D(\delta/\nu^b)$.

3.4.3 The $O_\varepsilon(\delta^{-\varepsilon})$ bound

We now prove that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. The structure of the argument is essentially the same as that in Section 3.2.3. Repeatedly iterating Corollary 3.4.9 gives the following result.

Lemma 3.4.10. *Let N be an integer chosen sufficiently large later and let δ be such that $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$. Then*

$$D(\delta) \lesssim D(\delta^{1-\frac{1}{2^N}}) + \delta^{-\frac{4}{3\cdot 2^N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}.$$

Proof. Iterating Corollary 3.4.9 N times gives that if δ and ν were such that $\nu^{2^N}\delta^{-1} \in \mathbb{N}$, then

$$\mathcal{M}_{1,1}(\delta, \nu) \lesssim \nu^{-1/3}\mathcal{M}_{2^N,2^N}(\delta, \nu)^{1/2^N} \cdot \prod_{j=0}^{N-1} D(\frac{\delta}{\nu^{2^j}})^{\frac{1}{2^{j+1}}}$$

Applying the trivial bound for the bilinear constant bounds gives that the above is

$$\lesssim \nu^{-1/3} D(\frac{\delta}{\nu^{2^N}})^{1/2^N} \prod_{j=0}^{N-1} D(\frac{\delta}{\nu^{2^j}})^{\frac{1}{2^{j+1}}}$$

Choosing $\nu = \delta^{1/2^N}$ shows that if $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$, then

$$\mathcal{M}_{1,1}(\delta, \delta^{1/2^N}) \lesssim \delta^{-\frac{1}{3\cdot 2^N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}.$$

By the bilinear reduction, if δ was such that $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$, then

$$D(\delta) \lesssim D(\delta^{1-\frac{1}{2^N}}) + \delta^{-\frac{4}{3\cdot 2^N}} \prod_{j=0}^{N-1} D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}.$$

This completes the proof of Lemma 3.4.10. □

Trivial bounds for $D(\delta)$ show that $1 \lesssim D(\delta) \lesssim \delta^{-1/2}$ for all $\delta \in \mathbb{N}^{-1}$. Let λ be the smallest real number such that $D(\delta) \lesssim_\varepsilon \delta^{-\lambda-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$. From the trivial bounds $\lambda \in [0, 1/2]$. We claim $\lambda = 0$. Suppose $\lambda > 0$.

Let N be a sufficiently large integer $\geqslant \frac{8}{3\lambda}$. This implies

$$1 + \frac{N}{2} - \frac{4}{3\lambda} \geqslant 1.$$

Lemma 3.4.10 then implies that for δ such that $\delta^{-1/2^N} \in \mathbb{N}$ and $0 < \delta < 100^{-2^N}$, we have

$$D(\delta) \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon} + \delta^{-\lambda(1-\frac{1}{2^N}(1+\frac{N}{2}-\frac{4}{3\lambda}))-\varepsilon} \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}$$

where the last inequality we have applied our choice of N. By almost multiplicity we then have the same estimate for all $\delta \in \mathbb{N}^{-1}$ (with a potentially larger constant depending on N). But this then contradicts minimality of λ. Therefore $\lambda = 0$.

3.5 Unifying the two styles of proof

We now attempt to unify the Bourgain-Demeter style of decoupling and the style of decoupling mentioned in the previous section. In view of Corollary 3.4.9, instead of having two integer parameters a and b we just have one integer parameter.

Let b be an integer $\geqslant 1$ and choose $s \in [2,3]$ any real number. Suppose $\delta \in \mathbb{N}^{-1}$ and $\nu \in \mathbb{N}^{-1} \cap (0, 1/100)$ were such that $\nu^b \delta^{-1} \in \mathbb{N}$. Let $\mathbf{M}_b^{(s)}(\delta, \nu)$ be the best constant such that

$$\begin{aligned} &\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(B)} \Big(\sum_{J \in P_{\nu^b}(I)} \|\mathcal{E}_J g\|_{L^2_\#(w_\Delta)}^2\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I')} \|\mathcal{E}_{J'} g\|_{L^2_\#(w_\Delta)}^2\Big)^{\frac{6-s}{2}} \\ &\leqslant \mathbf{M}_b^{(s)}(\delta,\nu)^6 \Big(\sum_{J \in P_\delta(I)} \|\mathcal{E}_J g\|_{L^2_\#(w_B)}^2\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_\delta(I')} \|\mathcal{E}_{J'} g\|_{L^2_\#(w_B)}^2\Big)^{\frac{6-s}{2}} \end{aligned} \tag{3.31}$$

for all squares B of side length δ^{-2}, $g : [0,1] \to \mathbb{C}$, and all intervals $I, I' \in P_\nu([0,1])$ which are ν-separated. Note that left hand side of the definition of $\mathbf{M}_b^{(3)}(\delta,\nu)$ is the same as $A_6(q, B^r, q)^6$ defined in [BD17] and from the uncertainty principle, $\mathbf{M}_1^{(2)}(\delta,\nu)$ is morally the same as $M_{1,1}(\delta,\nu)$ defined in (3.2) and $\mathcal{M}_{1,1}(\delta,\nu)$ defined in (3.27). The l^2 piece in the definition of $\mathbf{M}_b^{(s)}(\delta,\nu)$ is so that we can make the most out of applying l^2L^2 decoupling.

We will use $\mathbf{M}_b^{(s)}$ as our bilinear constant in this section to show that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$. The bilinear constant $\mathbf{M}_b^{(s)}$ obeys much the same lemmas as in the previous sections.

Lemma 3.5.1 (cf. Lemmas 3.2.3 and 3.4.3)**.** *If δ and ν were such that $\nu^b\delta^{-1} \in \mathbb{N}$, then*

$$\mathbf{M}_b^{(s)}(\delta,\nu) \lesssim D(\frac{\delta}{\nu^b}).$$

Proof. Fix arbitrary $I_1, I_2 \in P_\nu([0,1])$ which are ν-separated. Moving up from $L^2_\#$ to $L^6_\#$ followed by Hölder in the average over Δ bounds the left hand side of (3.31)

$$\Big(\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(B)} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|_{L^6_\#(w_\Delta)}^2\Big)^{\frac{6}{2}}\Big)^s \Big(\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(B)} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_J g\|_{L^6_\#(w_\Delta)}^2\Big)^{\frac{6}{2}}\Big)^{6-s}.$$

Using Minkowski to switch the l^2 and l^6 sum followed by $\sum_\Delta w_\Delta \lesssim w_B$ shows that this is

$$\lesssim \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|_{L^6_\#(w_B)}^2\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|_{L^6_\#(w_B)}^2\Big)^{\frac{6-s}{2}}.$$

Parabolic rescaling then completes the proof of Lemma 3.5.1. □

Lemma 3.5.2 (Bilinear reduction, cf. Lemmas 3.2.5 and 3.4.5)**.** *Suppose δ and ν were such that $\nu\delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \lesssim D(\frac{\delta}{\nu}) + \nu^{-1}\mathbf{M}_1^{(s)}(\delta, \nu).$$

Proof. Note that the left hand side of the definition of $\mathbf{M}_1^{(s)}(\delta, \nu)$ is

$$\operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \|\mathcal{E}_{I_1} g\|_{L^2_\#(w_\Delta)}^s \|\mathcal{E}_{I_2} g\|_{L^2_\#(w_\Delta)}^{6-s}.$$

Proceeding as in the proof of Lemmas 3.2.5 and 3.4.5, for $I_i, I_j \in P_\nu([0,1])$ which are ν-separated, we have

$$\||\mathcal{E}_{I_i} g||\mathcal{E}_{I_j} g|\|_{L^3_\#(B)}^{1/2} \leqslant \||\mathcal{E}_{I_i} g|^{\frac{s}{6}}|\mathcal{E}_{I_j} g|^{1-\frac{s}{6}}\|_{L^6_\#(B)}^{1/2} \||\mathcal{E}_{I_i} g|^{1-\frac{s}{6}}|\mathcal{E}_{I_j} g|^{\frac{s}{6}}\|_{L^6_\#(B)}^{1/2}. \tag{3.32}$$

We have

$$\begin{aligned}
\||\mathcal{E}_{I_i} g|^{\frac{s}{6}}|\mathcal{E}_{I_j} g|^{1-\frac{s}{6}}\|_{L^6_\#(B)}^6 &= \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_{I_i} g|^s |\mathcal{E}_{I_j} g|^{6-s} \\
&\leqslant \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \|\mathcal{E}_{I_i} g\|_{L^s_\#(\Delta)}^s \|\mathcal{E}_{I_j} g\|_{L^\infty(\Delta)}^{6-s} \\
&\lesssim \operatorname*{Avg}_{\Delta \in P_{\nu^{-1}}(B)} \|\mathcal{E}_{I_i} g\|_{L^2_\#(w_\Delta)}^s \|\mathcal{E}_{I_j} g\|_{L^2_\#(w_\Delta)}^{6-s}
\end{aligned}$$

where the last inequality we have used Bernstein. Inserting this into (3.32) and applying the definition of $\mathbf{M}_1^{(s)}(\delta, \nu)$ then completes the proof of Lemma 3.5.2. □

Lemma 3.5.3 (Ball inflation, cf. Lemma 3.4.6)**.** *Let $b \geqslant 1$ be a positive integer. Suppose I_1 and I_2 are ν-separated intervals of length ν. Then for any square Δ' of side length ν^{-2b} and any $\varepsilon > 0$, we have*

$$\begin{aligned}
&\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|_{L^s_\#(w_\Delta)}^2\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|_{L^{6-s}_\#(w_\Delta)}^2\Big)^{\frac{6-s}{2}} \\
&\qquad \lesssim_\varepsilon \nu^{-1-b\varepsilon} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|_{L^s_\#(w_{\Delta'})}^2\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|_{L^{6-s}_\#(w_{\Delta'})}^2\Big)^{\frac{6-s}{2}}
\end{aligned}$$

Proof. The $s = 2$ case be proven directly using Lemma 3.4.6 without any loss in $\nu^{-b\varepsilon}$. The proof for $s \in (2, 3]$ proceeds as in the proof of ball inflation in [BD17, Section 9.2] (see also Section 2.6 for more details and explicit constants).

From dyadic pigeonholing, since we can lose a $\nu^{-b\varepsilon}$, it suffices to restrict the sum over J and J' to families $\mathcal{F}_1$ and $\mathcal{F}_2$ such that for all $J \in \mathcal{F}_1$, $\|\mathcal{E}_J g\|_{L^s_{\#}(w_{\Delta'})}$ are comparable up to a factor of 2 and similarly for all $J' \in \mathcal{F}_2$. Hölder gives

$$\begin{aligned}
&\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \Big(\sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|^2_{L^s_{\#}(w_\Delta)}\Big)^{\frac{s}{2}} \Big(\sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|^2_{L^{6-s}_{\#}(w_\Delta)}\Big)^{\frac{6-s}{2}} \\
&\quad \leqslant (\#\mathcal{F}_1)^{\frac{s}{2}-1} (\#\mathcal{F}_2)^{\frac{6-s}{2}-1} \operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \Big(\sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|^s_{L^s_{\#}(w_\Delta)}\Big)\Big(\sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|^{6-s}_{L^{6-s}_{\#}(w_\Delta)}\Big).
\end{aligned}$$

The proof of Lemma 3.4.6 shows that this is

$$\lesssim \nu^{-1} (\#\mathcal{F}_1)^{\frac{s}{2}-1} (\#\mathcal{F}_2)^{\frac{6-s}{2}-1} \Big(\sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|^s_{L^s_{\#}(w_{\Delta'})}\Big)\Big(\sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|^{6-s}_{L^{6-s}_{\#}(w_{\Delta'})}\Big).$$

Since for $J \in \mathcal{F}_1$ the values of $\|\mathcal{E}_J g\|_{L^s_{\#}(w_{\Delta'})}$ are comparable and similarly for $J' \in \mathcal{F}_2$, the above is

$$\lesssim \nu^{-1} \Big(\sum_{J \in \mathcal{F}_1} \|\mathcal{E}_J g\|^2_{L^s_{\#}(w_{\Delta'})}\Big)^{\frac{s}{2}} \Big(\sum_{J' \in \mathcal{F}_2} \|\mathcal{E}_{J'} g\|^2_{L^{6-s}_{\#}(w_{\Delta'})}\Big)^{\frac{6-s}{2}}.$$

This completes the proof of Lemma 3.5.3. □

Lemma 3.5.4 (cf. Corollary 3.4.9)**.** *Suppose δ and ν were such that $\nu^{2b}\delta^{-1} \in \mathbb{N}$. Then for every $\varepsilon > 0$,*

$$\mathbf{M}_b^{(s)}(\delta, \nu) \lesssim_\varepsilon \nu^{-\frac{1}{6}(1+b\varepsilon)} \mathbf{M}_{2b}^{(s)}(\delta, \nu)^{1/2} D\Big(\frac{\delta}{\nu^b}\Big)^{1/2}.$$

Proof. Let θ and φ be such that $\frac{\theta}{2} + \frac{1-\theta}{6} = \frac{1}{s}$ and $\frac{\varphi}{2} + \frac{1-\varphi}{6} = \frac{1}{6-s}$. Then Hölder gives $\|f\|_{L^s} \leqslant \|f\|^{\theta}_{L^2} \|f\|^{1-\theta}_{L^6}$ and $\|f\|_{L^{6-s}} \leqslant \|f\|^{\varphi}_{L^2} \|f\|^{1-\varphi}_{L^6}$.

Fix arbitrary $I_1, I_2 \in P_\nu([0,1])$ which are ν-separated. We have

$$\begin{aligned}
&\operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(B)} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|^2_{L^2_{\#}(w_\Delta)}\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|^2_{L^2_{\#}(w_\Delta)}\Big)^{\frac{6-s}{2}} \\
&\leqslant \operatorname*{Avg}_{\Delta' \in P_{\nu^{-2b}}(B)} \operatorname*{Avg}_{\Delta \in P_{\nu^{-b}}(\Delta')} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|^2_{L^s_{\#}(w_\Delta)}\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|^2_{L^{6-s}_{\#}(w_\Delta)}\Big)^{\frac{6-s}{2}} \\
&\lesssim_\varepsilon \nu^{-1-b\varepsilon} \operatorname*{Avg}_{\Delta' \in P_{\nu^{-2b}}(B)} \Big(\sum_{J \in P_{\nu^b}(I_1)} \|\mathcal{E}_J g\|^2_{L^s_{\#}(w_{\Delta'})}\Big)^{\frac{s}{2}} \Big(\sum_{J' \in P_{\nu^b}(I_2)} \|\mathcal{E}_{J'} g\|^2_{L^{6-s}_{\#}(w_{\Delta'})}\Big)^{\frac{6-s}{2}}
\end{aligned}$$

where the first inequality is from Hölder and the second inequality is from ball inflation. We now use how θ and φ are defined to return to a piece which we control by l^2L^2 decoupling and a piece which we can control by parabolic rescaling. Hölder (as in the definition of θ and φ) gives that the average above is bounded by

$$\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^2_\#(w_{\Delta'})}^{2\theta}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2(1-\theta)}\Big)^{\frac{s}{2}}\times$$
$$\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^2_\#(w_{\Delta'})}^{2\varphi}\|\mathcal{E}_{J'} g\|_{L^6_\#(w_{\Delta'})}^{2(1-\varphi)}\Big)^{\frac{6-s}{2}}.$$

Hölder in the sum over J and J' shows that this is

$$\leqslant \operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\left(\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\theta}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{1-\theta}\right)^{\frac{s}{2}}\times$$
$$\left(\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\varphi}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{1-\varphi}\right)^{\frac{6-s}{2}}.$$

Since $\theta s = 3-\frac{s}{2}$ and $\varphi(6-s)=\frac{s}{2}$, rearranging the above gives

$$\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\left(\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\frac{1}{2}(3-\frac{s}{2})}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\frac{1}{2}\cdot\frac{s}{2}}\right)\times$$
$$\left(\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{1}{2}\cdot 3(\frac{s}{2}-1)}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{1}{2}\cdot 3(2-\frac{s}{2})}\right).$$

Cauchy-Schwarz in the average over Δ' then bounds the above by

$$\begin{aligned}&\left(\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\frac{6-s}{2}}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^2_\#(w_{\Delta'})}^{2}\Big)^{\frac{s}{2}}\right)^{\frac{1}{2}}\times\\&\left(\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{3(s-2)}{2}}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{3(4-s)}{2}}\right)^{\frac{1}{2}}.\end{aligned}\tag{3.33}$$

After l^2L^2 decoupling, the first term in (3.33) is

$$\lesssim \mathbf{M}^{(s)}_{2b}(\delta,\nu)^3\Big(\sum_{J\in P_\delta(I_1)}\|\mathcal{E}_J g\|_{L^2_\#(w_B)}^{2}\Big)^{\frac{1}{2}\cdot\frac{6-s}{2}}\Big(\sum_{J'\in P_\delta(I_2)}\|\mathcal{E}_{J'} g\|_{L^2_\#(w_B)}^{2}\Big)^{\frac{1}{2}\cdot\frac{s}{2}}.\tag{3.34}$$

Hölder in the average over Δ' bounds the second term in (3.33) by

$$\Big(\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{6}{2}}\Big)^{\frac{s-2}{4}}\Big(\operatorname*{Avg}_{\Delta'\in P_{\nu^{-2b}}(B)}\Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_{\Delta'})}^{2}\Big)^{\frac{6}{2}}\Big)^{\frac{4-s}{4}}.$$

Applying Minkowski to interchange the l^2 and l^6 norms shows that this is

$$\lesssim \Big(\sum_{J\in P_{\nu^b}(I_1)}\|\mathcal{E}_J g\|_{L^6_\#(w_B)}^{2}\Big)^{\frac{3(s-2)}{4}}\Big(\sum_{J'\in P_{\nu^b}(I_2)}\|\mathcal{E}_{J'} g\|_{L^6_\#(w_B)}^{2}\Big)^{\frac{3(4-s)}{4}}.$$

Parabolic rescaling bounds this by

$$D(\frac{\delta}{\nu^b})^3(\sum_{J\in P_\delta(I_1)}\|\mathcal{E}_J g\|^2_{L^6_\#(w_B)})^{\frac{1}{2}\cdot\frac{3(s-2)}{2}}(\sum_{J'\in P_\delta(I_2)}\|\mathcal{E}_{J'} g\|^2_{L^6_\#(w_B)})^{\frac{1}{2}\cdot\frac{3(4-s)}{2}}. \tag{3.35}$$

Combining (3.34) and (3.35) then completes the proof of Lemma 3.5.4. □

With Lemma 3.5.4, the same proof as Lemma 3.4.10 gives the following.

Lemma 3.5.5 (cf. Corollary 3.2.12 and Lemma 3.4.10)**.** *Let N be an integer chosen sufficient large later and let δ be such that $\delta^{-1/2^N}\in\mathbb{N}$ and $0<\delta<100^{-2^N}$. Then*

$$D(\delta)\lesssim_\varepsilon D(\delta^{1-\frac{1}{2^N}})+\delta^{-\frac{4}{3\cdot 2^N}-\frac{N\varepsilon}{6\cdot 2^N}}\prod_{j=0}^{N-1}D(\delta^{1-\frac{1}{2^{N-j}}})^{\frac{1}{2^{j+1}}}.$$

Proof. This follows from the proof of Lemma 3.4.10 and the observation that

$$\mathbf{M}_1^{(s)}(\delta,\nu)\lesssim_\varepsilon \nu^{-\frac{1}{3}-\frac{1}{6}N\varepsilon}\mathbf{M}_{2^N}^{(s)}(\delta,\nu)^{\frac{1}{2^N}}\prod_{j=0}^{N-1}D(\frac{\delta}{\nu^{2^j}})^{\frac{1}{2^{j+1}}}.$$

along with Lemmas 3.5.1 and 3.5.2. □

To finish, we proceed as at the end of the previous section. Let $\lambda\in[0,1/2]$ be the smallest real such that $D(\delta)\lesssim_\varepsilon\delta^{-\lambda-\varepsilon}$. Suppose $\lambda>0$. Choose N such that

$$1+\frac{N}{2}-\frac{4}{3\lambda}\geqslant 1.$$

Then for δ such that $\delta^{-1/2^N}\in\mathbb{N}$ and $0<\delta<100^{-2^N}$, Lemma 3.5.5 gives

$$D(\delta)\lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}+\delta^{-\lambda(1-\frac{1}{2^N}(1+\frac{N}{2}-\frac{4}{3\lambda}))-\varepsilon(1-\frac{1}{2^N})+\frac{N\varepsilon}{2\cdot 2^N}-\frac{N\varepsilon}{6\cdot 2^N}}\lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}.$$

Almost multiplicativity gives that $D(\delta)\lesssim_{N,\varepsilon}\delta^{-\lambda(1-\frac{1}{2^N})-\varepsilon}$ for all $\delta\in\mathbb{N}^{-1}$, contradicting the minimality of λ.

3.6 An efficient congruencing style proof of l^2L^4 decoupling for the parabola

3.6.1 Setup and some standard lemmas

Having compared the iteration from Bourgain-Demeter with an efficient congruencing style decoupling proof at L^6, we compare the two arguments for some $2<p<6$. We using techniques from the previous sections to prove an explicit upper bound for the l^2L^4 decoupling constant for the parabola. We will make use of the uncertainty principle at times, however the rigorous argument can easily be made in a similar manner as how we transitioned from Section 3.3 to Section 3.4.

Aside from the notation for the linear and bilinear decoupling constants, we adopt all notation from the previous sections. For simplicity, in this section we write $D(\delta)$ to be the l^2L^4 decoupling constant for the parabola. That is, for $\delta \in \mathbb{N}^{-1}$, let $D(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]}g\|_{L^4(B)} \leqslant D(\delta)(\sum_{J\in P_\delta([0,1])}\|\mathcal{E}_J g\|^2_{L^4(w_B)})^{1/2}$$

for all $g : [0,1] \to \mathbb{C}$ and all squares B of side length δ^{-2}.

Let geom be the standard geometric mean. We will assume that $\delta^{-1} \in \mathbb{N}$ and $\nu \in \mathbb{N}^{-1} \cap (0, 1/10000)$. Fix arbitrary integer $a \geqslant 1$, Suppose δ and ν was such that $\nu^a\delta^{-1} \in \mathbb{N}$. For this δ and ν, let $M_a(\delta, \nu)$ be the best constant such that

$$\|\operatorname{geom}|\mathcal{E}_{I_i}g|\|_{L^4(B)} \leqslant M_a(\delta,\nu)\operatorname{geom}(\sum_{J\in P_\delta(I_i)}\|\mathcal{E}_J g\|^2_{L^4(w_B)})^{1/2}$$

for all squares B of side length δ^{-2}, $g : [0,1] \to \mathbb{C}$, and all intervals $I_1, I_2 \in P_{\nu^a}([0,1])$ with $d(I_1, I_2) \geqslant 3\nu$.

In Chapter 2 we showed that $D(\delta) \lesssim \exp(O((\log \frac{1}{\delta})^{2/3}))$. In this section we will show that the methods from the previous section give

$$D(\delta) \lesssim \exp(O((\log \frac{1}{\delta})^{3/4})) \tag{3.36}$$

which is qualitatively the same as the bound we obtained in Chapter 2.

Remark 3.6.1. Since $4 = 2 + 2$, it turns out that we only need to have one frequency scale in $M_a(\delta, \nu)$. One could also define an alternative bilinear decoupling constant with two frequency scales $M_{a,b}(\delta, \nu)$ analogously as in (3.2). In this case, the key properties are $M_{a,b}(\delta, \nu) = M_{b,a}(\delta, \nu)$ and $M_{a,b}(\delta, \nu) \lesssim \nu^{-1/4}M_{b,2b}(\delta, \nu)$. In both definitions we obtain essentially the same iteration and that $D(\delta) \lesssim \exp(O((\log \frac{1}{\delta})^{3/4}))$.

We have the following standard lemmas which we will state without proof.

Lemma 3.6.2 (Parabolic rescaling)**.** *Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$. Let I be an arbitrary interval in $[0,1]$ of length σ. Then*

$$\|\mathcal{E}_I g\|_{L^4(B)} \lesssim D(\frac{\delta}{\sigma})(\sum_{J \in P_\delta(I)} \|\mathcal{E}_J g\|_{L^4(w_B)}^2)^{1/2}$$

for every $g : [0,1] \to \mathbb{C}$ and every square B of side length δ^{-2}.

Lemma 3.6.3 (Almost multiplicativity)**.** *Let $0 < \delta < \sigma < 1$ be such that $\sigma, \delta, \delta/\sigma \in \mathbb{N}^{-1}$, then*

$$D(\delta) \lesssim D(\sigma)D(\delta/\sigma).$$

Lemma 3.6.4 (Bilinear reduction)**.** *Suppose δ and ν were such that $\nu\delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \lesssim D(\frac{\delta}{\nu}) + \nu^{-1} M_1(\delta, \nu).$$

Lemma 3.6.5. *If δ and ν are such that $\nu^a\delta^{-1} \in \mathbb{N}$, then*

$$M_a(\delta, \nu) \lesssim D(\frac{\delta}{\nu^a}).$$

3.6.2 The key technical lemma

Much like how Lemma 3.2.8 was the key step in the previous section, the following key technical lemma drives our iteration.

Lemma 3.6.6. *Let a and b be integers such that $1 \leqslant a < b$. Suppose δ and ν are such that $\nu^b\delta^{-1} \in \mathbb{N}$. Then*

$$M_a(\delta, \nu) \lesssim M_b(\delta, \nu).$$

Proof. It suffices to assume that B is centered at the origin with side length δ^{-2}. Note that the integrality conditions imply that $\delta \leqslant \nu^b$ and since $\nu^{-1} \in \mathbb{N}$, $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$.

Fix arbitrary intervals $I_1 = [\alpha, \alpha + \nu^a]$ and $I_2 = [\beta, \beta + \nu^a]$ both in $P_{\nu^a}([0,1])$ and are 3ν-separated. Observe that

$$\| \operatorname{geom} |\mathcal{E}_{I_i} g| \|_{L^4(B)}^4 = \int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^2.$$

Let $g_\beta(x) := g(x+\beta)$, $T_\beta = \left(\begin{smallmatrix} 1 & 2\beta \\ 0 & 1 \end{smallmatrix}\right)$, and $d := \alpha - \beta$. Then shifting I_2 to $[0, \nu^a]$ gives that

$$\begin{aligned} \int_B |\mathcal{E}_{I_1} g|^2 |\mathcal{E}_{I_2} g|^2 &= \int_B |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(T_\beta x)|^2 |\mathcal{E}_{[0,\nu^a]} g_\beta)(T_\beta x)|^2 \, dx \\ &= \int_{T_\beta(B)} |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(x)|^2 |(\mathcal{E}_{[0,\nu^a]} g_\beta)(x)|^2 \, dx. \end{aligned} \tag{3.37}$$

Note that d can be negative, however since $g : [0,1] \to \mathbb{C}$ and $d = \alpha - \beta$, $\mathcal{E}_{[d,d+\nu^a]} g_\beta$ is defined. Since $|\beta| \leqslant 1/2$, $T_\beta(B) \subset 10B$. Combining this with $1_{10B} \leqslant \eta_{10B}$ gives that the above is

$$\begin{aligned} &\leqslant \int_{\mathbb{R}^2} |(\mathcal{E}_{[d,d+\nu^a]} g_\beta)(x)|^2 |(\mathcal{E}_{[0,\nu^a]} g_\beta)(x)|^2 \eta_{10B}(x) \, dx \\ &= \sum_{\substack{J_1, J_2 \in P_{\nu^b}([d, d+\nu^a]) \\ K_1, K_2 \in P_{\nu^b}([0,\nu^a])}} \int_{\mathbb{R}^2} \mathcal{E}_{J_1} g_\beta \overline{\mathcal{E}_{J_2} g_\beta \mathcal{E}_{K_1} g_\beta} \mathcal{E}_{K_2} g_\beta \eta_{10B} \, dx. \end{aligned} \tag{3.38}$$

We will show that the integral above is zero unless $d(J_1, J_2) \leqslant \nu^b$ and $d(K_1, K_2) \leqslant \nu^b$. If we can show this, then we can add these two conditions into the sum in (3.38) and hence Cauchy-Schwarz bounds (3.38) by

$$\sum_{\substack{J \in P_{\nu^b}([d, d+\nu^a]) \\ K \in P_{\nu^b}([0,\nu^a])}} \int_{\mathbb{R}^2} |\mathcal{E}_J g_\beta|^2 |\mathcal{E}_K g_\beta|^2 \eta_{10B} \, dx.$$

Undoing the change of variables as in (3.37) gives that the above is equal to

$$\sum_{\substack{J \in P_{\nu^b}(I_1) \\ K \in P_{\nu^b}(I_2)}} \int_{\mathbb{R}^2} |\mathcal{E}_J g|^2 |\mathcal{E}_K g|^2 \eta_{10B}(T_\beta x) \, dx.$$

The definition of M_b and the observation that $\eta_{10B}(T_\beta x) \lesssim w_B(x)$ gives that the above is bounded above by (here we will need a version of M_b with the left hand side with weight w_B, but such a constant is equivalent to M_b)

$$\begin{aligned} M_b(\delta, \nu)^4 &\sum_{\substack{J \in P_{\nu^b}(I_1) \\ K \in P_{\nu^b}(I_2)}} \Big(\sum_{J' \in P_\delta(J)} \|\mathcal{E}_{J'} g\|_{L^4(w_B)}^2\Big)\Big(\sum_{K' \in P_\delta(K)} \|\mathcal{E}_{K'} g\|_{L^4(w_B)}^2\Big) \\ &\leqslant M_b(\delta, \nu)^4 \Big(\sum_{J \in P_\delta(I_1)} \|\mathcal{E}_J g\|_{L^4(w_B)}^2\Big)\Big(\sum_{K \in P_\delta(I_2)} \|\mathcal{E}_K g\|_{L^4(w_B)}^2\Big). \end{aligned}$$

This then proves Lemma 3.6.6 provided we can add in the conditions $d(J_1, J_2) \leqslant \nu^b$ and $d(K_1, K_2) \leqslant \nu^b$ into (3.38).

Fix $J_1, J_2 \in P_{\nu^b}([d, d+\nu^a])$ and $K_1, K_2 \in P_{\nu^b}([0, \nu^a])$. Suppose $d(J_1, J_2) > \nu^b$. We claim that

$$\int_{\mathbb{R}^2} \mathcal{E}_{J_1} g_\beta \overline{\mathcal{E}_{J_2} g_\beta \mathcal{E}_{K_1} g_\beta} \mathcal{E}_{K_2} g_\beta \eta_{10B}\, dx = 0 \tag{3.39}$$

in this case. The case when $d(K_1, K_2) > \nu^b$ is similar. The left hand side is equal to

$$\int_{J_1 \times J_2 \times K_1 \times K_2} g_\beta(\xi_1) \overline{g_\beta(\xi_2) g_\beta(\xi_3)} g_\beta(\xi_4) \int_{\mathbb{R}^2} e(\cdots) \eta_{10B}(x)\, dx\, d\xi$$

where the expression in the $e(\cdots)$ is

$$((\xi_1 - \xi_2 - \xi_3 + \xi_4)x_1 + (\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2)x_2).$$

Therefore by the Fourier support of η_{10B}, (3.39) is equal to 0 unless

$$|\xi_1 - \xi_2 - \xi_3 + \xi_4| \leqslant \frac{\delta^2}{10}$$
$$|\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2| \leqslant \frac{\delta^2}{10}.$$

Since $d(J_1, J_2) > \nu^b$, $|\xi_1 - \xi_2| > \nu^b$ and since I_1 and I_2 are 3ν-separated, $|\xi_2 - \xi_4| > 3\nu$. Note that $|\xi_i| \leqslant 1$ and

$$\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = (\xi_1 - \xi_2 + \xi_3 - \xi_4)(\xi_2 - \xi_4) + (\xi_1 - \xi_2 - \xi_3 + \xi_4)(\xi_1 + \xi_3).$$

Therefore

$$|\xi_1 - \xi_2 + \xi_3 - \xi_4| \leqslant \frac{1}{10}\delta^2 \nu^{-1} \leqslant \frac{1}{10}\nu^{2b-1}.$$

We claim that the above inequalities are inconsistent. Since we are not given the relative positions of the ξ_i, we have the following two cases.

(i) $\xi_1 > \xi_2$ and $\xi_4 > \xi_3$ OR $\xi_2 > \xi_1$ and $\xi_3 > \xi_4$: We have

$$\frac{\delta^2}{10} \geqslant |\xi_1 - \xi_2 - \xi_3 + \xi_4| = |\xi_1 - \xi_2| + |\xi_4 - \xi_3| \geqslant |\xi_1 - \xi_2| > \nu^b.$$

Since $\delta \leqslant \nu^b$, we then have $\nu^b \leqslant \nu^{2b}/10$, a contradiction.

(*ii*) $\xi_1 > \xi_2$ and $\xi_3 > \xi_4$ OR $\xi_2 > \xi_1$ and $\xi_4 > \xi_3$: We have

$$\frac{1}{10}\nu^{2b-1} \geqslant |\xi_1 - \xi_2 + \xi_3 - \xi_4| = |\xi_1 - \xi_2| + |\xi_3 - \xi_4| \geqslant |\xi_1 - \xi_2| > \nu^b,$$

a contradiction since $b > 1$ and ν is sufficiently small.

Therefore in all cases (3.39) is equal to 0 when $d(J_1, J_2) > \nu^b$. This completes the proof of Lemma 3.6.6. □

The following alternate to Lemma 3.6.6 can also be used and is reminiscent of the proofs of Lemmas 3.3.4 and 3.3.5.

Lemma 3.6.7. *Let a be a positive integer. Suppose δ and ν are such that $\nu^{2a}\delta^{-1} \in \mathbb{N}$. Then*

$$M_a(\delta, \nu) \lesssim \nu^{-1/4} M_{2a}(\delta, \nu).$$

Proof. We will make use of the uncertainty principle in this proof, but this can be made rigorous through the same methods we used to make Section 3.3 rigorous.

It suffices to prove that

$$\int_B |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^2 \lesssim \nu^{-1} \sum_{\substack{J \in P_{\nu^{2a}}(I) \\ J' \in P_{\nu^{2a}}(I')}} \int_B |\mathcal{E}_J g|^2 |\mathcal{E}_{J'} g|^2 \tag{3.40}$$

for $I, I' \in P_{\nu^a}([0,1])$ with $d(I, I') \gtrsim \nu$.

Fix $I, I' \in P_{\nu^a}([0,1])$ with $d(I, I') \gtrsim \nu$. To show (3.40), it suffices to show that

$$\frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^2 \lesssim \nu^{-1} \sum_{\substack{J \in P_{\nu^{2a}}(I) \\ J' \in P_{\nu^{2a}}(I')}} \frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_J g|^2 |\mathcal{E}_{J'} g|^2 \tag{3.41}$$

for each $\Delta \in P_{\nu^{-2a}}(B)$.

Since the uncertainty principle implies that $|\mathcal{E}_J g|$ and $|\mathcal{E}_{J'} g|$ are essentially constant on Δ, combining this with l^2L^2 decoupling shows (3.41) reduces to showing that

$$\frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_I g|^2 |\mathcal{E}_{I'} g|^2 \lesssim \nu^{-1}(\frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_I g|^2)(\frac{1}{|\Delta|}\int_\Delta |\mathcal{E}_{I'} g|^2). \tag{3.42}$$

Now as in the proof of Lemma 3.3.4, the uncertainty principle says that on Δ, $|\mathcal{E}_I g| \sim \sum_T |c_T| 1_T$ and $|\mathcal{E}_{I'} g| \sim \sum_{T'} |c_{T'}| 1_{T'}$ where $\{T\}$ and $\{T'\}$ are $\nu^{-a} \times \nu^{-2a}$ rectangles covering Δ' and pointing in the normal direction of the cap on the parabola living above I and I', respectively.

Thus we would have (3.42) if we could show that for each pair of tubes T, T' associated to I, I', we have

$$(\frac{1}{|\Delta|}\int_{\Delta} 1_T 1_{T'}) \lesssim \nu^{-1}(\frac{1}{|\Delta|}\int_{\Delta} 1_T)(\frac{1}{|\Delta|}\int_{\Delta} 1_{T'}) \tag{3.43}$$

for some absolute constant C. But since $d(I, I') \gtrsim \nu$, the left hand side is equal to $\nu^{-1}(\nu^{2a})$ while the right hand side is $\nu^{-1}(\nu^a)^2$ which proves (3.43) and hence proves (3.40) which completes the proof of Lemma 3.6.7. □

3.6.3 The iteration and endgame

First applying Lemma 3.6.4 followed by Lemma 3.6.6 and then Lemma 3.6.5 then gives the following lemma.

Lemma 3.6.8. *Let $m > 10$. Suppose δ and ν were such that $\nu^m \delta^{-1} \in \mathbb{N}$. Then*

$$D(\delta) \lesssim D(\frac{\delta}{\nu}) + \nu^{-1} D(\frac{\delta}{\nu^m}).$$

Choosing $\nu = \delta^{1/m}$ (and recalling that we also require $\nu \in \mathbb{N}^{-1} \cap (0, 1/100)$) gives the following result.

Lemma 3.6.9. *Let $m > 10$. Suppose δ was such that $\delta^{-1/m} \in \mathbb{N}$ and $\delta < 100^{-m}$. Then*

$$D(\delta) \lesssim D(\delta^{1-1/m}) + \delta^{-1/m}$$

where the implied constant is independent of m.

We now give a proof that $D(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$ for all $\varepsilon > 0$.

Proposition 3.6.10. *For all $\delta \in \mathbb{N}^{-1}$, $D(\delta) \lesssim_{\varepsilon} \delta^{-\varepsilon}$.*

Proof. The trivial bounds for $D(\delta)$ are $1 \lesssim D(\delta) \lesssim \delta^{-1/2}$ for all $\delta \in \mathbb{N}^{-1}$. Let λ be the smallest real number such that $D(\delta) \lesssim_\varepsilon \delta^{-\lambda-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$. From the trivial bounds, $\lambda \in [0, 1/2]$. We claim that $\lambda = 0$. Suppose $\lambda > 0$.

Since $\lambda \leqslant 1/2$, choose m to be an integer such that

$$\frac{1}{m\lambda} < 1 - \frac{1}{m}$$

Then by Lemma 3.6.9, for $\delta^{-1/m} \in \mathbb{N}$ with $\delta < 100^{-m}$,

$$D(\delta) \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{m})-\varepsilon} + \delta^{-\lambda(\frac{1}{m\lambda})} \lesssim_\varepsilon \delta^{-\lambda(1-\frac{1}{m})-\varepsilon}.$$

Applying almost multiplicativity then shows that for all $\delta \in \mathbb{N}^{-1}$,

$$D(\delta) \lesssim_{m,\varepsilon} \delta^{-\lambda(1-\frac{1}{m})-\varepsilon},$$

contradicting minimality of λ. Therefore $\lambda = 0$. This completes the proof of Proposition 3.6.10. $\square$

Having shown that $D(\delta) \lesssim_\varepsilon \delta^{-\varepsilon}$, we now make this bound explicit. Fix arbitrary $0 < \varepsilon < 1/100$. Then $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ for all $\delta \in \mathbb{N}^{-1}$.

Lemma 3.6.11. *Fix arbitrary* $0 < \varepsilon < 1/100$. *Let* $m > 10$ *be such that*

$$\frac{1}{m\varepsilon} < 1 - \frac{1}{m}$$

and δ *such that* $\delta^{-1/m} \in \mathbb{N}$ *and* $\delta < 100^{-m}$. *Then*

$$D(\delta) \lesssim C_\varepsilon^{1-\varepsilon/m} \delta^{-\varepsilon}$$

where the implied is absolute.

Proof. Increasing C_ε, we may assume that $C_\varepsilon > 1$. Inserting $D(\delta) \leqslant C_\varepsilon \delta^{-\varepsilon}$ into Lemma 3.6.9 gives that for all integers $m > 1$ and δ such that $\delta^{-1/m} \in \mathbb{N}$ and $\delta < 100^{-m}$, we have

$$D(\delta) \lesssim (C_\varepsilon \delta^{\frac{\varepsilon}{m}} + \delta^{-\frac{1}{m}+\varepsilon})\delta^{-\varepsilon}. \tag{3.44}$$

If additionally $\delta < C_\varepsilon^{-1}$, then (3.44) becomes

$$D(\delta) \lesssim C_\varepsilon^{1-\frac{\varepsilon}{m}} \delta^{-\varepsilon}. \tag{3.45}$$

On the other hand if $\delta > C_\varepsilon^{-1}$, we can just apply the trivial bound $D(\delta) \lesssim \delta^{-1/2} \lesssim C_\varepsilon^{1/2}$ which is bounded above by the right hand side of (3.45). This completes the proof of Lemma 3.6.11. □

Using almost multiplicativity to get rid of the integrality conditions, we have the following lemma.

Lemma 3.6.12. *Fix arbitrary* $0 < \varepsilon < 1/100$. *For all* $\delta \in \mathbb{N}^{-1}$,

$$D(\delta) \lesssim \exp(O(\frac{1}{\varepsilon}))C_\varepsilon^{1-\varepsilon^2/2}\delta^{-\varepsilon}.$$

Thus if $P(C, \lambda)$ is the statement that $D(\delta) \leqslant C\delta^{-\lambda}$ for all $\delta \in \mathbb{N}^{-1}$, Lemma 3.6.12 implies that

$$P(C_\varepsilon, \varepsilon) \implies P(C\exp(O(\frac{1}{\varepsilon}))C_\varepsilon^{1-\varepsilon^2/2}, \varepsilon)$$

for an absolute constant C. Iterating this repeatedly then gives the following result.

Lemma 3.6.13. *Fix arbitrary* $0 < \varepsilon < 1/100$. *For all* $\delta \in \mathbb{N}^{-1}$,

$$D(\delta) \leqslant \exp(O(\frac{1}{\varepsilon^3}))\delta^{-\varepsilon}.$$

Optimizing in ε then proves (3.36).

3.7 A decoupling interpretation of efficient congruencing for the cubic moment curve

Having interpreted efficient congruencing for the quadratic Vinogradov conjecture in terms of l^2 decoupling, one immediate question is whether other works of efficient congruencing such as [Hea15] or [Woo19] can give a new and different proof of decoupling for the moment curve.

We sketch an argument that is ongoing work with Shaoming Guo and Po-Lam Yung in this direction. We reinterpret the iteration given in [Hea15] into decoupling language. To rigorously use the uncertainty principle, we use a slightly different formulation than what is below, however, the formulation below makes the connection to [Hea15] clearer. We are able to give a new proof of l^4L^{12} decoupling for the moment curve $t \mapsto (t, t^2, t^3)$ that is different from that given by Bourgain-Demeter-Guth in [BDG16] (who actually prove an l^2L^{12} decoupling theorem). In particular, we use a bilinear argument while [BDG16] uses a trilinear argument.

For the purposes of number theory, any l^pL^{12} decoupling theorem is sufficient. However our argument is only able to prove an l^pL^{12} decoupling theorem for the cubic moment curve for $p \geqslant 4$.

Let

$$(\mathcal{E}_I g)(x) := \int_I g(\xi)e(\xi x_1 + \xi^2 x_2 + \xi^3 x_3)\, d\xi.$$

We let $D(\delta)$ be the best constant such that

$$\|\mathcal{E}_{[0,1]}g\|_{L^{12}(B)} \leqslant D(\delta)(\sum_{J\in P_\delta([0,1])} \|\mathcal{E}_J g\|_{L^{12}(B)}^4)^{1/4}$$

for all functions $g : [0, 1] \to \mathbb{C}$ and all squares B of side length δ^{-3}. We prove that

$$D(\delta) \lesssim_\varepsilon \delta^{-1/4-\varepsilon}$$

which is the sharp l^4L^{12} decoupling theorem for the moment curve $t \mapsto (t, t^2, t^3)$.

Suppose $\nu \in 2^{-2^{\mathbb{N}}} \cap (0, 1/1000)$. We define two bilinear decoupling constants $\mathcal{M}_{1,a,b}(\delta, \nu)$ and $\mathcal{M}_{2,a,b}(\delta, \nu)$. Suppose a and b are integers and δ and ν are such that $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$. Let $\mathcal{M}_{1,a,b}(\delta, \nu)$ be the best constant such that

$$\int_B |\mathcal{E}_I g|^2|\mathcal{E}_{I'} g|^{10} \leqslant \mathcal{M}_{1,a,b}(\delta, \nu)^{12}(\sum_{J\in P_\delta(I)} \|\mathcal{E}_J g\|_{L^{12}(B)}^4)^{1/2}(\sum_{J'\in P_\delta(I')} \|\mathcal{E}_{J'} g\|_{L^{12}(B)}^4)^{5/2}$$

for all functions $g : [0, 1] \to \mathbb{C}$, cubes $B \subset \mathbb{R}^3$ of side length δ^{-3} and all pairs of intervals $I \in P_{\nu^a}([0, 1])$, $I' \in P_{\nu^b}([0, 1])$ with $d(I, I') \gtrsim \nu$. Similarly, let $\mathcal{M}_{2,a,b}(\delta, \nu)$ be the best constant such that

$$\int_B |\mathcal{E}_I g|^4|\mathcal{E}_{I'} g|^8 \leqslant \mathcal{M}_{2,a,b}(\delta, \nu)^{12}(\sum_{J\in P_\delta(I)} \|\mathcal{E}_J g\|_{L^{12}(B)}^4)(\sum_{J'\in P_\delta(I')} \|\mathcal{E}_{J'} g\|_{L^{12}(B)}^4)^2$$

for all functions $g : [0,1] \to \mathbb{C}$, cubes $B \subset \mathbb{R}^3$ of side length δ^{-3} and all pairs of intervals $I \in P_{\nu^a}([0,1])$, $I' \in P_{\nu^b}([0,1])$ with $d(I, I') \gtrsim \nu$. In addition to parabolic rescaling, our l^4L^{12} decoupling theorem is a consequence of the following five additional lemmas.

Lemma 3.7.1 (Bilinearization)**.** *If δ and ν were such that $\nu\delta^{-1} \in \mathbb{N}$, then*

$$D(\delta) \lesssim \nu^{-1/4} D(\frac{\delta}{\nu}) + \nu^{-1}\mathcal{M}_{2,1,1}(\delta, \nu).$$

Lemma 3.7.2. *If a and b are positive integers and δ and ν were such that $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$, then*

$$\mathcal{M}_{2,a,b}(\delta, \nu) \lesssim \mathcal{M}_{2,b,a}(\delta, \nu)^{1/3}\mathcal{M}_{1,a,b}(\delta, \nu)^{2/3}.$$

Lemma 3.7.3. *If a and b are positive integers and δ and ν were such that $\nu^a\delta^{-1}, \nu^b\delta^{-1} \in \mathbb{N}$, then*

$$\mathcal{M}_{1,a,b}(\delta, \nu) \lesssim \mathcal{M}_{2,b,a}(\delta, \nu)^{1/4} D(\frac{\delta}{\nu^b})^{3/4}.$$

Lemma 3.7.4. *Let a and b be integers such that $1 \leqslant a \leqslant 3b$. Suppose δ and ν were such that $\nu^{3b}\delta^{-1} \in \mathbb{N}$. Then*

$$\mathcal{M}_{1,a,b}(\delta, \nu) \lesssim_{a,b} \nu^{-\frac{1}{24}(3b-a)-C_0}\mathcal{M}_{1,3b,b}(\delta, \nu)$$

for some large absolute constant C_0.

Lemma 3.7.5. *Let a and b be integers such that $1 \leqslant a \leqslant b$. Suppose δ and ν were such that $\nu^{2b-a}\delta^{-1} \in \mathbb{N}$. Then for every $\varepsilon > 0$,*

$$\mathcal{M}_{2,a,b}(\delta, \nu) \lesssim_{a,b,\varepsilon} \nu^{-\frac{1}{6}(1+\varepsilon)(b-a)}\mathcal{M}_{2,2b-a,b}(\delta, \nu)$$

for some large absolute constant C_0.

The proof of Lemma 3.7.1 is similar to that of Lemma 3.2.5. The proof of Lemmas 3.7.2 and 3.7.3 essentially follow from the observations that

$$\int f^4 g^8 \leqslant (\int f^8 g^4)^{1/3} (\int f^2 g^{10})^{2/3}$$

and

$$\int f^2 g^{10} \leqslant (\int f^8 g^4)^{1/4} (\int g^{12})^{3/4}.$$

The proof of Lemma 3.7.4 relies on l^2L^2 decoupling and two ball inflation lemmas similar to that in Lemma 3.3.4. Bourgain-Demeter-Guth's proof of l^2L^{12} decoupling for the cubic moment curve will make use of l^2L^6 decoupling of the parabola as a lower dimensional input. It turns out that Lemma 3.7.5 will make use of the following lower dimensional decoupling theorem.

Lemma 3.7.6. *Let* $(E_I^{2D}g)(x) := \int_I g(\xi)e(\xi x_1 + \xi^2 x_2)\, d\xi$*. Then for every* $\varepsilon > 0$*,*

$$\|E_{[0,1]}^{2D} g\|_{L^4(B)} \lesssim_\varepsilon \delta^{-1/4-\varepsilon} (\sum_{J \in P_\delta([0,1])} \|E_J^{2D} g\|_{L^4(w_B)}^4)^{1/4}$$

for all functions $g : [0,1] \to \mathbb{C}$ *and squares* $B \subset \mathbb{R}^2$ *of side length* δ^{-1}*.*

The loss of $\delta^{-1/4}$ in Lemma 3.7.6 is sharp (up to $\delta^{-\varepsilon}$ losses) which can be seen by taking $g = 1_{[0,1]}$. Furthermore, the use of Lemma 3.7.6 is precisely why we were only able to prove an l^4L^{12} decoupling theorem rather than an l^2L^{12} decoupling theorem.

CHAPTER 4

More properties of the parabola decoupling constant

In this chapter, we collection some short stories about the parabola decoupling constant. First we prove some more equivalences of the parabola decoupling constant and show that these parabola decoupling constants are all monotonic. Among these parabola decoupling constants is the global decoupling constant that is used in [BD15]. Next, after having given iterative proofs of l^2L^4 decoupling for the parabola in Chapter 2 and Section 3.6, we give an elementary proof which shows that in the case of l^2L^4 decoupling for the parabola, the associated decoupling constant is $O(1)$. Finally in Section 4.4, we address a "small ball" l^2 decoupling theorem for the paraboloid that the author first learned from Hong Wang in January 2018.

4.1 Equivalence of some more parabola decoupling constants

In Section 2.3 (in particular (2.38)), we showed many spatially localized decoupling constants were all equivalent. Now we define a few more decoupling constants and show that they are equivalent. The decoupling constants we introduce are all of the type that involve an f with Fourier support in a δ^2 neighborhood of the parabola above $[0, 1]$. We then relate this to $\widehat{D}_{p,E}(\delta)$ from Definition 2.3.3 thus proving that a slew of local and global decoupling constants are equivalent. Here by local we mean spatially localized while by global we mean nonspatially localized. This section and Section 2.3 combined provide similar results that were stated (though not explicitly proven) in Remark 5.2 of [BD15].

As we stated in Remark 2.3.6, equivalence of various parabola decoupling constants is an extremely useful result. Because of the shear matrix, parabolic rescaling is easier using the global formulation rather than the local formulation. Thus by also showing that certain global decoupling constants are equivalent to some local decoupling constants we can apply parabolic rescaling using the global decoupling formulation and then pass this result to the local decoupling formulation. Also the result of this section shows that various local decoupling constants involving a function Fourier supported in some $O(\delta^2)$ neighborhood of the parabola are equivalent to each other regardless of decay E in the weight $w_{B,E}$ or thickness C of the $C\delta^2$ neighborhood of the parabola. The results in this section can be generalized to an arbitrary $h \in C^2$ satisfying: $h(0) = h'(0) = 0$, $0 < h'(t) \leqslant 1$ for $t \in (0,1]$, and $1/2 \leqslant h''(t) \leqslant 2$ for $t \in [0,1]$ but we do not pursue that here.

4.1.1 Basic tools and definitions

We first define two local and global decoupling constants. We show that these decoupling constants are equivalent by linearly approximating the regions where f has Fourier support and using that Fourier restriction to polygons are bounded in L^p.

For a square B centered at c with side length R, let $w_{B,E}(x) := (1 + \frac{|x-c|}{R})^{-E}$. Let η be a Schwartz function such that $\eta \geqslant 1_{B(0,1)}$ and $\mathrm{supp}(\widehat{\eta}) \subset B(0,1)$. For a square B centered at c of side length R, we let $\eta_B(x) := \eta(\frac{x-c}{R})$.

If $J \in P_\delta([0,1])$ and $n \in \mathbb{N}$, let

$$\theta_{J,n} := \{(s, s^2 + t) : s \in J, |t| \leqslant \frac{n}{2}\delta^2\} \tag{4.1}$$

and $\Theta_n := \bigcup_{J \in P_\delta([0,1/2])} \theta_{J,n}$. We now define the following two decoupling constants.

Definition 4.1.1. *Let $D^L_{p,n,E}(\delta)$ be the best constant such that*

$$\|f\|_{L^p(B)} \leqslant D^L_{p,n,E}(\delta)(\sum_{J \in P_\delta([0,1])} \|f_{\theta_{J,n}}\|^2_{L^p(w_{B,E})})^{1/2}$$

for all f with Fourier support in Θ_n and squares B of side length δ^{-2}.

Let $D^G_{p,n}(\delta)$ be the best constant such that

$$\|f\|_p \leqslant D^G_{p,n}(\delta)(\sum_{J \in P_\delta([0,1])} \|f_{\theta_{J,n}}\|^2_p)^{1/2}$$

for all f with Fourier support in Θ_n.

We reintroduce the parallelograms from the discussion above Lemma 2.3.1 though this time instead of a 10δ neighborhood we use an $n\delta^2$ neighborhood (we also have switched notation slightly so that $\delta^{1/2}$ and δ in Chapter 2 have become δ and δ^2, but this does not change any of our results). If $J = [n_J\delta, (n_J+1)\delta] \in P_\delta([0,1])$, let L_J be the line connecting the point $(n_J\delta, n_J^2\delta^2)$ and $((n_J+1)\delta, (n_J+1)^2\delta^2)$. Explicitly we have

$$L_J(x) := \delta(2n_J+1)(x - n_J\delta) + n_J^2\delta^2.$$

For $J \in P_\delta([0,1])$ and $n \in \mathbb{N}$, let

$$\theta'_{J,n} := \{(s, L_J(s) + t) : s \in J, |t| \leqslant \frac{n}{2}\delta^2\}.$$

Pictorially, $\theta'_{J,n}$ is a parallelogram with sides parallel to L_J of height $n\delta^2$. Finally, we let $\Theta'_n := \bigcup_{J \in P_\delta([0,1])} \theta'_{J,n}$.

We now define two more decoupling constants we will consider which are the parallelogram versions of Definition 4.1.1.

Definition 4.1.2. *Let $D^{par,L}_{p,n,E}(\delta)$ be the best constant such that*

$$\|f\|_{L^p(B)} \leqslant D^{par,L}_{p,n,E}(\delta)(\sum_{J \in P_\delta([0,1])} \|f_{\theta'_{J,n}}\|^2_{L^p(w_{B,E})})^{1/2}$$

for all f with Fourier support in Θ'_n and squares B of side length δ^{-2}.

Let $D^{par,G}_{p,n}(\delta)$ be the best constant such that

$$\|f\|_p \leqslant D^{par,G}_{p,n}(\delta)(\sum_{J \in P_\delta([0,1])} \|f_{\theta'_{J,n}}\|^2_p)^{1/2}$$

for all f with Fourier support in Θ'_n.

In Lemmas 4.1.5-4.1.6 we show that no matter how we modify the n and E parameter, the local and global decoupling constants defined in Definition 4.1.2 are equivalent. The proof will make use that $\theta'_{J,n}$ is a parallelogram, in particular, we will often make use that Fourier restriction to a parallelogram is bounded as an operator on L^p. We also have the following reverse triangle inequality which will prove to be useful.

Lemma 4.1.3 (Reverse triangle inequality)**.** *Let θ and θ' be two parallelograms with disjoint interior. Then for $1 < p < \infty$,*

$$\|f_\theta\|_p + \|f_{\theta'}\|_p \sim_p \|f_{\theta\cup\theta'}\|_p.$$

Proof. Since θ and θ' are disjoint, $f_{\theta\cup\theta'} = f_\theta + f_{\theta'}$ and hence $\|f_{\theta\cup\theta'}\|_p \leqslant \|f_\theta\|_p + \|f_{\theta'}\|_p$ from the triangle inequality. We observe that $f_\theta = (f_{\theta\cup\theta'})_\theta$ and $f_{\theta'} = (f_{\theta\cup\theta'})_{\theta'}$ and so since Fourier restriction to a parallelogram is bounded in L^p for $1 < p < \infty$,

$$\|f_\theta\|_p + \|f_{\theta'}\|_p = \|(f_{\theta\cup\theta'})_\theta\|_p + \|(f_{\theta\cup\theta'})_{\theta'}\|_p \lesssim \|f_{\theta\cup\theta'}\|_p.$$

This completes the proof of Lemma 4.1.3. □

4.1.2 Equivalence of parallelogram decoupling constants

We first show that we have many equivalences for the parallelogram decoupling constants. The restriction to $2 \leqslant p \leqslant 6$ is not important and is just there to get rid of the dependence on p.

Lemma 4.1.4 (Global equivalence for $n \neq m$)**.** *For $2 \leqslant p \leqslant 6$ and $n \neq m$,*

$$D_{p,n}^{par,G}(\delta) \sim_{n,m} D_{p,m}^{par,G}(\delta).$$

Proof. It suffices to show the case when $n = 1$. Since $m > 1$, $\Theta_1' \subset \Theta_m'$ and hence if f is Fourier supported in $\Theta_1'(h)$, we then have

$$\|f\|_p \leqslant D_{p,m}^{par,G}(\delta)(\sum_{J\in P_\delta([0,1])} \|f_{\theta'_{J,m}}\|_p^2)^{1/2}.$$

However since f is Fourier supported in Θ_1', $f_{\theta'_{J,m}} = f_{\theta'_{J,1}}$ and hence $D_{p,1}^{par,G}(\delta) \leqslant D_{p,m}^{par,G}(\delta)$.

The reverse inequality will make use of Lemma 4.1.3. The idea is to partition Θ_m' into m translates of Θ_1', apply $D_{p,1}^{par,G}(\delta)$ to each of these translates, and then sum them together using Lemma 4.1.3 (losing a constant depending on m).

Let f be Fourier supported in Θ_m'. For each $J \in P_\delta([0,1])$, we can write

$$\theta'_{J,m} = \bigcup_{i=1}^{m} \theta'_{J,1} + (0, c_i)$$

for some c_i and the union is a disjoint union (except at the boundary). Explicitly if m is odd, then we can take $\{c_i\} = \{\frac{k\delta^2}{2} : k \text{ even}, |k| \leqslant m-1\}$ and if m is even, then we can take $\{c_i\} = \{\frac{k\delta^2}{2} : k \text{ odd}, |k| \leqslant m-1\}$.

Next Lemma 4.1.3 implies that

$$\|f_{\theta'_{J,m}}\|_p \sim_m \sum_{i=1}^{m} \|f_{\theta'_{J,1}+(0,c_i)}\|_p$$

where here we have removed the dependence on p because $2 \leqslant p \leqslant 6$. Therefore

$$\sum_{i=1}^{m} \|f_{\theta'_{J,1}+(0,c_i)}\|_p^2 \lesssim (\sum_{i=1}^{m} \|f_{\theta'_{J,1}+(0,c_i)}\|_p)^2 \lesssim_m \|f_{\theta'_{J,m}}\|_p^2. \tag{4.2}$$

With this, we write $f = \sum_{i=1}^{m} f_{\Theta'_1+(0,c_i)}$ and estimate

$$\|f\|_p \lesssim_m (\sum_{i=1}^{m} \|f_{\Theta'_1+(0,c_i)}\|_p^2)^{1/2} \lesssim D_{p,1}^{par,G}(\delta)(\sum_{i=1}^{m} \sum_{J\in P_\delta([0,1])} \|f_{\theta'_{J,1}+(0,c_i)}\|_p^2)^{1/2}.$$

Interchanging sums and then applying (4.2) then shows $D_{p,m}^{par,G}(\delta) \lesssim_m D_{p,1}^{par,G}(\delta)$. This completes the proof of Lemma 4.1.4. □

Lemma 4.1.5 (Local-global equivalence for the same n)**.** *For* $2 \leqslant p \leqslant 6$,

$$D_{p,n,E}^{par,L}(\delta) \sim_{n,E} D_{p,n}^{par,G}(\delta).$$

Proof. We first show that $D_{p,n}^{par,G}(\delta) \lesssim_{n,E} D_{p,n,E}^{par,L}(\delta)$. Let $\mathcal{B}$ be a partition of $\mathbb{R}^2$ into squares of side length δ^{-2}. Since $\sum_{B\in\mathcal{B}} 1_B = 1$, convolving both sides with $w_{B(0,\delta^{-2}),E}$ and using convolution properties of $w_{B,E}$ (Lemma 2.2.1) shows that $\sum_{B\in\mathcal{B}} w_{B,E} \lesssim_E 1$.

Let f be Fourier supported in Θ'_n. Then

$$\|f\|_p^p = \sum_{B\in\mathcal{B}} \|f\|_{L^p(B)}^p \leqslant D_{p,n,E}^{par,L}(\delta)^p \sum_{B\in\mathcal{B}} (\sum_{J\in P_\delta([0,1])} \|f_{\theta'_{J,n}}\|_{L^p(w_{B,E})}^2)^{p/2},$$

Using Minkowski (and that $p \geqslant 2$) to interchange the l_J^2 and l_B^p bounds this by

$$(\sum_{J\in P_\delta([0,1])} \|f_{\theta'_{J,n}}\|_{L^p(\sum_{B\in\mathcal{B}} w_{B,E})}^2)^{p/2}.$$

Finally using that $\sum_{B\in\mathcal{B}} w_{B,E} \lesssim_E 1$ then shows that $D_{p,n}^{par,G}(\delta) \lesssim_E D_{p,n,E}^{par,L}(\delta)$ where here we have used that $p \leqslant 6$ to remove the dependence on p.

From Lemma 4.1.4, to show the reverse inequality, it suffices to show

$$D^{par,L}_{p,n,E}(\delta, h) \lesssim_{n,E} D^{par,G}_{p,10n}(\delta, h).$$

Let f be Fourier supported in Θ'_n. We have

$$\|f\|^2_{L^p(B)} \lesssim_E \|f_{\theta'_{[0,\delta],n}}\|^2_{L^p(w_{B,E})} + \|\eta_B f_{\theta'_{[\delta,1-\delta],n}}\|_p^2 + \|f_{\theta'_{[1-\delta,1],n}}\|^2_{L^p(w_{B,E})}.$$

Since $n/2+1 \leqslant 10n$, the Fourier transform of $\eta_B f_{\theta'_{[\delta,1-\delta],n}}$ is supported in Θ'_{10n}. Observe that for $J \in P_\delta([0,1])$,

$$(\eta_B f_{\theta'_{[\delta,1-\delta],n}})_{\theta'_{J,10n}} = \begin{cases} (\eta_B f_{\theta'_{J_r,n}})_{\theta'_{J,10n}} & \text{if } J = [0,\delta] \\ (\eta_B f_{\theta'_{J,n}})_{\theta'_{J,10n}} + (\eta_B f_{\theta'_{J_r,n}})_{\theta'_{J,10n}} & \text{if } J = [\delta, 2\delta] \\ \sum_{I \in \{J_\ell, J, J_r\}} (\eta_B f_{\theta'_{I,n}})_{\theta'_{J,10n}} & \text{if } J \in P_\delta([2\delta, 1-2\delta]) \\ (\eta_B f_{\theta'_{J_\ell,n}})_{\theta'_{J,10n}} + (\eta_B f_{\theta'_{J,n}})_{\theta'_{J,10n}} & \text{if } J = [1-2\delta, 1-\delta] \\ (\eta_B f_{\theta'_{J_\ell,n}})_{\theta'_{J,10n}} & \text{if } J = [1-\delta, 1] \end{cases} \tag{4.3}$$

where J_ℓ and J_r are the intervals to the left and right of J, respectively. Applying the definition of $D^{par,G}_{p,10n}(\delta)$ gives

$$\|\eta_B f_{\theta'_{[\delta,1-\delta],n}}\|_p^2 \leqslant D^{par,G}_{p,10n}(\delta)^2 \sum_{J \in P_\delta([0,1])} \|(\eta_B f_{\theta'_{[\delta,1-\delta],n}})_{\theta'_{J,10n}}\|_p^2.$$

Using (4.3) and the observations that $\theta'_{J,10n}(h)$ is a parallelogram and Fourier restriction to a parallelogram is bounded in L^p, the above is

$$\lesssim D^{par,G}_{p,10n}(\delta)^2 \sum_{J \in P_\delta([0,1])} \|f_{\theta'_{J,n}}\|^2_{L^p(\eta_B)}$$

where we have removed the dependence on p because $p \leqslant 6$. Since $\eta_B \lesssim_E w_{B,E}$, it then follows that $D^{par,L}_{p,n,E}(\delta) \lesssim_E D^{par,G}_{p,10n}(\delta)$. This completes the proof of Lemma 4.1.5. $\square$

Corollary 4.1.6 (Local equivalence for $n \neq m$, fixed E)**.** *For $2 \leqslant p \leqslant 6$ and $n \neq m$,*

$$D^{par,L}_{p,n,E}(\delta) \sim_{n,m,E} D^{par,L}_{p,m,E}(\delta).$$

Proof. From Lemma 4.1.5, $D^{par,L}_{p,n,E}(\delta) \sim_{n,E} D^{par,G}_{p,n}(\delta)$. From Lemma 4.1.4, $D^{par,G}_{p,n}(\delta) \sim_{n,m} D^{par,G}_{p,m}(\delta)$. Applying Lemma 4.1.5 again then completes the proof of Corollary 4.1.6. $\square$

Corollary 4.1.7 (Local equivalence for $n \neq m$, $E_1 \neq E_2$)**.** *For* $2 \leqslant p \leqslant 6$, $n \neq m$, $E_1 \neq E_2$,

$$D^{par,L}_{p,n,E_1}(\delta) \sim_{n,m,E_1,E_2} D^{par,L}_{p,m,E_2}(\delta).$$

Proof. Corollary 4.1.6 and Lemma 4.1.5 gives that

$$D^{par,L}_{p,n,E_1}(\delta) \sim_{n,m,E_1} D^{par,L}_{p,m,E_1}(\delta) \sim_{m,E_1} D^{par,G}_{p,m}(\delta) \sim_{m,E_2} D^{par,L}_{p,m,E_2}(\delta)$$

which completes the proof of Corollary 4.1.7. □

4.1.3 Equivalence of decoupling constants

We have the following lemma which will help us relate the parallelogram decoupling constants from Definition 4.1.2 to the decoupling constants we have defined in Definition 4.1.1.

Lemma 4.1.8. *For* $n \geqslant 2$, *we have*

$$\theta'_{J,1} \subset \theta_{J,n} \subset \theta'_{J,2n}.$$

Proof. For $s \in J$, recall from (2.35) that

$$|s^2 - L_J(s)| \leqslant \delta^2/4.$$

Since $n \geqslant 2$, for $s \in J$,

$$L_J(s) + \frac{\delta^2}{2} \leqslant s^2 + \frac{n\delta^2}{2} \leqslant L_J(s) + n\delta^2$$

which completes the proof of Lemma 4.1.8. □

Like the parallelogram decoupling constant equivalence, we have the following three equivalences. The purpose of introducing the parallelogram decoupling constants was because Fourier restriction to $\theta_{J,n}$ is not a bounded operator on L^p, however, Fourier restriction to $\theta'_{J,n}$ is a bounded operator on L^p.

Lemma 4.1.9 (Local-global equivalence for the same n)**.** *For* $2 \leqslant p \leqslant 6$ *and* $n \geqslant 2$,

$$D^{L}_{p,n,E}(\delta) \sim_{n,E} D^{G}_{p,n}(\delta).$$

Proof. Since $n \geqslant 2$, Lemma 4.1.8 implies $\Theta_1' \subset \Theta_n \subset \Theta_{2n}'$ and hence

$$D_{p,1,E}^{par,L}(\delta) \leqslant D_{p,n,E}^{L}(\delta) \leqslant D_{p,2n,E}^{par,L}(\delta) \lesssim_{n,E} D_{p,1,E}^{par,L}(\delta) \tag{4.4}$$

where the last inequality we have used Corollary 4.1.6. Using similar reasoning and Lemma 4.1.4 gives

$$D_{p,1}^{par,G}(\delta) \leqslant D_{p,n}^{G}(\delta) \leqslant D_{p,2n}^{par,G}(\delta) \lesssim_{n,E} D_{p,1}^{par,G}(\delta). \tag{4.5}$$

Finally combining these two estimates and Lemma 4.1.5 imply $D_{p,n,E}^{L}(\delta) \sim_{n,E} D_{p,n}^{G}(\delta)$ which completes the proof of Lemma 4.1.9. □

Corollary 4.1.10 (Global equivalence for $n \neq m$)**.** *For $2 \leqslant p \leqslant 6$ and $n \neq m$ with $n, m \geqslant 2$,*

$$D_{p,n}^{G}(\delta) \sim_{n,m} D_{p,m}^{G}(\delta).$$

Proof. It suffices to show that for each $n \geqslant 2$, $D_{p,n}^{G}(\delta) \sim_{n} D_{p,1}^{par,G}(\delta)$. But this exactly was shown in (4.5). □

Corollary 4.1.11 (Local equivalence for $n \neq m$, fixed E)**.** *For $2 \leqslant p \leqslant 6$ and $n \neq m$ with $n, m \geqslant 2$,*

$$D_{p,n,E}^{L}(\delta) \sim_{n,m,E} D_{p,m,E}^{L}(\delta).$$

Proof. For each $n \geqslant 2$, it is enough to show that $D_{p,n,E}^{L}(\delta) \sim_{n} D_{p,1,E}^{par,L}(\delta)$ but this is what was shown in (4.4). □

Corollary 4.1.12 (Local equivalence for $n \neq m$, $E_1 \neq E_2$)**.** *For $2 \leqslant p \leqslant 6$ and $n \neq m$ with $n, m \geqslant 2$,*

$$D_{p,n,E_1}^{L}(\delta) \sim_{n,m,E_1,E_2} D_{p,m,E_2}^{L}(\delta).$$

Proof. From Corollary 4.1.11, it is enough to show that $D_{p,m,E_1}^{L}(\delta) \sim_{m,E_1,E_2} D_{p,m,E_2}^{L}(\delta)$. But this follows immediately from Lemma 4.1.9. □

Note that $\widehat{D}_{p,E}(\delta)$ defined in Definition 2.3.3 is the same as $D_{p,10,E}^{par,L}(\delta)$ in this section. Therefore we have shown that for $2 \leqslant p \leqslant 6$, all the following constants are equivalent (up to constants that depend on all parameters of the constants involved except for p and δ):

(a) Extension operator based, spatially localized:

- $D_{p,E}(\delta)$, defined in (2.1), used in [BD17]
- $\widetilde{D}_{p,E}(\delta)$, defined in (2.2)
- $\overline{D}_p(\delta)$, defined in Definition 2.3.3

(b) Fourier based, spatially localized:

- $\widehat{D}_{p,E}(\delta)$, defined in Definition 2.3.3, equal to $D^{par,L}_{p,10,E}(\delta)$
- $D^{L}_{p,n,E}(\delta)$, defined in Definition 4.1.1
- $D^{par,L}_{p,n,E}(\delta)$, defined in Definition 4.1.2

(c) Fourier based, global:

- $D^{G}_{p,n}(\delta)$, defined in Definition 4.1.1, used in [BD15]
- $D^{par,G}_{p,n}(\delta)$, defined in Definition 4.1.2

That is, take any number of the eight above decoupling constants, for example, $D_{p,E_1}(\delta)$, $D^{par,G}_{p,n}(\delta)$, $\overline{D}_p(\delta)$, and $D^{L}_{p,m,E_2}(\delta)$ (also assume $n, m \geqslant 2$). Then our results show that for $2 \leqslant p \leqslant 6$,

$$D_{p,E_1}(\delta) \sim_{n,E_1} D^{par,G}_{p,n}(\delta) \sim_n \overline{D}_p(\delta) \sim_{m,E_2} D^{L}_{p,m,E_2}(\delta).$$

4.2 Monotonicity of the parabola decoupling constant

One immediate application of the results in Section 4.1, is that we can show that the decoupling constant, however defined in the list above is essentially a decreasing function of δ. The way we show Corollary 4.2.2 is not the most efficient way to show this for a particular decoupling constant. If one is willing to work with weight functions $w_{B,E}$, $\widetilde{w}_{B,E}$, η_B directly one can show the applicable monotonicity result using a calculation that is similar to the proof of parabolic rescaling (Section 2.4). However, having done the heavy lifting in Section 4.1 in showing many decoupling constants are equivalent we present a nice application of our work. This application of the equivalence of decoupling constants shows the power of such an equivalence since often certain calculations are easier with some decoupling constants while others are much more tedious.

The main proposition we claim is the following:

Proposition 4.2.1. *For $N \in \mathbb{N}$ and $2 \leqslant p \leqslant 6$, we have*

$$D^G_{p,2}(\frac{1}{N}) \leqslant D^G_{p,2}(\frac{1}{N+1})$$

where $D^G_{p,n}(\delta)$ is as in Definition 4.1.1.

Proof. This proof is a change of variables. To emphasize the interval and the scale δ, instead of using the notation $\theta_{J,2}$ from (4.1), we will let $T(\delta, I)$ be the piece of δ^2-tube living above $I \subset [0,1]$. That is

$$T(\delta, I) := \{(s, s^2 + t) : s \in I, |t| \leqslant \delta^2\}(= \theta_{I,2}).$$

Suppose f is Fourier supported in a $1/N^2$-tube of the parabola living above $[0,1]$. We have

$$\begin{aligned} f(x) &= \int_{T(\frac{1}{N},[0,1])} \widehat{f}(\xi) e(x \cdot \xi)\, d\xi \\ &= (\frac{N+1}{N})^3 \int_{T(\frac{1}{N+1},[0,\frac{N}{N+1}])} \widehat{f}(\frac{N+1}{N}\eta_1, \frac{(N+1)^2}{N^2}\eta_2) e(x_1 \frac{N+1}{N}\eta_1 + x_2 \frac{(N+1)^2}{N^2}\eta_2)\, d\eta \end{aligned}$$

Therefore

$$\|f\|_p = (\frac{N+1}{N})^{3-3/p} \|g_N\|_p \tag{4.6}$$

with

$$g_N(x) := \int_{\mathbb{R}^2} \widehat{f}(\frac{N+1}{N}\eta_1, \frac{(N+1)^2}{N^2}\eta_2) 1_{T(\frac{1}{N+1},[0,\frac{N}{N+1}])}(\eta) e(\eta \cdot x)\, d\eta.$$

Note that g_N is Fourier supported in a $1/(N+1)^2$-tube of the parabola living above $[0,1]$. Then

$$(\frac{N+1}{N})^{3-3/p} \|g_N\|_p \leqslant (\frac{N+1}{N})^{3-3/p} D^G_{p,2}(\frac{1}{N+1}) (\sum_{\substack{0 \leqslant i \leqslant N \\ \tau \in T(\frac{1}{N+1},[\frac{i}{N+1},\frac{i+1}{N+1}])}} \|(g_N)_\tau\|_p^2)^{1/2}. \tag{4.7}$$

For $i < N$,

$$\widehat{(g_N)}_\tau(\eta) = \widehat{f}(\frac{N+1}{N}\eta_1, \frac{(N+1)^2}{N^2}\eta_2)1_\tau(\eta)$$

and when $i = N$, $(g_N)_\tau = 0$. Undoing the change of variables used to obtain (4.6) gives that (4.7) is equal to

$$D^G_{p,2}(\frac{1}{N+1})(\sum_{\substack{0\leqslant j\leqslant N-1 \\ \tau\in T(\frac{1}{N},[\frac{i}{N},\frac{i+1}{N}])}} \|f_\tau\|_p^2)^{1/2}.$$

Applying the definition of $D^G_{p,2}(1/N)$ then completes the proof of Proposition 4.2.1. □

The following corollary follows from combining the above proposition and the results in Section 4.1.

Corollary 4.2.2. *For $N \in \mathbb{N}$ and $2 \leqslant p \leqslant 6$, the following eight inequalities are true:*

$$\begin{aligned}
D_{p,E}(\frac{1}{N}) &\lesssim_E D_{p,E}(\frac{1}{N+1}) \\
\widetilde{D}_{p,E}(\frac{1}{N}) &\lesssim_E D_{p,E}(\frac{1}{N+1}) \\
\widehat{D}_{p,E}(\frac{1}{N}) &\lesssim_E \widehat{D}_{p,E}(\frac{1}{N+1}) \\
D^{par,L}_{p,n,E}(\frac{1}{N}) &\lesssim_{n,E} D^{par,L}_{p,n,E}(\frac{1}{N+1}) \\
D^{L}_{p,n,E}(\frac{1}{N}) &\lesssim_{n,E} D^{L}_{p,n,E}(\frac{1}{N+1}) \\
\overline{D}_p(\frac{1}{N}) &\lesssim \overline{D}_p(\frac{1}{N+1}) \\
D^{par,G}_{p,n}(\frac{1}{N}) &\lesssim_{n,E} D^{par,G}_{p,n}(\frac{1}{N+1}) \\
D^{G}_{p,n}(\frac{1}{N}) &\lesssim_{n} D^{G}_{p,n}(\frac{1}{N+1}).
\end{aligned}$$

We can obtain a similar result when applying this idea to the observation that $D^G_{p,2}(\delta)$ is almost multiplicative, that is, for $\delta_1, \delta_2 \in \mathbb{N}^{-1}$, $D^G_{p,2}(\delta_1\delta_2) \leqslant D^G_{p,2}(\delta_1)D^G_{p,2}(\delta_2)$, however we omit the proof here.

4.3 An elementary proof of l^2L^4 decoupling for the parabola

Having seen two iterative proofs of l^2L^4 decoupling for the parabola, we now give a direct proof. This is the only nontrivial parabola decoupling theorem that can be proven directly (as far as the author knows). The proof is similar in spirit to the short proof of discrete Fourier restriction in L^4 for (n, n^2) that Bourgain gives in Proposition 2.1 of [Bou93].

For an interval $I \subset [0,1]$, let

$$(\mathcal{E}_I g)(x) = \int_I g(\xi)e(\xi x_1 + \xi^2 x_2)\, d\xi$$

where $e(x) = e^{2\pi i x}$. We will prove that not only can we decouple $[0,1]$ into intervals of length δ at some $O(1)$ cost, but also we can decouple $[0,1]$ into an arbitrary collection of intervals at an $O(1)$ cost. Let $\mathcal{I} = \{I_i\}_{i=1}^N$ be an arbitrary partition of $[0,1]$ into N intervals. Let

$$R = (\min_{I\in\mathcal{I}} |I|)^{-2}$$

and if B is a square of side length R centered at c_B, let

$$w_B(x) = (1 + \frac{|x - c_B|}{R})^{-100}.$$

Let η be a Schwartz function such that $\operatorname{supp}(\widehat{\eta}) \subset B(0,1)$ and $1_{B(0,1)} \leqslant \eta$. For a square $B = B(c_B, R)$, let $\eta_B(x) = \eta(\frac{x-c_B}{R})$.

Proposition 4.3.1. *For all $g : [0,1] \to \mathbb{C}$ and all squares B of side length R,*

$$\|\mathcal{E}_{[0,1]} g\|_{L^4(B)} \lesssim (\sum_{I\in\mathcal{I}} \|\mathcal{E}_I g\|^2_{L^4(w_B)})^{1/2} \tag{4.8}$$

where the implied constant is an absolute constant independent of the partition $\mathcal{I}$.

Remark 4.3.2. It is an open problem whether an analogous statement is true with L^4 replaced with L^p for some other $p < 6$ even if we accept an $(\#\mathcal{I})^\varepsilon$ loss.

Proof. Since $g : [0,1] \to \mathbb{C}$ is arbitrary, we may assume that B is centered at the origin. We have

$$\|\mathcal{E}_{[0,1]} g\|^4_{L^4(B)} = \|\mathcal{E}_{[0,1]} g \cdot \overline{\mathcal{E}_{[0,1]} g}\|^2_{L^2(B)}.$$

Then

$$\|\mathcal{E}_{[0,1]}g\|_{L^4(B)}^4 \lesssim \|\sum_{\substack{1\leqslant i,j\leqslant N\\ |i-j|\leqslant 1}} \mathcal{E}_{I_i}g\overline{\mathcal{E}_{I_j}g}\|_{L^2(B)}^2 + \|\sum_{\substack{1\leqslant i,j\leqslant N\\ |i-j|>1}} \mathcal{E}_{I_i}g\overline{\mathcal{E}_{I_j}g}\|_{L^2(B)}^2. \tag{4.9}$$

We analyze the first expression in (4.6). We have

$$\|\sum_{\substack{1\leqslant i,j\leqslant N\\ |i-j|\leqslant 1}} \mathcal{E}_{I_i}g\overline{\mathcal{E}_{I_j}g}\|_{L^2(B)}^2 \leqslant (\sum_{\substack{1\leqslant i,j\leqslant N\\ |i-j|\leqslant 1}} \|\mathcal{E}_{I_i}g\|_{L^4(B)}\|\mathcal{E}_{I_j}g\|_{L^4(B)})^2 \lesssim (\sum_{I\in\mathcal{I}}\|\mathcal{E}_I g\|_{L^4(w_B)}^2)^2 \tag{4.10}$$

where the last inequality is by Cauchy-Schwarz. We now analyze the second term in (4.9). Since $1_B \leqslant 1_{10B} \leqslant \eta_{10B}$, it suffices to analyze

$$\begin{aligned}&\|\sum_{\substack{1\leqslant i,j\leqslant N\\ |i-j|>1}} \mathcal{E}_{I_i}g\overline{\mathcal{E}_{I_j}g}\|_{L^2(\eta_{10B})}^2\\ &= \sum_{\substack{1\leqslant i,i',j,j'\leqslant N\\ |i-j|>1, |i'-j'|>1}} \int_{I_i\times I_j\times I_{i'}\times I_{j'}} g(\xi_1)\overline{g(\xi_2)g(\xi_3)}g(\xi_4)\int_{\mathbb{R}^2} e(\cdots)\eta_{10B}(x)\,dx\,d\xi\end{aligned} \tag{4.11}$$

where the expression in $e(\cdots)$ is

$$(\xi_1-\xi_2-\xi_3+\xi_4)x_1 + (\xi_1^2-\xi_2^2-\xi_3^2+\xi_4^2)x_2.$$

We claim the integral in ξ above is equal to 0 if $|i-i'|>1$ or $|j-j'|>1$ and so we can add the conditions that $|i-i'|\leqslant 1$ and $|j-j'|\leqslant 1$ to the sum in (4.11).

We only show that case when $|i-i'|>1$, the case when $|j-j'|>1$ is similar. Since η_{10B} has Fourier support on $B(0,1/(10R))$, for the integral in (4.11) to not be 0, it is necessary that

$$\begin{aligned}|\xi_1-\xi_2-\xi_3+\xi_4| &\leqslant \frac{1}{10R}\\ |\xi_1^2-\xi_2^2-\xi_3^2+\xi_4^2| &\leqslant \frac{1}{10R}\end{aligned} \tag{4.12}$$

for all $\xi_1\in I_i$, $\xi_2\in I_j$, $\xi_3\in I_{i'}$, and $\xi_4\in I_{j'}$ and therefore we can insert this condition into the integral in the ξ-variables. Since $|i-j|>1$, $|i'-j'|>1$, and $|i-i'|>1$, we have $|\xi_1-\xi_2|>R^{-1/2}$, $|\xi_3-\xi_4|>R^{-1/2}$, and $|\xi_1-\xi_3|>R^{-1/2}$, respectively. We claim that these inequalities are incompatible with (4.12).

Lemma 4.3.3. *Suppose* $0 \leqslant \xi_1, \xi_2, \xi_3, \xi_4 \leqslant 1$. *The system*

$$|\xi_1 - \xi_2 - \xi_3 + \xi_4| \leqslant \frac{1}{10R} \tag{4.13}$$

$$|\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2| \leqslant \frac{1}{10R} \tag{4.14}$$

$$|\xi_3 - \xi_4| > \frac{1}{R^{1/2}} \tag{4.15}$$

$$|\xi_1 - \xi_3| > \frac{1}{R^{1/2}} \tag{4.16}$$

has no solution.

Proof. Suppose there was a solution to the above system of inequalities. Note that

$$\xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = (\xi_1 - \xi_2 - \xi_3 + \xi_4)(\xi_1 + \xi_2) + (\xi_3 - \xi_4)(\xi_1 + \xi_2 - \xi_3 - \xi_4)$$

and so combining this with (4.13), (4.14), (4.15), the triangle inequality, and that $\xi_i \in [0, 1]$ gives

$$\frac{1}{R^{1/2}}|\xi_1 + \xi_2 - \xi_3 - \xi_4| \leqslant \frac{1}{10R} + 2|\xi_1 - \xi_2 - \xi_3 + \xi_4| \leqslant \frac{3}{10R}.$$

Therefore

$$|\xi_1 + \xi_2 - \xi_3 - \xi_4| \leqslant \frac{3}{10R^{1/2}}. \tag{4.17}$$

Since we are not given the relative positions of the ξ_i, we have the following four cases.

(*i*) $\xi_3 > \xi_1$ and $\xi_2 > \xi_4$: Using (4.13), positivity of $\xi_3 - \xi_1$ and $\xi_2 - \xi_4$, and (4.16) gives

$$\frac{1}{10R} \geqslant |\xi_3 - \xi_1 + \xi_4 - \xi_2| = |\xi_3 - \xi_1| + |\xi_4 - \xi_2| \geqslant |\xi_3 - \xi_1| > \frac{1}{R^{1/2}}$$

which is impossible.

(*ii*) $\xi_1 > \xi_3$ and $\xi_4 > \xi_2$: Using (4.13), positivity of $\xi_1 - \xi_3$ and $\xi_4 - \xi_2$, and (4.16) gives

$$\frac{1}{10R} \geqslant |\xi_1 - \xi_2 - \xi_3 + \xi_4| = |\xi_1 - \xi_3| + |\xi_4 - \xi_2| \geqslant |\xi_1 - \xi_3| > \frac{1}{R^{1/2}}$$

which is impossible.

(*iii*) $\xi_3 > \xi_1$ and $\xi_4 > \xi_2$: Using (4.17), positivity of $\xi_3 - \xi_1$ and $\xi_4 - \xi_2$, and (4.16) gives

$$\frac{3}{10R^{1/2}} \geqslant |\xi_3 - \xi_1 + \xi_4 - \xi_2| = |\xi_3 - \xi_1| + |\xi_4 - \xi_2| \geqslant |\xi_3 - \xi_1| > \frac{1}{R^{1/2}}$$

which is impossible.

(*iv*) $\xi_1 > \xi_3$ and $\xi_2 > \xi_4$: Using (4.17), positivity of $\xi_1 - \xi_3$ and $\xi_2 - \xi_4$, and (4.16) gives

$$\frac{3}{10R^{1/2}} \geqslant |\xi_1 - \xi_3 + \xi_2 - \xi_4| = |\xi_1 - \xi_3| + |\xi_2 - \xi_4| \geqslant |\xi_1 - \xi_3| > \frac{1}{R^{1/2}}$$

which is impossible.

Thus we have shown the inequalities (4.13)-(4.16) to be incompatible. This completes the proof of Lemma 4.3.3. □

Therefore Lemma 4.3.3 implies (4.11) is

$$\begin{aligned} &\leqslant \sum_{\substack{1\leqslant i,i',j,j'\leqslant N\\ |i-j|>1, |i'-j'|>1\\ |i-i'|\leqslant 1, |j-j'|\leqslant 1}} \int_{\mathbb{R}^2} |\mathcal{E}_{I_i}g\overline{\mathcal{E}_{I_j}g\mathcal{E}_{I_{i'}}g}\mathcal{E}_{I_{j'}}g|\eta_{10B}\,dx \\ &\lesssim \sum_{1\leqslant i,j\leqslant N}\int_{\mathbb{R}^2}|\mathcal{E}_{I_i}g|^2|\mathcal{E}_{I_j}g|^2 w_B\,dx \leqslant (\sum_{I\in\mathcal{I}}\|\mathcal{E}_I g\|^2_{L^4(w_B)})^2 \end{aligned} \tag{4.18}$$

where the second inequality is by Cauchy-Schwarz and that $\eta_{10B} \lesssim w_{10B} \lesssim w_B$ and the last inequality is by Hölder's inequality. Combining (4.9), (4.10), and (4.18) then proves (4.8). This completes the proof of Proposition 4.3.1. □

4.4 Small ball l^2 decoupling for the paraboloid

Decoupling for the paraboloid as stated in 1.2 has an $L^p(B)$ where B is a cube in $\mathbb{R}^n$ of side length δ^{-2}. This is a natural scale since we are decoupling into frequency cubes in $[0,1]^{n-1}$ of side length δ and hence the wavepackets that arise are of size $\delta^{-1} \times \cdots \times \delta^{-1} \times \delta^{-2}$.

One can ask perhaps what happens in l^2 decoupling for the paraboloid when we consider B to be a ball of radius δ^{-r} with $1 \leqslant r < 2$. The following result was communicated to the author by Hong Wang in January 2018. This a purely expository chapter and the author claims no originality in the argument below. All errors are my own misunderstanding of her argument.

For $Q \subset [0,1]^{n-1}$ and $g : [0,1]^{n-1} \to \mathbb{C}$, define the extension operator

$$(\mathcal{E}_Q g)(x) := \int_Q g(\xi)e(\xi_1 x_1 + \cdots \xi_{n-1}x_{n-1} + |\xi|^2 x_n)\, d\xi = \int_Q g(\xi)e(\xi \cdot \underline{x} + |\xi|^2 x_n)\, d\xi.$$

Also define $\mathcal{E}g := \mathcal{E}_{[0,1]^{n-1}}g$. We will ignore any weight functions or integrality issues that may arise in this analysis and freely make use of the uncertainty principle. Given a cube Q, let $P_\delta(Q)$ be the partition of Q into cubes of side length δ.

Fix $1 \leqslant r < 2$ and $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$, let $D_p(\delta, r)$ be the best constant such that

$$\|\mathcal{E}g\|_{L^p(B_r)} \leqslant D_p(\delta, r)(\sum_{Q \in P_\delta([0,1]^{n-1})} \|\mathcal{E}_Q g\|^2_{L^p(B_r)})^{1/2} \tag{4.19}$$

for all $g : [0,1]^{n-1} \to \mathbb{C}$ and all cubes $B_r \subset \mathbb{R}^n$ of side length δ^{-r}. Note that the standard Bourgain-Demeter decoupling for the paraboloid [BD15] gives that $1 \lessapprox D_p(\delta, 2) \lesssim_\varepsilon \delta^{-\varepsilon}$. We claim the following result.

Proposition 4.4.1. *For $1 \leqslant r < 2$ and $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$,*

$$\delta^{-(1-\frac{r}{2})(\frac{1}{2}-\frac{1}{p})(n-1)} \lessapprox D_p(\delta, r) \lesssim_\varepsilon \delta^{-(1-\frac{r}{2})(\frac{1}{2}-\frac{1}{p})(n-1)-\varepsilon}.$$

In particular, Proposition 4.4.1 implies that at spatial scales smaller than δ^{-2}, to decouple we must lose some negative power of δ. For the lower bound, we exhibit a specific g (in particular $g = 1_{[0,\delta^{r/2}]^{n-1}}$) and compute both sides of (4.19). For the upper bound, we reduce the problem using the uncertainty principle to be a problem about the Fourier transform.

4.4.1 The lower bound

Without loss of generality we may assume that $B_r = [0, \delta^{-r}]^n$. Let $g := 1_{[0,\delta^{r/2}]^{n-1}}$ (if B_r is a different cube in $\mathbb{R}^n$ of side length δ^{-r}, then we can multiply g by an appropriate phase). We then have

$$\begin{aligned}(\mathcal{E}1_{[0,\delta^{r/2}]^{n-1}})(x) &= \int_{[0,\delta^{r/2}]^{n-1}} e(\xi \cdot \underline{x} + |\xi|^2 x_n)\, d\xi \\ &= \delta^{(n-1)r/2} \int_{[0,1]^{n-1}} e(\eta \cdot \delta^{r/2}\underline{x} + \delta^r |\eta|^2 x_n)\, d\eta.\end{aligned}$$

Another change of variables then gives

$$\|\mathcal{E}1_{[0,\delta^{r/2}]^{n-1}}\|_{L^p(B_r)} = \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}\|\mathcal{E}1_{[0,1]^{n-1}}\|_{L^p([0,\delta^{-r/2}]^{n-1}\times[0,1])}. \tag{4.20}$$

Since $|\mathcal{E}1_{[0,1]^{n-1}}|$ is essentially constant on $1 \times 1 \times \cdots \times 1$ boxes, for $x \in [0,\delta^{-r/2}]^{n-1} \times [0,1]$ we can replace $|(\mathcal{E}1_{[0,1]^{n-1}})(x)|$ by $|(\mathcal{E}1_{[0,1]^{n-1}})(\underline{x},0)|$ and hence (4.20) is essentially the same as

$$\delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}\|\check{1}_{[0,1]^{n-1}}\|_{L^p([0,\delta^{-r/2}]^{n-1})} = \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}\|\check{1}_{[0,1]}\|^{n-1}_{L^p([0,\delta^{-r/2}])}.$$

The same computations give that the right hand side of (4.19) is

$$\begin{aligned}(\sum_{Q\in P_\delta([0,1]^{n-1})}\|\mathcal{E}_Q 1_{[0,\delta^{r/2}]^{n-1}}\|^2_{L^p(B_r)})^{1/2} &= (\sum_{Q\in P_\delta([0,\delta^{r/2}]^{n-1})}\|\mathcal{E}1_Q\|^2_{L^p(B_r)})^{1/2}\\ &= \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}(\sum_{Q\in P_{\delta^{1-r/2}}([0,1]^{n-1})}\|\mathcal{E}1_Q\|^2_{L^p([0,\delta^{-r/2}]^{n-1}\times[0,1])})^{1/2}.\end{aligned}$$

Note that here we have implicitly used that $r < 2$ since this implies $\delta^{1-r/2} < 1$. From the uncertainty principle, this is once again essentially

$$\begin{aligned}&\delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}(\sum_{Q\in P_{\delta^{1-r/2}}([0,1]^{n-1})}\|\check{1}_Q\|^2_{L^p([0,\delta^{-r/2}]^{n-1})})^{1/2}\\ &\quad= \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})}\delta^{-(1-\frac{r}{2})\frac{n-1}{2}}\|\check{1}_{[0,\delta^{1-r/2}]^{n-1}}\|_{L^p([0,\delta^{-r/2}]^{n-1})}\\ &\quad= \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})-(1-\frac{r}{2})\frac{n-1}{2}}\|\check{1}_{[0,\delta^{1-r/2}]}\|^{n-1}_{L^p([0,\delta^{-r/2}])}\\ &\quad= \delta^{\frac{r}{2}(n-1-\frac{n+1}{p})-(1-\frac{r}{2})\frac{n-1}{2}+(1-\frac{r}{2})(1-\frac{1}{p})(n-1)}\|\check{1}_{[0,1]}\|^{n-1}_{L^p([0,\delta^{1-r}])}.\end{aligned}$$

Therefore

$$\sup_{g,B_r}\frac{\|\mathcal{E}g\|_{L^p(B_r)}}{(\sum_{Q\in P_\delta([0,1]^{n-1})}\|\mathcal{E}_Q g\|^2_{L^p(B_r)})^{1/2}} \geqslant \delta^{-(1-\frac{r}{2})(\frac{1}{2}-\frac{1}{p})(n-1)}(\frac{\|\check{1}_{[0,1]}\|_{L^p([0,\delta^{-r/2}])}}{\|\check{1}_{[0,1]}\|_{L^p([0,\delta^{1-r}])}})^{n-1}.$$

Since $r/2 > r-1$, the ratio of L^p norms is $\geqslant 1$ which then proves the lower bound of Proposition 4.4.1.

4.4.2 The upper bound

As in the lower bound we will apply a (slightly different) change of variables and the uncertainty principle to transform the problem into a problem about the Fourier transform. We want to show that

$$\|\mathcal{E}g\|_{L^p(B_r)} \lesssim_\varepsilon \delta^{-(1-\frac{r}{2})(\frac{1}{2}-\frac{1}{p})(n-1)-\varepsilon}\Big(\sum_{Q\in P_\delta([0,1]^{n-1})}\|\mathcal{E}_Q g\|_{L^p(B_r)}^2\Big)^{1/2}$$

for all $g : [0,1]^{n-1} \to \mathbb{C}$ and all cubes $B_r \subset \mathbb{R}^n$ of side length δ^{-2}. Since $2 \leqslant p \leqslant \frac{2(n+1)}{n-1}$, decoupling for the paraboloid gives that

$$\|\mathcal{E}g\|_{L^p(B_r)} \lesssim_\varepsilon \delta^{-\varepsilon}\Big(\sum_{Q'\in P_{\delta^{r/2}}([0,1]^{n-1})}\|\mathcal{E}_{Q'} g\|_{L^p(B_r)}^2\Big)^{1/2}.$$

Therefore it remains to show that for each $Q' \in P_{\delta^{r/2}}([0,1]^{n-1})$,

$$\|\mathcal{E}_{Q'}g\|_{L^p(B_r)} \lesssim \delta^{-(1-\frac{r}{2})(\frac{1}{2}-\frac{1}{p})(n-1)}\Big(\sum_{Q\in P_\delta(Q')}\|\mathcal{E}_Q g\|_{L^p(B_r)}^2\Big)^{1/2}. \tag{4.21}$$

Without loss of generality (in particular ignoring issues with weights), we may assume that $Q' = [0,\delta^{r/2}]^{n-1}$. Let $g_\delta(x) := g(\delta x)$. A change of variables gives that

$$\begin{aligned}(\mathcal{E}_{[0,\delta^{r/2}]^{n-1}}g)(x) &= \int_{[0,\delta^{r/2}]^{n-1}} g(\xi)e(\xi\cdot\underline{x} + |\xi|^2x_n)\,d\xi \\ &= \delta^{n-1}\int_{[0,\delta^{-1+r/2}]^{n-1}} g_\delta(\eta)e(\eta\cdot\delta\underline{x} + |\eta|^2\delta^2x_n)\,d\eta\end{aligned}$$

and hence

$$\|\mathcal{E}_{[0,\delta^{r/2}]^{n-1}}g\|_{L^p(B_r)} = \delta^{(n-1)-\frac{n+1}{p}}\|\mathcal{E}_{[0,\delta^{-1+r/2}]^{n-1}}g_\delta\|_{L^p([0,\delta^{-r+1}]^{n-1}\times[0,\delta^{-r+2}])}. \tag{4.22}$$

From the uncertainty principle, $|(\mathcal{E}_{[0,\delta^{-1+r/2}]^{n-1}}g_\delta)(x)|$ is essentially constant on $\delta^{1-r/2}\times\cdots\times\delta^{1-r/2}\times\delta^{2-r}$ boxes. Therefore for $x \in [0,\delta^{-r+1}]^{n-1}\times[0,\delta^{-r+2}]$, $|(\mathcal{E}_{[0,\delta^{-1+r/2}]^{n-1}}g_\delta)(x)|$ is essentially equal to $|(\mathcal{E}_{[0,\delta^{-1+r/2}]^{n-1}}g_\delta)(\underline{x},0)|$ and hence (4.22) becomes essentially equal to

$$\delta^{\frac{2-r}{p}}\times\delta^{(n-1)-\frac{n+1}{p}}\Big\|\int_{[0,\delta^{-1+r/2}]^{n-1}} g_\delta(\eta)e(\eta\cdot y)\,d\eta\Big\|_{L^p_y([0,\delta^{-r+1}]^{n-1})}.$$

The same reasoning then shows that

$$\begin{aligned}&\Big(\sum_{Q\in P_\delta([0,\delta^{r/2}]^{n-1})}\|\mathcal{E}_Q g\|_{L^p(B_r)}^2\Big)^{1/2} \\ &\quad\approx \delta^{\frac{2-r}{p}}\times\delta^{(n-1)-\frac{n+1}{p}}\Big(\sum_{Q\in P_1([0,\delta^{-1+r/2}]^{n-1})}\Big\|\int_Q g_\delta(\eta)e(\eta\cdot y)\,d\eta\Big\|_{L^p_y([0,\delta^{-r+1}]^{n-1})}^2\Big)^{1/2}.\end{aligned}$$

Therefore since $r-1 \geqslant 0$, (4.21) then follows from the following lemma and parallel decoupling. The argument below basically is from Lecture 2 of Larry Guth's lectures notes on decoupling [Gut18].

Lemma 4.4.2. *Suppose $\widehat{f}$ is supported on $[0,N]^d$. Then*

$$\|f\|_{L^p([0,1]^d)} \lesssim N^{d(\frac{1}{2}-\frac{1}{p})}\Big(\sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^p([0,1]^d)}\Big)^{1/2}$$

where here $\widehat{f_Q} = \widehat{f}1_Q$.

To prove Lemma 4.4.2, we first recall Bernstein's inequality (and we ignore weight functions).

Lemma 4.4.3. *Suppose $\widehat{f}$ is supported on a cube of side length 1. Then for any cube B of side length 1, $\|f\|_{L^\infty(B)} \lesssim \|f\|_{L^1(B)}$.*

Proof of Lemma 4.4.2. Since $f = \sum_{Q\in P_1([0,N]^d)} f_Q$, almost orthogonality and ignoring weights gives that essentially

$$\|f\|^2_{L^2([0,1]^d)} \lesssim \sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^2([0,1]^d)}.$$

Observe that

$$\int_{[0,1]^d} |f|^p \leqslant \|f\|^{p-2}_{L^\infty([0,1]^d)} \int_{[0,1]^d} |f|^2 \lesssim \Big(\sum_{Q\in P_1([0,N]^d)} \|f_Q\|_{L^\infty([0,1]^d)}\Big)^{p-2} \sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^2([0,1]^d)}.$$

Hölder and Bernstein then bound the above by

$$\begin{aligned} &N^{d\frac{(p-2)}{2}}\Big(\sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^\infty([0,1]^d)}\Big)^{\frac{p-2}{2}}\Big(\sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^2([0,1]^d)}\Big) \\ &\lesssim N^{d\frac{(p-2)}{2}}\Big(\sum_{Q\in P_1([0,N]^d)} \|f_Q\|^2_{L^p([0,1]^d)}\Big)^{p/2}. \end{aligned}$$

Taking $1/p$ powers then completes the proof of Lemma 4.4.2. $\square$

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