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UNIVERSITY OF CALIFORNIA
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Deformations of Compact Holomorphic Poisson Manifolds and Algebraic Poisson
Schemes

A Dissertation submitted in partial satisfaction
of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

by

Chunghoon Kim

March 2014

Dissertation Committee:

Professor Ziv Ran, Chairperson
Professor Wee Liang Gan
Professor David Rush

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The Dissertation of Chunghoon Kim is approved:

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I have been fortunate to have great friends here in UCR. I thank John Dusel for being my friend. I learned a lot from him on the importance of writing. I truly appreciate his support, encouragement, and advice. I will miss our numerous conversations. I thank Adam Navas for being my friend. Probably he is the one whom I most hung out with in my graduate years. I will miss the times I spent with him and I appreciate his tremendous help. I don't want to imagine how my life in UCR would be without him. I thank Jorge Perez for his genuine kindness. He is the kindest person I have ever met in my whole life. His existence gave me a lot of thoughts on human nature, and I believe that he will find his meaning of life. I thank Jason Payne for being my friend. After he left UCR, sometimes I missed our conversations on mathematics, and I really appreciate his occasional emails. I also thank Daniel Majcherek and Oliver Thistlethwaite for spending our graduate years together as good friends. I thank Dr. Sung Rak Choi for his kindness and his help in my early graduate years while he was a postdoc here in UCR.

I thank my family whose constant encouragement and support helped me to finish my thesis. Lastly, I thank Professor Yongnam Lee who encouraged me to be a mathematician when I was an undergraduate student. I deeply appreciate his help and support.

*In memory of my grandfather Joo Young Kim, whose unconditional love will remain
forever in my heart.*

ABSTRACT OF THE DISSERTATION

Deformations of Compact Holomorphic Poisson Manifolds and Algebraic Poisson Schemes

by

Chunghoon Kim

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, March 2014
Professor Ziv Ran, Chairperson

In this thesis, we study deformations of compact holomorphic Poisson manifolds and algebraic Poisson schemes. Deformations of compact holomorphic Poisson manifolds is based on Kodaira-Spencer's analytic deformation theory of compact complex manifolds, and deformations of algebraic Poisson schemes is based on Grothendieck's algebraic deformation theory of algebraic schemes. The only difference is that we deform an additional structure, namely 'Poisson structures' as well as underlying complex structures or algebraic structures in a family of compact holomorphic Poisson manifolds or in a family of algebraic Poisson schemes. Hence when we ignore Poisson structures, the underlying deformation theory is same to ordinary deformation theory in the sense of Kodaira-Spencer, and Grothendieck.

In the part I of the thesis, we study deformations of compact holomorphic Poisson manifolds. We define a concept of a family of holomorphic Poisson manifolds, called a Poisson analytic family on the basis of Kodaira-Spencer's complex analytic family. We use the truncated holomorphic Poisson cohomology to study infinitesimal deformations of holomorphic Poisson manifolds and define a Poisson Kodaira Spencer map. We deduce the integrability condition. We study the 'theorem of existence' for holomorphic Poisson structures.

In the part II of the thesis, we describe a differential graded Lie algebra governing infinitesimal Poisson deformations of a compact holomorphic Poisson manifold (X, Λ_0) over a local artinian \mathbb{C} -algebra with the residue \mathbb{C} . We study an universal Poisson deformation of (X, Λ_0) when $HP^1(X, \Lambda_0) = 0$.

In the part III of the thesis, we study deformations of algebraic Poisson schemes. We focus on infinitesimal Poisson deformations of an algebraic Poisson scheme (X, Λ_0) over a local artinian k -algebra with the residue k , where k is a algebraically closed field. We study first order Poisson deformation, obstruction and Poisson deformation functor $PDef_{(X, \Lambda_0)}$. By following [LS67], we construct a Poisson cotangent complex for a Poisson k -algebra homomorphism $A \rightarrow B$ and a Poisson B -module M and define $PT^i(B/A, M)$. As an application to Poisson deformation, we show that for a

Poisson algebra B_0 , $PDef_{Spec(B_0)}(k[\epsilon])$ is a natural one to one correspondence with $PT^1(B_0/k, B_0)$. We also show that given a Poisson algebra B_0 and an Poisson ideal I of B_0 , deformations of a Poisson subscheme $Spec(C)$ of $Spec(B_0)$ over $Spec(k[\epsilon])$ is one to one correspondence with $PT^1(C/B_0, C)$ where $C = B_0/I$.

Contents

Preface	ix
I Deformations of compact holomorphic Poisson manifolds	1
1 Poisson analytic families	6
1.1 Families of holomorphic Poisson manifolds	6
1.2 Infinitesimal deformation	12
1.2.1 Infinitesimal deformation and truncated holomorphic Poisson cohomology . .	12
1.2.2 Trivial, locally trivial family and rigidity	16
1.2.3 Change of parameter	17
2 Integrability condition	20
2.1 Preliminaries	20
2.1.1 Identification of the deformations of complex structures with $\varphi(t)$	22
2.1.2 Identification of the deformations of Poisson structures with $\Lambda(t)$	23
2.2 Expression of infinitesimal deformations in terms of $\varphi(t)$ and $\Lambda(t)$	28
2.3 Integrability condition	32
3 Theorem of Existence for holomorphic Poisson structures	36
3.1 Theorem of existence	36
3.1.1 Statement of the theorem	36
3.1.2 Construction of $\alpha(t) = \varphi(t) + \Lambda(t)$	39
3.1.3 Construction of a Poisson analytic family	41
3.2 A concept of Kuranishi family in holomorphic Poisson category	43
II Infinitesimal Poisson deformations and Universal Poisson deformations of compact holomorphic Poisson manifolds	47
4 Jacobi complex	51
4.1 Preliminaries	51
4.1.1 Induced differential	53
4.2 Jacobi complex	56
4.3 Morphic elements	61
4.3.1 Morphic elements and ring homomorphism	63
4.3.2 Explicit description of a morphic element	63

5	Infinitesimal Poisson deformations	65
5.1	Bracket calculus	65
5.2	Deformations of a compact holomorphic Poisson manifold	67
5.3	Integrability condition	70
5.3.1	Integrability Condition	74
5.3.2	$[\epsilon(\alpha)] \in \mathbb{H}^0(J_n(\mathfrak{g}))$ as a canonical element associated with \mathcal{X}	79
6	Universal Poisson deformations	84
6.1	Isomorphism of two deformation functors $Def_{\mathfrak{g}} \cong PDef_{(X, \Lambda_0)}$	84
6.1.1	Deformation functors	84
6.1.2	The Poisson deformation functor $Def_{(X, \Lambda_0)}$ is controlled by the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -])$	85
6.2	Universal Poisson deformations	90
6.2.1	Independence of choices of morphic elements giving the same cohomology class	90
6.2.2	n -th Universal Poisson deformations	93
6.2.3	Formal Completion	94
III	Deformations of algebraic Poisson schemes	96
7	Deformations of algebraic Poisson schemes	99
7.1	Definitions of Poisson schemes, morphisms and cohomology	99
7.1.1	Characterization of a Poisson bracket $\{-, -\}$ of a Poisson algebra A over R	99
7.1.2	Affine Poisson Schemes	102
7.1.3	Poisson Schemes	105
7.1.4	(truncated) Lichnerowicz-Poisson cohomology	108
7.2	Deformations of algebraic Poisson schemes	109
7.2.1	Basic materials on deformations of algebraic Poisson schemes	109
7.2.2	Infinitesimal Poisson deformations	111
7.2.3	Higher-order Poisson deformation-obstructions	115
8	Poisson deformation functors	120
8.1	Schlessinger's criterion	120
8.2	Poisson Deformation functors	124
9	Poisson cotangent complex	132
9.1	Poisson modules and Poisson enveloping algebras	132
9.2	Poisson cotangent complex	135
9.3	First order Poisson deformations of affine Poisson schemes	153
9.4	First order deformations of a Poisson closed subscheme of an affine Poisson scheme	155
A	Basic materials on Poisson algebras and holomorphic Poisson manifolds	158
B	Hypercohomology	161
C	Differential graded Lie algebra structure on $\bigoplus_{i \geq 0} g^i = \bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(M, \wedge^q T_M)$	166
D	Ellipticity of the operator $\bar{\partial} + [\Lambda, -]$	170
	Bibliography	176

Preface

In this thesis, we study deformations of compact holomorphic Poisson manifolds¹ and algebraic Poisson schemes.² Deformations of compact holomorphic Poisson manifolds is based on Kodaira-Spencer's deformation theory of compact complex manifolds, and deformations of algebraic Poisson schemes is based on Grothendieck's deformation theory of algebraic schemes. The only difference is that we deform an additional structure, namely 'Poisson' structures' in a family of compact holomorphic Poisson manifolds or algebraic Poisson schemes. Hence when we ignore Poisson structures, the underlying deformation theory is same to ordinary deformation theory in the sense of Kodaira-Spencer, and Grothendieck. The relationship between deformations of compact complex manifolds and deformations of algebraic schemes is well described in the Introduction of Sernesi's book [Ser06]. I will briefly explain their relationship described in [Ser06], and then extend their relationship to the relationship between deformations of compact holomorphic Poisson manifolds, and algebraic Poisson schemes in the following.

Given a compact complex manifold X , a family of deformations of X is a commutative diagram of holomorphic maps between complex manifolds

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \xi \downarrow & & \downarrow \pi \\ \star & \longrightarrow & B \end{array}$$

with π proper and smooth, B connected and where \star denotes the singleton space. We denote by \mathcal{X}_t the fibre $\pi^{-1}(t)$, $t \in B$. We call (\mathcal{X}, B, π) a complex analytic family. Kodaira and Spencer started studying small deformations of X in a complex analytic family by defining, for every tangent vector $\frac{\partial}{\partial t} \in T_{t_0}B$, the derivative of the family along $\frac{\partial}{\partial t} \in T_{t_0}B$ as an element

$$\frac{\partial \mathcal{X}_t}{\partial t} \in H^1(X, \Theta)$$

¹For general information of Poisson geometry, see Appendix A. A holomorphic Poisson manifold is a complex manifold such that its structure sheaf is a sheaf of Poisson algebras. For more details of deformations of compact holomorphic Poisson manifolds, see the part I of the thesis.

²A Poisson algebraic scheme is an algebraic scheme over an algebraically closed field k such that its structure sheaf is a sheaf of Poisson algebras. For more details of the definition of Poisson schemes and deformations of algebraic Poisson schemes, see the part III of the thesis.

which gives a Kodaira Spencer map $\kappa : T_{t_0}B \rightarrow H^1(X, \Theta)$. They investigated the problem of classifying all small deformations of X , by constructing a “complete family” of deformations of X which roughly means that every small deformation of X is induced from the complete family. More precisely, they established the following theorems.

Theorem 0.1 (Theorem of Existence) *Let X be a compact complex manifold and suppose $H^2(X, \Theta) = 0$. Then there exists a complex analytic family (\mathcal{X}, B, π) with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:*

1. $\pi^{-1}(0) = M$
2. $\rho_0 : \frac{\partial}{\partial t} \rightarrow \left(\frac{\partial M_t}{\partial t}\right)_{t=0}$ with $M_t = \pi^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $H^1(M, \Theta) : T_0(B) \xrightarrow{\rho_0} H^1(M, \Theta)$.

Theorem 0.2 (Theorem of Completeness) *Let (\mathcal{X}, B, π) be a complex analytic family and $\pi^{-1}(0) = X$. If $\rho_0 : T_0B \rightarrow H^1(X, \Theta)$ is surjective, the complex analytic family (\mathcal{X}, B, π) is complete at $0 \in B$.*

By combing these two theorems, we get

Corollary 0.3 *If $H^2(X, \Theta) = 0$, then there exists a complete family of deformations of X whose Kodaira Spencer map is an isomorphism. If moreover, $H^0(X, \Theta) = 0$, then such complete family is universal.*

Later Kuranishi generalized this result without assumptions on $H^2(X, \Theta) = 0$ by relaxing the definition of a family of deformations of X in a way that B is allowed to be an analytic space.

On the other hand, Grothendieck’s algebraic deformation theory is to algebraically formalize Kodaira-Spencer’s analytic deformation theory. Let X be an algebraic scheme over k , where k is an algebraically closed field. A local deformation, or a local family of deformations of X is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & \mathcal{X} \\ \xi : \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \longrightarrow & S \end{array}$$

where π is a flat, $S = \text{Spec}(A)$ where A is a local k -algebra with residue field k , and X is identified with the fibre over the closed point. We can define a deformation functor

$$\text{Def}_X : \mathcal{A}^* \rightarrow (\text{Sets})$$

defined by $Def_X(A) = \{\text{local deformations of } X \text{ over } Spec(A)\}/(\text{isomorphisms})$, where \mathcal{A}^* is the category of noetherian local k -algebras with the residue k . To study the question of representability of the functor Def_X by some noetherian local k -algebra \mathcal{O} , the approach of Grothendieck was to formalize the method of Kodaira and Spencer, which consists in a formal construction followed by a proof of convergence. One of main problems is on prorepresentability of $Def_X : \mathbf{Art} \rightarrow (\mathbf{Sets})$, where \mathbf{Art} is the category of local artinian k -algebras with residue k .

As I said before, deformation theories of holomorphic Poisson manifolds and algebraic Poisson schemes are based on deformation theories of compact complex manifolds and algebraic schemes. The main difference is that we simply put one more structure on complex analytic families or algebraic families, namely ‘‘Poisson structures’’. So deformations of compact holomorphic Poisson manifolds, or algebraic Poisson schemes mean that we deform not only underlying complex or algebraic structures, but also Poisson structures. I will explain small deformations of compact holomorphic Poisson manifolds. Given a holomorphic Poisson manifold (X, Λ_0) , a family of deformations of (X, Λ_0) is a commutative diagram of holomorphic maps between a holomorphic Poisson manifold (\mathcal{X}, Λ) and a complex manifold B

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \xi : \downarrow & & \downarrow \pi \\ \star & \longrightarrow & B \end{array}$$

with π is proper and smooth, B is connected and where \star denotes the singleton space. We denote $(\mathcal{X}_t, \Lambda_t)$ the fiber $\pi^{-1}(t)$, $t \in B$ which is a compact holomorphic Poisson submanifold of (\mathcal{X}, Λ) . We call $(\mathcal{X}, \Lambda, \pi, B)$ a Poisson analytic family. As in a complex analytic family, we can define, for every tangent vector $\frac{\partial}{\partial t} \in T_{t_0}B$, the derivative of the family along $\frac{\partial}{\partial t}$ as an element

$$\frac{\partial(\mathcal{X}_t, \Lambda_t)}{\partial t} \in HP^2(X, \Lambda_0)$$

which gives a linear map

$$\varphi : T_{t_0}B \rightarrow HP^2(X, \Lambda_0)$$

called the Poisson Kodaira Spencer map of the family $(\mathcal{X}, \Lambda, \pi, B)$. We can also define the concept of a complete family as in deformations of compact complex manifolds. I was interested in the problem of classifying all small deformations of (X, Λ_0) , by constructing a ‘‘complete family’’ of deformations of (X, Λ_0) , but by some technical issues, I believe that I only proved the theorem of existence for holomorphic Poisson structures.

Theorem 0.4 (Theorem of Existence for holomorphic Poisson structures) *Let (M, Λ_0) be a compact holomorphic Poisson manifold satisfying some assumption and suppose that $HP^3(M, \Lambda_0) = 0$. Then there exists a Poisson analytic family $(\mathcal{X}, \Lambda, B, \pi)$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:*

1. $\pi^{-1}(0) = (X, \Lambda_0)$
2. $\varphi_0 : \frac{\partial}{\partial t} \rightarrow \left(\frac{\partial(M_t, \Lambda_t)}{\partial t} \right)_{t=0}$ with $(\mathcal{X}_t, \Lambda_t) = \pi^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $HP^2(M, \Lambda_0) : T_0B \xrightarrow{\rho_0} HP^2(M, \Lambda_0)$.

Conjecture 1 (Theorem of Completeness for holomorphic Poisson structures)

Let $(\mathcal{X}, \Lambda, B, \pi)$ be a Poisson analytic family and $\pi^{-1}(0) = (X, \Lambda_0)$. If $\varphi_0 : T_0B \rightarrow HP^2(X, \Lambda_0)$ is surjective, then the Poisson analytic family $(\mathcal{X}, \Lambda, B, \pi)$ is complete at $0 \in B$.

By combining the theorem and the conjecture, we get

Corollary 0.5 *If $HP^3(X, \Lambda_0) = 0$, then there exists a complete family of deformations of (X, Λ_0) whose Poisson Kodaira Spencer map is an isomorphism. Moreover if $HP^1(X, \Lambda_0) = 0$, then such complete family is universal.*

The natural question is the existence of Kuranishi family for deformations of a holomorphic Poisson manifold.

Conjecture 2 *A complete family of deformations of (X, Λ_0) such that the Poisson Kodaira Spencer map is an isomorphism exists without assumptions on $HP^3(X, \Lambda_0) = 0$ provided the base B is allowed to be an analytic space.*

While I worked on deformations of holomorphic Poisson structures, the reason why I focused on “theorem of existence”, “theorem of completeness” and “construction of Kuranishi family” for holomorphic Poisson structures is that I wanted to extend the relationship between Kodaira Spencer’s analytic deformation theory and Grothendieck’s algebraic deformation theory to the relationship between analytic Poisson deformation theory and algebraic deformation theory of Poisson schemes as presented in the book [Ser06]. Now I will explain deformations of algebraic Poisson schemes. Deformations of Poisson schemes have already been studied by Namikawa ([Nam08]), Ginzburg, and Kaledin ([GK04]). It seems that Ginzburg and Kaledin ([GK04]) defined firstly deformations of Poisson schemes in the context of Grothendieck’s deformation theory. Let’s fix an algebraically closed field k and consider an Poisson algebraic k -scheme (X, Λ_0) . A local Poisson deformation or a local family of Poisson deformations of (X, Λ_0) is a cartesian diagram

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \xi : \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \longrightarrow & S \end{array}$$

where π is a flat morphism, $S = \text{Spec } A$ and (\mathcal{X}, Λ) is a Poisson S -scheme via π where A is a local k -algebra with residue field k , and the Poisson k -scheme (X, Λ_0) is identified with the fiber over the closed point. Similarly we can define a Poisson deformation functor

$$Def_X : \mathcal{A}^* \rightarrow (\text{Sets})$$

defined by $Def_X(A) = \{\text{local Poisson deformations of } X \text{ over } \text{Spec}(A)\}/(\text{isomorphisms})$, where \mathcal{A}^* is the category of noetherian local k -algebras with the residue k . We can consider analogous problems coming from classical deformation theory of algebraic schemes.

I have been guided by the analytic and algebraic deformation theory originating from Kodaira-Spencer and Grothendieck in the context of Poisson category through books, articles and papers from senior mathematicians. My thesis on deformations of compact holomorphic Poisson manifolds and algebraic Poisson schemes is the sophistication of this general picture.

Part I

Deformations of compact holomorphic Poisson manifolds

In the first part of the thesis, we study deformations of holomorphic Poisson structures in the framework of Kodaira and Spencer's deformation theory of complex analytic structures ([KS58],[KS60]). The main difference from Kodaira and Spencer's deformation theory is that for deformations of a holomorphic Poisson manifold, we deform not only its complex structures, but also holomorphic Poisson structures. We thoroughly apply Kodaira and Spencer's ideas to holomorphic Poisson category.

Kodaira and Spencer's main idea of deformations of complex analytic structures is as follows [Kod05, p.182]. A n -dimensional compact complex manifold M^3 is obtained by glueing domains U_1, \dots, U_n in $\mathbb{C}^n : M = \cup_{j=1}^n U_j$ where $\mathfrak{U} = \{U_j | j = 1, \dots, n\}$ is a locally finite open covering of M , and that each U_j is a polydisk:

$$U_j = \{z_j \in \mathbb{C}^n \mid |z_j^1| < 1, \dots, |z_j^n| < 1\}$$

and for $p \in U_j \cap U_k$, the coordinate transformation

$$f_{jk} : z_k \rightarrow z_j = (z_j^1, \dots, z_j^n) = f_{jk}(z_k)$$

transforming the local coordinates $z_k = (z_k^1, \dots, z_k^n) = z_k(p)$ into the local coordinates $z_j = (z_j^1, \dots, z_j^n) = z_j(p)$ is biholomorphic. According to Kodaira,

“A deformation of M is considered to be the glueing of the same polydisks U_j via different identification. In other words, replacing $f_{jk}^\alpha(z_k)$ by the functions $f_{jk}^\alpha(z_k, t) = f_{jk}^\alpha(z_k, t_1, \dots, t_m)$, $f_{jk}(z_k, 0) = f_{jk}^\alpha(z_k)$ of z_k , and the parameter $t = (t_1, \dots, t_m)$, we obtain deformations M_t of $M = M_0$ by glueing the polydisks U_1, \dots, U_n by identifying $z_k \in U_k$ with $z_j = f_{jk}(z_k, t) \in U_j$ ”

A n -dimensional compact holomorphic Poisson manifold M is a compact complex manifold such that the structure sheaf \mathcal{O}_M is a sheaf of Poisson algebras.(See Appendix A) The holomorphic Poisson structure is encoded in a holomorphic section (a holomorphic bivector field) $\Lambda \in H^0(M, \wedge^2 \Theta_M)$ with $[\Lambda, \Lambda] = 0$.⁴ In the sequel a holomorphic Poisson manifold will be denoted by (M, Λ) . For deformations of a holomorphic Poisson manifold (M, Λ) , we use the ideas of Kodaira and Spencer. A n -dimensional holomorphic Poisson manifold is obtained by glueing the domains U_1, \dots, U_n in $\mathbb{C}^n : M = \bigcup_{j=1}^n U_j$ where $\mathfrak{U} = \{U_j | j = 1, \dots, n\}$ is a locally finite open covering of M and each U_j is a polydisk

$$U_j = \{z_j \in \mathbb{C}^n \mid |z_j^1| < 1, \dots, |z_j^n| < 1\}$$

equipped with a holomorphic bivector fields $\Lambda_j = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ ⁵ with $[\Lambda_j, \Lambda_j] = 0$ on

³In this thesis, we assume that a complex manifold is connected

⁴We denote by $T = T_M$ the holomorphic tangent bundle of M , by Θ_M the sheaf of holomorphic vector fields on M , by $T^* = T_M^*$ by the dual bundle of T_M , by $\bar{T} = \bar{T}_M$ the anti holomorphic tangent bundle, by $\bar{T}^* = \bar{T}_M^*$ the dual bundle of \bar{T}_M , by $T_{\mathbb{C}}M = T \oplus \bar{T}$ the complexified tangent bundle, and by $T_{\mathbb{C}}^*M = T^* \oplus \bar{T}^*$ the dual bundle of $T_{\mathbb{C}}M$ and the bracket $[-, -]$ is the Schouten bracket. See Appendix C.

⁵In this thesis, we always assume that $g_{\alpha\beta}^j(z) = -g_{\beta\alpha}^j(z)$

U_j and for $p \in U_j \cap U_k$, the coordinate transformation

$$f_{jk} : z_k \rightarrow z_j = (z_j^1, \dots, z_j^n) = f_{jk}(z_k)$$

transforming the local coordinates $z_k = (z_k^1, \dots, z_k^n) = z_k(p)$ into the local coordinates $z_j = (z_j^1, \dots, z_j^n) = z_j(p)$ is a biholomorphic Poisson map.⁶

Deformations of a holomorphic Poisson manifold (M, Λ) is the glueing of the Poisson polydisks $(U_j, \Lambda_j(t))$ parametrized by t via different identification. That is, replacing $f_{jk}^\alpha(z_k)$ by the functions $f_{jk}^\alpha(z_k, t)(f_{jk}(z_k, 0) = f_{jk}^\alpha(z_k)$ of z_k), replacing $\Lambda_j = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ by $\Lambda_j(t) = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ with $[\Lambda_j(t), \Lambda_j(t)] = 0$ and $\Lambda_j(0) = \Lambda_j$, and the parameter $t = (t_1, \dots, t_m)$, we obtain deformations (M_t, Λ_t) by glueing the Poisson polydisks $(U_1, \Lambda_1(t)), \dots, (U_n, \Lambda_n(t))$ by identifying $z_k \in U_k$ with $z_j = f_{jk}(z_k, t) \in U_j$. The work on deformations of holomorphic Poisson structures is based on this fundamental idea.

In chapter 1, we define a family of compact holomorphic Poisson manifolds, called a Poisson analytic family in the framework of Kodaira-Spencer deformation theory. In other words, when we ignore Poisson structures, a family of compact holomorphic Poisson manifolds is just a family of compact complex manifolds in the sense of Kodaira and Spencer. So deformations of holomorphic Poisson manifolds means that we deform complex structures as well as Poisson structures. And we show that infinitesimal deformation of a holomorphic Poisson manifold (M, Λ) in a Poisson analytic family is encoded in the truncated holomorphic Poisson cohomology. More precisely, an infinitesimal deformation is realized as an element in the second hypercohomology group⁷ $HP^2(M, \Lambda)$ of a complex of sheaves $0 \rightarrow \Theta_M \rightarrow \wedge^2 \Theta_M \rightarrow \dots \rightarrow \wedge^n \Theta_M \rightarrow 0$ induced by $[\Lambda, -]$. Analogously to deformations of complex structure, we define so called Poisson Kodaira Spencer map where the Kodaira Spencer map is realized as a component of the Poisson Kodaira Spencer map. We define a concept of a trivial family, locally trivial family, rigidity and pullback family, and raise some questions that I cannot answer at this stage.

In chapter 2, we study the integrability condition for a Poisson analytic family. Kodaira showed that given a family of deformations of a compact complex manifold M , locally the family is represented by a $C^\infty(0, 1)$ vectors $\varphi(t)$ with $\varphi(0) = 0$ satisfying $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$. And we show that given a family of deformations of a holomorphic Poisson manifold (M, Λ) , locally the family is represented by a $C^\infty(0, 1)$ vectors $\varphi(t)$ with $\varphi(0) = 0$ and a C^∞ bivectors $\Lambda(t)$ with $\Lambda(0) = \Lambda$ satisfying $[\Lambda(t), \Lambda(t)] = 0$, $\bar{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0$, and $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$. Replacing $\varphi(t)$ by $-\varphi(t)$, the integrability condition becomes $\bar{\partial}(\varphi(t) + \Lambda(t)) + \frac{1}{2}[\varphi(t) + \Lambda(t), \varphi(t) + \Lambda(t)] = 0$ which is a solution of the Maurer Cartan equation of the differential graded Lie algebra $(\mathfrak{g}, \bar{\partial}, [-, -])$. (See Appendix C). But we have another differential graded Lie algebra structure on \mathfrak{g} . (See Proposition C.14) If we take $\Lambda'(t) = \Lambda(t) - \Lambda$. Then we have $\Lambda'(0) = 0$ and the integrability condition is equivalent

⁶For the definition of Poisson map, See Appendix A.

⁷We adopt the notation from [Nam08] for the expression of the truncated holomorphic Poisson cohomology groups

to $L(\varphi(t) + \Lambda'(t)) + \frac{1}{2}[\varphi(t) + \Lambda'(t), \varphi(t) + \Lambda'(t)] = 0$ where $L = \bar{\partial} + [\Lambda, -]$.⁸ Then $\varphi(t) + \Lambda'(t)$ is a solution of the Maurer Cartan equation of the differential graded Lie algebra $(\mathfrak{g}, L, [-, -])$. In the part II of the thesis, we show that the differential graded Lie algebra $(\mathfrak{g}, L, [-, -])$ is a differential graded Lie algebra governing the holomorphic Poisson deformations of (M, Λ) in the language of functor of Artin rings.

In chapter 3, under some assumption, we establish an analogous theorem to the following theorem of Kodaira and Spencer.([KNS58],[Kod05] p.270)

Theorem 0.6 (Theorem of Existence) *Let M be a compact complex manifold and suppose $H^2(M, \Theta) = 0$. Then there exists a complex analytic family (\mathcal{M}, B, ω) with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:*

1. $\omega^{-1}(0) = M$
2. $\rho_0 : \frac{\partial}{\partial t} \rightarrow \left(\frac{\partial M_t}{\partial t}\right)_{t=0}$ with $M_t = \omega^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $H^1(M, \Theta) : T_0(B) \xrightarrow{\rho_0} H^1(M, \Theta)$.

Similiary, under the assumption (3.1), we prove the theorem of existence for deformations of holomorphic Poisson structures.(See Theorem 3.2)

Theorem 0.7 (Theorem of Existence for holomorphic Poisson structures) *Let (M, Λ_0) be a compact holomorphic Poisson manifold satisfying (3.1) and suppose that $HP^3(M, \Lambda_0) = 0$. Then there exists a Poisson analytic family $(\mathcal{M}, \Lambda, B, \omega)$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:*

1. $\omega^{-1}(0) = (M, \Lambda_0)$
2. $\varphi_0 : \frac{\partial}{\partial t} \rightarrow \left(\frac{\partial(M_t, \Lambda_t)}{\partial t}\right)_{t=0}$ with $(M_t, \Lambda_t) = \omega^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $HP^2(M, \Lambda_0) : T_0B \xrightarrow{\rho_0} HP^2(M, \Lambda_0)$.

Our proof is rather formal. We throughly follow the Kuranishi methods presented in [MK06]. The reason for the assumption is to apply their methods in the holomorphic Poisson context and my

⁸We remark that the integrability condition was proved in more general context in the language of generalized complex geometry (See [Gua11]). As $H^1(M, \Theta)$ is realized as a subspace of the generalized second cohomology group of a complex manifold M , $HP^2(M, \Lambda)$ is realized as a subspace of the generalized second cohomology group of a holomorphic Poisson manifold (M, Λ) . In this thesis, we deduce the integrability condition by following Kodaira's original approach, that is, by starting from a concept of a geometric family (a Poisson analytic family).

unfamiliarity with the analytic properties of the operator $\bar{\partial} + [\Lambda, -]$. I do not know that we could relax the assumption (3.1). Lastly, based on Kuranishi's lecture notes [Kur71], we define a Poisson analytic family over a complex space, a concept of pullback family, and complete family. We pose a problem on the existence of Kuranishi family in holomorphic Poisson context. I could not access to this problem for my unfamiliarity with analysis behind the operator $\bar{\partial} + [\Lambda, -]$.

Chapter 1

Poisson analytic families

1.1 Families of holomorphic Poisson manifolds

Definition 1.1 (compare [Kod05] p.59) Suppose given a domain $B \in \mathbb{C}^m$, and a set $\{(M_t, \Lambda_t) | t \in B\}$ of holomorphic Poisson manifolds (M_t, Λ_t) , depending on $t \in B$. We say that $\{(M_t, \Lambda_t) | t \in B\}$ is a family of compact holomorphic Poisson manifolds or a Poisson analytic family of compact holomorphic Poisson manifolds if (\mathcal{M}, Λ) is a holomorphic Poisson manifold and a holomorphic map $\pi : \mathcal{M} \rightarrow B$ satisfies the following properties

1. $\pi^{-1}(t)$ is a compact holomorphic Poisson submanifolds of (\mathcal{M}, Λ) .
2. $(M_t, \Lambda_t) = \pi^{-1}(t)$ has the induced Poisson holomorphic structure Λ_t from Λ .
3. The rank of Jacobian of π is equal to m at every point of \mathcal{M} .

Then we can choose a system of local complex coordinates $\{z_1, \dots, z_j, \dots\}, z_j : p \rightarrow z_j(p)$, and coordinate polydisks U_j with respect to z_j satisfying the following conditions.

1. $z_j(p) = (z_j^1(p), \dots, z_j^n(p), t_1, \dots, t_m), (t_1, \dots, t_m) = \omega(p)$
2. $\mathcal{U} = \{U_j | j = 1, 2, \dots\}$ is locally finite.

Then

$$\{p \mapsto (z_j^1(p), \dots, z_j^n(p)) \mid \mathcal{U}_j \cap M_t \neq \emptyset\}$$

gives a system of local complex coordinates on M_t . In terms of these coordinates, ω is the projection given by

$$\omega : (z_j^1, \dots, z_j^n, t_1, \dots, t_m) \rightarrow (t_1, \dots, t_m).$$

For j, k with $U_j \cap U_k \neq \emptyset$, we denote the coordinate transformations from z_k to z_j by

$$f_{jk} : (z_k^1, \dots, z_k^n, t) \rightarrow (z_j^1, \dots, z_j^n, t) = f_{jk}(z_k^1, \dots, z_k^n, t)$$

Note that t_1, \dots, t_m as part of local coordinates on \mathcal{M} do not change under these coordinate transformations. Thus f_{jk} is given by

$$z_j^\alpha = f_{jk}(z_k^1, \dots, z_k^n, t_1, \dots, t_m), \quad \alpha = 1, \dots, n.$$

We now discuss the holomorphic Poisson structures. Since $M_t \hookrightarrow \mathcal{M}$ is a holomorphic Poisson manifold induced from Λ and $\mathcal{M} = \cup M_t$, the Poisson structure Λ on \mathcal{M} can be expressed in terms of local coordinates as $\Lambda = g_{\alpha\beta}(z_j^1, \dots, z_j^n, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ on U_j . For fixed t^0 , the holomorphic Poisson structure Λ_{t^0} on M_{t^0} is given by $g_{\alpha\beta}(z_j^1, \dots, z_j^n, t^0) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ by restricting Λ to M_{t^0} and $g_{\alpha\beta}(z_j, t)$ is holomorphic with respect to z_j .

Of course, the definition can be extended to the case B is an arbitrary complex manifold. We also could define a family of compact holomorphic Poisson manifolds in the following way.

Definition 1.2 (compare [SU02] p.2) *A holomorphic map π of \mathcal{M} onto B is called a family of compact holomorphic Poisson manifolds or a Poisson analytic family of compact holomorphic poisson manifolds if there is a holomorphic Poisson manifold (\mathcal{M}, Λ) satisfying the following conditions*

1. π is proper. In other words, inverse image of a compact set in B is compact.
2. π is a submersion. In other words, for each point $x \in \mathcal{M}$, $d\pi_x : T_x \mathcal{M} \rightarrow T_{\pi(x)} B$ is surjective.

(The above two conditions imply that $\pi^{-1}(t)$ is a complex submanifold of \mathcal{M} for each $t \in B$.)

3. $\pi^{-1}(t)$ is a holomorphic Poisson submanifold of (\mathcal{M}, Λ) for each $t \in B$
4. $\pi^{-1}(t)$ is connected.

Example 1.3 (complex tori) ([KS58] p.408) Let S be the space of $n \times n$ matrices $s = (s_{\beta}^{\alpha})$ with $|\Im s| > 0$, where α denotes the row index and β the column index. For each matrix $s \in S$ we define an $n \times 2n$ matrix $\omega(s) = (\omega_j^{\alpha}(s))$ by

$$\omega_j^{\alpha}(s) = \begin{cases} \delta_j^{\alpha}, & \text{for } 1 \leq j \leq n \\ s_j^{\alpha}, & \text{for } j = n + \beta, 1 \leq \beta \leq n \end{cases} \quad (1.4)$$

Let \mathbb{C}^n be the space of n complex variables $z = (z^1, \dots, z^{\alpha}, \dots, z^n)$ and let G be the discontinuous abelian group of analytic automorphisms of $\mathbb{C}^n \times S$ generated by

$$g_j : (z, s) \rightarrow (z + \omega_j(s), s), \quad j = 1, \dots, 2n,$$

where $\omega_j(s) = (\omega_j^1(s), \dots, \omega_j^{\alpha}(s), \dots, \omega_j^n(s))$ is the j -th column vector of $\omega(s)$. The factor space

$$\mathcal{B} = \mathbb{C}^n \times S/G$$

is obviously a complex manifold and the canonical projection $\mathbb{C}^n \times S \rightarrow S$ induces a regular map $\omega : \mathcal{B} \rightarrow S$ such that $B_s = \omega^{-1}(s)$ is a complex torus of complex dimension n with the periods $\omega_j(s)$ ($j = 1, \dots, 2n$). Then $\mathcal{B} = \{B_s, s \in S\}$ forms a complex analytic family of complex tori.

We would like to describe a G -invariant holomorphic bivector field of the form $\sum_{i,j} f(z, s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ on $\mathbb{C}^n \times S$. Then this induces a holomorphic bivector field on \mathcal{B} . Any element of $g \in G$ is of the form $g : (z, s) \rightarrow (z + m_1 \omega_1(s) + \dots + m_n \omega_{2n}(s), s)$. Hence for $\sum_{i,j} f_{ij}(z, s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ to be invariant vector field, we have $f_{ij}(z, s) = f_{ij}(z + m_1 \omega_1(s) + \dots + m_n \omega_{2n}(s), s)$ for any inters m_1, \dots, m_{2n} . This means that $f(z, s)$ is independent of z . So $f(z, s) = f(s)$. Hence an invariant bivector field is of the form $\Lambda = \sum_{i,j} f_{ij}(s) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$. Since $f_{ij}(s)$ are independent of z , we have $[\Lambda, \Lambda] = 0$. So (\mathcal{B}, Λ) is a Poisson analytic family.

Example 1.5 (Hirzebruch-Nagata surface) ([SU02] p.13) Take two $\mathbb{C} \times \mathbb{P}^1$ and write the coordinates as $(u, (\xi_0 : \xi_1)), (v, (\eta_0 : \eta_1))$, respectively, where $(\xi_0 : \xi_1), (\eta_0 : \eta_1)$ are the homogeneous

coordinates of \mathbb{P}^1 . Patch $\mathbb{C} \times \mathbb{P}^1$ together by relation

$$\begin{cases} u = 1/v, \\ (\xi_0 : \xi_1) = (\eta_0 : v^m \eta_1) \end{cases} \quad (1.6)$$

Then we obtain a two dimensional compact complex manifold F_m . The complex manifold F_m is called the Hirzebruch-Nagata surface. Now we deform the patching by introducing a new patching relation with parameter $t \in \mathbb{C}$.

$$\begin{cases} u = 1/v, \\ (\xi_0 : \xi_1) = (\eta_0 : v^m \eta_1 + tv^k \eta_0), m - 2 \leq 2k \leq m \end{cases} \quad (1.7)$$

and patching two $\mathbb{C} \times \mathbb{P}^1$ by the relation, we obtain a surface S_t for each $t \in \mathbb{C}$. By the relation we have $S_0 = F_m$ and $\omega : \mathcal{S} = \{S_t\}_{t \in \mathbb{C}} \rightarrow \mathbb{C}$ is an complex analytic family.

We put a holomorphic Poisson structure Λ on \mathcal{S} so that (\mathcal{S}, Λ) is a Poisson analytic family.

For one $\mathbb{C} \times \mathbb{P}^1$ with coordinate $(u, (\xi_0 : \xi_1))$, we have two affine covers, namely, $\mathbb{C} \times \mathbb{C}$ and $\mathbb{C} \times \mathbb{C}$. They are glued via $\mathbb{C} \times (\mathbb{C} - \{0\})$ and $\mathbb{C} \times (\mathbb{C} - \{0\})$ by $(u, x = \frac{\xi_1}{\xi_0}) \mapsto (u, y = \frac{\xi_0}{\xi_1}) = (u, \frac{1}{x})$. Similarly for another $\mathbb{C} \times \mathbb{P}^1$ they are glued via $\mathbb{C} \times (\mathbb{C} - \{0\})$ and $\mathbb{C} \times (\mathbb{C} - \{0\})$ and $\mathbb{C} \times (\mathbb{C} - \{0\})$ via $(v, w = \frac{\eta_1}{\eta_0}) \mapsto (v, z) = (v, \frac{1}{w} = \frac{\eta_0}{\eta_1})$. We put holomorphic Poisson structures Λ with $[\Lambda, \Lambda] = 0$ on each patches and show that they are glued via the above relations to give a global bivector field on \mathcal{S} . On (u, x) coordinate, we give $g(t)x^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x}$. On (u, y) coordinate, we give $-g(t) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y}$. On (v, w) coordinate, we give $-g(t)v^{2k-m+2}(wv^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w}$. And on (v, z) coordinate, we give $g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z}$. In the following picture, we have

$$(u, x) = (\frac{1}{v}, v^m w + tv^k) = (\frac{1}{v}, \frac{v^m + tv^k z}{z}), g(t)x^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} \quad \longleftarrow \quad (v, w), -g(t)v^{2k-m+2}(wv^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w}$$

↓

↓

$$(u, y) = (u, \frac{1}{x}) = (\frac{1}{v}, \frac{z}{v^m + tv^k z}) = (\frac{1}{v}, \frac{1}{v^m w + tv^k}), -g(t) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y} \quad \longleftarrow \quad (v, z) = (\frac{1}{w}), g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z}$$

$$\begin{aligned} \text{we have } \frac{\partial}{\partial x} &= -\frac{1}{x^2} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial v} = -\frac{1}{v^2} \frac{\partial}{\partial u} + (mv^{m-1}w + ktv^{k-1}) \frac{\partial}{\partial x} = -\frac{1}{v^2} \frac{\partial}{\partial u} + \frac{mv^{m-1} + ktv^{k-1}z}{z} \frac{\partial}{\partial x} = \\ &= -\frac{1}{v^2} \frac{\partial}{\partial u} + \frac{-mv^{m-1} - ktv^{k-1}}{(v^m + tv^k)^2} \frac{\partial}{\partial y}. \end{aligned}$$

$$\frac{\partial}{\partial w} = v^m \frac{\partial}{\partial x} = \frac{-v^m}{(v^m w + tv^k)^2} \frac{\partial}{\partial y}, \quad \text{and } \frac{\partial}{\partial z} = \frac{-v^m}{z^2} \frac{\partial}{\partial x} = \frac{v^m}{(v^m + tv^k z)^2} \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial w} = -\frac{1}{w^2} \frac{\partial}{\partial z}.$$

Now we show that they are glued.

$$1. (u, x) \text{ and } (u, y). \quad g(t)x^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} = g(t)x^2 \left(-\frac{1}{x^2}\right) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y} = -g(t) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y}$$

2. (v, w) and (v, z) . $-g(t)v^{2k-m+2}(wv^{m-k} + t)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} = -g(t)v^{2k-m+2}(wv^{m-k} + t)^2 \left(\frac{-1}{w^2}\right) \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z} = g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z}$
3. (u, x) and (v, w) . $-g(t)v^{2k-m+2}(wv^{m-k} + t)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} = -g(t)v^{2k-m+2}(wv^{m-k} + t)^2 (-v^{m-2}) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} = g(t)x^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x}$
4. (u, y) and (v, z) . $g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z} = g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{v^m}{(v^m + tv^k z)^2} \left(-\frac{1}{v^2}\right) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y} = -g(t) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y}$
5. (u, x) and (v, z) . $g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial z} = g(t)v^{2k-m+2}(v^{m-k} + tz)^2 \frac{v^{m-2}}{z^2} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} = g(t) \left(\frac{v^m}{z} + tv^k\right)^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x} = g(t)x^2 \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial x}$
6. (u, y) and (v, w) . $-g(t)v^{2k-m+2}(wv^{m-k} + t)^2 \frac{\partial}{\partial v} \wedge \frac{\partial}{\partial w} = -g(t)v^{2k-m+2}(wv^{m-k} + t)^2 \frac{v^{m-2}}{(v^m w + tv^k z)^2} \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y} = -g(t) \frac{\partial}{\partial u} \wedge \frac{\partial}{\partial y}$

So (\mathcal{S}, Λ) is a Poisson analytic family.

Example 1.8 (Hopf surfaces) By a Hopf surface we mean any complex manifold homeomorphic to $S^1 \times S^3$. Let $W = \mathbb{C}^2 - \{0\}$.

Theorem 1.9 For every Hopf surface X there exist numbers $m \in \mathbb{N}, a, b, t, \in \mathbb{C}$ satisfying

$$0 < |a| \leq |b| < 1 \quad \text{and} \quad (b^m - a)t = 0$$

such that X is biholomorphic to $W / \langle \gamma \rangle$, where γ is an automorphism of W given by

$$\gamma(z_1, z_2) = (az_1 + tz_2^m, bz_2)$$

Conversely, for any m, a, b, t as above, the corresponding group $\langle \gamma \rangle$ acts freely and properly discontinuous on W and the complex manifold $W / \langle \gamma \rangle$ is a Hopf surface.

Proof. See [Kod66]. ■

We construct an one parameter Poisson analytic family of general Hopf surfaces. An automorphism of $W \times \mathbb{C}$ given by

$$g : (z_1, z_2, t) \rightarrow (az_1 + tz_2^m, bz_2, t)$$

where $0 < |a| \leq |b| < 1$ and $b^m - a = 0$ (i.e $a = b^m$), generates an infinitely cyclic group G , which properly discontinuous and fixed point free by Theorem 1.9. Hence $\mathcal{M} = W \times \mathbb{C}/G$ is a complex manifold. Since the projection of $W \times \mathbb{C}$ to \mathbb{C} commutes with g , it induces a holomorphic map ω of \mathcal{M} to \mathbb{C} . Clearly the rank of the Jacobian matrix of ω is equal to 1. Thus $(\mathcal{M}, \mathbb{C}, \omega)$ is a complex analytic family with $\omega^{-1}(t) = W/G_t = M_t$.

We give a holomorphic Poisson structure on \mathcal{M} . A holomorphic bivector field on \mathcal{M} is induced from a G -invariant holomorphic bivector field on $W \times \mathbb{C}$. In what follows we write $(z', t') = (z'_1, z'_2, t')$ instead of $(a^n z_1 + na^{n-1}tz_2^m, b^n z_2, t)$. In this notation we have

$$g^n : (z_1, z_2, t) \rightarrow (z'_1, z'_2, t'),$$

We consider a G -invariant holomorphic bivector field on $W \times \mathbb{C}$ of the form $f(z_1, z_2, t) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$, where $f(z_1, z_2, t)$ is a holomorphic function on $W \times \mathbb{C}$. Since

$$\frac{\partial}{\partial z_1} = a^n \frac{\partial}{\partial z'_1}, \quad \frac{\partial}{\partial z_2} = mna^{n-1}tz_2^{m-1} \frac{\partial}{\partial z'_1} + b^n \frac{\partial}{\partial z'_2}$$

the bivector field $f(z_1, z_2, t) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ is transformed by g^n into the bivector field

$$f(z_1, z_2, t) a^n b^n \frac{\partial}{\partial z'_1} \wedge \frac{\partial}{\partial z'_2}$$

Since $f(z_1, z_2, t) \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ is G -invariant, we have

$$f(z'_1, z'_2, t') = f(z_1, z_2, t) a^n b^n$$

By Hartog's theorem, holomorphic function $f(z_1, z_2, t)$ on $W \times \mathbb{C}$ are extended to holomorphic function on $\mathbb{C}^2 \times \mathbb{C}$. Therefore we may assume that $f(z_1, z_2, t)$ is holomorphic on all $\mathbb{C}^2 \times \mathbb{C}$. We have

$$f(z_1, z_2, t) = \frac{1}{a^n b^n} f(a^n z_1 + na^{n-1}tz_2^m, b^n z_2, t) = \frac{1}{b^{n(m+1)}} f(b^{nm} z_1 + nb^{m(n-1)}tz_2^m, b^n z_2, t)$$

Consequently, since $0 < |b| < 1$, letting

$$f(z_1, z_2, t) = \sum_{i,j,k}^{+\infty} c_{ijk} z_1^i z_2^j t^k$$

be the power series expansion of $f(z_1, z_2, t)$, we have

$$\begin{aligned} f(z_1, z_2, t) &= \lim_{n \rightarrow +\infty} \frac{1}{b^{n(m+1)}} \sum_{i,j,k} c_{ijk} (b^{nm} z_1 + n b^{m(n-1)} t z_2^m)^i (b^n z_2)^j t^k \\ &= (c_{0(m+1)0} + c_{0(m+1)1} t + c_{0(m+1)2} t^2 + \dots) z_2^{m+1} \end{aligned}$$

Hence $(\mathcal{M}, [(c_{0(m+1)0} + c_{0(m+1)1} t + c_{0(m+1)2} t^2 + \dots) z_2^{m+1}] \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2})$ is a Poisson analytic family of Hopf surfaces. For each t , we have a holomorphic Poisson structure $[(c_{0(m+1)0} + c_{0(m+1)1} t + c_{0(m+1)2} t^2 + \dots) z_2^{m+1}] \frac{\partial}{\partial z_1} \wedge \frac{\partial}{\partial z_2}$ on a Hopf surface M_t .

1.2 Infinitesimal deformation

1.2.1 Infinitesimal deformation and truncated holomorphic Poisson cohomology

In this section, we show that for a Poisson analytic family $(\mathcal{M}, \Lambda, B, \omega)$, the infinitesimal deformation of a holomorphic Poisson manifold (M_t, Λ_t) with dimension n is captured by the second Hypercohomology group of complex of sheaves $0 \rightarrow \Theta_{M_t} \rightarrow \wedge^2 \Theta_{M_t} \rightarrow \dots \rightarrow \wedge^n \Theta_{M_t} \rightarrow 0$ induced by $[\Lambda_t, -]^1$ analogously to how the infinitesimal deformation of a complex manifold M_t is captured by the first cohomology group $H^1(M_t, \Theta_t)$.

Let (M, Λ) be a holomorphic Poisson manifold and consider the complex of sheaves

$$0 \rightarrow \Theta_M \xrightarrow{[\Lambda, -]} \wedge^2 \Theta_M \xrightarrow{[\Lambda, -]} \dots \xrightarrow{[\Lambda, -]} \wedge^n \Theta_M \rightarrow 0$$

where Θ_M is the holomorphic tangent sheaf. Let $\mathcal{U} = \{U_j\}$ be sufficiently fine open covering of M such that $U_j = \{z_j \in \mathbb{C}^n \mid |z_j^\alpha| < r_j^\alpha, \alpha = 1, \dots, n\}$. Then we can compute the hypercohomology group of the above complex of sheaves by the following Čech resolution. (See Appendix B)

¹For the definition, see Appendix B

$$\begin{array}{ccccccc}
& & \uparrow [\Lambda, -] & & & & \\
& & C^0(\mathcal{U}, \wedge^3 \Theta_M) & \xrightarrow{-\delta} & \dots & & \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \\
& & C^0(\mathcal{U}, \wedge^2 \Theta_M) & \xrightarrow{\delta} & C^1(\mathcal{U}, \wedge^2 \Theta_M) & \xrightarrow{-\delta} & \dots \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] \\
& & C^0(\mathcal{U}, \Theta_M) & \xrightarrow{-\delta} & C^1(\mathcal{U}, \Theta_M) & \xrightarrow{\delta} & C^2(\mathcal{U}, \Theta_M) & \xrightarrow{-\delta} & \dots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

Definition 1.10 We say that the i -th truncated holomorphic Poisson cohomology group² of a holomorphic Poisson manifold (M, Λ) is the i -th hypercohomology group associated with the complex of sheaves $0 \rightarrow \Theta_M \xrightarrow{[\Lambda, -]} \wedge^2 \Theta_M \xrightarrow{[\Lambda, -]} \dots \xrightarrow{[\Lambda, -]} \wedge^n \Theta_M \rightarrow 0$ where Θ_M is the holomorphic tangent sheaf, and is denoted by $HP^i(X, \Lambda)$ ³

Now we relate the 2nd truncated holomorphic Poisson cohomology group $HP^2(M_t, \Lambda_t)$ to the infinitesimal deformation of (M_t, Λ_t) in a Poisson analytic family $(\mathcal{M}, \Lambda, B, \pi)$ for each t . Let $t_0 \in B$ and choose a sufficiently small polydisk Δ with $t_0 \in \Delta \subset B$. Then $\omega^{-1}(\Delta) = \mathcal{M}_\Delta = \bigcup_{j=1}^l \mathcal{U}_j$ where $\mathcal{U}_j := U_j \times \Delta$ such that U_j is a polydisk, and $(z_j, t) \in U_j \times \Delta$ and $(z_k, t) \in U_k \times \Delta$ are the same point on \mathcal{M}_Δ if

$$z_j^\alpha = f_{jk}^\alpha(z_k, t), \quad \alpha = 1, \dots, n$$

$z_j^\alpha = f_{jk}^\alpha(z_k^1, \dots, z_k^n, t_1, \dots, t_m)$ is a holomorphic transition function in $z_k^1, \dots, z_k^n, t_1, \dots, t_m$. And on each local complex coordinate system $U_j \times \Delta$, Λ can be expressed as $\sum_{\alpha, \beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ where $g_{\alpha\beta}^j(z, t)$ is a holomorphic function on $U_j \times \Delta$ and $g_{\alpha\beta}^j(f_{jk}^1(z_k, t), \dots, f_{jk}^n(z_k, t)) = \sum_{r, s} g_{rs}^k(z_k, t) \frac{\partial f_{jk}^\alpha}{\partial z_k^r} \frac{\partial f_{jk}^\beta}{\partial z_k^s}$ on $U_j \times \Delta \cap U_k \times \Delta$. We will denote $\mathcal{U}_j^t := U_j \times t$. For each $t \in \Delta$, $\mathcal{U}^t = \{\mathcal{U}_j^t\}$ be an open covering of M_t . Let $\frac{\partial}{\partial t} = \sum_{\lambda=1}^m c_\lambda \frac{\partial}{\partial t_\lambda}$, $c_\lambda \in \mathbb{C}$, of B . We show that

Proposition 1.11

$$\left\{ \{\theta_{jk}(t) = \sum_{\alpha=1}^n \frac{\partial f_{jk}^\alpha(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha}\}, \{\Lambda_j(t) = \sum_{\alpha, \beta} \frac{\partial g_{\alpha\beta}^j(z, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}\} \right\} \in C^1(\mathcal{U}^t, \Theta_{M_t}) \oplus C^0(\mathcal{U}^t, \wedge^2 \Theta_{M_t})$$

²In [ELW99], holomorphic Poisson cohomology for a holomorphic Poisson manifold (M, Λ) is defined by the i -th hypercohomology group of complex of sheaves $\mathcal{O}_M \rightarrow \Theta_M \rightarrow \wedge^2 \Theta_M \rightarrow \dots \rightarrow \wedge^n \Theta_M \rightarrow 0$ induced by $[\Lambda, -]$. However since there is no role of the structure sheaf \mathcal{O}_M in deformations of holomorphic Poisson manifolds, we truncate the complex of sheaves. See also [Nam08].

³We adopt the notation from [Nam08]. By general philosophy of deformation theory, it might be natural to shift the grading after truncation so that the 0-th cohomology group corresponds to infinitesimal automorphisms, the first cohomology group corresponds to infinitesimal deformations and third cohomology group corresponds to obstructions. However, we follow [Nam08]. So I put $0 \rightarrow 0 \rightarrow 0 \dots$ on the bottom of the complex.

define a 2-cocycle and call its cohomology class $\in HP^2(M_t, \Lambda_t)$ the infinitesimal (Poisson) deformation along $\frac{\partial}{\partial t}$ and this expression is independent of the choice of system of local coordinates.

Proof. First, $\delta(\{\theta_{jk}(t)\}) = 0$ (See [Kod05] p.201). Second, since $[\sum_{\alpha,\beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}, \sum_{\alpha,\beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}] = 0$, by taking the derivative with respect to t , we have $[\sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}^j(z, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}, \sum_{\alpha,\beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}] + [\sum_{\alpha,\beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}, \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}^j(z, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}] = 2[\sum_{\alpha,\beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}, \sum_{\alpha,\beta} \frac{\partial g_{\alpha\beta}^j(z, t)}{\partial t} \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}] = 0$. It remains to show that $\delta(\{\Lambda_j(t)\}) + [\Lambda_t, \{\theta_{jk}\}] = 0$. More precisely, on \mathcal{U}_{jk}^t , we show that $\Lambda_k(t) - \Lambda_j(t) + [\Lambda_t, \theta_{jk}(t)] = 0$. In other words,

$$(*) \sum_{r,s=1}^n \frac{\partial g_{rs}^k}{\partial t} \frac{\partial}{\partial z_k^r} \wedge \frac{\partial}{\partial z_k^s} - \sum_{\alpha,\beta=1}^n \frac{\partial g_{\alpha\beta}^j}{\partial t} \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} + [\sum_{r,s=1}^n g_{rs}^j(z, t) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s}, \sum_{c=1}^n \frac{\partial f_{jk}^c(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^c}] = 0$$

We note that since $z_j^\alpha = f_{jk}^\alpha(z_k^1, \dots, z_k^n, t_1, \dots, t_m)$ for $\alpha = 1, \dots, n$, $\frac{\partial}{\partial z_k^r} = \sum_{a=1}^n \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial}{\partial z_j^a}$ for $r = 1, \dots, n$. Hence

$$\sum_{r,s=1}^n \frac{\partial g_{rs}^k}{\partial t} \frac{\partial}{\partial z_k^r} \wedge \frac{\partial}{\partial z_k^s} = \sum_{r,s,a,b=1}^n \frac{\partial g_{rs}^k}{\partial t} \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s} \frac{\partial}{\partial z_j^a} \wedge \frac{\partial}{\partial z_j^b}$$

$$\begin{aligned} & [\sum_{r,s=1}^n g_{rs}^j(z, t) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s}, \sum_{c=1}^n \frac{\partial f_{jk}^c(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^c}] = \sum_{r,s,c=1}^n [g_{rs}^j(z, t) \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s}, \frac{\partial f_{jk}^c(z_k, t)}{\partial t} \frac{\partial}{\partial z_j^c}] \\ & = \sum_{r,s,c=1}^n [g_{rs}^j \frac{\partial}{\partial z_j^r}, \frac{\partial f_{jk}^c}{\partial t} \frac{\partial}{\partial z_j^c}] \wedge \frac{\partial}{\partial z_j^s} - g_{rs}^j [\frac{\partial}{\partial z_j^r}, \frac{\partial f_{jk}^c}{\partial t} \frac{\partial}{\partial z_j^c}] \wedge \frac{\partial}{\partial z_j^s} \\ & = \sum_{r,s,c=1}^n g_{rs}^j \frac{\partial}{\partial z_j^r} \left(\frac{\partial f_{jk}^c}{\partial t} \right) \frac{\partial}{\partial z_j^c} \wedge \frac{\partial}{\partial z_j^s} - \frac{\partial f_{jk}^c}{\partial t} \frac{\partial g_{rs}^j}{\partial z_j^c} \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} + g_{rs}^j \frac{\partial}{\partial z_j^r} \left(\frac{\partial f_{jk}^c}{\partial t} \right) \frac{\partial}{\partial z_j^c} \wedge \frac{\partial}{\partial z_j^s} \end{aligned}$$

By considering the coefficients of $\frac{\partial}{\partial z_j^a} \wedge \frac{\partial}{\partial z_j^b}$, (*) is equivalent to

$$(**) \sum_{r,s=1}^n \frac{\partial g_{rs}^k}{\partial t} \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s} - \frac{\partial g_{ab}^j}{\partial t} - \sum_{c=1}^n \frac{\partial g_{ab}^j}{\partial z_j^c} \frac{\partial f_{jk}^c}{\partial t} + \sum_{c=1}^n g_{cb}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^a}{\partial t} \right) + g_{ac}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^b}{\partial t} \right) = 0$$

On the other hand, since $\sum_{\alpha,\beta=1}^n g_{\alpha\beta}^j \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} = \sum_{r,s=1}^n g_{rs}^j \frac{\partial}{\partial z_k^r} \wedge \frac{\partial}{\partial z_k^s} = \sum_{r,s,a,b=1}^n g_{rs}^j \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s} \frac{\partial}{\partial z_j^a} \wedge \frac{\partial}{\partial z_j^b}$ on $\mathcal{U}_j^t \cap \mathcal{U}_k^t \neq \emptyset$, we have

$$g_{ab}^j(f_{jk}^1(z_k, t), \dots, f_{jk}^n(z_k, t), t_1, \dots, t_m) = \sum_{r,s=1}^n g_{rs}^j \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s}.$$

By taking the derivative with respect to t , we have

$$\frac{\partial g_{ab}^j}{\partial z_j^1} \frac{\partial f_{jk}^1}{\partial t} + \cdots + \frac{\partial g_{ab}^j}{\partial z_j^n} \frac{\partial f_{jk}^n}{\partial t} + \frac{\partial g_{ab}^j}{\partial t} = \sum_{r,s=1}^n \frac{\partial g_{rs}^k}{\partial t} \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s} + g_{rs}^k \left(\frac{\partial}{\partial z_k^r} \left(\frac{\partial f_{jk}^a}{\partial t} \right) \frac{\partial f_{jk}^b}{\partial z_k^s} + \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial}{\partial z_k^s} \left(\frac{\partial f_{jk}^b}{\partial t} \right) \right),$$

Hence (**) is equivalent to

$$\sum_{c=1}^n g_{cb}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^a}{\partial t} \right) + g_{ac}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^b}{\partial t} \right) = \sum_{r,s=1}^n g_{rs}^k \left(\frac{\partial}{\partial z_k^r} \left(\frac{\partial f_{jk}^a}{\partial t} \right) \frac{\partial f_{jk}^b}{\partial z_k^s} + \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial}{\partial z_k^s} \left(\frac{\partial f_{jk}^b}{\partial t} \right) \right)$$

Indeed,

$$\begin{aligned} \sum_{c=1}^n g_{cb}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^a}{\partial t} \right) + g_{ac}^j \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^b}{\partial t} \right) &= \sum_{r,s,c=1}^n g_{rs}^k \frac{\partial f_{jk}^c}{\partial z_k^r} \frac{\partial f_{jk}^b}{\partial z_k^s} \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^a}{\partial t} \right) + g_{rs}^k \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^c}{\partial z_k^s} \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^b}{\partial t} \right) \\ \sum_{r,s=1}^n g_{rs}^k \left(\frac{\partial}{\partial z_k^r} \left(\frac{\partial f_{jk}^a}{\partial t} \right) \frac{\partial f_{jk}^b}{\partial z_k^s} + \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial}{\partial z_k^s} \left(\frac{\partial f_{jk}^b}{\partial t} \right) \right) &= \sum_{r,s,c=1}^n g_{rs}^k \frac{\partial f_{jk}^c}{\partial z_k^r} \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^a}{\partial t} \right) \frac{\partial f_{jk}^b}{\partial z_k^s} + g_{rs}^k \frac{\partial f_{jk}^a}{\partial z_k^r} \frac{\partial f_{jk}^c}{\partial z_k^s} \frac{\partial}{\partial z_j^c} \left(\frac{\partial f_{jk}^b}{\partial t} \right) \end{aligned}$$

It remains to show that $(\Lambda(t), \theta(t))$ is independent of the choice of systems of local coordinates. We can show the infinitesimal deformation does not change under the refinement of the open covering (See [Kod05] page 190). Since we can choose a common refinement for two system of local coordinates, it is sufficient to show that given two local coordinates $x_j = (z_j, t)$ and $u_j = (w_j, t)$ on each \mathcal{U}_j , the infinitesimal deformation $(\eta(t), \Lambda'(t))$ with respect to $\{u_j\}$ coincides with $(\theta(t), \Lambda(t))$ with respect to $\{x_j\}$. Let

$$w_j^\alpha = g_j^\alpha(z_j^1, \dots, z_j^n, t)$$

the coordinate transformation from (z_j, t) to (w_j, t) which is holomorphic in z_j^1, \dots, z_j^n . So we have $\frac{\partial}{\partial z_j^r} = \sum_\alpha \frac{\partial g_j^\alpha}{\partial z_j^r} \frac{\partial}{\partial w_j^\alpha}$. And let

$$\theta_j(t) = \sum_{\alpha=1}^n \frac{\partial g_j^\alpha(z_j, t)}{\partial t} \frac{\partial}{\partial w_j^\alpha}, \quad w_j^\alpha = g_j^\alpha(z_j, t),$$

Then we claim that $(\theta_{jk}(t), \Lambda_j(t)) - (\eta_{jk}(t), \Lambda'_j(t)) = \theta_k(t) - \theta_j(t) - [\Lambda, \theta_j(t)] = -\delta(-\theta(t)) + [\Lambda, -\theta(t)]$.

Since $\delta(\theta_j(t)) = \{\theta_{jk}(t)\} - \{\eta_{jk}(t)\}$ (see page 192). We only need to see $\Lambda_j(t) - \Lambda'_j(t) + [\Lambda, \theta_j(t)] = 0$.

Equivalently,

$$\sum_{r,s} \frac{\partial \Lambda_j^{rs}(z_j, t)}{\partial t} \frac{\partial}{\partial z_j^r} \wedge \frac{\partial}{\partial z_j^s} - \sum_{\alpha,\beta} \frac{\partial \Lambda_j^{\alpha\beta}(w_j, t)}{\partial t} \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta} + [\sum_{\alpha,\beta} \Lambda_j^{\alpha\beta}(w_j, t) \frac{\partial}{\partial w_j^\alpha} \wedge \frac{\partial}{\partial w_j^\beta}, \sum_c \frac{\partial g_j^c(z_j, t)}{\partial t} \frac{\partial}{\partial w_j^c}] = 0$$

But the computation is essentially same to the above. ■

Definition 1.12 ((holomorphic) Poisson Kodaira-Spencer map) *Let $(\mathcal{M}, \Lambda, B, \pi)$ be a family of compact holomorphic Poisson manifolds, where B is a domain of \mathbb{C}^m , and (z, t) its system of*

local coordinates. Then each (z_j, t) on U_j is a local complex coordinate system of the complex manifold \mathcal{M} . And in the local complex coordinate system Λ can be expressed as $\sum_{\alpha, \beta} g_{\alpha\beta}^j(z, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ where $g_{\alpha\beta}^j(z, t)$ is a holomorphic function on U_j . For a tangent vector $\frac{\partial}{\partial t} = \sum_{\lambda=1}^m c_\lambda \frac{\partial}{\partial t_\lambda}$, $c_\lambda \in \mathbb{C}$, of B , we put

$$\frac{\partial \Lambda_t}{\partial t} = \sum_{\alpha, \beta} \left[\sum_{\lambda=1}^m c_\lambda \frac{\partial g_{\alpha\beta}^j(z, t)}{\partial t_\lambda} \right] \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$$

The (holomorphic) Poisson Kodaira-Spencer map is a \mathbb{C} -linear map of

$$\begin{aligned} \varphi_t : T_t(B) &\rightarrow HP^2(M_t, \Lambda_t) \\ \frac{\partial}{\partial t} &\mapsto \left[\rho_t \left(\frac{\partial}{\partial t} \right) \left(= \frac{\partial M_t}{\partial t} \right), \frac{\partial \Lambda_t}{\partial t} \right] = \frac{\partial(M_t, \Lambda_t)}{\partial t} \end{aligned}$$

where $\rho_t : T_t(B) \rightarrow H^1(M_t, \Theta_t)$ is the Kodaira-Spencer map. (See [Kod05] p.201)

1.2.2 Trivial, locally trivial family and rigidity

Definition 1.13 Two Poisson analytic families $(\mathcal{M}, \Lambda, B, \pi)$ and $(\mathcal{N}, \Lambda', B, \pi')$ are equivalent if there is a biholomorphic Poisson map Φ of (\mathcal{M}, Λ) onto (\mathcal{N}, Λ') such that $\pi = \pi' \circ \Phi$. Then (M_t, Λ_t) and (N_t, Λ'_t) are biholomorphic Poisson map.

Definition 1.14 A Poisson analytic family $(\mathcal{M}, \Lambda, B, \pi)$ is called trivial if it is equivalent to $(M \times B, \Lambda_{t_0} \oplus 0, B, \pi')$ with $M = \pi^{-1}(t^0)$ where t^0 is some point of B . Similarly we define the local triviality of $(\mathcal{M}, \Lambda, B, \pi)$: for each $t \in B$, there exists a neighborhood Δ of t such that $(\mathcal{M}_\Delta, \Lambda_\Delta, \Delta, \pi)$ is trivial.⁵

The following problem is an analogue of a question from deformations of complex structures.

⁴For the definition of product of holomorphic Poisson manifolds, see Appendix A. Here we consider B as a holomorphic Poisson manifold with trivial Poisson structure 0

⁵Let $(\mathcal{M}, \Lambda, B, \pi)$ be a Poisson analytic family. Let Δ be an open set of B . Then $(\mathcal{M}_\Delta = \pi^{-1}(\Delta), \Lambda|_{\mathcal{M}_\Delta}, \Delta, \pi|_{\mathcal{M}_\Delta})$ is an Poisson analytic family. We denote the family by $(\mathcal{M}_\Delta, \Lambda_\Delta, \Delta, \pi)$

Problem 3 *If $\dim HP^2(M_t, \Lambda_t)$ is independent of $t \in B$, and $\varphi_t = 0$ identically, then is the holomorphic Poisson family $(\mathcal{M}, \Lambda, B, \pi)$ locally trivial ?⁶*

Definition 1.15 *We say that a compact holomorphic Poisson manifold (M, Λ_0) is rigid if, for any Poisson analytic family $(\mathcal{M}, \Lambda, B, \pi)$ such that $M_{t_0} = M$, we can find a neighborhood Δ of t_0 such that $M_t = M_{t_0}$ for $t \in \Delta$. More precisely, $(\mathcal{M}_\Delta, \Lambda_\Delta, \Delta, \pi)$ is Poisson biholomorphic to $(M_{t_0} \times \Delta, \Lambda_{t_0} \oplus 0, \Delta, pr)$ where we consider Δ as a trivial holomorphic Poisson manifold $(\Delta, 0)$ and pr is the second projection.*

The following problem is an analogue of a question from deformations of complex structures.

Problem 4 *If $HP^2(M, \Lambda_0) = 0$, is (M, Λ_0) is rigid ?⁷*

1.2.3 Change of parameter

(compare [Kod05] p.205) Suppose given a Poisson analytic family $\{(M_t, \Lambda_t) | (M_t, \Lambda_t) = \omega^{-1}(t), t \in B\} = (\mathcal{M}, \Lambda, B, \pi)$ of compact holomorphic Poisson manifolds, where B is a domain of \mathbb{C}^m . Let D be a domain of \mathbb{C}^r and $h : s \rightarrow t = h(s), s \in D$, a holomorphic map of D into B . Then by changing the parameter from t to s , we construct a Poisson analytic family $\{(M_{h(t)}, \Lambda_{h(t)}) | s \in D\}$ on the parameter space D in the following way.

Let $\mathcal{M} \times_B D := \{(p, s) \in \mathcal{M} \times B | \omega(p) = h(s)\}$. Then we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{M} \times_B D & \xrightarrow{p} & \mathcal{M} \\ \pi \downarrow & & \downarrow \omega \\ D & \xrightarrow{h} & B \end{array}$$

Since ω is a submersion, $\mathcal{M} \times_B D$ is a complex submanifold of $\mathcal{M} \times D$ and π is a submersion. So $(\mathcal{M} \times_B D, D, \pi)$ is a complex analytic family in the sense of Kodaira and Spencer and we have $\pi^{-1}(s) = M_{h(s)}$. We show that it is naturally a Poisson analytic family such that $\pi^{-1}(s) = (M_{h(s)}, \Lambda_{h(s)})$. Note that D can be considered as a holomorphic Poisson manifold with trivial Poisson structure. In other words, $(D, 0)$ is a holomorphic Poisson manifold. Then $(\mathcal{M} \times D, \Lambda \oplus 0)$ is

⁶For the question of deformations of complex structures, we can find the proof in [Kod05]. But I could not access to this problem for unfamiliarity of analysis

⁷I verified that we can use Kodaira's methods presented in [MK06]. Actually the proof is the special case of theorem of completeness(See [Kod05]). But I could not prove the inductive step in Kodaira's methods. However, in the part III of the thesis, we prove that for a nonsingular Poisson variety (X, Λ_0) over an algebraically closed field k , if $HP^2(X, \Lambda_0) = 0$, then (X, Λ_0) is rigid in algebraic Poisson deformations (see Proposition 7.42). I could not find any example with $HP^2(M, \Lambda) = 0$. Even for complex projective plane $\mathbb{P}_{\mathbb{C}}^2$ with any holomorphic Poisson structure Λ , $HP^2(\mathbb{P}_{\mathbb{C}}^2, \Lambda) \neq 0$. See [HX11].

a holomorphic Poisson manifold.⁸ We show that $\mathcal{M} \times_B D$ is a holomorphic Poisson submanifold of $(\mathcal{M} \times D, \Lambda \oplus 0)$. We check locally by applying Proposition A.8 (3). Let $(p_0, s_0) \in \mathcal{M} \times_B D$. Taking a sufficiently small coordinate polydisk Δ with $h(s_0) \in \Delta$, we represent $(\mathcal{M}_\Delta, \Lambda_\Delta) = \omega^{-1}(\Delta)$ in the form of

$$(\mathcal{M}_\Delta, \Lambda_\Delta) = \left(\bigcup_{j=1}^l U_j \times \Delta, \sum_{\alpha, \beta} g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)$$

where each U_j is a polydisk independent of t , and $(z_j, t) \in U_j \times \Delta$ and $(z_k, t) \in U_k \times \Delta$ are the same point on \mathcal{M}_Δ if $z_j^\alpha = f_{jk}^\alpha(z_k, t)$, $\alpha = 1, \dots, n$. Let E be a sufficiently small polydisk of D with $s_0 \in E$ and $h(E) \subset \Delta$. Then we can represent $\mathcal{M} \times D$ locally in the form of

$$(\mathcal{M}_\Delta \times E, \Lambda_\Delta \oplus 0) = \left(\bigcup_{j=1}^l U_j \times \Delta \times E, \sum_{\alpha, \beta} g_{\alpha\beta}^j(z_j, t) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)$$

where $(z_j, t, s) \in U_j \times \Delta \times E$ and $(z_k, t, s) \in U_k \times \Delta \times E$ are the same point on $\mathcal{M}_\Delta \times E$ if $z_j = f_{jk}(z_k, t)$. And we can represent $\mathcal{M} \times_B D$ locally in the form of

$$\bigcup_{j=1}^l U_j \times G_E \subset \mathcal{M} \times \Delta$$

where $G_E = \{(h(s), s) | s \in E\} \subset \Delta \times E$ and $(z_j, h(s), s) \in U_j \times G_E$ and $(z_k, h(s), s) \in U_k \times G_E$ are the same points if $z_j = f_{jk}(z_k, h(s))$. We note that at $(p_0, s_0) \in \mathcal{M} \times_B D$, we have $(\Lambda \oplus 0)_{(p_0, s_0)} = \sum_{\alpha, \beta} g_{\alpha\beta}^j(p_0, h(s_0)) \frac{\partial}{\partial z_j^\alpha} |_{p_0} \wedge \frac{\partial}{\partial z_j^\beta} |_{p_0} \in \wedge^2 T_{\mathcal{M} \times_B D}$. Hence $\mathcal{M} \times_B D$ is a holomorphic Poisson submanifold of $(\mathcal{M} \times D, \Lambda \oplus 0)$. Since $i : \mathcal{M} \times_B D \hookrightarrow \mathcal{M} \times D$ is a Poisson map and $\mathcal{M} \times D \rightarrow \mathcal{M}$ is a Poisson map, $p : \mathcal{M} \times_B D \rightarrow \mathcal{M}$ is a Poisson map.

Since G_E is biholomorphic to E . The holomorphic Poisson manifold $\mathcal{M} \times_B D$ is represented locally by the form

$$\left(\bigcup_{j=1}^l U_j \times E, \sum_{\alpha, \beta} g_{\alpha\beta}^j(z_j, h(s)) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta} \right)$$

where $(z_k, s) \in U_k \times E$ and $(z_j, s) \in U_j \times E$ are the same points if $z_j = f_{jk}(z_k, h(s))$.

Definition 1.16 *The Poisson analytic family $(\mathcal{M} \times_B D, D, (\Lambda \oplus 0)|_{\mathcal{M} \times_B D}, \pi)$ is called the Poisson analytic family induced from $(\mathcal{M}, B, \Lambda, \omega)$ by the holomorphic map $h : D \rightarrow B$.*

Now we consider the change of variable formula in the infinitesimal deformations.

Theorem 1.17 *For any tangent vector $\frac{\partial}{\partial s} = c_1 \frac{\partial}{\partial s_1} + \dots + c_r \frac{\partial}{\partial s_r} \in T_s(D)$, the infinitesimal holomorphic poisson deformation of $(M_{h(s)}, \Lambda_{h(s)})$ along $\frac{\partial}{\partial s}$ is given by*

$$\frac{\partial(M_{h(s)}, \Lambda_{h(s)})}{\partial s} = \left(\sum_{\lambda=1}^m \frac{\partial t_\lambda}{\partial s} \frac{\partial M_t}{\partial t_\lambda}, \sum_{\lambda=1}^m \frac{\partial t_\lambda}{\partial s} \frac{\partial \Lambda_t}{\partial t_\lambda} \right)$$

⁸For the definition of product of two holomorphic Poisson manifolds, see Appendix A

Proof. We put

$$\begin{aligned}\theta_{\lambda jk}(t) &= \sum_{\alpha=1}^n \frac{\partial f_{jk}^{\alpha}(z_k, t_1, \dots, t_m)}{\partial t_{\lambda}} \frac{\partial}{\partial z_j^{\alpha}}, \\ \eta_{jk}(s) &= \sum_{\alpha=1}^n \frac{\partial f_{jk}^{\alpha}(z_k, h(s))}{\partial s} \frac{\partial}{\partial z_j^{\alpha}}, \\ \Lambda_{\lambda j}(t) &= \sum_{\alpha, \beta=1}^n \frac{\partial g_{\alpha\beta}^j(z_j, t_1, \dots, t_m)}{\partial t_{\lambda}} \frac{\partial}{\partial z_j^{\alpha}} \wedge \frac{\partial}{\partial z_j^{\beta}}, \\ \Lambda_j(s) &= \sum_{\alpha, \beta=1}^n \frac{\partial g_{\alpha\beta}^j(z_j, h(s))}{\partial s} \frac{\partial}{\partial z_j^{\alpha}} \wedge \frac{\partial}{\partial z_j^{\beta}}\end{aligned}$$

$\frac{\partial(M_t, \Lambda_t)}{\partial t_{\lambda}}$ is the cohomology class of the 1-cocycle $(\{\theta_{\lambda jk}(t)\}, \{\Lambda_j(s)\})$, and $\frac{\partial(M_{h(s)}, \Lambda_{h(s)})}{\partial s}$ is that of $(\{\eta_{jk}(s)\}, \{\Lambda_j(s)\})$. Since $h(s) = (t_1, \dots, t_m)$, we have

$$\begin{aligned}\frac{\partial f_{jk}^{\alpha}(z_k, h(s))}{\partial s} &= \sum_{\lambda=1}^m \frac{\partial t_{\lambda}}{\partial s} \frac{\partial f_{jk}^{\alpha}(z_k, t_1, \dots, t_m)}{\partial t_{\lambda}}, \\ \frac{\partial g_{\alpha\beta}^j(z_j, h(s))}{\partial s} &= \sum_{l=1}^r c_l \frac{\partial g_{\alpha\beta}^j(z_j, h(s))}{\partial s_l} = \sum_{l=1}^r \sum_{\lambda=1}^m c_l \frac{\partial t_{\lambda}}{\partial s_l} \frac{\partial g_{\alpha\beta}^j(z_j, t_1, \dots, t_m)}{\partial t_{\lambda}} = \sum_{\lambda=1}^m \frac{\partial t_{\lambda}}{\partial s} \frac{\partial g_{\alpha\beta}^j(z_j, t_1, \dots, t_m)}{\partial t_{\lambda}}\end{aligned}$$

Hence we get the theorem. ■

At this point, we discuss a concept of completeness in deformations of holomorphic Poisson manifolds. We define a complete family.

Definition 1.18 *Let $(\mathcal{M}, \Lambda, B, \omega)$ be a Poisson analytic family of compact holomorphic Poisson manifolds, and $t^0 \in B$. Then $(\mathcal{M}, \Lambda, B, \omega)$ is called complete at $t^0 \in B$ if for any Poisson analytic family $(\mathcal{N}, \Lambda', D, \pi)$ such that D is a domain of \mathbb{C}^l containing 0 and that $\pi^{-1}(0) = \omega^{-1}(t^0)$, there are a sufficiently small domain Δ with $0 \in \Delta \subset D$, and a holomorphic map $h : s \rightarrow t = h(s)$ with $h(0) = t^0$ such that $(\mathcal{N}_{\Delta}, \Lambda'_{\Delta}, \Delta, \pi)$ is the Poisson analytic family induced from $(\mathcal{M}, \Lambda, B, \omega)$ by h where $(\mathcal{N}_{\Delta}, \Lambda'_{\Delta}) = \pi^{-1}(\Delta)$.*

The following problem is an analogue of theorem of completeness from deformations of complex structures.

Problem 5 (Theorem of Completeness for deformations of holomorphic Poisson manifolds)

If $\varphi_0 : T_0 B \rightarrow HP^2(M, \Lambda_0)$ is surjective, is the Poisson analytic family $(\mathcal{M}, \Lambda, B, \omega)$ complete at $0 \in B$?⁹

⁹I verified that for this problem, we can use Kodaira's methods presented in [Kod05]. But I could not prove the inductive step in the Poisson direction.

Chapter 2

Integrability condition

In a family $(\mathcal{M}, B, \Lambda)$ of deformations of a complex manifold M , the deformations near M are represented by a C^∞ vector $(1, 0)$ -form $\varphi(t) \in A^{0,1}(M, T_M)$ on M with $\varphi(0) = 0$, $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$ where $t \in \Delta$ a sufficiently small polydisk in B . In this chapter, we show that in a family $(\mathcal{M}, B, \Lambda, \pi)$ of deformations of a holomorphic Poisson manifold (M, Λ_0) , the deformations near (M, Λ_0) are represented by C^∞ vector $(1, 0)$ -form $\varphi(t) \in A^{0,1}(M, T_M)$ and C^∞ bivector $\Lambda(t) \in A^{0,0}(M, \wedge^2 T_M)$ with $\varphi(0) = 0$, $\Lambda(0) = \Lambda_0$ and $\bar{\partial}(\varphi(t) + \Lambda(t)) + \frac{1}{2}[\varphi(t) + \Lambda(t), \varphi(t) + \Lambda(t)] = 0$. To deduce the integrability condition, we follow Kodaira's approach ([Kod05] section §5.3 (b) page 259) in the context of holomorphic Poisson deformations.

2.1 Preliminaries

Let $(\mathcal{M}, \Lambda, B, \omega)$ be a Poisson analytic family of compact Poisson holomorphic manifolds, and put $(M_t, \Lambda_t) = \omega^{-1}(t)$ where B is a domain of \mathbb{C}^m containing the origin 0. Define $|t| = \max_\lambda |t_\lambda|$ for $t = (t_1, \dots, t_m) \in \mathbb{C}^m$, and let $\Delta = \Delta_r = \{t \in \mathbb{C}^m \mid |t| < r\}$ the polydisk of radius $r > 0$. If we take a sufficiently small $\Delta \subset B$, $\mathcal{M}_\Delta = \omega^{-1}(\Delta)$ is represented in the form

$$\mathcal{M}_\Delta = \bigcup_j U_j \times \Delta$$

We denote a point of U_j by $\xi_j = (\xi_j^1, \dots, \xi_j^n)$ and its holomorphic Poisson structure $\Lambda_j = g_{\alpha\beta}^j(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}$ on $U_j \times \Delta$. For simplicity we assume that $U_j = \{\xi_j \in \mathbb{C}^m \mid |\xi_j| < 1\}$ where $|\xi| = \max_a |\xi_j^a|$. $(\xi_j, t) \in U_j \times \Delta$ and $(\xi_k, t) \in U_k \times \Delta$ are the same point on \mathcal{M}_Δ if $\xi_j^\alpha = f_{jk}^\alpha(\xi_k, t)$, $\alpha = 1, \dots, n$ where $f_{jk}^\alpha(\xi_k, t)$ is a poisson holomorphic map of $\xi_k^1, \dots, \xi_k^n, t_1, \dots, t_m$, defined on $U_k \times \Delta \cap U_j \times \Delta$ and we have $g_{\alpha\beta}^j(f_{jk}^1(\xi_k, t), \dots, f_{jk}^n(\xi_k, t)) = \sum_{r,s} g_{rs}^k(\xi_k, t) \frac{\partial f_{jk}^\alpha}{\partial \xi_k^r} \frac{\partial f_{jk}^\beta}{\partial \xi_k^s}$. So $(M_t, \Lambda_t) = \cup_j (U_j, \Lambda_j(t))$ is a

compact holomorphic Poisson manifold obtained by glueing a finite number of Poisson polydisks $(U_1, \Lambda_1(t)), \dots, (U_j, \Lambda_j(t)), \dots$ by identifying $\xi_j \in U_j$ and $\xi_k \in U_k$ if $\xi_j = f_{jk}(\xi_k, t)$, and that holomorphic Poisson structure of (M_t, Λ_t) varies since the manner of glueing and holomorphic Poisson structure vary with t . We note that by [Kod05] Theorem 2.3, when we ignore complex structures and Poisson structures M_t for any $t \in \Delta$ is a diffeomorphic to M_0 as differentiable manifolds.

By [Kod05] Theorem 2.5, if we take a sufficiently small Δ , there is a diffeomorphism Ψ of $M \times \Delta$ onto \mathcal{M}_Δ as differentiable manifolds such that $\omega \circ \Psi$ is the projection $M \times \Delta \rightarrow \Delta$, where we put $M = M_0$. If we denote a point of M by z , we have

$$\omega \circ \Psi(z, t) = t, \quad t \in \Delta.$$

Ψ is the identity of $M = M \times 0$ onto $M = M_0 \subset \mathcal{M}_\Delta$, namely $\Psi(z, 0) = z$. Put $\Psi(z, t) = (\xi, t) = (\xi_j, t)$ for $\Psi(z, t) \in U_j \times \Delta$. Then each component $\xi_j^\alpha = \xi_j^\alpha(z, t)$, $\alpha = 1, \dots, n$, of $\xi_j = (\xi_j^1, \dots, \xi_j^n)$ is a C^∞ function:

$$\Psi(z, t) = (\xi_j^1(z, t), \dots, \xi_j^n(z, t), t_1, \dots, t_m).$$

If we identify $\mathcal{M}_\Delta = \Psi(M \times \Delta)$ with $M_0 \times \Delta$ via Ψ , $(\mathcal{M}_\Delta, \Lambda)$ is considered as a holomorphic Poisson structure defined on the C^∞ manifold $M \times \Delta$ by the system of local coordinates

$$\{(\xi_j, t) | j = 1, 2, 3, \dots\}, \quad (\xi_j, t) = (\xi_j^1(z, t), \dots, \xi_j^n(z, t), t_1, \dots, t_m).$$

and local holomorphic Poisson structures on $U_j \times \Delta$

$$\left\{ \sum_{\alpha, \beta} g_{\alpha\beta}^j(\xi_j(z, t), t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta} | j = 1, 2, 3, \dots \right\}$$

Let (z^1, \dots, z^n) be arbitrary local complex coordinates of a point z of M_0 .

$$\xi_j^\alpha(z, t) = \xi_j^\alpha(z_1, \dots, z_n, t_1, \dots, t_m), \quad \alpha = 1, \dots, n,$$

are C^∞ functions of the complex variables $z^1, \dots, z^n, t_1, \dots, t_m$. Since for $t = 0$, both $(\xi_j^1(z, 0), \dots, \xi_j^n(z, 0))$ and (z_1, \dots, z_n) are local complex coordinates on the complex holomorphic manifold $M_0 = M$, $\xi_j^\alpha(z, 0)$ are holomorphic functions of z_1, \dots, z_n , and

$$\det \left(\frac{\partial \xi_j^\alpha(z, 0)}{\partial z_\lambda} \right)_{\alpha, \lambda=1, \dots, n} \neq 0$$

Hence, if we take Δ sufficiently small, it follows that

$$\det \left(\frac{\partial \xi_j^\alpha(z, t)}{\partial z_\lambda} \right)_{\alpha, \lambda=1, \dots, n} \neq 0$$

for any $t \in \Delta$.

With this preparation, we identify the holomorphic Poisson deformations near (M, Λ_0) , where $M = M_0$ in the analytic family $(\mathcal{M}, \Lambda, B, \Delta)$ with $\varphi(t) + \Lambda(t)$ where $\varphi(t)$ is a C^∞ vector $(0, 1)$ form and $\Lambda(t)$ is a C^∞ bivector on M .

2.1.1 Identification of the deformations of complex structures with $\varphi(t)$

We consider $\bar{\partial}\xi_j^\alpha(z, t) = \sum \bar{\partial}\xi_j^\alpha(z, t)d\bar{z}^v$. The domain $\mathcal{U}_j = \Psi^{-1}(U_j \times \Delta)$ of $\xi_j^\alpha(z, t)$ is a domain of $M \times \Delta$.

Since $\det\left(\frac{\partial\xi_j^\alpha(z, t)}{\partial z_\lambda}\right)_{\alpha, \lambda=1, \dots, n} \neq 0$, we define a $(0, 1)$ -form $\varphi_j^\lambda(z, t) = \sum_{v=1}^n \varphi_{jv}^\lambda(z, t)d\bar{z}_v$ in the following way:

$$\begin{pmatrix} \varphi_j^1(z, t) \\ \vdots \\ \varphi_j^n(z, t) \end{pmatrix} := \begin{pmatrix} \frac{\partial\xi_j^1}{\partial z_1} & \cdots & \frac{\partial\xi_j^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial\xi_j^n}{\partial z_1} & \cdots & \frac{\partial\xi_j^n}{\partial z_n} \end{pmatrix}^{-1} \begin{pmatrix} \bar{\partial}\xi_j^1 \\ \vdots \\ \bar{\partial}\xi_j^n \end{pmatrix}$$

Then we have

$$\begin{pmatrix} \frac{\partial\xi_j^1}{\partial z_1} & \cdots & \frac{\partial\xi_j^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial\xi_j^n}{\partial z_1} & \cdots & \frac{\partial\xi_j^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} \varphi_j^1(z, t) \\ \vdots \\ \varphi_j^n(z, t) \end{pmatrix} = \begin{pmatrix} \bar{\partial}\xi_j^1 \\ \vdots \\ \bar{\partial}\xi_j^n \end{pmatrix}$$

which is equivalent to

$$\begin{pmatrix} \frac{\partial\xi_j^1}{\partial z_1} & \cdots & \frac{\partial\xi_j^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial\xi_j^n}{\partial z_1} & \cdots & \frac{\partial\xi_j^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} \varphi_{j1}^1 & \cdots & \varphi_{jn}^1 \\ \vdots & \vdots & \vdots \\ \varphi_{j1}^n & \cdots & \varphi_{jn}^n \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix} = \begin{pmatrix} \frac{\partial\xi_j^1}{\partial z_1} & \cdots & \frac{\partial\xi_j^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial\xi_j^n}{\partial z_1} & \cdots & \frac{\partial\xi_j^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} d\bar{z}_1 \\ \vdots \\ d\bar{z}_n \end{pmatrix}$$

In other words, we have $(0, 1)$ -forms

$$\varphi_j^\lambda(z, t) = \sum_{v=1}^n \varphi_{jv}^\lambda(z, t)d\bar{z}_v$$

for each $\lambda = 1, \dots, n$, such that

$$\bar{\partial}\xi_j^\alpha(z, t) = \sum_{\lambda=1}^n \varphi_j^\lambda(z, t) \frac{\partial\xi_j^\alpha(z, t)}{\partial z_\lambda}, \quad \alpha = 1, \dots, n$$

The coefficients $\varphi_{jv}^\alpha(z, t)$ are C^∞ functions on \mathcal{U}_j .

Lemma 2.1 *On $\mathcal{U}_j \cap \mathcal{U}_k$, we have*

$$\sum_{\lambda=1}^n \varphi_j^\lambda(z, t) \frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \varphi_k^\lambda(z, t) \frac{\partial}{\partial z_\lambda}$$

Proof. See [Kod05] p.262. ■

If for $(z, t) \in \mathcal{U}_j$, we define

$$\varphi(z, t) := \sum_{\lambda=1}^n \varphi_j^\lambda(z, t) \frac{\partial}{\partial z_\lambda} = \sum_{v, \lambda} \varphi_{jv}^\lambda(z, t) d\bar{z}_v \frac{\partial}{\partial z_\lambda} \quad (2.2)$$

By Lemma 2.1, $\varphi(t) = \varphi(z, t)$ is a C^∞ vector $(0, 1)$ -form on M for every $t \in \Delta$.

Then since $\bar{\partial}\xi_j^\alpha(z, 0) = 0$, and $\det\left(\frac{\partial\xi_j^\alpha(z, t)}{\partial z_\lambda}\right)_{\alpha, \lambda=1, \dots, n} \neq 0$, we have $\varphi(0) = 0$. $\varphi(t)$ satisfies $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0^1$ and we have the following theorem.

Theorem 2.3 *If we take a sufficiently small polydisk Δ , then for $t \in \Delta$, al local C^∞ function f on M is holomorphic with respect to the complex structure M_t if and only if f satisfies the equation*

$$(\bar{\partial} - \varphi(t))f = 0$$

Proof. See [Kod05] Theorem 5.3 p.263. ■

2.1.2 Identification of the deformations of Poisson structures with $\Lambda(t)$

For, on each $U_j \times \Delta$, the holomorphic Poisson structure $\sum_{\alpha, \beta} g_{\alpha, \beta}^j(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}$, there exists the unique bivector field $\Lambda' = \sum_{r, s} f_{rs}^j(z, t) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ on $\mathcal{U}_j = \Psi^{-1}(U_j \times \Delta)$ such that $\sum_{r, s} f_{rs}^j(z, t) \frac{\partial \xi_j^\alpha}{\partial z_r} \frac{\partial \xi_j^\beta}{\partial z_s} = g_{\alpha\beta}^j(\xi_j(z, t), t)$. Indeed, since $\det\left(\frac{\partial \xi_j^\alpha(z, t)}{\partial z_\lambda}\right)_{\alpha, \lambda=1, \dots, n} \neq 0$, we set

$$\begin{pmatrix} f_{11}^j(z, t) & \dots & f_{1n}^j(z, t) \\ \vdots & \ddots & \vdots \\ f_{n1}^j(z, t) & \dots & f_{nn}^j(z, t) \end{pmatrix} := \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial z_1} & \dots & \frac{\partial \xi_j^n}{\partial z_n} \end{pmatrix}^{-1} \begin{pmatrix} g_{11}^j(\xi_j(z, t)) & \dots & g_{1n}^j(\xi_j(z, t)) \\ \vdots & \ddots & \vdots \\ g_{n1}^j(\xi_j(z, t)) & \dots & g_{nn}^j(\xi_j(z, t)) \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \end{pmatrix}^{-1}$$

Then we have the unique C^∞ bivector field $\Lambda'_j := \sum_{r, s} f_{rs}^j(z, t) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ on \mathcal{U}_j

Lemma 2.4 *On $\mathcal{U}_j \cap \mathcal{U}_k$, we have $f_{rs}^j(z, t) = f_{rs}^k(z, t)$.*

Proof. We first note the following identities.

$$\begin{aligned} & \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial z_1} & \dots & \frac{\partial \xi_j^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} f_{11}^j(z, t) & \dots & f_{1n}^j(z, t) \\ \vdots & \ddots & \vdots \\ f_{n1}^j(z, t) & \dots & f_{nn}^j(z, t) \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^1}{\partial z_1} & \dots & \frac{\partial \xi_j^1}{\partial z_n} \end{pmatrix} = \begin{pmatrix} g_{11}^j(\xi_j(z, t)) & \dots & g_{1n}^j(\xi_j(z, t)) \\ \vdots & \ddots & \vdots \\ g_{n1}^j(\xi_j(z, t)) & \dots & g_{nn}^j(\xi_j(z, t)) \end{pmatrix} \\ & \begin{pmatrix} \frac{\partial \xi_k^1}{\partial z_1} & \dots & \frac{\partial \xi_k^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_k^n}{\partial z_1} & \dots & \frac{\partial \xi_k^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} f_{11}^k(z, t) & \dots & f_{1n}^k(z, t) \\ \vdots & \ddots & \vdots \\ f_{n1}^k(z, t) & \dots & f_{nn}^k(z, t) \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_k^1}{\partial z_1} & \dots & \frac{\partial \xi_k^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_k^1}{\partial z_1} & \dots & \frac{\partial \xi_k^1}{\partial z_n} \end{pmatrix} = \begin{pmatrix} g_{11}^k(\xi_k(z, t)) & \dots & g_{1n}^k(\xi_k(z, t)) \\ \vdots & \ddots & \vdots \\ g_{n1}^k(\xi_k(z, t)) & \dots & g_{nn}^k(\xi_k(z, t)) \end{pmatrix} \\ & \begin{pmatrix} \frac{\partial \xi_j^1}{\partial \xi_k^1} & \dots & \frac{\partial \xi_j^1}{\partial \xi_k^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial \xi_k^1} & \dots & \frac{\partial \xi_j^n}{\partial \xi_k^n} \end{pmatrix} \begin{pmatrix} g_{11}^k(\xi_k(z, t)) & \dots & g_{1n}^k(\xi_k(z, t)) \\ \vdots & \ddots & \vdots \\ g_{n1}^k(\xi_k(z, t)) & \dots & g_{nn}^k(\xi_k(z, t)) \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_j^1}{\partial \xi_k^1} & \dots & \frac{\partial \xi_j^1}{\partial \xi_k^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^1}{\partial \xi_k^1} & \dots & \frac{\partial \xi_j^1}{\partial \xi_k^n} \end{pmatrix} = \begin{pmatrix} g_{11}^j(\xi_j(z, t)) & \dots & g_{1n}^j(\xi_j(z, t)) \\ \vdots & \ddots & \vdots \\ g_{n1}^j(\xi_j(z, t)) & \dots & g_{nn}^j(\xi_j(z, t)) \end{pmatrix} \end{aligned}$$

¹For the proof, see [Kod05] p.263,p.265

Since $\frac{\partial \xi_j^q}{\partial z_p} = \sum_{r=1}^n \frac{\partial \xi_j^q}{\partial \xi_k^r} \frac{\partial \xi_k^r}{\partial z_p}$, we have

$$\begin{pmatrix} \frac{\partial \xi_j^1}{\partial \xi_k^1} & \cdots & \frac{\partial \xi_j^1}{\partial \xi_k^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial \xi_k^1} & \cdots & \frac{\partial \xi_j^n}{\partial \xi_k^n} \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_k^1}{\partial z_1} & \cdots & \frac{\partial \xi_k^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_k^n}{\partial z_1} & \cdots & \frac{\partial \xi_k^n}{\partial z_n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \cdots & \frac{\partial \xi_j^1}{\partial z_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial z_1} & \cdots & \frac{\partial \xi_j^n}{\partial z_n} \end{pmatrix}$$

$$\begin{pmatrix} \frac{\partial \xi_k^1}{\partial z_1} & \cdots & \frac{\partial \xi_k^n}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_k^1}{\partial z_n} & \cdots & \frac{\partial \xi_k^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} \frac{\partial \xi_j^1}{\partial \xi_k^1} & \cdots & \frac{\partial \xi_j^1}{\partial \xi_k^n} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^n}{\partial \xi_k^1} & \cdots & \frac{\partial \xi_j^n}{\partial \xi_k^n} \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi_j^1}{\partial z_1} & \cdots & \frac{\partial \xi_j^n}{\partial z_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial \xi_j^1}{\partial z_n} & \cdots & \frac{\partial \xi_j^n}{\partial z_n} \end{pmatrix}$$

Since $\det \left(\frac{\partial \xi_j^\alpha(z,t)}{\partial z_\lambda} \right)_{\alpha,\lambda=1,\dots,n} \neq 0$, we have $f_{rs}^j(z,t) = f_{rs}^k(z,t)$. ■

If for $(z,t) \in \mathcal{U}_j$, we define

$$\Lambda(z,t) := \sum_{r,s} f_{rs}^j(z,t) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}. \quad (2.5)$$

By Lemma 2.4, $\Lambda(t) = \Lambda(z,t)$ is a C^∞ bivector field on M for every $t \in \Delta$.

Theorem 2.6 *If we take a sufficiently small polydisk Δ , then for the Poisson structure $\sum_{\alpha,\beta} g_{\alpha,\beta}^j(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}$ on $U_j \times \Delta$ for each j , there exists the unique bivector field $\Lambda'_j = \sum_{r,s} f_{rs}^j(z,t) \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s}$ on \mathcal{U}_j satisfying*

1. $\sum_{r,s} f_{rs}^j(z,t) \frac{\partial \xi_j^\alpha}{\partial z_r} \frac{\partial \xi_j^\beta}{\partial z_s} = g_{\alpha\beta}^j(\xi_j(z,t), t)$
2. Λ'_j are glued together to define a C^∞ bivector field Λ' on $M \times \Delta$
3. for each j , $[\Lambda'_j, \Lambda'_j] = 0$. Hence we have $[\Lambda', \Lambda'] = 0$

We need the following lemma to prove the theorem.

Lemma 2.7 *If $\rho = \sum_{p,q} \sigma_{pq} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q}$, then $[\sigma, \sigma] = 0$ is equivalent to*

$$\sum_{l=1}^n \sigma_{lk} \frac{\partial \sigma_{ij}}{\partial z_l} + \sigma_{li} \frac{\partial \sigma_{jk}}{\partial z_l} + \sigma_{lj} \frac{\partial \sigma_{ki}}{\partial z_l} = 0$$

for each $1 \leq i, j, k \leq n$.

Proof of Theorem 2.6.

We have already showed (1) and (2). It remains to show

(3). We note that $[\sum_{\alpha,\beta} g_{\alpha\beta}^j(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}, \sum_{\alpha,\beta} g_{\alpha\beta}^j(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}] = 0$ and $g_{\alpha\beta}^j(\xi_j(z,t), t) =$

$\sum_{a,b} f_{ab}^j(z,t) \frac{\partial \xi_j^\alpha}{\partial z_a} \frac{\partial \xi_j^\beta}{\partial z_b}$ is holomorphic with respect to $\xi_j = (\xi_j^\alpha), \alpha = 1, \dots, n$. In the following, for simplicity, we denote $\xi_j^\alpha(z_j, t)$ by ξ_α and $f_{ab}^j(z, t)$ by f_{ab} . By Lemma 2.7, we have

$$\begin{aligned}
0 &= \sum_{a,b,c,d,l} f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial}{\partial \xi_l} \left(f_{cd} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} \right) + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial}{\partial \xi_l} \left(f_{cd} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} \right) + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial}{\partial \xi_l} \left(f_{cd} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial}{\partial \bar{\xi}_l} \left(f_{cd} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} \right) + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial}{\partial \bar{\xi}_l} \left(f_{cd} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} \right) + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial}{\partial \bar{\xi}_l} \left(f_{cd} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} \right) \\
&= \sum_{a,b,c,d,l} f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial f_{cd}}{\partial \xi_l} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_i}{\partial z_c} \right) \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial \xi_i}{\partial z_c} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_j}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial f_{cd}}{\partial \xi_l} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_j}{\partial z_c} \right) \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial \xi_j}{\partial z_c} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_k}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial f_{cd}}{\partial \xi_l} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_k}{\partial z_c} \right) \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \xi_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial \xi_k}{\partial z_c} \frac{\partial}{\partial \xi_l} \left(\frac{\partial \xi_i}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial f_{cd}}{\partial \bar{\xi}_l} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_i}{\partial z_c} \right) \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial \xi_i}{\partial z_c} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_j}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial f_{cd}}{\partial \bar{\xi}_l} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_j}{\partial z_c} \right) \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial \xi_j}{\partial z_c} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_k}{\partial z_d} \right) \\
&+ \sum_{a,b,c,d,l} f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial f_{cd}}{\partial \bar{\xi}_l} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_k}{\partial z_c} \right) \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \bar{\xi}_l}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial \xi_k}{\partial z_c} \frac{\partial}{\partial \bar{\xi}_l} \left(\frac{\partial \xi_i}{\partial z_d} \right) \\
&= \sum_{a,b,c,d} f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial^2 \xi_i}{\partial z_a \partial z_c} \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial \xi_k}{\partial z_b} f_{cd} \frac{\partial \xi_i}{\partial z_c} \frac{\partial^2 \xi_j}{\partial z_a \partial z_d} \\
&+ \sum_{a,b,c,d} f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial^2 \xi_j}{\partial z_a \partial z_c} \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial \xi_i}{\partial z_b} f_{cd} \frac{\partial \xi_j}{\partial z_c} \frac{\partial^2 \xi_k}{\partial z_a \partial z_d} \\
&+ \sum_{a,b,c,d} f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial^2 \xi_k}{\partial z_a \partial z_c} \frac{\partial \xi_i}{\partial z_d} + f_{ab} \frac{\partial \xi_j}{\partial z_b} f_{cd} \frac{\partial \xi_k}{\partial z_c} \frac{\partial^2 \xi_i}{\partial z_a \partial z_d} \\
&= \sum_{a,b,c,d} f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_k}{\partial z_b} \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} + f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_i}{\partial z_b} \frac{\partial \xi_j}{\partial z_c} \frac{\partial \xi_k}{\partial z_d} + f_{ab} \frac{\partial f_{cd}}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial \xi_k}{\partial z_c} \frac{\partial \xi_i}{\partial z_d} \\
&= \sum_{a,b,c,d} \left(f_{ab} \frac{\partial f_{cd}}{\partial z_a} + f_{ac} \frac{\partial f_{db}}{\partial z_a} + f_{ad} \frac{\partial f_{bc}}{\partial z_a} \right) \frac{\partial \xi_i}{\partial z_c} \frac{\partial \xi_j}{\partial z_d} \frac{\partial \xi_k}{\partial z_b}
\end{aligned}$$

Since $\det \left(\frac{\partial \xi_j^\alpha(z,t)}{\partial z_\lambda} \right)_{\alpha,\lambda=1,\dots,n} \neq 0$, $\sum_a f_{ab} \frac{\partial f_{cd}}{\partial z_a} + f_{ac} \frac{\partial f_{db}}{\partial z_a} + f_{ad} \frac{\partial f_{bc}}{\partial z_a} = 0$. So by Lemma 2.7, $[\Lambda'_j, \Lambda'_j] = 0$.

■

Remark 2.8 In summary, for holomorphic Poisson manifold (M_t, Λ_t) for each $t \in \Delta$ in the Poisson analytic family, there exists a bivector field $\Lambda'(t)$ on M with $[\Lambda'(t), \Lambda'(t)] = 0$ for $t \in \Delta$. Conversely, $\Lambda'(t)$ induces Λ_t via diffeomorphism Ψ . More precisely, the $(2,0)$ -part of $\Psi_* \Lambda'(t)$ is Λ_t for $t \in \Delta$.²

²For the type of a bivector field, we mean the decomposition $\wedge^2 T_{\mathbb{C}} M = \wedge^2 T^{1,0} \oplus T^{1,0} \otimes T^{0,1} \oplus \wedge^2 T^{0,1}$ with respect

Now we discuss the condition when a C^∞ bivector field Λ on M with $[\Lambda, \Lambda] = 0$ gives a holomorphic bivector field $\Lambda^{2,0}$ with respect to the complex structure M_t when we restrict Λ to $(2,0)$ part. Before proceeding our discussion, we recall a bracket structure $[-, -]$ on $A = \bigoplus_{p+q=i, p \geq 0, q \geq 1} A^{0,p}(M, \wedge^q T_M)$ (See Appendix C), which we need for the computation of the integrability condition.

The bracket structure on A is defined in the following way.

$$[-, -] : A^{0,p}(M, \wedge^q T_M) \times A^{0,p'}(M, \wedge^{q'} T_M) \rightarrow A^{p+p'}(M, \wedge^{q+q'-1} T_M)$$

In local coordinates it is given by

$$[f dz_I \frac{\partial}{\partial z_J}, g dz_K \frac{\partial}{\partial z_L}] = (-1)^{|K|(|J|+1)} dz_I \wedge dz_K [f \frac{\partial}{\partial z_J}, g \frac{\partial}{\partial z_L}]$$

Then $(A[1], \bar{\partial}, [-, -])$ is a differential graded Lie algebra. So we have the following. For $a \in A^{0,p}(M, \wedge^q T_M)$, $b \in A^{0,p'}(M, \wedge^{q'} T_M)$,

1. $[a, b] = -(-1)^{(p+q+1)(p'+q'+1)} [b, a]$
2. $[a, [b, c]] = [[a, b], c] + (-1)^{(p+q+1)(p'+q'+1)} [b, [a, c]]$
3. $\bar{\partial}[a, b] = [\bar{\partial}a, b] + (-1)^{p+q+1} [a, \bar{\partial}b]$

Theorem 2.9 *If we take a sufficiently small polydisk Δ , then for $t \in \Delta$, a $(2,0)$ -part $\Lambda^{2,0}$ of a C^∞ bivector field $\Lambda = \sum_{\alpha, \beta=1}^n f_{\alpha\beta}(z) \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$ on M is holomorphic with respect to the complex structure M_t , if and only if it satisfies the equation*

$$\bar{\partial}\Lambda - [\Lambda, \varphi(t)] = 0$$

Moreover, if $[\Lambda, \Lambda] = 0$, then $[\Lambda^{2,0}, \Lambda^{2,0}] = 0$.

Proof. Since $(2,0)$ part of $\Lambda = \sum_{\alpha, \beta=1}^n f_{\alpha\beta}(z) \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$ is $\sum_{\alpha, \beta, i, j} f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} \frac{\partial}{\partial \xi^i} \wedge \frac{\partial}{\partial \xi^j}$, we have to show that for each i, j , $\sum_{\alpha, \beta} f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}$ is holomorphic with respect to the complex structure M_t , which is equivalent to $(\bar{\partial} - \varphi(t))(\sum_{\alpha, \beta} f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) = 0$ by Theorem 2.3, if and only if $\bar{\partial}\Lambda - [\Lambda, \varphi(t)] = 0$. First we compute $\bar{\partial}\Lambda - [\Lambda, \varphi(t)] = \sum_{\alpha, \beta, v} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} - [\sum_{\alpha, \beta} f_{\alpha\beta} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}, \sum_{v, \lambda} \varphi_v^\lambda(z, t) d\bar{z}_v \frac{\partial}{\partial z_\lambda}] = \sum_{\alpha, \beta, v} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} - \sum_{\alpha, \beta, v, \lambda} [f_{\alpha\beta} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}, \varphi_v^\lambda d\bar{z}_v \frac{\partial}{\partial z_\lambda}] = \sum_{\alpha, \beta, v} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} + \sum_{\alpha, \beta, v, \lambda} [f_{\alpha\beta} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}, \varphi_v^\lambda \frac{\partial}{\partial z_\lambda}] d\bar{z}_v = \sum_{\alpha, \beta, v} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} + \sum_{\alpha, \beta, v, \lambda} (f_{\alpha\beta} \frac{\partial \varphi_v^\lambda}{\partial z_\alpha} \frac{\partial}{\partial z_\beta} \wedge \frac{\partial}{\partial z_\lambda} - \varphi_v^\lambda \frac{\partial f_{\alpha\beta}}{\partial z_\lambda} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \varphi_v^\lambda}{\partial z_\beta} \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\lambda}) d\bar{z}_v$. By considering the coefficients of $d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$, $\bar{\partial}\Lambda - [\Lambda, \varphi(t)] = 0$ is equivalent to $\sum_{\alpha, \beta, v} [\frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} + \sum_c (f_{c\beta} \frac{\partial \varphi_v^c}{\partial z_c} - \varphi_v^c \frac{\partial f_{\alpha c}}{\partial z_c} + f_{\alpha c} \frac{\partial \varphi_v^c}{\partial z_c})] d\bar{z}_v \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta} = 0$ which is equivalent to

to the almost complex structure induced from the complex structure. If $\Lambda \in C^\infty(\wedge T_{\mathbb{C}}M)$, then we denote by $\Lambda^{2,0}$ the component of $C^\infty(\wedge^2 T^{1,0})$, by $\Lambda^{1,1}$ the component of $C^\infty(T^{0,1} \otimes T^{0,1})$, and by $\Lambda^{0,2}$ the component of $C^\infty(\wedge^2 T^{0,1})$. So we have $\Lambda = \Lambda^{2,0} + \Lambda^{1,1} + \Lambda^{0,2}$.

$$(*) \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} + \sum_c (f_{c\beta} \frac{\partial \varphi_v^\alpha}{\partial z_c} - \varphi_v^c \frac{\partial f_{\alpha\beta}}{\partial z_c} + f_{\alpha c} \frac{\partial \varphi_v^\beta}{\partial z_c}) = 0 \text{ for each } \alpha, \beta, v.$$

On the other hand, we compute $(\bar{\partial} - \varphi(t))(\sum_{\alpha,\beta} f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) = \sum_{\alpha,\beta} (\bar{\partial} - \varphi(t))(f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta})$ for each i, j . $\sum_{\alpha,\beta} (\bar{\partial} - \varphi(t))(f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) = \sum_{\alpha,\beta,v} (\frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial}{\partial \bar{z}_v} (\frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial}{\partial \bar{z}_v} (\frac{\partial \xi^j}{\partial z_\beta})) d\bar{z}_v - \sum_{\alpha,\beta,v,\lambda} \varphi_v^\lambda d\bar{z}_v (\frac{\partial f_{\alpha\beta}}{\partial z_\lambda} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial^2 \xi^i}{\partial z_\alpha \partial z_\lambda} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta \partial z_\lambda})$. So $(\bar{\partial} - \varphi(t))(\sum_{\alpha,\beta} f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) = 0$ is equivalent to

$$(**) \sum_{\alpha,\beta} (\frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial}{\partial \bar{z}_v} (\frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial}{\partial \bar{z}_v} (\frac{\partial \xi^j}{\partial z_\beta})) - \sum_{\alpha,\beta,c} \varphi_v^c (\frac{\partial f_{\alpha\beta}}{\partial z_c} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial^2 \xi^i}{\partial z_\alpha \partial z_c} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta \partial z_c}) = 0$$

for each i, j, v . Since

$$\begin{pmatrix} \frac{\partial \xi^1}{\partial z_1} & \cdots & \frac{\partial \xi^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \xi^n}{\partial z_1} & \cdots & \frac{\partial \xi^n}{\partial z_n} \end{pmatrix} \begin{pmatrix} \varphi_1^1 & \cdots & \varphi_1^n \\ \vdots & \vdots & \vdots \\ \varphi_n^1 & \cdots & \varphi_n^n \end{pmatrix} = \begin{pmatrix} \frac{\partial \xi^1}{\partial z_1} & \cdots & \frac{\partial \xi^1}{\partial z_n} \\ \vdots & \vdots & \vdots \\ \frac{\partial \xi^n}{\partial z_1} & \cdots & \frac{\partial \xi^n}{\partial z_n} \end{pmatrix},$$

we have $\frac{\partial \xi^i}{\partial \bar{z}_v} = \sum_c \frac{\partial \xi^i}{\partial z_c} \varphi_v^c$ and $\frac{\partial \xi^j}{\partial \bar{z}_v} = \sum_c \frac{\partial \xi^j}{\partial z_c} \varphi_v^c$. So $(**)$ is equivalent to

$$\sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + \sum_{\alpha,\beta,c} (f_{\alpha\beta} (\frac{\partial^2 \xi^i}{\partial z_\alpha \partial z_c} \varphi_v^c + \frac{\partial \xi_i}{\partial z_c} \frac{\partial \varphi_v^\alpha}{\partial z_\alpha}) \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} (\frac{\partial^2 \xi^j}{\partial z_\beta \partial z_c} \varphi_v^c + \frac{\partial \xi_j}{\partial z_c} \frac{\partial \varphi_v^\beta}{\partial z_\beta})) - \sum_{\alpha,\beta,c} \varphi_v^c (\frac{\partial f_{\alpha\beta}}{\partial z_c} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial^2 \xi^i}{\partial z_\alpha \partial z_c} \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta \partial z_c}) = 0.$$

which is equivalent to

$$\sum_{\alpha,\beta} \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} + \sum_{\alpha,\beta,c} (f_{\alpha\beta} (\frac{\partial \xi_i}{\partial z_c} \frac{\partial \varphi_v^\alpha}{\partial z_\alpha}) \frac{\partial \xi^j}{\partial z_\beta} + f_{\alpha\beta} \frac{\partial \xi^i}{\partial z_\alpha} (\frac{\partial \xi_j}{\partial z_c} \frac{\partial \varphi_v^\beta}{\partial z_\beta})) - \sum_{\alpha,\beta,c} \varphi_v^c (\frac{\partial f_{\alpha\beta}}{\partial z_c} \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta}) = 0$$

which is equivalent to

$$\sum_{\alpha,\beta} [\frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} + \sum_c (f_{c\beta} \frac{\partial \varphi_v^\alpha}{\partial z_c} - \varphi_v^c \frac{\partial f_{\alpha\beta}}{\partial z_c} + f_{\alpha c} \frac{\partial \varphi_v^\beta}{\partial z_c})] \frac{\partial \xi^i}{\partial z_\alpha} \frac{\partial \xi^j}{\partial z_\beta} = 0 \text{ for each } i, j, v.$$

Since $\det \left(\frac{\partial \xi_j^\alpha(z,t)}{\partial z_\lambda} \right)_{\alpha,\lambda=1,\dots,n} \neq 0$, this is equivalent to

$$(***) \frac{\partial f_{\alpha\beta}}{\partial \bar{z}_v} + \sum_c (f_{c\beta} \frac{\partial \varphi_v^\alpha}{\partial z_c} - \varphi_v^c \frac{\partial f_{\alpha\beta}}{\partial z_c} + f_{\alpha c} \frac{\partial \varphi_v^\beta}{\partial z_c}) = 0 \text{ for each } \alpha, \beta, v.$$

Note that $(*)$ is same to $(***)$.

For the second statement, we can write $\Lambda = \sum_{a,b} f_{ab} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b} = \sum_{a,b,i,j} f_{ab} \frac{\partial \xi_i}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} + 2f_{ab} \frac{\partial \xi_i}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} + f_{ab} \frac{\partial \xi_i}{\partial z_a} \frac{\partial \xi_j}{\partial z_b} \frac{\partial}{\partial \xi_i} \wedge \frac{\partial}{\partial \xi_j} = \Lambda^{2,0} + \Lambda^{1,1} + \Lambda^{2,0}$. Since $[\Lambda, \Lambda] = 0$, $(3,0)$ part of $[\Lambda, \Lambda] = 0$. But $(3,0)$ part happens in $[\Lambda^{2,0}, \Lambda^{2,0}] + [\Lambda^{2,0}, \Lambda^{1,1}]^{3,0}$. Since $\Lambda^{2,0}$ is holomorphic with respect to the complex structure induced by $\varphi(t)$, $[\Lambda^{2,0}, \Lambda^{1,1}]^{3,0} = 0$. ■

Remark 2.10 A C^∞ complex bivector field Λ on M with $[\Lambda, \Lambda] = 0$ gives a Poisson bracket on C^∞ complex valued functions on M . We point out that when we restrict to holomorphic functions with respect to the complex structure M_t , this is exactly the Poisson bracket induced from $\Lambda^{2,0}$.

2.2 Expression of infinitesimal deformations in terms of $\varphi(t)$

and $\Lambda(t)$

In this section, we study how the infinitesimal deformation of (M, Λ_0) in a family is represented in terms of $\varphi(t)$ and $\Lambda(t)$. Recall that $(\mathcal{M}, \Lambda, B, \pi)$ is a Poisson analytic family of compact holomorphic Poisson manifolds with $(M, \Lambda_0) = \omega^{-1}(0)$ and for sufficiently small polydisk $\Delta \subset B$, $M_\Delta = \omega^{-1}(\Delta)$ is represented in the form $\mathcal{M}_\Delta = \bigcup_j U_j \times \Delta$ where $U_j = \{x_j \in \mathbb{C}^m \mid |\xi_j| < 1\}$ and the holomorphic Poisson structures on U_j is $\sum_{\alpha\beta} g_{\alpha\beta}(\xi_j, t) \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta}$ and $\xi_j^\alpha = f_{jk}^\alpha(\xi_k, t)$, $\alpha = 1, \dots, m$ on $U_k \times \Delta \cap U_j \times \Delta$. We showed that the infinitesimal deformation at (M, Λ_0) is captured by the element $(\frac{(M_t, \Lambda_t)}{\partial t})_{t=0} \in H\mathcal{P}^2(M, \Lambda_0)$ of the complex of sheaves of $0 \rightarrow \Theta_M \rightarrow \wedge^2 \Theta_M \rightarrow \dots \rightarrow \wedge^n \Theta_M \rightarrow 0$ by using the following Čech hypercohomology resolution associated with the open covering $\mathcal{U}^0 = \{U_j^0 := U_j \times 0\}$. (See Proposition 1.11)

$$\begin{array}{ccccccc}
 & [\Lambda, -] \uparrow & & & & & \\
 C^0(\mathcal{U}^0, \wedge^3 \Theta_M) & \xrightarrow{-\delta} & \dots & & & & \\
 & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & & & \\
 C^0(\mathcal{U}^0, \wedge^2 \Theta_M) & \xrightarrow{\delta} & C^1(\mathcal{U}^0, \wedge^2 \Theta_M) & \xrightarrow{-\delta} & \dots & & \\
 & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & \\
 C^0(\mathcal{U}^0, \Theta_M) & \xrightarrow{-\delta} & C^1(\mathcal{U}^0, \Theta_M) & \xrightarrow{\delta} & C^2(\mathcal{U}^0, \Theta_M) & \xrightarrow{-\delta} & \dots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots
 \end{array}$$

And we can also compute the hypercohomology group of $0 \rightarrow \Theta_M \rightarrow \wedge^2 \Theta_M \rightarrow \dots \rightarrow \wedge^n \Theta_M \rightarrow 0$ by using the following Dolbeault type resolution. (See example B.6)

$$\begin{array}{ccccccc}
 & [\Lambda, -] \uparrow & & & & & \\
 A^{0,0}(M, \wedge^3 T_M) & \xrightarrow{\bar{\partial}} & \dots & & & & \\
 & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & & & \\
 A^{0,0}(M, \wedge^2 T_M) & \xrightarrow{\bar{\partial}} & A^{0,1}(M, \wedge^2 T_M) & \xrightarrow{\bar{\partial}} & \dots & & \\
 & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & & [\Lambda, -] \uparrow & \\
 A^{0,0}(M, T_M) & \xrightarrow{\bar{\partial}} & A^{0,1}(M, T_M) & \xrightarrow{\bar{\partial}} & A^{0,2}(M, T_M) & \xrightarrow{\bar{\partial}} & \dots \\
 \uparrow & & \uparrow & & \uparrow & & \uparrow \\
 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 \longrightarrow \dots
 \end{array}$$

We describe how the element in the Čech hypercohomology look like in the Dolbeault hypercohomology. In the picture below, we connect two resolutions. We only depict a part of resolutions that we need in the next page.

$$\begin{array}{ccccccc}
& & & \wedge^3 \Theta_M & \longrightarrow & C^0(\wedge^3 \Theta_M) & \\
& & & \swarrow & & \swarrow & \\
& & A^{0,0}(M, \wedge^3 T_M) & \longrightarrow & C^0(\mathscr{A}^{0,0}(\wedge^3 T_M)) & \longrightarrow & C^0(\wedge^3 \Theta_M) \\
& & \uparrow & \dashrightarrow & \uparrow & \dashrightarrow & \uparrow \\
& & A^{0,0}(M, \wedge^2 T_M) & \longrightarrow & C^0(\mathscr{A}^{0,0}(\wedge^2 T_M)) & \longrightarrow & C^0(\wedge^2 \Theta_M) \xrightarrow{\delta} C^1(\wedge^2 \Theta_X) \\
& & \uparrow & \dashrightarrow & \uparrow & \dashrightarrow & \uparrow \\
& & A^{0,1}(M, \wedge^2 T_M) & \longrightarrow & C^0(A^{0,1}(\wedge^2 T_M)) & \longrightarrow & C^1(\mathscr{A}^{0,0}(T_M)) \\
& & \uparrow & \dashrightarrow & \uparrow & \dashrightarrow & \uparrow \\
& & A^{0,0}(M, T_M) & \longrightarrow & C^0(\mathscr{A}^{0,0}(T_M)) & \longrightarrow & C^0(\Theta_M) \xrightarrow{-\delta} C^1(\Theta_X) \\
& & \uparrow & \dashrightarrow & \uparrow & \dashrightarrow & \uparrow \\
& & A^{0,1}(M, T_M) & \longrightarrow & C^0(\mathscr{A}^{0,1}(T_M)) & \longrightarrow & C^1(\mathscr{A}^{0,0}(T_M)) \\
& & & & \uparrow & & \uparrow \\
& & & & C^0(\Theta_M) & & C^1(\Theta_X)
\end{array}$$

Now we explicitly construct the isomorphism of second hypercohomology groups from Čech hyperresolution and Dolbeault hyperresolution, namely

$$HP^2(M, \Lambda_0) \cong \frac{\ker(A^{0,0}(M, \wedge^2 T_M) \oplus A^{1,0}(M, T_M) \rightarrow A^{0,0}(M, \wedge^3 T_M) \oplus A^{1,0}(M, \wedge^2 T_M) \oplus A^{2,0}(M, T_M))}{\text{im}(A^{0,0}(M, T_M) \rightarrow A^{0,0}(M, \wedge^2 T_M) \oplus A^{1,0}(M, \wedge T_M))}$$

Note that each horizontal complex is exact except for edges of the “real wall”.

We define the map in the following way: let $(b, a) \in C^0(\mathcal{U}, \wedge^2 \Theta_M) \oplus C^1(\mathcal{U}, \Theta_M)$ be a cohomology class of $HP^2(M, \Lambda)$. Since $\delta a = 0$, there exists a $c \in C^0(\mathcal{U}, \mathscr{A}^{0,0}(T_M))$ such that $-\delta c = a$. Since a is holomorphic ($\bar{\partial} a = 0$), by the commutativity $\bar{\partial} c \in A^{0,1}(M, T_M)$. And we claim that $[\Lambda, c] - b \in A^{0,0}(M, \wedge^2 T_M)$. Indeed $\delta([\Lambda, c] - b) = \delta([\Lambda, c]) - \delta b = -[\Lambda, -\delta c] - \delta b = -[\Lambda, a] - \delta b = 0$. Now we show that $(\bar{\partial} c, [\Lambda, c] - b)$ is a cohomology class of Dolbeault type resolution. Clearly $\bar{\partial}(\bar{\partial} c) = 0$. $[\Lambda, [\Lambda, c] - b] = 0$. And $\bar{\partial}([\Lambda, c] - b) + [\Lambda, \bar{\partial} c] = -[\Lambda, \bar{\partial} c] + [\Lambda, \bar{\partial} c] = 0$. We define the map by $(b, a) \mapsto ([\Lambda, c] - b, \bar{\partial} c)$.

Now we show that this map is well defined.

1. (independence of choice of c) let c' with $-\delta c' = a$. Then $-\delta(c - c') = 0$. So $d = c - c' \in A^{0,0}(M, \Theta_M)$. Then $([\Lambda, c] - b, \bar{\partial} c) - ([\Lambda, c'] - b, \bar{\partial} c') = ([\Lambda, c - c'], \bar{\partial}(c - c')) = \bar{\partial} d + [\Lambda, d]$
2. (independence of choice of (b, a)) Let (b, a) and (b', a') are in the same cohomology class. We show that $(b - b', a - a')$ is mapped to 0. Indeed, there exists $e \in C^0(\mathcal{U}, \Theta_M)$ such that $-\delta e + [\Lambda, e] = (a - a') - (b - b')$. We can use e as c . Then $(b - b', a - a')$ is mapped to $([\Lambda, e] - (b - b'), \bar{\partial} e) = (0, 0)$.

For the inverse map, let $(\beta, \alpha) \in A^{0,0}(M, \wedge^2 T_M) \oplus A^{0,1}(M, T_M)$ be the cohomology class of Dolbeault type resolution. Then there exists $c \in C^0(\mathcal{U}, \mathcal{A}^{0,0}(T_M))$ such that $\bar{\partial}c = \alpha$. We define the inverse map $(\beta, \alpha) \mapsto ([\Lambda, c] - \beta, -\delta c)$.

Theorem 2.11 $\left(\left(\frac{\partial \varphi(t)}{\partial t} \right)_{t=0}, - \left(\frac{\partial \Lambda(t)}{\partial t} \right)_{t=0} \right)$ satisfies

$$[\Lambda_0, - \left(\frac{\partial \Lambda(t)}{\partial t} \right)_{t=0}] = 0, \quad \bar{\partial} \left(- \left(\frac{\partial \Lambda(t)}{\partial t} \right)_{t=0} \right) + [\Lambda_0, \left(\frac{\partial \varphi(t)}{\partial t} \right)_{t=0}] = 0, \quad \bar{\partial} \left(\frac{\partial \varphi(t)}{\partial t} \right)_{t=0} = 0, \quad \text{and}$$

under the isomorphism

$$HP^2(M) \cong \frac{\ker(A^{0,0}(M, \wedge^2 T_M) \oplus A^{1,0}(M, \wedge T_M) \rightarrow A^{0,0}(M, \wedge^3 T_M) \oplus A^{1,0}(M, \wedge^2 T_M) \oplus A^{2,0}(M, T_M))}{\text{im}(A^{0,0}(M, T_M) \rightarrow A^{0,0}(M, \wedge^2 T_M) \oplus A^{1,0}(M, \wedge T_M))}$$

$$\left(\frac{\partial(M_t, \Lambda_t)}{\partial t} \right)_{t=0} \in HP^2(M, \Lambda_0) \text{ corresponds to } \left(\left(\frac{\partial \varphi(t)}{\partial t} \right)_{t=0}, - \left(\frac{\partial \Lambda(t)}{\partial t} \right)_{t=0} \right)$$

Proof. By taking the derivative of our integrability condition with respect to t and plugging 0, we get the first claim. Now we construct the isomorphism between two second cohomology groups and their correspondence. Put

$$\begin{aligned} \theta_{jk} &= \sum_{\alpha=1}^n \left(\frac{\partial f_{jk}^\alpha(\xi_k, t)}{\partial t} \right)_{t=0} \frac{\partial}{\partial \xi_j^\alpha} \\ \sigma_j &= \sum_{r,s=1}^n \left(\frac{\partial g_{rs}(\xi, t)}{\partial t} \right)_{t=0} \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s} \end{aligned}$$

The infinitesimal deformation $\left(\frac{\partial(M_t, \Lambda_t)}{\partial t} \right)_{t=0} \in HP^2(M, \Lambda_0)$ is the cohomology class of the $(\{\theta_{jk}\}, \{\sigma_j\}) \in C^1(\mathcal{U}^0, \Theta) \oplus C^0(\mathcal{U}^0, \wedge^2 \Theta)$. We fix a tangent vector $\frac{\partial}{\partial t} \in T_0(\Delta)$, denote $\left(\frac{\partial f(t)}{\partial t} \right)_{t=0}$ by \dot{f} . By differentiating

$$\xi_j^\alpha(z, t) = f_{jk}^\alpha(\xi_k(z, t), t) = f_{jk}^\alpha(\xi_k^1(z, t), \dots, \xi_k^n(z, t), t)$$

with respect to t and putting $t = 0$, we get

$$\dot{\xi}_j^\alpha := \sum_{\beta=1}^n \frac{\partial \xi_j^\alpha}{\partial \xi_k^\beta} \dot{\xi}_k^\beta + \left(\frac{\partial f_{jk}^\alpha(\xi_k, t)}{\partial t} \right)_{t=0}$$

where $\xi_j^\alpha = \xi_j^\alpha(z, 0)$ and $\xi_k^\beta = \xi_k^\beta(z, 0)$. Therefore putting

$$\xi_j = \sum_{\alpha=1}^n \dot{\xi}_j^\alpha \frac{\partial}{\partial \xi_j^\alpha}$$

for each j , we have

$$\theta_{jk} = \xi_j - \xi_k$$

Since $\dot{\xi}_j^\alpha = \left(\frac{\partial \xi_j^\alpha(z,t)}{\partial t}\right)_{t=0}$ is a C^∞ function on U_j , ξ_j is a C^∞ vector field on U_j . So we have $\{\xi_j\} \in C^0(\mathcal{U}_0, \mathcal{A}^{0,0}(\Theta))$ and $\delta\{\xi_j\} = -\{\theta_{jk}\}$ where δ is the usual Čech map.

We need the following lemma.

Lemma 2.12 $\bar{\partial}\xi_j = \sum_{\lambda=1}^n \left(\frac{\partial \varphi^\lambda(z,t)}{\partial t}\right)_{t=0} \frac{\partial}{\partial z_\lambda} = \sum_{\lambda=1}^n \dot{\varphi}^\lambda \frac{\partial}{\partial z_\lambda} = \dot{\varphi}$ and $\dot{\Lambda} - \sigma_j + [\Lambda_0, \xi_j] = 0$. More precisely,

$$\sum_{r,s=1}^n \left(\frac{\partial f_{rs}(z,t)}{\partial t}\right)_{t=0} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s} - \sum_{\alpha,\beta=1}^n \left(\frac{\partial g_{\alpha\beta}^j(\xi_j,t)}{\partial t}\right)_{t=0} \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta} + \left[\sum_{r,s=1}^n g_{rs}^j(\xi_j,0) \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s}, \sum_{c=1}^n \dot{\xi}_j^c \frac{\partial}{\partial \xi_j^c} \right] = 0$$

equivalently,

$$(*) \sum_{r,s} \dot{f}_{rs} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s} - \sum_{\alpha,\beta=1}^n \dot{g}_{\alpha\beta}^j \frac{\partial}{\partial \xi_j^\alpha} \wedge \frac{\partial}{\partial \xi_j^\beta} + \left[\sum_{r,s=1}^n g_{rs}^j(\xi_j,0) \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s}, \sum_{c=1}^n \dot{\xi}_j^c \frac{\partial}{\partial \xi_j^c} \right] = 0$$

Proof. By differentiating

$$\bar{\partial}\xi_j^\alpha(z,t) = \sum_{\lambda=1}^n \varphi^\lambda(z,t) \frac{\partial \xi_j^\alpha(z,t)}{\partial z_\lambda}$$

with respect to t , and putting $t = 0$, we obtain

$$\bar{\partial}\dot{\xi}_j^\alpha = \sum_{\lambda=1}^n \dot{\varphi}^\lambda \frac{\partial \xi_j^\alpha(z,0)}{\partial z_\lambda}$$

since $\varphi^\lambda(z,0) = 0$. Hence

$$\bar{\partial}\xi_j = \sum_{\alpha=1}^n \bar{\partial}\dot{\xi}_j^\alpha \frac{\partial}{\partial \xi_j^\alpha} = \sum_{\lambda=1}^n \sum_{\alpha=1}^n \dot{\varphi}^\lambda \frac{\partial \xi_j^\alpha(z,0)}{\partial z_\lambda} \frac{\partial}{\partial \xi_j^\alpha} = \sum_{\lambda=1}^n \dot{\varphi}^\lambda \frac{\partial}{\partial z_\lambda} = \dot{\varphi}$$

For (*), we note that

$$\sum_{r,s=1}^n \dot{f}_{rs} \frac{\partial}{\partial z_r} \wedge \frac{\partial}{\partial z_s} = \sum_{r,s,a,b=1}^n \dot{f}_{rs} \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} \frac{\partial}{\partial \xi_j^a} \wedge \frac{\partial}{\partial \xi_j^b}$$

$$\begin{aligned} & \sum_{r,s,c=1}^n [g_{rs}^j(\xi_j,0) \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s}, \dot{\xi}_j^c \frac{\partial}{\partial \xi_j^c}] = \sum_{r,s,c=1}^n [g_{rs}^j(\xi_j,0) \frac{\partial}{\partial \xi_j^r}, \dot{\xi}_j^c \frac{\partial}{\partial \xi_j^c}] \wedge \frac{\partial}{\partial \xi_j^s} - g_{rs}^j(\xi_j,0) [\frac{\partial}{\partial \xi_j^s}, \dot{\xi}_j^c \frac{\partial}{\partial \xi_j^c}] \wedge \frac{\partial}{\partial \xi_j^r} \\ & = \sum_{r,s,c=1}^n g_{rs}^j(\xi_j,0) \frac{\partial \dot{\xi}_j^c}{\partial \xi_j^r} \frac{\partial}{\partial \xi_j^c} \wedge \frac{\partial}{\partial \xi_j^s} - \dot{\xi}_j^c \frac{\partial g_{rs}^j(\xi_j,0)}{\partial \xi_j^c} \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^s} + g_{rs}^j(\xi_j,0) \frac{\partial \dot{\xi}_j^c}{\partial \xi_j^s} \frac{\partial}{\partial \xi_j^r} \wedge \frac{\partial}{\partial \xi_j^c} \end{aligned}$$

By considering the coefficients of $\frac{\partial}{\partial \xi_j^a} \wedge \frac{\partial}{\partial \xi_j^b}$, (*) is equivalent to

$$(**) \sum_{r,s=1}^n f_{rs} \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} - g_{ab}^j - \sum_{c=1}^n \xi_j^c \frac{\partial g_{ab}(\xi_j,0)}{\partial \xi_j^c} + \sum_{c=1}^n g_{cb}^j(\xi_j,0) \frac{\partial \xi_j^a}{\partial \xi_j^c} + g_{ac}^j(\xi_j,0) \frac{\partial \xi_j^b}{\partial \xi_j^c} = 0$$

On the other hand, we have

$$g_{ab}^j(\xi_j^1(z,t), \dots, \xi_j^n(z,t), t_1, \dots, t_m) = \sum_{r,s=1}^n f_{rs}(z,t) \frac{\partial \xi_j^a(z,t)}{\partial z_r} \frac{\partial \xi_j^b(z,t)}{\partial z_s}$$

By taking the derivative with respect to t and putting $t = 0$, we have

$$\frac{\partial g_{ab}^j(\xi_j,0)}{\partial \xi_j^1} \dot{\xi}_j^1 + \dots + \frac{\partial g_{ab}^j(\xi_j,0)}{\partial \xi_j^n} \dot{\xi}_j^n + \dot{g}_{ab}^j = \sum_{r,s=1}^n f_{rs} \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} + f_{rs}(z,0) \left(\frac{\partial \xi_j^a}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} + \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b}{\partial z_s} \right)$$

Hence (**) is equivalent to

$$\sum_{c=1}^n g_{cb}^j(\xi_j,0) \frac{\partial \xi_j^a}{\partial \xi_j^c} + g_{ac}^j(\xi_j,0) \frac{\partial \xi_j^b}{\partial \xi_j^c} = \sum_{r,s=1}^n f_{rs}(z,0) \frac{\partial \xi_j^a}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} + f_{rs}(z,0) \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b}{\partial z_s}$$

Indeed,

$$\begin{aligned} \sum_{c=1}^n g_{cb}^j(\xi_j,0) \frac{\partial \xi_j^a}{\partial \xi_j^c} + g_{ac}^j(\xi_j,0) \frac{\partial \xi_j^b}{\partial \xi_j^c} &= \sum_{r,s,c=1}^n f_{rs}(z,0) \frac{\partial \xi_j^c(z,0)}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} \frac{\partial \xi_j^a}{\partial \xi_j^c} + f_{rs}(z,0) \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^c(z,0)}{\partial z_s} \frac{\partial \xi_j^b}{\partial \xi_j^c} \\ &= \sum_{r,s=1}^n f_{rs}(z,0) \frac{\partial \xi_j^a}{\partial z_r} \frac{\partial \xi_j^b(z,0)}{\partial z_s} + f_{rs}(z,0) \frac{\partial \xi_j^a(z,0)}{\partial z_r} \frac{\partial \xi_j^b}{\partial z_s} \end{aligned}$$

■ Going back to our proof of Theorem 2.11, we defined the isomorphism between two hypercohomology groups $(b, a) \mapsto ([\Lambda, c] - b)$ where $-\delta c = a$ in the discussion above the Theorem 2.11. We take $(b, a) = (\{\sigma_j\}, \{\theta_{jk}\})$ and $c = \{\xi_j\}$. Since $-\delta\{\xi_j\} = \{\theta_{jk}\}$, we have $-\delta c = a$. Then by the isomorphism $(\{\sigma_j\}, \{\theta_{jk}\})$ is mapped to $([\Lambda, \{\xi_j\}] - \{\sigma_j\}, \bar{\partial}\{\xi_j\})$ which is $(-\dot{\Lambda}, \dot{\varphi})$ by Lemma 2.12. ■

2.3 Integrability condition

We showed that given a Poisson analytic family (M, Λ, B, ω) deformation (M_t, Λ_t) of M near (M_0, Λ_0) is represented by the vector (0,1)-form $\varphi(t)$ and the bivector field $\Lambda(t)$ on M with $\varphi(0) = 0$ and $\Lambda(0) = \Lambda_0$ satisfying the conditions: (1) $[\Lambda(t), \Lambda(t)] = 0$, (2) $\bar{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0$ and (3) $\bar{\partial}\varphi(t) - \frac{1}{2}[\varphi(t), \varphi(t)] = 0$

Conversely, we show that on the holomorphic Poisson manifold (M, Λ_0) , a vector (0,1)-form φ and a bivector field Λ on M such that φ and $\Lambda_0 + \Lambda$ satisfying the integrability condition define another holomorphic Poisson structure on M .

Let $\varphi = \sum_{\lambda=1}^n \varphi_v^\lambda(z) d\bar{z}^v \frac{\partial}{\partial z^\lambda}$ be a C^∞ vector $(0,1)$ -form and Λ be a C^∞ bivector field on a holomorphic Poisson manifold (M, Λ_0) and suppose $\det(\delta_v^\lambda - \sum_\mu \varphi_v^\mu \overline{\varphi_\mu^\lambda}) \neq 0$. We assume that φ and Λ satisfies the integrability condition³

$$\begin{aligned} (1) & [\Lambda_0 + \Lambda, \Lambda_0 + \Lambda] = 0 \\ (2) & \bar{\partial}(\Lambda_0 + \Lambda) - [\Lambda_0 + \Lambda, \varphi] = 0 \\ (3) & \bar{\partial}\varphi - \frac{1}{2}[\varphi, \varphi] = 0 \end{aligned}$$

Then by the Newlander-Nirenberg theorem([NN57],[Kod05]), the condition (3) gives a finite open covering $\{U_j\}$ of M and C^∞ -functions $\xi_j^\alpha = \xi_j^\alpha(z), \alpha = 1, \dots, n$ on each U_j such that $\xi_j : z \rightarrow \xi_j(z) = (\xi_j^1(z), \dots, \xi_j^n(z))$ gives complex coordinates on U_j and $\{\xi_1, \dots, \xi_j, \dots\}$ defines another complex structure on M , which we denote by M_φ . And by Theorem 2.9, the condition (1) and (2) gives another holomorphic Poisson structure $(\Lambda_0 + \Lambda)^{2,0}$ on M with respect to the complex structure induced by φ where $(\Lambda_0 + \Lambda)^{2,0}$ means the $(2,0)$ -part of $\Lambda_0 + \Lambda$ with respect to the complex structure induced by φ .

Example 2.13 (Hitchin-Goto family) *Hitchin showed the following theorem.*

Theorem 2.14 *Let (M, σ) be a holomorphic Poisson manifold which satisfies the $\partial\bar{\partial}$ -lemma. Then any class $\sigma([\omega]) \in H^1(M, T)$ for $[\omega] \in H^1(M, T^*)$ is tangent to a deformation of complex structure induced by $\phi(t) = \sigma(\alpha)$ where $\alpha = t\omega + \partial(t^2\beta_2 + t^3\beta_3 + \dots)$ for $(0,1)$ -forms β_i with respect to the original complex structure.*

Proof. See [Hit12] theorem 1. ■

And also Hitchin showed for each $\phi(t)$, there is a holomorphic Poisson structure σ_t with respect to $M_{\phi(t)}$. We construct a Poisson analytic family (\mathcal{M}, Λ) such that $(M_t, \Lambda_t) = (M_{\phi(t)}, \sigma_t)$ by showing that $(\phi(t), \sigma)$ satisfies the integrability condition.

³If we replace φ by $-\varphi$, then (1), (2), and (3) are equivalent to

$$L(\Lambda + \varphi) + \frac{1}{2}[\Lambda + \varphi, \Lambda + \varphi] = 0 \quad \text{where } L = \bar{\partial} + [\Lambda_0, -]$$

which is a solution of the Maurer-Cartan equation of a differential graded Lie algebra $(\mathfrak{g} = \bigoplus_{i \geq 0} g^i = \bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(M, \wedge^q T_M), L, [-, -])$ (See Appendix C). In the part II of the thesis, we prove that this differential graded Lie algebra controls deformations of a holomorphic Poisson manifold (M, Λ_0) in the language of functor of Artin rings.

Lemma 2.15 Let $\sigma \in C^\infty(\wedge^2 T_M)$ be a bivector of a complex manifold M such that $[\sigma, \sigma] = 0$. Then we have $[\sigma, \sigma(\partial\beta)] = 0$ where β is an $(0, 1)$ -form.

Proof. If we write $\sigma = \sum_{l,k=1}^n \sigma^{lk} \frac{\partial}{\partial z_l} \wedge \frac{\partial}{\partial z_k}$ and $\beta = \sum_{i=1}^n f_i d\bar{z}_i$. Then $\partial\beta = \sum_{i,j=1}^n \frac{\partial f_i}{\partial z_j} dz_j \wedge d\bar{z}_i$. Then $\sigma(\partial\beta) = \sum_{i,l,k} \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} d\bar{z}_i - \sigma^{lk} \frac{\partial f_i}{\partial z_k} \frac{\partial}{\partial z_l} d\bar{z}_i = \sum_{i,l,k} 2\sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} d\bar{z}_i$. So it is sufficient to show that

$$\begin{aligned} & \sum_{p,q,l,k} [\sigma^{pq} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q}, \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k}] = 0 \\ & \sum_{p,q,l,k} [\sigma^{pq} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q}, \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k}] = \sum_{p,q,l,k} [\sigma^{pq} \frac{\partial}{\partial z_p}, \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k}] \frac{\partial}{\partial z_q} - \sigma^{pq} [\frac{\partial}{\partial z_q}, \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k}] \frac{\partial}{\partial z_p} \\ & = \sum_{p,q,l,k} \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} + \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} - \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial \sigma^{pq}}{\partial z_k} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q} - \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_q} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \\ & \frac{\partial}{\partial z_p} - \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_q \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_p}. \end{aligned}$$

Let's consider $\sum_{p,q,l,k} \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} - \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_q \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_p} = 2 \sum_{p,q,l,k} \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q}$. By considering the coefficient of $\frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b}$ of $\sum_{p,q,l,k} \sigma^{pq} \sigma^{lk} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q}$, we have $\sum_{p,l} \sigma^{pb} \sigma^{la} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b} - \sum_{p,l} \sigma^{pa} \sigma^{lb} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b} = \sum_{p,l} \sigma^{pb} \sigma^{la} \frac{\partial^2 f_i}{\partial z_p \partial z_l} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b} - \sum_{p,l} \sigma^{la} \sigma^{pb} \frac{\partial^2 f_i}{\partial z_l \partial z_p} \frac{\partial}{\partial z_a} \wedge \frac{\partial}{\partial z_b} = 0$

So we have $\sum_{p,q,l,k} [\sigma^{pq} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q}, \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k}] = \sum_{p,q,l,k} \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} - \sigma^{lk} \frac{\partial f_i}{\partial z_l} \frac{\partial \sigma^{pq}}{\partial z_k} \frac{\partial}{\partial z_p} \wedge \frac{\partial}{\partial z_q} - \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_q} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_p} = \sum_{q,k,l} \left(\sum_p \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} - \sigma^{lp} \frac{\partial f_i}{\partial z_l} \frac{\partial \sigma^{kq}}{\partial z_p} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} - \sigma^{qp} \frac{\partial \sigma^{lk}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} \right) = \sum_{q,k,l} \left(\sum_p \sigma^{pq} \frac{\partial \sigma^{lk}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} + \sigma^{pl} \frac{\partial f_i}{\partial z_l} \frac{\partial \sigma^{kq}}{\partial z_p} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} + \sigma^{kp} \frac{\partial \sigma^{lq}}{\partial z_p} \frac{\partial f_i}{\partial z_l} \frac{\partial}{\partial z_k} \wedge \frac{\partial}{\partial z_q} \right) = 0$ by Lemma 2.7.

■

Now we assume that $\phi(t) = \sigma(\alpha)$ constructed by Hitchin converges for $t \in \Delta \subset \mathbb{C}$. And we know that $\psi(t) = -\phi(t)$ satisfies $\bar{\partial}\psi(t) - \frac{1}{2}[\psi(t), \psi(t)] = 0$. We can consider $\phi := \phi(t)$ as a $C^\infty(0, 1)$ -vector on $M \times \Delta$. Then by Newlander-Nirenberg theorem([NN57],[Kod05] p.268), we can give a holomorphic coordinate on $M \times \Delta$ induced by ϕ (more precisely $-\phi$). Let's denote the complex manifold induced by ϕ by \mathcal{M} . Then $\omega : \mathcal{M} \rightarrow \Delta$ is a family of compact complex manifolds. If we choose a sufficiently fine locally finite open covering $\{U_j\}$ of M , we have $n+1$ holomorphic coordinates $\xi_j^\beta(z, t), \beta = 1, \dots, n$ and t on each $U_j \times \Delta$, and the map

$$\xi_j : (z, t) \rightarrow (\xi_j^1(z, t), \dots, \xi_j^n(z, t), t)$$

gives local complex coordinates of \mathcal{M} on $U_j \times \Delta$.

And we can think of σ on M as a C^∞ bivector on $M \times \Delta$. (more precisely, $\sigma \oplus 0$). We note that a bivector $(2,0)$ part Λ of $\sigma \in C^\infty(\wedge^2 T_M)$ on $M \times \Delta$ is holomorphic with respect to the complex structure \mathcal{M} , in other words with respect to coordinates systems $\xi_j(z,t)$, if and only if it satisfies

$$\bar{\partial}\sigma + [\sigma, \phi(t)] = 0$$

Since σ satisfies the equation by Lemma 2.15, Λ is a holomorphic bivector field on \mathcal{M} and Λ_t induces Poisson holomorphic structure on $M_{\phi(t)}$ for each t . If we write $\sigma = \sum_{\alpha,\beta=1}^n \sigma^{\alpha\beta}(z_j) \frac{\partial}{\partial z_j^\alpha} \wedge \frac{\partial}{\partial z_j^\beta}$ on $U_j \times \Delta$, then in new complex coordinate systems, it becomes $\Lambda = \sum_{p,q,\alpha,\beta=1}^n \sigma^{\alpha\beta}(z_j) \frac{\partial \xi_j^p}{\partial z_j^\alpha} \frac{\partial \xi_j^q}{\partial z_j^\beta} \frac{\partial}{\partial \xi_j^p} \wedge \frac{\partial}{\partial \xi_j^q}$. So we have a Poisson analytic family (\mathcal{M}, Λ) . For each t , we have

$$\Lambda_t = \sum_{p,q,\alpha,\beta=1}^n \sigma^{\alpha\beta}(z_j) \frac{\partial \xi_j^p(z_j, t)}{\partial z_j^\alpha} \frac{\partial \xi_j^q(z_j, t)}{\partial z_j^\beta} \frac{\partial}{\partial \xi_j^p} \wedge \frac{\partial}{\partial \xi_j^q}$$

On the other hand, Hitchin defines a holomorphic Poisson structure on $M_{\phi(t)}$ in the following way:

$$\sigma_t(f, g) = \sigma(\partial f, \partial g)$$

for f, g local holomorphic functions with respect to the complex structure at t . So we have $\sigma_t(\xi_j^p(z, t), \xi_j^q(z, t)) = \sum_{\alpha,\beta=1}^n \sigma^{\alpha\beta}(z_j) \frac{\partial \xi_j^p(z_j, t)}{\partial z_j^\alpha} \frac{\partial \xi_j^q(z_j, t)}{\partial z_j^\beta}$. This implies that $(\mathcal{M}_t, \Lambda_t) = (M_{\phi(t)}, \sigma_t)$. And since Λ does not depend on t , we have 0 in the Poisson direction under the Poisson Kodaira Spencer map $\varphi_0 : T_0\Delta \rightarrow HP^2(M, \sigma)$ by Theorem 2.11. More precisely $\varphi_0(\frac{\partial}{\partial t}) = (\sigma([\omega]), 0)$.

Chapter 3

Theorem of Existence for holomorphic Poisson structures

3.1 Theorem of existence

In this chapter, we prove theorem of existence for holomorphic Poisson deformations under the assumption (3.1) as an analogue of theorem of existence for deformations of complex structure.

3.1.1 Statement of the theorem

Before we state the theorem of existence, we discuss the assumption (3.1)¹ that we use in the proof of theorem of existence. Let (M, Λ) be compact holomorphic Poisson manifold. First we note that the differential operator $L = \bar{\partial} + [\Lambda, -]$ is elliptic and so we have operators L^*, H, G and $\square = LL^* + L^*L$.(See Appendix D)

we introduce the Hölder norms in the spaces $A^p = A^{0,p-1}(M, T) \oplus \cdots \oplus A^{0,0}(M, \wedge^p T)$. To do this, we fix a finite open covering $\{U_j\}$ of M such that (z_j) are coordinates on U_j . Let $\varphi \in A^p$,

$$\varphi = \sum_{r+s=p, s \geq 1} \varphi_{j\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(z) d\bar{z}_j^{\alpha_1} \wedge \cdots \wedge d\bar{z}_j^{\alpha_r} \wedge \frac{\partial}{\partial z_j^{\beta_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_j^{\beta_s}}$$

¹For my unfamiliarity of analysis, I do not know that we can relax the assumption

Let $k \in \mathbb{Z}, k \geq 0, \alpha \in \mathbb{R}, 0 < \alpha < 1$. Let $h = (h_1, \dots, h_{2n}), h_i \geq 0, \sum_{i=1}^{2n} h_i = |h|$ where $n = \dim M$. Then denote

$$D_j^h = \left(\frac{\partial}{\partial x_j^1} \right)^{h_1} \cdots \left(\frac{\partial}{\partial x_j^{2n}} \right)^{h_{2n}}, \quad z_j^\alpha = x_j^{2\alpha-1} + ix_j^{2\alpha}$$

Then the Hölder norm $\|\varphi\|_{k+\alpha}$ is defined as follows:

$$\|\varphi\|_{k+\alpha} = \max_j \left\{ \sum_{h, |h| \leq k} \left(\sup_{z \in U_j} |D_j^h \varphi_{j\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(z)| \right) + \sup_{y, z \in U_j, |h|=k} \frac{|D_j^h \varphi_{j\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(y) - D_j^h \varphi_{j\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(z)|}{|y - z|^\alpha} \right\},$$

where the sup is over all $\alpha_1, \dots, \alpha_r, \beta_1, \dots, \beta_s$

Our assumption is the following. For any $\varphi \in A^2$,

$$\|\varphi\|_{k+\alpha} \leq C(\|\square\varphi\|_{k-2+\alpha} + \|\varphi\|_0) \quad (3.1)$$

where $k \geq 2$, C is a constant which is independent of φ

$$\text{and } \|\varphi\|_0 = \max_{j, \alpha_1, \dots, \beta_s} \sup_{z \in U_j} |\varphi_{j\alpha_1 \dots \alpha_r \beta_1 \dots \beta_s}(z)|.$$

Now we state the theorem of existence of holomorphic Poisson deformations.

Theorem 3.2 (Theorem of Existence) *Let (M, Λ_0) be a compact holomorphic Poisson manifold satisfying (3.1) and suppose that $HP^3(M, \Lambda_0) = 0$. Then there exists a Poisson analytic family $(\mathcal{M}, \Lambda, B, \omega)$ with $0 \in B \subset \mathbb{C}^m$ satisfying the following conditions:*

1. $\omega^{-1}(0) = (M, \Lambda_0)$
2. $\varphi_0 : \frac{\partial}{\partial t} \rightarrow \left(\frac{\partial(M_t, \Lambda_t)}{\partial t} \right)_{t=0}$ with $(M_t, \Lambda_t) = \omega^{-1}(t)$ is an isomorphism of $T_0(B)$ onto $HP^2(M, \Lambda_0) : T_0B \xrightarrow{\rho_0} HP^2(M, \Lambda_0)$.

Assume that there is a Poisson analytic family $(\mathcal{M}, \Lambda, B, \omega)$ satisfying (1) and (2). Take a sufficiently small Δ with $0 \in \Delta \subset B$, and the C^∞ vector $(0, 1)$ -form $\varphi(t) = \sum_{\lambda=1}^n \varphi^\lambda(z, t) \frac{\partial}{\partial z_\lambda}$ and $\Lambda(t)$ on M defined by (2.2) and (2.5) defined by $(\mathcal{M}_\Delta, \Lambda_\Delta, \Delta, \omega)$. Then we have $[\Lambda(t), \Lambda(t)] = 0, \bar{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0, \bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$ and we have $\varphi(0) = 0, \Lambda(0) = \Lambda_0$. If we put

$$(\dot{\varphi}_\lambda, -\dot{\Lambda}_\lambda) = \left(\left(\frac{\partial \varphi(t)}{\partial t_\lambda} \right)_{t=0}, - \left(\frac{\partial \Lambda(t)}{\partial t_\lambda} \right)_{t=0} \right), \quad \lambda = 1, \dots, m,$$

Then $\{(\dot{\varphi}_1, -\dot{\Lambda}_1), \dots, (\dot{\varphi}_m, -\dot{\Lambda}_m)\}$ forms a basis of $HP^2(M, \Lambda_0)$.

Conversely, given $(\beta_\lambda, \pi_\lambda) \in A^{0,1}(M, T) \oplus A^{2,0}(M, \wedge^2 T)$ for $\lambda = 1, \dots, m$, such that $\{(\eta_1, \pi_1), \dots, (\eta_m, \pi_m)\}$ forms a basis of $HP^2(M, \Lambda_0)$, assume that there is a family $\{(\varphi(t), \Lambda(t)) | t \in \Delta\}$ of C^∞ vector $(0, 1)$ -forms $\varphi(t)$ and $(2, 0)$ vector $\Lambda(t)$ on M with $0 \in \Delta \subset \mathbb{C}^m$, which satisfy

1. $[\Lambda(t), \Lambda(t)] = 0$
2. $\bar{\partial}\Lambda(t) - [\Lambda(t), \varphi(t)] = 0$
3. $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$

and the initial conditions

$$\varphi(0) = 0, \Lambda(0) = \Lambda_0, \left(\left(\frac{\partial\varphi(t)}{\partial t_\lambda} \right)_{t=0}, - \left(\frac{\partial\Lambda(t)}{\partial t_\lambda} \right)_{t=0} \right) = (\eta_\lambda, \pi_\lambda), \quad \lambda = 1, \dots, m,$$

Since Δ is assumed to be sufficiently small, we may assume that $\varphi(t) = \sum_\lambda \sum_v \varphi_v^\lambda d\bar{z}_v \frac{\partial}{\partial z_\lambda}$ satisfies

$$\det(\delta_v^\lambda - \sum_\mu \varphi_v^\mu(t) \overline{\varphi_\mu^\lambda(t)}) \neq 0.$$

Therefore, by the Newlander-Nirenberg theorem([NN57],[Kod05] p.268), each $\varphi(t)$ determines a complex structure $M_{\varphi(t)}$ on M . And condition (2),(3) implies (2,0)-part $\Lambda(t)^{2,0}$ of $\Lambda(t)$ is a holomorphic Poisson structure on $M_{\varphi(t)}$. If the family $\{M_{\varphi(t)}, \Lambda(t)^{2,0}\}$ is a Poisson analytic family, it satisfies the conditions (1) and (2) in Theorem 3.2 by our assumption and Theorem 2.11. we construct such a family $\{(\varphi(t), \Lambda(t)) | t \in \Delta\}$ and then show that $\{(M_{\varphi(t)}, \Lambda(t)^{2,0})\}$ is a Poisson analytic family.(See 3.1.3)

Remark 3.3 *Constructing $\{\varphi(t), \Lambda(t)\}$ is equivalent to constructing $\{-\varphi(t), \Lambda(t)\}$. By replacing $\varphi(t)$ by $-\varphi(t)$, we construct $\varphi(t)$ satisfying*

1. $[\Lambda(t), \Lambda(t)] = 0$
2. $\bar{\partial}\Lambda(t) + [\Lambda(t), \varphi(t)] = 0$
3. $\bar{\partial}\varphi(t) + \frac{1}{2}[\varphi(t), \varphi(t)] = 0$

and the initial conditions

$$\varphi(0) = 0, \Lambda(0) = \Lambda_0, \left(- \left(\frac{\partial\varphi(t)}{\partial t_\lambda} \right)_{t=0}, - \left(\frac{\partial\Lambda(t)}{\partial t_\lambda} \right)_{t=0} \right) = (-\eta_\lambda, \pi_\lambda), \quad \lambda = 1, \dots, m,$$

And we note that (1), (2), (3) are equivalent to

$$\bar{\partial}(\varphi(t) + \Lambda(t)) + \frac{1}{2}[\varphi(t) + \Lambda(t), \varphi(t) + \Lambda(t)] = 0$$

We construct $\alpha(t) = \varphi(t) + \Lambda(t)$ in the following section.

3.1.2 Construction of $\alpha(t) = \varphi(t) + \Lambda(t)$

We use the Kuranishi methods. We need the following lemmas.

Lemma 3.4 For $\varphi, \psi \in A^2$, we have $\|[\varphi, \psi]\|_{k+\alpha} \leq C\|\varphi\|_{k+1+\alpha}\|\psi\|_{k+1+\alpha}$, where C is independent of φ and ψ .

Lemma 3.5 For $\varphi \in A^2$, we have $\|G\varphi\|_{k+\alpha} \leq C\|\varphi\|_{k-2+\alpha}$, $k \geq 2$, where C depends only on k and α , not on φ .

Proof. This follows from the assumption 3.1. See [MK06] p.160 Proposition 2.3. ■

Let us now construct the $\varphi(t)$ and $\Lambda(t)$. We want to construct $\alpha(t) := \varphi(t) + \Lambda(t) = \Lambda_0 + \sum_{\mu=1}^{\infty} \varphi_{\mu}(t) + \Lambda_{\mu}(t)$, where

$$\varphi_{\mu}(t) + \Lambda_{\mu}(t) = \sum_{v_1 + \dots + v_m = \mu} (\varphi_{v_1 \dots v_m} + \Lambda_{v_1 \dots v_m}) t_1^{v_1} \dots t_m^{v_m}$$

where $\varphi_{v_1 \dots v_m} + \Lambda_{v_1 \dots v_m} \in A^{0,1}(M, T) \oplus A^{0,0}(M, \wedge^2 T)$ such that

$$\bar{\partial}\alpha(t) + \frac{1}{2}[\alpha(t), \alpha(t)] = 0 \tag{3.6}$$

$$\alpha_1(t) = \varphi_1(t) + \Lambda_1(t) = \sum_{v=1}^m (\eta_v + \pi_v) t_v, \tag{3.7}$$

where $\{\eta_v + \pi_v\}$ is a basis for $\mathbb{H}^2 \cong HP^2(M, \Lambda_0)$.

Let $\beta(t) = \alpha(t) - \Lambda_0$. Then (5.2.3) is equivalent to

$$L\beta(t) + \frac{1}{2}[\beta(t), \beta(t)] = 0.$$

Consider the equation

$$(*)\beta(t) = \beta_1(t) - \frac{1}{2}L^*G[\beta(t), \beta(t)],$$

where $\beta_1(t) = \alpha_1(t)$. (*) has a unique formal power series solution $\beta(t)$. Indeed,

$$\begin{aligned} \beta_2(t) &= -\frac{1}{2}L^*G[\beta_1(t), \beta_1(t)] \\ \beta_3(t) &= -\frac{1}{2}L^*G([\beta_1(t), \beta_2(t)] + [\beta_2(t), \beta_1(t)]) \\ \beta_{\mu}(t) &= -\frac{1}{2}L^*G\left(\sum_{\lambda=1}^{\mu-1} [\beta_{\lambda}(t), \beta_{\mu-\lambda}(t)]\right) \end{aligned}$$

Proposition 3.8 For small $|t|$, $\beta(t) = \sum_{\mu=1}^{\infty} \beta_{\mu}(t)$ converges in the norm $\|\cdot\|_{k+\alpha}$.

Proof. See [MK06] p.162 Proposition 2.4. ■

Proposition 3.9 The $\beta(t)$ satisfies $L\beta(t) + \frac{1}{2}[\beta(t), \beta(t)] = 0$ if and only if $H[\beta(t), \beta(t)] = 0$, where $H : A^3 = A^{0,2}(M, T) \oplus A^{0,1}(M, \wedge^2 T) \oplus A^{0,0}(M, \wedge^3 T) \rightarrow \mathbb{H}^3 \cong HP^3(M, \Lambda_0)$ is the orthogonal projection to the harmonic subspace of A^3 .

Proof. (\Rightarrow) $L\beta(t) = -\frac{1}{2}[\beta(t), \beta(t)]$. If we take H on both sides, we have

$$0 = HL\beta(t) = -\frac{1}{2}H[\beta(t), \beta(t)]$$

(\Leftarrow) Let $H[\beta(t), \beta(t)] = 0$.

$$\text{Set } \psi(t) = L\beta(t) + \frac{1}{2}[\beta(t), \beta(t)] = \bar{\partial}\beta(t) + [\Lambda_0, \beta(t)] + \frac{1}{2}[\beta(t), \beta(t)]$$

$$\begin{aligned} 2\psi(t) &= 2L\beta(t) + [\beta(t), \beta(t)] \\ &= -LL^*G[\beta(t), \beta(t)] + [\beta(t), \beta(t)] \\ &= -LL^*G[\beta(t), \beta(t)] + \square G[\beta(t), \beta(t)] \\ &= -LL^*G[\beta(t), \beta(t)] + (LL^* + L^*L)G[\beta(t), \beta(t)] \\ &= L^*LG[\beta(t), \beta(t)] = L^*GL[\beta(t), \beta(t)] \\ &= 2L^*G[L\beta(t), \beta(t)] \end{aligned}$$

since $(A[1], L, [-, -])$ is a differential graded Lie algebra (See Proposition C.14). So we have $\psi(t) = L^*G[\psi(t), \beta(t)]$. And by Lemma 3.4 and Lemma 3.5, we have

$$\begin{aligned} \|\psi(t)\|_{k+\alpha} &= \|L^*G[\psi(t), \beta(t)]\|_{k+\alpha} \\ &\leq C_1 \|G[\psi(t), \beta(t)]\|_{k+1+\alpha} \\ &\leq C_1 C_{k,\alpha} \|[\psi(t), \beta(t)]\|_{k-1+\alpha} \\ &\leq C_1 C_{k,\alpha} C \|\psi(t)\|_{k+\alpha} \|\beta(t)\|_{k+\alpha} \end{aligned}$$

Choose $|t|$ so small that $\|\beta(t)\|_{k+\alpha} C_1 C_{k,\alpha} C < 1$. Then we get the contradiction $\|\psi(t)\|_{k+\alpha} < \|\psi(t)\|_{k+\alpha}$ unless $\psi(t) = 0$ for all small t since $\beta(t)$ converges and $\beta(0) = 0$. ■

Proposition 3.10 $\alpha(t) = \varphi(t) + \Lambda(t)$ is C^∞ in (z, t) and holomorphic in t .

Proof. See [MK06] p.163 Proposition 2.6. ■

With the assumption $HP^3(M, \Lambda_0) = 0$, $\beta(t)$ solves the integrability condition for small $|t|$, where $t \in \Delta_\epsilon$.

3.1.3 Construction of a Poisson analytic family

We have constructed a family $\{(\varphi(t), \Lambda(t)) | t \in \Delta_\epsilon\}$ of C^∞ vector $(0, 1)$ -forms $\varphi(t) = \sum_{\lambda=1}^n \sum_{v=1}^n \varphi_v^\lambda(z, t) d\bar{z}^v \frac{\partial}{\partial z^\lambda}$ and C^∞ $(2, 0)$ bivector $\Lambda(t) = \sum_{\alpha, \beta=1}^n g_{\alpha\beta}(z, t) \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta}$ satisfying the integrability condition $[\Lambda(t), \Lambda(t)] = 0$, $\bar{\partial}\Lambda(t) = [\Lambda(t), \varphi(t)]$, $\bar{\partial}\varphi(t) = \frac{1}{2}[\varphi(t), \varphi(t)]$ and the initial conditions $\varphi(0) = 0$, $\Lambda(0) = \Lambda_0$, $\left(\frac{\partial\varphi(t)}{\partial t_\lambda}\right)_{t=0}, -\left(\frac{\partial\Lambda(t)}{\partial t_\lambda}\right)_{t=0} = (\beta_\lambda, \pi_\lambda)$, $\lambda = 1, \dots, m$, where $\varphi_v^\lambda(z, t)$ and $g_{\alpha\beta}(z, t)$ are C^∞ functions of $z^1, \dots, z^n, t_1, \dots, t_m$ and holomorphic in t_1, \dots, t_m .

Each $(\varphi(t), \Lambda(t))$ determines a holomorphic Poisson structure $(M_{\varphi(t)}, \Lambda(t))$ on M . In order to show that $\{(M_{\varphi(t)}, \Lambda(t)) | t \in \Delta_\epsilon\}$ is a Poisson analytic family, we consider $\varphi = \varphi(t)$ as a vector $(0, 1)$ -form on the complex manifold $M \times \Delta_\epsilon$ and $\Lambda = \Lambda(t)$ as a $(2, 0)$ bivector on $M \times \Delta_\epsilon$. Namely, we consider $\varphi(t)$ as

$$\varphi = \varphi(t) = \sum_{\lambda=1}^n \left(\sum_{v=1}^n \varphi_v^\lambda d\bar{z}^v + \sum_{\mu=1}^m \varphi_{n+\mu}^\lambda d\bar{t}_\mu \right) \frac{\partial}{\partial z^\lambda} + \sum_{\mu=1}^m \varphi^{n+\mu} \frac{\partial}{\partial t_\mu}$$

with $\varphi^{t+\mu} = \varphi_{n+\mu}^\lambda = 0$, $\mu = 1, \dots, m$. Then since $\varphi_v^\lambda = \varphi_v^\lambda(z, t)$ are holomorphic in t_1, \dots, t_m (Proposition 3.10), we have $\frac{\partial\varphi_v^\lambda}{\partial t_\mu} = 0$ in

$$\bar{\partial}\varphi = \sum_{\lambda, v=1}^n \left(\sum_{\beta=1}^n \frac{\partial\varphi_v^\lambda}{\partial \bar{z}^\beta} d\bar{z}^\beta + \sum_{\mu=1}^m \frac{\partial\varphi_v^\lambda}{\partial \bar{t}_\mu} d\bar{t}_\mu \right) \wedge d\bar{z}^v \frac{\partial}{\partial z^\lambda}$$

Similarly since $g_{\alpha\beta}(z, t)$ is holomorphic in t_1, \dots, t_m (Proposition 3.10), we have $\frac{\partial g_{\alpha\beta}}{\partial t_\mu} = 0$ in

$$\bar{\partial}\Lambda = \sum_{\alpha, \beta} \left(\sum_{v=1}^n \frac{\partial g_{\alpha\beta}}{\partial \bar{z}^v} d\bar{z}^v + \sum_{\mu=1}^m \frac{\partial g_{\alpha\beta}}{\partial \bar{t}_\mu} d\bar{t}_\mu \right) \frac{\partial}{\partial z^\alpha} \wedge \frac{\partial}{\partial z^\beta}$$

By $\bar{\partial}\varphi(t)$ we denote the exterior differential of $\varphi(t)$ as a vector $(0, 1)$ -form on M with fixed t . Then $\bar{\partial}\varphi$ coincides with $\bar{\partial}\varphi(t)$ and we obtain $[\varphi, \varphi] = [\varphi(t), \varphi(t)]$. Similarly $\bar{\partial}\Lambda(t)$ coincides with $\bar{\partial}\Lambda$ and we obtain $[\Lambda, \varphi] = [\Lambda(t), \varphi(t)]$ and $[\Lambda, \Lambda] = [\Lambda(t), \Lambda(t)]$. Therefore as a C^∞ vector $(0, 1)$ -form on $M \times \Delta_\epsilon$ and C^∞ $(2, 0)$ bivector on $M \times \Delta_\epsilon$, φ and Λ satisfies $\bar{\partial}\varphi = \frac{1}{2}[\varphi, \varphi]$, $\bar{\partial}\Lambda = [\Lambda, \varphi]$, and $[\Lambda, \Lambda] = 0$.

Then by the Newlander-Nirenberg theorem ([NN57], [Kod05] p.268), φ defines a complex structure \mathcal{M} on $M \times \Delta_\epsilon$ and $\bar{\partial}\Lambda = [\Lambda, \varphi]$, and $[\Lambda, \Lambda] = 0$ imply that $(2, 0)$ -part $\Lambda^{2,0}$ of Λ defines a holomorphic Poisson structure $(\mathcal{M}, \Lambda^{2,0})$. If we choose a sufficiently fine locally finite open covering $\{U_j\}$ of M , and take a sufficiently small Δ_ϵ , we have C^∞ functions $\xi_j^\beta(z, t)$, $\beta = 1, \dots, m+n$ on each $U_j \times \Delta_\epsilon$, and the map

$$\xi_j : (z, t) \rightarrow (\xi_j^1(z, t), \dots, \xi_j^{m+n}(z, t))$$

gives local complex coordinates of \mathcal{M} on $U_j \times \Delta_\epsilon$, and $\xi_j^{n+\mu}(z, t) = t_\mu$ for $\mu = 1, \dots, m$.

Then we have

$$\xi_j : (z, t) \rightarrow (\xi_j^1(z, t), \dots, \xi_j^n(z, t), t_1, \dots, t_m).$$

Therefore

$$\omega : (\xi_j^1(z, t), \dots, \xi_j^n(z, t), t_1, \dots, t_m) \rightarrow (t_1, \dots, t_m)$$

is a holomorphic map of \mathcal{M} onto Δ_ϵ . For each $t \in \Delta_\epsilon$, $\omega^{-1}(t)$ is a holomorphic Poisson manifold whose system of local complex coordinates is given by $\{\xi_j^1(z, t), \dots, \xi_j^n(z, t)\}$ and a holomorphic Poisson structure is given by $(2, 0)$ -part $\Lambda(t)^{2,0}$ of $\Lambda(t)$. So we have $\omega^{-1}(t) = (M_{\varphi(t)}, \Lambda(t)^{2,0})$. Thus $\{(M_{\varphi(t)}, \Lambda(t)^{2,0}) | t \in \Delta_\epsilon\}$ forms a Poisson analytic family $(\mathcal{M}, \Lambda^{2,0}, \Delta_\epsilon, \omega)$.

Example 3.11 Let $U_i = \{[z_0, z_1, z_2] | z_i \neq 0\}$ $i = 0, 1, 2$ be an open cover of complex projective plane $\mathbb{P}_{\mathbb{C}}^2$. Let $x = \frac{z_1}{z_0}$ and $w = \frac{z_2}{z_0}$ be coordinates on U_0 . Then the holomorphic Poisson structures on U_0 are parametrized by $t = (t_1, \dots, t_{10}) \in \mathbb{C}^{10}$

$$(t_1 + t_2x + t_3w + t_4x^2 + t_5xw + t_6w^2 + t_7x^3 + t_8x^2w + t_9xw^2 + t_{10}w^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$$

This parametrizes the whole holomorphic Poisson structures on $\mathbb{P}_{\mathbb{C}}^2$. (See [HX11] Proposition 2.2). Let $\Lambda_0 = x \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$ be the holomorphic Poisson structure on $\mathbb{P}_{\mathbb{C}}^2$. Then $HP^2(\mathbb{P}_{\mathbb{C}}^2, \Lambda_0) = 5$, $HP^3(\mathbb{P}_{\mathbb{C}}^2, \Lambda_0) = 0$. (See [HX11] Example 3.5) and $w^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$, $x^3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$, $x^2w \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$, $xw^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$ and $w^3 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$ are the representatives of the cohomology classes consisting of the basis of $HP^2(\mathbb{P}_{\mathbb{C}}^2, \Lambda_0)$. Let $t = (t_1, t_2, t_3, t_4, t_5) \in \mathbb{C}^5$. Let $\Lambda(t) = (t_1w^2 + x + t_2x^3 + t_3x^2w + t_4xw^2 + t_5w^3) \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial w}$ be the holomorphic Poisson structure on $\mathbb{P}_{\mathbb{C}}^2 \times \mathbb{C}^5$. Then $(\mathbb{P}_{\mathbb{C}}^2 \times \mathbb{C}^5, \Lambda(t), \mathbb{C}^5, \omega)$, where ω is the natural projection, is a Poisson analytic family with $\omega^{-1}(0) = (\mathbb{P}_{\mathbb{C}}^2, \Lambda_0)$. Since the complex structure does not change in the family, the Poisson Kodaira Spencer map is an isomorphism. Hence the family satisfies the theorem of existence for $(\mathbb{P}_{\mathbb{C}}^2, \Lambda_0)$.

3.2 A concept of Kuranishi family in holomorphic Poisson category

By following the lecture notes of Kuranishi [Kur71], we extend the definition of complex analytic family over a complex space to Poisson analytic family over a complex space and raise the question of existence of a complete family without assumption $HP^3(M, \Lambda) = 0$ where (M, Λ) is a holomorphic Poisson manifold.

In this section \mathbf{M} is a real C^∞ compact manifold. And an analytic set S is by definition a subset of a domain D (which is called the ambient space of S) defined by zero locus of finitely many holomorphic functions on D . A map $f : S \rightarrow S'$ between two analytic sets is called analytic if for each $s \in S$, f can be extended to a complex analytic map from an open neighborhood of s in D into the ambient space of S' .

Definition 3.12 *Let S be an analytic set. By a C^∞ family of holomorphic Poisson charts of \mathbf{M} with a parameter in S we mean*

1. a C^∞ map

$$\tilde{z} : U \times S' \rightarrow \mathbb{C}^n$$

where U (resp S') is an open subset of \mathbf{M} (resp. of S), such that the map $z^t : U \rightarrow \mathbb{C}^n$ defined by $z^t(p) = \tilde{z}(p, t)$ is a holomorphic complex chart of \mathbf{M} for each $t \in S'$.

2. A C^∞ -complex bivector field Λ of the form $\sum_{i,j} g_{ij}(x, t) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ on $M \times S^2$ such that $[\Lambda, \Lambda] = 0$ for each $t \in S$ and $(2, 0)$ -part $\Lambda_t^{2,0}$ of the restriction Λ_t of Λ on $\mathbf{M} \times t$ is a holomorphic bivector field with respect to the complex structure defined by z^t .

Let $\tilde{z}^\sharp : U \times S' \rightarrow \mathbb{C}^n \times S'$ defined by $\tilde{z}^\sharp = (\tilde{z}(p, t), t)$.

If \tilde{w} is another such family with domain V and over S'' , we mean by change of charts from \tilde{z} to \tilde{w} the map $\tilde{w}^\sharp \circ \tilde{z}^{\sharp-1}$ of $\tilde{z}^\sharp((U \cap V) \times (S' \cap S''))$ to $\tilde{w}^\sharp((U \cap V) \times (S' \cap S''))$. The domain and the image of the change of charts are open subsets of $\mathbb{C}^n \times S$, and hence we may ask if the change is complex analytic.

Now we can define the notion of a Poisson analytic family of holomorphic Poisson structures on \mathbf{M}

²Here we mean by a C^∞ bivector field Λ on $M \times S$ that for each $s \in S$, $g_{ij}(x, s)$ can be extended to be C^∞ functions on $M \times U$ where U is a neighborhood of s in the ambient space of S

Definition 3.13 Let S be an analytic set. By a Poisson analytic family of holomorphic Poisson structures on \mathbf{M} with a parameter in S we mean

1. A C^∞ complex bivector field Λ on $M \times S$ of the form $\sum_{i,j} g_{ij}(x,s) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$ with $[\Lambda, \Lambda] = 0$ for each $s \in S$.

2. a collection $(\tilde{\Phi}, \Lambda)$ of C^∞ families of holomorphic Poisson charts of \mathbf{M} with a parameter in S satisfying

(a) If $\tilde{z}, \tilde{w} \in \tilde{\Phi}$, then the change of charts from \tilde{z} to \tilde{w} is complex analytic.

(b) for any p in \mathbf{M} and any t in S there is a \tilde{z} in $\tilde{\Phi}$ with domain U and S' such that $(p, t) \in U \times S'$.

(c) If \tilde{u} is a C^∞ family of Poisson holomorphic charts of \mathbf{M} with a parameter in S and if the change of charts from \tilde{z} to \tilde{u} is complex analytic for any \tilde{z} in $\tilde{\Phi}$, then \tilde{u} is in $\tilde{\Phi}$.

If this is the case, for each fixed t , $\Phi_t = \{z^t : \tilde{z} \in \tilde{\Phi}\}$ is a chart covering of a holomorphic Poisson structure, say $(M_t, \Lambda_t^{2,0})$ on \mathbf{M} . $(M_t, \Lambda_t^{2,0})$ is called the holomorphic Poisson structure in $\tilde{\Phi}$ over t . Thus we have a collection $\{(M_t, \Lambda_t) : t \in S\}$ of holomorphic Poisson structures on \mathbf{M} .

Let B be an analytic set. Denote by τ an analytic map $B \rightarrow S$ and let $\Lambda \circ \tau = \sum_{i,j} g_{ij}(x, h(s)) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. If \tilde{z} is a C^∞ family of holomorphic Poisson chart of \mathbf{M} with domain U and over $S' \subset S$, then $\tilde{z} \circ (id \times \tau) : U \times B' \rightarrow \mathbb{C}^n$ where $B' = \tau^{-1}(S')$ and id is the identity map of U , is a C^∞ family of holomorphic Poisson charts of \mathbf{M} with a parameter in B with respect to $\Lambda \circ \tau = \sum_{i,j} g_{ij}(x, h(s)) \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j}$. It is clear that the collection $\{\tilde{z} \circ (id \times \tau) : \tilde{z} \in \tilde{\Phi}\}$ can be enlarged to a unique Poisson analytic family of holomorphic Poisson structures $(\tilde{\Phi} \circ \tau, \Lambda \circ \tau)$ on \mathbf{M} .

Definition 3.14 The above family $(\tilde{\Phi} \circ \tau, \Lambda \circ \tau)$ is called the Poisson analytic family induced from $(\tilde{\Phi}, \Lambda)$ by τ .

Let $(\tilde{\Phi}, \Lambda)$ and $(\tilde{\Psi}, \Lambda')$ be Poisson analytic family over an analytic set B . Denote by \mathbf{f} a family of diffeomorphism of \mathbf{M} parametrized by B , say $\{f^b : b \in B\}$

Definition 3.15 We say that \mathbf{f} induces an isomorphism from $(\tilde{\Phi}, \Lambda)$ to $(\tilde{\Psi}, \Lambda')$ over the identity map of B if the following conditions are satisfied:

1. for $\tilde{w} : V \times B' \rightarrow \mathbb{C}^n$ in $\tilde{\Psi}$, let U, B'' be open subsets such that $f^b(U) \subset V$ for all $b \in B''$.

Then $(p, b) \in U \times B'' \mapsto \tilde{w}(f^b(p), b) \in \mathbb{C}^n$ is an element of $\tilde{\Phi}$.

2. the map $(p, b) \in M \times B \mapsto f^b(p) \in \mathbf{M}$ is a C^∞ map, i.e. on a neighborhood of each point it is C^∞

3. let $F : \mathbf{M} \times B \rightarrow \mathbf{M} \times B$ defined by $(x, t) \rightarrow (f^t(x), t)$. Then $F_*\Lambda = \Lambda'$. In particular we have

$$f_*^b \Lambda_b^{2,0} = \Lambda_b'^{2,0} \text{ for each } b \in B.$$

Definition 3.16 Let (M, Λ_0) be a holomorphic Poisson structure on \mathbf{M} . We say that $(\tilde{\Phi}, \Lambda)$ is a Poisson analytic family of deformations of (M, Λ_0) over (S, s_0) if the holomorphic Poisson structure in $\tilde{\Phi}$ over s_0 is (M, Λ_0) .

Definition 3.17 A Poisson analytic family $(\tilde{\Phi}, \Lambda)$ of deformations of (M, Λ_0) over (S, s_0) is called complete at s_0 if for any pointed analytic set (B, b_0) and any Poisson analytic family $(\tilde{\Psi}, \Lambda')$ of deformations of (M, Λ) over (B, b_0) , we can find an open neighborhood B' of b_0 and a complex analytic map $\tau : (B', b_0) \rightarrow (S, s_0)$ such that $(\tilde{\Psi}|_{B'}, \Lambda'|_{B'})$ is isomorphic to the family $(\tilde{\Phi} \circ \tau, \Lambda \circ \tau)$.

The following problem is an analogue of Kuranishi's completeness theorem.

Problem 6 Let (M, Λ_0) be a compact holomorphic Poisson manifold. Then does there a complete Poisson analytic family of deformations of (M, Λ_0) exist?^{3 4}

³For my unfamiliarity of analysis, I could not access to this problem.

⁴ Let (M, Λ_0) be a compact holomorphic Poisson manifold. We fix a Hermitian metric on M and define L^*, \square, G, \dots , and so forth. Let $\{\eta_v + \pi_v | v = 1, \dots, m\}$ be a base for $\mathbb{H}^2 \cong H^2(M, \Lambda_0)$. Assume that we have a unique convergent power series solution $\beta(t)$ of

$$\beta(t) = \beta_1(t) + \frac{1}{2}L^*G[\beta(t), \beta(t)],$$

where $\beta_1(t) = \sum_{v=1}^m (\eta_v + \pi_v)t_v$. $\beta(t) = \alpha(t) - \Lambda_0$ satisfies

$$L\beta(t) + \frac{1}{2}[\beta(t), \beta(t)] = 0$$

if and only if $H[\beta(t), \beta(t)] = 0$. Let $\{\beta_\lambda | \lambda = 1, \dots, r\}$ be an orthonormal base of \mathbb{H}^3 and let $(-, -)$ be the inner product in $A^{0,0}(M, \wedge^3 T) \oplus A^{0,1}(M, \wedge^2 T) \oplus A^{0,2}(M, T)$. Then

$$H[\beta(t), \beta(t)] = \sum_{\lambda=1}^r ([\beta(t), \beta(t)], \beta_\lambda)\beta_\lambda$$

Hence $H[\beta(t), \beta(t)] = 0$ if and only if $([\beta(t), \beta(t)], \beta_\lambda) = 0$ for $\lambda = 1, \dots, r$. Since $\beta(t)$ is a power series in t so is $([\beta(t), \beta(t)], \beta_\lambda) = b_\lambda(t)$. Thus $b_\lambda(t)$ is holomorphic in t for $\lambda = 1, \dots, r$ and $|t|$ small ($|t| < \epsilon$). Also $b_\lambda(0) = 0$. Define an analytic set S as follows:

$$S = \{t | |t| < \epsilon, b_\lambda(t), \lambda = 1, \dots, r\}$$

Lastly, we pose some natural questions I can not answer at this stage.

Problem 7 *Can we establish the upper-semicontinuity theorem in a Poisson analytic family? (See [Kod05] page 200)*

Problem 8 *Let (\mathcal{M}, B, ω) be a complex analytic family. Let $b \in B$ and $\omega^{-1}(b) = M_b$ with a holomorphic Poisson structure Λ_b . What is the conditions or obstructions for the following?*

*“There exists an open neighborhood U of b in B such that $(\mathcal{M}|_U, U, \omega)$ can be extended to be a Poisson analytic family $(\mathcal{M}|_U, \Lambda, U, \omega)$ such that $\omega^{-1}(b) = (M_b, \Lambda_b)$ ”.*⁵

Problem 9 *Can we establish the stability theorem for holomorphic Poisson submanifolds in the holomorphic Poisson category? ([Kod63])*

Then S is an analytic subset of B_ϵ containing the origin. I believe that this would be the base space of Kuranishi family for holomorphic Poisson deformations of M .

⁵This problem is related to the operator $L = \bar{\partial} + [\Lambda, -]$

Part II

Infinitesimal Poisson deformations and Universal Poisson deformations of compact holomorphic Poisson manifolds

In the part II of the thesis, we present infinitesimal deformations of a compact holomorphic Poisson manifold (X, Λ_0) over an artinian local \mathbb{C} -algebra (R, \mathfrak{m}) with residue \mathbb{C} . We extend the method of [Ran00] to show that given an infinitesimal Poisson deformation of (X, Λ_0) , as in the case of the part I of the thesis, we can canonically associate an element $\phi + \Lambda \in (A^{0,0}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ satisfying the Maurer-Cartan equation $L(\phi + \Lambda) + \frac{1}{2}[\phi + \Lambda, \phi + \Lambda] = 0$ of the differential graded Lie algebra $\mathfrak{g} = \bigoplus_i g_i = (\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -])$. By using the language of functors of Artin rings, we show that the differential graded Lie algebra \mathfrak{g} controls infinitesimal Poisson deformations of (X, Λ_0) . In other words, we establish the following theorem.

Theorem 10 *Let (X, Λ_0) be a compact holomorphic Poisson manifold. Then the Poisson deformation functor $PDef_{(X, \Lambda_0)}$ is controlled by the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -])$. In other words, we have an isomorphism of two functors*

$$Def_{\mathfrak{g}} \cong PDef_{(X, \Lambda_0)}$$

We study universal Poisson deformation of a compact holomorphic Poisson manifold (X, Λ_0) with $HP^1(X, \Lambda_0) = 0$. Based on the method of [Ran00], we explicitly construct a n -th order universal Poisson deformation space P_n^u over an artinian \mathbb{C} -algebra $(R_n^u, \mathfrak{m}_n^u)$ with exponent n (i.e $\mathfrak{m}_n^{u, n+1} = 0$) such that any infinitesimal Poisson deformation of (X, Λ_0) over an artinian local \mathbb{C} -algebra (R, \mathfrak{m}) with exponent n (i.e $\mathfrak{m}^{n+1} = 0$) can be induced from the n -th order universal Poisson deformation space via base change from a canonical ring homomorphism $r : R_n^u \rightarrow R$ up to equivalence. By taking the limit, we have a universal Poisson formal space. The main ingredient of universal Poisson deformation is the Jacobi complex or Quillen standard complex associated with the differential graded Lie algebra $\mathfrak{g} = \bigoplus_i g_i = (\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T), L := \bar{\partial} + [\Lambda_0, -], [-, -])$. The base ring of an n -th universal Poisson deformation is given by $R_n^u := \mathbb{C} \oplus \mathfrak{m}_n^u$, where $\mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^*$ which is the dual of 0-th Jacobi cohomology group associated to \mathfrak{g} . For given an infinitesimal Poisson deformation, the associated element $\alpha := \phi + \Lambda \in (A^{0,0}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ gives an element $[\epsilon(\alpha)]$ in $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ which is the 0-th Jacobi cohomology group associated with the differential graded Lie algebra $\mathfrak{g} \otimes \mathfrak{m}$. The element $[\epsilon(\alpha)]$ gives a ring homomorphism $r : R_n^u \rightarrow R$. We will present the full detail of the construction of Jacobi complex since we need actual computations for our main theorem⁶ in the following:

⁶Similar result on universal Poisson deformation of a Poisson algebra is proved in [GK04] (Theorem 1.10). More precisely, for a Poisson algebra A with $HP^1(A)$ and $HP^2(A)$ a finite-dimensional vector space over \mathbb{C} , there is an universal formal Poisson deformation of the algebra A . We also prove in the part III of the thesis that the Poisson deformation functor $PDef_{(X, \Lambda_0)}$ is prorepresentable when (X, Λ_0) is a smooth projective Poisson scheme with $HP^1(X, \Lambda_0) = 0$.

Theorem 11 *Let (X, Λ_0) be a compact holomorphic Poisson manifold with $HP^1(X, \Lambda_0) = 0$ and J be the Jacobi complex associated with the differential graded Lie algebra*

$$\mathfrak{g} = \bigoplus_i g_i = \left(\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T), L := \bar{\partial} + [\Lambda_0, -], [-, -] \right)$$

where $[-, -]$ is the Schouten bracket. Then

1. For each $n \geq 1$, $R_n^u = \mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$ is a local artinian \mathbb{C} -algebra with the residue \mathbb{C} in a canonical way. The maximal ideal of R_n^u is given by $\mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^*$ and have exponent n (which means $\mathfrak{m}_n^{u, n+1} = 0$).
2. There is a n -th order universal Poisson deformation P_n^u of (X, Λ_0) over R_n^u in the following sense: for any artinian local \mathbb{C} -algebra (R, \mathfrak{m}) of exponent n (which means $\mathfrak{m}^{n+1} = 0$) and infinitesimal Poisson deformation P of (X, Λ_0) over R , there is a canonical homomorphism $r : R_n^u \rightarrow R$ and an isomorphism of Poisson analytic spaces over R

$$P/R \xrightarrow{\sim} r^* P_n^u := P_n^u \times_{\text{Spec}(R_n^u)} \text{Spec}(R);$$

3. For each $n \geq 1$, P_n^u/R_n^u fit together to form a direct system with limit, which give an universal formal Poisson doformation $\hat{P}^u/\hat{R}^u := \varinjlim_n P_n^u/R_n^u$ of (X, Λ_0) over \hat{R}^u in the following sense: if $\hat{R} = \varprojlim_n R_n$ is a complete local noetherian \mathbb{C} -algebra and $\hat{P}/\hat{R} = \varinjlim_n P_n/R_n$, then $\hat{r} = \varprojlim_n r_n : \hat{R}^u \rightarrow \hat{R}$ exists and $\hat{P}/\hat{R} \cong \hat{r}^*(\hat{P}^u/\hat{R}^u) := \hat{P}^u \times_{\text{Spec}(\hat{R}^u)} \text{Spec}(\hat{R})$.

In chapter 4, we present the construction of n -th Jacobi complex or Quillen standard complex associated to a differential graded Lie algebra \mathfrak{g} . We show that we can canonically define a local artinian \mathbb{C} -algebra structure on $\mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$ with residue \mathbb{C} and exponent n , where $\mathbb{H}^0(J_n(\mathfrak{g}))^*$ is the dual space of 0-th Jacobi cohomology group $\mathbb{H}^0(J_n(\mathfrak{g}))$ associated with the Jacobi complex $J_n(\mathfrak{g})$. We also describe a morphic element in $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ which gives an \mathbb{C} -algebra homomorphism $\mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow R$, where (R, \mathfrak{m}) is a local artinian \mathbb{C} -algebra with residue \mathbb{C} and exponent n .

In chapter 5, we study infinitesimal Poisson deformations of compact holomorphic Poisson manifolds. We define infinitesimal deformations of compact holomorphis Poisson manifolds over local artinian \mathbb{C} -algebras with residue \mathbb{C} which is an infinitesimal version of Poisson analytic families. We

deduce the same integrability condition as in the Part I of the thesis. The idea is essentially same to the Part I of the thesis. However, we use the infinitesimal language.

In chapter 6, we show that the differential graded Lie algebra \mathfrak{g} controls infinitesimal Poisson deformations in the language of functor of Artin rings. We complete the proof of Theorem 11 on universal Poisson deformations.

Chapter 4

Jacobi complex

We present the full details of the construction of the Jacobi complex or Quillen standard complex associated with a differential graded Lie algebra since we need actual computations for our infinitesimal Poisson deformations. See also [HS97] 2.2 Quillen standard complex.

4.1 Preliminaries

Let $\mathfrak{g} = \bigoplus_{i \geq 0} g_i$ be a graded complex with a differential d where g_i is a vector space over a field \mathbb{C} . In other words, we have the following complex $\mathfrak{g} : g_0 \xrightarrow{d} g_1 \xrightarrow{d} g_2 \xrightarrow{d} \dots$.

Definition 4.1 *The symmetric algebra of a graded complex (\mathfrak{g}, d) are defined as a graded complex $S(\mathfrak{g}) = T(\mathfrak{g})/I$, where $T(\mathfrak{g}) = \sum_{n \geq 0} \mathfrak{g}^{\otimes n}$ is the tensor algebra of \mathfrak{g} and I is the two sided ideal generated by elements of the form $a \otimes b - (-1)^{|a||b|} b \otimes a$ where a, b are homogeneous elements of \mathfrak{g} . We denote $\overline{S(\mathfrak{g})} = \overline{T(\mathfrak{g})}/I$, where $\overline{T(\mathfrak{g})} = \sum_{n \geq 1} \mathfrak{g}^{\otimes n}$. We will denote by $x_1 \odot \dots \odot x_n$ the image of $x_1 \otimes \dots \otimes x_n$.*

Definition 4.2 *The exterior algebra of a graded vector space \mathfrak{g} are defined as $\bigwedge \mathfrak{g} = T(\mathfrak{g})/I$ where $T(\mathfrak{g}) = \sum_{n \geq 0} \mathfrak{g}^{\otimes n}$ is the tensor algebra of \mathfrak{g} and I is the two sided ideal generated by elements of the form $a \otimes b + (-1)^{|a||b|} b \otimes a$ where a, b are homogeneous elements of \mathfrak{g} . We denote $\overline{\bigwedge \mathfrak{g}} = \overline{T(\mathfrak{g})}/I$,*

where $\overline{T(\mathfrak{g})} = \sum_{n \geq 1} \mathfrak{g}^{\otimes n}$. We will denote by $x_1 \wedge \cdots \wedge x_n$ the image of $x_1 \otimes \cdots \otimes x_n$.

Remark 4.3 *we have*

$$\begin{aligned} \wedge^n \mathfrak{g} &= \wedge^n (g_0 \oplus g_1 \oplus g_2 \oplus \cdots) \\ &= \bigoplus_{r_0+r_1+\cdots=n, r_i \geq 0} (\wedge^{r_0} g_0) \otimes (\text{sym}^{r_1} g_1) \otimes \cdots \otimes (\wedge^{r_{2k}} g_{2k}) \otimes (\text{sym}^{r_{2k+1}} g_{2k+1}) \otimes \cdots \end{aligned}$$

where $\wedge^k V$ is the usual k -th anti-symmetric product of a vector space V and $\text{sym}^k V$ is the usual k -th symmetric product of a vector space V when we ignore the grading.

Definition 4.4 *We can define the coalgebra structure Δ' and Δ on $\overline{S(\mathfrak{g})}$ and $\overline{\wedge} \mathfrak{g}$ in the following way:*

$$\begin{aligned} \Delta'(x_1 \odot \cdots \odot x_n) &= \sum_I (-1)^{s(I)} x_I \otimes x_{\bar{I}} \\ \Delta(v_1 \wedge \cdots \wedge v_n) &= \sum_I (-1)^{t(I)} v_I \otimes v_{\bar{I}} \end{aligned}$$

where the summation is over all subsets $I = \{r_1, \dots, r_p\}$, $r_1 < \cdots < r_p$ and $\bar{I} = \{s_1, \dots, s_q\}$ such that $s_1 < \cdots < s_q$ with $I \cup \bar{I} = \{1, \dots, n\}$, $x_I = x_{r_1} \odot \cdots \odot x_{r_p}$, $x_{\bar{I}} = x_{s_1} \odot \cdots \odot x_{s_q}$ and similary $v_I = v_{r_1} \wedge \cdots \wedge v_{r_p}$, $v_{\bar{I}} = v_{s_1} \wedge \cdots \wedge v_{s_q}$. Here $s(I)$ and $t(I)$ are determined in the following way:

$$\begin{aligned} x_1 \odot \cdots \odot x_n &= (-1)^{s(I)} x_I \odot x_{\bar{I}} \\ v_1 \wedge \cdots \wedge v_n &= (-1)^{t(I)} v_I \wedge v_{\bar{I}} \end{aligned}$$

Let (\mathfrak{g}, d) be a graded complex and we denote $(\mathfrak{g}[n], d)$ be a graded complex by shifting the degree by n , i.e. $\mathfrak{g}[n]^i = \mathfrak{g}_{n+i}$.

Remark 4.5 (d ecalage isomorphism) *We have an isomorphism*

$$\begin{aligned} \text{dec} : S^n(\mathfrak{g}[1]) &\cong \left(\bigwedge^n \mathfrak{g} \right)[n] \\ \bar{x}_1 \odot \cdots \odot \bar{x}_n &\mapsto (-1)^{\sum_{i=1}^{n-1} (n-i)p_i} x_1 \wedge \cdots \wedge x_n \end{aligned}$$

where x_i is an element of \mathfrak{g} , \bar{x}_i is an element of $\mathfrak{g}[1]$ via the natural map $\mathfrak{g} \rightarrow \mathfrak{g}[1]$, and p_i is the degree of x_i

Notation 12 Let $I = \{p_1, \dots, p_r\}$. A_I is defined by the following relation: $\text{dec}(\bar{x}_{p_1} \odot \dots \odot \bar{x}_{p_r}) = (-1)^{A_I} x_{p_1} \wedge \dots \wedge x_{p_r}$. We will denote $\bar{x}_I = x_{p_1} \odot \dots \odot x_{p_r}$ and $x_I = x_{p_1} \wedge \dots \wedge x_{p_r}$. We denote $|I| = r$ the cardinality of I and $|x_I| := \text{deg}(x_{p_1}) + \dots + \text{deg}(x_{p_r})$

Via this isomorphism dec , we have the following commutative diagram

$$\begin{array}{ccc} \overline{S(\mathfrak{g}[1])} & \xrightarrow{\text{dec}} & \overline{\wedge \mathfrak{g}} \\ \Delta' \downarrow & & \downarrow \Delta \\ \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{\tilde{\text{dec}}} & \overline{\wedge \mathfrak{g}} \otimes \overline{\wedge \mathfrak{g}} \end{array}$$

where $\tilde{\text{dec}}$ is defined in the following way: $\tilde{\text{dec}}(\bar{x}_I \otimes \bar{x}_J) = (-1)^{A_I + A_J + |x_I||J|} x_I \otimes x_J$.

4.1.1 Induced differential

Let \mathfrak{g} be a differential graded Lie algebra. We define the map $Q_n : \wedge^n \mathfrak{g} \rightarrow \wedge^{n-1} \mathfrak{g}$ by

$$Q_n(x_1 \wedge \dots \wedge x_n) = \sum_{i < j} (-1)^a [x_i, x_j] \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_n$$

where a is defined in the following way

$$x_1 \wedge \dots \wedge x_n = (-1)^a x_i \wedge x_j \wedge x_1 \wedge \dots \wedge \hat{x}_i \wedge \dots \wedge \hat{x}_j \wedge \dots \wedge x_n$$

More precisely,

$$a = \underbrace{i - 1 + p_i(p_1 + \dots + p_{i-1})}_{\text{first moving } x_i} + \underbrace{j - 2 + p_j(p_1 + \dots + \hat{p}_i + \dots + p_{j-1})}_{\text{second moving } x_j}$$

And we define $d_n : \wedge^n \mathfrak{g} \rightarrow \wedge^n \mathfrak{g}$ inductively on n by

$$d_n(x_1 \wedge \dots \wedge x_n) = dx_1 \wedge x_2 \wedge \dots \wedge x_n + \sum_{i=2}^n (-1)^{p_1 + \dots + p_{i-1}} x_1 \wedge \dots \wedge x_{i-1} \wedge dx_i \wedge x_{i+1} \wedge \dots \wedge x_n$$

First we show that $d^2 = 0$ and $Q^2 = 0$. We show $d^2 = 0$ by induction on k , where $x_1 \wedge \dots \wedge x_k$. For $k = 1$, the statement is true by the definition of d . Assume that the statement is true for $k = n - 1$.

$$\begin{aligned} d \circ d(x_1 \wedge \dots \wedge x_n) &= d(dx_1 \wedge x_2 \wedge \dots \wedge x_n + (-1)^{p_1} x_1 \wedge d(x_2 \wedge \dots \wedge x_n)) \\ &= dd x_1 \wedge x_2 \wedge \dots \wedge x_n + (-1)^{p_1+1} dx_1 \wedge d(x_2 \wedge \dots \wedge x_n) \\ &\quad + (-1)^{p_1} dx_1 \wedge d(x_2 \wedge \dots \wedge x_n) + (-1)^{p_1+p_1} x_1 \wedge dd(x_2 \wedge \dots \wedge x_n) = 0 \end{aligned}$$

by the induction hypothesis. For $Q^2 = 0$, we recall the definition of Q . Then $Q \circ Q(x_1 \wedge \cdots \wedge x_n)$ is of the following form

$$\sum_{i < j < k} ((-1)^a [[x_i, x_j], x_k] + (-1)^b [[x_j, x_k], x_i] + (-1)^c [[x_i, x_k], x_j]) x_1 \wedge \cdots \wedge \hat{x}_i \cdots \wedge \hat{x}_j \cdots \wedge \hat{x}_k \wedge \cdots \wedge x_n$$

where

$$a = \underbrace{i - 1 + p_i(p_1 + \cdots + p_{i-1})}_{\text{first moving } x_i} + \underbrace{j - 2 + p_j(p_1 + \cdots + \hat{p}_i + \cdots + p_{j-1})}_{\text{second moving } x_j} + \underbrace{k - 3 + p_k(p_1 + \cdots + \hat{p}_i + \cdots + \hat{p}_j + \cdots + p_{k-1})}_{\text{third moving } x_k}$$

$$b = \underbrace{j - 1 + p_j(p_1 + \cdots + p_{j-1})}_{\text{first moving } x_j} + \underbrace{k - 2 + p_k(p_1 + \cdots + \hat{p}_j + \cdots + p_k)}_{\text{second moving } x_k} + \underbrace{i - 1 + p_i(p_1 + \cdots + p_{i-1})}_{\text{third moving } x_i}$$

$$c = \underbrace{i - 1 + p_i(p_1 + \cdots + p_{i-1})}_{\text{first moving } x_i} + \underbrace{k - 2 + p_k(p_1 + \cdots + \hat{p}_i + \cdots + p_k)}_{\text{second moving } x_k} + \underbrace{j - 2 + p_j(p_1 + \cdots + \hat{p}_i + \cdots + p_{j-1})}_{\text{third moving } x_j}$$

Set

$$d = i + j + k + p_i(p_1 + \cdots + p_{i-1}) + p_j(p_1 + \cdots + p_{j-1}) + p_k(p_1 + \cdots + p_{k-1})$$

Then

$$\begin{aligned} (-1)^d &= (-1)^{a+p_j p_i + p_k p_i + p_k p_j} & (-1)^a &= (-1)^{d+p_j p_i + p_k p_i + p_k p_j} \\ (-1)^d &= (-1)^{b+p_k p_j} & (-1)^b &= (-1)^{d+p_k p_j} \\ (-1)^d &= (-1)^{c+1+p_k p_i + p_j p_i} & (-1)^c &= (-1)^{d+1+p_k p_i + p_j p_i} \end{aligned}$$

So we have

$$\begin{aligned} &(-1)^a [[x_i, x_j], x_k] + (-1)^b [[x_j, x_k], x_i] + (-1)^c [[x_i, x_k], x_j] \\ &= (-1)^d ((-1)^{p_j p_i + p_k p_i + p_k p_j} [[x_i, x_j], x_k] + (-1)^{p_k p_j} [[x_j, x_k], x_i] + (-1)^{1+p_k p_i + p_j p_i} [[x_i, x_k], x_j]) = 0 \end{aligned}$$

by the following relations and graded Jacobi identity

$$\begin{aligned} (-1)^{p_j p_i + p_k p_i + p_k p_j} [[x_i, x_j], x_k] &= (-1)^{p_j p_i + p_k p_j + p_k p_j} [x_k, [x_i, x_j]] \\ (-1)^{p_k p_j} [[x_j, x_k], x_i] &= (-1)^{p_k p_j + p_i p_j + p_i p_k} [x_i, [x_j, x_k]] \\ (-1)^{1+p_k p_i + p_j p_i} [[x_i, x_k], x_j] &= (-1)^{p_k p_j + p_j p_i + p_j p_i} [x_j, [x_k, x_i]] \end{aligned}$$

$$(-1)^{p_i p_k} [x_i, [x_j, x_k]] + (-1)^{p_j p_i} [x_j, [x_k, x_i]] + (-1)^{p_k p_j} [x_k, [x_i, x_j]] = 0$$

Hence we have $Q^2 = 0$.

Next we will show that $Qd = dQ$ by induction on k where $x_1 \wedge \cdots \wedge x_k$. For $k = 2$,

$$\begin{aligned} Q \circ d(x_1 \wedge x_2) &= Q(dx_1 \wedge x_2 + (-1)^{p_1} x_1 \wedge dx_2) = [dx_1, x_2] + (-1)^{p_1} [x_1, dx_2] \\ d \circ Q(x_1 \wedge x_2) &= d[x_1, x_2] = [dx_1, x_2] + (-1)^{p_1} [x_1, dx_2] \end{aligned}$$

So induction holds for $k = 2$. Now we assume that the statement holds for $k = n - 1$. We will prove that it holds for $k = n$.

$$\begin{aligned} d(x_1 \wedge \cdots \wedge x_n) &= dx_1 \wedge (x_2 \wedge \cdots \wedge x_n) + (-1)^{p_1} x_1 \wedge d(x_2 \wedge \cdots \wedge x_n) \\ dx_1 \wedge (x_2 \wedge \cdots \wedge x_n) &= (-1)^{n-1+(p_1+1)(p_2+\cdots+p_n)} (x_2 \wedge \cdots \wedge x_n) \wedge dx_1 \\ (-1)^{p_1} x_1 \wedge d(x_2 \wedge \cdots \wedge x_n) &= (-1)^{p_1+i-2+p_i(p_2+\cdots+p_{i-1})} x_1 \wedge d(x_i \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n) \\ &= (-1)^{p_1+i-2+p_i(p_2+\cdots+p_{i-1})} x_1 \wedge dx_i \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + (-1)^{p_1+i-2+p_i(p_2+\cdots+p_{i-1})+p_i} x_1 \wedge x_i \wedge d(x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n) \\ &= (-1)^{p_1+n-2+p_1(p_2+\cdots+p_n+1)} d(x_2 \wedge \cdots \wedge x_n) \wedge x_1 \end{aligned}$$

Then

$$\begin{aligned} Qd(x_1 \wedge \cdots \wedge x_n) &= \sum_{i=2}^n (-1)^{i-2+p_i(p_2+\cdots+p_{i-1})} [dx_1, x_i] \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + (-1)^{n-1+(p_1+1)(p_2+\cdots+p_n)} Q(x_2 \wedge \cdots \wedge x_n) \wedge dx_1 \\ &\quad + \sum_{i=2}^n (-1)^{p_1+i-2+p_i(p_2+\cdots+p_{i-1})} [x_1, dx_i] \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + \sum_{i=2}^n (-1)^{p_1+i-2+p_i(p_2+\cdots+p_{i-1})+p_i} [x_1, x_i] \wedge d(x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n) \\ &\quad + (-1)^{p_1+n-2+p_1(p_2+\cdots+p_n+1)} Qd(x_2 \wedge \cdots \wedge x_n) \wedge x_1 \end{aligned}$$

Now we compute dQ . First we note the following relations

$$\begin{aligned} x_1 \wedge (x_2 \wedge \cdots \wedge x_n) &= (-1)^{i-2+p_i(p_1+\cdots+p_{i-1})} x_1 \wedge x_i \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &= (-1)^{n-1+p_1(p_2+\cdots+p_n)} (x_2 \wedge \cdots \wedge x_n) \wedge x_1 \end{aligned}$$

Then

$$\begin{aligned} Q(x_1 \wedge (x_2 \wedge \cdots \wedge x_n)) &= \sum_{i=2}^n (-1)^{i-2+p_i(p_1+\cdots+p_{i-1})} [x_1, x_i] \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\ &\quad + (-1)^{n-1+p_1(p_2+\cdots+p_n)} Q(x_2 \wedge \cdots \wedge x_n) \wedge x_1 \end{aligned}$$

Hence we have

$$\begin{aligned}
dQ(x_1 \wedge \cdots \wedge x_n) &= \sum_{i=2}^n (-1)^{i-2+p_i(p_1+\cdots+p_{i-1})} [dx_1, x_i] \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\
&+ \sum_{i=2}^n (-1)^{i-2+p_i(p_1+\cdots+p_{i-1})+p_1} [x_1, dx_i] \wedge x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n \\
&+ \sum_{i=1}^n (-1)^{i-2+p_i(p_1+\cdots+p_{i-1})+p_1+p_i} [x_1, x_i] \wedge d(x_2 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge x_n) \\
&+ (-1)^{n-1+p_1(p_2+\cdots+p_n)} dQ(x_2 \wedge \cdots \wedge x_n) \wedge x_1 \\
&+ (-1)^{n-1+p_1(p_2+\cdots+p_n)+(p_2+\cdots+p_n)} Q(x_2 \wedge \cdots \wedge x_n) \wedge dx_1
\end{aligned}$$

By induction hypothesis for $k = n - 1$, we have $Qd(x_2 \wedge \cdots \wedge x_n) = dQ(x_2 \wedge \cdots \wedge x_n)$.

Hence we have

$$Qd(x_1 \wedge \cdots \wedge x_n) = dQ(x_1 \wedge \cdots \wedge x_n)$$

So the statement holds for $k = n$. Hence $Qd = dQ$. So if we we define Q'

$$Q'(x_1 \wedge \cdots \wedge x_n) = ((-1)^n d + Q)(x_1 \wedge \cdots \wedge x_n)$$

Then $Q' \circ Q' = 0$.

4.2 Jacobi complex

Definition 4.6 (un-degree shifted complex) *Let \mathfrak{g} be a differential graded Lie algebra. Let's*

consider the total complex of the following bicomplex with the differential Q' defined as above,

$$\begin{array}{ccccccc}
g_0 & \xrightarrow{-d} & g_1 & \xrightarrow{-d} & g_2 & \xrightarrow{-d} & g_3 & \longrightarrow & \cdots \\
Q \uparrow & & Q \uparrow & & Q \uparrow & & Q \uparrow & & \\
\Lambda^2 g_0 & \xrightarrow{d} & (g_0 \otimes g_1) & \xrightarrow{d} & (g_0 \otimes g_2) \oplus \text{sym}^2 g_1 & \xrightarrow{d} & (g_0 \otimes g_3) \oplus (g_1 \otimes g_2) & \longrightarrow & \cdots \\
Q \uparrow & & Q \uparrow & & Q \uparrow & & Q \uparrow & & \\
\Lambda^3 g_0 & \xrightarrow{-d} & (\Lambda^2 g_0) \otimes g_1 & \xrightarrow{-d} & (\Lambda^2 g_0 \otimes g_2) \oplus (g_0 \otimes \text{sym}^2 g_1) & \xrightarrow{-d} & (\Lambda^2 g_0 \otimes g_3) \oplus (g_0 \otimes g_1 \otimes g_2) & \longrightarrow & \cdots \\
& & & & & & \oplus \text{sym}^3 g_1 & & \\
Q \uparrow & & Q \uparrow & & Q \uparrow & & Q \uparrow & & \\
\Lambda^4 g_0 & \xrightarrow{d} & (\Lambda^3 g_0) \otimes g_1 & \xrightarrow{d} & (\Lambda^3 g_0 \otimes g_2) & \xrightarrow{d} & (\Lambda^3 g_0 \otimes g_3) & \longrightarrow & \cdots \\
& & & & \oplus (\Lambda^2 g_0 \otimes \text{sym}^2 g_1) & & \oplus (\Lambda^2 g_0 \otimes g_1 \otimes g_2) & & \\
& & & & & & \oplus (g_0 \otimes \text{sym}^3 g_1) & & \\
Q \uparrow & & Q \uparrow & & Q \uparrow & & Q \uparrow & & \\
\cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & \longrightarrow & \cdots & & \\
Q \uparrow & & Q \uparrow & & Q \uparrow & & Q \uparrow & & \\
\Lambda^n g_0 & \xrightarrow{(-1)^n d} & (\Lambda^{n-1} g_0) \otimes g_1 & \xrightarrow{(-1)^n d} & \cdots & \xrightarrow{(-1)^n d} & \cdots & \longrightarrow & \cdots
\end{array}$$

Now we will define n -th order Jacobi complex $J_n(\mathfrak{g})$. First we note that we can consider the direct sum of each components in the above whole complex as a subset S_n of $S = \overline{S(\mathfrak{g}[1])}$ via dec .

Definition 4.7 We define a differential \bar{Q} on $\overline{S(\mathfrak{g}[1])}$ in a way that the following commutative diagram commutes.

$$\begin{array}{ccc} \bar{x}_I & \xrightarrow{dec} & (-1)^{A_I} x_I \\ \bar{Q} \downarrow & & Q \downarrow \\ \bar{Q}(\bar{x}_I) & \xrightarrow{dec} & (-1)^{A_I} Q(x_I) \end{array}$$

In other words, $\bar{Q} := dec^{-1} \circ Q \circ dec$.

Definition 4.8 we define a differential \bar{d} in a way that the following diagram commutes

$$\begin{array}{ccc} \bar{x}_I & \xrightarrow{dec} & (-1)^{A_I} x_I \\ \bar{d} \downarrow & & (-1)^{|I|} d \downarrow \\ \bar{d}(\bar{x}_I) & \xrightarrow{dec} & (-1)^{A_I+|I|} d(x_I) \end{array}$$

We translate the above complex $(\wedge \mathfrak{g}, d, Q)$ (un degree shifted complex in Definition 4.6) in terms of $(\overline{S(\mathfrak{g}[1])}, \bar{d}, \bar{Q})$ via dec to define Jacobi complex.

Definition 4.9 (Jacobi complex) Let \mathfrak{g} be a differential graded Lie algebra. The n -th order Jacobi complex $J_n(\mathfrak{g})$ is the total complex of the following bicomplex (here we have to be careful of the grading: we are working on $S_n \subset \overline{S(\mathfrak{g}[1])}$, hence here the grading of g_i is actually $i - 1$)

$$\begin{array}{ccccccc} g_0 & \xrightarrow{\bar{d}} & g_1 & \xrightarrow{\bar{d}} & g_2 & \xrightarrow{\bar{d}} & g_3 & \longrightarrow & \dots \\ \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \\ \Lambda^2 g_0 & \xrightarrow{\bar{d}} & (g_0 \otimes g_1) & \xrightarrow{\bar{d}} & (g_0 \otimes g_2) \oplus \text{sym}^2 g_1 & \xrightarrow{\bar{d}} & (g_0 \otimes g_3) \oplus (g_1 \otimes g_2) & \longrightarrow & \dots \\ \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \\ \Lambda^3 g_0 & \xrightarrow{\bar{d}} & (\Lambda^2 g_0) \otimes g_1 & \xrightarrow{\bar{d}} & (\Lambda^2 g_0 \otimes g_2) \oplus (g_0 \otimes \text{sym}^2 g_1) & \xrightarrow{\bar{d}} & (\Lambda^2 g_0 \otimes g_3) \oplus (g_0 \otimes g_1 \otimes g_2) & \longrightarrow & \dots \\ & & & & & & \oplus \text{sym}^3 g_1 & & \\ \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \\ \Lambda^4 g_0 & \xrightarrow{\bar{d}} & (\Lambda^3 g_0) \otimes g_1 & \xrightarrow{\bar{d}} & (\Lambda^3 g_0 \otimes g_2) & \xrightarrow{\bar{d}} & (\Lambda^3 g_0 \otimes g_3) & \longrightarrow & \dots \\ & & & & \oplus (\Lambda^2 g_0 \otimes \text{sym}^2 g_1) & & \oplus (\Lambda^2 g_0 \otimes g_1 \otimes g_2) & & \\ & & & & & & \oplus (g_0 \otimes \text{sym}^3 g_1) & & \\ \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \\ \dots & \longrightarrow & \dots & \longrightarrow & \dots & \longrightarrow & \dots & & \\ \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \bar{Q} \uparrow & & \\ \wedge^n g_0 & \xrightarrow{\bar{d}} & (\wedge^{n-1} g_0) \otimes g_1 & \xrightarrow{\bar{d}} & \dots & \xrightarrow{\bar{d}} & \dots & \longrightarrow & \dots \end{array}$$

We give a bigrading on S_n by $S_n^{-r,s} \cong_{dec} (\wedge^r \mathfrak{g})^s$. In other words, if $x \in S_n^{-r,s}$, then x is of the form $x = \sum_i \bar{x}_{i_1} \odot \cdots \odot \bar{x}_{i_r}$ where $deg(x_{i_1}) + \cdots + deg(x_{i_r}) = s$ in \mathfrak{g} . Then we set $J_n(\mathfrak{g})^i = \bigoplus_{-r+s=i} S_n^{-r,s}$. For example, $J_n(\mathfrak{g})^0 = g_1 \oplus (g_0 \otimes g_2) \oplus sym^2 g_1 \oplus (\wedge^2 g_0 \otimes g_3) \oplus (g_0 \otimes g_1 \otimes g_2) \oplus sym^3 g_1 \oplus \cdots$. Then we define the Jacobi complex to be the following complex

$$\cdots \xrightarrow{\bar{d}+\bar{Q}} J_n(\mathfrak{g})^{i-1} \xrightarrow{\bar{d}+\bar{Q}} J_n(\mathfrak{g})^i \xrightarrow{\bar{d}+\bar{Q}} J_n(\mathfrak{g})^{i+1} \xrightarrow{\bar{d}+\bar{Q}} \cdots$$

We also note natural inclusions $J_1(\mathfrak{g}) \hookrightarrow J_2(\mathfrak{g}) \hookrightarrow \cdots \hookrightarrow J_n(\mathfrak{g}) \hookrightarrow \cdots$, which induces $\mathbb{H}^i(J_1(\mathfrak{g})) \rightarrow \mathbb{H}^i(J_2(\mathfrak{g})) \rightarrow \cdots \rightarrow \mathbb{H}^i(J_n(\mathfrak{g})) \rightarrow \cdots$.

Definition 4.10 We denote i -th cohomology group of the n -th order Jacobi complex associated with a differential graded Lie algebra \mathfrak{g} by $\mathbb{H}^i(J_n(\mathfrak{g}))$.

Remark 4.11 We can modify the Jacobi complex $J_n(\mathfrak{g})$ by tensoring $\otimes \mathfrak{m}$ for some local artinian \mathbb{C} -algebra (R, \mathfrak{m}) with residue \mathbb{C} to get $J_n(\mathfrak{g}) \otimes \mathfrak{m}$. Then its cohomology groups coincide with $\mathbb{H}^i(J_n(\mathfrak{g})) \otimes \mathfrak{m}$.

Remark 4.12 In practice, when we compute the Jacobi cohomology for our infinitesimal Poisson deformations, we use the first complex (un-degree shifted complex in Definition 4.6). In other words, we will work on $\overline{\wedge \mathfrak{g}}$ for actual computation of Jacobi cohomology groups. The reason why we pass from $\overline{\wedge \mathfrak{g}}$ to $\overline{S(\mathfrak{g}[1])}$ is that we want to give a commutative \mathbb{C} -algebra structure on $\mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$. We will explain this below.

Let \mathfrak{g} be a differential graded Lie algebra. Then $\overline{S(\mathfrak{g}[1])}$ has a symmetric coalgebra structure. Let Q be the differential induced from the bracket $[-, -]$ as above.

Lemma 4.13 We have $dec(\bar{Q}(\bar{x}_I) \odot \bar{x}_J) = (-1)^{A_I+A_J+|J||x_I|} Q(x_I) \wedge x_J$ and $dec(\bar{x}_I \odot \bar{Q}(\bar{x}_J)) = (-1)^{A_I+A_J+|x_I|(|J|-1)} x_I \wedge Q(x_J)$.

Proof. We simply note that $dec(\bar{x}_J) = (-1)^{A_J} x_J$, $dec \circ \bar{Q}(\bar{x}_I) = (-1)^{A_I} Q(x_I)$ and $dec \circ \bar{Q}(\bar{x}_J) = (-1)^{A_J} Q(x_J)$. ■

By the definition of \bar{Q} on $\overline{S(\mathfrak{g}[1])}$, we would like to show the following diagram commutes, (which means \bar{Q} is a coderivation)

$$\begin{array}{ccc}
\overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q}} & \overline{S(\mathfrak{g}[1])} \\
\Delta' \downarrow & & \Delta' \downarrow \\
\overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q} \otimes 1 + 1 \otimes \bar{Q}} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])}
\end{array}$$

Since $(\bar{Q} \otimes 1 + 1 \otimes \bar{Q}) \circ \Delta'(\bar{x}_1 \odot \cdots \odot \bar{x}_n) = \sum_{I,J} (-1)^{S(I,J)} (\bar{Q}(\bar{x}_I) \otimes \bar{x}_J + (-1)^{|\bar{x}_I|+|I|} \bar{x}_I \otimes \bar{Q}(\bar{x}_J))$, where $\bar{x}_1 \odot \cdots \odot \bar{x}_n = (-1)^{S(I,J)} \bar{x}_I \odot \bar{x}_J$. First we note the following commutative diagram

$$\begin{array}{ccccc}
\overline{S(g[1])} & \xrightarrow{\bar{Q}} & \overline{S(g[1])} & \xrightarrow{\Delta'} & \overline{S(g[1])} \otimes \overline{S(g[1])} \\
dec \downarrow & & dec \downarrow & & \tilde{dec} \downarrow \\
\overline{\Lambda g} & \xrightarrow{Q} & \overline{\Lambda g} & \xrightarrow{\Delta} & \overline{\Lambda g} \otimes \overline{\Lambda g}
\end{array}$$

By this commutativity, we are working on $\overline{\Lambda \mathfrak{g}}$ instead of $\overline{S(\mathfrak{g}[1])}$ to show that \bar{Q} is a coderivation on $\overline{S(\mathfrak{g}[1])}$.

so our claim is equivalent to

$$\sum_{I,J} (-1)^{S(I,J)} \tilde{dec}(\bar{Q}(\bar{x}_I) \otimes \bar{x}_J + (-1)^{|\bar{x}_I|+|I|} \bar{x}_I \otimes \bar{Q}(\bar{x}_J)) = \Delta \circ Q \circ dec(\bar{x}_1 \odot \cdots \odot \bar{x}_n) \quad (4.14)$$

Remark 4.15 Via the above commutative diagram, for any I and J , where $I \cup J = \{1, \dots, n\}$, $Q(x_I) \otimes x_J$ and $x_I \otimes Q(x_J)$ appear (up to sign) as terms in $\Delta \circ Q \circ dec(\bar{x}_1 \odot \cdots \odot \bar{x}_n)$. Conversely, since $Q(x_1 \wedge \cdots \wedge x_n) = \sum (-1)^{P(i,j)} [x_i, x_j] \wedge x_1 \wedge \cdots \wedge \hat{x}_i \wedge \cdots \wedge \hat{x}_j \wedge \cdots \wedge x_n$, $Q(x_I) \otimes x_J$ and $x_I \otimes Q(x_J)$ for all the pairs I, J (up to sign) exhaust all the terms in $\Delta \circ Q \circ dec(\bar{x}_1 \odot \cdots \odot \bar{x}_n)$. Hence in order to prove our claim (4.14), we only need to check that the signs for $Q(x_I) \otimes x_J$ and $x_I \otimes Q(x_J)$ in $\Delta \circ Q \circ dec(\bar{x}_1 \odot \cdots \odot \bar{x}_n)$ equal to the signs for $Q(x_I) \otimes x_J$ and $x_I \otimes Q(x_J)$ in $\tilde{dec}(\bar{Q}(\bar{x}_I) \otimes \bar{x}_J)$ and $(-1)^{|\bar{x}_I|+|I|} \tilde{dec}(\bar{x}_I \otimes \bar{Q}(\bar{x}_J))$.

We now prove the claim (4.14). We note that $\bar{x}_I \odot \bar{x}_J = (-1)^{(|I|+|x_I|)(|J|+|x_J|)} \bar{x}_J \odot \bar{x}_I$

$$\begin{array}{ccc}
\bar{x}_I \odot \bar{x}_J & \xrightarrow{dec} & (-1)^{A_I+A_J+|x_I||J|} x_I \wedge x_J \\
\bar{Q} \downarrow & & Q \downarrow \\
\bar{Q}(\bar{x}_I \odot \bar{x}_J) & \xrightarrow{dec} & (-1)^{A_I+A_J+|x_I||J|} Q(x_I) \wedge x_J + \cdots
\end{array}$$

Hence $(-1)^{A_I+A_J+|x_I||J|} Q(x_I) \otimes x_J$ corresponds to $\bar{Q}(\bar{x}_I) \otimes \bar{x}_J$ via \tilde{dec} .

$$\begin{array}{ccc}
(-1)^{(|I|+|x_I|)(|J|+|x_J|)} \bar{x}_J \odot \bar{x}_I & \xrightarrow{dec} & (-1)^{(|I|+|x_I|)(|J|+|x_J|)+A_I+A_J+|x_J||I|} x_J \wedge x_I \\
\bar{Q} \downarrow & & Q \downarrow \\
\bar{Q}(\bar{x}_I \odot \bar{x}_J) = (-1)^{(|I|+|x_I|)(|J|+|x_J|)} \bar{Q}(\bar{x}_J \odot \bar{x}_I) & \xrightarrow{dec} & (-1)^{(|I|+|x_I|)(|J|+|x_J|)+A_I+A_J+|x_J||I|} Q(x_J) \wedge x_I + \cdots
\end{array}$$

We note that

$$\begin{aligned} (-1)^{(|I|+|x_I|)(|J|+|x_J|)+A_I+A_J+|x_J||I|} Q(x_J) \wedge x_I &= (-1)^{(|I|+|x_I|)(|J|+|x_J|)+A_I+A_J+|x_J||I|+|I|(|J|-1)+|x_I||x_J|} x_I \wedge Q(x_J) \\ &= (-1)^{|x_I||J|+A_I+A_J+|I|} x_I \wedge Q(x_J) \end{aligned}$$

Hence $(-1)^{|x_I||J|+A_I+A_J+|I|} x_I \otimes Q(x_J)$ corresponds via \tilde{dec} to

$$(-1)^{|x_I||J|+A_I+A_J+|I|+A_I+A_J+|x_I|(|J|-1)} \bar{x}_I \otimes \bar{Q}(\bar{x}_J) = (-1)^{|I|+|x_I|} \bar{x}_I \otimes \bar{Q}(\bar{x}_J)$$

This completes the claim (4.14). Hence \bar{Q} is a coderivation of $\overline{S(\mathfrak{g}[1])}$.

On the other hand, would like to show that

$$\bar{d}(\bar{x}_I \odot \bar{x}_J) = \bar{d}(\bar{x}_I) \odot \bar{x}_J + (-1)^{|I|+|x_I|} \bar{x}_I \odot \bar{d}(\bar{x}_J)$$

which implies that \bar{d} defines a coderivation of $\overline{S(\mathfrak{g}[1])}$. In other words, the following commutative diagram holds

$$\begin{array}{ccc} \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{d}} & \overline{S(\mathfrak{g}[1])} \\ \Delta' \downarrow & & \Delta' \downarrow \\ \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{d} \otimes 1 + 1 \otimes \bar{d}} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \end{array}$$

First we note the following relations

$$\begin{array}{ccc} \bar{x}_I \odot \bar{x}_J & \xrightarrow{dec} & (-1)^{A_I+A_J+|x_I||J|} x_I \wedge x_J \\ \bar{d} \downarrow & & (-1)^{|I|+|J|} d \downarrow \\ \bar{d}(\bar{x}_I \odot \bar{x}_J) & \xrightarrow{dec} & (-1)^{A_I+A_J+|x_I||J|+|I|+|J|} d(x_I \wedge x_J) \end{array}$$

$$\bar{d}(\bar{x}_I) \odot \bar{x}_J \xrightarrow{dec} (-1)^{A_I+|I|+A_J+|J|(|x_I|+1)} dx_I \wedge x_J$$

$$\bar{x}_I \odot \bar{d}(\bar{x}_J) \xrightarrow{dec} (-1)^{A_I+A_J+|J|+|J||x_I|} x_I \wedge dx_J$$

$$(-1)^{|I|+|x_I|} \bar{x}_I \odot \bar{d}(\bar{x}_J) \xrightarrow{dec} (-1)^{A_I+A_J+|J|+|J||x_I|+|I|+|x_I|} x_I \wedge dx_J$$

$$(-1)^{A_I+A_J+|x_I||J|+|I|+|J|} d(x_I \wedge x_J) = (-1)^{A_I+A_J+|x_I||J|+|I|+|J|} dx_I \wedge x_J + (-1)^{A_I+A_J+|x_I||J|+|I|+|J|+|x_I|} x_I \wedge dx_J$$

Hence \bar{d} defines a coderivation of $\overline{S(\mathfrak{g}[1])}$.

In conclusion, $\bar{Q}' = \bar{d} + \bar{Q}$ is a coderivation of $\overline{S(\mathfrak{g}[1])}$. In other words, the following commutative diagram holds

$$\begin{array}{ccc} \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q}'} & \overline{S(\mathfrak{g}[1])} \\ \Delta' \downarrow & & \Delta' \downarrow \\ \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q}' \otimes 1 + 1 \otimes \bar{Q}'} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \end{array}$$

4.3 Morphic elements

We have a comultiplication map Δ' and coderivation \bar{Q}' which is induced from differential and bracket from a differential graded Lie algebra \mathfrak{g} .

$$\begin{array}{ccc} \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q}'} & \overline{S(\mathfrak{g}[1])} \\ \Delta' \downarrow & & \Delta' \downarrow \\ \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{\bar{Q}' \otimes id + id \otimes \bar{Q}'} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \end{array}$$

The following diagram commutes with the coderivation \bar{Q}'

$$\begin{array}{ccc} \overline{S(\mathfrak{g}[1])} & \xrightarrow{\Delta'} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \\ \Delta' \downarrow & & \Delta' \otimes id \downarrow \\ \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} & \xrightarrow{id \otimes \Delta'} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \end{array}$$

Hence we have the following commutative diagram

$$\begin{array}{ccccc} \mathbb{H}^0 & \longrightarrow & \bigoplus_{i+j=0} \mathbb{H}^i \otimes \mathbb{H}^j & \longrightarrow & \mathbb{H}^0 \otimes \mathbb{H}^0 \\ \downarrow & & \downarrow & \swarrow & \downarrow \\ \bigoplus_{i+j=0} \mathbb{H}^i \otimes \mathbb{H}^j & \longrightarrow & \bigoplus_{a+b+c=0} \mathbb{H}^a \otimes \mathbb{H}^b \otimes \mathbb{H}^c & & \\ \downarrow & \nearrow & & \searrow & \downarrow \\ \mathbb{H}^0 \otimes \mathbb{H}^0 & \longrightarrow & & \longrightarrow & \mathbb{H}^0 \otimes \mathbb{H}^0 \otimes \mathbb{H}^0 \end{array}$$

This induces a comultiplication map $\mathbb{H}^0(J_n(\mathfrak{g})) \rightarrow \sum_{i+j=0} \mathbb{H}^i(J_n(\mathfrak{g})) \otimes \mathbb{H}^j(J_n(\mathfrak{g})) \rightarrow \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathbb{H}^0(J_n(\mathfrak{g}))$, where the last map is a projection.

And we have

$$\begin{array}{ccc}
\overline{S(\mathfrak{g}[1])} & \xrightarrow{\Delta'} & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])} \\
& \searrow \Delta' & \swarrow \tau \\
& & \overline{S(\mathfrak{g}[1])} \otimes \overline{S(\mathfrak{g}[1])}
\end{array}$$

where $\tau(a \otimes b) = (-1)^{|a||b|}b \otimes a$.

This induces

$$\begin{array}{ccccc}
\mathbb{H}^0 & \longrightarrow & \bigoplus_{i+j=0} \mathbb{H}^i \otimes \mathbb{H}^j & \longrightarrow & \mathbb{H}^0 \otimes \mathbb{H}^0 \\
\downarrow & & \swarrow & & \nearrow \\
\bigoplus_{i+j=0} \mathbb{H}^i \otimes \mathbb{H}^j & & & & \\
\downarrow & & \swarrow & & \nearrow \\
\mathbb{H}^0 \otimes \mathbb{H}^0 & & & &
\end{array}$$

So the comultiplication map induces $\mathbb{H}^0(J_n(\mathfrak{g})) \rightarrow \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g}))) \cong \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g}))^*)^*$, where V^* is the dual space of a vector space V over \mathbb{C} .

Remark 4.16 When we assume that $H^0(\mathfrak{g}) = 0$ which is the main assumption for universal Poisson deformation, we have $\mathbb{H}^i = 0$ for $i < 0$. In this case, we have simply $\bigoplus_{i+j=0} \mathbb{H}^i \otimes \mathbb{H}^j = \mathbb{H}^0 \otimes \mathbb{H}^0$ and $\bigoplus_{a+b+c=0} \mathbb{H}^a \otimes \mathbb{H}^b \otimes \mathbb{H}^c = \mathbb{H}^0 \otimes \mathbb{H}^0 \otimes \mathbb{H}^0$.

Remark 4.17 The commultiplication Δ' of $S_n \subset \overline{S(\mathfrak{g}[1])}$ induces a map

$$* : \mathbb{H}^0(J_n(\mathfrak{g}))^* \times \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathbb{H}^0(J_n(\mathfrak{g}))^*$$

satisfying the associative and commutative laws. Hence $\mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$ is a commutative \mathbb{C} -algebra.

Moreover, $\underbrace{\Delta' \circ \dots \circ \Delta'}_{n+1}$ on S_n is 0. We have $\underbrace{\mathbb{H}^0(J_n(\mathfrak{g}))^* * \dots * \mathbb{H}^0(J_n(\mathfrak{g}))^*}_{n+1} = 0$.

In conclusion, set $\mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^*$. Then $\mathbb{C} \oplus \mathfrak{m}_n^u$ is a local commutative \mathbb{C} -algebra with the maximal ideal \mathfrak{m}_n^u and residue \mathbb{C} such that $\mathfrak{m}_n^{u, n+1} = 0$. We note that for our infinitesimal Poisson deformation, \mathfrak{m}_n^u is finite dimensional. Hence $\mathbb{C} \oplus \mathfrak{m}_n^u$ is artinian. This proves our main Theorem 11 (1) in the Introduction of the part II of the thesis.

4.3.1 Morphic elements and ring homomorphism

Lemma 4.18 *Let $A \xrightarrow{v} B \xrightarrow{w} C$ be a complex of vector spaces over \mathbb{C} and $C^* \xrightarrow{w^*} B^* \xrightarrow{v^*} A^*$ be induced complex of dual vector spaces. Then $[b_1] = [b_2] \in H(B)$ and $[f] = [g] \in H(B^*)$, where $[t]$ is a cohomology class for $t \in B$ or $t \in B^*$. Then $f(b_1) = g(b_2)$.*

Proof. First we note that $w(b_1) = w(b_2) = 0$ and $v^*(f) = f \circ v = 0, v^*(g) = g \circ v = 0$. Let $v(a) = b_1 - b_2$ and $w^*(h) = h \circ w = f - g$. $f(b_1) - g(b_2) = f(b_1) - g(b_1) + g(b_1) - g(b_2) = (f - g)(b_1) + g(b_1 - b_2) = h \circ w(b_1) + g \circ v(a) = 0$ ■

Let $[\phi] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ where \mathfrak{m} is the maximal ideal of some artinian local \mathbb{C} -algebra (R, \mathfrak{m}) with residue \mathbb{C} and $\mathfrak{m}^{n+1} = 0$. Then this $[\phi]$ defines a linear map $f : \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathfrak{m}$ by $a \rightarrow a(\phi)$. By Lemma 4.18, the linear map f is independent of choices of a and ϕ . Let's consider a bilinear map $\mathbb{H}^0(J_n(\mathfrak{g}))^* \times \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathfrak{m}^2$ defined by $(a, b) \mapsto f(a)f(b) = a(\phi)b(\phi)$. Since \mathfrak{m} is commutative, this induces a map

$$f \times f : \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g}))^*) \rightarrow \mathfrak{m}^2$$

Definition 4.19 (morphic elements) *We call $[\phi] \in \mathbb{H}^0(J_n(\mathfrak{g}))^*$ a morphic element if $f \times f$ defines a ring homomorphism.*

In order for $f \times f$ to be a ring homomorphism (equivalently for $[\phi]$ to be a morphic element), we have to have $f(a)f(b) = f(a \cdot b)$ where \cdot is induced from $\Delta' : \mathbb{H}^0(J_n(\mathfrak{g})) \rightarrow \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g}))^*)^*$. More precisely, by taking the dual of Δ' , we have the map $\cdot : \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g}))^*) \cong \text{sym}^2(\mathbb{H}^0(J_n(\mathfrak{g})))^* \rightarrow \mathbb{H}^0(J_n(\mathfrak{g}))^*$ by $a \cdot b = (a \otimes b) \circ \Delta'$. Hence $f(a)f(b) = f(a \cdot b)$ means that

$$(a \otimes b)(\phi \otimes \phi) = a(\phi)b(\phi) = (a \otimes b)(\Delta'(\phi)) = (b \otimes a)(\Delta'(\phi))$$

Hence $(a \otimes b)(\Delta'(\phi) - \phi \otimes \phi) = 0$ for all $a, b \in \mathbb{H}^0(J_n(\mathfrak{g}))^*$. This implies that $\Delta'(\phi) = \phi \otimes \phi$. Hence for any morphic element $[\phi] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$, we have $\Delta'(\phi) = \phi \otimes \phi$.

4.3.2 Explicit description of a morphic element

We describe a morphic element v in $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ for some local artinian \mathbb{C} -algebra (R, \mathfrak{m}) with residue \mathbb{C} and $\mathfrak{m}^{n+1} = 0$. When we consider the Jacobi bicomplex, a 0-th cohomology class v is of the form

$$v = v_1 + \cdots + v_n$$

where $v_i \in S^i(\mathfrak{g}[1]) \otimes \mathfrak{m}$. In particular $v_1 \in g_1 \otimes \mathfrak{m}$. For v to be a morphic element, we have to have $\Delta'(v) = v \otimes v$ where Δ' is the comultiplication map. So $\sum \Delta'(v_i) = \sum v_i \otimes v_j$. So we have the following relations

$$\begin{aligned}\Delta'(v_1) &= 0 \\ \Delta'(v_2) &= v_1 \otimes v_1 \\ \Delta'(v_3) &= v_1 \otimes v_2 + v_2 \otimes v_1 \\ \Delta'(v_4) &= v_1 \otimes v_3 + v_2 \otimes v_2 + v_3 \otimes v_1 \\ &\dots \\ \Delta'(v_n) &= v_1 \otimes v_{n-1} + \dots + v_{n-1} \otimes v_1\end{aligned}$$

So v_i is determined by v_1, \dots, v_{i-1} inductively, hence completely determined by v_1 . Since $\Delta'(v_2) = v_1 \otimes v_1$, we see that $v_2 = \frac{1}{2}v_1 \odot v_1$. Since $\Delta'(v_3) = \frac{1}{2}v_1 \otimes (v_1 \odot v_1) + \frac{1}{2}(v_1 \odot v_1) \otimes v_1$, we have $v_3 = \frac{1}{3!}v_1 \odot v_1 \odot v_1$. Inductively, since $\Delta'(v_n) = \frac{1}{(n-1)!}v_1 \otimes (v_1 \odot \dots \odot v_1) + \frac{1}{2!(n-2)!}(v_1 \odot v_1) \otimes (v_1 \odot \dots \odot v_n) + \frac{1}{3!(n-3)!}(v_1 \odot v_1 \odot v_1) \otimes (v_1 \odot \dots \odot v_1) + \dots + \frac{1}{(n-2)!2!}(v_1 \odot \dots \odot v_1) \otimes (v_1 \odot v_1) + \frac{1}{(n-1)!}(v_1 \odot \dots \odot v_1) \otimes v_1$, and $\binom{n}{i} = \frac{n!}{i!(n-i)!}$, we have $v_n = \frac{1}{n!}v_1 \odot \dots \odot v_1$. Hence a morphic element is of the form

$$v_1 + \frac{1}{2!}v_1 \odot v_1 + \dots + \frac{1}{n!}v_1 \odot \dots \odot v_1$$

where $\frac{1}{i!} \underbrace{v_1 \odot \dots \odot v_1}_i \in \text{sym}^i g_1 \otimes \mathfrak{m}^i$.

Chapter 5

Infinitesimal Poisson deformations

5.1 Bracket calculus

Let $(\mathfrak{g} = \bigoplus_i g_i, \bar{\partial}, [-, -])$ be a differential graded Lie algebra. Let R is a local artinian \mathbb{C} -algebra with residue \mathbb{C} . Then $\mathfrak{g} \otimes R$ is a differential graded Lie algebra with $(\mathfrak{g} \otimes R)_i = g_i \otimes R$, $\bar{\partial}(a \otimes r) = \bar{\partial}a \otimes r$ and $[a \otimes r_1, b \otimes r_2] = [a, b] \otimes r_1 r_2$ for $a, b \in \mathfrak{g}$. Let X be a complex manifold. Since $(\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T), \bar{\partial}, [-, -])$ is a differential graded Lie algebra (see Appendix C), this induces a differential graded Lie algebra structure on $\bigoplus_{p+q-1=i, p \geq 1, q \geq 0} A^{0,p}(X, \wedge^q T) \otimes_{\mathbb{C}} R$.

Definition 5.1 *Let X be a compact complex manifold. We consider any element of the differential graded Lie algebra $A = \bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T) \otimes_{\mathbb{C}} R$ as an operator acting on itself in the following way. Let $\psi \in A$. We define*

$$\begin{aligned}\psi &= [\psi, -] : A \rightarrow A \\ a &\mapsto [\psi, a]\end{aligned}$$

Definition 5.2 *Let $\bar{\partial}$ be the differential as an operator acting on A in \mathfrak{g} . We formally define that*

$[\bar{\partial}, a] := \bar{\partial}a$, i.e

$$\bar{\partial} := [\bar{\partial}, -] : A \rightarrow A$$

$$a \mapsto [\bar{\partial}, a] := \bar{\partial}a$$

Proposition 5.3 *As an operator acting on A , we have $\bar{\partial}a = \bar{\partial} \circ a - (-1)^{|a|}a \circ \bar{\partial}$*

Proof. $[\bar{\partial}, [a, b]] = \bar{\partial}[a, b] = [\bar{\partial}a, b] + (-1)^{|a|}[a, \bar{\partial}b]$ ■

Proposition 5.4 *As an operator acting on A , we have $[a, b] = a \circ b - (-1)^{|a||b|}b \circ a$*

Proof. $[[a, b], c] = [a, [b, c]] - (-1)^{|a||b|}[b, [a, c]]$. ■

Definition 5.5 *We would like to define a differential (which we will denote by still same $\bar{\partial}$) which generalizes Proposition 5.3 on the space of operators on L_A generated by $\{L_a = [a, -] | a \in A\}$, identity, and $\bar{\partial} := [\bar{\partial}, -]$ in compository manner. So L_A has an identity element, which we denote by id . We set $\deg(\bar{\partial}) = 1$ and $\deg(id) = 0$. We define $\bar{\partial}X := \bar{\partial} \circ X - (-1)^{|X|}X \circ \bar{\partial}$ for $X \in L_A$. Then $\bar{\partial}(id) = 0$ and $\bar{\partial}(\bar{\partial}) = \bar{\partial}([\bar{\partial}, -]) = 0$ (First $\bar{\partial}$ is the differential on L and second $\bar{\partial}$ is an operator in L_A). We would like to define the grading on L_A in the following way: $\deg(a_1 \circ \dots \circ a_n) = |a_1| + \dots + |a_n|$ for $a_i \in A$ as operators where $|a|$ is the degree of a in A . Then $|\bar{\partial}X| = |X| + 1$ since the degree of $[\bar{\partial}] = 1$ and $\bar{\partial}(\bar{\partial}X) = 0$ for $X \in L_A$. We define the bracket on L_A in the following way: $[X, Y] = X \circ Y - (-1)^{|X||Y|}Y \circ X$.*

So we have $[\bar{\partial}, a] = \bar{\partial}a = \bar{\partial} \circ a - (-1)^{|a|}a \circ \bar{\partial}$ and $[a, b] = a \circ b - (-1)^{|a||b|}b \circ a$ for $a, b \in A$ as operators. This coincides with Proposition 5.3 and Proposition 5.4.

Remark 5.6 *We have an embedding*

$$A \hookrightarrow L_A$$

$$a \mapsto a := [a, -]$$

which is bracket $([-, -])$ preserving, and differential $(\bar{\partial})$ preserving, in other words,

$$[a, b] \mapsto [[a, b], -] = [[a, [b, -]] - (-1)^{|a||b|}[b, [a, -]]] = a \circ b - (-1)^{|a||b|}b \circ a$$

$$\bar{\partial}a \mapsto [\bar{\partial}a, -] = \bar{\partial} \circ a - (-1)^{|a|}a \circ \bar{\partial}$$

Proposition 5.7 (Product Rule) For $X, Y \in L_A$, we have $\bar{\partial}(X \circ Y) = \bar{\partial}X \circ Y + (-1)^{|X|}X \circ \bar{\partial}Y$.

Proof. $\bar{\partial}(X \circ Y) = \bar{\partial} \circ X \circ Y - (-1)^{|X|+|Y|}X \circ Y \circ \bar{\partial}$. On the other hand $\bar{\partial}X \circ Y = \bar{\partial} \circ X \circ Y - (-1)^{|X|}X \circ \bar{\partial} \circ Y$, and $(-1)^{|X|}X \circ \bar{\partial}Y = (-1)^{|X|}X \circ \bar{\partial} \circ Y - (-1)^{|X|+|Y|}X \circ Y \circ \bar{\partial}$. ■

Example 5.8 $\exp(tu) \circ u = u \circ \exp(tu)$, $t \in \mathbb{R}$. Here $\exp(x) = id + x + \frac{x \circ x}{2!} + \dots$ for $x \in L_A$.

Example 5.9 $\frac{d}{dt}\exp(tu) = \exp(tu)u$ for $t \in \mathbb{R}$. Indeed, $\frac{d}{dt}\exp(tu) = \frac{d}{dt}(id + tu + \frac{(tu) \circ (tu)}{2!} + \dots) = \frac{d}{dt}(id + tu + \frac{1}{2}t^2u \circ u + \dots) = u + tu \circ u + \frac{1}{2}t^2u \circ u \circ u + \dots = u(id + tu + \frac{1}{2}tu \circ tu + \dots) = u\exp(tu)$

Example 5.10 If $a \in A$ is holomorphic, then as a operator $\bar{\partial}(\exp(a)) = 0$. Indeed, since $\bar{\partial}a = 0$, we have $\bar{\partial}(\exp(a)) = \bar{\partial}(id + a + \frac{1}{2}a \circ a + \dots) = 0$ by applying the product rule.

Example 5.11 Let $\deg(a) = 0$. Then as an operator

$$\exp(a) \circ \bar{\partial}(\exp(a)) = [D(ad(a))(\bar{\partial}a), -] = [\bar{\partial}a - \frac{[a, \bar{\partial}a]}{2!} + \frac{[a, [a, \bar{\partial}a]]}{3!} + \dots, -]$$

where

$$D(x) = \frac{\exp(x) - 1}{x} = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

5.2 Deformations of a compact holomorphic Poisson manifold

Let (X, Λ_0) be a compact holomorphic Poisson manifold. In other words, the structure sheaf \mathcal{O}_X is a sheaf of Poisson algebras induced by Λ_0 . We define an infinitesimal Poisson deformation of (X, Λ_0) over $\text{Spec } R$, where R is a local artinian \mathbb{C} -algebra with residue \mathbb{C} .

Definition 5.12 Let (X, Λ_0) be a compact holomorphic Poisson manifold. An infinitesimal Poisson deformation of (X, Λ_0) is a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & \mathcal{X} \\ \downarrow & & \downarrow \pi \\ \text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(R) \end{array}$$

where π is a proper flat morphism, A is a local artinian \mathbb{C} -algebra with the residue \mathbb{C} , $X \cong \mathcal{X} \times_{\text{Spec}(A)} \text{Spec}(\mathbb{C})$, and $\mathcal{O}_{\mathcal{X}}$ is a sheaf of Poisson R -algebra, which induces the Poisson \mathbb{C} -algebra \mathcal{O}_X given by Λ_0 .

Remark 5.13 When we ignore Poisson structures, an infinitesimal Poisson deformation is simply a infinitesimal deformation of a underlying compact complex manifold. As complex analytic spaces, X is a closed subspace of \mathcal{X} . Since $\text{Spec}(R)$ is a fat point (one point set with structure sheaf R itself) and $\mathcal{X} \times_{\text{Spec}(A)} \text{Spec}(\mathbb{C})$, we have $\mathcal{X} \cong X$ topologically.

Remark 5.14 We can assume that $i : X \rightarrow \mathcal{X}$ is an identity map as topological spaces. In other words, i is simply equivalent to that $\mathcal{O}_{\mathcal{X}}(U) \rightarrow \mathcal{O}_X(U)$ is surjective for any open set U of X . In the sequel, I assume that i is identity as topological spaces.

Remark 5.15 The cartesian diagram can be seen in the sheaf theoretic setting which we mainly use in the part II of the thesis. The cartesian diagram is equivalent to the following cartesian diagram of sheaves

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{X}} & \longrightarrow & \mathcal{O}_X \\ \uparrow & & \uparrow \\ R & \longrightarrow & R/\mathfrak{m} \cong \mathbb{C} \end{array}$$

where a sheaf morphism $\mathcal{O}_{\mathcal{X}} \rightarrow \mathcal{O}_X$ over \mathbb{C} on X such that $\mathcal{O}_{\mathcal{X}}$ is a sheaf of flat Poisson A -algebra with $\mathcal{O}_{\mathcal{X}} \otimes_R R/\mathfrak{m} \cong \mathcal{O}_X$ as sheaves of Poisson \mathbb{C} -algebras in the following sense: for any open set U of X , the Poisson bracket $\{-, -\}_{\mathcal{X}} : \mathcal{O}_{\mathcal{X}}(U) \times \mathcal{O}_{\mathcal{X}}(U) \rightarrow \mathcal{O}_{\mathcal{X}}(U)$ induces the Poisson bracket on $\mathcal{O}_X(U)$ in the following way. $\{-, -\}_X : (\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathbb{C}} R/\mathfrak{m}) \times (\mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathbb{C}} R/\mathfrak{m}) \rightarrow \mathcal{O}_{\mathcal{X}}(U) \otimes_{\mathbb{C}} R/\mathfrak{m}$ by $\{f \otimes r_1, g \otimes r_2\}_X = \{f, g\}_{\mathcal{X}} \otimes r_1 r_2$.

Proposition 5.16 Let U be an open ball containing $0 \in \mathbb{C}^n$ with a coordinate (z_1, \dots, z_n) . An infinitesimal deformation \mathcal{X} of U over $\text{Spec}(R)$ is rigid. In other words, $\mathcal{X} \cong U \times \text{Spec}(R)$.

Proof. We refer to [Ser06] Theorem 1.2.4. ■

Let \mathcal{X} be an infinitesimal Poisson deformation of a compact holomorphic Poisson manifold (X, Λ_0) over a local artinian \mathbb{C} -algebra (R, \mathfrak{m}) with residue \mathbb{C} . Let $\{U_i\}$ be an open covering of X ,

where U_i is isomorphic to an open ball. By Proposition 5.16, locally an infinitesimal deformation \mathcal{X} of the underlying complex manifold X is

$$\begin{array}{ccc} U_i & \longrightarrow & \mathcal{X}|_{U_i} \cong U_i \times \text{Spec}(R) \\ \downarrow & & \downarrow \pi \\ \text{Spec}(\mathbb{C}) & \longrightarrow & \text{Spec}(R) \end{array}$$

We call such an open cover $\{U_i\}$ locally trivial open covering of \mathcal{X} .

Remark 5.17 Let $\{U_i\}$ be a locally trivial open covering of \mathcal{X} . Since $\mathcal{O}_{\mathcal{X}}(U_i) \cong \mathcal{O}_X(U_i) \otimes_{\mathbb{C}} R$, the Poisson R -algebra structure on $\mathcal{O}_{\mathcal{X}}(U_i)$ is encoded in some holomorphic bivector field $\Lambda'_i \in \Gamma(U_i, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes_{\mathbb{C}} R$ with $\bar{\partial}\Lambda'_i = 0$, $[\Lambda'_i, \Lambda'_i] = 0$, where $\bar{\partial}$ on $\Gamma(U_i, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes_{\mathbb{C}} R$ is induced from $\bar{\partial}$ on $\Gamma(U_i, \mathcal{A}^{0,0}(\wedge^2 T_X))$ and $[-, -]$ is induced from the bracket on U_i by extending R -linearly. More precisely, let U_i be an open ball with coordinate $z = (z_1, \dots, z_n)$. Then the Poisson structure on $\mathcal{O}_{\mathcal{X}}(U_i)$ over R , where R is generated by $\langle 1, m_1, \dots, m_r \rangle$ over \mathbb{C} is encoded in the bivector field

$$\Lambda'_i = \sum_{\alpha, \beta} \Lambda_{\alpha\beta} \frac{\partial}{\partial z_{\alpha}} \wedge \frac{\partial}{\partial z_{\beta}} = \sum_{\alpha, \beta} (\Lambda_{\alpha\beta}^0 + m_1 \Lambda_{\alpha\beta}^1 + \dots + m_r \Lambda_{\alpha\beta}^r) \frac{\partial}{\partial z_{\alpha}} \wedge \frac{\partial}{\partial z_{\beta}}$$

with $[\Lambda'_i, \Lambda'_i] = 0$, where $\Lambda_{\alpha\beta}^k(z)$ is holomorphic on U_i for each k . Then

$$\{f, g\} = \langle \Lambda'_i, df \wedge dg \rangle = \langle \sum_{\alpha, \beta} \Lambda_{\alpha\beta} \frac{\partial}{\partial z_{\alpha}} \wedge \frac{\partial}{\partial z_{\beta}}, df \wedge dg \rangle = \sum_{\alpha, \beta} \Lambda_{\alpha\beta} \left(\frac{\partial f}{\partial z_{\alpha}} \frac{\partial g}{\partial z_{\beta}} - \frac{\partial g}{\partial z_{\alpha}} \frac{\partial f}{\partial z_{\beta}} \right)$$

On the other hand,

$$\begin{aligned} [[\Lambda'_i, f], g] &= \sum_{\alpha, \beta} [[\Lambda_{\alpha\beta} \frac{\partial}{\partial z_{\alpha}} \wedge \frac{\partial}{\partial z_{\beta}}, f], g] = \sum_{\alpha, \beta} [\Lambda_{\alpha\beta} \frac{\partial f}{\partial z_{\alpha}} \frac{\partial}{\partial z_{\beta}} - \Lambda_{\alpha\beta} \frac{\partial f}{\partial z_{\beta}} \frac{\partial}{\partial z_{\alpha}}, g] \\ &= \sum_{\alpha, \beta} \Lambda_{\alpha\beta} \left(\frac{\partial f}{\partial z_{\alpha}} \frac{\partial g}{\partial z_{\beta}} - \frac{\partial g}{\partial z_{\alpha}} \frac{\partial f}{\partial z_{\beta}} \right) \end{aligned}$$

So we have the expression

$$\{f, g\} = [[\Lambda'_i, f], g]$$

Note that since Λ'_i induces the Poisson structure of (U_i, Λ_0) , the Poisson structure of (X, Λ_0) on U_i is given by

$$\Lambda_0 = \sum_{\alpha\beta} \Lambda_{\alpha\beta}^0 \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial z_\beta}$$

Definition 5.18 (equivalent Poisson deformations) We say that an infinitesimal Poisson deformation \mathcal{X} and an infinitesimal Poisson deformation \mathcal{Y} of a holomorphic Poisson manifold (X, Λ_0) over a local artinian \mathbb{C} -algebra R are equivalent if the following diagram is commutative.

$$\begin{array}{ccc} & (X, \Lambda_0) & \\ \swarrow & & \searrow \\ \mathcal{X} & \xrightarrow{f \cong} & \mathcal{Y} \\ \searrow & & \swarrow \\ & \text{Spec } R & \end{array}$$

where $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a Poisson isomorphism. In other words, $f^\# : \mathcal{O}_{\mathcal{Y}} \rightarrow f_* \mathcal{O}_{\mathcal{X}}$ is an isomorphism of Poisson sheaves.

5.3 Integrability condition

Let \mathcal{X} be a infinitesimal Poisson deformation of a compact holomorphic Poisson manifold (X, Λ_0) over R , where (R, \mathfrak{m}) is an local artinian \mathbb{C} -algebra with residue \mathbb{C} .

Let $\mathcal{U} = \{U_\alpha\}$ be a locally trivial open covering of \mathcal{X} . Then we have a set of isomorphisms of Poisson R -algebra

$$\varphi_\alpha : \mathcal{O}_{\mathcal{X}}(U_\alpha) \xrightarrow{\sim} \mathcal{O}_X(U_\alpha) \otimes R$$

where the Poisson R -structure on $\mathcal{O}_X(U_\alpha) \otimes R$, which we denote by Λ'_α , is induced from the Poisson structure of $\mathcal{O}_{\mathcal{X}}(U_\alpha)$ via φ_α .

$$\theta_{\alpha\beta} := \varphi_\alpha \varphi_\beta^{-1} : (\mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R, \Lambda'_\beta) \rightarrow (\mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R, \Lambda'_\alpha)$$

are Poisson isomorphisms and satisfying cocycle condition $\theta_{\gamma\alpha} \circ \theta_{\alpha\beta} = \theta_{\gamma\beta}$.

Lemma 5.19 *In the above, if (R, \mathfrak{m}) has the exponent n (i.e. $\mathfrak{m}^{n+1} = 0$), then $\theta_{\alpha\beta} = \exp(t_{\alpha\beta}) = I + t_{\alpha\beta} + \frac{1}{2}(t_{\alpha\beta})^2 + \cdots + \frac{1}{n!}(t_{\alpha\beta})^n$ for some $t_{\alpha\beta} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}$.*

Proof. We prove this by induction on the exponent n of the maximal ideal \mathfrak{m} of a local artinian \mathbb{C} -algebra (R, \mathfrak{m}) .

Let's prove that the proposition holds for $n = 1$. We assume that $\mathfrak{m}^2 = 0$. Let $f := f \otimes 1 \in \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R$, which we consider an element in $\mathcal{O}_X(U_\alpha \cap U_\beta)$. Note that $\theta_{\alpha\beta}$ is completely determined by $\mathcal{O}_X(U_\alpha \cap U_\beta)$ since θ is R -algebra map. Let $\dim_{\mathbb{C}} \mathfrak{m} = r$ with $\mathfrak{m} = \langle e_1, \dots, e_r \rangle$. Since $R = \mathbb{C} \oplus \mathfrak{m}$, $\theta_{\alpha\beta}(f) = f + e_1 h_1(f) + \cdots + e_r h_r(f)$. Then $t_{\alpha\beta} := \sum_i e_i h_i$ is a derivation. Indeed, for $f := f \otimes 1, g := g \otimes 1 \in \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R$, we have $\theta_{\alpha\beta}(fg) = \theta_{\alpha\beta}(f)\theta_{\alpha\beta}(g) = fg + e_1 h_1(fg) + \cdots + e_r h_r(fg) = (f + e_1 h_1(f) + \cdots + e_r h_r(f))(g + e_1 h_1(g) + \cdots + e_r h_r(g)) = fg + e_1(fh_1(g) + gh_1(f)) + \cdots + e_r(fh_r(g) + gh_r(f))$ since $\mathfrak{m}^2 = 0$, so we have $\{t_{\alpha\beta} = \sum_i h_i e_i\} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}$ and $\theta_{\alpha\beta} = \exp(t_{\alpha\beta}) = \exp(\sum h_i e_i)$.

Now we assume that the proposition holds for up to $n-1$. Let (R, \mathfrak{m}) with $\mathfrak{m}^{n+1} = 0$. Then $(R/\mathfrak{m}^n, \mathfrak{m}/\mathfrak{m}^n)$ is a local artinian \mathbb{C} -algebra with residue \mathbb{C} and exponent $n-1$. We note that $\theta_{\alpha\beta} : \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R \rightarrow \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R$ induces $\bar{\theta}_{\alpha\beta} : \mathcal{O}(U_\alpha \cap U_\beta) \otimes (R/\mathfrak{m}^n) \rightarrow \mathcal{O}(U_\alpha \cap U_\beta) \otimes (R/\mathfrak{m}^n)$. Then by the induction hypothesis, $\bar{\theta}_{\alpha\beta} = \exp(\bar{t}_{\alpha\beta})$ for some $\bar{t}_{\alpha\beta} \in C^1(\mathcal{U}, \Theta_X) \otimes (\mathfrak{m}/\mathfrak{m}^n)$. Let $t_{\alpha\beta}$ be an arbitrary lifting of $\bar{t}_{\alpha\beta}$ via $\mathfrak{m} \rightarrow \mathfrak{m}/\mathfrak{m}^n$. Let $\eta_{\alpha\beta} := \theta_{\alpha\beta} - \exp(t_{\alpha\beta})$. We claim that $\{\eta_{\alpha\beta}\} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}^n$. Indeed, let f, g as above. Then $\eta_{\alpha\beta}(fg) = \theta_{\alpha\beta}(f)\theta_{\alpha\beta}(g) - (\exp(t_{\alpha\beta})f)(\exp(t_{\alpha\beta})g) = \theta_{\alpha\beta}(f)\theta_{\alpha\beta}(g) - \theta_{\alpha\beta}(f)(\exp(t_{\alpha\beta})g) + \theta_{\alpha\beta}(f)(\exp(t_{\alpha\beta})g) - (\exp(t_{\alpha\beta})f)(\exp(t_{\alpha\beta})g) = \theta_{\alpha\beta}(f)(\theta_{\alpha\beta} - \exp(t_{\alpha\beta}))(g) + (\exp(t_{\alpha\beta})g)(\theta_{\alpha\beta} - \exp(t_{\alpha\beta}))(f) = f\eta_{\alpha\beta}(g) + g\eta_{\alpha\beta}(f)$ since $\eta_{\alpha\beta}(g), \eta_{\alpha\beta}(f) \in \mathcal{O}(U_\alpha \cap U_\beta) \otimes \mathfrak{m}^n$ and $\mathfrak{m}^{n+1} = 0$. So $\eta_{\alpha\beta} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}^n$. Hence we have

$$\theta_{\alpha\beta} = \exp(t_{\alpha\beta}) + \eta_{\alpha\beta} = \exp(t_{\alpha\beta} + \eta_{\alpha\beta})$$

■

Proposition 5.20 (Compare [Kod05] Theorem 2.4 page 64) *Let $\mathcal{X} \rightarrow \text{Spec } R$ be an infinitesimal deformation of a compact complex manifold X . Then we have an isomorphism of sheaves $\mathcal{A}_{\mathcal{X}}^{0,0} \cong \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R$. In other words, a locally trivial infinitesimal C^∞ -deformation of a compact complex manifold X over a local artinian \mathbb{C} -algebra R with residue \mathbb{C} is (globally) trivial.¹ We call a map defining $\mathcal{A}_{\mathcal{X}}^{0,0} \cong \mathcal{A}_X^{0,0} \otimes R$ a C^∞ -trivialization.*

Remark 5.21 *Before the proof of Proposition 5.20, we will clarify the meaning of $\mathcal{A}_{\mathcal{X}}^{0,0} \cong \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R$.*

We define a sheaf, denoted by $\mathcal{A}_X^{0,0}$ on X in the following way. Let $\mathcal{U} = \{U_\alpha\}$ be a locally trivial open

¹The proposition is the infinitesimal version of [Kod05] Theorem 2.4 (page 64). It took a lot of time for me to prove this proposition. Eventually I got the idea of the proof of the proposition from [Kod05] Theorem 2.4.

covering of \mathcal{X} . So we have $\varphi_\alpha : \mathcal{O}_{\mathcal{X}}(U_\alpha) \cong \mathcal{O}_X(U_\alpha) \otimes R$ and $\varphi_\alpha \varphi_\beta^{-1} = \exp(t_{\alpha\beta})$ on $\mathcal{O}_X(U_\alpha \cap U_\beta)$ satisfying cocycle conditions where $\{t_{\alpha\beta}\} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}$. Let $\mathcal{A}_X^{0,0}$ be a sheaf of complex valued C^∞ -functions on X . We will denote its section on U by $\mathcal{A}_X^{0,0}(U) := \Gamma(U, \mathcal{A}_X^\infty)$ for an open set U of X . Then $\exp(t_{\alpha\beta}) : \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R \rightarrow \mathcal{O}_X(U_\alpha \cap U_\beta) \otimes R$ induces $\exp(t_{\alpha\beta}) : \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes R \rightarrow \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes R$. Since $\{\exp(t_{\alpha\beta})\}$ satisfies cocycle conditions, this defines a sheaf locally isomorphic to $\mathcal{A}_X^{0,0}|_{U_\alpha} \otimes R$ on U_α , which is the sheaf $\mathcal{A}_X^{0,0}$. We will denote its section on U by $\mathcal{A}_X^{0,0}(U) := \Gamma(U, \mathcal{A}_X^{0,0})$. We will use the same φ_α for local trivialization of $\mathcal{A}_X^{0,0}$ (i.e. $\varphi_\alpha : \mathcal{A}_X^{0,0}(U_\alpha) \rightarrow \mathcal{A}_X^{0,0}(U_\alpha) \otimes R$). We would like to construct an explicit morphism of sheaves $\mathcal{A}_X^{0,0} \rightarrow \mathcal{A}_X^{0,0} \otimes R$.

Proof of Proposition 5.20.

We will prove by induction on the exponent of the maximal ideal \mathfrak{m} of R . First we show that the proposition holds for (R, \mathfrak{m}) with $\mathfrak{m}^2 = 0$. Let $\{U_\alpha\}$ be an locally trivial open covering of \mathcal{X} . Then locally we have $\mathcal{A}_X^{0,0}(U_\alpha) \cong \mathcal{A}_X^{0,0}(U_\alpha) \otimes_{\mathbb{C}} R$. There exist $\exp(t_{\alpha\beta}) : \mathcal{A}_X^{0,0}(U_\beta \cap U_\alpha) \otimes_{\mathbb{C}} R \rightarrow \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R$ and we have $\exp(t_{\alpha\gamma}) = \exp(t_{\alpha\beta}) \circ \exp(t_{\beta\gamma})$ which is same to $1 + t_{\alpha\gamma} = (1 + t_{\alpha\beta}) \circ (1 + t_{\beta\gamma}) = 1 + t_{\alpha\beta} + t_{\beta\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. We also note that $t_{\alpha\alpha} = 0$ and $t_{\alpha\beta} = -t_{\beta\alpha}$. We will show that $\mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R \cong \mathcal{A}_X^{0,0}$ by defining a map which is compatible with $\exp(t_{\alpha\beta})$. Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$. Then for each α , we set $s_\alpha := \sum_\gamma \rho_\gamma t_{\gamma\alpha} \in \Gamma(U_\alpha, \mathcal{A}_X^{0,0}(T_X)) \otimes \mathfrak{m}$ ($U_\gamma \cap U_\alpha \neq \emptyset$). Then we have

$$\begin{aligned} \exp(s_\alpha) \circ \exp(t_{\alpha\beta}) &= 1 + t_{\alpha\beta} + \sum_\gamma \rho_\gamma t_{\gamma\alpha} = 1 + \sum_\gamma \rho_\gamma t_{\alpha\beta} + \sum_\gamma \rho_\gamma t_{\gamma\alpha} \\ &= 1 + \sum_\gamma \rho_\gamma t_{\gamma\beta} = \exp(s_\beta) \end{aligned}$$

So we have the following commutative diagram

$$\begin{array}{ccc} & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R & \\ \exp(s_\beta) \swarrow & & \searrow \exp(s_\alpha) \\ \mathcal{A}_X^{0,0}(U_\beta \cap U_\alpha) \otimes_{\mathbb{C}} R & \xrightarrow{\exp(t_{\alpha\beta}) = 1 + t_{\alpha\beta}} & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R \end{array}$$

This compatibly shows that $\mathcal{A}_{\mathcal{X}}^{0,0} \cong \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R$. So we proved the proposition for R with the exponent 1.

Now we assume that the proposition holds for R with exponent up to $n - 1$. Let (R, \mathfrak{m}) be an artinian local \mathbb{C} -algebra with the residue \mathbb{C} and exponent n (i.e. $\mathfrak{m}^{n+1} = 0$), and \mathcal{X} be an infinitesimal deformation of X over R . Then $(R/\mathfrak{m}^n, \mathfrak{m}/\mathfrak{m}^n)$ is a local artinian \mathbb{C} -algebra with exponent $n - 1$. Let $\{U_\alpha\}$ be a locally trivializing open covering of \mathcal{X} . There exist $exp(t_{\alpha\beta}) : \mathcal{A}_X^{0,0}(U_\beta \cap U_\alpha) \otimes_{\mathbb{C}} R \rightarrow \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R$ and we have $exp(t_{\alpha\gamma}) = exp(t_{\alpha\beta}) \circ exp(t_{\beta\gamma})$. Then by the induction hypothesis, we have $\mathcal{A}_X^{0,0} \otimes_R R/\mathfrak{m}^n \cong \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R/\mathfrak{m}^n$ and so there exist $\bar{s}_\alpha \in \Gamma(U_\alpha, \mathcal{A}_X^{0,0}(T_X)) \otimes \mathfrak{m}/\mathfrak{m}^n$ such that $exp(\bar{s}_\alpha) = exp(\bar{t}_{\alpha\beta}) \circ exp(\bar{s}_\beta)$, where $\{\bar{t}_{\alpha\beta}\}$ is the image of the natural map $C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m} \rightarrow C^1(\mathcal{U}, \Theta_X) \otimes R/\mathfrak{m}^n$. Let s_α be a lifting of \bar{s}_α by the natural map $\Gamma(U_\alpha, \mathcal{A}_X^{0,0}(T_X)) \otimes \mathfrak{m} \rightarrow \Gamma(U_\alpha, \mathcal{A}_X^{0,0}(T_X)) \otimes \mathfrak{m}/\mathfrak{m}^n$. Then $exp(t_{\alpha\beta}) \circ exp(s_\beta) - exp(s_\alpha)$ is 0 modulo \mathfrak{m}^n and is a derivation. Indeed, set $A = exp(t_{\alpha\beta}) \circ exp(s_\beta)$ and $B = exp(s_\alpha)$. Then $(A - B)(fg) = A(f)(A - B)(g) + B(g)(A - B)(f) = f(A - B)(g) + g(A - B)(f)$ since $A - B = 0$ modulo \mathfrak{m}^n and $\mathfrak{m}^{n+1} = 0$. Set $r_{\alpha\beta} := exp(t_{\alpha\beta}) \circ exp(s_\beta) - exp(s_\alpha)$. Since $exp(-t_{\alpha\beta}) = exp(t_{\beta\alpha})$, we have $exp(t_{\beta\alpha}) \circ r_{\alpha\beta} = exp(s_\beta) - exp(t_{\beta\alpha}) \circ exp(s_\alpha)$. So we get $exp(t_{\beta\alpha}) \circ r_{\alpha\beta} = -r_{\beta\alpha}$. Since $r_{\alpha\beta}$ is 0 modulo \mathfrak{m}^n , we have $r_{\alpha\beta} = -r_{\beta\alpha}$. Next we claim that $r_{\alpha\beta} = r_{\gamma\beta} + r_{\alpha\gamma}$. Indeed, since $r_{\alpha\beta} = exp(t_{\alpha\beta}) \circ exp(s_\beta) - exp(s_\alpha)$ and $r_{\alpha\beta} = 0$ modulo \mathfrak{m}^n , by applying $exp(t_{\gamma\alpha})$ on both sides,

$$\begin{aligned} r_{\alpha\beta} &= exp(t_{\gamma\alpha}) \circ r_{\alpha\beta} = exp(t_{\gamma\alpha}) \circ exp(t_{\alpha\beta}) \circ exp(s_\beta) - exp(t_{\gamma\alpha}) \circ exp(s_\alpha) \\ &= exp(t_{\gamma\beta}) \circ exp(s_\beta) - r_{\gamma\alpha} - exp(s_\gamma) = r_{\gamma\beta} - r_{\gamma\alpha} = r_{\gamma\beta} + r_{\alpha\gamma} \end{aligned}$$

Let $\{\rho_\alpha\}$ be a partition of unity subordinate to the open covering $\{U_\alpha\}$. Then for each α , we set $e_\alpha := \sum_\gamma \rho_\gamma r_{\alpha\gamma} \in \Gamma(U_\alpha, \mathcal{A}_X^{0,0}(T_X)) \otimes \mathfrak{m}^n$. Then we have

$$\begin{aligned} exp(t_{\alpha\beta}) \circ exp(s_\beta + e_\beta) &= exp(t_{\alpha\beta}) \circ exp(s_\beta) + e_\beta = exp(s_\alpha) + r_{\alpha\beta} + e_\beta \\ &= exp(s_\alpha) + \sum_\gamma \rho_\gamma r_{\alpha\beta} + \sum_\gamma \rho_\gamma r_{\beta\gamma} = exp(s_\alpha) + \sum_\gamma \rho_\gamma (r_{\alpha\beta} + r_{\beta\gamma}) \\ &= exp(s_\alpha) + \sum_\gamma \rho_\gamma r_{\alpha\gamma} = exp(s_\alpha) + e_\alpha = exp(s_\alpha + e_\alpha) \end{aligned}$$

So we have the following commutative diagram

$$\begin{array}{ccc} & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R & \\ & \swarrow \quad \quad \searrow & \\ exp(s_\beta + e_\beta) & & exp(s_\alpha + e_\alpha) \\ & \swarrow \quad \quad \searrow & \\ \mathcal{A}_X^{0,0}(U_\beta \cap U_\alpha) \otimes_{\mathbb{C}} R & \xrightarrow{exp(t_{\alpha\beta})} & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes_{\mathbb{C}} R \end{array}$$

This compatibly shows that $\mathcal{A}_X^{0,0} \cong \mathcal{A}_X^{0,0} \otimes_{\mathbb{C}} R$. This completes Proposition 5.20.

■

As in the proof of Proposition 5.20, there is a global C^∞ trivialization $C : \mathcal{A}_X^{0,0} \otimes R \cong \mathcal{A}_X^{0,0}$. This induces $C : \mathcal{A}_X^{0,0}(U_\alpha) \otimes R \rightarrow \mathcal{A}_X^{0,0}(U_\alpha)$ such that $\varphi_\alpha \circ C$ is of the form $\exp(s_\alpha)$, where $s_\alpha \in \Gamma(U_\alpha, \mathcal{A}^{0,0}(T_X)) \otimes \mathfrak{m}$. Here $\varphi_\alpha : \mathcal{A}_X^{0,0}(U_\alpha) \cong \mathcal{A}_X^{0,0}(U_\alpha) \otimes R$ is a local trivialization of $\mathcal{A}_X^{0,0}(U_\alpha)$. (See Remark 5.21)

So we have

$$\begin{aligned} \exp(s_\alpha) &= \varphi_\alpha \circ C : \mathcal{A}_X^{0,0}(U_\alpha) \otimes R \rightarrow \mathcal{A}_X^{0,0}(U_\alpha) \otimes R \\ \exp(-s_\beta) &= C^{-1} \circ \varphi_\beta^{-1} \end{aligned}$$

So we have $\exp(s_\alpha)\exp(-s_\beta) = \varphi_\alpha \circ C \circ C^{-1} \circ \varphi_\beta^{-1} = \exp(t_{\alpha\beta})$.

So we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{A}_X^{0,0}(U_\beta \cap U_\alpha) \otimes R & \xrightarrow{\exp(t_{\alpha\beta})} & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes R \\ & \swarrow \varphi_\beta & \searrow \varphi_\alpha \\ & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) & \\ \exp(s_\beta) \swarrow & \uparrow C & \searrow \exp(s_\alpha) \\ & \mathcal{A}_X^{0,0}(U_\alpha \cap U_\beta) \otimes R & \end{array}$$

5.3.1 Integrability Condition

We set

$$\phi_\alpha := \exp(-s_\alpha)\bar{\partial}(\exp(s_\alpha)) = [D(\text{ad}(-s_\alpha))(\bar{\partial}s_\alpha), -] \text{ as an operator acting on } A$$

where D is the function

$$D(x) = \frac{\exp(x) - 1}{x} = \sum_{i=0}^{\infty} \frac{x^i}{(i+1)!}$$

We have

$$D(\text{ad}(s_\alpha))(\bar{\partial}s_\alpha) = \bar{\partial}s_\alpha - \frac{[s_\alpha, \bar{\partial}s_\alpha]}{2!} + \frac{[s_\alpha, [s_\alpha, \bar{\partial}s_\alpha]]}{3!} + \dots \in \Gamma(U_\alpha, \mathcal{A}^{0,1}(T)) \otimes \mathfrak{m}$$

Note that we have

$$0 = \bar{\partial}\exp(t_{\alpha\beta}) = \bar{\partial}(\exp(s_\alpha)\exp(-s_\beta)) = \bar{\partial}(\exp(s_\alpha))\exp(-s_\beta) + \exp(s_\alpha)\bar{\partial}(\exp(-s_\beta))$$

So we have

$$\exp(-s_\alpha)\bar{\partial}(\exp(s_\alpha)) = -\bar{\partial}(\exp(-s_\beta))\exp(s_\beta)$$

Since $0 = \bar{\partial}(\exp(-s_\beta)\exp(s_\beta)) = \bar{\partial}(\exp(-s_\beta))\exp(s_\beta) + \exp(-s_\beta)\bar{\partial}(\exp(s_\beta))$, we have

$$-\bar{\partial}(\exp(-s_\beta))\exp(s_\beta) = \exp(-s_\beta)\bar{\partial}(\exp(s_\beta))$$

Hence

$$\exp(-s_\alpha)\bar{\partial}(\exp(s_\alpha)) = \exp(-s_\beta)\bar{\partial}(\exp(s_\beta))$$

So $D(\text{ad}(s_\alpha))(\bar{\partial}s_\alpha)$ glue together to give a global section

$$\phi \in A^{0,1}(X, T) \otimes \mathfrak{m}$$

Since

$$\bar{\partial}\phi_\alpha = \bar{\partial}\exp(-s_\alpha)\bar{\partial}\exp(-s_\alpha) = \bar{\partial}\exp(-s_\alpha)\exp(s_\alpha)\exp(-s_\alpha)\bar{\partial}\exp(s_\alpha) = -\phi_\alpha \circ \phi_\alpha = -\frac{1}{2}[\phi_\alpha, \phi_\alpha]$$

We have

$$\bar{\partial}\phi = -\frac{1}{2}[\phi, \phi]$$

Now we consider the Poisson structures. We need the following lemma.

Lemma 5.22 *We have $[\exp(a)f, \exp(a)g] = \exp(a)[f, g]$, where $\deg(a) = 0$.*

Proof. We claim that

$$\sum_{l+m=n} \frac{1}{l!m!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]}_l, \underbrace{[[a, [a, [\dots [a, g] \dots]]]}_m = \frac{1}{n!} \underbrace{[a, [a, [\dots [a, [f, g]] \dots]]]}_n$$

Then this implies the lemma. We will prove this by induction. First we note that $[[a, f], g] = [a, [f, g]] - [f, [a, g]]$. So we have

$$[a, [f, g]] = [[a, f], g] + [f, [a, g]]$$

Hence the claim holds for $n = 1$. Now let's assume that the claim holds for $n - 1$. We will prove that it holds for n .

$$\begin{aligned}
& \frac{1}{n!} [a, \underbrace{[a, [\dots [a, [f, g]] \dots]]}_n] = \frac{1}{n} [a, \frac{1}{(n-1)!} \underbrace{[a, [a, [\dots [a, [f, g]] \dots]]}_{n-1}] \\
&= \frac{1}{n} [a, \sum_{p+q=n-1} \frac{1}{p!q!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_p}_{p} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_q}_{q}] \text{ (by induction hypothesis)} \\
&= \frac{1}{n} \sum_{p+q=n-1} \frac{1}{p!q!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_{p+1}}_{p+1} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_q}_{q} \\
&+ \frac{1}{n} \sum_{p+q=n-1} \frac{1}{p!q!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_p}_{p} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_{q+1}}_{q+1} \\
&= \frac{1}{n} \sum_{l+m=n, l \geq 1} \frac{l}{l!m!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_l}_{l} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_m}_{m} \text{ (by taking } p+1=l, q=m) \\
&+ \frac{1}{n} \sum_{l+m=n, m \geq 1} \frac{m}{l!m!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_l}_{l} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_m}_{m} \text{ (by taking } p=l, q+1=m) \\
&= \frac{1}{n} \sum_{l+m=n, l, m \geq 1} \frac{l+m}{l!m!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_l}_{l} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_m}_{m} \\
&+ \frac{1}{n} \frac{n}{n!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_n}_{n} + \frac{1}{n} \frac{n}{n!} [f, \underbrace{[[a, [a, [\dots [a, g] \dots]]]_n}_{n}] \\
&= \sum_{l+m=n} \frac{1}{l!m!} \underbrace{[[a, [a, [\dots [a, f] \dots]]]_l}_{l} \underbrace{[[a, [a, [\dots [a, g] \dots]]]_m}_{m}
\end{aligned}$$

■

Definition 5.23 Let $\mathcal{X} \rightarrow \text{Spec } R$ be an infinitesimal deformation of X , where R is generated by $\langle 1, m_1, \dots, m_r \rangle$. We define a sheaf $\mathcal{A}^{0,p}(\wedge^q T_{\mathcal{X}/R})$ which is locally isomorphic to $\mathcal{A}^{0,p}(\wedge^q T_X)|_{U_\alpha} \otimes R$ on U_α in the following way: we define this sheaf by gluing sheaves $\mathcal{A}^{0,p}(\wedge^q T_X)|_{U_\alpha} \otimes R$ locally defined on each U_α . On $U_\alpha \cap U_\beta$, we have isomorphisms $\exp(t_{\alpha\beta}) : (\mathcal{A}^{0,p}(\wedge^q T_X)|_{U_\beta})|_{U_\alpha \cap U_\beta} \rightarrow (\mathcal{A}^{0,p}(\wedge^q T_X)|_{U_\alpha})|_{U_\alpha \cap U_\beta}$. Since $\exp(t_{\alpha\beta}) = \exp(t_{\alpha\gamma})\exp(t_{\gamma\beta})$ on $U_\alpha \cap U_\beta \cap U_\gamma$, we can glue these

sheaves. By Lemma 5.22, we can define a bracket $[-, -]$ on $\oplus_{p \geq 1, q \geq 0} \mathcal{A}^{0,p}(\wedge^q T_{\mathcal{X}/R})$ locally induced from the bracket on $[-, -]$ on $\oplus_{p \geq 1, q \geq 0} \Gamma(U_\alpha, \mathcal{A}^{0,p}(\wedge^q T_X)) \otimes R$.

Now we define the dolbeault differential $\bar{\partial}_{\mathcal{X}} : \mathcal{A}^{0,p}(\wedge^q T_{\mathcal{X}/R}) \rightarrow \mathcal{A}^{0,p+1}(\wedge^q T_{\mathcal{X}/R})$ of morphism of sheaves. Locally for $f \in \Gamma(U_\alpha, \mathcal{A}^{0,p}(\wedge^q T_X)) \otimes R$, which is of the form $f = f_0 + m_1 f_1 + \dots + m_r f_r$, where $f_i \in \Gamma(U_\alpha, \mathcal{A}^{0,p}(\wedge^q T_X))$, we define $\bar{\partial}_{\mathcal{X}} f = \bar{\partial} f_0 + m_1 \bar{\partial} f_1 + \dots + m_r \bar{\partial} f_r$. We show that this is glued together. It is equivalent to show that $\exp(t_{\alpha\beta}) \circ \bar{\partial}_{\mathcal{X}} f = \bar{\partial}_{\mathcal{X}} \circ \exp(t_{\alpha\beta}) f$. It is enough to show that $t_{\alpha\beta} \circ \bar{\partial}_{\mathcal{X}} = \bar{\partial}_{\mathcal{X}} \circ t_{\alpha\beta}$, which is equivalent to $\bar{\partial}_{\mathcal{X}} t_{\alpha\beta} = 0$ (more precisely, let $t_{\alpha\beta} = t_0 + m_1 t_1 + \dots + m_r t_r$, then above condition is equivalent to $\bar{\partial} t_i = 0$, which is equivalent to $\{t_{\alpha\beta}\} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}$). So $\bar{\partial}_{\mathcal{X}} : \mathcal{A}^{0,p}(\wedge^q T_{\mathcal{X}/R}) \rightarrow \mathcal{A}^{0,p+1}(\wedge^q T_{\mathcal{X}/R})$ is well defined with $\bar{\partial}_{\mathcal{X}} \circ \bar{\partial}_{\mathcal{X}} = 0$. We will denote $\bar{\partial}_{\mathcal{X}}$ simply by $\bar{\partial}$.

Remark 5.24 Let $\mathcal{X} \rightarrow \text{Spec } R$ be an infinitesimal Poisson deformation of (X, Λ_0) . Let $\{U_\alpha\}$ be a locally trivial open covering of \mathcal{X} . Then $\mathcal{O}_{\mathcal{X}}(U_\alpha) \cong \mathcal{O}_X(U_\alpha) \otimes R$. The Poisson structure is encoded in $\Lambda'_\alpha \in \Gamma(U_\alpha, \mathcal{A}^{0,0}(\wedge^2 T)) \otimes R$ with $[\Lambda'_\alpha, \Lambda'_\alpha] = 0$ and $\bar{\partial} \Lambda'_\alpha = 0$. On $U_\alpha \cap U_\beta$, $\exp(t_{\alpha\beta})[[\Lambda'_\beta, f], g] = [[\Lambda'_\alpha, \exp(t_{\alpha\beta})f], \exp(t_{\alpha\beta})g]$. Since $\exp(t_{\alpha\beta})[[\Lambda'_\beta, f], g] = [[\exp(t_{\alpha\beta})\Lambda'_\beta, \exp(t_{\alpha\beta})f], \exp(t_{\alpha\beta})g]$ by Lemma 5.22, we have $\exp(t_{\alpha\beta})\Lambda'_\beta = \Lambda'_\alpha$. Hence the Poisson R -structure of \mathcal{X} can be identified with the existence of a global section Λ' of the sheaf $\mathcal{A}^{0,0}(\wedge^2 T_{\mathcal{X}/R})$ with $[\Lambda', \Lambda'] = 0$ and $\bar{\partial} \Lambda' = 0$ such that Λ' induces Λ_0 .

Going back to our discussion, since $\exp(-s_\alpha) = C^{-1} \circ (\varphi_\alpha)^{-1}$ induces an isomorphism

$$\Gamma(U_\alpha, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes R \cong \Gamma(U_\alpha, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes R$$

which is compatible with each α ,

$$\begin{array}{ccc}
& \Gamma(U_\alpha \cap U_\beta, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes_{\mathbb{C}} R & \\
& \nearrow \text{exp}(-s_\beta) & \nwarrow \text{exp}(-s_\alpha) \\
\Gamma(U_\alpha \cap U_\beta, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes_{\mathbb{C}} R & \xrightarrow{\text{exp}(t_{\alpha\beta})} & \Gamma(U_\alpha \cap U_\beta, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes_{\mathbb{C}} R
\end{array}$$

we have an isomorphism of sheaves $\mathcal{A}^{0,0}(\wedge^2 T_{X/R}) \cong \mathcal{A}^{0,0}(\wedge^2 T) \otimes R$. Since $\mathcal{O}_X(U_\alpha) \cong \mathcal{O}_X(U_\alpha) \otimes R$ locally, the Poisson structure is locally encoded in $\Lambda'_\alpha \in A^{0,0}(\wedge^2 T)(U_\alpha) \otimes R$ with $[\Lambda'_\alpha, \Lambda'_\alpha] = 0$, $\bar{\partial}\Lambda'_\alpha = 0$. We define $\Lambda''_\alpha := \text{exp}(-s_\alpha)\Lambda'_\alpha$. Since $\Lambda'_\alpha = \text{exp}(s_\alpha)\text{exp}(-s_\beta)\Lambda'_\beta$, we have $\Lambda''_\alpha = \text{exp}(-s_\alpha)\Lambda'_\alpha = \text{exp}(-s_\beta)\Lambda'_\beta = \Lambda''_\beta$. So Λ''_α glue together to make a global section

$$\Lambda'' \in A^{0,0}(X, \wedge^2 T) \otimes R$$

By Lemma 5.22, $[\Lambda''_\alpha, \Lambda''_\alpha] = [\text{exp}(-s_\alpha)\Lambda'_\alpha, \text{exp}(-s_\alpha)\Lambda'_\alpha] = \text{exp}(-s_\alpha)[\Lambda'_\alpha, \Lambda'_\alpha] = 0$.

$$\begin{aligned}
& \bar{\partial}(\Lambda''_\alpha) + [D(\text{ad}(-s_\alpha))(\bar{\partial}s_\alpha), \Lambda''_\alpha] \\
&= \bar{\partial}(\text{exp}(-s_\alpha)\Lambda'_\alpha) + \text{exp}(-s_\alpha) \circ \bar{\partial}(\text{exp}(s_\alpha))\text{exp}(-s_\alpha)\Lambda'_\alpha \\
&= \bar{\partial}(\text{exp}(-s_\alpha)\Lambda'_\alpha) + (\text{exp}(-s_\alpha) \circ \bar{\partial} \circ \text{exp}(s_\alpha) - \bar{\partial})\text{exp}(-s_\alpha)\Lambda'_\alpha \\
&= \bar{\partial}(\text{exp}(-s_\alpha)\Lambda'_\alpha) + \text{exp}(-s_\alpha) \circ \bar{\partial}\Lambda'_\alpha - \bar{\partial}(\text{exp}(-s_\alpha)\Lambda'_\alpha) = 0
\end{aligned}$$

since $\bar{\partial}\Lambda'_\alpha = 0$.

In conclusion, we have

1. $\bar{\partial}\phi + \frac{1}{2}[\phi, \phi] = 0$
2. $[\Lambda'', \Lambda''] = 0$
3. $\bar{\partial}\Lambda'' + [\phi, \Lambda''] = 0$

By taking $\Lambda = \Lambda'' - \Lambda_0 \in A^{0,0}(X, \wedge^2 T) \otimes \mathfrak{m}$, the above equations are equivalent to

$$L(\phi + \Lambda) + \frac{1}{2}[\phi + \Lambda, \phi + \Lambda] = 0$$

where $L = \bar{\partial} + [\Lambda_0, -]$, which is a solution of Maurer Cartan equation of the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, p \geq 0, q \geq 1} A^{0,p}(X, \wedge^q T) \otimes \mathfrak{m}, L = \bar{\partial} + [\Lambda_0, -], [-, -])$

Now set $\alpha = \phi + \Lambda$. Then we have $L\alpha = -\frac{1}{2}[\alpha, \alpha]$ and $\alpha \wedge L\alpha = -L\alpha \wedge \alpha$ (note that $\deg(\alpha) = 1$). We denote $\bar{\alpha}$ be the corresponding element in $\mathfrak{g}[1]$. Now we assume that $\mathfrak{m}^{n+1} = 0$. We claim that

$$\epsilon(\alpha) := (\bar{\alpha}, \frac{1}{2}\bar{\alpha} \odot \bar{\alpha}, \dots, \frac{1}{n!} \underbrace{\bar{\alpha} \odot \dots \odot \bar{\alpha}}_n) \in \bigoplus_{i=1}^n \text{sym}^i \mathfrak{g}_1 \otimes \mathfrak{m} \subset J_n(\mathfrak{g}) \otimes \mathfrak{m} \quad (5.25)$$

is a hypercycle in $S_n \subset \overline{S(\mathfrak{g}[1])}$, which corresponds to via *dec*

$$(\alpha, \dots, (-1)^{\frac{i(i-1)}{2}} \frac{1}{2!} \underbrace{\alpha \wedge \dots \wedge \alpha}_i, \dots, (-1)^{\frac{n(n-1)}{2}} \frac{1}{n!} \underbrace{\alpha \wedge \dots \wedge \alpha}_n)$$

in $\overline{\wedge \mathfrak{g}}$. For the claim (5.25), we have to show that $(-1)^i L((-1)^{\frac{(i-1)i}{2}} \frac{1}{i!} \underbrace{\alpha \wedge \dots \wedge \alpha}_i) + (-1)^{\frac{i(i+1)}{2}} \frac{1}{(i+1)!} \binom{i+1}{2} [\alpha, \alpha] \wedge \underbrace{\alpha \wedge \dots \wedge \alpha}_{i-1} = 0$. In other words, $L(\frac{1}{i!} \underbrace{\alpha \wedge \dots \wedge \alpha}_i) + \frac{1}{(i+1)!} \binom{i+1}{2} [\alpha, \alpha] \wedge \underbrace{\alpha \wedge \dots \wedge \alpha}_{i-1} = 0$. Indeed,

$$\begin{aligned} L(\frac{1}{i!} \underbrace{\alpha \wedge \dots \wedge \alpha}_i) &= \frac{1}{i!} (L\alpha \wedge \alpha \wedge \dots \wedge \alpha - \alpha \wedge L\alpha \wedge \alpha \wedge \dots \wedge \alpha + \dots + (-1)^{i-1} \alpha \wedge \dots \wedge \alpha \wedge L\alpha) \\ &= \frac{1}{(i-1)!} L\alpha \wedge \alpha \wedge \dots \wedge \alpha \\ \frac{1}{(i+1)!} \frac{(i+1)i}{2} [\alpha, \alpha] \wedge \alpha \wedge \dots \wedge \alpha &= \frac{1}{(i-1)!} \frac{1}{2} [\alpha, \alpha] \wedge \alpha \wedge \dots \wedge \alpha \end{aligned}$$

So we get the claim (5.25) by the condition $L\alpha + \frac{1}{2}[\alpha, \alpha] = 0$. So $\epsilon(\alpha)$ is a hypercycle in $S_n \subset \overline{S(\mathfrak{g}[1])}$. So $[\epsilon(\alpha)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$.

5.3.2 $[\epsilon(\alpha)] \in \mathbb{H}^0(J_n(\mathfrak{g}))$ as a canonical element associated with \mathcal{X}

For given an infinitesimal Poisson deformation \mathcal{X} of (\mathcal{X}, Λ_0) over (R, \mathfrak{m}) with $\mathfrak{m}^{n+1} = 0$, we could find a cohomology class $[\epsilon(\alpha)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$, where $\alpha = \phi + \Lambda \in A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)$ for given an locally trivial open covering $\{U_\alpha\}$, φ_α and C^∞ -trivialization. In this subsection, we show that the cohomology class

$$[\epsilon(\alpha)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$$

is independent of choices of C^∞ -trivialization and local trivialization for fixed locally trivial open covering $\{U_\alpha\}$. Then for given two locally trivial open covering, by choosing refinement of these two open coverings, we conclude that the cohomology class $[\epsilon(\alpha)]$ is canonically associated to the infinitesimal Poisson deformation \mathcal{X} of (X, Λ_0) .

Independence of choices of local trivialization

For given an locally trivial open covering $\mathcal{U} = \{U_\alpha\}$, let's assume that we have two local trivialization $\varphi_\alpha : \mathcal{O}_{\mathcal{X}}(U_\alpha) \rightarrow \mathcal{O}_X(U_\alpha) \otimes R$ and $\varphi'_\alpha : \mathcal{O}_n(U_\alpha) \rightarrow \mathcal{O}_X(U_\alpha) \otimes R$. We will show that $v = \phi + \Lambda \in A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)$ associated with φ_α and C^∞ -trivialization C is same to $w = \psi + \Pi \in A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)$ associated with φ'_α and same C . We have the following commutative diagram

$$\begin{array}{ccccc}
\mathcal{A}_X^{0,0}(U_\alpha) \otimes R & \xrightarrow{C} & \mathcal{A}_X^{0,0}(U_\alpha) & \xrightarrow{\varphi_\alpha} & \mathcal{A}_X^{0,0}(U_\alpha) \otimes R \\
& & \searrow \varphi'_\alpha & & \downarrow \exp(t_\alpha) \\
& & & & \mathcal{A}_X^{0,0}(U_\alpha) \otimes R
\end{array}$$

for some $\{t_\alpha\} \in C^1(\mathcal{U}, \Theta_X) \otimes \mathfrak{m}$ and $\exp(s_\alpha) = \varphi_\alpha \circ C$, $\exp(s'_\alpha) = \varphi'_\alpha \circ C$. Then we have, by our construction as above,

$$\begin{aligned}
\psi_\alpha &= \exp(-s'_\alpha) \bar{\partial}(\exp(s'_\alpha)) = \exp(-s_\alpha) \circ \exp(-t_\alpha) \bar{\partial}(\exp(t_\alpha) \circ \exp(s_\alpha)) \\
&= \exp(-s_\alpha) \circ \exp(-t_\alpha) \circ \bar{\partial}(\exp(t_\alpha)) \circ \exp(s_\alpha) + \exp(-s_\alpha) \circ \exp(-t_\alpha) \circ \exp(t_\alpha) \circ \bar{\partial} \exp(s_\alpha) \\
&= \exp(-s_\alpha) \bar{\partial}(\exp(s_\alpha)) = \phi_\alpha
\end{aligned}$$

since $\bar{\partial} t_\alpha = 0$. On the other hand,

$$\Lambda_0 + \Pi_\alpha = \exp(-s'_\alpha) \omega'_\alpha = \exp(-s_\alpha) \circ \exp(t_\alpha) \omega'_\alpha = \exp(-s_\alpha) \Lambda'_\alpha = \Lambda_0 + \Lambda_\alpha.$$

Hence we have $\Pi = \Lambda$. So we have $v = w$. So $[\epsilon(v)] = [\epsilon(w)]$.

Independence of choices of C^∞ -trivialization

Now, we show that the class $[\epsilon(v)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ is independent of choice of C^∞ -trivialization. Let's assume that we have two C^∞ -trivialization $C : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,0}$ and $\tilde{C} : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,0}$. Then we have the following commutative diagram

$$\begin{array}{ccc}
\mathcal{A}_X^{0,0}(X) \otimes R & \xrightarrow{\tilde{C}} & \mathcal{A}_X^{0,0}(X) \\
\searrow \exp(u) = C^{-1} \circ \tilde{C} & & \nearrow C \\
& & \mathcal{A}_X^{0,0}(X) \otimes R
\end{array}$$

$C^{-1} \circ \tilde{C}$ is a homomorphism inducing identity up to \mathfrak{m} . As in the proof of Lemma 5.19, we can show that there is an element $u \in A^{0,0}(T) \otimes \mathfrak{m}$ such that $C^{-1} \circ \tilde{C} = \exp(u)$.

For given an locally trivial open covering $\{U_\alpha\}$ and local trivializations $\varphi_\alpha : \mathcal{O}_X(U_\alpha) \rightarrow \mathcal{O}_X(U_\alpha) \otimes R$, let $\tilde{v} = \tilde{\phi} + \tilde{\Lambda}$ be the element of $(A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ induced from \tilde{C} and $v = \phi + \Lambda$ be the element of $(A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ induced from C . We want to show that $[\epsilon(v_1)] = [\epsilon(v_0)]$.

Since $\exp(s_\alpha) = \varphi_\alpha \circ C$, we have $\exp(s_\alpha) \circ \exp(u) = \varphi_\alpha \circ \tilde{C}$. Hence we have

$$\begin{aligned} \tilde{\phi} &= [\exp(s_\alpha) \circ \exp(u)]^{-1} \bar{\partial}(\exp(s_\alpha) \circ \exp(u)) \\ &= \exp(-u) \circ \exp(-s_\alpha) \bar{\partial} \exp(s_\alpha) \exp(u) + \exp(-u) \circ \exp(-s_\alpha) \exp(s_\alpha) \bar{\partial} \exp(u) \\ &= \exp(-u) \bar{\partial} \exp(u) + \exp(-u) \phi \exp(u) \\ \Lambda_0 + \tilde{\Lambda} &= \exp(-u) \circ \exp(-s_\alpha) (\Lambda'_\alpha) \\ &= \exp(-u) (\Lambda_0 + \Lambda) \end{aligned}$$

We set for $t \in [0, 1] \subset \mathbb{R}$,

$$\begin{aligned} \phi_t &= \exp(-tu) \bar{\partial} \exp(tu) + \exp(-tu) \phi \exp(tu) \\ \Lambda^t &= \exp(-tu) (\Lambda_0 + \Lambda) - \Lambda_0 \end{aligned}$$

Note that $\phi_0 = \phi, \Lambda^0 = \Lambda$ and $\phi_1 = \tilde{\phi}, \Lambda^1 = \tilde{\Lambda}$. We would like to show that $[\epsilon(v)] = [\epsilon(\phi + \Lambda)] = [\epsilon(\tilde{\phi} + \tilde{\Lambda})] = [\epsilon(\tilde{v})]$ by showing that $[\epsilon(v_t)] := [\epsilon(\phi_t + \Lambda^t)]$ is constant independent of t . To this end, we note that since the operator $L = \bar{\partial} + [\Lambda_0, -]$ is elliptic, $\mathbb{H}^0(J_n(\mathfrak{g}))$ is finite dimensional vector space over \mathbb{C} . Since \mathfrak{g} is a global section of a complex vector bundle over X , \mathfrak{g} is endowed with a suitable metric, which induces a metric on $\mathbb{H}^0(J_n(\mathfrak{g}))$ by hodge theory which coincides with the standard Euclidean topology and likewise for $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$. So differentiation of a function $\mathbb{R} \rightarrow \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ makes sense if the derivative exists with respect to the metric on $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$. Now we claim that $\frac{d}{dt}([\epsilon(v_t)]) = \frac{d}{dt}([\epsilon(\phi_t + \Lambda^t)]) = [\frac{d}{dt}(\epsilon(\phi_t + \Lambda^t))] = 0$, which implies that $[\epsilon(v_t)]$ is constant, and so $[\epsilon(\tilde{v})] = [\epsilon(v)]$. We note that

$$\begin{aligned} \phi'_t &:= \frac{d}{dt}(\phi_t) = -u \exp(-tu) \bar{\partial}(\exp(tu)) + \exp(-tu) \bar{\partial}(\exp(tu)u) \\ &\quad - u \exp(-tu) \phi \exp(tu) + \exp(-tu) \phi \exp(tu)u \\ &= -u \exp(-tu) \bar{\partial}(\exp(tu)) + \exp(-tu) \bar{\partial}(\exp(tu))u + \bar{\partial}u \\ &\quad - u \exp(-tu) \phi \exp(tu) + \exp(-tu) \phi \exp(tu)u \\ &= -u \phi_t + \phi_t u + \bar{\partial}u = \bar{\partial}u - [u, \phi_t] \\ (\Lambda^t)' &:= \frac{d}{dt}(\Lambda^t) = -u \exp(-tu) (\Lambda_0 + \Lambda) = -u (\Lambda^t + \Lambda_0) \\ &= -[u, \Lambda^t] + [\Lambda_0, u] \end{aligned}$$

So $v'_t := \frac{d}{dt}v_t = \frac{d}{dt}(\phi_t + \Lambda^t) = \phi'_t + (\Lambda^t)' = Lu - [u, \phi_t + \Lambda^t] = Lu - [u, v_t]$. And we have

$$\frac{d}{dt}(\epsilon(v_t)) = (\bar{v}'_t, \bar{v}'_t \odot \bar{v}_t, \dots, \frac{1}{(n-1)!} \bar{v}'_t \odot \underbrace{\bar{v}_t \odot \dots \odot \bar{v}_t}_{n-1}) \in \bigoplus_{i=1}^n \text{sym}^i g_1 \otimes \mathfrak{m}$$

which corresponds to

$$(v'_t, \dots, (-1)^{\frac{(i-1)i}{2}} \frac{1}{(i-1)!} v'_t \wedge \underbrace{v_t \wedge \dots \wedge v_t}_{i-1}, \dots, (-1)^{\frac{(n-1)n}{2}} \frac{1}{(n-1)!} v'_t \wedge \underbrace{v_t \wedge \dots \wedge v_t}_{n-1})$$

in $\overline{\bigwedge \mathfrak{g}}$.

Note that $Lv_t = -\frac{1}{2}[v_t, v_t]$.² Let

$$(-\bar{u}, -\bar{u} \odot \bar{\alpha}_t, \dots, \frac{1}{(n-1)!} (-\bar{u}) \odot \underbrace{\bar{\alpha}_t \odot \dots \odot \bar{\alpha}_t}_{n-1}) \in \bigoplus_{i=1}^{n-1} g_0 \otimes \text{sym}^i g_1 \otimes \mathfrak{m}$$

which corresponds to

$$(-u, \dots, (-1)^{\frac{(i-1)i}{2}} \frac{1}{i!} (-u) \wedge \underbrace{\alpha_t \wedge \dots \wedge \alpha_t}_i, \dots, (-1)^{\frac{(n-2)(n-1)}{2}} \frac{1}{(n-1)!} (-u) \wedge \underbrace{\alpha_t \wedge \dots \wedge \alpha_t}_{n-1})$$

in $\overline{\bigwedge \mathfrak{g}}$.

We claim that this is the coboundary of $\frac{d}{dt}(\epsilon(v_t))$. Indeed, (note that $\text{deg}(u) = 0$ and

²For given the locally trivial open covering $\{U_\alpha\}$ and local trivialization $\varphi_\alpha : \mathcal{O}_{\mathcal{X}}(U_\alpha) \rightarrow \mathcal{O}_X(U_\alpha) \otimes R$, let $C_t : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,0}$ be C^∞ -trivialization defined by $C_t = C \circ \exp(tu)$. Then we have $\varphi_\alpha \circ C = \exp(s_\alpha) \circ \exp(tu)$. Then we have

$$\begin{aligned} \phi_t &= (\exp(s_\alpha) \circ \exp(tu))^{-1} \bar{\partial} (\exp(s_\alpha) \circ \exp(tu)) \\ \Lambda_0 + \Lambda_t &= (\exp(s_\alpha) \circ \exp(tu))^{-1} \Lambda'_\alpha \end{aligned}$$

Hence by the construction, we have $Lv_t + \frac{1}{2}[v_t, v_t] = 0$.

$\deg(v_t) = 1.$)

$$\begin{aligned}
& (-1)^{i+1} L \left((-1)^{\frac{(i-1)i}{2}} \frac{1}{i!} (-u) \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i \right) + (-1)^{\frac{i(i+1)}{2}} \frac{i+1}{(i+1)!} [-u, v_t] \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i \\
& + (-1)^{\frac{i(i+1)}{2}} \frac{1}{(i+1)!} \binom{i+1}{2} [v_t, v_t] \wedge (-u) \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} \\
& = (-1)^{\frac{i(i+1)}{2}} \frac{1}{i!} (Lu \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i + u \wedge Lv_t \wedge v_t \wedge \cdots \wedge v_t - u \wedge v_t \wedge Lv_t \wedge v_t \wedge \cdots \wedge v_t + \cdots \\
& + (-1)^{i-1} u \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} \wedge Lv_t) - (-1)^{\frac{i(i+1)}{2}} \frac{1}{i!} [u, v_t] \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i \\
& + (-1)^{\frac{i(i+1)}{2}} \frac{1}{(i-1)!} \frac{1}{2} u \wedge [v_t, v_t] \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} \\
& = (-1)^{\frac{i(i+1)}{2}} \left(\frac{1}{i!} Lu \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i + \frac{1}{(i-1)!} u \wedge Lv_t \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} - \frac{1}{i!} [u, v_t] \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i \right) \\
& + (-1)^{\frac{i(i+1)}{2}} \frac{1}{(i-1)!} \frac{1}{2} u \wedge [v_t, v_t] \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} \\
& = (-1)^{\frac{i(i+1)}{2}} \frac{1}{i!} (Lu - [u, v_t]) \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_{i-1} \\
& = (-1)^{\frac{i(i+1)}{2}} \frac{1}{i!} v_t' \wedge \underbrace{v_t \wedge \cdots \wedge v_t}_i.
\end{aligned}$$

So we have $\left[\frac{d}{dt} (\epsilon(v_t)) \right] = 0.$

In conclusion, for given an infinitesimal Poisson deformation \mathcal{X} of (X, Λ_0) , we can canonically associate the Poisson deformation \mathcal{X} up to equivalence with the cohomology class $[\epsilon(v)]$. In the next chapter, we will show that under the assumption $HP^1(X, \Lambda_0) = 0$, for given a choice $v, w \in (A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ with $[\epsilon(v)] = [\epsilon(w)]$, the Poisson deformation associated with v is equivalent to the Poisson deformation associated with w .

Chapter 6

Universal Poisson deformations

6.1 Isomorphism of two deformation functors $Def_{\mathfrak{g}} \cong PDef_{(X, \Lambda_0)}$

In this section, we will show that two functors of Artin rings are isomorphic: namely the Poisson deformation functor $Def_{(X, \Lambda_0)}$ is isomorphic to the deformation functor $Def_{\mathfrak{g}}$ associated to the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -])$. So this shows that deformations of a compact holomorphic Poisson manifold are controlled by the differential graded Lie algebra \mathfrak{g} . For deformation functors associated with a differential graded Lie algebra, we refer to [Man04].

6.1.1 Deformation functors

Definition 6.1 *A functor of Artin rings is a covariant functor*

$$F : \mathbf{Art} \rightarrow \mathbf{Sets}$$

such that $F(\mathbb{C})$ has only one element, where \mathbf{Art} is the category of local artinian \mathbb{C} -algebra with residue \mathbb{C} , and \mathbf{Sets} is the category of sets.

Definition 6.2 (functors associated with a DGLA L) *Let $L = (\bigoplus_{i \geq 0} L_i, d, [-, -])$ be a differential graded Lie algebra and $(R, \mathfrak{m}) \in \mathbf{Art}$. Let $MC(L)(R)$ be the set of all Maurer-Cartan*

elements of $L \otimes \mathfrak{m}$, i.e

$$MC_L(R) = \{x \in L_1 \otimes \mathfrak{m} \mid dx + \frac{1}{2}[x, x] = 0\}$$

Then $MC_L : \mathbf{Art} \rightarrow \mathbf{Sets}$ is a functor.

Let $g = L \otimes \mathfrak{m}$ be the induced differential graded Lie algebra from L . Then $g_0 = L_0 \otimes \mathfrak{m}$ is a nilpotent Lie algebra. Then the set $exp(g_0) = \{e^x \mid x \in g_0\}$ forms a group by the Campbell-Hausdorff formula. We have a group action of $exp(g_0)$ on $g_1 = L_1 \otimes \mathfrak{m}$ given by

$$e^a \cdot x := x + \sum_{n \geq 1} \frac{(ada)^{n-1}}{n!} ([a, x] - da) \quad \text{where } ada : g_1 \rightarrow g_1 \text{ defined by } b \mapsto [a, b]$$

The action is known as the gauge action of $exp(g_0)$ on g_1 and the set of Maurer-Cartan elements is stable under the gauge action.

Definition 6.3 Let $x, y \in MC_L(R)$. We say that x is gauge equivalent to y if there exists $e^a \in exp(g_0)$ such that $e^a \cdot x = y$. Let $Def_L(R)$ be the set of all gauge equivalence classes of elements of $MC_L(R)$. Then the functor $Def_L : \mathbf{Art} \rightarrow \mathbf{Sets}$ is called the deformation functor associated to the differential graded Lie algebra L .

Definition 6.4 We say that a functor of Artin rings F is controlled by a differential graded Lie algebra L if $F \cong Def_L$.

Definition 6.5 (Poisson deformation functor) Let (X, Λ_0) be a compact holomorphic Poisson manifold. The Poisson deformation functor $Def_{(X, \Lambda_0)} : \mathbf{Art} \rightarrow \mathbf{Sets}$ is defined by

$$PDef_{(X, \Lambda_0)}(R) = \{\text{equivalent classes of infinitesimal Poisson deformations of } (X, \Lambda_0) \text{ over } R\}$$

In the next subsection, we will prove that $Def_{\mathfrak{g}} \cong PDef_{(X, \Lambda_0)}$.

6.1.2 The Poisson deformation functor $Def_{(X, \Lambda_0)}$ is controlled by the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -])$

Let $\mathcal{X} \rightarrow \text{Spec } R$ be an infinitesimal Poisson deformation of (X, Λ_0) , where (R, \mathfrak{m}) is generated by $\langle 1, m_1, \dots, m_r \rangle$ with exponent n . Let $\{U_\alpha\}$ be a locally trivial open cover of \mathcal{X} . We defined a sheaf $\mathcal{A}^{0,p}(\wedge^q T_{\mathcal{X}/R})$, a bracket $[-, -]$ and the deaulbault differential $\bar{\partial}_{\mathcal{X}}$. For $q = 0$, we will denote this sheaf by $\mathcal{A}_{\mathcal{X}}^{0,p}$ and similarly for $\mathcal{A}_X^{0,p}$.

Given an element $v := \phi + \Lambda \in (A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ with $L(\phi + \Lambda) = -\frac{1}{2}[\phi + \Lambda, \phi + \Lambda]$. In particular, we have $\bar{\partial}\phi = -\frac{1}{2}[\phi, \phi]$. We define an operator $\bar{\partial} + \phi := \bar{\partial} \otimes 1 + \phi$ on $\mathcal{A}^{0,p} \otimes R$ and a sequence

$$0 \rightarrow \mathcal{A}_X^{0,0} \otimes R \xrightarrow{\bar{\partial} + \phi} \mathcal{A}_X^{0,1} \otimes R \xrightarrow{\bar{\partial} + \phi} \dots \xrightarrow{\bar{\partial} + \phi} \mathcal{A}_X^{0,p} \otimes R \xrightarrow{\bar{\partial} + \phi} \quad (6.6)$$

which is a complex by the condition $\bar{\partial}\phi = -\frac{1}{2}[\phi, \phi]$. By tensoring $\otimes_{R/\mathfrak{m}}$, we have

$$0 \rightarrow \mathcal{A}_X^{0,0} \otimes \mathbb{C} \xrightarrow{\bar{\partial}} \mathcal{A}_X^{0,1} \otimes \mathbb{C} \xrightarrow{\bar{\partial}} \dots$$

which is a acyclic resolution of \mathcal{O}_X . Hence the complex (6.6) is exact in positive degree.

We define

$$\mathcal{O}(v) := \ker(\bar{\partial} + \phi : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,1} \otimes R)$$

which is a flat R -sheaf on X .¹ $\mathcal{O}(v)$ has a Poisson bracket induced by $\Lambda_0 + \Lambda$. We define on $\{-, -\}$ on $\mathcal{O}(v)$ by

$$\{f, g\} := [[\Lambda_0 + \Lambda, f], g]$$

for local sections $f, g \in \mathcal{O}(v)$. Then $\{-, -\}$ defines a biderivation since $d(gh) = gdh + hdg$ and R -bilinear since for $a = a_0 + a_1 m_1 + \dots + a_r m_r \in R$ where $a_i \in \mathbb{C}$, we have $da = 0$. $[\Lambda_0 + \Lambda, \Lambda_0 + \Lambda] = 0$ shows that $\{-, -\}$ satisfies the Jacobi identity. So it remains to show that $\mathcal{O}(v)$ is closed under $\{-, -\}$. Note that for $f, g \in \mathcal{O}(v)$, we have $\bar{\partial}f + [\phi, f] = 0$, $\bar{\partial}g + [\phi, g] = 0$. We set $\omega = \Lambda_0 + \Lambda$. Then we have $\bar{\partial}\omega + [\phi, \omega] = 0$

$$\begin{aligned} \bar{\partial}[[\omega, f], g] + [\phi, [[\omega, f], g]] &= [\bar{\partial}[\omega, f], g] + [[\omega, f], \bar{\partial}g] + [[\phi, [\omega, f]], g] + [[\omega, f], [\phi, g]] \\ &= [[\bar{\partial}\omega, f], g] - [[\omega, \bar{\partial}f], g] + [[[\phi, \omega], f], g] - [[\omega, [\phi, f]], g] \\ &= 0 \end{aligned}$$

So we have $\{f, g\} \in \mathcal{O}(v)$. Hence $\mathcal{O}(v)$ is a sheaf of Poisson R -algebras. We also have $\mathcal{O}(v) \otimes_{R/\mathfrak{m}} \cong \mathcal{O}_X$ as Poisson sheaves over \mathbb{C} . So $\mathcal{O}(v)$ defines an infinitesimal Poisson deformation of (X, Λ_0) over R .

¹In general, let R be an artinian local ring with residue field k and M be an R -module. Let

$$M \rightarrow N^0 \rightarrow N^1 \rightarrow \dots$$

be a flat resolution such that this induces a resolution

$$M \otimes k \rightarrow N^0 \otimes k \rightarrow N^1 \otimes k \dots$$

Then M is a R -flat.

Now we define the map

$$\begin{aligned}\mathcal{O} : Def_{\mathfrak{g}}(R) &\rightarrow PDef_{(X, \Lambda_0)}(R) \\ v = \phi + \Lambda &\mapsto \mathcal{O}(v)\end{aligned}$$

We claim that v is gauge equivalent to w if and only if $\mathcal{O}(v) \cong \mathcal{O}(w)$ as Poisson R -sheaves. This claim shows that the map \mathcal{O} is well-defined and injective. Let $v = \phi + \Lambda, w = \psi + \Pi \in (A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$. Since v is gauge equivalent to w , for some $a \in A^{0,0}(X, T) \otimes \mathfrak{m}$, we have

$$\begin{aligned}(1)\psi &= \phi + \sum_{n \geq 1} \frac{(ada)^{n-1}}{n!} ([a, \phi] - \bar{\partial}a) \\ (2)\Pi &= \Lambda + \sum_{n \geq 1} \frac{(ada)^{n-1}}{n!} ([a, \Lambda] - [\Lambda_0, a]) = \exp(a)(\Lambda_0 + \Lambda) - \Lambda_0\end{aligned}$$

(1) is equivalent that the following commutative diagram commutes:

$$\begin{array}{ccc}\mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \phi} & \mathcal{A}_X^{0,1} \otimes R \\ \exp(a) \downarrow & & \downarrow \exp(a) \\ \mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \phi'} & \mathcal{A}_X^{0,1} \otimes R\end{array}$$

which implies $\mathcal{O}(v) \cong \mathcal{O}(w)$ as sheaves of R -algebras.

(2) means $\Lambda_0 + \Pi = \exp(a)(\Lambda_0 + \Lambda)$ which means $\mathcal{O}(v) \cong \mathcal{O}(w)$ as sheaves of Poisson R -algebras. So we get the claim.

Now we show that $\mathcal{O} : Def_{\mathfrak{g}}(R) \rightarrow Def_{(X, \Lambda_0)}(R)$ is surjective. For given an infinitesimal Poisson deformation of (X, Λ_0) over (R, \mathfrak{m}) , we showed that there is a canonically associated element $v = \phi + \Lambda \in (A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}$ with $L(\phi + \Lambda) + \frac{1}{2}[\phi + \Lambda, \phi + \Lambda] = 0$. We claim that for each α , the following diagram is commutative.

$$\begin{array}{ccc}\mathcal{A}_X^{0,0}(U_\alpha) \otimes R & \xrightarrow{\bar{\partial} + \phi} & \mathcal{A}_X^{0,1}(U_\alpha) \otimes R \\ \exp(s_\alpha) \downarrow & & \downarrow \exp(s_\alpha) \\ \mathcal{A}_X^{0,0}(U_\alpha) \otimes R & \xrightarrow{\bar{\partial}_X = \bar{\partial}} & \mathcal{A}_X^{0,1}(U_\alpha) \otimes R\end{array}$$

Note that $\exp(s_\alpha) = \varphi_\alpha \circ C$. Indeed, the commutativity means that

$$\begin{aligned}\bar{\partial}f + \phi(f) &= \bar{\partial}f + \exp(-s_\alpha) \circ \bar{\partial}(\exp(s_\alpha))f \\ &= \bar{\partial}f + \exp(-s_\alpha) \circ (\bar{\partial} \circ \exp(s_\alpha) - \exp(s_\alpha) \circ \bar{\partial})f \\ &= \exp(-s_\alpha) \circ \bar{\partial} \circ \exp(s_\alpha)f\end{aligned}$$

Since the diagram is compatible with each α , we have the following commutative diagram of sheaves

$$\begin{array}{ccc}
\mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \phi} & \mathcal{A}_X^{0,1} \otimes R \\
\cong \downarrow & & \downarrow \cong \\
\mathcal{A}_X^{0,0} & \xrightarrow{\bar{\partial}_X} & \mathcal{A}_X^{0,1}
\end{array}$$

So we have isomorphism of sheaves

$$\mathcal{O}(v) := \ker(\bar{\partial} + \phi : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,1} \otimes R) \cong \mathcal{O}_X = \ker(\bar{\partial}_X : \mathcal{A}_X^{0,0} \rightarrow \mathcal{A}_X^{0,1})$$

$\mathcal{O}(v)$ is a sheaf of Poisson R -algebras as above defined by

$$\{f, g\} := [[\Lambda_0 + \Lambda, f], g] \quad \text{for local sections } f, g \in \mathcal{O}(v)$$

Now we claim that as Poisson R -sheaves, we have

$$\mathcal{O}(v) \cong \mathcal{O}_X$$

We check this locally on U_α : for $f, g \in \Gamma(U_\alpha, \mathcal{O}(v)) = \ker(\bar{\partial} + \phi : \mathcal{A}_X^{0,0}(U_\alpha) \otimes R \rightarrow \mathcal{A}_X^{0,1}(U_\alpha) \otimes R)$,

$$\begin{aligned}
\exp(s_\alpha)\{f, g\} &= \exp(s_\alpha)[[\Lambda_0 + \Lambda, f], g] = \exp(s_\alpha)[[\Lambda''_\alpha, f], g] = [[\exp(s_\alpha)\Lambda''_\alpha, \exp(s_\alpha)f], \exp(s_\alpha)g] \\
&= [[\exp(s_\alpha)\exp(-s_\alpha)\Lambda'_\alpha, \exp(s_\alpha)f], \exp(s_\alpha)g] = [[\Lambda'_\alpha, \exp(s_\alpha)f], \exp(s_\alpha)g]
\end{aligned}$$

where $\Lambda'_\alpha \in \Gamma(U_\alpha, \mathcal{A}^{0,0}(\wedge^2 T_X)) \otimes R$ is the Poisson structure on $\mathcal{O}_X(U_\alpha) \otimes R \cong \mathcal{O}_X(U_\alpha)$ and $\Lambda''_\alpha = \exp(-s_\alpha)\Lambda'_\alpha$. (See Remark 5.24 for notations)

Hence the infinitesimal Poisson deformation \mathcal{X} of (X, Λ_0) over R is equivalent to $\mathcal{O}(v) := \ker(\bar{\partial} + \phi : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,1} \otimes R)$ equipped with the Poisson structure $\Lambda_0 + \Lambda$. This shows that the map $\mathcal{O} : Def_{\mathfrak{g}}(R) \rightarrow Def_{(X, \Lambda_0)}(R)$ is surjective.

So we proved that for an artinian local \mathbb{C} -algebra R with residue \mathbb{C} , we have an isomorphism $\mathcal{O} : MC_{\mathfrak{g}}(R) \rightarrow PDef_{(X, \Lambda_0)}(R)$. To show that $Def_{\mathfrak{g}} \cong PDef_{(x, \Lambda_0)}$, we have to show that \mathcal{O} is a morphism of functors of Artin rings, in other words, \mathcal{O} is compatible with any local homomorphism $R \rightarrow S$ in **Art**.

Definition 6.7 (Base change) *Given an infinitesimal Poisson deformation \mathcal{X} of (X, Λ) over R , and a local \mathbb{C} -algebra homomorphism $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$, we can define an infinitesimal Poisson deformation $\mathcal{X} \times_{Spec R} Spec S$ of (X, Λ) over S by base change.*

$$\begin{array}{ccccc}
X & \longrightarrow & \mathcal{X} \times_{Spec R} Spec S & \longrightarrow & \mathcal{X} \\
\downarrow & & \downarrow & & \downarrow \\
Spec \mathbb{C} & \longrightarrow & Spec S & \longrightarrow & Spec R
\end{array}$$

We only need to explain the induced Poisson structure of $\mathcal{X}_S := \mathcal{X} \times_{\text{Spec } R} \text{Spec } S$ over S . For any open set U of X , $\mathcal{O}_{\mathcal{X}_S}(U) = \mathcal{O}_{\mathcal{X}}(U) \otimes_R S$. We define the Poisson bracket $\{-, -\}_S$ on $\mathcal{O}_{\mathcal{X}_S}(U)$ by

$$\{f \otimes s_1, g \otimes s_2\}_S = \{f, g\}_R \otimes s_1 s_2$$

where $\{-, -\}_R$ is the Poisson bracket on $\mathcal{O}_{\mathcal{X}}(U)$.

We note that the induced infinitesimal Poisson deformation by the base change $(R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ can be interpreted in terms of a Maurer-Cartan element of \mathcal{X} . Let $\phi + \Lambda \in (A^{0,1}(X, T) \oplus A^{0,0}(X, \wedge^2 T)) \otimes \mathfrak{m}_R$ be a Maurer-Cartan element of \mathcal{X} . Hence $\mathcal{O}_{\mathcal{X}}$ is equivalent to $\ker(\bar{\partial} + \phi : \mathcal{A}^{0,0} \otimes R \rightarrow \mathcal{A}^{0,1} \otimes R)$ with the Poisson structure $\Lambda_0 + \Lambda$. The homomorphism $g : (R, \mathfrak{m}_R) \rightarrow (S, \mathfrak{m}_S)$ induces the homomorphisms $A^{0,p}(X, \wedge^q T) \otimes \mathfrak{m}_R \rightarrow A^{0,p}(X, \wedge^q T) \otimes \mathfrak{m}_S$. Let $\phi_S + \Lambda_S$ be the image of $\phi + \Lambda$, which also satisfy $L_S(\phi_S + \Lambda_S) + \frac{1}{2}[\phi_S + \Lambda_S, \phi_S + \Lambda_S] = 0$. We have the following commutative diagram

$$\begin{array}{ccc} (\mathcal{A}_X^{0,0} \otimes R, \Lambda_0 + \Lambda_R) & \xrightarrow{\bar{\partial} + \phi} & \mathcal{A}_X^{0,1} \otimes R \\ \downarrow & & \downarrow \\ (\mathcal{A}_X^{0,0} \otimes S, \Lambda_0 + \Lambda_S) & \xrightarrow{\bar{\partial} + \phi_S} & \mathcal{A}_X^{0,1} \otimes S \end{array}$$

We claim that

Proposition 6.8 $\mathcal{O}_{\mathcal{X}_S}$ is equivalent to $(\phi_S + \Lambda_S)$.

Proof. We recall that for given locally trivial open covering $\{U_\alpha\}$, and local trivialization $\varphi_\alpha : \mathcal{O}_{\mathcal{X}}(U_\alpha) \rightarrow \mathcal{O}_X(U_\alpha) \otimes R$ and C^∞ -trivialization $C : \mathcal{A}_X^{0,0} \otimes R \rightarrow \mathcal{A}_X^{0,0}$ for the family \mathcal{X} , we have $\phi = \exp(-s_\alpha) \bar{\partial} \exp(s_\alpha) \in A^{0,1}(X, T) \otimes \mathfrak{m}_R$ and $\Lambda = \exp(-s_\alpha)(\Lambda_0 + \Lambda_\alpha) - \Lambda_0 \in A^{0,0}(X, \wedge^2 T) \otimes \mathfrak{m}_R$. Now we consider the family \mathcal{X}_S over S . For the same open covering $\{U_\alpha\}$, the local trivialization is induced from ψ_α by tensoring $\otimes_R S$ and C^∞ trivialization is also induced from C by tensoring $\otimes_R S$. This observation gives the proposition. ■

Hence we have the following commutative diagram

$$\begin{array}{ccc} Def_{\mathfrak{g}}(R) & \xrightarrow{\mathcal{O}} & PDef_{(X, \Lambda_0)}(R) \\ \downarrow & & \downarrow \\ Def_{\mathfrak{g}}(S) & \xrightarrow{\mathcal{O}} & PDef_{(X, \Lambda_0)}(S) \end{array}$$

Hence we proved the following theorem.

Theorem 6.9 Let (X, Λ_0) be a compact holomorphic Poisson manifold. Then the Poisson deformation functor $Def_{(X, \Lambda_0)}$ is controlled by the differential graded Lie algebra $\mathfrak{g} = (\bigoplus_{p+q-1=i, p \geq 1, q \geq 1}$

$A^{0,p}(X, \wedge^q T), L = \bar{\partial} + [\Lambda_0, -], [-, -]$. In other words, we have an isomorphism of two functors

$$Def_{\mathfrak{g}} \cong PDef_{(X, \Lambda_0)}$$

6.2 Universal Poisson deformations

Now we assume that for a holomorphic Poisson manifold (X, Λ_0) , $HP^1(X, \Lambda_0) = 0$.

6.2.1 Independence of choices of morphic elements giving the same cohomology class

Let \mathfrak{m} be the maximal ideal of a local artinian \mathbb{C} -algebra R with residue such that $\mathfrak{m}^{n+1} = 0$. Our goal is that for given $v = \phi + \Lambda$, $w = \psi + \Pi \in g_1 \otimes \mathfrak{m}$ such that $[\epsilon(v)] = [\epsilon(w)]$ in $\mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$, where $\epsilon(v) = (\bar{v}, \frac{1}{2}\bar{v} \odot \bar{v}, \dots, \frac{1}{n!} \underbrace{\bar{v} \odot \dots \odot \bar{v}}_n) \cong_{dec} (v, \dots, (-1)^{\frac{(n-1)n}{2}} \frac{1}{n!} \underbrace{v \wedge \dots \wedge v}_n) \in \bigoplus_{i=1}^n sym^i g_1 \otimes \mathfrak{m}^i$ and $Lv = -\frac{1}{2}[v, v]$, and same for w , we want to show that the Poisson deformation $\mathcal{O}(v)$ is equivalent to the Poisson deformation $\mathcal{O}(w)$. In other words, we want to show that

1. there exists $u_0 \in g_0 \otimes \mathfrak{m}$ such that $exp(u_0)(\bar{\partial} + \phi)exp(-u_0) = \bar{\partial} + \psi$.

$$\begin{array}{ccc} \mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \phi} & \mathcal{A}_X^{0,1} \otimes R \\ \exp(u_0) \downarrow & & \downarrow \exp(u_0) \\ \mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \psi} & \mathcal{A}_X^{0,1} \otimes R \end{array}$$

2. $exp(u_0)(\Lambda_0 + \Lambda) = \Lambda_0 + \Pi$

We will prove the statement by induction on the exponent k of maximal ideal of artinian local \mathbb{C} -algebra with residue \mathbb{C} . Let $k = 1$. So we have $\mathfrak{m}^2 = 0$. Let $[\epsilon(v)] = [\epsilon(w)] \in \mathbb{H}^0(J_1(\mathfrak{g})) \otimes \mathfrak{m}$. Then there exists $u_0 \in g_0 \otimes \mathfrak{m}$ such that $(-1)^1 Lu_0 = w - v$. So we have $Lu_0 = \bar{\partial}u_0 + [\Lambda_0, u_0] = v - w$. In other words,

$$\begin{aligned} \bar{\partial}u_0 &= \phi - \psi \\ [\Lambda_0, u_0] &= \Lambda - \Pi \end{aligned}$$

We note that $exp(u_0) = 1 + u_0$ and $exp(-u_0) = 1 - u_0$. Then since $u_0 \in g_0 \otimes \mathfrak{m}$, $\phi, \psi, \Lambda, \Pi \in g_1 \otimes \mathfrak{m}$ and $\mathfrak{m}^2 = 0$, we have

1.

$$\begin{aligned}
\exp(u_0)(\bar{\partial} + \phi)\exp(-u_0) &= (1 + u_0)(\bar{\partial} + \phi)(1 - u_0) = (\bar{\partial} + \phi + u_0\bar{\partial} + u_0\phi)(1 - u_0) \\
&= \bar{\partial} + \phi + u_0\bar{\partial} + u_0\phi - \bar{\partial} \cdot u_0 - \phi u_0 - u_0\bar{\partial} \cdot u_0 - u_0\phi u_0 \\
&= \bar{\partial} + \phi + u_0\bar{\partial} - \bar{\partial} \cdot u_0 = \bar{\partial} + \phi - \bar{\partial}u_0 \\
&= \bar{\partial} + \psi
\end{aligned}$$

2.

$$\begin{aligned}
\exp(u_0)(\Lambda_0 + \Lambda) &= (1 + u_0)(\Lambda_0 + \Lambda) = \Lambda_0 + \Lambda + [u_0, \Lambda_0 + \Lambda] \\
&= \Lambda_0 + \Lambda - [\Lambda_0, u_0] = \Lambda_0 + \Lambda - (\Lambda - \Pi) \\
&= \Lambda_0 + \Pi
\end{aligned}$$

So the statement holds for $k = 1$.

Now let's assume that the statement holds for $k = n - 1$. Now let (R, \mathfrak{m}) be a local artinian \mathbb{C} -algebra with exponent n and let $[\epsilon(v)] = [\epsilon(w)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ where $v, w \in g_1 \otimes \mathfrak{m}$. We have the following exact sequence of finite dimensional vector spaces

$$0 \rightarrow \mathfrak{m}^n \rightarrow R \rightarrow R/\mathfrak{m}^n \rightarrow 0$$

So we have the following splitting as vector spaces

$$R \cong (R/\mathfrak{m}^n) \oplus \mathfrak{m}^n$$

Note that $(R/\mathfrak{m}^n, \mathfrak{m}/\mathfrak{m}^n)$ is a local artinian \mathbb{C} -algebra with exponent $n - 1$. Now let $v = v_1 + v_2 = (\phi_1 + \Lambda_1) + (\phi_2 + \Lambda_2)$ and $w = w_1 + w_2 = (\psi_1 + \Pi_1) + (\psi_2 + \Pi_2)$, where $v_1, w_1 \in g_1 \otimes (R/\mathfrak{m}^n)$ and $v_2, w_2 \in \mathfrak{m}^n$. Then we have $[\epsilon(v_1)] = [\epsilon(w_1)] \in \mathbb{H}^0(J_{n-1}(\mathfrak{g})) \otimes \mathfrak{m}/\mathfrak{m}^n$, where $v_1, w_1 \in g_1 \otimes \mathfrak{m}/\mathfrak{m}^n$. By the induction hypothesis,

1. we have the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{A}_X^{0,0} \otimes R/\mathfrak{m}^n & \xrightarrow{\bar{\partial} + \phi_1} & \mathcal{A}_X^{0,1} \otimes R/\mathfrak{m}^n \\
\exp(u) \downarrow & & \downarrow \exp(u) \\
\mathcal{A}_X^{0,0} \otimes R/\mathfrak{m}^n & \xrightarrow{\bar{\partial} + \psi_1} & \mathcal{A}_X^{0,1} \otimes R/\mathfrak{m}^n
\end{array}$$

where some $u \in g_0 \otimes \mathfrak{m}/\mathfrak{m}^n$.

2. $\exp(u)(\Lambda_0 + \Lambda_1) = \Lambda_0 + \Pi_1$.

Let's consider the natural projection $g_0 \otimes R \cong g_0 \otimes (R/\mathfrak{m}^n \oplus \mathfrak{m}^n) \rightarrow g_0 \otimes R/\mathfrak{m}^n$. Choose the lifting of u to be $u + 0 \in g_0 \otimes (R/\mathfrak{m}^n \oplus \mathfrak{m}^n) \cong g_0 \otimes R$. Then

1. we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \phi_1 + \phi_2} & \mathcal{A}_X^{0,1} \otimes R \\ \exp(u+0) \downarrow & & \downarrow \exp(u+0) \\ \mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \psi_1 + \phi_2} & \mathcal{A}_X^{0,1} \otimes R \end{array}$$

since $\phi_2 = \phi_2 \circ \exp(u) = \exp(u) \circ \phi_2$. (note that $\phi_2 \in A^{0,1}(X, T) \otimes \mathfrak{m}^n$, $u \in \mathfrak{g}_0 \otimes \mathfrak{m}$ and $\mathfrak{m}^{n+1} = 0$.)

2. $\exp(u)(\Lambda_0 + \Lambda_1 + \Lambda_2) = \Lambda_0 + \Pi_1 + \Lambda_2$ since $\exp(u)(\Lambda_2) = \Lambda_2$ (note that $\Lambda_2 \in A^{0,0}(X, \wedge^2 T) \otimes \mathfrak{m}^n$).

Since $\mathcal{O}(v_1 + v_2)$ and $\mathcal{O}(w_1 + v_2)$ are equivalent Poisson deformations, we have $[\epsilon(v_1 + v_2)] = [\epsilon(w_1 + v_2)]$. If we show that $\mathcal{O}(w_1 + v_2)$ is equivalent to $\mathcal{O}(w_1 + w_2)$, then this means that $\mathcal{O}(v)$ is equivalent to $\mathcal{O}(w)$.

Since $[\epsilon(w_1 + v_2)] = [\epsilon(w_1 + w_2)]$, there exists $(u_0, \dots, u_{n-1}) \in \bigoplus_{i=0}^{n-1} \mathfrak{g}_0 \otimes \text{sym}^i \mathfrak{g}_1 \otimes \mathfrak{m}^{i+1}$ such that

$$\begin{array}{ccc} u_0 & \xrightarrow{(-1)^1 L} & w_2 - v_2 \\ \delta \uparrow & & \\ u_1 & \xrightarrow{(-1)^2 L} & -\frac{1}{2}((w_1 + w_2)^2 - (w_1 + v_2)^2) = 0 \\ & & \delta \uparrow \\ & & \dots & \longrightarrow & \dots \\ & & & & \delta \uparrow \\ & & & & u_{n-1} & \xrightarrow{(-1)^n L} & (-1)^{\frac{(n-1)n}{2}} \frac{1}{n!} ((w_1 + w_2)^n - (w_1 + v_2)^n) = 0 \end{array}$$

Since $v_2, w_2 \in \mathfrak{g}_1 \otimes \mathfrak{m}^n$ and $w_1 \in \mathfrak{g}_1 \otimes \mathfrak{m}$, we have

$$(-1)^{\frac{(i-1)i}{2}} \frac{1}{i!} ((w_1 + w_2)^i - (w_1 + v_2)^i) = 0 \quad \text{for } i > 1.$$

Let's consider $u_{n-1} \in \mathfrak{g}_0 \otimes \text{sym}^{n-1} \mathfrak{g}_1 \otimes \mathfrak{m}^n$. Write $u_{n-1} = \sum_k a_k \otimes b_k$ where $a_k \in \mathfrak{g}_0$ and b_k are linearly independent in $\text{sym}^{n-1} \mathfrak{g}_1 \otimes \mathfrak{m}^n$. Then since $(-1)^n L u_{n-1} = 0$, we have $\sum_k L a_k \otimes b_k + a_k \otimes L b_k = 0$. So $L a_k = 0$. Since b_k are linearly independent and, $L a_k \otimes b_k$ and $a_k \otimes L b_k$ live in different spaces, we have $L a_k = 0$. Since $H^0(\mathfrak{g}) = H^1(X, \Lambda_0) = 0$, we have $a_k = 0$. $u_{n-1} = 0$. In this way we can show that $u_1 = \dots = u_{n-1} = 0$. Hence we have $(-1)^1 L u_0 = w_2 - v_2 \in \mathfrak{g}_1 \otimes \mathfrak{m}^n$. So $L u_0 = v_2 - w_2$. In other words, we have

$$\begin{aligned} \bar{\partial} u_0 &= \phi_2 - \psi_2 \\ [\Lambda_0, u_0] &= \Lambda_2 - \Pi_2 \end{aligned}$$

Let $u_0 := x_1 + x_2 \in \mathfrak{g}_0 \otimes R \cong \mathfrak{g}_0 \otimes (R/\mathfrak{m}^n \oplus \mathfrak{m}^n)$. Since $L x_1 = 0$ and $L x_2 = v_2 - w_2$, we have $x_1 = 0$ by $H^0(\mathfrak{g}) = 0$. So $u_0 \in \mathfrak{g}_0 \otimes \mathfrak{m}^n$. Then we have the following commutative diagram.

$$\begin{array}{ccc}
\mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \psi_1 + \phi_2} & \mathcal{A}_X^{0,1} \otimes R \\
\exp(u_0) \downarrow & & \downarrow \exp(u_0) \\
\mathcal{A}_X^{0,0} \otimes R & \xrightarrow{\bar{\partial} + \psi_1 + \psi_2} & \mathcal{A}_X^{0,1} \otimes R
\end{array}$$

Indeed,

$$\begin{aligned}
\exp(u_0)(\bar{\partial} + \psi_1 + \phi_2)\exp(-u_0) &= (1 + u_0)(\bar{\partial} + \psi_1 + \phi_2)(1 - u_0) \\
&= \bar{\partial} + \psi_1 + \phi_2 + u_0\bar{\partial} - \bar{\partial}u_0 \\
&= \bar{\partial} + \psi_1 + \phi_2 - \bar{\partial}u_0 \\
&= \bar{\partial} + \psi_1 + \psi_2
\end{aligned}$$

And

$$\begin{aligned}
\exp(u_0)(\Lambda_0 + \Pi_1 + \Lambda_2) &= (1 + u_0)(\Lambda_0 + \Pi_1 + \Lambda_2) \\
&= \Lambda_0 + \Pi_1 + \Lambda_2 + [u_0, \Lambda_0] = \Lambda_0 + \Pi_1 + \Lambda_2 - [\Lambda_0, u_0] \\
&= \Lambda_0 + \Pi_1 + \Pi_2
\end{aligned}$$

So the induction holds for $k = n$.

6.2.2 n -th Universal Poisson deformations

Recall that $R_n^u = \mathbb{C} \oplus \mathfrak{m}_n^u := \mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$ is a local artinian \mathbb{C} -algebra with residue \mathbb{C} and exponent n (i.e. $\mathfrak{m}_n^{u, n+1} = 0$).

Definition 6.10 (n -th universal Poisson deformation) *Since the identity map $\mathbb{H}^*(J_n(\mathfrak{g})) \rightarrow \mathbb{H}^*(J_n(\mathfrak{g}))$ is a homomorphism, it corresponds to a morphic element*

$$[\epsilon(v_u)] = [(\bar{v}_u, \frac{1}{2}\bar{v}_u \odot v_u, \dots, \frac{1}{n!} \underbrace{\bar{v}_u \odot \dots \odot \bar{v}_u}_n)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}_n^u$$

where $v_u := \phi_u + \Lambda_u \in \mathfrak{g}_1 \otimes \mathbb{H}^0(J_n(\mathfrak{g}))^*$. Then v_u defines an infinitesimal Poisson deformation $P_n^u := \mathcal{O}(v_u)$ over a local artinian \mathbb{C} -algebra $R_n^u := \mathbb{C} \oplus \mathbb{H}^0(J_n(\mathfrak{g}))^*$. We will call P_n^u be a n -th order universal Poisson deformation of (X, Λ_0) over R_n^u .

Let P be an infinitesimal Poisson deformation of (X, Λ_0) over (R, \mathfrak{m}) with $\mathfrak{m}^{n+1} = 0$. Assume that $HP^1(X, \Lambda_0) = 0$. Let $v = \phi + \Lambda$ be an Maurer Cartan element corresponding to the infinitesimal Poisson deformation P of (X, Λ_0) over R . Then $[\epsilon(v)] = [(\bar{v}, \frac{1}{2}\bar{v} \odot \bar{v}, \dots, \frac{1}{n!}\bar{v} \odot \dots \odot \bar{v})] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$, which induces a homomorphism $[\epsilon(v)] : \mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathfrak{m}$. Via the morphism $r := [\epsilon(v)]$, $v_u \in \mathfrak{g}_1 \otimes \mathfrak{m}_n^u$ is sent to $\tilde{v}_u \in \mathfrak{g}_1 \otimes \mathfrak{m}$. Then \tilde{v}_u satisfies the Maurer Cartan equation since v_u

does. Hence $[\epsilon(\tilde{v}_u)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}$ defines a morphic element and we have the corresponding homomorphism $[\epsilon(\tilde{v}_u)] : \mathbb{H}^0(J_n(\mathfrak{g}))^* \xrightarrow{id=[\epsilon(v_u)]} \mathbb{H}^0(J_n(\mathfrak{g}))^* \xrightarrow{r} \mathfrak{m}$, which is exactly $[\epsilon(v)]$. Hence we have $[\epsilon(v)] = [\epsilon(\tilde{v}_u)]$. Hence by the assumption of $HP^1(X, \Lambda_0) = 0$, the induced deformation $\tilde{v}_u = \tilde{\phi}_u + \tilde{\Lambda}_u$ from the deformation v_u by the base change $[\epsilon(v)] : \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathfrak{m}$

$$\begin{array}{ccc} (\mathcal{A}_X^{0,0} \otimes R_n^u, \Lambda_0 + \Lambda_u) & \xrightarrow{\bar{\partial} + \phi_u} & \mathcal{A}_X^{0,1} \otimes R_n^u \\ \downarrow & & \downarrow \\ (\mathcal{A}_X^{0,0} \otimes R, \Lambda_0 + \tilde{\Lambda}_u) & \xrightarrow{\bar{\partial} + \tilde{\phi}_u} & \mathcal{A}_X^{0,1} \otimes R \end{array}$$

is equivalent to $v = \phi + \Lambda$, which represents the infinitesimal Poisson deformation P of (X, Λ_0) over R . Then we have $P/R \cong r^*P_n^u = P_n^u \times_{\text{Spec}(R_n^u)} \text{Spec}(R)$. This proves our main Theorem 11 (2) in the Introduction of the part II of the thesis.

6.2.3 Formal Completion

The natural map $\mathbb{H}^0(J_{n-1}(\mathfrak{g})) \rightarrow \mathbb{H}^0(J_n(\mathfrak{g}))$ gives dually the homomorphism $\mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathbb{H}^0(J_{n-1}(\mathfrak{g}))^*$. Set $\mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^*$ and $R_n^u := \mathbb{C} \oplus \mathfrak{m}_n^u$. Take the inverse limit $\mathfrak{m}^u := \varprojlim_n \mathfrak{m}_n^u$. Then we have

$$\hat{R}^u := \mathbb{C} \oplus \mathfrak{m}^u = \mathbb{C} \oplus \varprojlim_n \mathfrak{m}_n^u = \varprojlim_n (\mathbb{C} \oplus \mathfrak{m}_n^u) = \varprojlim_n R_n^u$$

By our construction of $J_n(\mathfrak{g})$, we have $\mathbb{C} \oplus \mathfrak{m}^u / \mathfrak{m}^{u, n+1} = \mathbb{C} \oplus \mathfrak{m}_n^u$. Hence $(\hat{R}^u, \mathfrak{m}^u) = \varprojlim (R_n^u, \mathfrak{m}_n^u)$ is a complete local noetherian \mathbb{C} -algebra with respect to the \mathfrak{m}^u -adic topology.

From $\mathfrak{m}_n^u = \mathbb{H}^0(J_n(\mathfrak{g}))^* \rightarrow \mathfrak{m}_{n-1}^u = \mathbb{H}^0(J_{n-1}(\mathfrak{g}))^*$, the morphic element $[\epsilon(v_u = \phi_u + \Lambda_u)] = [(\bar{v}_u, \frac{1}{2}\bar{v}_u \odot \bar{v}_u, \dots, \frac{1}{n!}\bar{v}_u \odot \dots \odot \bar{v}_u)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}_n^u$ inducing the identity map on $\mathbb{H}^0(J_n(\mathfrak{g}))^*$, gives a morphic element $[\epsilon(\tilde{v}_u = \tilde{\phi}_u + \tilde{\Lambda}_u)] = [(\tilde{v}_u, \dots, \frac{1}{(n-1)!}\tilde{v}_u \odot \dots \odot \tilde{v}_u, 0)] \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}_{n-1}^u$ (via $\mathfrak{m}_n^u \rightarrow \mathfrak{m}_{n-1}^u$) which can be considered as an element in $\mathbb{H}^0(J_{n-1}(\mathfrak{g})) \otimes \mathfrak{m}_{n-1}^u$ inducing the identity map on $\mathbb{H}^0(J_{n-1}(\mathfrak{g}))^*$ and so we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}^0(J_n(\mathfrak{g}))^* & \xrightarrow{id=[\epsilon(v_u)]} & \mathbb{H}^0(J_n(\mathfrak{g}))^* \\ \downarrow & & \downarrow \\ \mathbb{H}^0(J_{n-1}(\mathfrak{g}))^* & \xrightarrow{id=[\epsilon(\tilde{v}_u)]} & \mathbb{H}^0(J_{n-1}(\mathfrak{g}))^* \end{array}$$

Hence we have the following commutative diagram

$$\begin{array}{ccc} (\mathcal{A}_X^{0,0} \otimes R_n^u, \Lambda_0 + \Lambda_u) & \xrightarrow{\bar{\partial} + \phi_u} & \mathcal{A}_X^{0,1} \otimes R_n^u \\ \downarrow & & \downarrow \\ (\mathcal{A}_X^{0,0} \otimes R_{n-1}^u, \Lambda_0 + \tilde{\Lambda}_u) & \xrightarrow{\bar{\partial} + \tilde{\phi}_u} & \mathcal{A}_X^{0,1} \otimes R_{n-1}^u \end{array}$$

So n -th universal Poisson deformations P_n^u/R_n^u fit together to form a direct system with limit

$$\hat{P}^u/\hat{R}^u = \varinjlim P_n^u/R_n^u$$

Now we set $\hat{R} = \varprojlim (R_n, \mathfrak{m}_n)$ is a complete local noetherian \mathbb{C} -algebra where (R_n, \mathfrak{m}_n) is a local artinian \mathbb{C} -algebra with residue \mathbb{C} and $\hat{P}/\hat{R} = \varinjlim_n P_n/R_n$ is a formal Poisson analytic space over \hat{R} , where P_n/R_n is an infinitesimal Poisson deformation of (X, Λ_0) , which can be interpreted as a sequence $\{r_n\}$ of morphic elements where $r_n \in \mathbb{H}^0(J_n(\mathfrak{g})) \otimes \mathfrak{m}_n$ such that r_n induces $r_{n-1} \in \mathbb{H}^0(J_{n-1}(\mathfrak{g})) \otimes \mathfrak{m}_{n-1}$ by the natural map $\mathfrak{m}_n \rightarrow \mathfrak{m}_{n-1}$. Hence we have the following commutative diagram:

$$\begin{array}{ccc} \mathbb{H}^0(J_n(\mathfrak{g}))^* & \xrightarrow{r_n} & \mathfrak{m}_n \\ \downarrow & & \downarrow \\ \mathbb{H}^0(J_{n-1}(\mathfrak{g}))^* & \xrightarrow{r_{n-1}} & \mathfrak{m}_{n-1} \end{array}$$

So we have the map $\hat{r} = \lim_n r_n : \hat{R}^u \rightarrow \hat{R}$ which induces $\hat{P}/\hat{R} = \hat{r}^*(\hat{P}^u/\hat{R}^u)$. This proves our main Theorem 11 (3) in the Introduction of the part II of the thesis. So we complete the Theorem 11.

Part III

Deformations of algebraic Poisson schemes

In the third part of the thesis, we study deformations of algebraic Poisson schemes over an algebraic closed field k , which is an algebraic version of the first part of the thesis. In chapter 7, we discuss the definition of Poisson schemes, morphisms and cohomology. A Poisson scheme X is a scheme whose structure sheaf \mathcal{O}_X is a sheaf of Poisson k -algebras. Equivalently, a Poisson structure on a scheme X is characterized by an element $\Lambda_0 \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ with $[\Lambda_0, \Lambda_0] = 0$. By a deformation of a Poisson scheme (X, Λ_0) we mean a commutative diagram

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \xi : \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \xrightarrow{s} & S \end{array}$$

where π is flat and surjective, and S is connected, (\mathcal{X}, Λ) is a Poisson scheme over S defined by $\Lambda \in \Gamma(\mathcal{X}, \mathcal{H}om(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}}))$ with $X \cong \mathcal{X} \times_S \text{Spec}(k)$ as a Poisson isomorphism. Note that when we ignore Poisson structures, a Poisson deformation is an usual flat deformation of an algebraic scheme X . By following Sernesi's book [Ser06], we extend the formalism of ordinary flat deformations to Poisson deformations. We show that given a Poisson scheme (X, Λ_0) , first order Poisson deformation (i.e Poisson deformations over a dual number $k[\epsilon]$) whose underlying flat deformation (when we ignore Poisson structures) is locally trivial, is naturally in one to one correspondence with $HP^2(X, \Lambda_0)$ which is the second (truncated) Lichnerowicz-Poisson cohomology group, in other words 2nd hypercohomology of the following complex of sheaves induced by $[\Lambda_0, -]$.

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda_0, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda_0, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^3 \Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda_0, -]} \dots$$

We also show that for a smooth Poisson algebraic scheme over k , any small extension $e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ (i.e (A, \mathfrak{m}) , $(\tilde{A}, \tilde{\mathfrak{m}})$ are local artinian k -algebras with residue k and $t \cdot \tilde{\mathfrak{m}} = 0$), and an infinitesimal Poisson deformation ξ of (X, Λ_0) over $\text{Spec}(A)$, we can associate an element $o_\xi(e) \in HP^3(X, \Lambda_0)$ such that $o_\xi(e)$ is 0 if and only if a lifting of ξ to \tilde{A} exists. So $HP^3(X, \Lambda_0)$ is an obstruction space. We also show that if $HP^2(X, \Lambda_0) = 0$, then (X, Λ_0) is rigid, which means that any infinitesimal Poisson deformation of (X, Λ_0) over A is trivial for all local artinian k -algebra A .

In chapter 8, we discuss Poisson deformation functor $PDef_{(X, \Lambda_0)}$ which is a functor of Artin rings. For a local artinian k -algebra A with residue k , $PDef_{(X, \Lambda_0)}(A)$ is the set of Poisson deformations over $\text{Spec}(A)$ up to Poisson equivalence. We show that for a smooth projective Poisson scheme (X, Λ_0) , $PDef_{(X, \Lambda_0)}$ satisfies Schlessinger's criterion $(H_0), (H_1), (H_2), (H_3)$ and so $PDef_{(X, \Lambda_0)}$ has a miniversal family. We also show that in addition if $HP^1(X, \Lambda_0) = 0$, $PDef_{(X, \Lambda_0)}$ is pro-representable.

In chapter 9, we extend the construction of a cotangent complex ([LS67]) to Poisson cases. Let $A \rightarrow B$ be a Poisson homomorphism of Poisson k -algebras, and M be a Poisson B -module. We construct $PT^i(B/A, M)$ in a similar way to construct $T^i(B/A, M)$ in [LS67]. As an application to Poisson deformation, we show that for a Poisson algebra B_0 , $PDef_{\text{Spec}(B_0)}(k[\epsilon])$ is a natural one to one correspondence with $PT^1(B_0/k, B_0)$. We also show that given a Poisson algebra B_0 and an

Poisson ideal I of B_0 , deformations of a Poisson subscheme $\text{Spec}(C)$ of $\text{Spec}(B_0)$ over $\text{Spec}(k[\epsilon])$ is one to one correspondence with $PT^1(C/B_0, C)$ where $C = B_0/I$.

Chapter 7

Deformations of algebraic Poisson schemes

7.1 Definitions of Poisson schemes, morphisms and cohomology

In this section, every algebra is a commutative k -algebra, where k is a field. Our reference is [LGPV13] Chapter 3. For algebraic geometry, we refer to [Har77], [Liu02].

7.1.1 Characterization of a Poisson bracket $\{-, -\}$ of a Poisson algebra A over R

In this subsection, we will characterize a Poisson structure of a commutative algebra A over R in terms of an element $\Lambda \in \text{Hom}_A(\Omega_{R/A}^1, A)$ with $[\Lambda, \Lambda] = 0$ where $[-, -]$ is the Schouten bracket on $\bigoplus_{p \geq 1} \text{Hom}_A(\wedge^p \Omega_{A/R}^1, A)$.

Definition 7.1 *Let A be a commutative R -algebra and let $p \geq 1$. A skew symmetric p -linear map $P \in \text{Hom}_R(\wedge^p A, A)$ is called a skew symmetric p -derivation of A over R , if P is a derivation in each of its components.*

Let $\Omega_{A/R}^1$ be the A -module of relative kähler differential forms of A over R . Then the R -module of all skew symmetric p -linear maps in $Hom_R(\wedge^p A, A)$ is identified with $Hom_A(\wedge^p \Omega_{A/R}^1, A)$. Let P be a skew symmetric p linear map in $Hom_R(\wedge^p A, A)$. Then the associated $\tilde{P} \in Hom_A(\wedge^p \Omega_{A/R}^1, A)$ is defined in the following way: $\tilde{P}(da_1 \wedge \cdots \wedge da_p) := P(a_1, \cdots, a_p)$ where $d : A \rightarrow \Omega_{A/R}^1$ is the canonical map.

The Shouten bracket on $\bigoplus_{p \geq 1} Hom_A(\wedge^p \Omega_{A/R}^1, A)$ and characterization of a Poisson bracket on A

Definition 7.2 For $p, q \in \mathbb{N}$, a (p, q) -shuffle is a permutation σ of the set $\{1, \dots, p+q\}$, such that $\sigma(1) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(p+q)$. The set of all (p, q) -shuffles is denoted by $S_{p,q}$. For a shuffle $\sigma \in S_{p,q}$, we denote the signature of σ by $sgn(\sigma)$. By convention, $S_{p,-1} := \emptyset$ and $S_{-1,q} := \emptyset$ for $p, q \in \mathbb{N}$.

Definition 7.3 We define the Schouten bracket $[-, -]$ on $\bigoplus_{p \geq 1} Hom_A(\wedge^p \Omega_{A/R}^1, A)$, namely a family of maps

$$[-, -] : Hom_A(\wedge^p \Omega_{A/R}^1, A) \times Hom_A(\wedge^q \Omega_{A/R}^1, A) \rightarrow Hom_A(\wedge^{p+q-1} \Omega_{A/R}^1, A)$$

for $p, q \in \mathbb{N}$ in the following way: let $P \in Hom_A(\wedge^p \Omega_{A/R}^1, A)$ and $Q \in Hom_A(\wedge^q \Omega_{A/R}^1, A)$, and for $F_1, \dots, F_{p+q-1} \in A$ by

$$\begin{aligned} [P, Q](dF_1 \wedge \cdots \wedge dF_{p+q-1}) &= \sum_{\sigma \in S_{q,p-1}} sgn(\sigma) P(d(Q(dF_{\sigma(1)} \wedge \cdots \wedge dF_{\sigma(q)}))) \wedge dF_{\sigma(q+1)} \cdots \wedge dF_{\sigma(p+q-1)} \\ &\quad - (-1)^{(p-1)(q-1)} \sum_{\sigma \in S_{p,q-1}} sgn(\sigma) Q(d(P(dF_{\sigma(1)} \wedge \cdots \wedge dF_{\sigma(p)}))) \wedge dF_{\sigma(p+1)} \wedge \cdots \wedge dF_{\sigma(p+q-1)} \end{aligned}$$

Example 7.4 Let $P \in Hom_A(\wedge^2 \Omega_{A/R}^1, A)$ and $Q \in Hom_A(\Omega_{A/R}^1, A)$. Then

$$[P, Q](dF_1 \wedge dF_2) = P(dQ(F_1) \wedge dF_2) - P(d(Q(F_2)) \wedge dF_1) - Q(d(P(dF_1 \wedge dF_2)))$$

Proposition 7.5 Let A be a commutative algebra over R . If Λ is a skew symmetric biderivation of A over R , i.e $\Lambda \in Hom_A(\wedge^2 \Omega_{A/R}^1, A)$, then P defines a Poisson bracket (i.e Jacobi identity holds) if and only if $[\Lambda, \Lambda] = 0$.

Proof. See [LGPV13] Proposition 3.5 page 80. ■

Notation 13 Let A be a Poisson algebra over R with a Poisson bracket $\{-, -\}$. Let Λ be the associated biderivation with the Poisson bracket $\{-, -\}$. Then we will denote by (A, Λ) the Poisson algebra A over R with the Poisson bracket $\{-, -\}$.

Remark 7.6 Let (A, Λ) be a Poisson algebra over R with the Poisson structure $\Lambda \in \text{Hom}_A(\wedge^2, \Omega_{A/R}^1, A)$ with $[\Lambda, \Lambda] = 0$. If we let $\mathfrak{g} = \bigoplus_{i \geq 0} g_i$, where $g_i = \text{Hom}_A(\wedge^{i+1} \Omega_{A/R}^1, A)$. Then $\mathfrak{g} = (\bigoplus_{i \geq 0} g_i, [-, -], [\Lambda, -])$ is a differential graded Lie algebra with the differential $[\Lambda, -]$. In other words, we have the following properties: for $P \in \text{Hom}_A(\wedge^p \Omega_{A/R}^1, A)$ and $Q \in \text{Hom}_A(\wedge^q \Omega_{A/R}^1, A)$ and $S \in \text{Hom}_A(\wedge^r \Omega_{A/R}^1, A)$,

1. $[\Lambda, [\Lambda, P]] = 0$ and $[\Lambda, P] \in \text{Hom}_A(\wedge^{p+1} \Omega_{A/R}^1, A)$
2. $[P, Q] = -(-1)^{(p-1)(q-1)}[Q, P]$
3. $[[P, Q], S] = [P, [Q, S]] - (-1)^{(p-1)(q-1)}[Q, [P, S]]$
4. $[\Lambda, [P, Q]] = [[\Lambda, P], Q] + (-1)^{p-1}[P, [\Lambda, Q]]$

Definition 7.7 Let (A, Λ) be a Poisson algebra over R . We define i -th truncated Lichnerowicz Poisson cohomology of (A, Λ) to be the i -th cohomology group of the following complex

$$0 \rightarrow \text{Hom}_A(\Omega_{A/R}^1, A) \xrightarrow{[\Lambda, -]} \text{Hom}_A(\wedge^2 \Omega_{A/R}^1, A) \xrightarrow{[\Lambda, -]} \text{Hom}_A(\Omega_{A/R}^1, A) \xrightarrow{[\Lambda, -]} \dots$$

We will denote i -th Lichnerowicz Poisson cohomology group by $HP^i(A, \Lambda)$.

Characterization of Poisson morphisms

Let $f : A \rightarrow B$ be a R -homomorphism. Then we have the following commutative diagram

$$\begin{array}{ccc} \Omega_{A/R} \otimes_A B & \longrightarrow & \Omega_{B/R} \\ d_A \uparrow & & d_B \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

So we have a canonical homomorphism $\wedge^2 \Omega_{A/R} \otimes_A B \rightarrow \wedge^2 \Omega_{B/R}$. This induces $f^* : \text{Hom}_B(\wedge^2 \Omega_{B/R}, B) \rightarrow \text{Hom}_B(\wedge^2 \Omega_{A/R} \otimes_A B, B) \cong \text{Hom}_A(\wedge^2 \Omega_{A/R}, B)$

Proposition 7.8 *Let (A, P) and (B, Q) be two Poisson R -algebras. Then a homomorphism $A \rightarrow B$ of R -algebras is a Poisson homomorphism if and only if $f^*Q = f \circ P$.*

Proof. Let f be a Poisson R -homomorphism. In other words, $f(\{a, b\}) = \{f(a), f(b)\}$. Then $f(P(d_A a, d_A b)) = Q(d_B f(a), d_B f(b)) = f^*Q(d_A a, d_A b)$. Hence we get $f^*Q = f \circ P$. ■

Example 7.9 (Poisson ideals) *Let I be an ideal of a commutative R -algebra A and set $B = A/I$. Let $\Lambda \in \text{Hom}_A(\wedge^2 \Omega_{A/R}, A)$ be a Poisson structure on A over R . The map $A \rightarrow B$ induces $\text{Hom}_B(\wedge^2 \Omega_{B/R}, B) \rightarrow \text{Hom}_A(\wedge^2 \Omega_{A/R}, B)$ which is injective since $\Omega_{A/R}^1 \otimes_A B \rightarrow \Omega_{B/R}^1$ is surjective. Let $\bar{\Lambda}$ be the composition of Λ followed by $A \rightarrow B$. If $\bar{\Lambda}$ has pre image P , then P defines a Poisson structure on B , which makes I to be a Poisson ideal of A . Indeed, we show that $[P, P] = 0$. Since $\text{Hom}_B(\wedge^3 \Omega_{B/R}, B) \rightarrow \text{Hom}_A(\wedge^3 \Omega_{A/R}, B)$ is injective and $[P, P]$ is sent to $[\bar{\Lambda}, \bar{\Lambda}] = 0$, where $[\bar{\Lambda}, \bar{\Lambda}]$ is the composition of $[\Lambda, \Lambda]$ followed by $A \rightarrow B$. We have $[P, P] = 0$.*

7.1.2 Affine Poisson Schemes

Poisson (k) -sheaves on a topological space X

Definition 7.10 *Let X be a topological space and let k be a field. A Poisson presheaf \mathcal{F} on X consists of the following data:*

1. *An Poisson k -algebra $\mathcal{F}(U)$ for every open subset U of X , and*
2. *a Poisson k -algebra homomorphism $\rho_{UV} : \mathcal{F}(U) \rightarrow \mathcal{F}(V)$ for every inclusion wof open subset $V \subset U$.*

which satisfy the following conditions:

1. *$\mathcal{F}(\emptyset) = 0$ for the empty set \emptyset .*
2. *ρ_{UU} is the identity map $\mathcal{F}(U) \rightarrow \mathcal{F}(U)$*
3. *If we have three open subsets $W \subset V \subset U$, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.*

We call ρ_{UV} restriction maps, and we write $s|_V$ instead of $\rho_{UV}(s)$ for $s \in \mathcal{F}(U)$. We refer to $\mathcal{F}(U)$ as the sections of \mathcal{F} over U .

Definition 7.11 *We say that a Poisson presheaf \mathcal{F} is a Poisson sheaf if we have the following properties:*

1. (*Uniqueness*) *Let $s \in \mathcal{F}(U)$ for an open subset U of X , and $\{U_i\}$ be a open covering of U . If $s|_{U_i} = 0$ for every i , then $s = 0$.*
2. (*Glueing local sections*) *Let U be an open subset of X and $\{U_i\}$ be a open covering of U . Let $s_i \in \mathcal{F}(U_i)$ such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$. Then there exists $s \in \mathcal{F}(U)$ such that $s|_{U_i} = s_i$.*

Definition 7.12 *Let \mathcal{F} and \mathcal{G} be Poisson presheaves on X . A morphism $f : \mathcal{F} \rightarrow \mathcal{G}$ is called Poisson morphism if $f(U) : \mathcal{F}(U) \rightarrow \mathcal{G}(U)$ is a Poisson homomorphism for any open set $U \subset X$.*

Remark 7.13

1. *Let \mathcal{F} be Poisson presheaf on X , and let $x \in X$. The stalk \mathcal{F}_x at x is a Poisson k -algebra.*
2. *Let \mathcal{F} be a Poisson presheaf on X . There exists a Poisson sheaf \mathcal{F}^+ associated to \mathcal{F} and a morphism of Poisson presheaves $\theta : \mathcal{F} \rightarrow \mathcal{F}^+$ verifying the following universal property: for every Poisson morphism $\alpha : \mathcal{F} \rightarrow \mathcal{G}$, where \mathcal{G} is a Poisson sheaf, there exists a unique Poisson morphism $\tilde{\alpha} : \mathcal{F}^+ \rightarrow \mathcal{G}$ such that $\alpha = \tilde{\alpha} \circ \theta$.*

Definition 7.14 (Poisson locally ringed spaces) *A Poisson ringed topological space consists of a topological space X endowed with a Poisson sheaf (a sheaf of Poisson k -algebras) \mathcal{O}_X on X such that $\mathcal{O}_{X,x}$ is a local ring for every $x \in X$ which is a Poisson k -algebra. We denote it by (X, \mathcal{O}_X) .*

Definition 7.15 *A Poisson morphism of Poisson ringed topological spaces*

$$(f, f^\#) : (X, \mathcal{O}_X) \rightarrow (Y, \mathcal{O}_Y)$$

consists of a continuous map $f : X \rightarrow Y$ and a morphism of Poisson sheaves $f^\# : \mathcal{O}_Y \rightarrow f_\mathcal{O}_X$ such that for every $x \in X$, the induced Poisson homomorphism $f_x^\# : \mathcal{O}_{Y,f(x)} \rightarrow \mathcal{O}_{X,x}$ is a local Poisson*

homomorphism. We define the compositions of two Poisson morphisms of Poisson ringed topological spaces in an obvious manner.

Affine Poisson schemes

We recall the following facts.

Lemma 7.16 *Let A be an commutative R -algebra, $S \subset A$ a multiplicatively closed systems, and A_S the corresponding localization of A . Then the module of relative differential forms $\Omega_{A_S/R}^1$ is given by the localization $(\Omega_{A/R}^1)_S$ and that the map*

$$d : A_S \rightarrow (\Omega_{A/R}^1)_S, \quad \frac{f}{s} \mapsto \frac{sd_{A/R}(f) - fd_{A/R}(s)}{s^2}$$

serves as the exterior differential of A_S over R where $d_{A/R} : A \rightarrow \Omega_{A/R}^1$ denotes the exterior differential of A . And we have $\Omega_{A/R}^1 \otimes_A A_S \rightarrow \Omega_{A_S/R}^1$ and $\Omega_{A_S/A}^1 = 0$.

Proof. See [Bos13] page 354 exercise 2. ■

Let A be a Poisson algebra over k . Now let $\Lambda \in \text{Hom}_A(\wedge^2 \Omega_{A/k}, A)$ be the Poisson k -structure on A , denoted by $\{-, -\}$. Let S be a multiplicative system of A . Then Λ induces a Poisson structure on A_S , denoted by $\{-, -\}_S$ from the natural map $\text{Hom}_A(\wedge^2 \Omega_{A/k}, A) \rightarrow \text{Hom}_A(\wedge^2 \Omega_{A/k}, A_S) \cong \text{Hom}_{A_S}(\wedge^2 A_S/k, A_S)$. More precisely, we have

$$\begin{aligned} \left\{ \frac{a_1}{s_1}, \frac{a_2}{s_2} \right\}_S &= \Lambda \left(d \left(\frac{a_1}{s_1} \right), d \left(\frac{a_2}{s_2} \right) \right) = \Lambda \left(\frac{s_1 da_1 - a_1 ds_1}{s_1^2}, \frac{s_2 da_2 - a_2 ds_2}{s_2^2} \right) \\ &= \frac{\{a_1, a_2\}}{s_1 s_2} - \frac{a_2 \{a_1, s_2\}}{s_1 s_2^2} - \frac{a_1 \{s_1, a_2\}}{s_1^2 s_2} + \frac{a_1 a_2 \{s_1, s_2\}}{s_1^2 s_2^2} \end{aligned}$$

Proposition 7.17 *$X = \text{Spec}(A)$ for a Poisson k -algebra (A, Λ) is a Poisson ringed topological space.*

Proof. Let \mathfrak{p} be a prime ideal of A . Then $A_{\mathfrak{p}}$ has a natural Poisson structure induced from (A, Λ) with the Poisson bracket $\{-, -\}_{\mathfrak{p}}$. For any open set U of $\text{Spec}(A)$ and $f, g \in \mathcal{O}_X(U)$. Then a, b can be identified with $a, b : U \rightarrow \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ locally defined by an element of A_f for $D(f) \subset U$. We define $\{a, b\} : U \rightarrow \bigcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ by $\mathfrak{p} \rightarrow \{a_{\mathfrak{p}}, b_{\mathfrak{p}}\}_{\mathfrak{p}}$. Since for each principle open set of the form $D(f)$, the Poisson structure on $D(f)$ are all induced from Λ , we have $\{a, b\} \in \mathcal{O}_X(U)$. Hence the structure sheaf \mathcal{O}_X is a Poisson sheaf. Hence X is a Poisson ringed topological space.

■

Definition 7.18 (affine Poisson schemes) We define an affine Poisson scheme to be a Poisson ringed topological space isomorphic to some $(\text{Spec } A, \mathcal{O}_{\text{Spec } A})$ for a Poisson k -algebra (A, Λ) .

Definition 7.19 (Poisson schemes) A Poisson scheme is a Poisson ringed topological space (X, \mathcal{O}_X) admitting an open covering $\{U_i\}$ of X such that $(U_i, \mathcal{O}_X|_{U_i})$ is an affine Poisson k -scheme for every i .

Note that any k -scheme can be considered to be a Poisson scheme since any k -algebra A has trivial Poisson structure, i.e. $\{f, g\} = 0$ for any $f, g \in A$. We will consider a scheme without Poisson structure to be a scheme with trivial Poisson structure.

7.1.3 Poisson Schemes

Definition 7.20 Let $X \rightarrow S$ be a morphism of schemes. There is an operation

$$[-, -] : \mathcal{H}om_{\mathcal{O}_X}(\wedge^p \Omega_{X/S}, \mathcal{O}_X) \times \mathcal{H}om_{\mathcal{O}_X}(\wedge^q \Omega_{X/S}, \mathcal{O}_X) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\wedge^{p+q-1} \Omega_{X/S}, \mathcal{O}_X)$$

which is called the Schouten bracket on a scheme X over S .

The bracket $[-, -]$ is defined in the following way: $\Gamma(U, \mathcal{H}om_{\mathcal{O}_X}(\wedge^p \Omega_{X/S}, \mathcal{O}_X))$ is the set of elements of the form $\beta : U \rightarrow \bigcup_{x \in U} \text{Hom}_{\mathcal{O}_{X,x}}(\wedge^p \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}, \mathcal{O}_{X,x})$ such that for any x , there exists an affine open neighborhood V of $s = f(x)$ and an affine open neighborhood $U \subset f^{-1}(V)$ of x and $\alpha \in \text{Hom}_{\mathcal{O}_X(U)}(\wedge^p \Omega_{\mathcal{O}_X(U)/\mathcal{O}_S(V)}, \mathcal{O}_X(U))$ with $\beta(x) = \alpha_x$. So on U , we define $[\beta_1, \beta_2] := U \rightarrow \bigcup_{x \in U} \text{Hom}_{\mathcal{O}_{X,x}}(\wedge^p \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}, \mathcal{O}_{X,x}), x \mapsto [\beta_1(x), \beta_2(x)]_x$, where $[-, -]_x$ is the Schouten bracket on $\bigoplus_p \text{Hom}_{\mathcal{O}_{X,x}}(\wedge^p \Omega_{\mathcal{O}_{X,x}/\mathcal{O}_{S,s}}, \mathcal{O}_{X,x})$. Hence to show the existence of Schouten bracket on X over S , we only need to check the following lemma.

Lemma 7.21 Let A be a commutative R -algebra $(f : R \rightarrow A)$, and \mathfrak{p} be a prime ideal of A . Let $\mathfrak{q} = f^{-1}(\mathfrak{p})$. The following diagram commutes

$$\begin{array}{ccc}
Hom_A(\wedge^p \Omega_{A/R}^1, A) \times Hom_A(\wedge^q \Omega_{A/R}^1, A) & \xrightarrow{[-, -]} & Hom_A(\wedge^{p+q-1} \Omega_{A/R}^1, A) \\
\downarrow & & \downarrow \\
Hom_A(\wedge^p \Omega_{A/R}^1, A_{\mathfrak{p}}) \times Hom_A(\wedge^q \Omega_{A/R}^1, A_{\mathfrak{p}}) & \xrightarrow{[-, -]} & Hom_A(\wedge^{p+q-1} \Omega_{A/R}^1, A_{\mathfrak{p}}) \\
\cong \downarrow & & \downarrow \cong \\
Hom_{A_{\mathfrak{p}}}(\wedge^p \Omega_{A_{\mathfrak{p}}/R}^1, A_{\mathfrak{p}}) \times Hom_{A_{\mathfrak{p}}}(\wedge^q \Omega_{A_{\mathfrak{p}}/R}^1, A_{\mathfrak{p}}) & \xrightarrow{[-, -]} & Hom_{A_{\mathfrak{p}}}(\wedge^{p+q-1} \Omega_{A_{\mathfrak{p}}/R}^1, A_{\mathfrak{p}}) \\
\cong \downarrow & & \downarrow \cong \\
Hom_{A_{\mathfrak{p}}}(\wedge^p \Omega_{A_{\mathfrak{p}}/R_{\mathfrak{q}}}^1, A_{\mathfrak{p}}) \times Hom_{A_{\mathfrak{p}}}(\wedge^q \Omega_{A_{\mathfrak{p}}/R_{\mathfrak{q}}}^1, A_{\mathfrak{p}}) & \xrightarrow{[-, -]} & Hom_{A_{\mathfrak{p}}}(\wedge^{p+q-1} \Omega_{A_{\mathfrak{p}}/R_{\mathfrak{q}}}^1, A_{\mathfrak{p}})
\end{array}$$

Example 7.22 Let $B \otimes_k A$ be a A -algebra where k is a field and B is a finitely generated k -algebra (Hence $\Omega_{B/k}^1$ is finitely presented). Then $Hom_{B \otimes_k A}(\wedge^p \Omega_{B \otimes_k A/A}^1, B \otimes_k A) \cong Hom_{B \otimes_k A}(\wedge^p \Omega_{B/k}^1 \otimes A, B \otimes_k A) \cong Hom_B(\wedge^p \Omega_{B/k}, B \otimes_k A) \cong Hom_k(\wedge^p \Omega_{B/k}, B) \otimes_k A$. So the Schouten bracket $[-, -]_{B \otimes_k A}$ on $Hom_B(\wedge^p \Omega_{B/k}, B) \otimes A$ over A can be seen as

$$[P \otimes a, Q \otimes b]_{B \otimes_k A} = [P, Q]_B \otimes ab$$

Let X be a scheme over k . We would like to characterize a Poisson structure on X by an element $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{\mathcal{O}_X/k}^1, \mathcal{O}_X))$ with $[\Lambda, \Lambda] = 0$.

Proposition 7.23 Let X be a scheme over k . The following are equivalent

1. X is a Poisson scheme over k .
2. There exists a global section $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ with $[\Lambda, \Lambda] = 0$

Proof. Let X be a Poisson scheme over k . Then for each x , $\mathcal{O}_{X,x}$ is a Poisson k -algebra. So we have $\Lambda_x \in Hom_{\mathcal{O}_{X,x}}(\wedge^2 \Omega_{\mathcal{O}_{X,x}/k}^1, \mathcal{O}_{X,x})$ with $[\Lambda_x, \Lambda_x] = 0$. We define $\Lambda : X \rightarrow \bigcup_{x \in X} Hom_{\mathcal{O}_{X,x}}(\wedge^2 \Omega_{\mathcal{O}_{X,x}/k}^1, \mathcal{O}_{X,x})$. Since X is locally defined by affine Poisson schemes, for each $x \in X$, there exists an affine neighborhood $Spec(A)$ of x with $\Lambda_A \in Hom_A(\wedge^2 \Omega_{A/k}^1, A)$ with $[\Lambda_A, \Lambda_A] = 0$ which induces Λ_x for $x \in Spec(A)$. Hence $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ with $[\Lambda, \Lambda] = 0$.

Conversely, we assume that we have a global section $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ with $[\Lambda, \Lambda] = 0$. Then Λ can be identified with $\Lambda : X \rightarrow \bigcup_{x \in X} Hom_{\mathcal{O}_{X,x}}(\wedge^2 \Omega_{\mathcal{O}_{X,x}/k}^1, \mathcal{O}_{X,x})$ as above. Hence $\mathcal{O}_{X,x}$ is a Poisson k -algebra induced from Λ_x with the Poisson bracket $\{-, -\}_x$. We show that \mathcal{O}_X is a sheaf of Poisson k -algebra. Let U be open set of X . Let $f, g \in \mathcal{O}_X(U)$. Then f, g can be identified with $f, g : U \rightarrow \bigcup_{x \in U} \mathcal{O}_{X,x}$ which are locally defined by elements of sections of affine open sets. We define $\{f, g\}$ by $U \rightarrow \bigcup_{x \in X} \mathcal{O}_{X,x}, x \mapsto \{f_x, g_x\}_x$. This makes \mathcal{O}_X to be a

sheaf of Poisson k -algebras. For each x , there exists an affine open set $\text{Spec}(A)$ of x such that Λ on $\text{Spec}(A)$ is induced from a $\Lambda_A \in \text{Hom}_A(\wedge^2 \Omega_{A/k}^1, A)$ with $[\Lambda_A, \Lambda_A] = 0$. So Λ_A defines a Poisson structure on $\text{Spec}(A)$. Hence X is locally defined by affine Poisson schemes. Hence X is a Poisson scheme over k . ■

Definition 7.24 Let X be a Poisson scheme over k . Let $f : X \rightarrow S$ be a morphism of schemes.

We say that X is Poisson over S or a Poisson S -scheme if for any open set U of S and $\mathcal{O}_S(U) \rightarrow \mathcal{O}_X(f^{-1}(U))$, $\mathcal{O}_X(f^{-1}(U))$ is a Poisson $\mathcal{O}_S(U)$ -algebra. In other words, $\{s, a\} = 0$ for any $s \in \mathcal{O}_S(U)$ and $a \in \mathcal{O}_X(f^{-1}(U))$

Proposition 7.25 Let X be a scheme over k and $f : X \rightarrow S$ be a morphism of schemes. The following are equivalent.

1. X is a Poisson scheme over S .
2. There exists a global section $P \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/S}, \mathcal{O}_X))$ with $[P, P] = 0$

Proof. Let $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ be the (k) -Poisson structure on X . Now we assume that X is a Poisson scheme over S via $f : X \rightarrow S$. We note that we have an exact sequence

$$0 \rightarrow \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/S}, \mathcal{O}_X)) \rightarrow \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X)).$$

We will show that Λ is actually in $\Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/S}, \mathcal{O}_X))$. For $x \in X$, via $f_* : \mathcal{O}_{S, f(x)} \rightarrow \mathcal{O}_{X, x}$, $\mathcal{O}_{X, x}$ is a Poisson $\mathcal{O}_{S, f(x)}$ -algebra. Since Λ_x is $\mathcal{O}_{S, f(x)}$ -linear, we have actually

$$\Lambda_x \in \text{Hom}_{\mathcal{O}_{X, x}}(\Omega_{\mathcal{O}_{X, x}/\mathcal{O}_{S, f(x)}}^1, \mathcal{O}_{X, x})$$

with $[\Lambda_x, \Lambda_x] = 0$. Hence $P =: \Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/S}, \mathcal{O}_X))$ with $[P, P] = 0$.

Conversely, assume there exists a global section $P \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/S}, \mathcal{O}_X))$ with $[P, P] = 0$. Then P defines a Poisson scheme over k by the above exactness. Since for each $x \in X$, $\mathcal{O}_{X, x}$ is a Poisson $\mathcal{O}_{S, f(x)}$ -algebra, X is a Poisson scheme over S .

■

Definition 7.26 Let (X, P) and (Y, Q) be Poisson schemes over S with $g : X \rightarrow S$ and $h : Y \rightarrow S$.

Then a morphism $f : X \rightarrow Y$ of schemes over S is called a morphism of Poisson schemes over S if for any open set U of S and any open set V of $h^{-1}(U)$ and any open set W of $f^{-1}(V)$, and $f^\# : \mathcal{O}_Y(V) \rightarrow \mathcal{O}_X(W)$ is a Poisson $\mathcal{O}_S(U)$ -homomorphism.

Let $f : X \rightarrow Y$ be a morphism of schemes over S . Then we have $f^*\Omega_{Y/S}^1 \rightarrow \Omega_{X/S}^1$. Since $\Omega_{Y/S}^1$ is quasi-coherent, we have $f^*(\wedge^2\Omega_{Y/S}^1) \cong \wedge^2 f^*\Omega_{Y/S}^1$. So we have $\mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/S}^1, \mathcal{O}_X) \rightarrow \mathcal{H}om_{\mathcal{O}_X}(f^*(\wedge^2\Omega_{Y/S}^1), f^*\mathcal{O}_Y)$. On the other hand, we have a natural sheaf morphism

$$\mathcal{H}om_{\mathcal{O}_Y}(\wedge^2\Omega_{Y/S}^1, \mathcal{O}_Y) \rightarrow f_*f^*\mathcal{H}om_{\mathcal{O}_Y}(\wedge^2\Omega_{Y/S}^1, \mathcal{O}_Y) \rightarrow f_*\mathcal{H}om_{\mathcal{O}_X}(f^*(\wedge^2\Omega_{Y/S}^1), f^*\mathcal{O}_Y)$$

By taking the global sections, we have two morphisms

$$\begin{aligned} \alpha : \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/S}^1, \mathcal{O}_X)) &\rightarrow \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(f^*(\wedge^2\Omega_{Y/S}^1), f^*\mathcal{O}_Y)) \\ \beta : \Gamma(Y, \mathcal{H}om_{\mathcal{O}_Y}(\wedge^2\Omega_{Y/S}^1, \mathcal{O}_Y)) &\rightarrow \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(f^*(\wedge^2\Omega_{Y/S}^1), f^*\mathcal{O}_Y) \end{aligned}$$

If $f : (X, P) \rightarrow (Y, Q)$ is a morphism of Poisson schemes over S , we have $\alpha(P) = \beta(Q)$.

Proposition 7.27 (Glueing Poisson schemes) *Let S be a k -scheme. Let us consider a family $\{X_i\}$ of Poisson schemes over S . We suppose given open subschemes X_{ij} of X_i (which is necessarily Poisson S -scheme) and Poisson isomorphisms of S -schemes $f_{ij} : X_{ij} \rightarrow X_{ji}$ such that $f_{ii} = Id_{X_i}$, $f_{ij}(X_{ij} \cap X_{ji}) = X_{ji} \cap X_{jk}$, and $f_{ik} = f_{jk} \circ f_{ij}$ on $X_{ij} \cap X_{ik}$. Then there exists an Poisson S -scheme X , unique up to isomorphism, with Poisson open immersion of S -schemes $g_i : X_i \rightarrow X$ such that $g_i : X_i \rightarrow X$ such that $g_i = g_j \circ f_{ij}$ on X_{ij} , and $X = \cup_i g_i(X_i)$.*

7.1.4 (truncated) Lichnerowicz-Poisson cohomology

Definition 7.28 *Let (X, Λ) be a Poisson scheme over S . Then we define i -th (truncated) Lichnerowicz-Poisson cohomology is the i -th hypercohomology group of the following complex of sheaves*

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/S}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/S}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^3\Omega_{X/S}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \dots$$

We denote i -th cohomology group by $HP^i(X, \Lambda)$.

Remark 7.29 *Let $X = \text{Spec}(A)$ be an affine scheme with a Poisson structure $\Lambda \in \text{Hom}_A(\wedge^2\Omega_{A/S}, A) = \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/S}^1, \mathcal{O}_X))$. Since $\mathcal{H}om_{\mathcal{O}_X}(\wedge^i\Omega_{X/S}^1, \mathcal{O}_X)$ are quasi coherent, and so its higher cohomology vanishes. Hence (truncated) Lichnerowicz-Poisson cohomology of (X, Λ) is same to the (truncated) Lichnerowicz-Poisson cohomology of a Poisson algebra (A, Λ) .*

7.2 Deformations of algebraic Poisson schemes

7.2.1 Basic materials on deformations of algebraic Poisson schemes

In this section, we discuss deformations of algebraic Poisson schemes by following [Ser06](see Chapter 1) in the Poisson context. We will always denote by k a fixed algebraically closed field. All schemes will be assumed to be defined over k , locally noetherian and separated. If S is a scheme and $s \in S$, we denote $k(s) = \mathcal{O}_{S,s}/\mathfrak{m}_s$ the residue field of S at s . We denote by **Art** the category of local artinian k -algebras with residue field k .

Definition 7.30 *Let (X, Λ_0) be an algebraic Poisson scheme. A cartesian diagram of morphisms of schemes*

$$\eta : \begin{array}{ccc} (X, \Lambda_0) & \xrightarrow{i} & (\mathcal{X}, \Lambda) \\ \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \xrightarrow{s} & S \end{array}$$

is called a family of Poisson deformations or a Poisson deformation of X parametrized by S where π is flat and surjective, and S is connected, (\mathcal{X}, Λ) is a Poisson S -scheme with $\Lambda \in \Gamma(\mathcal{X}, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}}))$ and $X \cong \mathcal{X} \times_S \text{Spec}(k)$ as Poisson isomorphism: in other words, Λ_0 is induced from Λ . We call S and (\mathcal{X}, Λ) respectively the parameter scheme and the total Poisson S -scheme of the Poisson deformation η . If S is algebraic, for each k -rational point $t \in S$ the scheme theoretic fiber $(\mathcal{X}(t), \Lambda(t))$ with the induced Poisson structure $\Lambda(t)$ from Λ is also called a Poisson deformation of (X, Λ_0) . A Poisson deformation η over $\text{Spec}(A)$ is called infinitesimal (reps. first-order) if $A \in \mathbf{Art}$ (reps. if $A = k[\epsilon]$).

Remark 7.31 *We will explain more in detail that Λ_0 is induced from Λ . Since $\Omega_{X/k}^1 \cong i^* \Omega_{\mathcal{X}/S}^1$, canonical map $\mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}}) \rightarrow i_* i^* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}}) \rightarrow i_* \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_{\mathcal{X}})$ induces $\Gamma(\mathcal{X}, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_{\mathcal{X}})) \rightarrow \Gamma(X, \mathcal{H}om_{\mathcal{O}_{\mathcal{X}}}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_{\mathcal{X}}))$. Via this map Λ is sent to Λ_0 . So (X, Λ_0) is a closed Poisson subscheme of (\mathcal{X}, Λ) since i is a Poisson morphism.*

Given another Poisson deformation

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{Y}, \Lambda') \\ \xi : \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & S \end{array}$$

of (X, Λ_0) over S , an isomorphism of η with ξ is an Poisson S -isomorphism $\phi : (\mathcal{X}, \Lambda) \rightarrow (\mathcal{Y}, \Lambda')$ inducing the identity on (X, Λ_0) , i.e. such that the following diagram is commutative.

$$\begin{array}{ccc} & (X, \Lambda_0) & \\ & \swarrow & \searrow \\ (\mathcal{X}, \Lambda) & \xrightarrow{\phi} & (\mathcal{Y}, \Lambda') \\ & \searrow & \swarrow \\ & S & \end{array}$$

By a pointed scheme, we will mean a pair (S, s) where S is a scheme and $s \in S$. If K is a field, we call (S, s) a K -pointed scheme if $K \cong k(s)$.

Definition 7.32 (trivial Poisson deformation) *Let (X, Λ_0) be a algebraic Poisson scheme, and (S, s) be a k -pointed scheme (S, s) . We define a trivial Poisson family induced by (X, Λ_0) and (S, s) to be the following Poisson deformation of (X, Λ_0) ,*

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (X \times_{\text{Spec}(k)} S, \Lambda_0 \oplus 0) \\ \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \xrightarrow{s} & S \end{array}$$

Here $(\Lambda_0 \oplus 0)$ is the Poisson structure on $X \times_{\text{Spec}(k)} S$ over S induced from the Poisson structure Λ_0 on X via the following diagram.

$$\begin{array}{ccc} X \times_{\text{Spec}(k)} S & \longrightarrow & X \\ \downarrow & & \downarrow \\ S & \longrightarrow & \text{Spec}(k) \end{array}$$

Poisson deformation of (X, Λ_0) over S is called trivial if it is isomorphic to the trivial Poisson family as above.

Definition 7.33 (rigid Poisson deformations) *An algebraic Poisson scheme (X, Λ_0) is called rigid if every infinitesimal Poisson deformations of X over A is trivial for every $\text{Spec}(A)$ in **Art**.*

Given a Poisson deformation η of (X, Λ_0) over S as above and a morphism $(S', s') \rightarrow (S, s)$ of k -pointed schemes there is induced a commutative diagram by base change

$$\eta' : \begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X} \times_S S', \Lambda \oplus 0) \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & S' \end{array}$$

which is a Poisson deformation of (X, Λ_0) over S' . This operation is functorial, in the sense that it commutes with composition of morphisms and the identity morphism does not change η . Moreover, it carries isomorphic Poisson deformations to isomorphic ones.

Definition 7.34 (locally trivial Poisson deformations) *An infinitesimal Poisson deformation η of (X, Λ_0) is called locally trivial if for every point $x \in X$ has an open neighborhood $U_x \subset X$ such that*

$$\begin{array}{ccc} (U_x, \Lambda_0|_{U_x}) & \longrightarrow & (\mathcal{X}|_{U_x}, \Lambda|_{U_x}) \\ \downarrow & & \downarrow \pi \\ \text{Spec}(k) & \xrightarrow{s} & S \end{array}$$

is a trivial Poisson deformation of U_x . In other words, $(\mathcal{X}|_{U_x}, \Lambda|_{U_x}) \cong (U_x \times_{\text{Spec}(k)} S, \Lambda_0 \oplus 0)$ as Poisson schemes.

7.2.2 Infinitesimal Poisson deformations

Definition 7.35 (small extension) *We say that for $(\tilde{A}, \tilde{\mathfrak{m}}), (A, \mathfrak{m}) \in \mathbf{Art}$, an exact sequence of the form $0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$ is a small extension if $t \in \tilde{\mathfrak{m}}$ is annihilated by $\tilde{\mathfrak{m}}$. (i.e $t \cdot \tilde{\mathfrak{m}} = 0$) so that (t) is an one dimensional k -vector space.*

Lemma 7.36 (compare [Ser06] Lemma 1.2.6 page 26) *Let B_0 be a Poisson k -algebra with the Poisson structure $\Lambda \in \text{Hom}_{B_0}(\wedge^2 \Omega_{B_0/k}, B_0)$, and*

$$e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$$

a small extension in \mathbf{Art} . Let $\Lambda_0 \in \text{Hom}_{B_0}(\wedge^2 \Omega_{B_0/k}, B_0) \otimes_k A$ be a Poisson structure on $B_0 \otimes_k A$ over A inducing Λ . Let $\Lambda_1, \Lambda_2 \in \text{Hom}_{B_0}(\wedge^2 \Omega_{B_0/k}, B_0) \otimes_k \tilde{A}$ be Poisson structures on $B_0 \otimes_k \tilde{A}$

over \tilde{A} which induces Λ_0 . This implies that there exists a $\Lambda' \in \text{Hom}_{B_0}(\wedge^2 \Omega_{B_0/k}, B_0)$ such that $\Lambda_1 - \Lambda_2 = t\Lambda'$. Then there is one to one correspondence

$$\begin{aligned} & \{ \text{Poisson isomorphisms between } (B_0 \otimes_k \tilde{A}, \Lambda_1) \text{ and } (B_0 \otimes_k \tilde{A}, \Lambda_2) \\ & \quad \text{inducing the identity on } (B_0 \otimes_k A, \Lambda_0) \} \\ & \rightarrow \{ P \in \text{Der}_k(B_0, B_0) = \text{Hom}_{B_0}(\Omega_{B_0/k}, B_0) \mid \Lambda' - [\Lambda, P] = \Lambda' + [P, \Lambda] = 0 \} \end{aligned}$$

In particular, when $\Lambda_1 = \Lambda_2$, there is a canonical isomorphism of groups

$$\begin{aligned} & \{ \text{Poisson automorphisms between } (B_0 \otimes_k \tilde{A}, \Lambda_1) \text{ and } (B_0 \otimes_k \tilde{A}, \Lambda_1) \\ & \quad \text{inducing the identity on } (B_0 \otimes_k A, \Lambda_0) \} \rightarrow \text{PDer}_k(B_0, B_0) = \text{HP}^1(B_0, \Lambda) \end{aligned}$$

Proof. Let $\theta : (B_0 \otimes_k \tilde{A}, \Lambda_1) \rightarrow (B_0 \otimes_k \tilde{A}, \Lambda_2)$ be a Poisson isomorphism. Then θ is \tilde{A} -linear and induces the identity modulo by t . We have $\theta(x) = x + tPx$, where $P \in \text{Der}_{\tilde{A}}(B_0 \otimes_k \tilde{A}, B_0) = \text{Der}_k(B_0, B_0) = \text{Hom}_{B_0}(\Omega_{B_0/k}^1, B_0)$. When we think of P as an element of $\text{Hom}_{B_0}(\Omega_{B_0/k}^1, B_0)$, we have $\theta(x) = x + tP(dx)$. We define the correspondence by $\theta \mapsto P$. Now we check that $\Lambda' - [\Lambda, P] = 0$. Since θ is a Poisson map, for $x, y \in B_0$, we have by Example (7.4),

$$\begin{aligned} \theta(\Lambda_1(dx \wedge dy)) &= \Lambda_2(d(\theta x) \wedge d(\theta y)) \\ \Lambda_1(dx \wedge dy) + tP(d(\Lambda_1(dx \wedge dy))) &= \Lambda_2((dx + td(P(dx))) \wedge (dy + td(P(dy)))) \\ \Lambda_1(dx \wedge dy) + tP(d(\Lambda(dx \wedge dy))) &= \Lambda_2(dx \wedge dy) + t\Lambda(dx \wedge d(P(dy))) + t\Lambda(d(P(dx)) \wedge dy) \\ t[\Lambda'(dx \wedge dy) + P(d(\Lambda(dx \wedge dy)))] &- \Lambda(dx \wedge d(P(dy))) - \Lambda(d(P(dx)) \wedge dy) = 0 \\ \Lambda' - [\Lambda, P] &= 0 \end{aligned}$$

Since θ is determined by P , the correspondence is one to one.

Now we assume that $\Lambda_1 = \Lambda_2$. So θ corresponds to P with $[\Lambda, P] = 0$. First we note that $P \in \text{Hom}_{B_0}(\Omega_{B_0/k}^1, B_0)$ with $[\Lambda, P] = 0$ is a Poisson derivation. In other words, $P(\{x, y\}) = \{Px, y\} + \{x, Py\}$. Indeed, $0 = [\Lambda, P](dx \wedge dy) = \Lambda(d(Px) \wedge dy) - \Lambda(d(Py) \wedge dx) - P(d(\Lambda(dx \wedge dy)))$.

Now we show that the correspondence is a group isomorphism. Indeed, let $\theta(x) = x + tPx$ and $\sigma(x) = x + tQx$ with $[\Lambda, P] = [\Lambda, Q] = 0$. Then $\sigma(\theta(x)) = \theta(x) + tQ(\theta(x)) = x + tPx + tQ(x + tPx) = x + tPx + tQx = x + t(P + Q)x$. Hence $\theta + \sigma$ corresponds to $P + Q$. Since $[\Lambda, P + Q] = 0$ and identity map corresponds to 0, the correspondence is group isomorphism. ■

Proposition 7.37 (compare [Ser06] Proposition 1.2.9 page 29 and see also [Nam08] Proposition 8)

Let (X, Λ_0) be an Poisson algebraic variety with $\Lambda_0 \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}, \mathcal{O}_X))$. There is a 1-1

correspondence:

$$\begin{aligned} \kappa : \{ & \text{Poisson isomorphism classes of first order Poisson deformations of } (X, \Lambda_0) \\ & \text{whose underlying flat deformation of } X \text{ is locally trivial} \} \rightarrow HP^2(X, \Lambda_0) \end{aligned}$$

such that $\kappa(\xi) = 0$ if and only if ξ is the trivial Poisson deformation class. In particular, if X is nonsingular, then we have 1 – 1 correspondence

$$\kappa : \{ \text{Poisson isomorphism classes of first order Poisson deformations of } (X, \Lambda_0) \} \rightarrow HP^2(X, \Lambda_0)$$

Proof. Given a first-order Poisson deformation of (X, Λ_0) whose underlying flat deformation is locally trivial,

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(k[\epsilon]) \end{array}$$

we choose an affine open cover $\mathcal{U} = \{U_i = \text{Spec}(B_i)\}_{i \in I}$ of X such that $\mathcal{X}|_{U_i} \cong U_i \times \text{Spec}(k[\epsilon]) = \text{Spec}(B_i) \times \text{Spec}(k[\epsilon])$ is trivial for all i with the induced Poisson structure $\Lambda_0 + \epsilon \Lambda_i = \in \text{Hom}_{B_i}(\wedge^2 \Omega_{B_i/k}^1, B_i) \otimes_k k[\epsilon]$ on $U_i \times \text{Spec}(k[\epsilon])$ from Λ . For each i , we have a Poisson isomorphism

$$\theta_i : (U_i \times \text{Spec}(k[\epsilon]), \Lambda_i) \rightarrow (\mathcal{X}|_{U_i}, \Lambda|_{U_i})$$

Then for each $i, j \in I$, $\theta_{ij} := \theta_i^{-1} \theta_j : (U_{ij} \times \text{Spec}(k[\epsilon]), \Lambda_j) \rightarrow (U_{ij} \times \text{Spec}(k[\epsilon]), \Lambda_i)$ is an Poisson isomorphism inducing the identity on (U_{ij}, Λ_0) by modulo ϵ . Hence by Lemma 7.36, θ_{ij} corresponds to a $p_{ij} \in \Gamma(U_{ij}, T_X)$ where $T_X = \mathcal{H}om(\Omega_X^1, \mathcal{O}_X) = \text{Der}_k(\mathcal{O}_X, \mathcal{O}_X)$ such that $\Lambda_i - \Lambda_j - [\Lambda_0, p_{ij}] = 0$. We claim that $(\{p_{ij}\}, \{\Lambda'_i\}) \in C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \oplus C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}, \mathcal{O}_X))$ represents a cohomology class in the following diagram (see Appendix B):

$$\begin{array}{ccccc} & & & & [\Lambda_0, -] \uparrow \\ & & & & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^3 \Omega_{X/k}, \mathcal{O}_X)) \xrightarrow{-\delta} \dots \\ & & & & [\Lambda_0, -] \uparrow \\ & & & & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}, \mathcal{O}_X)) \xrightarrow{\delta} C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}, \mathcal{O}_X)) \xrightarrow{-\delta} \dots \\ & & & & [\Lambda_0, -] \uparrow \\ & & & & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \xrightarrow{-\delta} C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \xrightarrow{\delta} C^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \\ & & & & \uparrow \qquad \qquad \qquad \uparrow \qquad \qquad \qquad \uparrow \\ & & & & 0 \qquad \qquad \qquad 0 \qquad \qquad \qquad 0 \end{array}$$

Since $[\Lambda_0 + \epsilon\Lambda_i, \Lambda_0 + \epsilon\Lambda_i] = 0$, we have $[\Lambda_0, \Lambda_i] = 0$. Since on each U_{ijk} we have $\theta_{ij}\theta_{jk}\theta_{ik}^{-1} = 1_{U_{ijk} \times \text{Spec}(k[\epsilon])}$, we have $p_{ij} + p_{jk} - p_{ik} = 0$, and so $\delta(\{p_{ij}\}) = 0$. Since $\Lambda_i - \Lambda_j - [\Lambda_0, p_{ij}] = 0$, we have $\delta(\{\Lambda_i\}) + [\Lambda_0, p_{ij}] = 0$. Hence $(\{p_{ij}\}, \{\Lambda_i\})$ defines a cohomology class.

Now we show that for two equivalent Poisson deformations of (X, Λ_0) , the cohomology class is same. If we have another Poisson deformation

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}', \Lambda') \\ \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(k[\epsilon]) \end{array}$$

and $\Phi : (\mathcal{X}, \Lambda) \rightarrow (\mathcal{X}', \Lambda')$ is an Poisson isomorphism of deformations, then for each $i \in I$ there is an induced Poisson isomorphism:

$$\alpha_i : (U_i \times \text{Spec}(k[\epsilon]), \Lambda_0 + \epsilon\Lambda_i) \xrightarrow{\theta_i} (\mathcal{X}|_{U_i}, \Lambda|_{U_i}) \xrightarrow{\Phi|_{U_i}} (\mathcal{X}'|_{U_i}, \Lambda'|_{U_i}) \xrightarrow{\theta_i'^{-1}} (U_i \times \text{Spec}(k[\epsilon]), \Lambda_0 + \epsilon\Lambda_i')$$

So α_i corresponds to $a_i \in \Gamma(U_i, T_X)$ such that $\Lambda_i' - \Lambda_i - [\Lambda_0, a_i] = 0$ by Lemma 7.36. Since $-\delta(\{a_i\}) = a_i - a_j = p'_{ij} - p_{ij}$ and $\Lambda_i' - \Lambda_i = [\Lambda_0, a_i]$, $(\{p_{ij}\}, \{\Lambda_i\})$ and $(\{p'_{ij}\}, \{\Lambda_i'\})$ are cohomologous.

Now we define an inverse map. Given an element in $HP^2(X, \Lambda_0)$, we represent it by a hyper Cech 1-cocycle $(\{p_{ij}\}, \{\Lambda_i\})$ for an affine open cover $\mathcal{U} = \{U_i\}$ of X . So we have $[\Lambda_0, \Lambda_i] = 0$, $p_{ij} + p_{jk} - p_{ik} = 0$ and $\Lambda_j - \Lambda_i = [\Lambda_0, p_{ij}] = 0$. By reversing the above process, the cohomology class gives a glueing condition to make a Poisson deformation of (X, Λ_0) whose underlying deformation is a locally trivial flat deformation. ■

Definition 7.38 (Poisson Kodaira-Spencer map in an algebraic Poisson family) *For every first-order Poisson deformation ξ of a Poisson algebraic variety (X, Λ_0) whose underlying flat deformation is locally trivial, the cohomology class $\kappa(\xi) \in HP^2(X, \Lambda_0)$ is called the Poisson Kodaira-Spencer class of ξ . Now we assume that (X, Λ_0) is a nonsingular Poisson variety. Let's consider a Poisson deformation of (X, Λ_0)*

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \xi : \downarrow & & \downarrow f \\ \text{Spec}(k) & \xrightarrow{s} & S \end{array}$$

where the base space S is a connected algebraic k -scheme and \mathcal{X} is a Poisson scheme over S defined by $\Lambda \in \Gamma(\mathcal{X}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{\mathcal{X}/S}^1, \mathcal{O}_X))$ We define k -linear map, called Poisson Kodaira-Spencer map of the family ξ at $s \in S$,

$$\kappa_{\xi, s} : T_{S, s} \rightarrow HP^2(X, \Lambda_0)$$

in the following way: let U be an affine open neighborhood of $s \in S$ and $d \in T_{S,s} = \text{Der}_k(\mathcal{O}_{S,s}, \mathcal{O}_{S,s})$. Let $\bar{d} : \mathcal{O}_{S,s} \rightarrow \mathcal{O}_{S,s}/\mathfrak{m}_s$ induced by d and the canonical surjection $\mathcal{O}_{S,s} \rightarrow \mathcal{O}_{S,s}/\mathfrak{m}_s, a \mapsto \bar{a}$. Let's consider the following homomorphisms

$$\mathcal{O}_S(U) \rightarrow (\mathcal{O}_{S,s}/\mathfrak{m}_s) \oplus \epsilon(\mathcal{O}_{S,s}/\mathfrak{m}_s) \cong k[\epsilon] \rightarrow \mathcal{O}_{S,s}/\mathfrak{m}_s \cong k, (a \mapsto \bar{a} + \epsilon \bar{d}(a) \mapsto \bar{a})$$

This defines a morphism $\text{Spec}(k) \rightarrow \text{Spec}(k[\epsilon]) \rightarrow U \hookrightarrow S$. We pullback (\mathcal{X}, Λ) over S to a first order Poisson deformation over $\text{Spec}(k[\epsilon])$ via the map $\text{Spec}(k[\epsilon]) \rightarrow S$. Then by Proposition 7.37, we can find a cohomology class in $HP^2(X, \Lambda_0)$.

7.2.3 Higher-order Poisson deformation-obstructions

Let (X, Λ_0) be a nonsingular Poisson algebraic variety. Consider a small extension

$$e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$$

in **Art.** let

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\mathcal{X}, \Lambda) \\ \xi : \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(A) \end{array}$$

be an infinitesimal Poisson deformation of (X, Λ_0) over A . A lifting of ξ to \tilde{A} is a infinitesimal Poisson deformation $\tilde{\xi}$ over \tilde{A} inducing ξ . In other words,

$$\begin{array}{ccc} (X, \Lambda_0) & \longrightarrow & (\tilde{\mathcal{X}}, \tilde{\Lambda}) \\ \tilde{\xi} : \downarrow & & \downarrow \\ \text{Spec}(k) & \longrightarrow & \text{Spec}(\tilde{A}) \end{array}$$

and an Poisson isomorphism ϕ of Poisson deformations such that the following diagram commutes

$$\begin{array}{ccccc} & & (X, \Lambda_0) & & \\ & \swarrow & & \searrow & \\ (\mathcal{X}, \Lambda) & \xrightarrow{\phi} & (\tilde{\mathcal{X}}, \tilde{\Lambda}) \times_{\text{Spec}(\tilde{A})} \text{Spec}(A) & & \\ & \searrow & & \swarrow & \\ & & \text{Spec}(A) & & \end{array}$$

Proposition 7.39 (compare [Ser06] Proposition 1.2.12) *Let (X, Λ_0) be a nonsingular Poisson variety. Let $A \in \mathbf{Art}$ and an infinitesimal Poisson deformation ξ of (X, Λ_0) over A . To every small extension $e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, there is associated an element $o_\xi(e) \in H\mathcal{P}^3(X, \Lambda_0)$ called the obstruction lifting ξ to \tilde{A} , which is 0 if and only if a lifting of ξ to \tilde{A} exists.*

Proof. Let $\mathcal{U} = \{U_i = \text{Spec}(B_i)\}_{i \in I}$ be an affine open cover of X . We have Poisson isomorphisms $\theta_i : (U_i \times \text{Spec}(A), \Lambda_i) \rightarrow (\mathcal{X}|_{U_i}, \Lambda|_{U_i})$ and $\theta_{ij} := \theta_i^{-1}\theta_j$ is a Poisson isomorphism. We have $\theta_{ij}\theta_{jk} = \theta_{ik}$ on $U_{ijk} \times \text{Spec}(A)$. To give a lifting $\tilde{\xi}$ of ξ to \tilde{A} is equivalent to give a collection of $\{\tilde{\Lambda}_i\}$ where $\tilde{\Lambda}_i \in \text{Hom}_{B_i}(\Omega_{B_i/k}^1, B_i) \otimes_k \tilde{A}$ with $[\tilde{\Lambda}_i, \tilde{\Lambda}_i] = 0$ is a Poisson structure on $U_i \times \text{Spec}(\tilde{A})$ and a collection of Poisson isomorphisms $\{\tilde{\theta}_{ij}\}$ where $\tilde{\theta}_{ij} : (U_{ij} \times \text{Spec}(\tilde{A}), \tilde{\Lambda}_j) \rightarrow (U_{ij} \times \text{Spec}(\tilde{A}), \tilde{\Lambda}_i)$ such that

1. $\tilde{\theta}_{ij}\tilde{\theta}_{jk} = \tilde{\theta}_{ik}$ as a Poisson isomorphism.
2. $\tilde{\theta}_{ij}$ restricts to θ_{ij} on $U_{ij} \times \text{Spec}(A)$.
3. $\tilde{\Lambda}_i$ restrits to Λ_i .

From such data, we can glue together $(U_i \times \text{Spec}(\tilde{A}), \tilde{\Lambda}_i)$ to make a Poisson deformation $(\tilde{\mathcal{X}}, \tilde{\Lambda})$. Now given a Poisson deformation $\xi = (\mathcal{X}, \Lambda)$ over A and a small extension $e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, we associate an element $o_\xi(e) \in H\mathcal{P}^3(X, \Lambda_0)$. Choose arbitrary automorphisms $\{\tilde{\theta}_{ij}\}$ satisfying (2) (for the existence of lifting, see [Ser06] Lemma 1.2.8) and arbitrary $\tilde{\Lambda}_i \in \text{Hom}_{B_i}(\Omega_{B_i/k}^1, B_i) \otimes_k \tilde{A}$ satisfying (3) (not necessarily $[\tilde{\Lambda}_i, \tilde{\Lambda}_i] = 0$). The lifting exists since $\text{Hom}_{B_i}(\Omega_{B_i/k}^1, B_i) \otimes_k \tilde{A} \rightarrow \text{Hom}_{B_i}(\Omega_{B_i/k}^1, B_i) \otimes_k A$ is surjective. Let $\tilde{\theta}_{ijk} = \tilde{\theta}_{ij}\tilde{\theta}_{jk}\tilde{\theta}_{ik}^{-1}$. Since $\tilde{\theta}_{ijk}$ is an automorphism on $U_{ijk} \times \text{Spec}(\tilde{A})$ inducing the identity on $U_{ijk} \times \text{Spec}(A)$, $\tilde{\theta}_{ijk}$ corresponds to $\tilde{d}_{ijk} \in \Gamma(U_{ijk}, T_X)$ and $d_{jkl} - d_{ikl} + d_{ijl} - d_{jkl} = 0$. So we have $-\delta(\{d_{ijk}\}) = 0$. Since $[\tilde{\Lambda}_i, \tilde{\Lambda}_i]$ is zero modulo (t) by $[\Lambda_i, \Lambda_i] = 0$, there exists $\Pi_i \in \text{Hom}_{B_i}(\wedge^3 \Omega_{B_i/k}^1, B_i)$ such that $[\tilde{\Lambda}_i, \tilde{\Lambda}_i] = t\Pi_i$. Since $0 = [\tilde{\Lambda}_i, [\tilde{\Lambda}_i, \tilde{\Lambda}_i]] = [\tilde{\Lambda}_i, t\Pi_i] = t[\Lambda_0, \Pi_i] = 0$, we have $[\Lambda_0, \Pi_i] = 0$.

Let $\tilde{f}_{ij} : \mathcal{O}_X(U_{ij}) \otimes_k \tilde{A} \rightarrow \mathcal{O}_X(U_{ji}) \otimes_k \tilde{A}$ be the ring homomorphism corresponding to $\tilde{\theta}_{ij}$. We will denote by $\tilde{f}_{ij}\Lambda_i$ be the induced biderivation structure on $\mathcal{O}_X(U_{ji}) \otimes_k \tilde{A}$ such that $\tilde{f}_{ij} : (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_{ji}) \otimes_k \tilde{A}, \tilde{f}_{ij}\Lambda_i)$ is biderivation-preserving. Since $\tilde{f}_{ij}\tilde{\Lambda}_i$ and $\tilde{\Lambda}_j$ are same modulo (t) by (3), there exists $\Lambda'_{ij} \in \Gamma(U_{ij}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ such that $t\Lambda'_{ij} = \tilde{f}_{ij}\tilde{\Lambda}_i - \tilde{\Lambda}_j$. Then $t\Lambda'_{ji} = \tilde{f}_{ji}\Lambda_j - \Lambda_i$. By applying \tilde{f}_{ij} on both sides, we have $t\Lambda'_{ji} = \tilde{\Lambda}_j - \tilde{f}_{ij}\tilde{\Lambda}_i = -t\Lambda'_{ij}$. Hence $\Lambda'_{ji} = -\Lambda'_{ij}$. Then $t\Pi_i - t\Pi_j = \tilde{f}_{ij}(t\Pi_i) - t\Pi_j = \tilde{f}_{ij}[\tilde{\Lambda}_i, \tilde{\Lambda}_i] - [\tilde{\Lambda}_j, \tilde{\Lambda}_j] = [\tilde{f}_{ij}\Lambda_i, \tilde{f}_{ij}\Lambda_i] - [\tilde{\Lambda}_j, \tilde{\Lambda}_j] = [\tilde{\Lambda}_j + t\Lambda'_{ij}, \tilde{\Lambda}_j + t\Lambda'_{ij}] - [\tilde{\Lambda}_j, \tilde{\Lambda}_j] = t[\Lambda_0, 2\Lambda'_{ij}]$. Hence we have $-\Pi_i - (-\Pi_j) + [\Lambda_0, 2\Lambda'_{ij}] = 0$. So we have $-\delta(\{-\Pi_i\}) + [\Lambda_0, \{2\Lambda'_{ij}\}] = 0$. In the following isomorphism

$$\tilde{\alpha}_{ijk} : U_{ijk} \times \text{Spec}(\tilde{A}) \xrightarrow{\tilde{\theta}_{ki}} U_{ijk} \times \text{Spec}(\tilde{A}) \xrightarrow{\tilde{\theta}_{jk}} U_{ijk} \times \text{Spec}(\tilde{A}) \xrightarrow{\tilde{\theta}_{ij}} U_{ijk} \times \text{Spec}(\tilde{A})$$

which corresponds to a $\tilde{d}_{ijk} \in \Gamma(U_{ijk}, T_X)$. Then we have

$$Id + t\tilde{d}_{ijk} : \mathcal{O}_X(U_{ijk}) \otimes_k \tilde{A} \xrightarrow{\tilde{f}_{ij}} \mathcal{O}_X(U_{ijk}) \otimes_k \tilde{A} \xrightarrow{\tilde{f}_{jk}} \mathcal{O}_X(U_{ijk}) \otimes_k \tilde{A} \xrightarrow{\tilde{f}_{ki}} \mathcal{O}_X(U_{ijk}) \otimes_k \tilde{A}$$

$Id + t\tilde{d}_{ijk} : (\mathcal{O}_X(U_{ijk}) \otimes \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_{ijk}) \otimes_k \tilde{A}, \tilde{f}_{ki}\tilde{f}_{jk}\tilde{f}_{ij}\tilde{\Lambda}_i)$ is an isomorphism compatible with bidervations. We note that $\tilde{\Lambda}_i - \tilde{f}_{ki}\tilde{f}_{jk}\tilde{f}_{ij}\tilde{\Lambda}_i = \tilde{\Lambda}_i - \tilde{f}_{ki}\tilde{f}_{jk}(\tilde{\Lambda}_j + t\Lambda'_{ij}) = \tilde{\Lambda}_i - \tilde{f}_{ki}(\tilde{\Lambda}_k + t\Lambda'_{jk} + \Lambda'_{ij}) = \tilde{\Lambda}_i - (\tilde{\Lambda}_i + t\Lambda'_{ki} + t\Lambda'_{jk} + t\Lambda'_{ij}) = -t(\Lambda'_{ki} + \Lambda'_{jk} + \Lambda'_{ij})$. Hence by Lemma 7.36, we have $\Lambda'_{ki} + \Lambda'_{jk} + \Lambda'_{ij} - [\Lambda_0, \tilde{d}_{ijk}] = 0$. So we have $-\delta(\{\Lambda'_{ij}\}) + [\Lambda_0, \{\tilde{d}_{ijk}\}] = 0$. Hence $\alpha = (\{-\Pi_i\}, \{2\Lambda'_{ij}\}, \{2\tilde{d}_{ijk}\}) \in C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^3\Omega_{X/k}, \mathcal{O}_X)) \oplus C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/k}, \mathcal{O}_X)) \oplus C^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X))$ is a cocycle in the following diagram (see Appendix B).

$$\begin{array}{ccccc} & & [\Lambda_0, -] \uparrow & & \\ & & \uparrow & & \\ C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^3\Omega_{X/k}, \mathcal{O}_X)) & \xrightarrow{-\delta} & \dots & & \\ & & [\Lambda_0, -] \uparrow & & \\ C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/k}, \mathcal{O}_X)) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/k}, \mathcal{O}_X)) & \xrightarrow{-\delta} & \dots \\ & & [\Lambda_0, -] \uparrow & & \\ C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{-\delta} & C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \\ & & \uparrow & & \uparrow \\ 0 & \longrightarrow & 0 & \longrightarrow & 0 \end{array}$$

We claim that given a different choice $\{\tilde{\theta}'_{ij}\}$ and $\{\tilde{\Lambda}'_i\}$ satisfying (1), (2), (3), the associated cocycle $\beta = (\{-\Pi'_i\}, \{2\Lambda''_{ij}\}, \{2\tilde{d}'_{ijk}\}) \in C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^3\Omega_{X/k}, \mathcal{O}_X)) \oplus C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2\Omega_{X/k}, \mathcal{O}_X)) \oplus C^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X))$ is cohomologous to the cocycle associated with $\{\tilde{\theta}_{ij}\}$ and $\{\tilde{\Lambda}_i\}$. Let $\tilde{f}_{ij} : \mathcal{O}_X(U_{ij}) \otimes \tilde{A} \rightarrow \mathcal{O}_X(U_{ij}) \otimes \tilde{A}$ corresponding to $\tilde{\theta}_{ij}$, and $\tilde{f}'_{ij} : \mathcal{O}_X(U_{ij}) \otimes \tilde{A} \rightarrow \mathcal{O}_X(U_{ij}) \otimes \tilde{A}$ corresponding to $\tilde{\theta}'_{ij}$. Then $\tilde{f}'_{ij} = \tilde{f}_{ij} + tp_{ij}$ for some $p_{ij} \in \Gamma(U_{ij}, T_X)$ ¹ and $\tilde{\Lambda}'_i = \tilde{\Lambda}_i + t\Lambda'_i$ for some $\Lambda'_i \in \mathcal{H}om_{B_i}(\wedge^2\Omega_{B_i/k}^1, B_i)$. For each i, j, k , $\tilde{\theta}'_{ij}\tilde{\theta}'_{jk}\tilde{\theta}'_{ik}{}^{-1}$ corresponds to the derivation $\tilde{d}'_{ijk} = \tilde{d}_{ijk} + (p_{ij} + p_{jk} - p_{ik})$. Hence $\delta(\{2p_{ij}\}) = \{2\tilde{d}'_{ijk} - (2\tilde{d}_{ijk})\}$. We also note that $t\Pi'_i = [\tilde{\Lambda}'_i, \tilde{\Lambda}'_i] = [\tilde{\Lambda}_i + t\Lambda'_i, \tilde{\Lambda}_i + t\Lambda'_i] = [\tilde{\Lambda}_i, \tilde{\Lambda}_i] + t[2\Lambda'_i, \Lambda_0] = t\Pi_i + t[2\Lambda'_i, \Lambda_0]$. Hence we have $[\Lambda_0, 2\Lambda'_i] = -\Pi_i - (-\Pi'_i)$. Since $t\Lambda'_{ij} = \tilde{f}_{ij}\tilde{\Lambda}_i - \tilde{\Lambda}_j$, $t\Lambda''_{ij} = \tilde{f}'_{ij}\tilde{\Lambda}'_i - \tilde{\Lambda}'_j = \tilde{f}_{ij}\tilde{\Lambda}'_i + t[p_{ij}, \tilde{\Lambda}'_i] - \tilde{\Lambda}'_j = \tilde{f}_{ij}\tilde{\Lambda}_i + t\Lambda'_i + t[p_{ij}, \Lambda_0] - \tilde{\Lambda}_j - t\Lambda'_j$, we have $\Lambda'_{ij} - \Lambda''_{ij} = -\Lambda'_i + [\Lambda_0, p_{ij}] + \Lambda'_j$. So $\delta(\{2\Lambda'_i\}) + [\Lambda_0, \{2p_{ij}\}] = \Lambda'_{ij} - \Lambda''_{ij}$. Hence $(\{2\Lambda'_i\}, \{-2p_{ij}\})$ is mapped to $\alpha - \beta$. Hence α and β are cohomologous. So given a deformation ξ and a small extension $e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, we can associate an element $o_\xi(e) :=$ the cohomology class of $\alpha \in H^3(X, \Lambda_0)$. We also note that $o_\xi(e) = 0$ if and only if there exists a collection of $\{\tilde{\theta}_{ij}\}$ and $\{\tilde{\Lambda}_i\}$ satisfying (2), (3) with $[\tilde{\Lambda}_i, \tilde{\Lambda}_i] = 0$ (which means $\tilde{\Lambda}_i$ defines a Poisson structure), $\Lambda'_{ij} = 0$ (which implies $\tilde{f}_{ij}\tilde{\Lambda}_i = \tilde{\Lambda}_j$) and $\tilde{d}_{ijk} = 0$ (which means (1)) if and only if there is a lifting $\tilde{\xi}$.

¹Since $\tilde{f}'_{ij} - \tilde{f}_{ij}$ is zero modulo t , we have $(\tilde{f}'_{ij} - \tilde{f}_{ij})(x) = 0 + tp_{ij}(x)$ for some map p_{ij} . We show that p_{ij} is a derivation. Indeed $tp_{ij}(xy) = (\tilde{f}'_{ij} - \tilde{f}_{ij})(xy) = \tilde{f}'_{ij}(x)(\tilde{f}'_{ij} - \tilde{f}_{ij})(y) + (\tilde{f}'_{ij} - \tilde{f}_{ij})(x)f_{ij}(y) = \tilde{f}'_{ij}(x)tp_{ij}(y) + tp_{ij}(y)f_{ij}(y) = t(xp_{ij}(y) + yp_{ij}(x))$. So p_{ij} is a derivation and so an element in $\Gamma(U_{ij}, T_X)$.

■

Definition 7.40 *The Poisson deformation ξ is called unobstructed if o_ξ is the zero map, otherwise ξ is called obstructed. (X, Λ_0) is unobstructed if every infinitesimal deformation of (X, Λ_0) is unobstructed, otherwise (X, Λ_0) is obstructed.*

Corollary 7.41 *A nonsingular Poisson variety (X, Λ_0) is unobstructed if $HP^3(X, \Lambda_0) = 0$.*

Proposition 7.42 *A nonsingular Poisson variety (X, Λ_0) is rigid if and only if $HP^2(X, \Lambda_0) = 0$.*

Proof. Assume that (X, Λ_0) is rigid. Since any infinitesimal Poisson deformation (in particular, any first order Poisson deformations) are trivial, $HP^2(X, \Lambda_0) = 0$ by Proposition 7.37. Assume that $HP^2(X, \Lambda_0) = 0$. First we claim that given an infinitesimal Poisson deformation η of (X, Λ_0) over $A \in \mathbf{Art}$ and a small extension $e : 0 \rightarrow (t) \rightarrow \tilde{A} \rightarrow A \rightarrow 0$, any two liftings $\xi, \tilde{\xi}$ to \tilde{A} are equivalent. Let $\{U_i\}$ be an affine open covering of $\xi = (\mathcal{X}, \Lambda)$ and $\tilde{\xi} = (\tilde{\mathcal{X}}, \tilde{\Lambda})$. Let $\{\theta_i\}$ where $\theta_i : U_i \times \text{Spec}(\tilde{A}) \rightarrow \mathcal{X}|_{U_i}$, $\{\Lambda_i\}$ where Λ_i is the Poisson structure on $U_i \times \text{Spec}(\tilde{A})$ induced from $\Lambda|_{U_i}$ and let $\theta_{ij} = \theta_i^{-1}\theta_j$ which corresponds to a $d_{ij} \in \Gamma(U_{ij}, T_X)$. Let $\{\tilde{\theta}_i\}$ where $\tilde{\theta}_i : U_i \times \text{Spec}(\tilde{A}) \rightarrow \tilde{\mathcal{X}}|_{U_i}$, $\{\tilde{\Lambda}_i\}$ where the induced Poisson structure from $\tilde{\Lambda}$ on $U_i \times \text{Spec}(\tilde{A})$ and let $\tilde{\theta}_{ij} = \tilde{\theta}_i^{-1}\tilde{\theta}_j$. Let $f_{ij} : (\mathcal{O}_X(U_{ij}) \otimes \tilde{A}, \Lambda_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes \tilde{A}, \Lambda_j)$ be the homomorphism corresponding to θ_{ij} and $\tilde{f}_{ij} : (\mathcal{O}_X(U_{ij}) \otimes \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes \tilde{A}, \tilde{\Lambda}_j)$ corresponding to $\tilde{\theta}_{ij}$. Since $\xi, \tilde{\xi}$ induce same Poisson deformation η over A , we have

$$\tilde{f}_{ij} = f_{ij} + tp_{ij}, \quad \tilde{\Lambda}_i = \Lambda_i + t\Lambda'_i$$

for some $p_{ij} \in \Gamma(U_{ij}, T_X)$. Since for all i, j, k we have $0 = \tilde{d}_{ijk} = p_{ij} + p_{jk} - p_{ik}$. Since $0 = [\tilde{\Lambda}_i, \tilde{\Lambda}_i] = [\Lambda_i + t\Lambda'_i, \Lambda_i + t\Lambda'_i] = 2t[\Lambda'_i, \Lambda_0]$. Since $f_{ij}\Lambda_i = \Lambda_j$ and $\tilde{f}_{ij}\tilde{\Lambda}_i = \tilde{\Lambda}_j$, we have $\Lambda_j + t\Lambda'_j = \tilde{\Lambda}_j = \tilde{f}_{ij}\tilde{\Lambda}_i = (f_{ij} + tp_{ij})(\Lambda_i + t\Lambda'_i) = \Lambda_j - t[\Lambda_0, p_{ij}] + t\Lambda'_i$. Hence we have $\Lambda'_j - \Lambda'_i + [\Lambda_0, p_{ij}] = 0$. Hence $(\{\Lambda'_i\}, \{p_{ij}\})$ defines a cocycle. Since $HP^2(X, \Lambda_0) = 0$, there exists $\{a_i\} \in \mathcal{C}^0(\mathcal{U}, T_X)$ such that $[\Lambda_0, a_i] = \Lambda'_i$ and $a_i - a_j = p_{ij}$. Now we explicitly construct a Poisson isomorphism $(\tilde{\mathcal{X}}, \tilde{\Lambda}) \cong (\mathcal{X}, \Lambda)$. We define a Poisson isomorphism locally on $U_i \times \text{Spec}(\tilde{A})$, and show that each map glue together to give an Poisson isomorphism $(\tilde{\mathcal{X}}, \tilde{\Lambda}) \cong (\mathcal{X}, \Lambda)$. We claim that $(U_i \times \text{Spec}(\tilde{A}), \Lambda_i) \rightarrow (U_i \times \text{Spec}(\tilde{A}), \tilde{\Lambda}_i)$ is a Poisson isomorphism induced from $Id + ta_i : (\mathcal{O}_X(U_i) \otimes_k \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_i) \otimes_k \tilde{A}, \Lambda_i)$. The inverse map is $Id - ta_i$. Since $\tilde{\Lambda}_i + t[a_i, \tilde{\Lambda}_i] = \tilde{\Lambda}_i + t[a_i, \Lambda_0] = \Lambda_i + t\Lambda'_i + t[a_i, \Lambda_0] = \Lambda_i$, $Id + ta_i$ is Poisson. We show that each Poisson isomorphism $\{Id + ta_i\}$ glues together to give a Poisson isomorphism $(\tilde{\mathcal{X}}, \tilde{\Lambda}) \cong (\mathcal{X}, \Lambda)$. Indeed, it is sufficient to show that the following diagram commutes.

$$\begin{array}{ccc} (\mathcal{O}_X(U_{ij}) \otimes_k \text{Spec}(\tilde{A}), \tilde{\Lambda}_j) & \xrightarrow{Id+ta_j} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \Lambda_j) \\ \tilde{f}_{ij} \uparrow & & \uparrow f_{ij} \\ (\mathcal{O}_X(U_{ij}) \otimes \text{Spec}(\tilde{A}), \tilde{\Lambda}_i) & \xrightarrow{Id+ta_i} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \Lambda_i) \end{array}$$

Indeed, the diagram commutes if and only if $(Id + ta_j) \circ \tilde{f}_{ij} = f_{ij} \circ (Id + ta_i)$ if and only if $\tilde{f}_{ij} + ta_j = f_{ij} + ta_i$ if and only if $p_{ij} = a_i - a_j$. Hence there is at most one lifting of ξ .

Now we prove that if $HP^2(X, \Lambda_0) = 0$, then (X, Λ_0) is rigid. We will prove by induction on the dimension on $(A, \mathfrak{m}) \in \mathbf{Art}$. For A with $\dim_k A = 2$, then any first order Poisson deformation is trivial. Let' assume that any infinitesimal Poisson deformation of (X, Λ_0) over A with $\dim_k A \leq n-1$ is trivial. Let ξ be an infinitesimal Poisson deformation of (X, Λ_0) over A with $\dim_k A = n$ such that $\mathfrak{m}^{p-1} \neq 0$ and $\mathfrak{m}^p = 0$. Choose an element $t \neq 0 \in \mathfrak{m}^{p-1}$. Then $0 \rightarrow (t) \rightarrow A \rightarrow A/(t) \rightarrow 0$ is a small extension and $\dim_k A/(t) \leq n-1$. Hence induced Poisson deformation $\bar{\xi}$ over $A/(t)$ from ξ is trivial by induction hypothesis. Since ξ is a lifting of $\bar{\xi}$, and trivial Poisson deformation over A is also a lifting of $\bar{\xi}$, ξ is trivial since we have at most one lifting of $\bar{\xi}$. ■

Chapter 8

Poisson deformation functors

8.1 Schlessinger's criterion

¹ We discuss Functor of Artin rings in more detail and Schlessinger's criterions. We recall that **Art** is the category of local artinian k -algebras with residue field k . Before the discussion, we note that following: let $f : (A', \mathfrak{m}') \rightarrow (A, \mathfrak{m})$ and $g : (A'', \mathfrak{m}'') \rightarrow (A, \mathfrak{m})$ be two local homomorphisms of local artinian k -algebras with the residue k . So we have $f^{-1}(\mathfrak{m}) = \mathfrak{m}'$ and $g^{-1}(\mathfrak{m}) = \mathfrak{m}''$. Let's consider the fiber product $\bar{A} = A' \times_A A'' = \{(a, b) | a \in A', b \in A'', f(a) = g(b)\}$ is also a local artinian k -algebra, which is defined by $(a_1, b_1) \cdot (a_2, b_2) = (a_1 a_2, b_1 b_2)$, $(a_1, b_1) + (a_2, b_2) = (a_1 + a_2, b_1 + b_2)$. We will show that $A' \times_A A''$ is a local Artinian k -algebra with the residue k . The maximal ideal is given by $\bar{\mathfrak{m}} = \{(m, n) | m \in \mathfrak{m}', n \in \mathfrak{m}'', f(m) = g(n)\}$. We will show that $\bar{\mathfrak{m}}$ is the unique maximal ideal. It is enough to show that $(a, b) \in \bar{A} - \bar{\mathfrak{m}}$ is an unit. Indeed, we have $a \in A - \mathfrak{m}'$ and $b \in A' - \mathfrak{m}''$. So a, b are units. Hence (a^{-1}, b^{-1}) is a inverse of (a, b) . Now we show that $\bar{A}/\bar{\mathfrak{m}} \cong k$. We define a map $\varphi : \bar{A} \rightarrow A'/\mathfrak{m}' \times_{A/\mathfrak{m}} A''/\mathfrak{m}''$ by $(a, b) \mapsto (\bar{a}, \bar{b}) \cong k \times_k k \cong k$. Let $\varphi(a, b) = (\bar{a}, \bar{b}) = 0$. Then $a \in \mathfrak{m}'$ and $b \in \mathfrak{m}''$. Hence $\ker \varphi = \bar{\mathfrak{m}}$. So the natural maps $f' : (\bar{A}, \bar{\mathfrak{m}}) \rightarrow (A', \mathfrak{m}')$ and $g' : (\bar{A}, \bar{\mathfrak{m}}) \rightarrow (A'', \mathfrak{m}'')$ are local homomorphisms of local Artinian k -algebras and $f'^{-1}(\mathfrak{m}') = \bar{\mathfrak{m}}$, $g'^{-1}(\mathfrak{m}'') = \bar{\mathfrak{m}}$.

Definition 8.1 *Let R be a complete local k -algebra, and for each $A \in \mathbf{Art}$, we define h_R be a*

¹For details, see [Har10] page 106-117.

functor of Artin rings in the following way

$$h_R : \mathbf{Art} \rightarrow \mathbf{Sets}$$

$$A \mapsto h(R) := \text{Hom}_k(R, A)$$

A covariant functor $F : \mathbf{Art} \rightarrow \mathbf{Set}$ that is isomorphic to a functor of the form h_R for some complete local k -algebra R is called *pro-representable*.

Let (R, \mathfrak{m}) be a complete local k -algebra. Let $\varphi : h_R \rightarrow F$ be a morphism of functors of Artin rings, then for each n , we have the following commutative diagram from the canonical map $R/\mathfrak{m}^{n+1} \rightarrow R/\mathfrak{m}^n$.

$$\begin{array}{ccc} \text{Hom}(R, R/\mathfrak{m}^{n+1}) & \xrightarrow{\varphi_{n+1}} & F(R/\mathfrak{m}^{n+1}) \\ \downarrow & & \downarrow \\ \text{Hom}(R, R/\mathfrak{m}^n) & \xrightarrow{\varphi_n} & F(R/\mathfrak{m}^n) \end{array}$$

Let $\pi_n : R \rightarrow R/\mathfrak{m}^n$ be the canonical surjection. Then we have $\xi = \{\xi_n\} := \{\varphi_n(\pi_n)\} \in \varprojlim F(R/\mathfrak{m}^n)$. We call $\xi = \{\xi_n\}$ a formal family of F over R .

Definition 8.2 Let \mathbf{C} be the category of complete local k -algebras with residue field k . Let F be a functor of Artin rings. We define

$$\hat{F} : \mathbf{C} \rightarrow \mathbf{Sets}$$

$$(R, \mathfrak{m}) \mapsto \hat{F}(R) = \varprojlim F(R/\mathfrak{m}^n)$$

Let $\xi = \{\xi_n\} \in \hat{F}(R)$ be a formal family. Then from this, we can define a morphism of functors $h_R \rightarrow F$ of functor of Artin rings as follows. For $A \in \mathbf{Art}$, we define

$$\begin{aligned} h_R(A) &= \text{Hom}_k(R, A) \rightarrow F(A) \\ f &\mapsto F(g)(\xi_n) \end{aligned}$$

Here g is defined in the following way. Since A is artinian, $f : R \rightarrow A$ factor through R/\mathfrak{m}^n by $g : R/\mathfrak{m}^n \rightarrow A$ for some n .

Remark 8.3 If F is a functor of Artin rings, and R is a complete local k -algebra with residue k , then there is a natural bijection between $\hat{F}(R)$ of formal families $\{\xi_n | \xi_n \in F(R/\mathfrak{m}^n)\}$ and the set of homomorphisms of functors $h_R \rightarrow F$. So if F is pro-representable, there is an isomorphism

$\xi : h_R \rightarrow F$ for some R , we can think of ξ as an element of $\hat{F}(R)$. We say that the pair (R, ξ) pro-represents the functor F .

Definition 8.4 Let F be a functor of Artin rings.

1. A pair (R, ξ) with $R \in \mathbf{C}$ and $\xi \in \hat{F}(R)$ is called a versal family for R if the associated map $h_R \rightarrow F$ is smooth. In other words, for every surjection $B \rightarrow A$ in \mathbf{Art} , the natural map $h_R(B) \rightarrow h_R(A) \times_{F(A)} F(B)$ is surjective. This implies that given a map $R \rightarrow A$ inducing an element $\eta \in F(A)$, given $\theta \in F(B)$ mapping to η , one can lift the map $R \rightarrow A$ to a map $R \rightarrow B$ inducing θ .
2. A versal family (R, ξ) with $R \in \mathbf{C}$ and $\xi \in \hat{F}(R)$ is called a miniversal family or F has a pro-representable hull (R, ξ) if $h_R(k[\epsilon]) \rightarrow F(k[\epsilon])$ is bijective.
3. A pair (R, ξ) with $R \in \mathbf{C}$ and $\xi \in \hat{F}(R)$ is called a universal family if it pro-represents the functor F .

Theorem 8.5 (Schlessinger's criterion) A functor of Artin rings has a miniversal family if and only if

- (H_0) $F(k)$ has one element.
- (H_1) The natural map $F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'')$ is surjective for every small extension $A'' \rightarrow A$.
- (H_2) The natural map $F(A' \times_A A'') \rightarrow F(A') \times_{F(A)} F(A'')$ is bijective when $A'' = k[\epsilon]$ and $A = k$.
- (H_3) $t_F := F(k[\epsilon])$ is a finite-dimensional k -vector space.

Proof. See [Har10] Theorem 16.2. ■

Remark 8.6 We explain in (H_3) why $t_F := F(k[\epsilon])$ is a k -vector space. Let F be a functor of Artin rings satisfying (H_0) and (H_2) . Then $F(k[\epsilon])$ can be considered to be k -vector space in the following

way. Let's consider the following map

$$\begin{aligned}\alpha : k[\epsilon] \times_k k[\epsilon] &\rightarrow k[\epsilon] \\ (a + b\epsilon, a + b'\epsilon) &\mapsto a + (b + b')\epsilon\end{aligned}$$

Then $\alpha((a+b\epsilon, a+b'\epsilon) \cdot (c+d\epsilon, c+d'\epsilon)) = \alpha(ac+(ad+bc)\epsilon, ac+(ad'+b'c)\epsilon) = ac+(ad+bc+ad'+b'c)\epsilon$.

On the other hand, $\alpha((a+b\epsilon, a+b'\epsilon)) \cdot \alpha((c+d\epsilon, c+d'\epsilon)) = (a+(b+b')\epsilon)(c+(d+d')\epsilon) = ac+(ad+ad'+bc+b'c)\epsilon$. Hence α is a homomorphism. So α induces $F(\alpha) : F(k[\epsilon] \times_k k[\epsilon]) \rightarrow F(k[\epsilon])$. Since F satisfies (H_3) , we have $F(k[\epsilon]) \times F(k[\epsilon]) \cong F(k[\epsilon] \times_k k[\epsilon])$. So We have a map $F(k[\epsilon]) \times F(k[\epsilon]) \rightarrow F(k[\epsilon])$. This defines an addition. By the following commutativity of homomorphisms and the property (H_3) , the operation satisfies associativity:

$$\begin{array}{ccc} k[\epsilon] \times_k k[\epsilon] \times_k k[\epsilon]((a + b\epsilon, a + b'\epsilon, a + b''\epsilon)) & \longrightarrow & k[\epsilon] \times_k k[\epsilon]((a + (b + b')\epsilon, a + b''\epsilon)) \\ \downarrow & & \downarrow \\ k[\epsilon] \times_k k[\epsilon]((a + b\epsilon, a + (b + b')\epsilon)) & \longrightarrow & k[\epsilon](a + (b + b' + b'')\epsilon) \end{array}$$

The zero element is the image of $F(k) \rightarrow F(k[\epsilon])$ induced from $k \rightarrow k[\epsilon], k \rightarrow k + \epsilon \cdot 0$. The scalar multiplication by $c \in k$ is defined by $F(k[\epsilon]) \rightarrow F(k[\epsilon])$ induced from $k[\epsilon] \rightarrow k[\epsilon], a + b\epsilon \mapsto a + (cb)\epsilon$. Then the inverse is defined by the map $F(k[\epsilon]) \rightarrow F(k[\epsilon])$ induced by $k[\epsilon] \rightarrow k[\epsilon], a + b\epsilon \mapsto a - b\epsilon$.

Let's assume that the functor F satisfies H_0, H_1 and H_2 . We claim that for any small extension $0 \rightarrow (t) \rightarrow A' \xrightarrow{p} A \rightarrow 0$ and any element $\eta \in F(A)$, there is a transitive group action of the vector space t_F on the set $p^{-1}(\eta)$ if it is nonempty. Here $p := F(p) : F(A') \rightarrow F(A)$. Indeed, we have an isomorphism

$$\begin{aligned}\gamma : k[\epsilon] \times_k A' &\rightarrow A' \times_A A' \\ (x + y\epsilon, a') &\mapsto (a' + yt, a') \\ \gamma^{-1} : A' \times_A A' &\rightarrow k[\epsilon] \times_k A' \\ (b', a') &\mapsto (\overline{a'} + \overline{(b' - a')}\epsilon, a')\end{aligned}$$

where $\overline{a'}$ and $\overline{b' - a'} \in k$ are the residues of a' and $b' - a'$ modulo by the maximal ideal of A' . Let $\beta : k[\epsilon] \times_k A' \rightarrow A, (x + y\epsilon, a') \mapsto a' + yt$. Then we have the following commutative diagram

$$\begin{array}{ccc} k[\epsilon] \times_k A' & \xrightarrow{\gamma} & A' \times_A A' \\ \beta \searrow & & \swarrow pr_1 \\ & A' & \end{array}$$

Since F satisfies H_2 , we have a bijection $\alpha^{-1} : F(k[\epsilon]) \times F(A') \rightarrow F(k[\epsilon] \times_k A')$, so a bijection $F(k[\epsilon]) \times F(A') \rightarrow F(A' \times_A A')$ induced from $F(\gamma) \circ \alpha^{-1}$. Since we have the following commutative diagram

$$\begin{array}{ccc}
 & t_F \times F(A') & \\
 & \downarrow \alpha^{-1} & \\
 & F(k[\epsilon] \times_k A') & \\
 & \downarrow F(\gamma) & \\
 & F(A' \times_A A') & \\
 \swarrow F(\beta) \circ \alpha^{-1} & & \searrow p^r \\
 F(A') & & F(A') \\
 \swarrow & \downarrow & \searrow \\
 & F(A') \times_{F(A)} F(A') &
 \end{array}$$

The map $t_F \times F(A') \rightarrow F(A') \times_{F(A)} F(A')$ is surjective and an isomorphism on the second factor since F satisfies H_1 . If we take $\eta \in F(A)$ and fix $\eta' \in p^{-1}(\eta)$, then we get a surjective map $t_F \times \{\eta'\} \rightarrow p^{-1}(\eta) \times \{\eta'\}$, and hence there is a transitive group action of t_F on $p^{-1}(\eta)$.

Theorem 8.7 (Schlessinger’s criterion) *Let F be a functor of Artin rings. The functor F is prorepresentable if and only if F satisfies $(H_0), (H_1), (H_2), (H_3)$ and*

- (H_4) *For every small extension $p : A'' \rightarrow A$ and every $\eta \in F(A)$ for which $p^{-1}(\eta)$ is nonempty, the group action of t_F on $p^{-1}(\eta)$ is bijective.*

Proof. See [Har10] Theorem 16.2. ■

8.2 Poisson Deformation functors

Definition 8.8 *Let (X, Λ_0) be a Poisson algebraic scheme. For $A \in \mathbf{Art}$, we let*

$PDef_{(X, \Lambda_0)} = \{\text{infinitesimal Poisson deformations of } (X, \Lambda_0) \text{ over } A\} / \text{Poisson isomorphisms}$

Then $PDef_{(X, \Lambda_0)}$ is a functor of Artin rings.

We will prove that $PDef_{(X, \Lambda_0)}$ has a miniversal family when (X, Λ_0) is a smooth projective Poisson scheme. Before the proof, we note the following: let B, B', B'' be Poisson algebras over local Artinian k -algebras A, A', A'' with the residue k as above. If A' -Poisson algebra homomorphisms $p : B' \rightarrow B$, and A'' -Poisson algebra homomorphism $q : B'' \rightarrow B$, then $\bar{B} = B' \times_B B'' = \{(m, n) | m \in B', n \in B'', p(m) = q(n)\}$ is a Poisson \bar{A} -algebra defined in the following way: $\bar{A} \times \bar{B} \rightarrow \bar{B}$, $(a, b) \times (m, n) \mapsto (am, bn)$. Bracket is defined by $\{(m, n), (r, s)\} = (\{m, r\}, \{n, s\})$. Then $\bar{B} \rightarrow B'$ is a Poisson homomorphism over \bar{A} and $\bar{B} \rightarrow B''$ is a Poisson homomorphism over \bar{A} . The Poisson algebra satisfies an universal mapping property in the following sense: let C be a Poisson algebra over \bar{A} and assume that we have a Poisson \bar{A} -homomorphism $f : C \rightarrow B'$ and $g : C \rightarrow B''$ such that $p \circ f = q \circ g$. Then there is a unique Poisson homomorphism $h : C \rightarrow B' \times_B B''$ defined by $h(c) = (f(c), g(c))$ which is \bar{A} -Poisson homomorphism since $h(\{c_1, c_2\}) = (f(\{c_1, c_2\}), g(\{c_1, c_2\})) = (\{f(c_1), f(c_2)\}, \{g(c_1), g(c_2)\}) = \{(f(c_1), g(c_1)), (f(c_2), g(c_2))\} = \{h(c_1), h(c_2)\}$, and for $(a, b) \in \bar{A}$, we have $h((a, b)c) = (f((a, b)c), g((a, b)c)) = (af(c), bg(c)) = (a, b)(f(c), g(c))$

Lemma 8.9 *Let A, A', A'' be local artinian k -algebras, and let $\bar{A} = A' \times_A A''$. Let B, B', B'' be algebras over A, A', A'' with compatible maps $B' \rightarrow B$ and $B'' \rightarrow B$, and assume that $B' \otimes_{A'} A \rightarrow B$ and $B'' \otimes_{A''} A \rightarrow B$ are isomorphisms. Let $\bar{B} = B' \times_B B''$.*

1. *Assume $A'' \rightarrow A$ is surjective. Then $\bar{B} \otimes_{\bar{A}} A' \rightarrow B'$ is an isomorphism and so $\bar{B} \otimes_{\bar{A}} A \rightarrow B$ is an isomorphism.*
2. *Now assume furthermore that $J = \ker(A'' \rightarrow A)$ is an ideal of square zeros and that B', B'' are flat over A', A'' respectively. Then \bar{B} is flat over \bar{A} and also $\bar{B} \otimes_{\bar{A}} A'' \rightarrow B''$ is an isomorphism.*

Proof. See [Har10] Proposition 16.4. ■

Theorem 8.10 *Let (X, Λ_0) be a smooth projective Poisson scheme over k . Then the Poisson deformation functor $PDef_{(X, \Lambda_0)}$ of Poisson deformations of (X, Λ_0) over local artinian rings has a miniversal family.*

Proof. We check Schlessinger's Criterion in Theorem 8.5. First $PDef_{(X, \Lambda_0)}(k)$ is one point set. So (H_0) is satisfied. Now we prove (H_2) . Let's consider the following commutative diagram of local artinian k -algebras with residue k . Assume that $A'' \rightarrow A$ is a small extension.

$$\begin{array}{ccc}
\bar{A} = A' \times_A A'' & \longrightarrow & A' \\
\downarrow & & \downarrow \\
A'' & \longrightarrow & A
\end{array}$$

Let (\mathcal{X}, Λ) be an infinitesimal Poisson deformation of (X, Λ_0) over A . Let (\mathcal{X}', Λ') be an infinitesimal Poisson deformation of (X, Λ_0) over A' and $(\mathcal{X}'', \Lambda'')$ be an infinitesimal Poisson deformation of (X, Λ_0) inducing (\mathcal{X}, Λ) via the above diagram. So we have the following fiber product of Poisson algebraic schemes.

$$\begin{array}{ccccccc}
(\mathcal{X}, \Lambda) & \longrightarrow & (\mathcal{X}', \Lambda') & & (\mathcal{X}, \Lambda) & \longrightarrow & (\mathcal{X}'', \Lambda'') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Spec}(A) & \longrightarrow & \text{Spec}(A') & & \text{Spec}(A) & \longrightarrow & \text{Spec}(A')
\end{array}$$

Then we will define an infinitesimal Poisson deformation $(\bar{\mathcal{X}}, \bar{\Lambda})$ of (X, Λ_0) inducing (\mathcal{X}, Λ) and (\mathcal{X}', Λ') , which implies (H_1) . Since $\text{Spec}(A), \text{Spec}(A')$ and $\text{Spec}(A'')$ are one point sets, $\mathcal{X}, \mathcal{X}'$ and \mathcal{X}'' have the same topological space of X . For any open set $U \subset X$, $\mathcal{O}_{\mathcal{X}'}(U) \rightarrow \mathcal{O}_{\mathcal{X}}(U)$ is a Poisson A' -algebra homomorphism and $\mathcal{O}_{\mathcal{X}''}(U) \rightarrow \mathcal{O}_{\mathcal{X}}(U)$ is a Poisson A'' -algebra homomorphism. Choose an affine open cover $\{U_i = \text{Spec}(B_i)\}$ of X . Then U_i are all affine open sets of $\mathcal{X}, \mathcal{X}', \mathcal{X}''$.² For affine open set U of X , let $\mathcal{O}_{\mathcal{X}}(U) = B, \mathcal{O}_{\mathcal{X}'}(U) = B'$ and $\mathcal{O}_{\mathcal{X}''}(U) = B''$. Then we have a Poisson homomorphism $p : B' \rightarrow B$ and $q : B'' \rightarrow B$. Then $B' \otimes_{A'} A \rightarrow B$ is a Poisson isomorphism and $B'' \otimes_{A''} A \rightarrow B$ is a Poisson isomorphism. Let $B' \times_B B''$ which is a Poisson algebra over \bar{A} . Since $\text{Spec}(B), \text{Spec}(B')$ and $\text{Spec}(B'')$ has same topological spaces. $\text{Spec}(B) \rightarrow \text{Spec}(B')$ induced from $p : B' \rightarrow B$ is bijective and $\text{Spec}(B) \rightarrow \text{Spec}(B'')$ induced from $q : B'' \rightarrow B$ is bijective. We would like to describe prime ideals of $\bar{B} = B' \times_B B''$. Since $A'' \rightarrow A$ is surjective, by Lemma 8.9, $\varphi : \bar{B} \otimes_{\bar{A}} A' \rightarrow B'$ induced from $\bar{B} \rightarrow B'$ is an Poisson isomorphism. Then $\text{Spec}(B')$ is topologically homeomorphic to $\text{Spec}(\bar{B})$. Hence all prime ideal of $\text{Spec}(\bar{B})$ is induced from $\text{Spec}(B)$ from the map $\bar{B} \rightarrow B$. Let \mathfrak{p} be the prime ideal of \bar{B} . Then $\mathfrak{p} = p'^{-1}q^{-1}(\mathfrak{q})$ for the unique prime ideal $\mathfrak{q} \in B$. Then we have $\bar{B}_{\mathfrak{p}} \cong B'_{p^{-1}(\mathfrak{q})} \times_{B_{\mathfrak{q}}} B''_{q^{-1}(\mathfrak{q})}$ which is a Poisson isomorphism induced from the natural fibered sum of Poisson algebras

$$\begin{array}{ccc}
\bar{B}_{\mathfrak{p}} & \longrightarrow & B'_{p^{-1}(\mathfrak{q})} \\
\downarrow & & \downarrow \\
B''_{q^{-1}(\mathfrak{q})} & \longrightarrow & B_{\mathfrak{q}}
\end{array}$$

Now we globalize our arguments. We would like to construct a Poisson scheme $\bar{\mathcal{X}}$ which is an infinitesimal Poisson deformation over \bar{A} . Our local model of \mathcal{X} for affine open set U of X will be isomorphic to Poisson affine scheme $\mathcal{O}_{\mathcal{X}'}(U) \times_{\mathcal{O}_{\mathcal{X}}(U)} \mathcal{O}_{\mathcal{X}''}(U)$ with naturally induced Poisson structure. We assume that $\mathcal{X} \rightarrow \mathcal{X}'$ and $\mathcal{X} \rightarrow \mathcal{X}''$ are identity maps on X when we ignore sheaf structures. Now we will define a Poisson sheaf $\mathcal{O}_{\bar{\mathcal{X}}}$ on X which is a Poisson scheme $\bar{\mathcal{X}}$ over \bar{A} in

²Such open set exists: let Z_0 be a closed subscheme of a scheme Z , defined by a sheaf of nilpotent ideals $N \subset \mathcal{O}_Z$. If Z_0 is affine, then Z is affine as well (See [Ser06] Lemma 1.2.3 page 23)

the following way: for open set U , $\mathcal{O}_{\bar{\mathcal{X}}}(U)$ is a set of elements $\phi : U \rightarrow \bigcup_{x \in U} \mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}$ such that for each $x \in X$, there exists an affine neighborhood U_x of x and an element $(a, b) \in \mathcal{O}_{\mathcal{X}''}(U_x) \times_{\mathcal{O}_{\mathcal{X}}(U_x)} \mathcal{O}_{\mathcal{X}'}(U_x)$ such that ϕ is canonically induced from (a, b) from the natural Poisson map $\mathcal{O}_{\mathcal{X}'}(U_x) \times_{\mathcal{O}_{\mathcal{X}}(U_x)} \mathcal{O}_{\mathcal{X}''}(U_x) \rightarrow \mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}$. Then $\mathcal{O}_{\bar{\mathcal{X}}}$ is a Poisson sheaf where the Poisson structure is induced from the Poisson structure $\mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}$. Then $\mathcal{O}_{\bar{\mathcal{X}}}$ defines a Poisson scheme $\bar{\mathcal{X}}$. Indeed, for any affine open set U of X , let $C = \mathcal{O}_{\mathcal{X}'}(U) \times_{\mathcal{O}_{\mathcal{X}}(U)} \mathcal{O}_{\mathcal{X}''}(U)$. Then we have a natural sheaf homomorphism $\mathcal{O}_{\text{Spec}(C)} \rightarrow \mathcal{O}_{\bar{\mathcal{X}}}|_U$ which is a Poisson isomorphism since it is isomorphic at each stalk by the compatibility of localization as shown above.

We claim that $\bar{\mathcal{X}} \times_{\text{Spec}(\bar{A})} \text{Spec}(A') \cong \mathcal{X}'$ and $\bar{\mathcal{X}} \times_{\text{Spec}(\bar{A})} \text{Spec}(A'') \cong \mathcal{X}''$ as Poisson schemes. We simply note that $(\mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}) \otimes_{\bar{A}} A' \cong \mathcal{O}_{\mathcal{X}',x}$ and $(\mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}) \otimes_{\bar{A}} A'' \cong \mathcal{O}_{\mathcal{X}'',x}$. $\bar{\mathcal{X}}$ is flat by Lemma 8.9.

Now we prove (H_2) . We assume that $A'' = k[\epsilon]$ and $A = k$. Let \mathcal{Y} be a infinitesimal Poisson deformation of (X, Λ_0) over \bar{A} inducing (\mathcal{X}, Λ') and $(\mathcal{X}'', \Lambda'')$, i.e $\mathcal{Y} \times_{\text{Spec}(\bar{A})} \text{Spec}(A') \cong \mathcal{X}'$ and $\mathcal{Y} \times_{\text{Spec}(\bar{A})} \text{Spec}(A'') \cong \mathcal{X}''$. We will construct an Poisson isomorphism $\bar{\mathcal{X}} \rightarrow \mathcal{Y}$. Equivalently, we will show an isomorphism of Poisson sheaves $\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\bar{\mathcal{X}}}$. Since $A' \rightarrow k$ and $k[\epsilon] \rightarrow k$ are surjective, $\mathcal{X}' \rightarrow \mathcal{Y}$ and $\mathcal{X}'' \rightarrow \mathcal{Y}$ are closed immersions. Since \mathcal{Y} is also an infinitesimal Poisson deformation of $\mathcal{X} = (X, \Lambda_0)$, we have the following commutative diagram.

$$\begin{array}{ccc} \mathcal{Y} & \longleftarrow & \mathcal{X}' \\ \uparrow & & \uparrow \\ \mathcal{X}'' & \longleftarrow & \mathcal{X} = (X, \Lambda_0) \end{array}$$

For each affine open set of X , we have the following commutative diagram of Poisson homomorphisms

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}}(U) & \longrightarrow & \mathcal{O}_{\mathcal{X}'}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{X}''}(U) & \longrightarrow & \mathcal{O}_{\mathcal{X}}(U) \end{array}$$

which is compatible with localization at each $x \in U$. So we have the following commutative diagram

$$\begin{array}{ccc} \mathcal{O}_{\mathcal{Y}}(U) & \longrightarrow & \mathcal{O}_{\mathcal{X}}(U) \times_{\mathcal{O}_{\mathcal{X}}(U)} \mathcal{O}_{\mathcal{X}''}(U) \\ \downarrow & & \downarrow \\ \mathcal{O}_{\mathcal{Y},x} & \longrightarrow & \mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x} \end{array}$$

This induces a natural Poisson homomorphism $\mathcal{O}_{\mathcal{Y},x} \rightarrow \mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}$ which is necessarily isomorphism. Now we define a sheaf map. For each open set U , we can identify $\mathcal{O}_{\mathcal{Y}}(U)$ is the set of elements $\alpha : U \rightarrow \bigcup_{x \in U} \mathcal{O}_{\mathcal{Y},x}$, locally coming from an element of section of affine open set of U . Then the map $\mathcal{O}_{\mathcal{Y}}(U) \rightarrow \mathcal{O}_{\bar{\mathcal{X}}}(U)$ induced from $\mathcal{O}_{\mathcal{X}'',x} \rightarrow \mathcal{O}_{\mathcal{X}',x} \times_{\mathcal{O}_{\mathcal{X},x}} \mathcal{O}_{\mathcal{X}'',x}$ is well defined. $\mathcal{O}_{\mathcal{Y}} \rightarrow \mathcal{O}_{\bar{\mathcal{X}}}$ is isomorphism since it is isomorphic at each stalk.

Lastly, since (X, Λ_0) is a smooth projective Poisson scheme, $HP^2(X, \Lambda_0)$ is a finite dimensional k -vector space. We have $\kappa : t_{PD\text{ef}(X, \Lambda_0)} \cong HP^2(X, \Lambda_0)$ as a map in Proposition 7.37. We

show that κ is isomorphism as k -vector spaces. We simply note that for given $\xi, \eta \in PDef(k[\epsilon])$ represented by $(Id + \epsilon p_{ij}, \Lambda_0 + t\Lambda'_i)$ and $(Id + \epsilon p'_{ij}, \Lambda_0 + t\Lambda''_i)$, $\xi + \eta$ is given by the data $(Id + \epsilon(p_{ij} + p'_{ij}), \Lambda_0 + \epsilon(\Lambda'_i + \Lambda''_i))$. So $t_{PDef_{(X, \Lambda_0)}} = PDef_{(X, \Lambda_0)}(k[\epsilon])$ is a finite dimensional k -vector space. Hence by Schlessinger's criterion (Theorem 8.5), $PDef_{(X, \Lambda_0)}$ has a miniversal family.

■

Lemma 8.11 *Let (X, Λ_0) be smooth projective Poisson scheme with $HP^1(X, \Lambda_0) = 0$. Then for any infinitesimal Poisson deformation (\mathcal{X}, Λ) of (X, Λ_0) over A for any $A \in \mathbf{Art}$, we have $PAut((\mathcal{X}, \Lambda)/(X, \Lambda_0)) = Id$, where $PAut((\mathcal{X}, \Lambda)/(X, \Lambda_0)) :=$ the set of Poisson automorphisms of (\mathcal{X}, Λ) restricting to the identity Poisson automorphism of (X, Λ_0) .*

Proof. We prove by induction on the dimension of A . Let $dim_k A = 1$. Then $A = k$. So we have nothing to prove. Let's assume that the lemma holds for A with $dim_k A \leq n - 1$. Let $dim_k A = n$ and (\mathcal{X}, Λ) be an infinitesimal Poisson deformation of (X, Λ_0) over A . Assume that the maximal ideal \mathfrak{m} of A satisfies $\mathfrak{m}^{p-1} \neq 0$ and $\mathfrak{m}^p = 0$. Choose $t \neq 0 \in \mathfrak{m}^{p-1}$. Then $A/(t) \in \mathbf{Art}$ with $dim_k A/(t) \leq n - 1$ and $0 \rightarrow (t) \rightarrow A \rightarrow A/(t) \rightarrow 0$ is a small extension. Now let $g : (\mathcal{X}, \Lambda) \rightarrow (\mathcal{X}, \Lambda)$ be a Poisson automorphism restricting to the identity Poisson automorphism of (X, Λ_0) . Let $\{U_i = Spec(B_i)\}$ be an affine open cover of X . Let $\{\theta_i\}$ where $\theta_i : U_i \times Spec(A) \rightarrow \mathcal{X}|_{U_i}$, $\{\Lambda_i\}$ where Λ_i is the Poisson structure on $U_i \times Spec(A)$ induced from $\Lambda|_{U_i}$ via θ and let $\theta_{ij} = \theta_i^{-1}\theta_j$ which corresponds to a Poisson homomorphism $f_{ij} : (\mathcal{O}_X(U_{ij}) \otimes_k A, \Lambda_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes_k A, \Lambda_j)$. Then f can be described by the data $g_i : (\mathcal{O}_X(U_i) \otimes A, \Lambda_i) \rightarrow (\mathcal{O}_X(U_i) \otimes A, \Lambda_i)$ which is a Poisson automorphism and the following commutative diagram

$$\begin{array}{ccc} (\mathcal{O}_X(U_{ij}) \otimes A, \Lambda_j) & \xrightarrow{g_j} & (\mathcal{O}_X(U_{ij}) \otimes A, \Lambda_j) \\ f_{ij} \uparrow & & \uparrow f_{ij} \\ (\mathcal{O}_X(U_{ij}) \otimes A, \Lambda_i) & \xrightarrow{g_i} & (\mathcal{O}_X(U_{ij}) \otimes A, \Lambda_i) \end{array}$$

Since by the induction hypothesis g_i induce the identity on $\mathcal{O}_X(U_i) \otimes A/(t)$. g_i is of the form $g_i = Id + td_i x$, where $d_i \in Der_k(\mathcal{O}_X(U_i), \mathcal{O}_X(U_i))$ with $[\Lambda_0, d_i] = 0$ by Lemma 7.36. Hence $\{d_i\} \in HP^1(X, \Lambda_0)$. Since $HP^1(X, \Lambda_0) = 0$, we have $d_i = 0$. Hence g_i is the identity. So g is the identity. This proves the lemma. ■

Proposition 8.12 *Let (X, Λ_0) be a projective smooth Poisson scheme with $HP^1(X, \Lambda_0) = 0$. Then the functor $PDef_{(X, \Lambda_0)}$ is pro-representable.*

Proof. Since $PDef_{(X_0, \Lambda_0)}$ satisfies $(H_0), (H_1), (H_2), (H_3)$ by Theorem 8.10, we will check (H_4) to show that the Poisson deformation functor $PDef_{(X, \Lambda_0)}$ is pro-representable by Theorem 8.7.

Let $0 \rightarrow (t) \rightarrow \tilde{A} \xrightarrow{\mu} A \rightarrow 0$ be a small extension in **Art**. Let $\xi = (\mathcal{X}, \Lambda) \in PDef_{(X, \Lambda_0)}(A)$ be an infinitesimal Poisson deformation of (X, Λ_0) over A . Let $p := PDef_{(X, \Lambda_0)}(\mu) : PDef_{(X, \Lambda_0)}(\tilde{A}) \rightarrow PDef_{(X, \Lambda_0)}(A)$ induced from $\mu : \tilde{A} \rightarrow A$. We will define a map $G : HP^2(X, \Lambda_0) \times p^{-1}(\xi) \rightarrow p^{-1}(\xi)$ which is a group action of $HP^2(X, \Lambda_0)$ acting on $p^{-1}(\xi)$. Let $\tilde{\xi} = (\tilde{\mathcal{X}}, \tilde{\Lambda}) \in p^{-1}(\xi)$ be a lifting of ξ over \tilde{A} . Let $\{U_i = Spec(B_i)\}$ be an affine cover of $\xi = (\tilde{\mathcal{X}}, \tilde{\Lambda})$. Let $\{\tilde{\theta}_i\}$ where $\tilde{\theta}_i : U_i \times Spec(\tilde{A}) \rightarrow \tilde{\mathcal{X}}|_{U_i}$, $\{\tilde{\Lambda}_i\}$ where $\tilde{\Lambda}_i$ is the Poisson structure on $U_i \times Spec(\tilde{A})$ induced from $\tilde{\Lambda}|_{U_i}$ via $\tilde{\theta}_i$ and let $\tilde{\theta}_{ij} = \tilde{\theta}_i^{-1}\tilde{\theta}_j$ which corresponds to a Poisson homomorphism $\tilde{f}_{ij} : (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_j)$. We note that $t\tilde{f}_{ij} : \mathcal{O}_X(U_{ij}) \otimes \tilde{A} \rightarrow \mathcal{O}_X(U_{ij}) \otimes \tilde{A}$ is same to tId since \tilde{f}_{ij} is identity map modulo by maximal ideal $\tilde{\mathfrak{m}}$ of \tilde{A} (i.e $\tilde{f}_{ij}(x) - Id(x) \in \tilde{\mathfrak{m}}$), and the maximal ideal $\tilde{\mathfrak{m}}$ is killed by (t) , and so we have $t(\tilde{f}_{ij} - Id)(x) = 0$. Let $(\{\Lambda'_i\}, \{p_{ij}\}) \in HP^2(X, \Lambda_0)$, where $\Lambda'_i \in Hom_{B_i}(\wedge^2 \Omega_{B_i/k}, B_i)$ and $p_{ij} \in \Gamma(U_i, T_X) = Der_k(B_i, B_i)$. So we have $[\Lambda_0, \Lambda'_i] = 0$, $\Lambda'_j - \Lambda'_i + [\Lambda_0, p_{ij}] = 0$ and $p_{ij} + p_{jk} - p_{ik} = 0$. Then we will define another lifting $\tilde{\xi}$ of ξ from $(\{\Lambda'_i\}, \{p_{ij}\})$ and $\tilde{\xi}$ by gluing $U_i \times Spec(\tilde{A})$ equipped with $\tilde{\Lambda}_i + t\Lambda'_i$ in the following way: $\tilde{f}_{ij} + tp_{ij} : \mathcal{O}_X(U_{ij}) \otimes_k \tilde{A} \rightarrow \mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}$ which is an isomorphism with the inverse $\tilde{f}_{ij} - tp_{ij}$. Indeed, $(\tilde{f}_{ij} - tp_{ij}) \circ (\tilde{f}_{ij} + tp_{ij}) : \mathcal{O}_X(U_i) \otimes_k \tilde{A} \rightarrow \mathcal{O}_X(U_i) \otimes_k \tilde{A}$, $Id + t(\tilde{f}_{ij}p_{ij} - p_{ij}\tilde{f}_{ij}) = Id + t(p_{ij} - p_{ij}) = Id$. $\{\tilde{f}_{ij} + tp_{ij}\}$ satisfies cocycle condition:

$$(\tilde{f}_{ki} + tp_{ki}) \circ (\tilde{f}_{jk} + tp_{jk}) \circ (\tilde{f}_{ij} + tp_{ij}) = \tilde{f}_{ki} \circ \tilde{f}_{jk} \circ \tilde{f}_{ij} + t(p_{ij} + p_{jk} + p_{ki}) = Id$$

Now let's consider the Poisson structures. Since $[\Lambda_0, \Lambda'_i] = 0$, we have $[\tilde{\Lambda}_i + t\Lambda'_i, \tilde{\Lambda}_i + t\Lambda'_i] = 0$. Hence $\tilde{\Lambda}_i + t\Lambda'_i$ defines a Poisson structure on $\mathcal{O}_X(U_i) \otimes_k \tilde{A}$. We claim that $\tilde{f}_{ij} + tp_{ij} : (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i + t\Lambda'_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_j + t\Lambda'_j)$ is a Poisson isomorphism. First we note that

$$\begin{aligned} & (\Lambda'_j - \Lambda'_i + [\Lambda_0, p_{ij}])(dx \wedge dy) \\ &= \Lambda'_j(dx \wedge dy) - \Lambda'_i(dx \wedge dy) + \Lambda_0(d(p_{ij}(x)) \wedge dy) - \Lambda_0(d(p_{ij}(y))) \wedge dx - p_{ij}(\Lambda_0(dx \wedge dy)) = 0 \end{aligned}$$

$$\begin{aligned} & (\tilde{f}_{ij} + tp_{ij})((\tilde{\Lambda}_i + t\Lambda'_i)(dx \wedge dy)) = (\tilde{f}_{ij} + tp_{ij})(\tilde{\Lambda}_i(dx \wedge dy) + t\Lambda'_i(dx \wedge dy)) \\ &= \tilde{f}_{ij}(\tilde{\Lambda}_i(dx \wedge dy)) + t\Lambda'_i(dx \wedge dy) + tp_{ij}(\Lambda_0(dx \wedge dy)) = \tilde{\Lambda}_j(d\tilde{f}_{ij}(x) \wedge d\tilde{f}_{ij}(y)) + t\Lambda'_j(dx \wedge dy) \\ &+ t\Lambda_0(d(p_{ij}(x)) \wedge dy) - t\Lambda_0(d(p_{ij}(y))) \wedge dx - tp_{ij}(\Lambda_0(dx \wedge dy)) + tp_{ij}(\Lambda_0(dx \wedge dy)) \\ &= \tilde{\Lambda}_j(d\tilde{f}_{ij}(x) \wedge d\tilde{f}_{ij}(y)) + t\Lambda'_j(dx \wedge dy) + t\Lambda_0(d(p_{ij}(x)) \wedge dy) - t\Lambda_0(d(p_{ij}(y))) \wedge dx \end{aligned}$$

On the other hand,

$$\begin{aligned} & (\tilde{\Lambda}_j + t\Lambda'_j)(d(\tilde{f}_{ij} + tp_{ij})(x)) \wedge d((\tilde{f}_{ij} + tp_{ij})(y)) \\ &= \tilde{\Lambda}_j(d(\tilde{f}_{ij} + tp_{ij})(x)) \wedge d((\tilde{f}_{ij} + tp_{ij})(y)) + t\Lambda'_j(d(\tilde{f}_{ij}(x)) \wedge d(\tilde{f}_{ij}(y))) \\ &= \tilde{\Lambda}_j(d\tilde{f}_{ij}(x) \wedge d\tilde{f}_{ij}(y)) + t\Lambda_0(d(p_{ij}(x)) \wedge dy) + t\Lambda_0(dx \wedge d(p_{ij}(y))) + t\Lambda'_j(dx \wedge dy) \end{aligned}$$

Hence we can define another lifting $\bar{\xi}$ from $\tilde{\xi}$ and $(\{\Lambda'_i\}, \{p'_{ij}\}) \in HP^2(X, \Lambda_0)$. Now we show that the map

$$G : HP^2(X, \Lambda_0) \times p^{-1}(\xi) \rightarrow p^{-1}(\xi)$$

$$((\{\Lambda'_i\}, \{p'_{ij}\}), \tilde{\xi}) \mapsto \bar{\xi}$$

is well-defined. Let $(\{\Lambda''_i\}, \{p''_{ij}\})$ represent the same cohomology class with $(\{\Lambda_i\}, \{p_{ij}\})$ in $HP^2(X, \Lambda_0)$. Then we have to show that the lifting $\bar{\xi}$ defined by $(\{\Lambda_i\}, \{p_{ij}\})$ is an equivalent infinitesimal Poisson deformation to the lifting $\bar{\xi}'$ defined by $(\{\Lambda''_i\}, \{p''_{ij}\})$. There exists $\{a_i\}$ where $a_i \in \Gamma(U_i, T_X)$ such that $[\Lambda_0, a_i] = \Lambda'_i - \Lambda''_i$ and $-\delta(\{a_i\}) = a_i - a_j = p_{ij} - p'_{ij}$. To show $\bar{\xi} \cong \bar{\xi}'$, it is sufficient to show that the following diagram commutes and $Id + ta_i$ defines Poisson isomorphisms.

$$\begin{array}{ccc} (\mathcal{O}_X(U_{ij}) \otimes_k \text{Spec}(\tilde{A}), \tilde{\Lambda}_j + t\Lambda'_j) & \xrightarrow{Id+ta_j} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_j + t\Lambda''_j) \\ \tilde{f}_{ij} + tp_{ij} \uparrow & & \uparrow \tilde{f}_{ij} + tp'_{ij} \\ (\mathcal{O}_X(U_{ij}) \otimes_k \text{Spec}(\tilde{A}), \tilde{\Lambda}_i + t\Lambda'_i) & \xrightarrow{Id+ta_i} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i + t\Lambda''_i) \end{array}$$

Indeed, $(Id + ta_j) \circ (\tilde{f}_{ij} + tp_{ij}) - (\tilde{f}_{ij} + tp'_{ij}) \circ (Id + ta_i) = \tilde{f}_{ij} + tp_{ij} + ta_j - (\tilde{f}_{ij} + ta_i + tp'_{ij}) = t(a_j - a_i + p_{ij} - p'_{ij}) = 0$. $\tilde{\Lambda}_i + t\Lambda'_i - (\tilde{\Lambda}_i + t\Lambda''_i) = t(\Lambda_i - \Lambda''_i)$ and $\Lambda_i - \Lambda''_i - [\Lambda_0, a_i] = 0$. So by Lemma 7.36, $\{Id + ta_i\}$ are Poisson isomorphisms.

Next, we show that the group action $HP^2(X, \Lambda_0) \times p^{-1}(\xi) \rightarrow p^{-1}(\xi)$ is transitive. Let $\tilde{\xi} \in p^{-1}(\xi)$ be a lifting of ξ as above. Choose arbitrary lifting $\eta \in p^{-1}(\xi)$ of ξ . We have to find $(\{\Lambda'_i\}, \{p'_{ij}\}) \in HP^2(X, \Lambda_0)$ such that $((\{\Lambda'_i\}, \{p'_{ij}\}), \tilde{\xi})$ is mapped to η under the action. Let η consist of the data $f'_{ij} : (\mathcal{O}_X(U_{ij}), \otimes \tilde{A}, \tilde{\Lambda}'_i) \rightarrow (\mathcal{O}_X(U_{ij}) \otimes \tilde{A}, \tilde{\Lambda}'_i)$. Since $\tilde{\xi}$ and η both induce ξ , we have $\tilde{f}'_{ij} = f_{ij} + tp_{ij}$ and $\tilde{\Lambda}'_i = \tilde{\Lambda}_i + t\Lambda'_i$ for some $p_{ij} \in \Gamma(U_{ij}, T_X)$ and $\Lambda'_i \in \text{Hom}_{B_0}(\wedge^2 \Omega_{B_i/k}^1, B_i)$. Then $(\{\Lambda'_i\}, \{p'_{ij}\})$ defines a cohomology class in $HP^2(X, \Lambda_0)$. So the group action is transitive.

Next, we show that the group action is free. Assume that for given $v = (\{\Lambda'_i\}, \{p'_{ij}\}) \in HP^2(X, \Lambda_0)$, we have $G(v, \tilde{\xi}) = \tilde{\xi}$. We have to show that $v = 0 \in HP^2(X, \Lambda_0)$. Let $\tilde{\xi}$ and $G(v, \tilde{\xi})$ as above. Then $G(v, \tilde{\xi}) = \tilde{\xi}$ implies that we have Poisson isomorphisms $g_i : (\mathcal{O}_X(U_i) \otimes_k \tilde{A}, \tilde{\Lambda}_i) \rightarrow (\mathcal{O}_X(U_i) \otimes_k \tilde{A}, \tilde{\Lambda}_i + t\Lambda'_i)$ such that the following diagram is commutative.

$$\begin{array}{ccc} (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_j) & \xrightarrow{g_j} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_j + t\Lambda'_j) \\ \tilde{f}_{ij} \uparrow & & \uparrow \tilde{f}_{ij} + tp_{ij} \\ (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i) & \xrightarrow{g_i} & (\mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}, \tilde{\Lambda}_i + t\Lambda'_i) \end{array}$$

Since g_i induces a Poisson automorphism on $\xi = (\mathcal{X}, \Lambda)$ by modulo (t) , which is necessarily identity by Lemma 8.11. Hence g_i is the form of $g_i = Id + tq_i$, where $q_i \in \text{Der}_k(\mathcal{O}_X(U_i), \mathcal{O}_X(U_i))$ with $-\Lambda'_i - [\Lambda_0, q_i] = 0$ by Lemma 7.36. Since the diagram commutes, we have $0 = g_j \tilde{f}_{ij} - (\tilde{f}_{ij} + tp_{ij})g_i = (Id + tq_j)\tilde{f}_{ij} - (\tilde{f}_{ij} + tp_{ij})(Id + tq_i) = t(q_j - p_{ij} - q_i)$. Hence $p_{ij} = q_j - q_i$. Hence $v = 0 \in HP^2(X, \Lambda_0)$.

Lastly, we claim that G is exactly same to $t_{PDef_{(\mathcal{X}, \Lambda_0)}} \times p^{-1}(\xi) \rightarrow p^{-1}(\xi)$ that we defined in section 8.1. We set $F := PDef_{(\mathcal{X}, \Lambda_0)}$. Let's describe $t_F \times p^{-1}(\xi) \rightarrow p^{-1}(\xi)$ by following the definition

in section 8.1. We note that here $A' = \tilde{A}$. We now describe $\alpha^{-1} : t_F \times F(\tilde{A}) \rightarrow F(k[\epsilon] \times_k \tilde{A})$. Given an element $v = (\{\Lambda'_i\}, \{p_{ij}\}) \in HP^2(X, \Lambda_0)$ gives a first order Poisson deformation $(\mathcal{X}_\epsilon, \Lambda_\epsilon)$ over $k[\epsilon]$ by Proposition 7.37. Let $(\mathcal{X}, \Lambda) \in F(\tilde{A})$ which is a lifting of $\xi \in F(A)$. Let $\{U_i\}$ be an affine open set of X . $(\mathcal{X}_\epsilon, \Lambda_\epsilon)$ can be described by the data $(\{Id + \epsilon p_{ij}\}, \{\Lambda_0 + \epsilon \Lambda'_i\})$, where $\Lambda_0 + \epsilon \Lambda'_i$ is a Poisson structure on $\mathcal{O}_X(U_i) \otimes_k k[\epsilon]$ and $Id + \epsilon p_{ij} : \mathcal{O}_X(U_{ij}) \otimes_k k[\epsilon] \rightarrow \mathcal{O}_X(U_{ij}) \otimes_k k[\epsilon]$ an Poisson isomorphism. Similarly (\mathcal{X}, Λ) can be described by the data $(\{f_{ij}\}, \{\Lambda_i\})$, where Λ_i is a Poisson structure on $\mathcal{O}_X(U_i) \otimes_k \tilde{A}$ and $f_{ij} : \mathcal{O}_X(U_{ij}) \otimes_k \tilde{A} \rightarrow \mathcal{O}_X(U_{ij}) \otimes_k \tilde{A}$ an Poisson isomorphism. Then $\alpha^{-1}((\mathcal{X}_\epsilon, \Lambda_\epsilon), (\mathcal{X}, \Lambda))$ is a fibered sum $\mathcal{X}_\epsilon \times_k \mathcal{X}$ which can be described as $(\{Id + \epsilon p_{ij}, f_{ij}\}, \{\Lambda_0 + \epsilon \Lambda'_i, \Lambda_i\})$. We note that

$$\begin{array}{ccc} k[\epsilon] \otimes_k \tilde{A}, (x + y\epsilon, a') & \longrightarrow & \tilde{A} \times_A \tilde{A}, (a' + yt, a') \longrightarrow \tilde{A}, (a' + yt) \\ & & \downarrow \\ & & \tilde{A}, (a') \end{array}$$

In the map $t_F \times F(\tilde{A}) \rightarrow F(\tilde{A}) \times_{F(A)} F(\tilde{A}), (v, \xi) \mapsto (\tau(v, \xi), \xi), \tau(v, (\mathcal{X}, \Lambda)) = (\mathcal{X}_\epsilon \times_k \mathcal{X}) \otimes_{k[\epsilon] \times_k \tilde{A}} \tilde{A}$ is induced from $k[\epsilon] \times_k \tilde{A} \rightarrow \tilde{A}, (x + y\epsilon, a') \mapsto (a' + yt)$. Hence $(\mathcal{X}_\epsilon \times_k \mathcal{X}) \otimes_{k[\epsilon] \times_k \tilde{A}} \tilde{A}$ can be described as $(f_{ij} + \epsilon p_{ij}, \Lambda_i + \epsilon \Lambda'_i)$ which is exactly same to $G(v, (\mathcal{X}, \Lambda))$.

■

Chapter 9

Poisson cotangent complex

In this chapter, we extend the construction of Schlessinger and Lichtenbaum's cotangent complex (See [LS67]) in terms of Poisson algebras.¹ We follow their arguments in Poisson context. In this chapter, every algebra is a k -algebra for a field k .

9.1 Poisson modules and Poisson enveloping algebras

(See [Fre06])

Definition 9.1 *Let A be a Poisson algebra. A Poisson module over A is a A -module M equipped with a bracket $\{-, -\} : A \otimes_k M \rightarrow M$ such that*

1. $\{a, bm\} = \{a, b\} + b\{a, m\}$
2. $\{ab, m\} = a\{b, m\} + b\{a, m\}$
3. $\{a, \{b, m\}\} = \{\{a, b\}, m\} + \{b, \{a, m\}\}$

for $a, b \in A$ and $m \in M$.

¹My original goal was to generalize Hartshorne's construction of ' T^i Functors' presented in [Har10] page 18-25 and apply to deformation problems for not necessarily smooth Poisson schemes. Hartshorne's book [Har10] led me to Lichtenbaum and Schlessinger's original paper [LS67]. I followed Schlessinger and Lichtenbaum [LS67] in the context of Poisson algebras in this chapter. However I could not succeed in globalization. See Remark 9.34. There is also a general approach to cotangent complex for algebras over an operad (see [LV12]). If our construction turns out to be correct, I expect that our construction PT^i for $i = 0, 1, 2$ is equivalent to the general construction in the language of operads.

Definition 9.2 Let M and N be Poisson modules of a Poisson algebra A . A map $\phi : M \rightarrow N$ is a morphism of Poisson modules if

$$\begin{aligned}\phi(am) &= a\phi(m) \\ \phi(\{a, x\}) &= \{a, \phi(m)\}\end{aligned}$$

for $a \in A$ and $m \in M$. We denote by $\text{Hom}_{\mathcal{U}_{\text{Pois}}(A)}(M, N)$ or $\text{PHom}_A(M, N)$ the k -module of morphisms of Poisson modules from M to N .

We will construct the Poisson enveloping algebra $\mathcal{U}_{\text{Pois}}(A)$ of a Poisson k -algebra A . This is an associative k -algebra which is characterized by the following property: “The category of left $\mathcal{U}_{\text{Pois}}(A)$ -modules is equivalent to the category of Poisson modules over A ” ([Fre06]).

Definition 9.3 The Poisson enveloping algebra $\mathcal{U}_{\text{Pois}}(A)$ is the associated A -algebra with unit generated by the symbols $X_a, a \in A$ by the quotient of the following relations

1. $X_a \cdot b = \{a, b\} + b \cdot X_a$
2. $X_{ab} = a \cdot X_b + b \cdot X_a$
3. $X_a \cdot X_b = X_{\{a, b\}} + X_b \cdot X_a$

for $a, b \in A$

Let M be a Poisson module over A (so M is a left $\mathcal{U}_{\text{Pois}}(A)$ -module), then M is also a right $\mathcal{U}_{\text{Pois}}(A)$ -module defined by

$$m \cdot a := a \cdot m, \quad m \cdot X_a = -\{a, m\}$$

So given a Poisson module M over A , by abuse of notation, we define $\{m, a\} := m \cdot X_a$. Then in practice, we treat the bracket $\{-, -\}$ on M like a bracket of a Poisson algebra.

Definition 9.4 Let $S \rightarrow A$ be a homomorphism of Poisson algebras. A Poisson S -derivation is a map $d : A \rightarrow M$ from a Poisson k -algebra A to a Poisson A -module M which is a S -linear derivation with respect to multiplication and with respect to the bracket $\{-, -\}$ of A . Explicitly, a

Poisson S -derivation d satisfies the following identities

$$d(sf) = sd(f)$$

$$d(fg) = df \cdot g + f \cdot dg$$

$$d\{f, g\} = \{df, g\} + \{f, dg\} = -\{g, df\} + \{f, dg\}$$

for $s \in S$ and $f, g \in A$. We denote $PDer_S(A, M)$ the A -module of Poisson S -derivations from $A \rightarrow M$. The functor $PDer_S(A, -)$ is representable : there exists a Poisson representation denoted by $\Omega^1_{\mathcal{U}_{Pois(A)/S}}$ such that

$$PDer_S(A, M) = Hom_{\mathcal{U}_{Pois(A)}}(\Omega^1_{\mathcal{U}_{Pois(A)/S}}, M)$$

To construct $\Omega^1_{\mathcal{U}_{Pois(A)/S}}$, let F be the free left $\mathcal{U}_{Pois(A)}$ -module generated by the symbols $df, f \in A$. Let E be the $\mathcal{U}_{Pois(A)}$ -submodule of F generated by the elements of the form $ds, s \in S, d(f+g) - df - dg, d(fg) = df \cdot g + f \cdot dg$ and $d\{f, g\} = -X_g \cdot df + X_f \cdot dg$. $\Omega^1_{\mathcal{U}_{Pois(A)/S}}$ is defined to be E/F . We have a natural map $d : A \rightarrow \Omega^1_{\mathcal{U}_{Pois(A)/S}}$. Via this map, the functor $PDer_S(A, -)$ is represented by $\Omega^1_{\mathcal{U}_{Pois(A)/S}}$.

Let $\rho : B \rightarrow C$ be a Poisson homomorphism compatible with A . In other words, $A \rightarrow B \rightarrow C$ is a Poisson homomorphism. Then there exist canonical homomorphism of $\mathcal{U}_{Pois(C)}$ -modules

$$\alpha : \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} \Omega^1_{Pois(B)/A} \rightarrow \Omega^1_{Pois(C)/A}$$

Proposition 9.5 *Let $A \rightarrow B$ be a homomorphism of Poisson k -algebras. Let I be a Poisson ideal of B .*

If $C = B/I$, we have an exact sequence of $\mathcal{U}_{Pois(C)}$ -modules

$$I/(I^2 \oplus \{I, I\}) \xrightarrow{\delta} \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} \Omega^1_{Pois(B)/A} \rightarrow \Omega^1_{Pois(C)/A} \rightarrow 0 \quad (9.6)$$

where for any $i \in I$, let \bar{i} be the image of i in $I/(I^2 \oplus \{I, I\})$. We define $\delta(\bar{i}) = 1 \otimes di$.

Proof.

We show that $I/(I^2 \oplus \{I, I\})$ is a Poisson B/I -module, defined by $\{\bar{b}, \bar{i}\} = \overline{\{b, i\}}$. Let $b_1, b_2 \in B$ with $b_1 - b_2 \in I$ and $i_1, i_2 \in I$ with $i_1 - i_2 \in I^2 \oplus \{I, I\}$. Then $\{b_1, i_1\} - \{b_2, i_2\} =$

$\{b_1, i_1 - i_2\} + \{b_1 - b_2, i_2\} \in I^2 \oplus \{I, I\}$. Hence the bracket is well-defined. We claim that

$$I/(I^2 \oplus \{I, I\}) \cong \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} I$$

as $\mathcal{U}_{Pois(C)}$ -modules. We define a map $\varphi : I/I^2 \oplus \{I, I\} \rightarrow \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} I$ where $\varphi(\bar{i}) = 1 \otimes i$. First we show that this map is well-defined. Let $i, j \in I$ with $i - j \in I^2 \oplus \{I, I\}$. Then $i - j = \sum_n f_n \cdot g_n$, where $f_n \in I$ or f_n is of the form $b \cdot X_a$ where $a \in I, b \in \mathcal{U}_{Pois(B)}$, and $g_n \in I$ as $\mathcal{U}_{Pois(B)}$ -module action. Since $\bar{f}_n = 0$ for $f_n \in I$ and $X_{\bar{a}} = 0$ for $a \in I$, we have $1 \otimes (i - j) = 0$. Hence φ is well-defined. Second, we show that φ is a $\mathcal{U}_{Pois(C)}$ -module homomorphism. Let $c \in \mathcal{U}_{Pois(C)}$. Then there exists $b \in \mathcal{U}_{Pois(B)}$ with $\bar{b} = c$. Then $\varphi(c \cdot \bar{i}) = \varphi(\bar{b} \cdot \bar{i}) = \varphi(\overline{b \cdot i}) = 1 \otimes b \cdot i = \bar{b} \otimes i = \bar{b}(1 \otimes i) = \bar{b}\varphi(\bar{i}) = c\varphi(\bar{i})$. Lastly we can define an inverse map $\varphi^{-1} : \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} I \rightarrow I/(I^2 \oplus \{I, I\})$ by $\varphi^{-1}(c \otimes i) = c \cdot \bar{i}$.

Now we prove the proposition. The sequence in (9.6) is equivalent to

$$\mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} I \xrightarrow{\delta} \mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} \Omega_{Pois(B)/A}^1 \xrightarrow{\alpha} \Omega_{Pois(C)/A}^1 \rightarrow 0$$

where $\delta(\bar{b} \otimes i) := \bar{b} \otimes di$. Since $B \rightarrow B/I$ is surjective, α is surjective. Exactness of the sequence in (9.6) is equivalent to the exactness of the following sequence

$$\begin{aligned} 0 \rightarrow \text{Hom}_{\mathcal{U}_{Pois(C)}}(\Omega_{Pois(C)/A}^1, N) &\rightarrow \text{Hom}_{\mathcal{U}_{Pois(C)}}(\mathcal{U}_{Pois(C)} \otimes_{\mathcal{U}_{Pois(B)}} \Omega_{Pois(B)/A}^1, N) \\ &\rightarrow \text{Hom}_{\mathcal{U}_{Pois(C)}}(I/(I^2 \oplus \{I, I\}), N) \end{aligned}$$

for any left $\mathcal{U}_{Pois(C)}$ -module N . Equivalently we have to show that the following sequence is exact.

$$0 \rightarrow PDer_A(C, N) \rightarrow PDer_A(B, N) \xrightarrow{\phi} \text{Hom}_{\mathcal{U}_{Pois(B)}}(I, N)$$

where for $D \in PDer_A(B, N)$, $\phi(D)$ is defined by restricting D to I . Now we assume that $\phi(D)$ is zero, then $D(I) = 0$ so D factor through B/I , hence $D \in PDer_A(C, N)$. This proves the proposition.

■

9.2 Poisson cotangent complex

Definition 9.7 (See [Umi12]) *We construct a free Poisson algebra generated by $\{x_i\}$ over k . Let g be a free Lie algebra generated by $\{x_i\}$ with the Lie bracket $[-, -]$. Free Poisson algebra generated by $\{x_i\}$ with the Poisson bracket $\{-, -\}$ is the polynomial algebra $k[g]$ with the Poisson bracket defined by $\{x_i, x_j\} := [x_i, x_j]$. We will denote the free Poisson algebra by $k\{x_i\}$.*

Definition 9.8 *Let A be a Poisson algebra with the bracket $\{-, -\}_A$ over k . Let's consider the free Poisson algebra P generated by A and $\{x_i\}$. We denote by $A\{x_i\}$ the quotient of P by Poisson ideals*

generated by $\{a, b\} - \{a, b\}_A$ and $a \cdot b = a * b$ for $a, b \in A$ where \cdot is the multiplication in P and $*$ is the multiplication in A . We call $A\{x_i\}$ a free Poisson algebra over A generated by $\{x_i\}$. We also note that $\Omega_{\mathcal{U}_{\text{Pois}(A\{x_i\})/A}}^1$ is a free $\mathcal{U}_{\text{Pois}(A\{x_i\})}$ -module generated by dx_i .

Remark 9.9 We have the following universal mapping property: let $X = \{x_i\}$ and $A \rightarrow B$ be a Poisson homomorphism. let $j : X \rightarrow B$ be a map. Then there exists an unique Poisson homomorphism $A\{x_i\} \rightarrow B$ such that the following diagram commutes

$$\begin{array}{ccccc}
 A & \longrightarrow & A\{x_i\} & \xrightarrow{u} & B \\
 & & & \swarrow & \uparrow j \\
 & & & & X
 \end{array}$$

Definition 9.10 Let $A \rightarrow B$ be a Poisson homomorphism. By a Poisson extension of B over A we mean an exact sequence

$$(\mathcal{E}) : 0 \rightarrow E_2 \xrightarrow{e_2} E_1 \xrightarrow{e_1} R \xrightarrow{e_0} B \rightarrow 0$$

where R is a Poisson algebra and $e_0 : R \rightarrow B$ is a Poisson homomorphism such that $A \rightarrow B$ factor through $e_0 : R \rightarrow B$, E_1 and E_2 are $\mathcal{U}_{\text{Pois}(R)}$ -modules, e_1 and e_2 are $\mathcal{U}_{\text{Pois}(R)}$ -module homomorphisms, and we have the following relation

$$e_1(x)y = e_1(y)x, \quad X_{e_1(x)}y + X_{e_1(y)}x = 0$$

for $x, y \in E_1$. We claim that E_2 is a Poisson B -module. Indeed, let I be a kernel of e_0 , which is a Poisson ideal of R . Now we give a Poisson R/I -module structure on E_2 . First we note that E_2 is a $\mathcal{U}_{\text{Pois}(R)}$ -module, and so Poisson R -module. Let $a \in B$ and $x \in E_2$. We define a Poisson B -module structure on E_2 by setting $b \cdot x := r \cdot x$ and $\{b, x\} := \{r, x\}$ where r is a lifting of b under $e_0 : R \rightarrow R/I \cong B$. This is well-defined since given two lifting r_1 and r_2 of b (i.e. $r_1 - r_2 \in I$), choose $y \in E_1$ with $e_1(y) = r_1 - r_2$. Then $e_2(r_1x - r_2x) = (r_1 - r_2)e_2(x) = e_1(y)e_2(x) = e_1(e_2(x))y = 0$

and so we have $r_1x = r_2x$. On the other hand, $e_2(\{r_1, x\} - \{r_2, x\}) = X_{r_1-r_2}e_2(x) = X_{e_1(y)}e_2(x) = -X_{e_1(e_2(yx))}y = 0$. So we have $\{r_1, x\} = \{r_2, x\}$. Then other property of a Poisson module trivially follows.

Let $A \rightarrow A' \rightarrow B'$ be a homomorphism of Poisson algebras, and \mathcal{E}' an Poisson extension of B' over A' . By a homomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ of Poisson extensions we mean a collection $(b, \alpha_0, \alpha_1, \alpha_2)$ with the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E'_2 & \xrightarrow{e'_2} & E'_1 & \xrightarrow{e'_1} & R' & \xrightarrow{e'_0} & B' & \longrightarrow & 0 \\ & & \uparrow \alpha_2 & & \uparrow \alpha_1 & & \uparrow \alpha_0 & & \uparrow b & & \\ 0 & \longrightarrow & E_2 & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & R & \xrightarrow{e_0} & B & \longrightarrow & 0 \end{array}$$

where b, α_0 is a Poisson homomorphism such that the following diagram commutes as Poisson homomorphisms

$$\begin{array}{ccccc} A' & \longrightarrow & R' & \xrightarrow{e'_0} & B' \\ \uparrow & & \uparrow \alpha_0 & & \uparrow b \\ A & \longrightarrow & R & \xrightarrow{e_0} & B \end{array}$$

and α_1 and α_2 are homomorphisms of $\mathcal{U}_{\text{Pois}(R)}$ -modules. We consider E'_1 and E'_2 are $\mathcal{U}_{\text{Pois}(R)}$ -modules via α_0 .

Definition 9.11 Let \mathcal{E} be an Poisson extension of B over A , we define a complex $PL^\bullet(\mathcal{E})$ of $\mathcal{U}_{\text{Pois}(B)}$ -modules or equivalently Poisson B -modules:

$$PL^\bullet(\mathcal{E}) : 0 \rightarrow E_2 \xrightarrow{d_2} \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} E_1 \xrightarrow{d_1} \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} \Omega^1_{\mathcal{U}_{\text{Pois}(R)/A}} \rightarrow 0$$

where d_2 is induced from e_2 , and d_1 is defined in the following way: let $I = \text{Ker}(e_0)$ (a Poisson ideal of R defining B). We also note that we have the canonical map $d : R \rightarrow \Omega^1_{\mathcal{U}_{\text{Pois}(R)/A}}$, and by restricting d on I , we have $d : I \rightarrow \Omega^1_{\mathcal{U}_{\text{Pois}(R)/A}}$ and then by tensoring $\mathcal{U}_{\text{Pois}(B)}$, we have $d : I/I^2 \oplus \{I, I\} \cong \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} I \rightarrow \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} \Omega^1_{\mathcal{U}_{\text{Pois}(R)/A}}$. We define $d_2 = d \circ (\mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} e_1)$. This is well-defined since $\text{im}(E_1) = I$. $PL^\bullet(\mathcal{E})$ is a complex since $e_1 \circ e_2 = 0$

Remark 9.12 Any Poisson extension of B over A are all obtained in the following way. Choose a surjection $e : R \rightarrow B$ with $A \rightarrow R \rightarrow B$ Poisson homomorphisms. Let $I = \text{Ker } e_0$ be a Poisson

ideal of R . Then choose an exact sequence of $\mathcal{U}_{\text{Pois}(R)}$ -modules $0 \rightarrow U \xrightarrow{i} F \xrightarrow{j} I \rightarrow 0$. Let U_0 be $\mathcal{U}_{\text{Pois}(R)}$ -submodule of F generated by $j(x)y - j(y)x$ and $X_{j(x)}y + X_{j(y)}x$, where $x, y \in F$. Then $j(U_0) = 0$ since $j(x)j(y) - j(y)j(x) = 0$, $X_{j(x)}j(y) + X_{j(y)}j(x) = \{j(x), j(y)\} + \{j(y), j(x)\} = 0$. Hence U_0 is also a submodule of U . We take $e_2 : U/U_0 \rightarrow F/U_0$ and $e_1 : F/U_0 \rightarrow R$ which is well-defined since $j(U_0) = 0$. Then $0 \rightarrow U/U_0 \rightarrow F/U_0 \rightarrow R \rightarrow B \rightarrow 0$ is a Poisson extension of B over A . Conversely, given a Poisson extension $(\mathcal{E}) : 0 \rightarrow E_2 \xrightarrow{e_2} E_1 \xrightarrow{e_1} R \xrightarrow{e_0} B \rightarrow 0$, we have $U_0 = 0$.

Definition 9.13 (Free Poisson extension) A Poisson extension of B over A is of the form $0 \rightarrow U/U_0 \rightarrow F/U_0 \rightarrow R \rightarrow B \rightarrow 0$ where R is a free Poisson algebra over A and F is a free $\mathcal{U}_{\text{Pois}(R)}$ -module is called a free Poisson extension of B over A .

Remark 9.14 For a free Poisson extension of B over A , $\mathcal{E} : 0 \rightarrow U/U_0 \rightarrow F/U_0 \xrightarrow{j} R = A\{x_i\} \xrightarrow{e_0} B \rightarrow 0$, we have $\mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} (F/U_0) \cong \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} F$. Hence $PL^1(\mathcal{E})$ is free. Indeed, let's consider the natural map $\alpha : \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} F \rightarrow \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(R)}} F/U_0$, defined by $\sum b_i \otimes f_i \mapsto \sum b_i \otimes \bar{f}_i$, which is surjective. Let $\sum b_i \otimes \bar{f}_i = 0$. Since $\mathcal{U}(e_0) : \mathcal{U}_{\text{Pois}(R)} \rightarrow \mathcal{U}_{\text{Pois}(B)}$ is surjective, we have $\sum b_i \otimes \bar{f} = \sum 1 \otimes a_i \bar{f} = 0$, where $\mathcal{U}(e_0)(a_i) = b_i$. Hence $\sum a_i \bar{f}_i = 0$. So $\sum a_i f_i \in U_0$. Hence $\sum_i a_i f_i = \sum a'_j g_j$, where g_j is of the form $j(x)y - j(y)x$ or $X_{j(x)}y + X_{j(y)}x$ where $x, y \in F$. Let $t \in \mathcal{U}_{\text{Pois}(R)}$. Since $1 \otimes t(j(x)y - j(y)x) = \mathcal{U}(e_0)(t)e_0(j(x)) \otimes y - \mathcal{U}(e_0)(t)e_0(j(y)) \otimes x = 0$ and $1 \otimes t(X_{j(x)}y + X_{j(y)}x) = \mathcal{U}(e_0)(t)X_{e_0(j(x))} \otimes y + \mathcal{U}(e_0)(t)X_{e_0(j(y))} \otimes x = 0$, we have $\sum 1 \otimes a_i f_i = \sum 1 \otimes a'_j g_j = 0$. Hence α is an isomorphism.

Consider the following commutative diagram of Poisson homomorphisms,

$$\begin{array}{ccc} B & \xrightarrow{b} & B' \\ \uparrow & & \uparrow \\ A & \xrightarrow{a} & A' \end{array}$$

Let \mathcal{E} be a free Poisson extension of B over A and \mathcal{E}' arbitrary Poisson extension of B' over A' . Then there exists a homomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ extending b .

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E'_2 & \xrightarrow{e'_2} & E'_1 & \xrightarrow{e'_1} & R' & \xrightarrow{e'_0} & B' & \longrightarrow & 0 \\
& & \uparrow \alpha_2 & & \uparrow \alpha_1 & & \uparrow \alpha_0 & & \uparrow b & & \\
0 & \longrightarrow & U/U_0 & \xrightarrow{e_2} & F/U_0 = (\oplus_k \mathcal{U}_{Pois(R)})/U_0 & \xrightarrow{e_1=j} & R = A\{x_i\} & \xrightarrow{e_0} & B & \longrightarrow & 0
\end{array}$$

where α_0, α_1 and α_2 are defined in the following way: for α_0 , we send x_i to an arbitrary lifting of $b(e_0(x_i))$. Let $\{v_k\}$ be the canonical basis of F (i.e v_k has 1 in the k -th component, and 0 in other components). For α_1 , first we define a map $\alpha'_1 : F \rightarrow E'_1$ by sending v_k to an arbitrary lifting w'_k of $\alpha_0(e_1(v_k))$. So we have $e'_1(w'_k) = \alpha_0(e_1(v_k)) = \alpha_0(j(v_k))$. We show that $\alpha'_2(U_0) = 0$, and so α_1 is well-defined. Indeed, we claim that $\alpha'_1(j(x)y - j(y)x) = j(x)\alpha'_1(y) - j(y)\alpha'_1(x) = \alpha_0(j(x))\alpha'_1(y) - \alpha_0(j(y))\alpha'_1(x) = 0$. Let $x = \sum_i a_i v_i$ and $y = \sum_k b_k v_k$. Then $j(x)y - j(y)x = \sum_{i,k} a_i j(v_i) b_k v_k - b_k j(v_k) a_i v_i$. Hence $\alpha'_1(j(x)y - j(y)x) = \sum_{i,k} \alpha_0(a_i) \alpha_0(j(v_i)) \alpha_0(b_k) \alpha'_1(v_k) - \alpha_0(b_k) \alpha_0(j(v_k)) \alpha_0(a_i) \alpha'_1(v_i)$. It is sufficient to show that

$$\alpha_0(a_i) \alpha_0(j(v_i)) \alpha_0(b_k) \alpha'_1(v_k) - \alpha_0(b_k) \alpha_0(j(v_k)) \alpha_0(a_i) \alpha'_1(v_i) = 0.$$

Since $e'_1(\alpha_0(b_k) \alpha'_1(v_k)) = \alpha_0(b_k) \alpha_0(j(v_k))$ and $e'_1(\alpha_0(a_i) \alpha'_1(v_i)) = \alpha_0(a_i) \alpha_0(j(v_i))$, we get the claim. On the other hand, we claim that $\alpha'_1(X_{j(x)}y + X_{j(y)}x) = 0$. Indeed, $X_{j(x)}y + X_{j(y)}x = \sum_{i,k} X_{a_i j(v_i)} b_k v_k + X_{b_k j(v_k)} a_i v_i$. So we have $\alpha'_1(X_{j(x)}y + X_{j(y)}x) = \sum_{i,k} X_{\alpha_0(a_i) \alpha_0(j(v_i))} \alpha_0(b_k) \alpha'_1(v_k) + X_{\alpha_0(b_k) \alpha_0(j(v_k))} \alpha_0(a_i) \alpha'_1(v_i)$. So it is sufficient to show that

$$X_{\alpha_0(a_i) \alpha_0(j(v_i))} \alpha_0(b_k) \alpha'_1(v_k) + X_{\alpha_0(b_k) \alpha_0(j(v_k))} \alpha_0(a_i) \alpha'_1(v_i) = 0.$$

Since $e'_1(\alpha_0(b_k) \alpha'_1(v_k)) = \alpha_0(b_k) \alpha_0(j(v_k))$ and $e'_1(\alpha_0(a_i) \alpha'_1(v_i)) = \alpha_0(a_i) \alpha_0(j(v_i))$, we get the claim. For α_2 , we simply see that U/U_0 is sent to E'_2 via α_1 .

Next we claim that we have a homomorphism $\mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(B)}} PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$.

$$\begin{array}{ccccccccccc}
0 & \longrightarrow & E'_2 & \xrightarrow{1 \otimes d_2} & \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R')}} E'_1 & \xrightarrow{1 \otimes d_1} & \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R')}} \Omega^1_{\mathcal{U}_{Pois(R')}/A'} & \longrightarrow & 0 \\
& & \uparrow \alpha_2 & & \uparrow \alpha'_1 & & \uparrow \alpha'_0 & & \\
0 & \longrightarrow & \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(B)}} U/U_0 & \xrightarrow{1 \otimes d_2} & \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R)}} F/U_0 & \xrightarrow{1 \otimes d_1} & \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R)}} \Omega^1_{\mathcal{U}_{Pois(R)}/A} & \longrightarrow & 0
\end{array}$$

where α'_2 and α'_1 are canonical induced from α_2, α_1 . For α'_0 , we note that we have a canonical commutative diagram

$$\begin{array}{ccc}
\Omega^1_{\mathcal{U}_{Pois(R)}/A} & \longrightarrow & \Omega^1_{\mathcal{U}_{Pois(R')}/A'} \\
d \uparrow & & \uparrow d' \\
R & \xrightarrow{\alpha_0} & R'
\end{array}$$

where d and d' are the canonical map. Let $I' = Ker(e'_0)$ and $I = Ker(e_0)$. Then $\alpha_0(I) \subset I'$. Hence we have

$$\begin{array}{ccc}
\Omega_{\mathcal{U}_{Pois(R)/A}}^1 & \longrightarrow & \Omega_{\mathcal{U}_{Pois(R')/A'}}^1 \\
d \uparrow & & \uparrow d' \\
I & \xrightarrow{\alpha_0} & I' \\
\uparrow & & \uparrow \\
F/U_0 & \xrightarrow{\alpha_1} & E'_1
\end{array}$$

By tensoring $\mathcal{U}_{Pois(B')}$, we get α'_0 .

Definition 9.15 *a complex of the form $PL^\bullet(\mathcal{E})$, where \mathcal{E} is a free Poisson extension of B over A is called a Poisson cotangent complex of B over A .*

Definition 9.16 *We say that a Poisson homomorphism $A \rightarrow R$ has property (L) if the following condition holds: let $A \rightarrow S$ be a Poisson homomorphism and $u : M \rightarrow S$ a homomorphism of $\mathcal{U}_{Pois(S)}$ -modules such that $u(x)y = u(y)x$ and $X_{u(x)}y + X_{u(y)}x = 0$ for $x, y \in M$. Then for any Poisson homomorphism $f, g : R \rightarrow S$ such that the following commutative diagram holds and $Im(f - g) \subset Im(u)$, there exists a Poisson biderivation $\lambda : R \rightarrow M$ such that $u \circ \lambda = f - g$.*

$$\begin{array}{ccc}
M & \xrightarrow{u} & S \\
\swarrow \lambda & & \uparrow f \quad \uparrow g \\
& & R
\end{array}$$

Here a Poisson biderivation means $\lambda(xy) = \lambda(x)g(y) + f(g)\lambda(y)$ and $\lambda(\{x, y\}) = \{\lambda(x), g(y)\} + \{f(x), \lambda(y)\}$. Recall that M is a $(\mathcal{U}_{Pois(S)} - \mathcal{U}_{Pois(S)})$ -bimodule, where right-module structure is defined by $m \cdot s := s \cdot m$ and $m \cdot X_s := -X_s \cdot m$ for $s \in S$.

Proposition 9.17 *Consider the following commutative diagram of Poisson homomorphisms,*

$$\begin{array}{ccc}
B & \xrightarrow{b} & B' \\
\uparrow & & \uparrow \\
A & \xrightarrow{a} & A'
\end{array}$$

Let \mathcal{E} be an Poisson extension of B over A and \mathcal{E}' be a Poisson extension of B' over A' . Let $\alpha, \beta : \mathcal{E} \rightarrow \mathcal{E}'$ be a homomorphism extending b :

$$\begin{array}{ccccccccc}
0 & \longrightarrow & E'_2 & \xrightarrow{e'_2} & E'_1 & \xrightarrow{e'_1} & R' & \xrightarrow{e'_0} & B' & \longrightarrow & 0 \\
& & \uparrow \beta_2 & \uparrow \alpha_2 & \uparrow \beta_1 & \uparrow \alpha_1 & \uparrow \beta_0 & \uparrow \alpha_0 & \uparrow b & & \\
0 & \longrightarrow & E_2 & \xrightarrow{e_2} & E_1 & \xrightarrow{e_1} & R & \xrightarrow{e_0} & B & \longrightarrow & 0
\end{array}$$

If R has property (L), then $\bar{\alpha}$ and $\bar{\beta}$ are homotopic maps of $\mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(B)}} PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$.

Proof. There exists a Poisson biderivation $\lambda : R \rightarrow E'_1$ such that $e'_1 \circ \lambda = \beta_0 - \alpha_0$. Let $\theta : E_1 \rightarrow E'_1$ be the map $\theta = (\beta_1 - \alpha_1) - \lambda \circ e_1$. We note that $e'_1 \circ \theta = e'_1 \circ (\beta_1 - \alpha_1 - \lambda \circ e_1) = (\beta_0 - \alpha_0) \circ e_1 - (\beta_0 - \alpha_0) \circ e_1 = 0$. Thus $Im(\theta)$ is in the $\mathcal{U}_{Pois(B')}$ -module $Im(e'_2) \cong E'_2$. Then $(e'_0 \circ \alpha_0(r) - e'_0 \circ \beta_0(r))x = (e_0 \circ b(r) - e_0 \circ b(r))x = 0$ and $\{e'_0 \circ \alpha_0(r) - e'_0 \circ \beta_0(r), x\} = 0$ for $x \in E'_2$. Hence on E'_2 (so $Im(\theta)$) as an $\mathcal{U}_{Pois(R)}$ -module, the action of R via α_0 and β_0 coincides. We claim that θ is Poisson R -linear or equivalently $\mathcal{U}_{Pois(R)}$ -linear. In other words, $\theta(rx) = \beta_0(r)\theta(x) = \alpha_0(r)\theta(x)$ and $\theta(\{r, x\}) = \{\beta_0(r), \theta(x)\} = \{\alpha_0(r), \theta(x)\}$ for $r \in R$ and $x \in E_1$. Indeed,

$$\begin{aligned}
\theta(rx) &= \beta_0(r)\beta_1(x) - \alpha_0(r)\alpha_1(x) - \lambda(re_1(x)) \\
&= \beta_0(r)\beta_1(x) - \alpha_0(r)\alpha_1(x) - \lambda(r)\alpha_0(e_1(x)) - \beta_0(r)\lambda(e_1(x)) \\
\beta_0(r)\theta(x) &= \beta_0(r)\beta_1(x) - \beta_0(r)\alpha_1(x) - \beta_0(r)\lambda(e_1(x)) \\
\theta(rx) - \beta_0(r)\theta(x) &= (\beta_0(r) - \alpha_0(r))\alpha_1(x) - \lambda(r)\alpha_0(e_1(x)) = e'_1(\lambda(r))\alpha_1(x) - \lambda(r)e'_1(\alpha_1(x)) = 0
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\theta(\{r, x\}) &= \{\beta_0(r), \beta_1(x)\} - \{\alpha_0(r), \alpha_1(x)\} - \lambda(\{r, e_1(x)\}) \\
&= \{\beta_0(r), \beta_1(x)\} - \{\alpha_0(r), \alpha_1(x)\} - \{\lambda r, \alpha_0(e_1(x))\} + \{\beta_0(r), \lambda(e_1(x))\} \\
\{\beta_0(r), \theta(x)\} &= \{\beta_0(r), \beta_1(x)\} - \{\beta_0(r), \alpha_1(x)\} - \{\beta_0(r), \lambda(e_1(x))\} \\
\theta(\{r, x\}) - \{\beta_0(r), \theta(x)\} &= \{\beta_0(r) - \alpha_0(r), \alpha_1(x)\} - \{\lambda r, \alpha_0(e_1(x))\} = X_{e'_1(\lambda r)}\alpha_1(x) + X_{e'_1(\alpha_1(x))}\lambda r = 0
\end{aligned}$$

Note that the Poisson biderivation $\lambda : R \rightarrow E'_1$ induces a Poisson derivation

$$1 \otimes \lambda : R \rightarrow \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R')}} E'_1$$

since the action induced by α_0 and β_0 coincides. Then by the universal mapping property of $\Omega^1_{Pois(R)/A}$, there is a $\bar{\lambda} : \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R)}} \Omega^1_{\mathcal{U}_{Pois(R)/A}} \rightarrow \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R')}} E'_1$ such that $\bar{\lambda}(b \otimes dx) = b \otimes \lambda x$. On the other hand, the Poisson R -module map $\theta : E_1 \rightarrow Im(e'_2) \cong E'_2$ induces a $\bar{\theta} : \mathcal{U}_{Pois(B')} \otimes_{\mathcal{U}_{Pois(R)}} E_1 \rightarrow E'_2$

$$\begin{array}{ccccc}
E'_2 & \xrightarrow{e'_2} & \mathcal{U}_{Pois(B')} \otimes \mathcal{U}_{Pois(R')} & E'_1 & \xrightarrow{f' := 1 \otimes d' \circ e'_1} & \mathcal{U}_{Pois(B')} \otimes \mathcal{U}_{Pois(R')} & \Omega^1_{\mathcal{U}_{Pois(R)}/A} \\
\uparrow \beta_2 - \bar{\alpha}_2 & \nearrow \bar{\theta} & \uparrow \beta_1 - \bar{\alpha}_1 & \nearrow \bar{\lambda} & \uparrow \beta_0 - \bar{\alpha}_0 & & \\
\mathcal{U}_{Pois(B')} \otimes \mathcal{U}_{Pois(R)} & E_2 & \xrightarrow{1 \otimes e_2} & \mathcal{U}_{Pois(B')} \otimes \mathcal{U}_{Pois(R)} & E_1 & \xrightarrow{f := 1 \otimes d \circ e_1} & \mathcal{U}_{Pois(B')} \otimes \mathcal{U}_{Pois(R)} & \Omega^1_{\mathcal{U}_{Pois(R)}/A}
\end{array}$$

where $d : R \rightarrow \mathcal{U}_{Pois(R)/A}$ and $d' : R' \rightarrow \mathcal{U}_{Pois(R')/A}$ are the canonical maps. Now we claim that $\bar{\beta}_2 - \bar{\alpha}_2 = \bar{\theta} \circ (1 \otimes e_2)$, $\bar{\beta}_1 - \bar{\alpha}_1 = e'_2 \circ \bar{\theta} + \bar{\lambda} \circ f'$ and $\bar{\beta}_0 - \bar{\alpha}_0 = f' \circ \bar{\lambda}$. For $\bar{\beta}_2 - \bar{\alpha}_2 = \bar{\theta} \circ (1 \otimes e_2)$, we note that for $x \in E_2$, $\theta(e_2(x)) = (\beta_1 - \alpha_1)(e_2(x)) - \lambda(e_1(e_2(x))) = e'_2 \circ (\beta_2 - \alpha_2)(x)$. For $\bar{\beta}_0 - \bar{\alpha}_0 = f' \circ \bar{\lambda}$, we note that since $e'_1 \circ \lambda = \beta_0 - \alpha_0$, we have $f' \circ \bar{\lambda}(b \otimes dx) = f'(b \otimes \lambda x) = b \otimes (d' \circ e'(\lambda x)) = b \otimes d'((\beta_0 - \alpha_0)(x)) = (\bar{\beta}_0 - \bar{\alpha}_0)(b \otimes dx)$. For $\bar{\beta}_1 - \bar{\alpha}_1 = e'_2 \circ \bar{\theta} + \bar{\lambda} \circ f'$, we note that for $x \in E_1$, $\theta(x) = (\beta_1 - \alpha_1)(x) - \lambda(e_1(x))$. ■

Lemma 9.18 *A free Poisson algebra $A\{x_i\}$ over a Poisson algebra A generated by $\{x_i\}$ satisfies the property (L).*

Proof. Let $A \rightarrow S$ be a Poisson homomorphism of Poisson algebras and $u : M \rightarrow S$ be a homomorphism of $\mathcal{U}_{Pois(S)}$ such that $u(x)y = u(y)x$ and $X_{u(x)}y + X_{u(y)}x = 0$, for all $x, y \in M$. Let $f, g : R \rightarrow S$ be Poisson homomorphism compatible with A such that $Im(f - g) \subset Im(u)$. We would like to define a Poisson biderivation $d : R \rightarrow M$ such that $u \circ d = f - g$. Let $R = A\{x_i\}$. Since $Im(f - g) \subset Im(u)$, we define $d(x_i)$ in M satisfying $u(d(x_i)) = f(x_i) - g(x_i)$ and $d(a) = 0$. In general we define d in the following way: for example

$$\begin{aligned}
d(x_1 x_2 [x_3, a x_4 x_5]) &:= d(x_2)g(x_2)[g(x_3), g(a)g(x_4)g(x_5)] + f(x_1)d(x_2)[g(x_3), g(a)g(x_4)g(x_5)] \\
&\quad + f(x_1)f(x_2)[d(x_3), g(a)g(x_4)g(x_5)] + f(x_1)f(x_2)[f(x_3), d(a), g(x_4)g(x_5)] \\
&\quad + f(x_1)f(x_2)[f(x_3), f(a)d(x_2)g(x_5)] + f(x_1)f(x_2)[f(x_3), f(a)f(x_4)d(x_5)]
\end{aligned}$$

where $[-, -]$ is the Poisson bracket on $A\{x_i\}$. We will show that this is well-defined and so by definition, d is a Poisson biderivation. Let L to be the free Lie algebra generated by A and x_i . First we show that d is well-defined on L . Simply we define d on free algebra generated by A and x_i by the above relation, where the operation is $[-, -]$. We show that $d([x, x]) = 0$ and

$d([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) = 0$ where $x, y, z \in A \cup \{x_i\}$. Then d is well-defined on L . Indeed, for $[x, x]$, we have to show $[dx, g(x)] + [f(x), dx] = 0$. Since $X_{u(dx)}dx + X_{ud(x)}dx = 0$, we have $[f(x) - g(x), dx] + [f(x) - g(x), dx] = 0$. Hence we have $[f(x) - g(x), dx] = 0$. So $[dx, g(x)] + [f(x), dx] = 0$.

For $[x, [y, z]] + [y, [z, x]] + [z, [x, y]]$, we want to show that $d([x, [y, z]] + [y, [z, x]] + [z, [x, y]]) = 0$, equivalently,

$$\begin{aligned} & [dx, [g(y), g(z)]] + [f(x), [dy, g(z)]] + [f(x), [f(y), dz]] \\ & + [dy, [g(z), g(x)]] + [f(y), [dz, g(x)]] + [f(y), [f(z), dx]] \\ & + [dz, [g(x), g(y)]] + [f(z), [dx, g(y)]] + [f(z), [f(x), dy]] = 0 \end{aligned}$$

For $[x, [y, z]]$, we note that $X_{u(d(x))}d([y, z]) + X_{u(d([y, z]))}d(x) = [f(x) - g(x), [dy, g(z)]] + [f(y), dz] + [[f(y) - g(y), g(z)], dx] + [[f(y), f(z) - g(z)], dx] = [f(x), [dy, g(z)]] - [g(x), [dy, g(z)]] + [f(x), [f(y), dz]] - [g(x), [f(y), dz]] + [[f(y), g(z)], dx] - [[g(y), g(z)], dx] + [[f(y), f(z)], dx] - [[f(y), g(z)], dx] = [f(x), [dy, g(z)]] - [g(x), [dy, g(z)]] + [f(x), [f(y), dz]] - [g(x), [f(y), dz]] - [[g(y), g(z)], dx] + [[f(y), f(z)], dx]. Hence we get the following. (however, we do not use this relationship for $[x, [y, z]]$. We do this for the symmetric arguments $[y, [z, x]]$ and $[z, [x, y]]$ in the below.)$

$$\begin{aligned} X_{u(d(x))}d([y, z]) + X_{u(d([y, z]))}d(x) &= [f(x), [dy, g(z)]] - [g(x), [dy, g(z)]] \\ &+ [f(x), [f(y), dz]] - [g(x), [f(y), dz]] \\ &- [[g(y), g(z)], dx] + [[f(y), f(z)], dx] = 0 \end{aligned}$$

For $[y, [z, x]]$, by symmetry we have

$$\begin{aligned} X_{u(d(y))}d([z, x]) + X_{u(d([z, x]))}d(y) &= [f(y), [dz, g(x)]] - [g(y), [dz, g(x)]] \\ &+ [f(y), [f(z), dx]] - [g(y), [f(z), dx]] \\ &- [[g(z), g(x)], dy] + [[f(z), f(x)], dy] = 0 \end{aligned}$$

For $[z, [x, y]]$, by symmetry we have

$$\begin{aligned} X_{u(d(z))}d([x, y]) + X_{u(d([x, y]))}d(x) &= [f(z), [dx, g(y)]] - [g(z), [dx, g(y)]] \\ &+ [f(z), [f(x), dy]] - [g(z), [f(x), dy]] \\ &- [[g(x), g(y)], dz] + [[f(x), f(y)], dz] = 0 \end{aligned}$$

Then we have

$$\begin{aligned}
& [dx, [g(y), g(z)]] + [f(x), [dy, g(z)]] + [f(x), [f(y), dz]] \\
& + [dy, [g(z), g(x)]] + [f(y), [dz, g(x)]] + [f(y), [f(z), dx]] \\
& + [dz, [g(x), g(y)]] + [f(z), [dx, g(y)]] + [f(z), [f(x), dy]] \\
& = [dx, [g(y), g(z)]] + [f(x), [dy, g(z)]] + [f(x), [f(y), dz]] \\
& + [g(y), [dz, g(x)]] + [g(y), [f(z), dx]] - [[f(z), f(x)], dy] \text{ from } X_{u(d(z))}d([z, y]) + X_{u(d([x, y]))}d(x) = 0 \\
& + [g(z), [dx, g(y)]] + [g(z), [f(x), dy]] - [[f(x), f(y)], dz] \text{ from } X_{u(d(z))}d([z, y]) + X_{u(d([x, y]))}d(x) = 0 \\
& = - [g(y), [g(z), dx]] + [g(z), [g(y), dx]] - [dy, [g(z), f(x)]] - [g(z), [f(x), dy]] - [f(y), [dz, f(x)]] - [dz, [f(x), f(y)]] \\
& + [g(y), [dz, g(x)]] + [g(y), [f(z), dx]] - [[f(z), f(x)], dy] \\
& + [g(z), [dx, g(y)]] + [g(z), [f(x), dy]] - [[f(x), f(y)], dz] \\
& = - [g(y), [g(z), dx]] - [dy, [g(z), f(x)]] - [f(y), [dz, f(x)]] \\
& + [g(y), [dz, g(x)]] + [g(y), [f(z), dx]] - [[f(z), f(x)], dy] \\
& = [g(y), [f(z) - g(z), dx]] + [dy, [f(z) - g(z), f(x)]] - [[f(y), [dz, f(x)]] + [g(y), [dz, g(x)]] \\
& = [g(y), [f(z) - g(z), dx]] - [f(z) - g(z), [f(x), dy]] - [f(x), [dy, f(z) - g(z)]] - [[f(y), [dz, f(x)]] + [g(y), [dz, g(x)]]
\end{aligned}$$

On the other hand, from $[g(y), X_{u(dz)}dx + X_{u(dx)}dz] = 0$, we have

$$[g(y), [f(z) - g(z), dx]] + [g(y), [f(x) - g(x), dz]] = 0$$

From $[f(x), X_{u(dz)}dy + X_{u(dy)}dz] = 0$, we have

$$[f(x), [f(z) - g(z), dy]] + [f(x), [f(y) - g(y), dz]] = 0$$

Hence we have

$$\begin{aligned}
& [g(y), [f(z) - g(z), dx]] - [f(z) - g(z), [f(x), dy]] - [f(x), [dy, f(z) - g(z)]] - [[f(y), [dz, f(x)]] + [g(y), [dz, g(x)]] \\
& = - [g(y), [f(x) - g(x), dz]] - [f(z) - g(z), [f(x), dy]] - [f(x), [f(y) - g(y), dz]] \\
& \quad - [[f(y), [dz, f(x)]] + [g(y), [dz, g(x)]] \\
& = - [g(y), [f(x), dz]] - [f(z) - g(z), [f(x), dy]] - [f(x), [f(y) - g(y), dz]] - [[f(y), [dz, f(x)]] \\
& = - [g(y), [f(x), dz]] - [f(z) - g(z), [f(x), dy]] - [f(x), [f(y), dz]] + [f(x), [g(y), dz]] - [[f(y), [dz, f(x)]] \\
& = [[f(x), g(y)], dz] - [f(z) - g(z), [f(x), dy]] - [[f(x), f(y)], dz]
\end{aligned}$$

From $X_{u(dz)}[f(x), dy] + X_{u([f(x), dy])}dz = 0$, we have

$$\begin{aligned}
& [f(z) - g(z), [f(x), dy]] + [[f(x), f(y) - g(y)], dz] \\
& = [f(z) - g(z), [f(x), dy]] + [[f(x), f(y)], dz] - [[f(x), g(y)], dz] = 0
\end{aligned}$$

Hence we have

$$[[f(x), g(y)], dz] - [f(z) - g(z), [f(x), dy]] - [[f(x), f(y)], dz] = 0$$

Hence d is well-defined on L . Since $(f - g)([x, y]) = [f(x), (f - g)(x)] + [(f - g)(x), g(x)]$, we have $u \circ d = f - g$ on L . Now we show that d is well-defined on the free commutative algebra S generated by L . Let $V = \{T_j\}$ be a basis of L over k . Then we define d on S by the following formula, for a monomial $T_{i_1} \cdots T_{i_n}$,

$$d(T_{i_1} \cdots T_{i_n}) = \sum_{k=1}^n f(T_{i_1} \cdots T_{i_{k-1}})d(T_{i_k})g(T_{i_{k+1}} \cdots T_{i_n}).$$

Then d is well-defined on S and we have $u \circ d = f - g$ on S (See [LS67] Lemma 2.1.6). We define a bracket $[-, -]_S$ on S such that $[T_i, T_j]_S := [T_i, T_j]$ and we use the relation $[x, yz] = y[x, z] + z[x, y]$ for $x, y, z \in L$. Then from $[x, yz] - y[x, z] - z[x, y]$ for $x, y, z \in S$, we want to show that $d([x, yz]) = d(y[x, z] + z[x, y])$, equivalently,

$$\begin{aligned} [dx, g(y)g(z)] + [f(x), dyg(z)] + [f(x), f(y)dz] &= dy[g(x), g(z)] + f(y)[dx, g(z)] + f(y)[f(x), dz] \\ &\quad + dz[g(x), g(y)] + f(z)[dx, g(y)] + f(z)[f(x), dy] \end{aligned}$$

Equivalently,

$$\begin{aligned} g(y)[dx, g(z)] + g(z)[dx, g(y)] + dy[f(x), g(z)] + g(z)[f(x), dy] + dz[f(x), f(y)] + f(y)[f(x), dz] \\ = dy[g(x), g(z)] + f(y)[dx, g(z)] + f(y)[f(x), dz] + dz[g(x), g(y)] + f(z)[dx, g(y)] + f(z)[f(x), dy] \end{aligned}$$

We note that

$$\begin{aligned} u(dy)d[x, z] - u(d[z, x])dy &= (f(y) - g(y))([dx, g(z)] + [f(x), dz]) - ([f(x) - g(x), g(z)] + [f(x), f(z) - g(z)])dy \\ &= f(y)[dx, g(z)] - g(y)[dx, g(z)] + f(y)[f(x), dz] - g(y)[f(x), dz] + [g(x), g(z)]dy - [f(x), f(z)]dy = 0 \end{aligned}$$

We also note that

$$\begin{aligned} u(dz)d[x, y] - u(d[y, x])dz &= (f(z) - g(z))([dx, g(y)] + [f(x), dy]) - ([f(x) - g(x), g(y)] + [f(x), f(y) - g(y)])dz \\ &= f(z)[dx, g(y)] - g(z)[dx, g(y)] + f(z)[f(x), dy] - g(z)[f(x), dy] + [g(x), g(y)]dz - [f(x), f(y)]dz = 0 \end{aligned}$$

So we have

$$\begin{aligned} dy[g(x), g(z)] + f(y)[dx, g(z)] + f(y)[f(x), dz] + dz[g(x), g(y)] + f(z)[dx, g(y)] + f(z)[f(x), dy] \\ = g(y)[dx, g(z)] + g(y)[f(x), dz] + [f(x), f(z)]dy + g(z)[dx, g(y)] + g(z)[f(x), dy] + [f(x), f(y)]dz \end{aligned}$$

$$\begin{aligned}
& g(y)[dx, g(z)] + g(z)[dx, g(y)] + dy[f(x), g(z)] + g(z)[f(x), dy] + dz[f(x), f(y)] + f(y)[f(x), dz] \\
& -g(y)[dx, g(z)] - g(y)[f(x), dz] - [f(x), f(z)]dy - g(z)[dx, g(y)] - g(z)[f(x), dy] - [f(x), f(y)]dz \\
& = dy[f(x), g(z)] + f(y)[f(x), dz] - g(y)[f(x), dz] - [f(x), f(z)]dy \\
& = (f(y) - g(y))[f(x), dz] - [(f(x), (f(z) - g(z)))]dy \\
& = u(dy)[f(x), dz] - u([f(x), dz])dy = 0
\end{aligned}$$

Lastly since $da = 0$ for all $a \in A$, d is well-defined on $A\{x_i\}$, and d is a Poisson biderivation and $u \circ d = f - g$.

■

Corollary 9.19 *Consider the following commutative diagram of Poisson homomorphisms*

$$\begin{array}{ccc}
B & \xrightarrow{b} & B' \\
\uparrow & & \uparrow \\
A & \xrightarrow{a} & A'
\end{array}$$

Let \mathcal{E} an free Poisson extension of B over A and \mathcal{E}' be a Poisson extension of B' over A' . Then there exists a homomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ extending b . If $\beta : \mathcal{E} \rightarrow \mathcal{E}'$ is any other homomorphism extending b , then $\bar{\alpha}$ and $\bar{\beta}$ are homotopic maps of $\mathcal{U}_{\text{Pois}(B')} \otimes_{\mathcal{U}_{\text{Pois}(B)}} PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$.

Definition 9.20 *Let $A \rightarrow B$ a Poisson homomorphism, \mathcal{E} be a free extension of B over A , and M be a Poisson B -module. We define $PT^i(B/A, M) := H^i(\text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^\bullet(\mathcal{E}), M)), i = 0, 1, 2$. Since $\text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(-, M)$ is an contravariant additive functor and any two Poisson cotangent complexes $PL^\bullet(\mathcal{E})$ and $PL^\bullet(\mathcal{F})$ induced from two free Poisson extensions \mathcal{E} and \mathcal{F} of B over A are homotopically equivalent, and so $\text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^\bullet(\mathcal{E}), M)$ is homotopically equivalent to $\text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^\bullet(\mathcal{F}), M)$. Hence $PT^i(B/A, M)$ is well-defined.*

Proposition 9.21 *Let $A \rightarrow B$ be Poisson homomorphism of Poisson algebras over k . Then for $i = 0, 1, 2$, $PT^i(B/A, \cdot)$ is a covariant, additive functor from the category of Poisson B -modules to category of B -modules. If $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$ is a short exact sequence of Poisson B -modules,*

then there is a long exact sequence

$$\begin{aligned}
0 &\rightarrow PT^0(B/A, M') \rightarrow PT^0(B/A, M) \rightarrow PT^0(B/A, M'') \rightarrow \\
&\rightarrow PT^1(B/A, M') \rightarrow PT^1(B/A, M) \rightarrow PT^1(B/A, M'') \rightarrow \\
&\rightarrow PT^2(B/A, M') \rightarrow PT^2(B/A, M) \rightarrow PT^2(B/A, M'')
\end{aligned}$$

Proof. By construction, $PT^i(B/A, \cdot)$ is a covariant additive functor. Let $\mathcal{E} : 0 \rightarrow U/U_0 \rightarrow F/U_0 \rightarrow R = A\{x_i\} \rightarrow B \rightarrow 0$ be a free Poisson extension of B over A . Then $PL^0 = \mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(R)/A}} \Omega^1_{\mathcal{U}_{Pois(R)/A}}$ which is free $\mathcal{U}_{Pois(B)}$ -module and $PL^1 = \mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(R)}} F$ which is a free $\mathcal{U}_{Pois(B)}$ -module and $PL^2 = U/U_0$. Let's consider the induced diagram.

$$\begin{array}{ccccccc}
0 & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^0, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^0, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^0, M'') \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^1, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^1, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^1, M'') \longrightarrow 0 \\
& & \downarrow & & \downarrow & & \downarrow \\
0 & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^2, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^2, M') & \longrightarrow & Hom_{\mathcal{U}_{Pois(B)}}(PL^2, M'')
\end{array}$$

First row and second row are exact since PL^0 and PL^1 are free, and third row is exact for the first two terms by the right exactness of $Hom_{\mathcal{U}_{Pois(B)}}(PL^2, \cdot)$. Hence we get the proposition. ■

Proposition 9.22 For any Poisson homomorphism $A \rightarrow B$ and any Poisson B -module M , $PT^0(B/A, M) = Hom_{\mathcal{U}_{Pois(B)}}(\Omega^1_{\mathcal{U}_{Pois(B)/A}}, M) = PDer_A(B, M)$. In particular, $PT^0(B/A, B) = Hom_{\mathcal{U}_{Pois(B)}}(\Omega^1_{\mathcal{U}_{Pois(B)/A}}, B) = PDer_A(B, B)$

Proof. Let's consider an exact sequence $0 \rightarrow I \rightarrow A\{x_i\} \xrightarrow{\phi} B \rightarrow 0$ for some free Poisson algebra $A\{x_i\}$ over A and ϕ is a Poisson homomorphism compatible with A . Then we have an exact sequence

$$\mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(A\{x_i\})/A}} I \cong I/(I^2 \oplus \{I, I\}) \rightarrow \mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(A\{x_i\})}} \Omega^1_{\mathcal{U}_{Pois(A\{x_i\})/A}} \rightarrow \Omega^1_{\mathcal{U}_{Pois(B)/A}}$$

Let's choose a free Poisson $\mathcal{U}_{Pois(A\{x_i\})}$ -module F such that $F \rightarrow I \rightarrow 0$. Then we have an exact sequence

$$PL^1 \rightarrow PL^0 \rightarrow \Omega^1_{\mathcal{U}_{Pois(B)/A}}$$

By taking $Hom_{\mathcal{U}_{Pois(B)}}(\cdot, M)$ which is a right exact functor, we have

$$PT^0(B/A, M) = Hom_{\mathcal{U}_{Pois(B)}}(\Omega^1_{\mathcal{U}_{Pois(B)/A}}, M) = PDer_A(B, M)$$

■

Proposition 9.23 *If $A \rightarrow B$ is a surjective Poisson algebra homomorphism with kernel I , then $PT^0(B/A, M) = 0$ for all M , and $PT^1(B/A, M) = \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I/I^2 \oplus \{I, I\}, M)$. In particular $PT^1(B/A, B) = \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I/I^2 \oplus \{I, I\}, B)$*

Proof. We take $R = A$ as a free Poisson extension with no generating set over A . Since $\Omega^1_{\mathcal{U}_{\text{Pois}(A)}/A} = 0$, we have $PT^0(B/A, M) = 0$. Let's consider the exact sequence $0 \rightarrow U \rightarrow F \xrightarrow{j} I \rightarrow 0$. Let U_0 be $\mathcal{U}_{\text{Pois}(R)}$ -submodule of F generated by $j(x)y - j(y)x$ and $X_{j(x)}y + X_{j(y)}x$, where $x, y \in F$. Then $0 \rightarrow U/U_0 \rightarrow F/U_0 \rightarrow I \rightarrow 0$ is exact. So we have an exact sequence.

$$\mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(A)}} U/U_0 \rightarrow \mathcal{U}_{\text{Pois}(B)} \otimes_{\mathcal{U}_{\text{Pois}(A)}} F/U_0 \rightarrow I/I^2 \oplus \{I, I\} \rightarrow 0$$

Hence we have $PT^1(B/A, M) = \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I/I^2 \oplus \{I, I\}, M)$. ■

Proposition 9.24 *Given a commutative diagram*

$$\begin{array}{ccc} B & \xrightarrow{b} & B' \\ \uparrow & & \uparrow \\ A & \xrightarrow{a} & A' \end{array}$$

where every morphisms are Poisson homomorphisms. Let M' be a Poisson B' -module. Then there are natural homomorphism

$$PT^i(B'/A', M') \rightarrow PT^i(B/A, M')$$

Proof. Let $\mathcal{E} : 0 \rightarrow F_2 \rightarrow F_1 \rightarrow P \rightarrow B \rightarrow 0$ be a free Poisson extension of B over A and $\mathcal{E}' : 0 \rightarrow F'_2 \rightarrow F'_1 \rightarrow P' \rightarrow B' \rightarrow 0$ be a free Poisson extension of B' over A' . Then we have a homomorphism $\alpha : \mathcal{E} \rightarrow \mathcal{E}'$ extending $b : B \rightarrow B'$ and this induces $\bar{\alpha} : PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$ (hence $\mathcal{U}_{\text{Pois}(B')} \otimes_{\mathcal{U}_{\text{Pois}(B)}} PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$) and so we have

$$\begin{array}{ccccc} \text{Hom}_{\mathcal{U}_{\text{Pois}(B')}}(PL^0, M') & \longrightarrow & \text{Hom}_{\mathcal{U}_{\text{Pois}(B')}}(PL^1, M') & \longrightarrow & \text{Hom}_{\mathcal{U}_{\text{Pois}(B')}}(PL^2, M') \\ \downarrow & & \downarrow & & \downarrow \\ \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^0, M') & \longrightarrow & \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^1, M') & \longrightarrow & \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(PL^2, M') \end{array}$$

So this induces $PT^i(B'/A', M') \rightarrow PT^i(B/A, M')$. The construction is independent of the choices of \mathcal{E} and \mathcal{E}' . Indeed, Let \mathcal{F} be another free Poisson extension of B over A and \mathcal{F}' be another free Poisson extension of B' over A' . Let $\bar{\beta} : PL^\bullet(\mathcal{F}) \rightarrow PL^\bullet(\mathcal{F}')$ a homomorphism extending $B \rightarrow B'$ similarly to $PL^\bullet(\mathcal{E}) \rightarrow PL^\bullet(\mathcal{E}')$ as above. Let $\bar{\gamma} : PL^\bullet(\mathcal{F}) \rightarrow PL^\bullet(\mathcal{E})$ and $\bar{\gamma}' : PL^\bullet(\mathcal{E}') \rightarrow PL^\bullet(\mathcal{F}')$ be homotopically equivalent maps. Then in the diagram

$$\begin{array}{ccc} PL^\bullet(\mathcal{E}) & \xrightarrow{\bar{\alpha}} & PL^\bullet(\mathcal{E}') \\ \bar{\gamma} \uparrow & & \downarrow \bar{\gamma}' \\ PL^\bullet(\mathcal{F}) & \xrightarrow[\bar{\gamma}' \circ \bar{\alpha} \circ \bar{\gamma}]{\bar{\beta}} & PL^\bullet(\mathcal{F}') \end{array}$$

$\bar{\beta}$ and $\bar{\gamma}' \circ \bar{\alpha} \circ \bar{\gamma}$ are homotopic maps since \mathcal{F} is a free Poisson extension. Hence the induced maps $PT^i(B'/A', M') \rightarrow PT^i(B/A, M)$ are equal.

■

Definition 9.25 (Short Poisson extension) *Let $A \rightarrow B$ a Poisson homomorphism of Poisson k -algebras, k a field, and M be a Poisson B -module. By a short Poisson extension of B over A by M , we mean an exact sequence:*

$$0 \rightarrow M \xrightarrow{i} E \xrightarrow{k} B \rightarrow 0$$

where $A \rightarrow E$ is an Poisson homomorphism of Poisson algebras and $k : E \rightarrow B$ a Poisson homomorphism compatible with A (i.e $A \rightarrow B$ factor through E) and M is regarded as a square zero Poisson ideal in E , i.e $i(M) \cdot i(M) = 0$ and $\{i(M), i(M)\} = 0$. We note that a short Poisson extension is a Poisson extension since M is a $\mathcal{U}_{\text{Pois}(E)}$ -module (the $\mathcal{U}_{\text{Pois}(E)}$ -module structure is induced from $\mathcal{U}_{\text{Pois}(B)}$ -module structure via k), and so $i(x)y - i(y)x$ and $X_{i(x)}y + X_{i(y)}x$ are trivially 0 because $ki(x) = 0$ for all $x \in M$. Hence $0 \rightarrow 0 \rightarrow M \rightarrow E \rightarrow B \rightarrow 0$ is a Poisson extension. If E' is another short Poisson extension of B over A by M , we say that E and E' are equivalent if there exists a Poisson homomorphism $\theta : E \rightarrow E'$ compatible with A inducing the following commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M & \xrightarrow{i} & E & \xrightarrow{k} & B & \longrightarrow & 0 \\ & & & & \parallel & & \parallel & & \\ & & & & \theta \downarrow & & & & \\ 0 & \longrightarrow & M & \xrightarrow{i'} & E' & \xrightarrow{k'} & B & \longrightarrow & 0 \end{array}$$

Definition 9.26 *We define $PEx^1(B/A, M)$ be the set of equivalence classes of short Poisson extensions of B over A by M .*

Lemma 9.27 *Let $0 \rightarrow M \xrightarrow{i} E \xrightarrow{k} B \rightarrow 0$ be a short Poisson extension of B over A by M as above. Given an Poisson homomorphism $A \rightarrow C$ and two Poisson homomorphisms $f_1, f_2 : C \rightarrow E$ compatible with A such that $kf_1 = kf_2$, the induced map $f_2 - f_1 : C \rightarrow M$ is an Poisson A -derivation.*

Proof. We assume that i is an inclusion. First we note that M is an Poisson B -module. We define C -module structure on M by setting $c \cdot m := f_1(c)m = f_2(c)m$ which is well-defined

since $M^2 = 0$. We define an Poisson C -module structure on M by setting $\{c, m\} := \{f_1(c), m\} = \{f_2(c), m\}$ which is well-defined since $\{M, M\} = 0$. Clearly $f_2 - f_1$ is A -linear. We note that

$$\begin{aligned}
(f_2 - f_1)(c_1 c_2) &= f_2(c_1) f_2(c_2) - f_1(c_1) f_1(c_2) \\
&= f_2(c_1) f_2(c_2) - f_2(c_1) f_1(c_2) + f_2(c_1) f_1(c_2) - f_1(c_1) f_1(c_2) \\
&= f_2(c_1)(f_2(c_2) - f_1(c_2)) + f_1(c_2)(f_2(c_1) - f_1(c_1)) \\
&= c_1 \cdot (f_2 - f_1)(c_2) + c_2 \cdot (f_2 - f_1)(c_1) \\
(f_2 - f_1)(\{c_1, c_2\}) &= \{f_2(c_1), f_2(c_2)\} - \{f_1(c_1), f_1(c_2)\} \\
&= \{f_2(c_1), f_2(c_2)\} - \{f_2(c_1), f_1(c_2)\} + \{f_2(c_1), f_1(c_2)\} - \{f_1(c_1), f_1(c_2)\} \\
&= \{f_2(c_1), (f_2 - f_1)(c_2)\} + \{(f_2 - f_1)(c_1), f_1(c_2)\} \\
&= \{c_1, (f_2 - f_1)(c_2)\} - \{c_2, (f_2 - f_1)(c_1)\}
\end{aligned}$$

■

Definition 9.28 *The short Poisson extension of B over A by $M : 0 \rightarrow M \xrightarrow{i} E \xrightarrow{k} B \rightarrow 0$ is called trivial if it has a section, that is if there exists a Poisson homomorphism $\sigma : B \rightarrow E$ such that $k\sigma = 1_B$ and σ is compatible with A .*

Given an Poisson B -module M , a trivial short Poisson extension of B over A by M can be constructed by considering the Poisson algebra $B \tilde{\oplus} M$ whose underlying A -module is $B \oplus M$ with multiplication and bracket defined by

$$\begin{aligned}
(b_1, m_1)(b_2, m_2) &= (b_1 b_2, b_1 m_2 + b_2 m_1) \\
\{(b_1, m_1), (b_2, m_2)\} &= (\{b_1, b_2\}, -\{b_2, m_1\} + \{b_1, m_2\})
\end{aligned}$$

The first projection

$$p : B \tilde{\oplus} M \rightarrow B$$

is a Poisson homomorphism compatible with A and defines an Poisson extension of B over A by M .

A section of p can be identified with a Poisson A -derivations $d : B \rightarrow M$. Indeed, if we have a section $\sigma : B \rightarrow B \tilde{\oplus} M$ with $\sigma(b) = (b, d(b))$, then for all $b, b' \in B$

$$\begin{aligned}
\sigma(bb') &= (bb', d(bb')) = \sigma(b)\sigma(b') = (b, d(b))(b', d(b')) = (bb', bd(b') + b'd(b)) \\
\sigma(\{b, b'\}) &= (\{b, b'\}, d(\{b, b'\})) = \{\sigma(b), \sigma(b')\} = \{(b, d(b)), (b', d(b'))\} = (\{b, b'\}, -\{b', d(b)\} + \{b, d(b')\})
\end{aligned}$$

and if $a \in A$ then $\sigma(ab) = (ab, d(ab)) = a\sigma(b) = a(b, d(b)) = (ab, ad(b))$. Hence $d : B \rightarrow M$ is a Poisson A -derivation. Conversely, every Poisson A -derivation $d : B \rightarrow M$ defines a section $\sigma_d : B \rightarrow B \tilde{\oplus} M$ by $\sigma_d(b) = (b, d(b))$.

Proposition 9.29 *Every trivial short Poisson extension E of B by M is isomorphic to $(B\tilde{\oplus}M, p)$.*

Proof. If $\sigma : B \rightarrow E$ is a section, an isomorphism $\xi : B\tilde{\oplus}M \rightarrow E$ is given by $\xi((b, m)) = \sigma(b) + m$. We check only Poisson compatibility. $\xi(\{(b_1, m_1), (b_2, m_2)\}) = \sigma(\{b_1, b_2\}) - \{b_2, m_1\} + \{b_1, m_2\} = \{\sigma(b_1), \sigma(b_2)\} - \{b_2, m_1\} + \{b_1, m_2\}$. On the other hand, $\{(\xi((b_1, m_1))), \xi((b_2, m_2))\} = \{\sigma(b_1) + m_1, \sigma(b_2) + m_2\} = \{\sigma(b_1), \sigma(b_2)\} - \{b_2, m_1\} + \{b_1, m_2\}$. We define an inverse map $\xi^{-1}(e') = (k(e'), e' - \sigma k(e'))$. ■

Let $\mathcal{E} : 0 \rightarrow M \rightarrow E \rightarrow B \rightarrow 0$ be a short Poisson extension of B over A by M as above. Let $h : A \rightarrow E$ be the Poisson homomorphism from \mathcal{E} . Then we have by Proposition 9.23,

$$PT^1(B/E, M) = Hom_{\mathcal{U}_{Pois(B)}}(M/M^2 \oplus \{M, M\}, M) = Hom_{\mathcal{U}_{Pois(B)}}(M, M).$$

Then by Proposition 9.24, h induces

$$h^* := Hom_{\mathcal{U}_{Pois(B)}}(M, M) = PT^1(B/E, M) \rightarrow PT^1(B/A, M)$$

Theorem 14 *The assignment $\mathcal{E} \rightarrow h^*(id)$ induces a bijection*

$$\rho : PE^1(B/A, M) \rightarrow PT^1(B/A, M)$$

in which the class of the trivial short Poisson extension corresponds to 0.

Proof. Let

$$\mathcal{F} : 0 \rightarrow F_2 \rightarrow F_1 \rightarrow P \rightarrow B \rightarrow 0$$

be a fixed free Poisson extension of B over A . Given an short extension $\mathcal{E} : 0 \rightarrow M \rightarrow E \rightarrow B \rightarrow 0$, which is also an Poisson extension, we have a homomorphism $\alpha : \mathcal{F} \rightarrow \mathcal{E}$ extending the identity $B \rightarrow B$, unique up to homotopy,

$$\begin{array}{ccccccccc} 0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{j} & E & \xrightarrow{k} & B & \longrightarrow & 0 \\ & & \uparrow \alpha_2 & & \uparrow \alpha_1 & & \uparrow \alpha_0 & & \uparrow id & & \\ 0 & \longrightarrow & F_2 & \xrightarrow{\bar{i}} & F_1 & \xrightarrow{\bar{j}} & P & \xrightarrow{\bar{k}} & B & \longrightarrow & 0 \end{array}$$

Let's consider the cotangent complex of B over A

$$0 \rightarrow F_2 \xrightarrow{\bar{i}} \mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(R)}} F_1 \xrightarrow{d\circ\bar{j}} \mathcal{U}_{Pois(B)} \otimes_{\mathcal{U}_{Pois(P)}} \Omega_{\mathcal{U}_{Pois(P)/A}}^1 \rightarrow 0$$

Then we have

$$Hom_{\mathcal{U}_{Pois(P)}}(\Omega_{\mathcal{U}_{Pois(P)/A}}^1, M) \rightarrow Hom_{\mathcal{U}_{Pois(P)}}(F_1, M) \rightarrow Hom_{\mathcal{U}_{Pois(B)}}(F_2, M)$$

The class of $\alpha_1 : F_1 \rightarrow M$ is $h^*(id)$. Let $\mathcal{E}' : 0 \rightarrow M \rightarrow E' \rightarrow B \rightarrow 0$ be an equivalent short Poisson extension with \mathcal{E} given by $\theta : E \rightarrow E'$. Then we have a homomorphism $\mathcal{F} \rightarrow \mathcal{E}'$ extending the identity $B \rightarrow B$,

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{j'} & E & \xrightarrow{k'} & B & \longrightarrow & 0 \\
& & \uparrow \alpha_2 & & \uparrow \alpha_1 & & \uparrow \theta \circ \alpha_0 & & \uparrow id & & \\
0 & \longrightarrow & F_2 & \xrightarrow{\bar{i}} & F_1 & \xrightarrow{\bar{j}} & P & \xrightarrow{\bar{k}} & B & \longrightarrow & 0
\end{array}$$

We also have the same $\alpha_1 : F_1 \rightarrow M$. Hence ρ is well-defined.

Now we define the inverse map ρ^{-1} of ρ in the following way: given $e \in PT^1(B/A, M)$, choose $f : F_1 \rightarrow M$ inducing e . We put $J = (P \oplus M)/K$ where K is the Poisson ideal of $(P \oplus M)$ generated by the elements of the form $(\bar{j}(x), -f(x))$ for $x \in F_1$. We note that we have actually $K = \{(\bar{j}(x), -f(x)) | x \in F_1\}$. Indeed, let $(p, m) \in P \oplus M$. Then $(p, m) \cdot (\bar{j}(x), -f(x)) = (p \cdot \bar{j}(x), -pf(x) + \bar{j}(x)m) = (\bar{j}(px), -f(px))$ since M is a Poisson P -module via $P \xrightarrow{\bar{k}} B$, so $\bar{j}(x)m$ means $\bar{k}(\bar{j}(x))m = 0$. On the other hand $\{(p, m), (\bar{j}(x), -f(x))\} = (\{p, \bar{j}(x)\}_P, -\{p, f(x)\}_M - \{\bar{j}(x), m\}_M) = (\bar{j}(\{p, x\}_{F_1}), f(\{p, x\}_{F_1}))$ since $\{\bar{j}(x), m\}_M$ means $\{\bar{k}\bar{j}(x), m\}_M = 0$. Hence $K = \{(\bar{j}(x), -f(x)) | x \in F_1\}$.

Now we claim that the following sequence is a short Poisson extension of B over A by M ,

$$\mathcal{E} : 0 \rightarrow M \xrightarrow{j} (P \oplus M)/K \xrightarrow{k} B \rightarrow 0$$

where $j(m) :=$ the class of $(0, m) = \overline{(0, m)}$, and $k(\overline{(p, m)}) := \bar{k}(p) \in B$. k is well-defined and a surjective Poisson map since for $(p, m) = (\bar{x}, -f(x)) \in K = \{(\bar{j}(x), -f(x)) | x \in F_1\}$, $\bar{k}(p) = \bar{k}(\bar{j}(x)) = 0$. Now let $k(\overline{(p, m)}) = \bar{k}(p) = 0$. Then there exists $x \in F_1$ such that $\bar{j}(x) = p$. Then $(p, m) - (0, m + f(x)) = (p, -f(x)) = (\bar{j}(x), -f(x)) \in K$. Hence $j(m + f(x)) = \overline{(0, m + f(x))} = \overline{(p, m)}$. Hence $\ker(k) \subset \text{im}(j)$ and clearly $\text{im}(j) \subset \ker(k)$. Hence $\text{im}(j) = \ker(k)$. Now we show that j is injective. Let $\overline{(0, m)} = 0$. Then $(0, m) \in K$. So we have $0 = \bar{j}(x)$ and $m = -f(x)$ for some $x \in F_1$. Hence $x = \bar{i}(y)$ for some $y \in F_2$. Note that under the map $\text{Hom}_{\mathcal{U}_{\text{Pois}(P)}}(F_1, M) \rightarrow \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(F_2, M)$, f goes to 0 since f defines the cohomology class e , and so $f \circ \bar{i} = 0$. Hence $m = -f(x) = -f(\bar{i}(y)) = 0$. So j is injective. We have the following commutative diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & M & \xrightarrow{j} & (P \oplus M)/K & \xrightarrow{k} & B & \longrightarrow & 0 \\
& & \uparrow & & \uparrow f & & \uparrow \alpha'_0 & & \uparrow id & & \\
0 & \longrightarrow & F_2 & \xrightarrow{\bar{i}} & F_1 & \xrightarrow{\bar{j}} & P & \xrightarrow{\bar{k}} & B & \longrightarrow & 0
\end{array}$$

where $\alpha'_0(p) = \overline{(p, 0)}$. Indeed, let $x \in F_1$. $\alpha'_0(\bar{j}(x)) = \overline{(\bar{j}(x), 0)}$ and $j(f(x)) = \overline{(0, f(x))}$. $(\bar{j}(x), 0) - (0, f(x)) = (\bar{j}(x), -f(x)) \in K$. Thus $e =$ the class of $f = h^*(id)$.

Now we show that ρ^{-1} is well-defined. In other words, $\rho^{-1}(e)$ is independent of choices of f inducing $e \in T^1(B/A, M)$ up to equivalence of short Poisson extensions. Let $f, f' : F_1 \rightarrow M$ inducing e . Then there exists $v : \Omega_{\mathcal{U}_{\text{Pois}(P)/A}}^1 \rightarrow M$ such that $f' - f = v \circ d \circ \bar{j}$ where $F_1 \xrightarrow{\bar{j}} P \xrightarrow{d} \Omega_{\mathcal{U}_{\text{Pois}(P)/A}}^1 \xrightarrow{v} M$. Let \mathcal{E}' be the Poisson extension constructed from $f' : F_1 \rightarrow M$ as above,

$$0 \rightarrow M \xrightarrow{j'} (P \oplus M)/K' \xrightarrow{k'} B \rightarrow 0$$

where K' is the Poisson ideal $\{(\bar{j}(x), -f'(x)) | x \in F_1\}$. Consider an endomorphism $P \oplus M \rightarrow P \oplus M$ defined by $\varphi : (p, m) \mapsto (p, -v(d(p)) + m)$, which is one to one and onto with the inverse $\varphi^{-1}(p, m) = (p, v(d(p)) + m)$. We show that φ is a Poisson homomorphism. Indeed,

$$\begin{aligned}
\varphi((p_1, m_1)(p_2, m_2)) &= \varphi(p_1 p_2, p_1 m_2 + p_2 m_1) = (p_1 p_2, -v(p_1 d p_2 + p_2 d p_1) + p_1 m_2 + p_2 m_1) \\
&= (p_1 p_2, p_1(-v(d p_2) + m_2) + p_2(-v(d p_1) + m_1)) \\
&= (p_1, -v(d p_1) + m_1)(p_2, -v(d p_2) + m_2) \\
&= \varphi(p_1, m_1)\varphi(p_2, m_2) \\
\varphi(\{(p_1, m_1), (p_2, m_2)\}) &= \varphi(\{p_1, p_2\}_P, \{p_1, m_2\}_M - \{p_2, m_1\}_M) \\
&= (\{p_1, p_2\}_P, -v(-\{p_2, d p_1\}_M + \{p_1, d p_2\}_M) + \{p_1, m_2\}_M - \{p_2, m_1\}_M) \\
&= (\{p_1, p_2\}_P, -\{p_2, m_1 - v(d p_1)\}_M + \{p_1, m_2 - v(d p_2)\}_M) \\
&= (\{(p_1, m_1 - v(d p_1)), (p_2, m_2 - v(d p_2))\}) \\
&= \{\varphi(p_1, m_1), \varphi(p_2, m_2)\}
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\varphi((\bar{j}(x), -f(x))) &= (\bar{j}(x), -v(d(\bar{j}(x))) - f(x)) = (\bar{j}(x), -f'(x)) \\
\varphi^{-1}(-\bar{j}(x), f'(x)) &= (-\bar{j}(x), -v(d(\bar{j}(x))) + f'(x)) = (-\bar{j}(x), f(x))
\end{aligned}$$

for $x \in F_1$. Hence φ maps K to K' . Hence φ induces an isomorphism $(P \oplus M)/K \rightarrow (P \oplus M)/K'$. Hence \mathcal{E} is equivalent to \mathcal{E}' . Hence ρ^{-1} is well-defined.

Lastly we show that the class of trivial short Poisson extension $0 \rightarrow M \rightarrow B \tilde{\oplus} M \rightarrow B \rightarrow 0$ corresponds to $0 \in PT^1(B/A, M)$ via ρ . In the following diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & M & \longrightarrow & B \tilde{\oplus} M & \xrightarrow{p} & B & \longrightarrow & 0 \\
& & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
0 & \longrightarrow & F_2 & \xrightarrow{\bar{i}} & F_1 & \xrightarrow{\bar{j}} & P & \xrightarrow{\bar{k}} & B & \longrightarrow & 0
\end{array}$$

Let $q : B \tilde{\oplus} M \rightarrow M$ be the projection. Then $q \circ \alpha_0 : P \rightarrow M$ be a Poisson A -derivation with $\alpha_1 = q \circ \alpha_0 \circ \bar{j}$. Hence there exists a map $v : \Omega_{\mathcal{U}_{Pois}(P)/A}^1 \rightarrow M$ such that $q \circ \alpha_0 = v \circ d$. So we have $\alpha_1 = v \circ d \circ \bar{j}$. Hence the class of α_1 is 0. ■

9.3 First order Poisson deformations of affine Poisson schemes

Let's explain in more detail a short Poisson extension of R over A by $I : 0 \rightarrow I \xrightarrow{j} R' \xrightarrow{\phi} R \rightarrow 0$, where I is a Poisson R -module with $j(I) \cdot j(I) = 0$ and $\{j(I), j(I)\} = 0$. Here ϕ is a Poisson homomorphism of Poisson k -algebras compatible with a Poisson algebra A . Then we assume that I is a Poisson R' -module via ϕ , and so j is a Poisson R' -module homomorphism.

When we say that a short Poisson extension of R over A by R means that we have an exact sequence $0 \rightarrow R \xrightarrow{j} R' \xrightarrow{\phi} R \rightarrow 0$, where R is a natural Poisson R -module, the image of j satisfies $j(R)^2 = 0$ and $\{j(R), j(R)\} = 0$, R also have a R' -module structure via ϕ . By these R' -module structure, and j is a Poisson R' -module homomorphism. Now we show that this extension gives a $k[\epsilon]$ -Poisson algebra structure on R' . Note that j is completely determined by 1 since $j(f) = j(f \cdot 1) = j(f' \cdot 1) = f' \cdot j(1)$ where f' is a lift of f . We will give a $k[\epsilon]$ -algebra structure on R' by $\epsilon \rightarrow j(1)$ since $j(1)^2 = 0$. To show $k[\epsilon]$ -Poisson algebra structure on R' , we have to show that $\{R', \epsilon\} = 0$, equivalently $\{r, j(1)\} = 0$ for all $r \in R'$. Since j is a R' -module homomorphism, $\{r, j(1)\}_{R'} = j(X_r \cdot 1) = j(\{\phi(r), 1\}_R) = 0$.

Proposition 9.30 (compare [Har10] Theorem 5.1) *Let B_0 be a Poisson k -algebra, and let $X_0 = \text{Spec}(B_0)$. Then there is a natural isomorphism*

$$PDef_{B_0}(k[\epsilon]) \cong PEx^1(B_0/k, B_0)$$

where the class of trivial Poisson deformation corresponds to $0 \in PT_{B_0}^1$.

Proof. A first order Poisson deformation of B_0 consists of a flat Poisson $k[\epsilon]$ -algebra B with Poisson k -isomorphism $B \otimes_{k[\epsilon]} k \cong B_0$ with the following commutative diagram

$$\begin{array}{ccc} B & \xrightarrow{\phi} & B_0 \\ \uparrow & & \uparrow \\ k[\epsilon] & \longrightarrow & k \end{array}$$

where ϕ is a Poisson homomorphism over k . We note that a algebra B is flat over $k[\epsilon]$ if and only if $0 \rightarrow B_0 \otimes_k (\epsilon) \cong B_0 \rightarrow B$ is exact. (see [Har10] Proposition 2.2). So given a first order Poisson deformation of B_0 , we have an exact sequence $0 \rightarrow B_0 \xrightarrow{j=\epsilon} B \xrightarrow{\phi} B/\epsilon B \cong B_0 \rightarrow 0$, where the first map $\epsilon(b_0) := \epsilon \cdot b$, where b is a lifting of b_0 via ϕ . This is a short Poisson extension of B_0 over k by B_0 since B_0 is a Poisson B -module and $\epsilon(B_0)^2 = \{\epsilon(B_0), \epsilon(B_0)\} = 0$ and the induced B_0 -module structure on B_0 via ϕ is given by the multiplication on B_0 . Let B' be an equivalent Poisson deformation of B_0 over $k[\epsilon]$ with the first order Poisson deformation B . Then we would like to show that the following diagram commutes

$$\begin{array}{ccccccc} 0 & \longrightarrow & B_0 & \xrightarrow{j} & B & \xrightarrow{\phi} & B_0 \longrightarrow 0 \\ & & \parallel & & \cong \Phi \downarrow & & \parallel \\ 0 & \longrightarrow & B_0 & \xrightarrow{j'} & B' & \xrightarrow{\phi'} & B_0 \longrightarrow 0 \end{array}$$

where Φ is a Poisson homomorphism over $k[\epsilon]$ defining equivalent first order Poisson deformations of B_0 . Since right diagram commutes by definition of equivalence, we only check that the left diagram

commutes. $j(b_0) = \epsilon b$ where $\phi(b) = b_0$. Since $\phi'(\Phi(b)) = b_0$, $j'(b_0) = \epsilon\Phi(b)$. Hence the diagram commutes. So equivalent first order Poisson deformations corresponds to equivalent short Poisson extensions B_0 over k by B_0 .

Conversely, let $0 \rightarrow B_0 \xrightarrow{j} B \xrightarrow{\phi} B_0 \rightarrow 0$ be a Poisson extension. Then B is a Poisson $k[\epsilon]$ -algebra by the above discussion. In this case, we can identify B_0 with $j(B_0) = \epsilon B = B \otimes_{k[\epsilon]} (\epsilon)$. Hence B is flat over $k[\epsilon]$ and $B/\epsilon B = B \otimes_{k[\epsilon]} k \cong B_0$. Since ϵB is a Poisson ideal (ϕ is a Poisson map), ϕ induces a Poisson isomorphism $B \otimes_{k[\epsilon]} k \cong B_0$. Given a equivalent Poisson extension, since the right diagram in the above commutes, it gives an equivalent Poisson deformations of B_0 .

Let $B_0 \otimes_k k[\epsilon] = B_0 \oplus \epsilon B_0$ be the trivial Poisson deformation of B_0 . Then the associated Poisson extension is trivial $0 \rightarrow B_0 \rightarrow B_0 \oplus \epsilon B_0 \rightarrow B_0$: ■

Corollary 9.31 *Let B_0 be a Poisson k -algebra. Then the set of first order Poisson deformations of B_0 is in natural one to one correspondence $PT^1(B_0/k, B_0)$.*

$$PDef_{B_0}(k[\epsilon]) \cong PT^1(B_0/k, B_0)$$

Proof. By Theorem 14 and Proposition 9.30, we have $PDef_{B_0}(k[\epsilon]) \cong PEx^1(B_0/k, B_0) \cong PT^1(B_0/k, B_0)$. ■

9.4 First order deformations of a Poisson closed subscheme of an affine Poisson scheme

Let X be a Poisson scheme over k and let Y be a closed Poisson subscheme of (X, Λ_0) . We will define a Poisson deformation of Y over $Spec k[\epsilon]$ in X to be a Poisson subscheme $Y' \subset (X \times Spec(k[\epsilon]), \Lambda_0 \oplus 0)$ where $(X \times Spec(k[\epsilon]), \Lambda_0 \oplus 0)$ is the trivial Poisson deformation of X over $Spec(k[\epsilon])$, $Y' \times_{Spec k[\epsilon]} k = Y$ and Y' is flat over $Spec(k[\epsilon])$.

We discuss deformations of Poisson subschemes when X is an affine Poisson scheme. Then X corresponds to a Poisson k -algebra (B, Λ_0) , and Y is defined by an Poisson ideal $I \subset B$. We would like to find Poisson ideals of $k[\epsilon]$ -Poisson algebra $(B' = B \otimes_k k[\epsilon], \Lambda_0 \oplus 0)$ such that the image of I' in $B = B'/\epsilon B'$ is I and B'/I' is flat over $k[\epsilon]$. We note that the flatness of B'/I' over $k[\epsilon]$ is equivalent to the exactness of $0 \rightarrow B/I \xrightarrow{\epsilon} B'/I' \rightarrow B/I \rightarrow 0$ where ϵ is the multiplication by ϵ if and only if $0 \rightarrow I \xrightarrow{\epsilon} I' \rightarrow I \rightarrow 0$ is exact. (see [Har10] page 11)

Proposition 9.32 (compare [Har10] Proposition 2.3) *To give a Poisson ideal $I' \subset B' = B \oplus$*

ϵB such that B'/I' is flat over $k[\epsilon]$ and the image of I' in B is I is equivalent to an element

$$\varphi \in \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I, B/I) = \text{Hom}_{\mathcal{U}_{\text{Pois}(B/I)}}(I/I^2 \oplus \{I, I\}, B/I).$$

In particular, $\varphi = 0$ corresponds to the trivial deformation given by $I' = I \oplus \epsilon I$ inside $B' = B \oplus \epsilon B$.

Proof. $B' = B \oplus \epsilon B$ is naturally a Poisson B -algebra in the following way: $a \cdot (b + \epsilon c) = ab + \epsilon ac$ and $\{a, b + \epsilon c\} = \{a, b\} + \epsilon\{a, c\}$. Now let I' be a Poisson ideal of $B' = B \oplus \epsilon B$ such that B'/I' is flat over $k[\epsilon]$ and $\pi(I') = I$, where $\pi : B \oplus \epsilon B \rightarrow B$ be the projection. Let $x \in I$ and choose a lifting x' of x via π . Then $x' = x + \epsilon y \in I'$ for some $y \in B$. Let $x'' = x + \epsilon y' \in I'$ be an another lifting of x . Then $y - y' \in I$ by the flatness of B'/I' over $k[\epsilon]$. So the image \bar{y} in B/I is uniquely determined. So $\varphi : I \rightarrow B/I, x \rightarrow \bar{y}$ is well-defined. We claim that this is a Poisson B -module homomorphism. Indeed, let $\varphi(x) = \bar{y}$. Since $x + \epsilon y \in I'$, we have $bx + \epsilon by \in I'$ for $b \in B$. So $\varphi(b \cdot x) = \overline{by} = b\bar{y} = b \cdot \varphi(x)$. On the other hand, since $\{b, x\} + \epsilon\{b, y\} \in I'$, we have $\varphi(\{b, x\}) = \overline{\{b, x\}} = \{b, \bar{x}\} = \{b, \varphi(x)\}$.

Conversely, let $\varphi \in \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I, B/I)$. Define

$$I' = \{x + \epsilon y | x \in I, y \in B, \text{ the image of } \bar{y} \text{ of } y \text{ in } B/I \text{ is equal to } \varphi(x)\}$$

We claim that I' is a Poisson ideal of B' . Let $x + \epsilon y \in I'$ and $a + \epsilon b \in B'$. Then $x \in I$ and $\varphi(x) = \bar{y}$. Since $(a + \epsilon b)(x + \epsilon y) = ax + \epsilon(bx + ay)$ and $\overline{bx + ay} = a\bar{y}$, we have $\varphi(ax) = a\varphi(x) = a\bar{y}$. On the other hand, since $\{a + \epsilon b, x + \epsilon y\} = \{a, x\} + \epsilon(\{b, x\} + \{a, y\})$ and $\overline{\{b, x\} + \{a, y\}} = \{a, \bar{y}\}$, we have $\varphi(\{a, x\}) = \overline{\{a, y\}} = \{a, \varphi(x)\}$. We have a natural exact sequence $0 \rightarrow I \xrightarrow{\epsilon} I' \rightarrow I \rightarrow 0$, where ϵ means multiplication by ϵ . Since the exactness means the exactness of $0 \rightarrow B/I \xrightarrow{\epsilon} B'/I' \rightarrow B/I \rightarrow 0$, B'/I' is a flat over $k[\epsilon]$.

These two construction is one to one correspondence. When $\varphi = 0 \in \text{Hom}_{\mathcal{U}_{\text{Pois}(B)}}(I, B/I)$, we have $I' = I \oplus \epsilon I$. ■

Corollary 9.33 *Let B_0 be a Poisson k -algebra and I be a Poisson ideal of B_0 . Let $C = B_0/I$. Then the set of first order deformations of Poisson closed subscheme $\text{Spec}(B_0/I)$ of an affine Poisson scheme $\text{Spec}(B_0)$ is in natural one to one correspondence with $PT^1(C/B_0, C)$.*

Proof. This follows from Proposition 9.23 and Proposition 9.32 ■

Remark 9.34 *If our construction of Poisson cotangent complex may turn out to be correct and use right languages, there is a “globalization” problem, which I cannot solve at this point. In [LS67],*

Lichtenbaum and Schlessinger actually constructed a quasi-coherent sheaf $\mathcal{T}^i(X/Y, \mathcal{F})$ for a morphism of schemes $f : X \rightarrow Y$ and a quasi-coherent sheaf \mathcal{F} of \mathcal{O}_X -module where X is separated, Y is noetherian and f is locally of finite type. I could not show that $PT^i(B/A, M)$ commutes with localization and so that we can define a sheaf $\mathcal{PT}^i(X/Y, \mathcal{F})$ where $f : X \rightarrow Y$ is a morphism of Poisson schemes satisfying suitable finiteness conditions and \mathcal{F} is a quasi-coherent Poisson \mathcal{O}_X -module.

Appendix A

Basic materials on Poisson algebras and holomorphic Poisson manifolds

In this section, we present basic facts about holomorphic Poisson manifolds relevant to our discussions. Our reference is [LGPV13].

Definition A.1 *A commutative \mathbb{C} -algebra A with identity is called a Poisson algebra if there is a Poisson bracket $\{-, -\}$ such that*

1. $(A, \{-, -\})$ is a Lie algebra over \mathbb{C}
2. The multiplications are compatible in the sense that

$$\{a \cdot b, c\} = a \cdot \{b, c\} + b \cdot \{a, c\}$$

for any $a, b, c \in A$

Definition A.2 *Let $(A, \{-, -\}_1)$ and $(B, \{-, -\}_2)$ be Poisson algebras. A \mathbb{C} -algebra homomorphism $\phi : A \rightarrow B$ is called a Poisson map if it is compatible with Poisson brackets*

$$\phi(\{a, b\}) = \{\phi(a), \phi(b)\}$$

for any $a, b \in A$.

Definition A.3 A holomorphic Poisson manifold M is a complex manifold such that the structure sheaf is a sheaf of Poisson algebra. In other words, for any open set $U \in M$, $\mathcal{O}_M(U)$ is a Poisson algebra and for any open set $V \subset U$, the restriction map $\mathcal{O}_M(U) \rightarrow \mathcal{O}_M(V)$ is a Poisson map.

We recall a Schouten bracket, denoted by $[-, -]_{Sch}$, on holomorphic polyvector fields $\bigoplus_{i \geq 0} H^0(M, \wedge^i T_M)$ on M .

Proposition A.4 Let M be a n -dimensional holomorphic Poisson manifold. Then there exist a holomorphic bivector field $\Lambda \in H^0(M, \wedge^2 T_M)$ with $[\Lambda, \Lambda]_{Sch} = 0$ and $\{-, -\}$ is defined in the following way: let $U = (z_1, \dots, z_n)$ be a coordinate neighborhood of M and $\Lambda = \sum_{i,j=1}^n \Lambda_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ on U , then for

$$(*) \{f, g\} = \Lambda(df, dg) = \Lambda(\partial f, \partial g) = \sum_{i,j=1}^n \Lambda_{ij}(z) \left(\frac{\partial f}{\partial z_i} \frac{\partial g}{\partial z_j} - \frac{\partial g}{\partial z_i} \frac{\partial f}{\partial z_j} \right)$$

for any holomorphic functions $f, g \in \mathcal{O}_M(U)$.

Conversely, a holomorphic bivector field Λ in $H^0(M, \wedge^2 T_M)$ with $[\Lambda, \Lambda]_{Sch} = 0$ makes M a holomorphic Poisson manifold by (*).

We denote a holomorphic Poisson manifold by (M, Λ) .

Definition A.5 A holomorphic map $f : (M, \Lambda) \rightarrow (N, \Lambda')$ between holomorphic Poisson manifolds is called a Poisson map if for all open sets $U \subset M$ and $V \subset N$ with $f(U) \subset V$, the induced map (defined by pullback) $f^* : \mathcal{O}_N(V) \rightarrow \mathcal{O}_M(U)$ is a Poisson map.

We recall that for each $p \in M$, we have a linear $f_* : T_p M \rightarrow T_{f(p)} N$. This extends to $f_* : \wedge^2 T_p M \rightarrow \wedge^2 T_{f(p)} N$.

Proposition A.6 Let $f : (M, \Lambda) \rightarrow (N, \Lambda')$ be a holomorphic map. Then f is a Poisson map if and only if

$$f_* \Lambda_p = \Lambda'_{f(p)}$$

for all $p \in M$.

Definition A.7 Let (M, Λ) be a holomorphic Poisson manifold and N be a complex submanifold of M . Assume that N is a holomorphic Poisson manifold with a holomorphic bivector field Λ' . Then (N, Λ') is called a Poisson submanifold if the inclusion map $i : (N, \Lambda') \rightarrow (M, \Lambda)$ is a Poisson map. Hence Λ' is uniquely determined by Λ by restricting Λ to N .

Proposition A.8 Let (M, Λ) be a holomorphic Poisson manifold and N be a complex submanifold of M . Then the following are equivalent.

1. N is a holomorphic Poisson submanifold of (M, Λ) .
2. The ideal sheaf $\mathcal{I}_N \subset \mathcal{O}_M$ defined by $\mathcal{I}_N(U) = \{f \in \mathcal{O}_M(U) \mid f|_{N \cap U} = 0\}$ for open sets $U \subset M$ is a Poisson ideal of $\mathcal{O}_M(U)$. That is, $\mathcal{I}_N(U)$ is an ideal under the Poisson bracket: if $f \in \mathcal{I}_M(U)$ and $g \in \mathcal{O}_N(U)$, then $\{f, g\} \in \mathcal{I}_M(U)$.
3. For every $p \in N$, the bivector Λ_p belongs to $\wedge^2 T_p N$.

Definition A.9 If (M, Λ) and (N, Λ') are holomorphic Poisson manifolds, then we can make $M \times N$ a holomorphic Poisson manifold $(M \times N, \Lambda \oplus \Lambda')$ induced from Λ and Λ' in the following way. Let $U = (z_1, \dots, z_n)$ and $V = (w_1, \dots, w_m)$ be coordinate neighborhoods of M and N respectively. Then $U \times V = (z_1, \dots, z_n, w_1, \dots, w_m)$ is a coordinate neighborhood of $M \times N$. Let $\Lambda = \sum_{i,j} g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ and $\Lambda' = \sum_{r,s} h_{rs}(w) \frac{\partial}{\partial w_r} \wedge \frac{\partial}{\partial w_s}$. We define

$$\Lambda \oplus \Lambda' = \sum_{i,j=1}^n g_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j} + \sum_{r,s=1}^m h_{rs}(w) \frac{\partial}{\partial w_r} \wedge \frac{\partial}{\partial w_s}$$

Then $\Lambda \oplus \Lambda'$ is a holomorphic bivector field on M and $[\Lambda \oplus \Lambda', \Lambda \oplus \Lambda']_{Sch} = 0$.

Appendix B

Hypercohomology

In this appendix, we present the basic materials on hypercohomology relevant to our discussion. Our reference is [EV92] Appendix. In this section we fix M as a complex manifold. We denote $T_M = T$ by the holomorphic tangent bundle of M and $\Theta_M = \Theta$ be the sheaves of germs of holomorphic section of T .

Definition B.1 *The map $\sigma : \mathcal{F}^\bullet \rightarrow \mathcal{I}^\bullet$ between two complexes of sheaves of \mathbb{C} -module on M is called an injective resolution of \mathcal{F}^\bullet if \mathcal{I}^\bullet is a complex of \mathbb{C} -module bounded below, σ is a quasi isomorphism, and the sheaves \mathcal{I}^i are injective for all i .*

Remark B.2 *Every complex of \mathbb{C} -modules on M which is bounded below admits an injective resolution.*

Definition B.3 *Let \mathcal{F}^\bullet be a complex of \mathbb{C} -modules on M which is bounded below. Then the hypercohomology group $\mathbb{H}^a(M, \mathcal{F})$ is defined to be the \mathbb{C} -module*

$$\mathbb{H}^a(M, \mathcal{F}^\bullet) := \frac{\ker \Gamma(M, \mathcal{I}^a) \rightarrow \Gamma(M, \mathcal{I}^{a+1})}{\text{im } \Gamma(M, \mathcal{I}^{a-1}) \rightarrow \Gamma(M, \mathcal{I}^a)}$$

Remark B.4 *This definition does not depend on the choice of injective resolution.*

Proposition B.5 *If $\sigma : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$ is a quasi-isomorphism and if $H^a(M, \mathcal{G}^i) = 0$ for all $a > 0$ and*

all i , then

$$\mathbb{H}^a(M, \mathcal{F}^\bullet) = \frac{\ker \Gamma(M, \mathcal{G}^a) \rightarrow \Gamma(X, \mathcal{G}^{a+1})}{\operatorname{im} \Gamma(M, \mathcal{G}^{a-1}) \rightarrow \Gamma(M, \mathcal{G}^a)}$$

We call \mathcal{G}^\bullet an acyclic resolution of \mathcal{F}^\bullet .

Example B.6 Let (M, Λ) be a n -dimensional compact holomorphic Poisson manifold. Let \mathcal{F}^\bullet be the complex of sheaves $0 \rightarrow \Theta \rightarrow \wedge^2 \Theta \rightarrow \dots \rightarrow \wedge^n \Theta \rightarrow 0$ induced by $[\Lambda, -]$. We denote by $\mathcal{A}^{0,p}(\wedge^q T)$ the sheaf of germs of C^∞ -section of $\wedge^p \bar{T}^* \otimes \wedge^q T$ and by $A^{0,p}(M, \wedge^q T)$ the global section of $\mathcal{A}^{0,p}(\wedge^q T)$. Let's consider the following bicomplex of sheaves.

$$\begin{array}{ccccccc} & & \uparrow [\Lambda, -] & & & & \\ & & \mathcal{A}^{0,0}(\wedge^3 T) & \xrightarrow{\bar{\partial}} & \dots & & \\ & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \\ & & \mathcal{A}^{0,0}(\wedge^2 T) & \xrightarrow{\bar{\partial}} & \mathcal{A}^{0,1}(\wedge^2 T) & \xrightarrow{\bar{\partial}} & \dots \\ & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] \\ & & \mathcal{A}^{0,0}(T) & \xrightarrow{\bar{\partial}} & \mathcal{A}^{0,1}(T) & \xrightarrow{\bar{\partial}} & \mathcal{A}^{0,2}(T) & \xrightarrow{\bar{\partial}} & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

Each rows is a resolution of $\wedge^i \Theta$. Hence \mathcal{F}^\bullet is quasi-isomorphic to the total complex of the above bicomplex of sheaves. Hence by Proposition B.5, the hypercohomology $\mathbb{H}^a(M, \mathcal{F}^\bullet)$ is the a -th cohomology of the total complex of the following bicomplex

$$\begin{array}{ccccccc} & & \uparrow [\Lambda, -] & & & & \\ & & A^{0,0}(M, \wedge^3 T) & \xrightarrow{\bar{\partial}} & \dots & & \\ & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \\ & & A^{0,0}(M, \wedge^2 T) & \xrightarrow{\bar{\partial}} & A^{0,1}(M, \wedge^2 T) & \xrightarrow{\bar{\partial}} & \dots \\ & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] \\ & & A^{0,0}(M, T) & \xrightarrow{\bar{\partial}} & A^{0,1}(M, T) & \xrightarrow{\bar{\partial}} & A^{0,2}(M, T) & \xrightarrow{\bar{\partial}} & \dots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots \end{array}$$

Now we will consider a Čech resolution of a complex of sheaves. Let $\mathcal{U} = \{U_\alpha : \alpha \in A\}$, for $A \in \mathbb{N}$ be a locally finite open covering of a complex manifold or some open covering of an algebraic

scheme X . For

$$U_{\alpha_0 \dots \alpha_a} := U_{\alpha_0} \cap \dots \cap U_{\alpha_a} \quad \alpha_0 < \alpha_1 < \dots < \alpha_a.$$

ρ denotes the open embedding

$$\rho = \rho_{\alpha_0 \dots \alpha_a} : U_{\alpha_0 \dots \alpha_a} \rightarrow X,$$

To a bounded below complex \mathcal{F}^\bullet we associates its Čech complex \mathcal{G}^\bullet such that

$$\mathcal{G}^i := \bigoplus_{a \geq 0} \mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-a})$$

where

$$\mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-a}) = \prod_{\alpha_0 < \alpha_1 < \dots < \alpha_a} \rho_* (\mathcal{F}^{i-1}|_{U_{\alpha_0 \dots \alpha_a}}).$$

The differential Δ of \mathcal{G}^\bullet is defined by

$$\Delta(s) = (-1)^i \delta s + d_{\mathcal{F}^\bullet} s \quad s \in \mathcal{C}^a(\mathcal{U}, \mathcal{F}^{i-1}).$$

and $d_{\mathcal{F}^\bullet}$ is the differential of \mathcal{F}^\bullet . Then the natural map

$$\sigma : \mathcal{F}^\bullet \rightarrow \mathcal{G}^\bullet$$

defined by

$$\mathcal{F}^i \xrightarrow{\rho} \prod_{\alpha \in A} \rho_* (\mathcal{F}^i|_{U_\alpha}) = \mathcal{C}^0(\mathcal{U}, \mathcal{F}^i)$$

is a quasi isomorphism.

Example B.7 Now let (M, Λ) be a n -dimensional compact holomorphic Poisson manifold. let \mathcal{F}^\bullet be the complex of sheaves $0 \rightarrow \Theta \rightarrow \wedge^2 \Theta \rightarrow \dots \rightarrow \wedge^n \Theta \rightarrow 0$ induced by $[\Lambda, -]$. And assume that for each $U_j \in \mathcal{U} = \{U_\alpha : \alpha \in A\}$, we have

$$U_j = \{z_j \in \mathbb{C}^n \mid |z_j^\alpha| < r_j^\alpha, \alpha = 1, \dots, n\}$$

Then by Proposition B.5, the hypercohomology $\mathbb{H}^a(M, \mathcal{F}^\bullet)$ is the a -th cohomology of the total complex of the following bicomplex

$$\begin{array}{ccccccc}
& & \uparrow [\Lambda, -] & & & & \\
& & C^0(\mathcal{U}, \wedge^3 \Theta) & \xrightarrow{-\delta} & \dots & & \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \\
& & C^0(\mathcal{U}, \wedge^2 \Theta) & \xrightarrow{\delta} & C^1(\mathcal{U}, \wedge^2 \Theta) & \xrightarrow{-\delta} & \dots \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] \\
& & C^0(\mathcal{U}, \Theta) & \xrightarrow{-\delta} & C^1(\mathcal{U}, \Theta) & \xrightarrow{\delta} & C^2(\mathcal{U}, \Theta) & \xrightarrow{-\delta} & \dots \\
& & \uparrow & & \uparrow & & \uparrow & & \uparrow \\
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \dots
\end{array}$$

where $C^i(\mathcal{U}, \wedge^j \Theta) = H^0(M, C^i(\mathcal{U}, \wedge^j \Theta)) = \bigoplus_{\alpha_0 < \dots < \alpha_i} H^0(\mathcal{U}_{\alpha_0 \dots \alpha_i}, \wedge^j \Theta)$.

Example B.8 Now let (X, Λ) be an algebraic Poisson scheme over k , where k is an algebraically closed field and $\Lambda \in \Gamma(X, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X))$ with $[\Lambda, \Lambda] = 0$ (for the definition, see chapter 7).

Let \mathcal{F}^\bullet be the complex of sheaves

$$0 \rightarrow \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \mathcal{H}om_{\mathcal{O}_X}(\wedge^3 \Omega_{X/k}^1, \mathcal{O}_X) \xrightarrow{[\Lambda, -]} \dots$$

which is the (truncated) Lichnerowicz-Poisson complex of sheaves of (X, Λ_0) (See Definition 7.28).

And assume that for each $U_j \in \mathcal{U} = \{U_\alpha, \alpha \in A\}$, U_j is affine. Then by Remark 7.29 and Proposition B.5, the hypercohomology $\mathbb{H}^a(X, \mathcal{F}^\bullet)$ is the a -th cohomology of the total complex of the following bicomplex

$$\begin{array}{ccccccc}
& & \uparrow [\Lambda, -] & & & & \\
& & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^3 \Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{-\delta} & \dots & & \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \\
& & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{\delta} & C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^2 \Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{-\delta} & \dots \\
& & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] & & \uparrow [\Lambda, -] \\
& & C^0(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{-\delta} & C^1(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) & \xrightarrow{\delta} & C^2(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\Omega_{X/k}^1, \mathcal{O}_X)) \\
& & \uparrow & & \uparrow & & \uparrow \\
& & 0 & \longrightarrow & 0 & \longrightarrow & 0
\end{array}$$

where

$$\begin{aligned} C^i(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^j \Omega_{X/k}^1, \mathcal{O}_X)) &= H^0(X, C^i(\mathcal{U}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^j \Omega_{X/k}^1, \mathcal{O}_X))) \\ &= \bigoplus_{\alpha_0 < \dots < \alpha_i} H^0(\mathcal{U}_{\alpha_0 \dots \alpha_i}, \mathcal{H}om_{\mathcal{O}_X}(\wedge^j \Omega_{X/k}^1, \mathcal{O}_X)). \end{aligned}$$

Appendix C

Differential graded Lie algebra structure on

$$\bigoplus_{i \geq 0} g^i = \bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(M, \wedge^q T_M)$$

In this section, we fix M as a n -dimensional complex manifold. Let $F = \bar{T}^* \oplus T$ be the direct sum of antiholomorphic cotangent bundle and holomorphic tangent bundle on M . We consider the bundle $\bigoplus_{i \geq 1} F^i = \bigoplus_{i \geq 1} \wedge^i F$ where $F^i = \bigoplus_{p+q=i, p, q \geq 0} \wedge^p \bar{T}^* \otimes \wedge^q T$. We denote by \mathcal{F} the sheaf of germs of C^∞ section of F , by \mathcal{F}^i by the sheaf of germs of C^∞ section of $\bigoplus_{p+q=i, p, q \geq 0} \wedge^p \bar{T}^* \otimes \wedge^q T$ and by $\mathcal{A}^{0,p}(\wedge^q T)$ the sheaf of germs of C^∞ -section of $\wedge^p \bar{T}^* \otimes \wedge^q T$. Then we have $\mathcal{F} = \bigoplus_{i \geq 1} \mathcal{F}^i = \bigoplus_{p+q=i, p, q \geq 0} \mathcal{A}^{0,p}(\wedge^q T)$. If we denote by $A^{0,p}(M, \wedge^q T)$ the global section of $\mathcal{A}^{0,p}(\wedge^q T)$, then the global section of \mathcal{F} is $\bigoplus_{p+q=i, p, q \geq 0} A^{0,p}(M, \wedge^q T)$. We set $A^i := \bigoplus_{p+q=i} A^{0,p}(M, \wedge^q T)$ and $A = \bigoplus_{i \geq 0} A^i = \bigoplus_{p+q=i, p, q \geq 0} A^{0,p}(M, \wedge^q T)$. Then A is a graded vector space over \mathbb{C} . In this appendix, we discuss the differential Gerstenhaber algebra structure on A and, by shifting the degree 1, a differential grade Lie algebra structure on $A[1] = \bigoplus_{i \geq 0} g^i = \bigoplus_{p+q-1=i} A^{0,p}(M, \wedge^q T_M)$. Our main references are [Mac05] and [Man04].

Definition C.1 *A differential Gerstenhaber algebra $(B, [,], \wedge, \bar{\partial})$ is the data of a graded vector space $B = \sum_{i \in \mathbb{Z}} B^i$ equipped with a bilinear bracket $[-, -] : B \times B \rightarrow B$, a wedge product \wedge and a linear*

map $\bar{\partial} : B \rightarrow B$ satisfying the following properties (here \bar{x} is the grading of the homogeneous element x .)

$$[B^i, B^j] \subset B^{i+j-1}. \quad (\text{C.2})$$

$$[a, b] = (-1)^{\bar{a}\bar{b}+\bar{a}+\bar{b}}[b, a] \text{ for homogeneous elements } a, b \in B. \quad (\text{C.3})$$

$$[a, [b, c]] = [[a, b], c] - (-1)^{\bar{a}\bar{b}+\bar{a}+\bar{b}}[b, [a, c]] \text{ for homogeneous elements } a, b, c \in B. \quad (\text{C.4})$$

$$B^i \wedge B^j \subset B^{i+j}. \quad (\text{C.5})$$

$$a \wedge b = (-1)^{\bar{a}\bar{b}}b \wedge a \quad (\text{C.6})$$

$$[a \wedge b, c] = a \wedge [b, c] + (-1)^{\bar{a}\bar{b}}b \wedge [a, c] \quad (\text{C.7})$$

$$\bar{\partial}B^i \subset B^{i+1}. \quad (\text{C.8})$$

$$\bar{\partial} \circ \bar{\partial} = 0. \quad (\text{C.9})$$

$$\bar{\partial}[a, b] = [\bar{\partial}a, b] - (-1)^{\bar{a}}[a, \bar{\partial}b] \quad (\text{C.10})$$

$$\bar{\partial}(a \wedge b) = \bar{\partial}a \wedge b + (-1)^{\bar{a}}a \wedge \bar{\partial}b. \quad (\text{C.11})$$

We discuss a differential Gerstenhaber algebra structure on $A = \bigoplus_{p+q=i, p, q \geq 0} A^{0,p}(M, \wedge^q T)$.

1. Let $f \in A^{0,0}(M, \mathcal{O}_M)$, $g_i \frac{\partial}{\partial z_i} \in A^{0,0}(M, T)$ and $h_j d\bar{z}^j \in A^{0,1}(M, \mathcal{O}_M)$. We define $\bar{\partial}$ in the following way.

- (a) $\bar{\partial}f := \frac{\partial f}{\partial \bar{z}^i} d\bar{z}^i$
- (b) $\bar{\partial}(g_i \frac{\partial}{\partial z_i}) := \frac{\partial g_i}{\partial \bar{z}^k} d\bar{z}^k \wedge \frac{\partial}{\partial z^i}$
- (c) $\bar{\partial}(h_j d\bar{z}^j) := \frac{\partial h_j}{\partial \bar{z}^k} d\bar{z}^k \wedge d\bar{z}^j$

These definitions are independent of coordinate transformations. By the rule C.11, we extend $\bar{\partial}$ to A .

2. We define $[-, -]$ in the following way

- (a) $[g_i \frac{\partial}{\partial z_i}, f] := g_i \frac{\partial f}{\partial z_i} = L_{g_i \frac{\partial}{\partial z_i}} f$ (Lie derivative of f in the direction of $g_i \frac{\partial}{\partial z_i}$).
- (b) $[g_i \frac{\partial}{\partial z_i}, a_k \frac{\partial}{\partial z_k}] := g_i \frac{\partial a_k}{\partial z_i} - a_k \frac{\partial g_i}{\partial z_k}$
- (c) $[g_i \frac{\partial}{\partial z_i}, h_j d\bar{z}^j] := g_i \frac{\partial h_j}{\partial z_i} d\bar{z}^j$
- (d) $[h_j d\bar{z}^j, f] := 0$

$$(e) [h_j d\bar{z}_j, b_l d\bar{z}_l] := 0$$

These definitions are independent of coordinate transformations. By the rule C.3 and C.7, we extend $[-, -]$ to A .

Then $(A, \wedge, \bar{\partial})$ is a differential Gerstenhaber algebra and have the following property: For $v_i = f_{ik} \frac{\partial}{\partial z_k}$ and $w_j = f'_{jk} \frac{\partial}{\partial z_k}$, then we have

$$[v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_m] = \sum_{i=1}^n \sum_{j=1}^m (-1)^{i+j} [v_i, w_j] \wedge v_1 \wedge \cdots \wedge \hat{v}_i \wedge \cdots \wedge v_n \wedge w_1 \wedge \cdots \wedge \hat{w}_j \wedge \cdots \wedge w_m$$

Remark C.12 *polyvector fields* $A' = \bigoplus_{i \geq 0} A^{0,0}(M, \wedge^i T)$ is a subalgebra of L . When we restrict the bracket $[-, -]$ on A' , we get the Schouten-Nijenhuis bracket $[-, -]_{Sch}$.

More generally, for $\phi_p = g_{pk} d\bar{z}_k$ and $\psi_q = g'_{qk} d\bar{z}_k$, we have

$$\begin{aligned} & [\phi_1 \wedge \cdots \wedge \phi_l \wedge v_1 \wedge \cdots \wedge v_n, \psi_1 \wedge \cdots \wedge \psi_k \wedge w_1 \wedge \cdots \wedge w_m] \\ &= \phi_1 \wedge \cdots \wedge \phi_l \wedge [v_1 \wedge \cdots \wedge v_n, \psi_1 \wedge \cdots \wedge \psi_k] \wedge w_1 \wedge \cdots \wedge w_m \\ &+ (-1)^{(l+n)(k+m)+(l+n)+(k+m)} \psi_1 \wedge \cdots \wedge \psi_k \wedge [w_1 \wedge \cdots \wedge w_m, \phi_1 \wedge \cdots \wedge \phi_l] \wedge v_1 \wedge \cdots \wedge v_n \\ &+ (-1)^{k(n+1)} \phi_1 \wedge \cdots \wedge \phi_l \wedge \psi_1 \wedge \cdots \wedge \psi_k \wedge [v_1 \wedge \cdots \wedge v_n, w_1 \wedge \cdots \wedge w_m] \end{aligned}$$

In particular, it is practical to have the following formula

$$\begin{aligned} [f dz_I \frac{\partial}{\partial z_J}, g dz_K \frac{\partial}{\partial z_L}] &= (-1)^{|K|(|J|+1)} dz_I \wedge dz_K [f \frac{\partial}{\partial z_J}, g \frac{\partial}{\partial z_L}]_{Sch} \\ [f \frac{\partial}{\partial z_I} d\bar{z}_J, g \frac{\partial}{\partial z_H} d\bar{z}_K] &= (-1)^{|J|(|H|-1)} [f \frac{\partial}{\partial z_I}, g \frac{\partial}{\partial z_H}]_{Sch} d\bar{z}_J \wedge d\bar{z}_K \end{aligned}$$

Now we discuss a differential graded Lie algebra structure on $A[1]$.

Definition C.13 A differential graded Lie algebra $(C, [-, -], \bar{\partial})$ is the data of a graded vector space $C = \bigoplus_{i \in \mathbb{Z}} C^i$ over \mathbb{C} together with a bilinear bracket $[-, -] : C \times C \rightarrow C$ and a linear map $\bar{\partial} : C \rightarrow C$ satisfying the following properties

1. $[C^i, C^j] \subset C^{i+j}$
2. $[c, d] = -(-1)^{\bar{c}\bar{d}} [d, c]$ for homogeneous elements c, d (here \bar{x} is the grading of the homogeneous element x)
3. $[a, [b, c]] = [[a, b], c] + (-1)^{\bar{a}\bar{b}} [b, [a, c]]$ for homogeneous elements a, b, c .

$$4. \bar{\partial}C^i \subset C^{i+1}$$

$$5. \bar{\partial} \circ \bar{\partial} = 0$$

$$6. \bar{\partial}[a, b] = [\bar{\partial}a, b] + (-1)^{\bar{a}}[a, \bar{\partial}b]$$

It is clear that if L is a differential Gerstenhaber algebra, then $L[1]$ ¹ is a differential graded lie algebra. Hence $(A[1], [-, -], \bar{\partial})$ is a differential graded lie algebra and so satisfies the following properties. But we have to mention that on $A[1]$, we have other differential graded Lie algebra structures by changing the differential into $L = \bar{\partial} + [\Lambda, -]$, where Λ is a holomorphic bivector field such that $[\Lambda, \Lambda] = 0$. (i.e $L\Lambda = 0$).

Proposition C.14 $(A[1], L, [-, -])$ is a differential graded Lie algebra.

Proof. Let $a \in A[1]$. We only need to see the properties (4), (5) and (6). (4) is clear by definition, and we have $L \circ L = 0$ by simple computation. For (6), note that by plugging Λ into a in (3) and combining (6) in the definition, we get $L[a, b] = [La, b] + (-1)^{\bar{a}}[a, Lb]$. ■

In holomorphic Poisson deformations there is no role of structure sheaf \mathcal{O}_M , we define the following sub differential graded Lie algebra of $(A[1], L, [-, -])$ on a holomorphic Poisson manifold (M, Λ) .

Definition C.15 $(\mathfrak{g} = \bigoplus_{i \geq 0} \mathfrak{g}^i = \bigoplus_{p+q-1=i, q \geq 1} A^{0,p}(M, \wedge^q T_M), L, [-, -])$.

Definition C.16 The Maurer-Cartan equation of a differential graded Lie algebra $(C, [-, -], \bar{\partial})$ is

$$\bar{\partial}a + \frac{1}{2}[a, a] = 0, \quad a \in C^1$$

The solutions $MC(L) \subset C^1$ of the Maurer-Cartan equation are called the Maurer Cartan elements of the differential graded Lie algebra C .

¹ $L[1]$ is the algebra L but the grading is shifted by 1. Hence $L[1]^i = L^{i+1}$

Appendix D

Ellipticity of the operator $\bar{\partial} + [\Lambda, -]$

Let (M, Λ) be a holomorphic Poisson manifolds. In this appendix, we discuss the operator $\bar{\partial} + [\Lambda, -]$ on M . Our main reference is [Wel08]. Let $\pi : E \rightarrow X$ be a differentiable complex vector bundle on a differentiable manifold. For an open set $U \subset X$, we denote C^∞ functions on U by $\mathcal{E}(U)$, C^∞ section of E on X by $\mathcal{E}(U, E)$ and $E_x := \pi^{-1}(x)$, $x \in X$ by a \mathbb{C} vector space fiber over x .

Definition D.1 *Let E and F be differentiable complex vector bundles over a differentiable manifold X . Let $L : \mathcal{E}(X, E) \rightarrow \mathcal{E}(X, F)$ be linear. We say that L is a differentiable operator if for any choice of local coordinates and local trivializations, there exists a linear partial differential operator \tilde{L} such that the diagram*

$$\begin{array}{ccc}
 [\mathcal{E}(U)]^p & \xrightarrow{\tilde{L}} & [\mathcal{E}(U)]^q \\
 \cong \uparrow & & \uparrow \cong \\
 \mathcal{E}(U, U \times \mathbb{C}^p) & \longrightarrow & \mathcal{E}(U, U \times \mathbb{C}^q) \\
 \cup \uparrow & & \uparrow \cup \\
 \mathcal{E}(X, E)|_U & \xrightarrow{L} & \mathcal{E}(X, F)|_U
 \end{array}$$

commutes. That is, for $f = (f_1, \dots, f_p) \in [\mathcal{E}(U)]^p$

$$\tilde{L}(f)_i = \sum_{j=1, |\alpha| \leq k}^p a_\alpha^{ij} D^\alpha f_j, \quad i = 1, \dots, q.$$

A differential operator is said to be of order k if there are no derivatives of order $\geq k + 1$ appearing in a local representation. Let $\text{Diff}_k(E, F)$ denote the vector space of all differential operators of order

k mapping $\mathcal{E}(X, E)$ to $\mathcal{E}(X, F)$.

Example D.2 If (M, Λ_0) is a holomorphic Poisson manifold, then

$$L = \bar{\partial} + [\Lambda_0, -] : A^{0,p-1}(M, T) \oplus \cdots \oplus A^{0,0}(M, \wedge^p T) \rightarrow A^{0,p}(M, T) \oplus A^{0,0}(M, \wedge^{p+1} T)$$

is a differential operator of order 1.

Definition D.3 Let $L : \mathcal{E}(X, E) \rightarrow \mathcal{E}(X, F)$ is differential operator of order k . Let $T_{\mathbb{R}}^*(X)$ be the real cotangent bundle to a differentiable manifold X , let $T'_{\mathbb{R}}(X)$ denote $T_{\mathbb{R}}^*(X)$ with the zero section deleted (the bundle of nonzero cotangent vectors), and let $T'_{\mathbb{R}}(X) \xrightarrow{\pi} X$ denote the projection mapping. Then π^*E and π^*F denote the pullback of E and F over $T'_{\mathbb{R}}(X)$. We set, for any $k \in \mathbb{Z}$,

$$\text{Smb}_k(E, F) := \{\sigma \in \text{Hom}(\pi^*E, \pi^*F) : \sigma(x, \rho v) = \rho^k \sigma(x, v), (x, v) \in T'_{\mathbb{R}}(X), \rho > 0\}$$

We now define a linear map

$$\sigma_k : \text{Diff}_k(E, F) \rightarrow \text{Smb}_k(E, F)$$

where $\sigma_k(L)$ is called the k -symbol of the differential operator L . To define $\sigma_k(L)$, we first note that $\sigma_k(L)(x, v)$ is to be a linear mapping from E_x to F_x , where $(x, v) \in T'_{\mathbb{R}}(X)$. Therefore let $(x, v) \in T'_{\mathbb{R}}(X)$ and $e \in E_x$ be given. Find $g \in \mathcal{E}(X)$ and $f \in \mathcal{E}(X, E)$ such that $dg_x = v$, and $f(x) = e$. Then we define

$$\sigma_k(L)(x, v)e = L \left(\frac{i^k}{k!} (g - g(x))^k f \right) (x) \in F_x.$$

Then this defines a linear mapping

$$\sigma_k(L)(x, v) : E_x \rightarrow F_x$$

which then defines an element of $\text{Symb}_k(E, F)$ and $\sigma_k(L)$ is independent of the choices made. We call $\sigma_k(L)$ the k -symbol of L .

Example D.4 Let (M, Λ) be a holomorphic Poisson manifold. Here we compute a symbol of

$$L = [\Lambda, -] : A^{0,p}(M, \wedge^q T) \rightarrow A^{0,p}(M, \wedge^{q+1} T)$$

First we note that L has order 1. Let $(x, v) \in T'_\mathbb{R} M$ and $e \in (\wedge^p \bar{T}^* \otimes \wedge^q T)_x$ be given. Find $g \in \mathcal{E}(M)$ and $f \in A^{0,p}(M, \wedge^q T)$ such that $dg_x = v$, and $f(x) = e$. Let $\Lambda = \sum_{i,j} \Lambda_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}$ around x and $f = \sum_{r_a, s_b} f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}$ around x . Then

$$\begin{aligned} \sigma_1(L)(x, v)e &= L(i(g - g(x))f)(x) = iL((g - g(x))f)(x) \\ &= i \sum_{r_a, s_b, i, j} [\Lambda_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, (g - g(x))f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}](x) \\ &= i \sum_{r_a, s_b, i, j} [\Lambda_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, (g - g(x))f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}} \wedge \dots \wedge \frac{\partial}{\partial z_{r_q}}] \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}(x) \\ &= i \sum_{r_a, s_b, i, j} [\Lambda_{ij}(z) \frac{\partial}{\partial z_i}, (g - g(x))f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}}] \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_{r_2}} \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}(x) \\ &\quad - \Lambda_{ij}(z) [\frac{\partial}{\partial z_j}, (g - g(x))f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}}] \wedge \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_{r_2}} \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}(x) \\ &= i \sum_{r_a, s_b, i, j} \Lambda_{ij}(z) \frac{\partial g}{\partial z_i} f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}} \wedge \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_{r_2}} \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}(x) \\ &\quad - \Lambda_{ij}(z) \frac{\partial g}{\partial z_j} f_{r_1 \dots r_q s_1 \dots s_p}(z) \frac{\partial}{\partial z_{r_1}} \wedge \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_{r_2}} \dots \wedge \frac{\partial}{\partial z_{r_q}} \wedge d\bar{z}_{s_1} \dots \wedge d\bar{z}_{s_p}(x) \\ &= -i \left(\sum_{i,j} \Lambda_{ij}(z) \frac{\partial g}{\partial z_i} \frac{\partial}{\partial z_j} - \Lambda_{ij}(z) \frac{\partial g}{\partial z_j} \frac{\partial}{\partial z_i} \right) \wedge f(x) \\ &= -i \Lambda_x(v^{1,0}) \wedge e \end{aligned}$$

where $v^{1,0} = \sum_{k=1}^n \frac{\partial g}{\partial z_k} dz_k|_x$.

Example D.5 Let (M, Λ) be a holomorphic Poisson manifold. Here we compute a symbol of

$$L = \bar{\partial} : A^{0,p}(M, \wedge^q T) \rightarrow A^{0,p}(M, \wedge^{q+1} T)$$

First we note that L has order 1. Let $(x, v) \in T'_\mathbb{R}(M)$ and $e \in (\wedge^p \bar{T}^* \otimes \wedge^q T)_x$ be given. Find $g \in A^{0,0}(M)$ and $f \in A^{0,p}(M, \wedge^q T)$ such that $dg_x = v$, and $f(x) = e$. Let $f = \sum_{r_a, s_b} f_{r_1 \dots r_q s_1 \dots s_p}(z) d\bar{z}_{s_1} \dots \wedge$

$d\bar{z}_{s_p} \wedge \frac{\partial}{\partial z_{r_1}} \wedge \cdots \wedge \frac{\partial}{\partial z_{r_q}}$ around x . Then

$$\begin{aligned} \sigma_1(L)(x, v)e &= L(i(g - g(x))f)(x) = iL((g - g(x))f)(x) \\ &= i \sum_{r_a, s_b, i} \frac{\partial g}{\partial \bar{z}_i} d\bar{z}_i \wedge f(x) \\ &= iv^{0,1} \wedge e \end{aligned}$$

Remark D.6 Let (M, Λ) be a holomorphic Poisson manifold and let

$$\begin{aligned} L = \bar{\partial} + [\Lambda_0, -] : A^p &= A^{0, p-1}(M, T) \oplus \cdots \oplus A^{0, 0}(M, \wedge^p T) \rightarrow A^{p+1} = \\ &A^{0, p}(M, T) \oplus \cdots \oplus A^{0, 0}(M, \wedge^{p+1} T) \end{aligned}$$

Let $(x, v) \in T'_\mathbb{R}(M)$ and $g \in \mathcal{E}(M)$ such that $dg_x = v$. Then $\sigma_1(L)(x, v)e = iv^{0,1} \wedge e - i\Lambda_x(v^{1,0}) \wedge e$

Definition D.7 Let E_0, \dots, E_N be a sequence of differentiable vector bundles defined over a compact differentiable manifold X . Suppose that there is a sequence of differential operators, of fixed order k , L_0, \dots, L_{N-1} mapping as in the following sequence

$$\mathcal{E}(X, E_0) \xrightarrow{L_0} \mathcal{E}(X, E_1) \xrightarrow{L_1} \mathcal{E}(X, E_2) \rightarrow \cdots \xrightarrow{L_{N-1}} \mathcal{E}(X, E_N).$$

Associated with the sequence is the associated symbol sequence

$$0 \rightarrow \pi^* E_0 \xrightarrow{\sigma(L_0)} \pi^* E_1 \xrightarrow{\sigma(L_1)} \pi^* E_2 \rightarrow \cdots \xrightarrow{\sigma(L_{N-1})} \pi^* E_N \rightarrow 0$$

Here we denote by $\sigma(L_i)$ the k -symbol of the operator L_j . The sequence of operators and vector bundle E_i , the sequence is called a complex if $L_i \circ L_{i-1} = 0, i = 1, \dots, N-1$. Such a complex is called an elliptic complex if the associated symbol sequence is exact.

Notation 15 Denote by

$$\mathcal{E} = \bigoplus_{j=0}^N \mathcal{E}(E_j)$$

the graded vector space so obtained with the natural grading. We define a operator L by letting

$$L(\xi) = L(\xi_0 + \cdots + \xi_N) = L_0\xi_0 + \cdots + L_N\xi_N$$

where $\xi = \xi_0 + \dots + \xi_N$ is the decomposition of $\xi \in \mathcal{E}(E)$ into homogeneous component corresponding to the above grading.

Proposition D.8 *Let (M, Λ) be a holomorphic Poisson manifold. Let $A^p = A^{0,p-1}(M, T) \oplus \dots \oplus A^{0,0}(M, \wedge^p T)$ and $L = \bar{\partial} + [\Lambda, -]$. Then*

$$0 \xrightarrow{L} A^1 \xrightarrow{L} \dots \rightarrow A^n \xrightarrow{L} A^{n+1} \rightarrow 0$$

is elliptic.

Proof. Let $(x, v) \in T_{\mathbb{R}}^1(X)$ and $g \in \mathcal{E}(M)$ such that $dg_x = v$. Then the associated symbol sequence is a total complex of the following bicomplex

$$\begin{array}{ccccccc}
& & -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow \\
& & \wedge^3 T_x & \xrightarrow{iv^{0,1}\wedge} & \bar{T}_x^* \otimes \wedge^3 T_x & \xrightarrow{iv^{0,1}\wedge} & \wedge^2 \bar{T}_x^* \otimes \wedge^3 T_x & \xrightarrow{iv^{0,1}\wedge} \\
& -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow & & \\
& & \wedge^2 T_x & \xrightarrow{iv^{0,1}\wedge} & \bar{T}_x^* \otimes \wedge^2 T_x & \xrightarrow{iv^{0,1}\wedge} & \wedge^2 \bar{T}_x^* \otimes \wedge^2 T_x & \xrightarrow{iv^{0,1}\wedge} \\
& -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow & & -i\Lambda_x(v^{1,0})\wedge \uparrow & & \\
& & T_x & \xrightarrow{iv^{0,1}\wedge} & \bar{T}_x^* \otimes T_x & \xrightarrow{iv^{0,1}\wedge} & \wedge^2 \bar{T}_x^* \otimes T_x & \xrightarrow{iv^{0,1}\wedge} \\
& & \uparrow & & \uparrow & & \uparrow & \\
& & 0 & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow
\end{array}$$

Indeed, the above diagram is bicomplex since $-i\Lambda_x(v^{1,0}) \wedge iv^{0,1} + iv^{0,1} \wedge (-i\Lambda_x(v^{1,0})) = 0$. Note that each row is exact. So the total complex is exact by the following lemma. Hence the sequence is elliptic. ■

Lemma D.9 (Acyclic Assembly Lemma) *Let C be a bounded double complex in the category of \mathbb{C} -vector spaces. Then the total complex is an acyclic chain complex if C is an upper half-plane complex with exact rows.*

Proof. See [Wei94] Lemma 2.7.3 p.59. ■

Theorem D.10 *Let $(\mathcal{E}(E) = \bigoplus_{j=0}^N \mathcal{E}(E_j), L)$ be an elliptic complex equipped with an inner product. Then*

1. There is an orthogonal decomposition

$$\mathcal{E} = \mathbb{H}(E) \oplus LL^*G\mathcal{E}(E) \oplus L^*LG\mathcal{E}(E)$$

2. The following commutation relations are valid:

$$(a) I = H + \square G = H + G\square$$

$$(b) HG = GH = H\square = \square H = 0$$

$$(c) L\square = \square L, L^*\square = \square L^*$$

$$(d) LG = GL, L^*G = GL^*$$

3. $\dim_{\mathbb{C}} \mathbb{H}(E) < \infty$, there is a canonical isomorphism

$$\mathbb{H}^j = \mathbb{H}(E_j) \cong H^j(E) : j\text{-th cohomology group with coefficient } E$$

where $\square = LL^* + L^*L$ and $\mathbb{H}(E) = \bigoplus \mathbb{H}(E_j)$ is the total space of \square -harmonic sections. G is a Green operator and H is the orthogonal projection onto the closed subspace $\mathbb{H}(E)$.

Proof. See [Wel08] Theorem 5.2 p.147. ■

Proposition D.11 Let $\xi \in \mathcal{E}(E)$. Then $\square\xi = 0$ if and only if $L\xi = L^*\xi = 0$; moreover, $LH = HL = L^*H = HL^* = 0$.

Proof. See [Wel08] Proposition 5.3 p.147. ■

Remark D.12 Let (M, Λ) be a holomorphic Poisson manifold. Let $L = \bar{\partial} + [\Lambda, -]$. And we have a Hermitian inner product on A^p . Since

$$0 \xrightarrow{L} A^1 \xrightarrow{L} \cdots \rightarrow A^n \xrightarrow{L} A^{n+1} \rightarrow 0$$

is elliptic, we can apply the above Theorem D.10 and Proposition D.11 .

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