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Cohomological Invariants of Algebraic Tori

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Cohomological Invariants of Algebraic Tori

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ABSTRACT OF THE DISSERTATION

Cohomological Invariants of Algebraic Tori

by

Semyon Blinstein

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2012

Professor Alexander Merkurjev, Chair

Given a field F of arbitrary characteristic and an algebraic torus T/F, we calculate degree 2 and 3 cohomological invariants of T with values in $\mathbb{Q}/\mathbb{Z}(1)$ and $\mathbb{Q}_p/\mathbb{Z}_p(2)$, respectively, the latter for $p \neq 2, \mathrm{char}(F)$, and generalize the former to other algebraic groups. Moreover, we obtain descriptions of the corresponding unramified cohomology groups, and in particular of $H^3_{\mathrm{nr}}(F(T), \mu_n^{\otimes 2})$ for n prime to 2 and $\mathrm{char}(F)$. In the process, we construct a useful short exact sequence for cohomological invariants and make connections with recent results on Chow groups of codimension 2.

The dissertation of Semyon Blinstein is approved.

Michael Gutperle
Christian Haesemeyer
Richard Elman
Alexander Merkurjev, Committee Chair

University of California, Los Angeles
2012

To my parents, for everything

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My experience in graduate study was fraught with difficultly and stress, and the introspection they induced, punctuated occasionally by moments of true understanding and the temporary elation that follows. This dual existence was lovingly and resolutely handled by my wife Sarah, a woman of incredible character, sharp intellect, and unmatched beauty. Her love keeps me tethered to the Earth and I owe my sanity to the space we have selfishly dug for ourselves in this world. My academic achievements pale in comparison to having convinced you to share your time with me.

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Vita

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Publications and Presentations

S. Blinstein and A. Merkurjev. "Cohomological Invariants of Algebraic Tori," preprint, 2012.

J. Pollanen, K.R. Shirer, S. Blinstein, J.P. Davis, H. Choi, T.M. Lippman, W.P. Halperin, and L.B. Lurio. "Globally Anisotropic High Porosity Silica Aerogels." *J. Non-Crystalline Solids* **354**:4668–4674, 2008.

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Introduction

Fix a base field F of characteristic $q \geq 0$ and let F_{sep} be a separable closure. Write Γ_F, or simply Γ, for the absolute Galois group $\mathrm{Gal}(F_{\mathrm{sep}}/F)$ and $H^i(F, C)$ for the Galois cohomology of a Γ-module C (see §1.1 for the precise definition). Let Fields_F be the category of field extensions of F with F-homomorphisms. If G/F is an algebraic group, i.e. a connected smooth affine group scheme of finite-type over F, and K/F any field extension, we write $H^1(K, G)$ for the (pointed) Galois cohomology set $H^1(K, G(K_{\mathrm{sep}}))$, which classifies G_K-torsors over K (= principal homogeneous spaces) up to isomorphism [Mil80, §III.4]. The distinguished element corresponds to the (isomorphism class of the) trivial torsor, which is represented by G itself with G acting by multiplication. A (degree i) *cohomological invariant*, or simply *invariant*, η of G with values in C is a natural transformation of functors

$$\eta : H^1(-, G) \longrightarrow H^i(-, C),$$

viewing both functors from Fields_F to PSets, the category of pointed sets. In particular, any such natural transformation maps the distinguished element of $H^1(K, G)$ to the trivial element in $H^i(K, C)$ (see Remark 0.1(b) below). Thus, for any extension K/F, η assigns to each G_K-torsor a degree i cohomology class with coefficients in C, and for any F-embedding $K \longrightarrow L$ one has a commutative diagram

$$\begin{tikzcd}
H^1(K, G) \arrow[r, "\eta_K"] \arrow[d] & H^i(K, C) \arrow[d] \\
H^1(L, G) \arrow[r, "\eta_L"] & H^i(L, C).
\end{tikzcd}$$

We denote by $\mathrm{Inv}^i(G, C)$ the set of degree i cohomological invariants of G with values in C, which has the natural structure of an abelian group under the pointwise operation. Although invariants have been known for some time, their study did not appear in the literature until [GMS03], which we follow throughout this section for some of the basic constructions and examples. We refer to [GMS03, Part 1, App. B] for details on their historical development and context.

Remark 0.1.

(a) Because the functor $H^1(-,G)$ commutes with direct limits, a cohomological invariant is determined by its restriction to the category of finitely generated extensions of F and we will see in §1.2 that there always exists a finitely generated extension E/F and a G-torsor $\mathscr{T} \in H^1(E,G)$ such that every cohomological invariant is determined by its value at $\mathscr{T}$. In fact, even more is true.

(b) Although it is possible to consider natural transformations of the given functors as functors taking values in simply Sets, this offers no new insight. In this case, there is an embedding $H^i(F,C) \hookrightarrow \mathrm{Inv}^i(G,C)$ given by $h \longmapsto a_h$, where the natural transformation a_h is defined as follows: given an extension K/F and a torsor $\mathscr{T} \in H^1(K,G)$, $a_h(\mathscr{T}) = h_K$, where h_K is the image of h under the natural map $H^i(F,G) \longrightarrow H^i(K,G)$. Such invariants are called *constant* and the subgroup of constant invariants is identified with $H^i(F,C)$. In this setting, an invariant is said to be *normalized* if it vanishes on the distinguished element of $H^1(F,G)$, which is equivalent to it being a natural transformation of functors with values in PSets. It is clear then that the invariants group would decompose as a direct sum of $H^i(F,C)$ and the normalized invariants, so it suffices to consider only normalized invariants as we do.

(c) More generally, invariants can be defined for any two functors $A : \mathsf{Fields}_F \longrightarrow \mathsf{Sets}$ and $H : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$ and there are interesting examples of functors A that are not of the form $H^1(-,G)$ for any algebraic group G and functors H which are not Galois cohomology (*cf.* [GMS03, Part 1, §3 and §4]). See [Mer99] for an investigation of invariants when $A = G$ is itself an algebraic group over F viewed as a functor of points and H varies over a large class of functors (discussed in §1.3) which includes Galois cohomology as a special case.

Example 0.1.

(a) For certain choices of G the pointed set $H^1(K,G)$ classifies well-known objects. When G is the constant group scheme of the symmetric group S_n, $H^1(K,G)$ classifies étale

K-algebras of rank n, i.e. products $\prod_i L_i$ of finite separable extensions L_i/K such that $\sum_i [L_i : K] = n$. For $G = \mathbf{PGL}_n$, $H^1(K, G)$ classifies central simple algebras over K of degree n, i.e. K-algebras of dimension n^2 with center K $(= K \cdot 1)$ and no nontrivial two-sided ideals. In both cases, the maps $H^1(K, G) \longrightarrow H^1(L, G)$ for an embedding of fields $K \longrightarrow L$ over F correspond to "extension of scalars," i.e., tensoring with L over K. For more examples see [GMS03, Part 1, §3] and [KMR98, §29].

(b) Let n be prime to q and set $G = \mathbf{PGL}_n$. If $C = \mu_n$ is the group of n^{th}-roots of unity in $F_{\text{sep}}^{\times}$, the Kummer sequence and Hilbert's Theorem 90 imply that $H^2(K, \mu_n) \cong {}_n\text{Br}(K)$, the elements of exponent dividing n in $\text{Br}(K)$. One can consider the invariant $a \in \text{Inv}^2(\mathbf{PGL}_n, \mu_n)$ given by mapping a central simple algebra to its corresponding class in $\text{Br}(K)$.

(c) Let A be a central simple algebra over F of exponent e in $\text{Br}(F)$ and let $\mathbf{SL}_1(A)$ be the kernel of the reduced norm homomorphism $\text{Nrd} : \mathbf{GL}_1(A) \longrightarrow \mathbf{G}_m$ (*cf.* [KMR98, §20]). Torsors are given by $H^1(F, \mathbf{SL}_1(A)) \cong F^{\times}/\,\text{Nrd}(A^{\times})$. One can define an invariant $r \in \text{Inv}^3(\mathbf{SL}_1(A), \mu_e^{\otimes 2})$ by the formula

$$r(a \cdot \text{Nrd}(A^{\times})) = \overline{a} \cup [A] \in H^3(F, \mu_e^{\otimes 2}),$$

where $\overline{a}$ is the class of a in $F^{\times}/F^{\times e} \cong H^1(F, \mu_e)$, $[A] \in {}_e\text{Br}(F) \cong H^2(F, \mu_e)$ is the Brauer class of A, and the cup-product is induced by the natural pairing $\mu_e \times \mu_e \longrightarrow \mu_e^{\otimes 2}$. This invariant is called the *Rost invariant* and is of particular interest because it is, in fact, the canonical generator of $\text{Inv}^3(\mathbf{SL}_1(A), \mu_e^{\otimes 2})$, a finite cyclic group of order e (*cf.* [KMR98, §31], [Mer02, App. B], and [GMS03, Part 2, §9 and §10] for details). See [Mer99, §5] for the Rost invariant in the context of invariants of the functors mentioned at the end of Remark 0.1(c) above.

Our motivation for considering invariants of tori comes from their connection with unramified cohomology (defined in §1.4 below). Specifically, this work began as an investigation of a question posed by Colliot-Thélène in [Col95, p. 39]: for n prime to q and $i \geq 0$, calculate $H^i_{\text{nr}}(F(T), \mu_n^{\otimes(i-1)})$, where $F(T)$ is the function field of a torus T/F and $H^i_{\text{nr}}(F(T), \mu_n^{\otimes i})$

is its unramified cohomology. In this work, we recover (Theorem 3.2) and elaborate on (Theorem 3.1 and its Corollary) the previously known case $i = 2$ and derive a new formula when $i = 3$ (§5, esp. Theorem 5.12 and Remark 5.6), the latter up to 2-torsion. We also illustrate how the $i = 2$ case easily generalizes to arbitrary groups in characteristic 0 and to reductive groups in positive characteristic (Remark 3.3) thanks to more recent work by Colliot-Thélène. The connection with invariants is provided by a result originally due to Rost (Theorem 1.12), which states that for a particular smooth variety X/F, one has an isomorphism $H^i(F, C) \oplus \mathrm{Inv}^i(G, C) \cong A^0(X, H^i[C])$. Here, $A^0(X, H^i[C])$ is the subgroup of $H^i(F(X), C)$ consisting of classes which are "unramified along every divisor of X":

$$A^0(X, H^i[C]) = \bigcap_{x \in X^{(1)}} \ker\left[\, H^i(F(X), C) \xrightarrow{\partial_x} H^{i-1}(F(x), C(-1)) \,\right],$$

where $F(X)$ is the function field of X, ∂_x is the residue homomorphism corresponding to the discrete valuation associated to a codimension one point $x \in X$ (*cf.* §1.3), $F(x)$ is the residue field at x, and $C(-1) = \mathrm{Hom}(\mu_n, C)$ when C is finite of exponent n and $\mathrm{Hom}(\varprojlim \mu_n, C)$ when C is torsion with all elements of order prime to q (see Remark 1.3(a) for restrictions on C). For a construction of this group in the larger context of cycle modules, see §1.3 below. The unramified cohomology group $H^i_{\mathrm{nr}}(F(X), C)$ is the subgroup of $A^0(X, H^i[C])$ consisting of classes which vanish under the residue homomorphism corresponding to *any* DVR A which contains F and has quotient field $F(X)$. Therefore, we sometimes refer to the groups $A^0(X, H^i[C])$ as *partially unramified* groups. The isomorphism $H^i(F, C) \oplus \mathrm{Inv}^i(G, C) \cong A^0(X, H^i[C])$ mentioned above then allows one to define unramified invariants $\mathrm{Inv}^i_{\mathrm{nr}}(G, C) \subseteq \mathrm{Inv}^i(G, C)$ as the subgroup corresponding to $H^i_{\mathrm{nr}}(F(X), C) \subseteq A^0(X, H^i[C])$ (see Construction 1.13 for details). Although by definition unramified invariants calculate unramified cohomology of (the function field of) X, for an arbitrary torus T one can in fact find a torus S such that $H^i_{\mathrm{nr}}(F(T), C) \cong H^i(F, C) \oplus \mathrm{Inv}^i_{\mathrm{nr}}(S, C)$ (Proposition 1.14). Moreover, S satisfies $\mathrm{Inv}^i_{\mathrm{nr}}(S, C) \cong \mathrm{Inv}^i(S, C)$ (Proposition 1.15). Turning this result around, we will see that $\mathrm{Inv}^i_{\mathrm{nr}}(T, C) \cong \mathrm{Inv}^i(N, C)$ for a torus N with the same properties as S (Proposition 1.16). Therefore, calculation of unramified cohomology and unramified invariants of tori is reduced to the determination of (ordinary) invariants of auxiliary tori.

In §4 we will see that computing this last group involves working with partially unramified groups of auxiliary varieties, hence reducing questions regarding (totally) unramified groups to partially unramified ones.

The structure of this work is as follows. In §1 we review essential constructions and define $\mathbb{Q}/\mathbb{Z}(i)$ and related modules. In particular, in §1.4, we define unramified cohomology and unramified invariants and investigate the relationship between the two in the case of tori (Propositions 1.14, 1.15, 1.16). In §2, we calculate $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ for a an algebraic group G/F, assumed reductive if F is not perfect (Theorem 2.2), recovering [KMR98, Prop. 31.19] when $q = 0$ (*cf.* Remark 2.1(b)). In §3 we apply the results of the previous section to determine $H^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1))$ (Theorem 3.1 and its Corollary) and $\mathrm{Inv}^2_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(1))$ for a torus T/F. We also recover the characterization originally given in [Col95, p. 39] (Theorem 3.2) and generalize the result to arbitrary groups in characteristic 0 and reductive groups in positive characteristic (Remark 3.3). In §4 we construct a short exact sequence relating (ordinary) invariants to the partially unramified groups of some auxiliary varieties and obtain the results of §2 (and therefore also of §3) for tori as a corollary (Remark 4.1). This sequence is then used in §5 to calculate $\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for $p \neq 2, q$ (Theorem 5.12), leading to formulas for degree 3 unramified cohomology of tori (Remark 5.6). In §6 we use the results on degree 3 invariants to obtain a new equivalence statement for tori. Finally, we conclude in §7 with connections to Pirutka's recent results [Pir11] on the failure of surjectivity of the natural map $CH^2(X) \longrightarrow CH^2(X_{\mathrm{sep}})^{\Gamma}$, where for an F-variety X we write X_{sep} for $X \times \mathrm{Spec}(F_{\mathrm{sep}})$.

Notation: Along with the notation already introduced above, we will use the following. By an algebraic variety X/F we mean an integral separated scheme of finite-type over F and denote by $X^{(i)}$ the points of codimension i in X. For any point $x \in X$ we write $F(x)$ for the residue field at x. We write $F[X]$ for the ring of regular functions of X and $F(X)$ for its function field. We let $F_{\mathrm{sep}}[X]$ and $F_{\mathrm{sep}}(X)$ denote the ring of regular functions and the function field of X_{sep}, respectively. For any field extension K/F we let $X_K = X \times \mathrm{Spec}(K)$

(except we write simply X_{sep} when $K = F_{\text{sep}}$). For any scheme Y, $\text{Br}(Y)$ will mean the cohomological Brauer group $H^2_{\text{ét}}(Y, \mathbf{G}_m)$. We write $\mu_n \subset F^{\times}_{\text{sep}}$ for the group of n^{th}-roots of unity. For an abelian group A and a prime p we write $A\{p\}$ for the p-primary component and $A_{(p)}$ for the localization of A at p. If A is a torsion abelian group then $A_{(p)}$ can be identified with $A\{p\}$. We write ${}_{\text{n}}A$ for the n-torsion of A, i.e., the kernel of the multiplication by n map $A \longrightarrow A$. We will abbreviate by A' the subgroup of A of all elements of (finite) order prime to q and refer to the q-component $A\{q\}$ as the *characteristic component*. Tensor products without a subscript are assumed to be taken over $\mathbb{Z}$. We tend to use $\Gamma, \mathfrak{g}, \mathfrak{h}$ for profinite groups and G, H for finite (abstract) groups when discussing both. We also use the latter symbols for algebraic groups. Most other notations are introduced in §1 as specific constructions are reviewed.

1 Preliminaries

1.1 Algebraic Tori and Resolutions

All the constructions and results in this subsection can be found in [CS77], but we review them here briefly for completeness. Let $\mathfrak{g}$ be a profinite group. By a $\mathfrak{g}$-module A we will mean a discrete abelian group with a $\mathfrak{g}$-action which satisfies any of the following equivalent statements: (i) the action map $\mathfrak{g} \times A \longrightarrow A$ is continuous for the product topology, (ii) $A = \bigcup_{\mathfrak{h}} A^{\mathfrak{h}}$, where the union is taken over all open subgroups $\mathfrak{h} \leq \mathfrak{g}$ and $A^{\mathfrak{h}} \leq A$ is the subgroup of elements fixed by $\mathfrak{h}$, or (iii) the stabilizer for each $a \in A$ is an open subgroup of $\mathfrak{g}$. Note also that the action respects A's group structure: $g \cdot (a + b) = g \cdot a + g \cdot b$ for all $g \in \mathfrak{g}$ and $a, b \in A$. All morphisms between $\mathfrak{g}$-modules are assumed $\mathfrak{g}$-equivariant. We write $\mathscr{L}_{\mathfrak{g}}$ for the category of finitely generated torsion-free $\mathfrak{g}$-modules, sometimes referred to as $\mathfrak{g}$-lattices. For $M, N \in \mathscr{L}_{\mathfrak{g}}$, write $\text{Hom}(M, N)$ for the $\mathfrak{g}$-module $\text{Hom}_{\mathbb{Z}}(M, N)$ with action $\sigma \cdot f = \sigma f \sigma^{-1}$ for $\sigma \in \mathfrak{g}$, and let $M \otimes N$ be the $\mathfrak{g}$-module with diagonal action. Denote by M^0 the *dual* module $\text{Hom}(M, \mathbb{Z})$, where $\mathbb{Z}$ has trivial $\mathfrak{g}$-action. A $\mathfrak{g}$-module P is called a *permutation* module if it has a $\mathbb{Z}$-basis permuted by $\mathfrak{g}$. Considering the stabilizers of a basis element from each orbit, it is clear that such a module is of the form $P \cong \bigoplus_i \mathbb{Z}[\mathfrak{g}/\mathfrak{h}_i]$ for some open subgroups $\mathfrak{h}_i \leq \mathfrak{g}$, hence $P \cong P^0$. A module I is called *invertible* if it is a direct summand (as a $\mathfrak{g}$-module) of a permutation module. Two modules $M, N \in \mathscr{L}_\mathfrak{g}$ are said to be *similar* if they are isomorphic up to addition of permutation modules: $M \oplus P_1 \cong N \oplus P_2$ for P_1, P_2 permutation modules; we write $[M]$ for the similarity class of M. The similarity classes of $\mathscr{L}_\mathfrak{g}$ form (under direct sum) a commutative monoid $S_\mathfrak{g}$ which contains the subgroup $U_\mathfrak{g}$ of invertible modules. The duality $M \longmapsto M^0$ induces an involution of $S_\mathfrak{g}$ which preserves $U_\mathfrak{g}$. If $[M] = [0]$ we say that M is *stably permutation*.

For the next construction we start with a finite group G. A G-module M is called *flasque* (resp. *coflasque*) if $H^1(H, M^0) = 0$ (resp. $H^1(H, M) = 0$) for all subgroups $H \leq G$; in particular, M is flasque if and only if M^0 is coflasque. A more common characterization is that M is flasque if and only if $\widehat{H}^{-1}(H, M) = 0$ for all subgroups $H \leq G$, where $\widehat{H}^i$ denotes the Tate cohomology groups (*cf.* [Ser79, Ch. VIII, §1]). For these and other characterizations of flasque and coflasque modules, see [CS77, Lemme 1]. Permutation modules are both flasque and coflasque, hence so are invertible modules. In particular, the groups $H^1(H, M)$ and $\widehat{H}^{-1}(H, M)$ are independent of the choice of representative of $[M]$ so we write F_G (resp. F_G^0) for the sub-monoid of S_G consisting of similarity classes of flasque (resp. coflasque) modules. Taking duals induces an isomorphism $F_G \cong F_G^0$.

A *flasque resolution* (resp. *coflasque resolution*) of a module $M \in \mathscr{L}_G$ is an exact sequence of G-modules $0 \longrightarrow M \longrightarrow P \longrightarrow S \longrightarrow 0$ (resp. $0 \longrightarrow Q \longrightarrow P \longrightarrow M \longrightarrow 0$) with P permutation and S flasque (resp. Q coflasque). Therefore, any coflasque resolution of M defines, by duality, a flasque resolution, and vice versa. Coflasque, hence flasque, resolutions exist by [CS77, Lemme 3].

For a profinite group $\mathfrak{g}$ and $\mathfrak{g}$-lattice M we can extend the above definition by saying that M is flasque (resp. coflasque) if $H^1(\mathfrak{h}, M^0) = 0$ (resp. $H^1(\mathfrak{h}, M) = 0$) for all open subgroups $\mathfrak{h} \leq \mathfrak{g}$. Moreover, the Galois cohomology groups satisfy ([Ser97, Ch. I, §2, Cor. 1])

$$H^i(\mathfrak{g}, M) = \operatorname*{colim}_{\mathfrak{n}} H^i(\mathfrak{g}/\mathfrak{n}, M^\mathfrak{n}), \quad i \geq 0,$$

where the colimit is taken over all open normal subgroups $\mathfrak{n} \leq \mathfrak{g}$. Since M is finitely generated and a continuous $\mathfrak{g}$-module, there exists an open normal subgroup $\mathfrak{h} \leq \mathfrak{g}$ fixing M, i.e. $M^{\mathfrak{h}} = M$, hence $H^i(\mathfrak{g}, M) = H^i(\mathfrak{g}/\mathfrak{h}, M)$. Furthermore, it is equivalent for a $\mathfrak{g}$-module with trivial $\mathfrak{h}$-action to satisfy the above properties (permutation, invertible, flasque, coflasque, etc.) in $\mathscr{L}_{\mathfrak{g}}$ or $\mathscr{L}_{\mathfrak{g}/\mathfrak{h}}$ and thus any flasque (resp. coflasque) resolution of M in $\mathscr{L}_{\mathfrak{g}/\mathfrak{h}}$ is a flasque (resp. coflasque) resolution of M in $\mathscr{L}_{\mathfrak{g}}$. So, it suffices to consider the finite case (see [CS77, §1, esp. Lemme 2]).

We recall the following simple but important result.

Lemma 1.1. ([CS77, Lemme 4 and Lemme 5]) *Let $M \in \mathscr{L}_{\mathfrak{g}}$ and let $0 \longrightarrow M \xrightarrow{i} P \longrightarrow S \longrightarrow 0$ be a flasque resolution and $0 \longrightarrow Q \longrightarrow R \xrightarrow{j} M \longrightarrow 0$ a coflasque resolution. Any morphism from M to a permutation module P' factors through i and any morphism from P' to M factors through j. Moreover, the similarity classes of S and Q do not depend on the representative of $[M]$.*

By the Lemma, the map $p : S_G \longrightarrow F_G$ given by $M \longmapsto [S]$ is well-defined and we call $p(M)$ the *Picard class* of M.

By an algebraic torus T/F we mean a group scheme T of finite type over F which upon base extension to the separable closure becomes isomorphic to a particular diagonalizable group:

$$T_{\mathrm{sep}} \cong \mathrm{Spec}(F_{\mathrm{sep}}[\mathbb{Z}^n]) \cong \mathbf{G}_m \times \cdots \times \mathbf{G}_m,$$

where $F_{\mathrm{sep}}[\mathbb{Z}^n]$ is the group ring. In particular, tori are smooth. Because they are of finite-type over F, one can in fact find a finite Galois extension L/F which *splits* T, i.e., such that $T_L \cong \mathrm{Spec}(L[\mathbb{Z}^n])$. If $G = \mathrm{Gal}(L/F)$, then the category of algebraic tori split by L is anti-equivalent to the category $\mathscr{L}_G$ by the mapping $T \longmapsto \mathrm{Hom}_{L-\mathrm{gp}}(T_L, \mathbf{G}_m)$ and $M \longmapsto D(M) := \mathrm{Spec}(L[M]^G)$. We say that $D(M)$ is the torus *dual* to the G-module M. We call $\widehat{T}_{\mathrm{sep}} := \mathrm{Hom}_{F_{\mathrm{sep}}-\mathrm{gp}}(T_{\mathrm{sep}}, \mathbf{G}_m)$ the *character module* of T and we write $\widehat{T} = \mathrm{Hom}_{F-\mathrm{gp}}(T, \mathbf{G}_m)$ for the characters over the ground field; we have that $\widehat{T} = \widehat{T}_{\mathrm{sep}}^{\Gamma}$. We call $\widehat{T}_{\mathrm{sep}}^{0} = \mathrm{Hom}(\widehat{T}_{\mathrm{sep}}, \mathbb{Z})$

the *cocharacter module.* If T is split by L/F then $\widehat{T}_{\mathrm{sep}} \cong \mathrm{Hom}_{L-\mathrm{gp}}(T_L, \mathbf{G}_m)$. In this case, $\mathrm{Gal}(F_{\mathrm{sep}}/L)$ acts trivially on $\widehat{T}_{\mathrm{sep}}$ hence the inflation-restriction exact sequence ([Ser79, Ch. VII, §6, Prop. 4]) implies that $H^1(F, \widehat{T}_{\mathrm{sep}}) \cong H^1(G, \widehat{T}_{\mathrm{sep}})$. Given a torus T/F, we call $T^0 := D(\widehat{T}^0_{\mathrm{sep}})$ the *dual torus*; it will always be clear whether we mean the torus $D(M)$ dual to a G-module M or the dual torus T^0 of a given torus T.

A torus T/F is called quasitrivial (resp. invertible, resp. flasque, resp. coflasque) if its character module $\widehat{T}_{\mathrm{sep}}$ is permutation (resp. invertible, resp. flasque, resp. coflasque). Therefore, we define a flasque resolution (resp. coflasque resolution) of a torus T as the exact sequence of tori dual to a flasque resolution (resp. coflasque resolution) of $\widehat{T}_{\mathrm{sep}}$. In particular, if T is split by L/F then all tori in such a resolution are assumed to be as well. If

$$1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$$

is a flasque resolution of T then $[\widehat{S}_{\mathrm{sep}}] = p(\widehat{T}_{\mathrm{sep}})$ so we sometimes refer to S as a *Picard torus* of T; it is defined up to a product with a quasitrivial torus. See Remark 3.3 for generalizations to other groups.

Example 1.2.

(a) We have that $D(\mathbb{Z}) = \mathbf{G}_m$. If $\mathfrak{g} = \mathrm{Gal}(L/F)$ then for any open subgroup $\mathfrak{h} \leq \mathfrak{g}$ corresponding to a Galois extension L/K, $D(\mathbb{Z}[\mathfrak{g}/\mathfrak{h}]) = R_{K/F}(\mathbf{G}_m)$, the Weil restriction. Moreover, $R_{K/F}(\mathbf{G}_m)$ is an open set in $\mathbb{A}^m_F$ with $m = [K : F]$ (*cf.* [Vos98, §3.12]).

(b) Since any permutation module is of the form $\bigoplus_i \mathbb{Z}[\mathfrak{g}/\mathfrak{h}_i]$, we see that a quasitrivial torus is isomorphic to a product $\prod_i R_{K_i/F}(\mathbf{G}_m)$ hence is rational since it can be identified with an open set in the affine space $\mathbb{A}^n_F$ for $n = \sum_i [K_i : F]$. In fact, it is a principle open set in $\mathbb{A}^n_F$. Moreover, by Hilbert's Theorem 90 and the Faddeev-Shapiro-Eckmann Lemma [KMR98, Thm. 29.2 and Lemma 29.6] we see that quasitrivial tori have no nontrivial torsors over any field extension K/F: $H^1(K, P) = 0$. Algebraic groups satisfying this property are called *universally special.*

(c) Consider the exact sequence $0 \longrightarrow I_{\mathfrak{g}/\mathfrak{h}} \longrightarrow \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$, with $\varepsilon\big(\sum_i n_i g_i \mathfrak{h}\big) = \sum_i n_i$ and $\{g_i\mathfrak{h}\}$ a fixed set of cosets; we call ε augmentation. Taking duals we have the sequence $0 \longrightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z}[\mathfrak{g}/\mathfrak{h}] \longrightarrow J_{\mathfrak{g}/\mathfrak{h}} \longrightarrow 0$, where N is determined by $N(1) = \sum_i g_i\mathfrak{h}$. In particular, the $\mathfrak{g}$-invariants $(J_{\mathfrak{g}/\mathfrak{h}})^{\mathfrak{g}}$ are trivial. We use the standard notation $R^{(1)}_{K/F}(\mathbf{G}_m) = D(J_{\mathfrak{g}/\mathfrak{h}})$ and refer to it as the torus of norm 1 elements in the extension K/F. Thus, we have an exact sequence

$$1 \longrightarrow R^{(1)}_{K/F}(\mathbf{G}_m) \longrightarrow R_{K/F}(\mathbf{G}_m) \longrightarrow \mathbf{G}_m \longrightarrow 1.$$

We make critical use of the module $J_{\mathfrak{g}/\mathfrak{h}}$ in Examples 6.4(b) and 7.3.

1.2 Versal Torsors and Classifying Varieties

The following is the key object in the study of cohomological invariants.

Definition 1.3. Let G/F be an algebraic group. A *versal G-torsor* is a G-torsor $\mathcal{E} \longrightarrow X$ over a smooth F-variety X such that for every extension K/F and every G-torsor $\mathscr{T}$ over K there exists an $x \in X(K)$ such that $\mathscr{T} \cong \mathcal{E}_x$, where $\mathcal{E}_x := \mathcal{E} \times_X \operatorname{Spec}(K)$ is the fiber of $\mathcal{E}$ over x. We call X the *classifying variety for G* and for $\xi \in X$ the generic point we refer to $\mathcal{E}_\xi$ as the *generic torsor*. That is, we have the pullback diagram

$$\begin{tikzcd}
\mathscr{T} \cong \mathcal{E}_x \arrow[r] \arrow[d] & \mathcal{E} \arrow[d] \\
\operatorname{Spec}(K) \arrow[r, "x"'] & X.
\end{tikzcd}$$

Remark 1.1. This definition agrees with [GMS03, Part 1, Def. 5.1], except we don't require that for K infinite the set of $x \in X(K)$ such that $\mathscr{T} \cong \mathcal{E}_x$ is dense in X. Also, the nomenclature is slightly different: there, our generic torsor $\mathcal{E}_\xi \in H^1(F(X), G)$ is called the versal torsor.

We reproduce the following construction of versal torsors because the details illuminate the classifying nature of the variety X and allow one to easily determine when certain torsors are versal. For a slightly different construction, see [GMS03, Part 1, Example 5.4].

Lemma 1.4. (*cf.* [GMS03, Part 1, §5.3]) *Versal torsors exist.*

Proof. Choose an embedding $G \hookrightarrow S$ into a universally special (*cf.* Example 1.2(b)) algebraic group S (e.g., $S = \mathbf{GL}_n, \mathbf{SL}_n$) and let $X = S/G$ be the homogeneous space. We have that for any extension K/F

$$X(K) = \Big[S(K_{\mathrm{sep}})\big/G(K_{\mathrm{sep}})\Big]^{\Gamma_K},$$

the Γ_K-invariants of the set of left cosets. We claim that the natural map $S \longrightarrow X$ is a versal G-torsor. It is obviously a G-torsor. By [Ser97, §I.5.4, Prop. 36] and the fact that S is universally special, we have the exact sequence

$$0 \longrightarrow G(K) \longrightarrow S(K) \longrightarrow X(K) \longrightarrow H^1(K, G) \longrightarrow 0,$$

hence for any $\mathscr{T} \in H^1(K, G)$, $\mathscr{T} \cong S_x$ for some $x \in X(K)$ and, in fact, such an x is unique up to the action of $S(K)$ on $X(K)$, which is given by left coset translation. ☐

Remark 1.2. We will use the construction in the previous lemma exclusively to obtain versal torsors. Moreover, we will have occasion for the associated action map

$$S \times X \xrightarrow{a} X.$$

Indeed, we let the G-torsor structure of the natural map $\mu : S \longrightarrow X$ be right multiplication, hence if we let G act on $S \times S$ trivially in the first factor and by right multiplication in the second then we have

$$(S \times S)\big/G \cong S \times X.$$

But the composition

$$S \times S \xrightarrow{m} S \xrightarrow{\mu} X$$

is constant on the G-orbits of this action hence by the universal property of the homogeneous space $(S \times S)/G$, it descends to the action map

$$S \times X \cong (S \times S)\big/G \xrightarrow{a} X.$$

In particular, we see that S acts on X on the opposite side that G acts on S by multiplication.

In our work with tori we will be interested in realizing a given torus as a classifying variety of some auxiliary torus. This is provided by the following observation.

Lemma 1.5. If $1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$ is an exact sequence of tori with P quasitrivial (e.g., a flasque resolution of T or a coflasque resolution of S) then $P \longrightarrow T$ is a versal S-torsor.

Proof. It is obvious that $P \longrightarrow T$ is an S-torsor. Versality follows as in the proof of the previous lemma because P is quasitrivial hence universally special (*cf.* Example 1.2(b)). □

The connection to cohomological invariants is provided by the following result due to Rost.

Theorem 1.6. ([GMS03, Part 1, §11 and §12]) *Let $\mathcal{E} \longrightarrow X$ be a versal G-torsor and $\mathcal{E}_\xi \in H^1(F(X), G)$ its generic torsor. Then the map $r : \mathrm{Inv}^i(G, C) \longrightarrow H^i(F(X), C)$ given by $a \longmapsto a(\mathcal{E}_\xi)$ is an embedding. Moreover, the image of r is contained in the subgroup $A^0(X, H^i[C])$.*

For a definition of $A^0(X, H^i[C])$, see the introduction or §1.3 below. See also Remark 1.4 below for a more accurate description of the image of r. See [Mer02, Thm. 3.2] for a proof of the above injection for a particular type of versal torsor.

1.3 Cycle Modules

We briefly recall some basic constructions from [Ros96], where all details can be found. For a field E, let $K_n(E)$ be *Milnor's K-group* (*cf.* [Mil70]). A *cycle module* over F is an object function

$$M : \mathsf{Fields}_F \longrightarrow \mathsf{Abelian\ Groups}$$

together with a $\mathbb{Z}$-grading $M = \coprod_n M_n$ and some data satisfying certain rules. The data includes:

(i) For every F-embedding $\varphi : E \longrightarrow L$, a degree 0 restriction homomorphism $\varphi_* : M(E) \longrightarrow M(L)$.

(ii) If $[L : E]$ is finite in (i), a degree 0 corestriction homomorphism $\varphi^* : M(L) \longrightarrow M(E)$.

(iii) For every extension E/F, a left $K_*(E)$-module structure on $M(E)$ that respects the gradings: $K_m(E) \cdot M_n(E) \subset M_{m+n}(E)$.

(iv) For every extension E/F having a discrete valuation v which is trivial on F and has residue field κ_v, a degree -1 residue homomorphism $\partial_v : M(E) \longrightarrow M(\kappa_v)$.

Let X/F be an algebraic variety. Write $M(x)$ for $M(F(x))$, where $F(x)$ is the residue field of a point $x \in X$. Using the various rules ([Ros96, §1]) and some additional data ([Ros96, Def. 2.1]) one can construct, for each $n \in \mathbb{Z}$, complexes

$$\cdots \longrightarrow \coprod_{x \in X^{(i-1)}} M_{n-(i-1)}(x) \longrightarrow \coprod_{x \in X^{(i)}} M_{n-i}(x) \longrightarrow \coprod_{x \in X^{(i+1)}} M_{n-(i+1)}(x) \longrightarrow \cdots,$$

and we denote the i^{th} homology group by $A^i(X, M_n)$, which is referred to as the *Chow group (of i-codimensional cycles) with coefficients in* M_n. The name is motivated by the next example.

Example 1.7.

(a) The most basic example of a cycle module over F is Milnor's K-ring K_* (*cf.* [Ros96, Thm. 1.4 and Rem. 2.4]). Moreover, when X is smooth, the group $A^p(X, K_p)$ is just the Chow group $CH^p(X)$ of p-codimensional cycles on X modulo rational equivalence. If X is normal, the group $A^0(X, K_1)$ is naturally isomorphic to the group $F[X]^\times$ of invertible regular functions on X.

(b) For each n relatively prime to q, let $\mu_n \subset F_{\text{sep}}^\times$ be the group of n^{th}-roots of unity. If C is a Γ-module of exponent n, define the i^{th} *Tate twist of* C to be

$$C(i) = \begin{cases} \mu_n^{\otimes i} \otimes_{\mathbb{Z}/n\mathbb{Z}} C & i \geq 0 \\ \operatorname{Hom}(\mu_n^{\otimes -i}, C) & i < 0, \end{cases}$$

where $\mu_n^{\otimes 0} = \mathbb{Z}/n\mathbb{Z}$. If C is infinite torsion with all elements of order prime to q, write $C = \varinjlim C'$, where C' are the finite submodules of C, and set $C(i) = \varinjlim C'(i)$. We then have that for *any* integers i, j, $C(i)(j) = C(i+j)$, hence in particular, $C(-i)(i) = C$. For any integer j and any extension K/F, the assignment

$$K \longmapsto H^*[C(j)](K) := \coprod_{i \geq 0} H^i(K, C(i-j))$$

defines a (cohomological) cycle module $H^*[C(j)]$ over F ([Ros96, Rem. 1.11 and Rem. 2.5]). In particular, if v is a discrete valuation on K which is trivial on F and has residue field κ_v, one constructs (using the Hochschild-Serre spectral sequence, *cf.* [Ros96, Rem. 1.11], [GMS03, Part 1, §7], or [Col95, §3.3]) residue homomorphisms

$$\partial_v : H^i(K, C(i-j)) \longrightarrow H^{i-1}(\kappa_v, C(i-j-1)).$$

We write simply $H^i[C]$ for $H^i[C(i)](-) = H^i(-, C)$ so that for a variety X/F we have

$$A^0(X, H^i[C]) = \ker \left[\; H^i(F(X), C) \xrightarrow{\coprod \partial_x} \coprod_{x \in X^{(1)}} H^{i-1}(F(x), C(-1)) \; \right],$$

which agrees with the notation in the introduction, where we write ∂_x for the residue homomorphism corresponding to the discrete valuation associated to a codimension one point $x \in X$.

Remark 1.3.

(a) The above residue homomorphisms are not defined on the characteristic component, so we always assume that C is either torsion-free or only has torsion prime to q when writing $A^0(X, H^i[C])$ or $H^i_{\mathrm{nr}}(F, C)$ (defined in the next section). Issues related to this restriction are discussed in Remark 2.1(c) and the comments after Remark 5.6 below. See the upcoming paper [BM12] for a more general definition of unramified cohomology which make sense for all components. The authors also associate to particular points on a variety a generalized residue homomorphism which makes sense on all components.

(b) It is possible to define residue homomorphisms ∂_v in the context of étale cohomology, as is done in [Col95, §3.3] for $C = \mu_n^{\otimes j}$, but they may differ by a sign from those defined above. This may require changing ∂_v by a sign to ensure $H^*[C(j)]$ satisfies all the rules for a cycle module, which obviously does not affect $A^0(X, H^i[C])$ or the groups $H^i_{\mathrm{nr}}(K, C)$ defined in the next section. See the footnote in [Ros96, p. 336] for implicit choices in the above construction that could also affect the sign of ∂_v.

The following properties will be essential.

Theorem 1.8. ([Ros96, §3.5, Prop. 8.6, §12]) *Let X and Y be smooth F-varieties and M a cycle module over F. Then*

(i) (*Functoriality*) $A^*(-, M_*)$ are contravariant functors from the category of smooth F-varieties to Abelian Groups. For $g : X \longrightarrow Y$ we call

$$g^* := A^i(g, M_n) : A^i(Y, M_n) \longrightarrow A^i(X, M_n)$$

the *pullback*. Moreover, the pullback induced by $\xi \in X(F(X))$ the generic point,

$$\xi^* : A^0(X, M_n) \longrightarrow A^0(\mathrm{Spec}(F(X)), M_n) = M_n(F(X)),$$

is the natural inclusion.

(ii) (*Homotopy invariance*) The pullback along the projection $p : X \times \mathbb{A}^n_F \longrightarrow X$ induces an isomorphism

$$A^*(X, M_*) \cong A^*(X \times \mathbb{A}^n_F, M_*).$$

Remark 1.4.

(a) Let $\mathscr{F}$ be a contravariant functor from the category of F-varieties to Abelian Groups and let X/F be a variety with structure morphism $s : X \longrightarrow \mathrm{Spec}(F)$. If $x : \mathrm{Spec}(F) \longrightarrow X$ is a rational point then it induces a splitting

$$\mathscr{F}(\mathrm{Spec}(F)) \xrightarrow{\mathscr{F}(s)} \mathscr{F}(X) \xrightarrow{\mathscr{F}(x)} \mathscr{F}(\mathrm{Spec}(F))$$

hence we have a decomposition

$$\mathscr{F}(X) \cong \mathscr{F}(\mathrm{Spec}(F)) \oplus \overline{\mathscr{F}}(X).$$

We refer to $\overline{\mathscr{F}}(X)$ as the *normalized* part of $\mathscr{F}(X)$. In particular, for any cycle module M,

$$A^0(X, M_n) \cong M_n(F) \oplus \overline{A}^0(X, M_n).$$

If $X = G$ is an algebraic group (respectively, $X = G/H$ for some subgroup $H \leq G$) then we always take the decomposition induced by the group identity in $G(F)$ (respectively, by the canonical element in $X(F) = [G(F_{\text{sep}})/H(F_{\text{sep}})]^\Gamma$).

(b) Since all our invariants are functors of *pointed* sets, i.e., they map the distinguished element of $H^1(F, G)$ to the trivial element of $H^i(F, C)$, we see that the Rost embedding of Theorem 1.6 in fact has image in $\overline{A}^0(X, H^i[C])$. Moreover, pullback respects the decomposition so that if $g : Y \longrightarrow X$ and $y \in Y(F)$ (hence $g(y) \in X(F)$) then $g^* : \overline{A}^0(X, M_n) \longrightarrow \overline{A}^0(Y, M_n)$.

(c) In the case $M_n = K_n$, we can describe this splitting explicitly. By construction (*cf.* §1.3)

$$A^0(X, K_n) = \ker \Big[\, K_n(F(X)) \xrightarrow{\coprod \partial_x} \textstyle\coprod_{x \in X^{(1)}} K_{n-1}(F(x)) \, \Big],$$

and for all $y \in F^\times$ we have that $v_x(y) = 0$, where v_x is the discrete valuation associated with a codimension 1 point $x \in X^{(1)}_{\text{sep}}$. Therefore, by [Mil70, Lemma 2.1], we have a factorization

$$\begin{tikzcd}
K_n(F) \arrow[r] \arrow[dr, "i"'] & K_n(F(X)) \\
 & A^0(X, K_n), \arrow[u, hook]
\end{tikzcd}$$

where the unlabeled maps are the natural ones. Since $A^0(\text{Spec}(F_{\text{sep}}), M_n) = M_n(F_{\text{sep}})$ for any cycle module M_*, any point in $X(F)$ splits i:

$$K_n(F) \xrightarrow{i} A^0(X, K_n) \longrightarrow K_n(F).$$

1.4 Unramified Cohomology

The following is the motivating construction for this work and was first defined in full generality (for $C = \mu_n^{\otimes j}$) by Colliot-Thélène and Ojanguren in [CO89].

Definition 1.9. Let K/F be an extension and C a Γ-module. The *unramified cohomology groups* are defined by

$$H^i_{\mathrm{nr}}(K,C) = \bigcap_v \ker\left[\, H^i(K,C) \xrightarrow{\partial_v} H^{i-1}(\kappa_v, C(-1)) \,\right],$$

where the intersection is taken over all discrete valuations v on K which are trivial on F and have residue field κ_v, and the ∂_v are the residue homomorphisms discussed in §1.3 above. If X/F is a variety we sometimes write $H^i_{\mathrm{nr}}(X,C)$ for $H^i_{\mathrm{nr}}(F(X),C)$ hence, with this notation, $H^i_{\mathrm{nr}}(-,C)$ is a birational invariant of F-varieties (in contrast to $A^0(-,H^i[C])$). In fact, more is true (*cf.* Theorem 1.10).

By construction we have the inclusions $H^i_{\mathrm{nr}}(X,C) \subseteq A^0(X,H^i[C]) \subseteq H^i(F(X),C)$, hence we sometimes refer to the groups $A^0(X,H^i[C])$ as *partially unramified.* Moreover, since we only consider discrete valuations which are trivial on F, for all extensions K/F the natural map $H^i(F,C) \longrightarrow H^i(K,C)$ has image in $H^i_{\mathrm{nr}}(K,C)$ (*cf.* [CS07, Lemma 5.4]). When we have a decomposition $A^0(X,H^i[C]) \cong H^i(F,C) \oplus \overline{A}^0(X,H^i[C])$ induced by some rational point $x \in X(F)$, the composition

$$H^i(F,C) \longrightarrow H^i_{\mathrm{nr}}(X,C) \subseteq A^0(X,H^i[C]) \longrightarrow H^i(F,C),$$

with the last map the projection, implies that we have an analogous decomposition

$$H^i_{\mathrm{nr}}(X,C) \cong H^i(F,C) \oplus \overline{H}^i_{\mathrm{nr}}(X,C),$$

so to calculate unramified cohomology groups of X it suffices to determine the normalized ones.

Remark 1.5. The usefulness of unramified cohomology groups was first demonstrated by Artin and Mumford [AM72] when they constructed unirational varieties (over $\mathbb{C}$) whose non-rationality was determined by showing that a particular cohomology group was non-trivial; later, it was observed that in the unirational case their group could be identified with the unramified Brauer group (*cf.* the discussion after Remark 1.6) of the variety's function field. The unramified point of view (in degree 2) was further developed by Saltman and Bogomolov in relation to Noether's problem, and we refer to the survey [CS07] for more details on that work. Colliot-Thélène and Ojanguren [CO89] were the first to use degree 3 unramified cohomology groups to prove non-rationality of unirational varieties whose degree 2 unramified cohomology groups vanished, and this approach was continued by Peyre in [Pey93]. The latter author also used higher unramified cohomology groups in further investigations of Noether's problem [Pey08]. But the power of unramified cohomology is not restricted to rationality problems: Pirutka's result [Pir11] was based on the construction of nontrivial elements in an unramified cohomology group of degree 3, and our parallel to her work relies on constructing a nontrivial cokernel into an unramified cohomology group (*cf.* §7).

The following property will be useful.

Theorem 1.10. (*cf.* [Col95, Thm. 4.1.5]) *Let K/F be a finitely generated extension. Then the natural map $H^i(K,C) \longrightarrow H^i(K(t_1,\dots,t_m),C)$ induces an isomoprhism*

$$H^i_{\mathrm{nr}}(K,C) \cong H^i_{\mathrm{nr}}(K(t_1,\dots,t_m),C).$$

In particular, $H^i(F,C) \cong H^i_{\mathrm{nr}}(F(t_1,\dots,t_n),C)$.

The next result will allow us to take advantage of smooth compactifications in our calculations.

Theorem 1.11. (*cf.* [Col95, Prop. 2.1.8 and Thm. 4.1.1]) *Let X/F be a proper smooth variety. Then the natural inclusion $H^i_{\mathrm{nr}}(X,C) \hookrightarrow A^0(X,H^i[C])$ is an isomorphism for any Γ-module C.*

The following result, due to Rost, allows us to define unramified invariants and use them to calculate (normalized) unramified cohomology groups of classifying varieties. See also [GMS03, Part 1, App. C]. For a construction in a specific case, see Theorem 2.1.

Theorem 1.12. ([Mer02, Thm. 3.2]) *Let G/F be an algebraic group and C a Γ-module. Then there exists a versal G-torsor $\mathcal{S} \longrightarrow Y$ such that the Rost embedding $r : \mathrm{Inv}^i(G,C) \longrightarrow \overline{A}^0(Y,H^i[C])$ of Theorem 1.6 is an isomorphism.*

Construction 1.13. Using this G-torsor we define the subgroup $\mathrm{Inv}^i_{\mathrm{nr}}(G,C) \subseteq \mathrm{Inv}^i(G,C)$ as the subgroup corresponding to $\overline{H}^i_{\mathrm{nr}}(Y,C)$ under the isomorphism $\mathrm{Inv}^i(G,C) \cong \overline{A}^0(Y, H^i[C])$. The versal G-torsor in the previous result is of the form $\mathcal{S} \longrightarrow Y = \mathcal{S}/\rho(G)$, where $\rho : G \hookrightarrow \mathcal{S}$ is an embedding of G into a split semisimple simply connected rational universally special algebraic group $\mathcal{S}$ (*cf.* Example 1.2(b) for the definition of universally special); for example, one can take $\mathcal{S} = \mathbf{SL}_n$ (*cf.* Theorem 2.1 below).

Suppose now that $\rho : G \hookrightarrow \mathcal{T}$ is an embedding of G into a merely rational universally special algebraic group $\mathcal{T}$ and let $X = \mathcal{T}/\rho(G)$; for example, one can take $\mathcal{T} = \mathbf{GL}_n$. It is shown in [Mer02, §2.1] that the stable birationality class of X is independent of ρ and $\mathcal{T}$ (this is where the rational hypothesis on $\mathcal{T}$ is used), hence by Theorem 1.10, $\overline{H}^i_{\mathrm{nr}}(F(X),C) \cong \overline{H}^i_{\mathrm{nr}}(F(Y),C)$. Moreover, the natural map $\mathcal{T} \longrightarrow X$ is a G-torsor and $\mathcal{T}$ universally special implies that it is in fact versal (*cf.* the proof of Lemma 1.4). Therefore, the Rost embedding $r : \mathrm{Inv}^i(G,C) \hookrightarrow \overline{A}^0(X, H^i[C])$, which exists for *any* versal torsor with classifying space X, maps $\mathrm{Inv}^i_{\mathrm{nr}}(G,C) \cong \overline{H}^i_{\mathrm{nr}}(F(Y),C)$ isomorphically onto $\overline{H}^i_{\mathrm{nr}}(F(X),C)$, even though r may not surject. This is essential because sometimes the most convenient versal G-torsor is one for which the Rost embedding is *not* an isomorphism; nonetheless, this will allow us to calculate (normalized) unramified cohomology groups of the corresponding classifying variety using the unramified invariants defined in the previous paragraph. In summary, for any rational universally special algebraic group $\mathcal{T}$ and embedding $\rho : G \hookrightarrow \mathcal{T}$, we have the following commutative diagram for $X = \mathcal{T}/\rho(G)$:

$$\begin{tikzcd}
\mathrm{Inv}^i_{\mathrm{nr}}(G,C) \arrow[r, "\cong"] \arrow[d, hook] & \overline{H}^i_{\mathrm{nr}}(F(X),C) \arrow[d, hook] \\
\mathrm{Inv}^i(G,C) \arrow[r, hook, "r"] & \overline{A}^0(X, H^i[C]).
\end{tikzcd} \tag{1.1}$$

Moreover, if X' is another classifying variety for G which is stably birational to such an X then we have the same diagram with X' in place of X.

Applying this to tori we obtain the following useful result.

Proposition 1.14. *Let*

$$1 \longrightarrow S \xrightarrow{i} P \longrightarrow T \longrightarrow 1$$

be an exact sequence of tori with P quasitrivial (e.g., a flasque resolution of T or a coflasque resolution of S). Then

$$\overline{H}^i_{\mathrm{nr}}(F(T), C) \cong \mathrm{Inv}^i_{\mathrm{nr}}(S, C).$$

If, moreover, S is flasque, then the isomorphism holds with the ordinary invariants $\mathrm{Inv}^i(S, C)$ *on the right.*

Proof. By Example 1.2(b), quasitrivial tori are both rational and universally special hence $T = P/i(S)$ is of the form considered above. Thus diagram (1.1) holds with $G = S$ and $X = T$. If S is flasque, the last statement follows from the next result. □

Proposition 1.15. *If S is a flasque torus then* $\mathrm{Inv}^i_{\mathrm{nr}}(S, C) \cong \mathrm{Inv}^i(S, C)$.

Proof. The main ingredient is the result [CS87, Thm. 2.2(i)] which gives flasque tori their name: if X/F is a smooth variety and $U \subset X$ an open set, the natural map $H^1(X, S) \longrightarrow H^1(U, S)$ is surjective (in fact, the hypotheses on X can be much weaker). Explicitly, every S-torsor $\mathcal{E} \longrightarrow U$ is the pullback of some S-torsor $\mathcal{E}' \longrightarrow X$ along the inclusion $U \hookrightarrow X$. Moreover, if $\mathcal{E} \longrightarrow U$ is a versal S-torsor then so is $\mathcal{E}' \longrightarrow X$: if K/F is a field extension and $\mathscr{T} \in H^1(K, S)$ an S-torsor then by assumption there exists a K-point $\mathrm{Spec}(K) \longrightarrow U$ such that $\mathscr{T}$ is the pullback of $\mathcal{E}$ along this point. This gives the sequence of pullbacks

$$\begin{tikzcd}
\mathscr{T} \arrow[r] \arrow[d] & \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E}' \arrow[d] \\
\mathrm{Spec}(K) \arrow[r] & U \arrow[r] & X
\end{tikzcd}$$

so indeed $\mathcal{E}' \longrightarrow X$ is also versal. Now, let

$$1 \longrightarrow S \longrightarrow P \longrightarrow Q \longrightarrow 1$$

be an exact sequence of tori with P quasitrivial (e.g., a coflasque resolution of S). By Lemma 1.5, $P \longrightarrow Q$ is a versal S-torsor. Let X be a toric model for Q ([CHS05]), so that in particular X/F is a smooth proper variety admitting an open embedding $Q \hookrightarrow X$. By the above discussion, there exists a versal S-torsor $\mathcal{E}' \longrightarrow X$, i.e., X is a classifying variety for S. Since X is birational to Q by construction, $\overline{H}^i_{\mathrm{nr}}(F(X), C) \cong \overline{H}^i_{\mathrm{nr}}(F(Q), C)$ and (as in the proof of the previous proposition, where Q was called T) diagram (1.1) implies that the Rost embedding maps $\mathrm{Inv}^i_{\mathrm{nr}}(S, C)$ isomorphically onto the latter. Since X/F is proper, Theorem 1.11 implies that the natural inclusion $\overline{H}^i_{\mathrm{nr}}(F(X), C) \hookrightarrow \overline{A}^0(X, H^i[C])$ is an isomorphism hence we have the diagram

$$\begin{tikzcd}
\mathrm{Inv}^i_{\mathrm{nr}}(S, C) \arrow[r, "\cong"] \arrow[d, hook] & \overline{H}^i_{\mathrm{nr}}(F(X), C) \arrow[d, hook, "\cong"] \\
\mathrm{Inv}^i(S, C) \arrow[r, hook, "r"] & \overline{A}^0(X, H^i[C]),
\end{tikzcd}$$

which implies that the natural inclusion $\mathrm{Inv}^i_{\mathrm{nr}}(S, C) \hookrightarrow \mathrm{Inv}^i(S, C)$ is an isomorphism. □

As stated, by taking flasque resolutions, Proposition 1.14 reduces the calculation of unramified cohomology of a torus T to the determination of (ordinary) invariants of an auxiliary torus. But, we can turn things around by starting with a short exact sequence involving T

$$1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$$

with P quasitrivial (e.g., a coflasque resolution of T) and view the result as a calculation of unramified *invariants* of T in terms of unramified *cohomology* of Q. One more application of the Proposition (this time to Q) then expresses the unramified invariants of T in terms of ordinary invariants of (yet another) auxiliary torus; in fact, a Picard torus of Q. We summarize this in the next result.

Proposition 1.16. *Let T/F be an algebraic torus. Then there exists an exact sequence of tori*

$$1 \longrightarrow T \longrightarrow N \longrightarrow P \longrightarrow 1$$

with P quasitrivial and N flasque. For any such sequence,

$$\mathrm{Inv}^i_{\mathrm{nr}}(T, C) \cong \mathrm{Inv}^i(N, C).$$

Proof. Let

$$1 \longrightarrow T \longrightarrow P' \longrightarrow Q \longrightarrow 1$$

be an exact sequence of tori with P' quasitrivial (e.g., a coflasque resolution of T) and

$$1 \longrightarrow N' \longrightarrow P \longrightarrow Q \longrightarrow 1$$

a flasque resolution of Q. Two applications of Proposition 1.14 to these sequences gives the isomorphisms

$$\mathrm{Inv}^i_{\mathrm{nr}}(T, C) \cong \overline{H}^i_{\mathrm{nr}}(F(Q), C) \cong \mathrm{Inv}^i_{\mathrm{nr}}(N', C),$$

respectively, and Proposition 1.15 implies that the last group is isomorphic to $\mathrm{Inv}^i(N', C)$. The proof of [CS87, Lemma 0.6] shows (for the dual modules) that the torus N' fits into an exact sequence

$$1 \longrightarrow T \longrightarrow P' \times N' \longrightarrow P \longrightarrow 1,$$

so we let $N = P' \times N'$, which incidentally shows that $[\widehat{N}_{\mathrm{sep}}] = [\widehat{N}'_{\mathrm{sep}}] = p(\widehat{Q}_{\mathrm{sep}})$, i.e., that N is a Picard torus of Q. Since P' is quasitrivial we have that $\mathrm{Inv}^i(N', C) \cong \mathrm{Inv}^i(N, C)$ (*cf.* Example 1.2(b)). □

Remark 1.6. Unramified invariants should be viewed as an intermediary device connecting (by the previous propositions) unramified cohomology to ordinary invariants, and it is for the latter that computational tools with be developed.

We will be primarily interested in the modules $\mathbb{Q}/\mathbb{Z}(i) = \varinjlim \mu_n^{\otimes i}$ and $\mathbb{Q}_p/\mathbb{Z}_p(i) = \varinjlim \mu_{p^m}^{\otimes i}$, for n and p prime to q, with the direct limits over n and m, respectively. Thus, $\mathbb{Q}/\mathbb{Z}(0) = \mathbb{Q}/\mathbb{Z}'$. (See §5 for a discussion of constructions of characteristic components for $\mathbb{Q}/\mathbb{Z}(i)$). In particular, the Kummer sequence and Hilbert's Theorem 90 imply that $H^2(K, \mathbb{Q}/\mathbb{Z}(1))$ is isomorphic to the subgroup of $\mathrm{Br}(K)$ consisting of all elements of exponent prime to q, i.e. $H^2(K, \mathbb{Q}/\mathbb{Z}(1)) = \mathrm{Br}(K)'$. We always have $H^2(K, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong \mathrm{Br}(K)\{p\}$ and $H^2(K, \mu_n) \cong {}_n\mathrm{Br}(K)$ for n and p prime to q. We define the *unramified Brauer group* $\mathrm{Br}_{\mathrm{nr}}(K)$ as $H^2_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(1))$, which in characteristic 0 agrees with constructions found in the literature (e.g. [Pey93], [Col95], [CS07]). Explicitly, when $q = 0$ one has that $\mathrm{Br}(K) =$

$\varinjlim H^2(K, \mu_n)$ and since $H^1(\kappa_v, \mathbb{Q}/\mathbb{Z}) = \varinjlim H^1(\kappa_v, \mathbb{Z}/n\mathbb{Z})$ for each discrete valuation v on K which is trivial on F and has residue field κ_v, one can define the residue homomorphism $\partial_v : \mathrm{Br}(K) \longrightarrow H^1(\kappa_v, \mathbb{Q}/\mathbb{Z})$ to be the union of the ordinary residue homomorphisms $H^2(K, \mu_n) \longrightarrow H^1(\kappa_v, \mathbb{Z}/n\mathbb{Z})$ discussed in §1.3. The intersection $\bigcap_v \ker \partial_v$ of their kernels then coincides with our construction of $H^2_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(1))$.

Remark 1.7. In [Col95, Rem. 3.3.2], Colliot-Thélène describes how one can in fact define residue homomorphisms $\partial_v : \mathrm{Br}(K) \longrightarrow H^1(\kappa_v, \mathbb{Q}/\mathbb{Z})$ under the sole assumption that K is perfect. In characteristic 0, these maps agree up to a sign with those constructed by taking the union of residue homomorphisms from §1.3, hence defining the unramified Brauer group as $\bigcap_v \ker \partial_v$ with these maps would agree with the above in this case. But, in arbitrary characteristic this definition of the unramified Brauer group may not, *a priori*, agree with our definition using $H^2_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(1))$ since by construction $H^2_{\mathrm{nr}}(K, \mathbb{Q}/\mathbb{Z}(1))\{q\}$ is trivial.

2 Degree 2 Invariants

In this section we compute $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ for an algebraic group, assumed reductive if F is not perfect. The heart of the calculation is the following result, originally due to Rost, which will be useful in its own right later on. The proof reduces to the same situation as in the proof of the more general result [Mer02, Thm. 3.2] but then exploits a property specific to $\mathbf{SL}_n$, hence is simpler insofar as it doesn't make use of spectral sequences and is self-contained.

Theorem 2.1. (*cf.* [Mer02, Thm. 3.2]) *Let G/F be an algebraic group. Choose an embedding $G \hookrightarrow \mathbf{SL}_n$ for some n and let $X = \mathbf{SL}_n/G$ be the homogenous space. Then* $\mathrm{Inv}^i(G, C) \cong \overline{A}^0(X, H^i[C])$.

Proof. The proof of Lemma 1.4 shows that the natural map $\mathbf{SL}_n \longrightarrow X$ is a versal G-torsor. Therefore, by Theorem 1.6 we have the Rost embedding $r : \mathrm{Inv}^i(G, C) \hookrightarrow \overline{A}^0(X, H^i[C])$, for which we will construct an inverse. As in the proof of Lemma 1.4, for all K/F

$$X(K)/\mathbf{SL}_n(K) \cong H^1(K,G),$$

i.e., for each torsor $\mathscr{T} \in H^1(K,G)$ there exists an $x \in X(K)$ such that $\mathscr{T} \cong \mathbf{SL}_{n,x}$ is given by the fiber of $\mathbf{SL}_n$ over x and such an x is unique up to the action of $\mathbf{SL}_n(K)$ on $X(K)$, which is given by left coset translation since $X(K) = [\mathbf{SL}_n(K_{\text{sep}})/G(K_{\text{sep}})]^{\Gamma_K}$ and $\mathbf{SL}_n(K) = \mathbf{SL}_n(K_{\text{sep}})^{\Gamma_K}$.

We now define a map $m : \overline{A}^0(X, H^i[C]) \longrightarrow \operatorname{Inv}^i(G,C)$. Given $\alpha \in \overline{A}^0(X, H^i[C])$ we let $m(\alpha) : H^1(-,G) \longrightarrow H^i(-,C)$ be the invariant defined as follows. For an extension K/F and a torsor $\mathscr{T} \in H^1(K,G)$ consider any element $x \in X(K)$ such that $\mathscr{T} \cong \mathbf{SL}_{n,x}$ as described above. Since $x : \operatorname{Spec}(K) \longrightarrow X$, it induces the pullback

$$x^* : \overline{A}^0(X, H^i[C]) \longrightarrow \overline{A}^0(\operatorname{Spec}(K), H^i[C]) = H^i(K,C),$$

and we set $m(\alpha)_K(\mathscr{T}) = x^*(\alpha) \in H^i(K,C)$. Once it is well-defined it is necessarily a homomorphism since if $\alpha, \alpha' \in \overline{A}^0(X, H^i[C])$ then $m(\alpha+\alpha')_K(\mathscr{T}) = x^*(\alpha+\alpha') = x^*(\alpha) + x^*(\alpha')$ by Theorem 1.8(i). To see that it is well-defined, we must check that it is independent of the choice of x, i.e., that it is independent of the action of $\mathbf{SL}_n(K)$ on $X(K)$: given $s \in S(K)$ and $x, x' \in X(K)$ such that $s \cdot x = x'$, we must show that $x^*(\alpha) = (x')^*(\alpha)$.

By definition (*cf.* Remark 1.2), we have that

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r, "{(s,x)}"] \arrow[rr, bend right, "x'"'] & \mathbf{SL}_n \times X \arrow[r, "a"] & X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r, "{(s,x)}"] \arrow[rr, bend right, "x"'] & \mathbf{SL}_n \times X \arrow[r, "\pi"] & X,
\end{tikzcd}$$

both commute, where π is the projection, hence it suffices to show that a and π induce the same pullback homomorphisms $\overline{A}^0(X, H^i[C]) \longrightarrow \overline{A}^0(\mathbf{SL}_n \times X, H^i[C])$. If we let $i : X \longrightarrow \mathbf{SL}_n \times X$ be the natural map $x \longmapsto (1,x)$ then $\pi \circ i = \mathrm{Id}_X = a \circ i$ implies (Theorem 1.8) that $i^* \circ \pi^* = \mathrm{Id}_{\overline{A}^0(X,H^i[C])} = i^* \circ a^*$. Therefore, it suffices to show that π^* induces an isomorphism.

For all $n \geq 2$ we have that $\mathbf{G}_a \leq \mathbf{SL}_n$ as algebraic groups by embedding $\mathbf{G}_a(R) = R$ as elementary matrices in $\mathbf{SL}_n(R)$ for every F-algebra R. Therefore, the action map can be restricted to $a : \mathbf{G}_a \times X \longrightarrow X$. Since $\mathbf{SL}_n(K)$ is generated by the elementary matrices, which can be identified with $K = \mathbf{G}_a(K)$, we can further assume that $s \in \mathbf{G}_a(K)$. Hence, it suffices to show that the pullback of the restricted projection $\pi : \mathbf{G}_a \times X \longrightarrow X$ is an isomorphism $\overline{A}^0(X, H^i[C]) \longrightarrow \overline{A}^0(\mathbf{G}_a \times X, H^i[C])$. But, this is precisely the homotopy invariance property of Theorem 1.8. Hence, our map $m : \overline{A}^0(X, H^i[C]) \longrightarrow \mathrm{Inv}^i(G, C)$ is well-defined.

To finish, we need to show m is an inverse to the Rost embedding. Given $\eta \in \mathrm{Inv}^i(G, C)$ we have that $r(\eta) = \eta(\mathbf{SL}_{n,\xi})$ where $\xi \in X(F(X))$ is the generic point. To see that $\eta = m(r(\eta))$ it suffices to show that they agree on the generic torsor $\mathbf{SL}_{n,\xi} \in H^1(F(X), G)$ since all invariants are determined by their value on the generic torsor, since r is injective by Theorem 1.6). But, by definition of m,

$$m(r(\eta))(\mathbf{SL}_{n,\xi}) = \xi^*(r(\eta)) = \xi^*(\eta(\mathbf{SL}_{n,\xi})) = \eta(\mathbf{SL}_{n,\xi}),$$

where the last equality follows by Theorem 1.8(i), which says that ξ^* is the identity on $A^0(X, H^i[C]) \subseteq H^i(F(X), C)$. Therefore $m \circ r = \mathrm{Id}$. Conversely, if $\alpha \in A^0(X, H^i[C])$ then

$$r(m(\alpha)) = m(\alpha)(\mathbf{SL}_{n,\xi}) = \xi^*(\alpha) = \alpha,$$

by the same result. Therefore $r \circ m = \mathrm{Id}$ as well and r is an isomorphism. □

The main result of this section now follows as a straightforward Corollary (see also Remark 4.1).

Theorem 2.2. *Let G/F be an algebraic group and assume that G is reductive if F is not perfect. Then* $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1)) \cong \mathrm{Pic}(G)'$.

Proof. With X as in the previous Theorem, it suffices to show that $\overline{A}^0(X, H^2[\mathbb{Q}_p/\mathbb{Z}_p(1)]) \cong \mathrm{Pic}(G)\{p\}$ for all $p \neq q$. Since $\mathbf{SL}_n \longrightarrow X$ is a G-torsor, by [San81, Prop. 6.10] we have the exact sequence

$$0 = \mathrm{Pic}(\mathbf{SL}_n) \longrightarrow \mathrm{Pic}(G) \longrightarrow \mathrm{Br}(X) \longrightarrow \mathrm{Br}(\mathbf{SL}_n) = \mathrm{Br}(F),$$

where the two equalities follow from [Vos98, §4.3, Theorem 1(3)]; see [Mer99, p. 144] or [GMS03, Part 2, §6] for a different proof of the first equality using K-theory. Since $\mathrm{Br}(X) \longrightarrow \mathrm{Br}(F)$ above splits the natural map $\mathrm{Br}(F) \longrightarrow \mathrm{Br}(X)$, we see that $\mathrm{Pic}(G) \cong \overline{\mathrm{Br}}(X) := \mathrm{Br}(X)/\mathrm{Br}(F)$. By (the component version of) [Col95, §3.4, Sequence (3.9)] we have the exact sequence

$$0 \longrightarrow \mathrm{Br}(X)\{p\} \longrightarrow \mathrm{Br}(F(X))\{p\} \xrightarrow{\oplus\partial_x} \bigoplus_{x\in X^{(1)}} H^1(F(x), \mathbb{Q}_p/\mathbb{Z}_p),$$

thus $\overline{A}^0(X, H^2[\mathbb{Q}_p/\mathbb{Z}_p(1)]) \cong \overline{\mathrm{Br}}(X)\{p\} \cong \mathrm{Pic}(G)\{p\}$. □

Remark 2.1.

(a) The reductive hypothesis in the previous theorem when F is not perfect is forced by Sansuc's result.

(b) [KMR98, Prop. 31.19] constructs a direct map $\mathrm{Pic}(G) \longrightarrow \mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$ as follows: to every element of $\mathrm{Pic}(G)$ one can associate an exact sequence of algebraic groups

$$1 \longrightarrow \mathbf{G}_m \longrightarrow G' \longrightarrow G \longrightarrow 1,$$

hence for any field extension K/F one has the connecting homomorphism $H^1(K, G) \longrightarrow H^2(K, \mathbf{G}_m) = \mathrm{Br}(K)$. Moreover, the definition of $\mathbb{Q}/\mathbb{Z}(1)$ there is assumed to include an appropriate q-component so that $H^2(K, \mathbb{Q}/\mathbb{Z}(1)) \cong \mathrm{Br}(K)$ in all characteristics (*cf.* the beginning of §5 or [GMS03, Part 2, App. A]) . Therefore, such an exact sequence gives rise to an invariant in $\mathrm{Inv}^2(G, \mathbb{Q}/\mathbb{Z}(1))$. The citation asserts (without proof) that this is an isomorphism, hence it claims the result of the previous theorem with the full $\mathrm{Pic}(G)$ in all characteristics and with arbitrary (connected) G (though with a modified $\mathbb{Q}/\mathbb{Z}(1)$). This is confirmed, for example, in the specific case $G = \mathbf{PGL}_n$. The natural invariant in $\mathrm{Inv}^2(\mathbf{PGL}_n, \mu_n)$ constructed in Example 0.1(c) is in fact a generator of $\mathrm{Inv}^2(\mathbf{PGL}_n, \mathbb{Q}/\mathbb{Z}(1))$ of order n. Since $\mathrm{Pic}(\mathbf{PGL}_n) \cong \mathbb{Z}/n\mathbb{Z}$, we get an isomorphism with the full Picard group.

(c) In this work we do not compute the characteristic component of the invariants in either degree 2 or 3 only because we crucially appeal to results and constructions in both cases which explicitly exclude characteristic components, for example, in the degree 2 case, the last sequence of the previous theorem ([Col95, §3.4]) and in the degree 3 case, the isomorphism in (5.9) ([MS90]). Moreover, our reliance on Rost's Chow groups with values in Galois cohomology also restricts us to $p \neq q$ because the residue homomorphisms we use are not defined on the characteristic component and in fact will also force us to impose the condition $p \neq 2$ later on (*cf.* Remark 1.3(a) and §5 and the discussion after Remark 5.6).

Corollary. *Let T/F be an algebraic torus. Then* $\mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \cong H^1(F, \widehat{T}_{\mathrm{sep}})'$.

Proof. It suffices to show that $\mathrm{Pic}(T) \cong H^1(F, \widehat{T}_{\mathrm{sep}})$. There is an exact sequence

$$1 \longrightarrow F_{\mathrm{sep}}[T]^\times \longrightarrow F_{\mathrm{sep}}(T)^\times \longrightarrow \mathrm{Div}(T_{\mathrm{sep}}) \longrightarrow \mathrm{Pic}(T_{\mathrm{sep}}) = 0, \tag{2.1}$$

where $\mathrm{Div}(T_{\mathrm{sep}})$ is the group of divisors. By [Ros61, Thm. 3], $F_{\mathrm{sep}}[T]^\times = F_{\mathrm{sep}}^\times \oplus \widehat{T}_{\mathrm{sep}}$ hence taking cohomology gives the exact sequence

$$0 \longrightarrow F[T]^\times \longrightarrow F(T)^\times \longrightarrow \mathrm{Div}(T) \longrightarrow H^1(F, \widehat{T}_{\mathrm{sep}}) \longrightarrow 0.$$

Since, by construction, we have the same exact sequence with $\mathrm{Pic}(T)$ in place of $H^1(F, \widehat{T}_{\mathrm{sep}})$, the two are necessarily isomorphic. □

3 Degree 2 Unramified Cohomology

When combined with Proposition 1.14 and 1.16, the results of the previous section give full, albeit indirect, descriptions of the groups $\overline{H}^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1))$ and $\mathrm{Inv}^2_{\mathrm{nr}}(T, \mathbb{Q}/\mathbb{Z}(1))$ for an arbitrary torus T/F. By Proposition 1.14 and the Corollary to Theorem 2.2, we have the following.

Theorem 3.1. *Let T/F be an algebraic torus and let $1 \longrightarrow S \longrightarrow P \longrightarrow T \longrightarrow 1$ be a flasque resolution of T. Then*

$$\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T)) \overset{\mathrm{def}}{=} \overline{H}^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1)) \cong H^1(F, \widehat{S}_{\mathrm{sep}})'.$$

Although we obtain this isomorphism indirectly via invariants, we can at least describe the embedding $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T)) \hookrightarrow H^1(F, \widehat{S}_{\mathrm{sep}})'$. Taking cohomology of the exact sequence (2.1) and using the fact $F_{\mathrm{sep}}[T]^\times = F^\times_{\mathrm{sep}} \oplus \widehat{T}_{\mathrm{sep}}$ mentioned there gives the exact sequence

$$H^1(F, \mathrm{Div}(T_{\mathrm{sep}})) \longrightarrow \mathrm{Br}(F) \oplus H^2(F, \widehat{T}_{\mathrm{sep}}) \longrightarrow H^2(F, F_{\mathrm{sep}}(T)^\times) \longrightarrow H^2(F, \mathrm{Div}(T_{\mathrm{sep}})).$$

By [Vos98, p. 19], we have that

$$H^i(F, \mathrm{Div}(T_{\mathrm{sep}})) = \begin{cases} 0 & i = 1 \\ \bigoplus_{t \in T^{(1)}} H^1(F(t), \mathbb{Q}/\mathbb{Z}) & i = 2, \end{cases}$$

so we obtain the exact sequence (of primed torsion abelian groups)

$$0 \longrightarrow \mathrm{Br}(F)' \oplus H^2(F, \widehat{T}_{\mathrm{sep}})' \xrightarrow{f} H^2(F, F_{\mathrm{sep}}(T)^\times)' \longrightarrow \bigoplus_{t \in T^{(1)}} H^1(F(t), \mathbb{Q}/\mathbb{Z})'. \tag{3.1}$$

Next, we can view $\mathrm{Br}_{\mathrm{nr}}(F(T)) \subseteq H^2(F, F_{\mathrm{sep}}(T)^\times)' \subseteq \mathrm{Br}(F(T))'$. The first inclusion follows from the inflation-restriction exact sequence ([Ser79, Ch. X, §5, Prop. 6]),

$$\begin{tikzcd}
0 \arrow[r] & H^2(\mathrm{Gal}(F_{\mathrm{sep}}(T)/F(T)), F_{\mathrm{sep}}(T)^\times) \arrow[r, "\mathrm{inf}"] & \mathrm{Br}(F(T)) \arrow[r, "\mathrm{res}"] & \mathrm{Br}(F_{\mathrm{sep}}(T)) \\
& & \mathrm{Br}_{\mathrm{nr}}(F(T)) \arrow[r] \arrow[u, hook] & \mathrm{Br}_{\mathrm{nr}}(F_{\mathrm{sep}}(T)), \arrow[u, hook]
\end{tikzcd}$$

where the bottom map and commutativity of the square follow from [CS07, §5, Lemma 5.5]). T_{sep} is a split torus hence $F_{\mathrm{sep}}(T) \cong F_{\mathrm{sep}}(t_1, \ldots, t_n)$ so by Theorem 1.10, or more specifically [CS07, §5, Prop. 5.7]), $\mathrm{Br}_{\mathrm{nr}}(F_{\mathrm{sep}}(T)) = \mathrm{Br}(F_{\mathrm{sep}}) = 0$. Since $\mathrm{Br}_{\mathrm{nr}}(F(T))\{q\} = 0$ by definition, exactness implies that $\mathrm{Br}_{\mathrm{nr}}(F(T))$ can be identified with a subgroup of $H^2(\mathrm{Gal}(F_{\mathrm{sep}}(T)/F(T)), F_{\mathrm{sep}}(T)^\times)'$. But, $\mathrm{Gal}(F_{\mathrm{sep}}(T)/F(T))$ can be identified with Γ, hence the latter cohomology group is $H^2(F, F_{\mathrm{sep}}(T)^\times)'$ and we have the first inclusion; the exact row of the diagram then shows that the second inclusion is given by the inflation map.

In particular, the second map in (3.1) is given by the residue homomorphisms corresponding to the codimension one points t of T restricted to the subgroup

$$H^2(F, F_{\mathrm{sep}}(T)^\times)' \subseteq \mathrm{Br}(F(T))' \cong H^2(F(T), \mathbb{Q}/\mathbb{Z}(1)).$$

Recall (§1.4, esp. Remark 1.7), the residue homomorphisms are not, in general, defined on all of $\mathrm{Br}(F(T))$, hence the necessity of using primed subgroups in (3.1). Since

$$\mathrm{Br}_{\mathrm{nr}}(F(T)) = H^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1)) \subseteq A^0(T, H^2[\mathbb{Q}/\mathbb{Z}(1)]) = \ker(\oplus\partial_t)$$

by definition, (3.1) implies that $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T))$ can be identified with a subgroup of $H^2(F, \widehat{T}_{\mathrm{sep}})'$. Therefore, we have the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
 & H^1(F, \widehat{S}_{\mathrm{sep}})' \arrow[d] \\
\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T)) \arrow[r, hook, "g"] \arrow[d] & H^2(F, \widehat{T}_{\mathrm{sep}})' \arrow[d] \\
\overline{\mathrm{Br}}_{\mathrm{nr}}(F(P)) \arrow[r, hook] & H^2(F, \widehat{P}_{\mathrm{sep}})',
\end{tikzcd}$$

with the right-hand vertical row exact and the square commutative. But P is a quasitrivial torus, hence is rational (Example 1.2(b)). Therefore, just as for T_{sep} above, we have that $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(P)) = 0$ and therefore g identifies $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T))$ with a subgroup of $\ker\left[H^2(F, \widehat{T}_{\mathrm{sep}})' \longrightarrow H^2(F, \widehat{P}_{\mathrm{sep}})'\right] = H^1(F, \widehat{S}_{\mathrm{sep}})'$. By construction, g is a section of f because it was obtained by showing that $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T))$ is contained in $\ker(\oplus\partial_t)$, the second map in (3.1): $f \circ g = \mathrm{Id}_{\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T))}$.

The universal property of flasque resolutions gives the following corollary to Theorem 3.1.

Corollary. *With the hypotheses of the previous theorem,*

$$\overline{H}^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1)) \;\cong \bigcap_{\widehat{T}_{\mathrm{sep}} \to \widehat{R}} \ker\left[H^2(F, \widehat{T}_{\mathrm{sep}})' \longrightarrow H^2(F, \widehat{R})'\right],$$

where the intersection is over all Γ-homomorphisms $\widehat{T}_{\mathrm{sep}} \longrightarrow \widehat{R}$ for every permutation Γ-module $\widehat{R}$.

Proof. Exactness of the dual sequence $0 \longrightarrow \widehat{S}_{\mathrm{sep}} \longrightarrow \widehat{P}_{\mathrm{sep}} \longrightarrow \widehat{T}_{\mathrm{sep}} \longrightarrow 0$ implies that $H^1(F, \widehat{S}_{\mathrm{sep}})' \cong \ker[H^2(F, \widehat{T}_{\mathrm{sep}})' \longrightarrow H^2(F, \widehat{P}_{\mathrm{sep}})']$, hence the intersection is necessarily contained in $H^1(F, \widehat{S}_{\mathrm{sep}})'$. Conversely, if $\widehat{T}_{\mathrm{sep}} \longrightarrow \widehat{R}$ is any Γ-homomorphism with $\widehat{R}$ permutation, then by Lemma 1.1 we have the diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{T}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{P}_{\mathrm{sep}} \arrow[r] \arrow[dl, dashed] & \widehat{S}_{\mathrm{sep}} \arrow[r] & 0 \\
& \widehat{R} & & &
\end{tikzcd}$$

hence we obtain the corresponding diagram in cohomology

$$\begin{tikzcd}
H^2(F, \widehat{T}_{\mathrm{sep}})' \arrow[r] \arrow[d] & H^2(F, \widehat{P}_{\mathrm{sep}})' \arrow[dl] \\
H^2(F, \widehat{R})' &
\end{tikzcd}$$

and so any element vanishing under the horizontal map must also vanish under the vertical. □

Remark 3.1. Since the modules appearing in the flasque resolution of $\widehat{T}_{\mathrm{sep}}$ can be chosen as G-modules, the corollary also holds if we consider only $\widehat{R}$ which are permutation G-modules.

The following formulates the result as originally stated in [Col95, p. 39] (which was in characteristic 0; see Remark 1.7). Because we will appeal to results specific to finite groups, we assume all tori involved are split by some fixed finite Galois extension L/F with Galois group G so that all degree 1 Galois cohomology groups above can be viewed over G by the inflation-restriction exact sequence ([Ser79, Ch. VII, §6, Prop. 4]). Note the small typo in the citation in which the full unramified Brauer group $\mathrm{Br}_{\mathrm{nr}}(F(T))$ was used instead of the normalized one $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T))$.

Theorem 3.2. *With the notation of the previous theorem,*

$$\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T)) \cong \{\alpha \in H^2(G, \widehat{T}_{\mathrm{sep}})' \mid \alpha \in \ker\left[\, H^2(G, \widehat{T}_{\mathrm{sep}}) \xrightarrow{\mathrm{res}} H^2(H, \widehat{T}_{\mathrm{sep}}) \,\right],\ \forall\, H \leq G \text{ cyclic}\}.$$

Proof. By the previous theorem, it suffices to show that the group on the right is isomorphic to $H^1(G, \widehat{S}_{\text{sep}})$. By exactness, the latter is isomprhic to ker $\left[H^2(G, \widehat{T}_{\text{sep}}) \xrightarrow{\phi} H^2(G, \widehat{P}_{\text{sep}})\right]$, which we call K for brevity.

Let $\alpha \in K$ and let $H \leq G$ be an arbitrary cyclic subgroup. We then have the diagram with exact rows

$$\begin{tikzcd}
H^1(G, \widehat{S}_{\text{sep}}) \arrow[r] \arrow[d, "\text{res}"'] & H^2(G, \widehat{T}_{\text{sep}}) \arrow[r, "\phi"] \arrow[d, "\text{res}"'] & H^2(G, \widehat{P}_{\text{sep}}) \arrow[d, "\text{res}"'] \\
H^1(H, \widehat{S}_{\text{sep}}) \arrow[r] & H^2(H, \widehat{T}_{\text{sep}}) \arrow[r] & H^2(H, \widehat{P}_{\text{sep}}).
\end{tikzcd} \tag{3.2}$$

The Endo-Miyata Theorem (*cf.* [CS77, Prop. 2, p. 184]) states that a group G is metacyclic (= all Sylow subgroup are cyclic) if and only if all flasque G-modules are invertible; in particular, $H^1(H, \widehat{S}_{\text{sep}}) = 0$ (hence the reason for reducing our Galois cohomology groups to finite group cohomology). Therefore, $\alpha \in \ker \left[H^2(G, \widehat{T}_{\text{sep}}) \longrightarrow H^2(H, \widehat{T}_{\text{sep}})\right]$.

Conversely, suppose that α is in the kernel of restriction to every cyclic subgroup of G. By Remark 1.2(b), we can write

$$\widehat{P}_{\text{sep}} \cong \bigoplus_i \mathbb{Z}[G/L_i],$$

so we can assume $\widehat{P}_{\text{sep}} = \mathbb{Z}[G/L]$ for some subgroup $L \leq G$. By the Faddeev-Shapiro Lemma ([Ser97, Ch. I, §2.5, Prop. 10]) we then have that

$$H^2(G, \widehat{P}_{\text{sep}}) \cong H^2(L, \mathbb{Z}) \cong H^1(L, \mathbb{Q}/\mathbb{Z}) = \text{Hom}(L, \mathbb{Q}/\mathbb{Z}).$$

If $\phi(\alpha) = \beta \neq 0$ then $\text{Im}(\beta) = \left\langle \frac{1}{n} \right\rangle$ for some n. Let $l \in L$ be such that $\beta(l) = \frac{1}{n}$ and let $H = \langle l \rangle \leq L \leq G$. Diagram (3.2) can be augmented to

$$\begin{tikzcd}
H^2(G, \widehat{T}_{\text{sep}}) \arrow[r, "\phi"] \arrow[d, "\text{res}"'] & H^2(G, \widehat{P}_{\text{sep}}) \arrow[r, "\cong"] \arrow[d, "\text{res}"'] & \text{Hom}(L, \mathbb{Q}/\mathbb{Z}) \arrow[d] \\
H^2(H, \widehat{T}_{\text{sep}}) \arrow[r] & H^2(H, \widehat{P}_{\text{sep}}) \arrow[r, "\psi"] & H^2(H, \mathbb{Z}) \cong \text{Hom}(H, \mathbb{Q}/\mathbb{Z}),
\end{tikzcd}$$

where the last vertical map is literal restriction and ψ is induced by the map of modules $\mathbb{Z}[G/L] \longrightarrow H$ which sends the identity coset to l. If $\gamma : H \longrightarrow \mathbb{Q}/\mathbb{Z}$ is the restriction of

β, then by construction $\gamma(l) \neq 0$, contradicting that $\operatorname{res}(\alpha) = 0$. Therefore we must have $\phi(\alpha) = \beta = 0$ and $\alpha \in K$. $\square$

Remark 3.2. Note that one obtains $\overline{H}^2_{\mathrm{nr}}(F(T), \mu_n)$ as the n-torsion subgroup of $\overline{\mathrm{Br}}_{\mathrm{nr}}(F(T) = \overline{H}^2_{\mathrm{nr}}(F(T), \mathbb{Q}/\mathbb{Z}(1))$.

For the sake of focus, we relegate the easy generalization of Theorem 3.1 to the following Remark.

Remark 3.3. In the more recent work [Col08], Colliot-Thélène generalized the notion of flasque resolutions to arbitrary algebraic groups (in our sense) when the characteristic is 0, and to reductive groups when the characteristic is positive. Explicitly, he defines an algebraic group H/F, assumed reductive in positive characteristic, to be *quasitrivial* if it is an extension of a quasitrivial torus by a simply connected group. This is equivalent to the two conditions $F_{\mathrm{sep}}[H]^\times / F_{\mathrm{sep}}^\times$ is a permutaiton Γ-module and $\operatorname{Pic}(H_{\mathrm{sep}}) = 0$ [Col08, §2]. If G/F is an algebraic group, assumed reductive in positive characteristic, there exists an exact sequence, called a *flasque resolution* of G,

$$1 \longrightarrow S \longrightarrow H \longrightarrow G \longrightarrow 1,$$

with S a flasque torus and H a quasitrivial group as defined above [Col08, §3]. If X is a smooth F-compactification of G then by [Col08, Thm. 7.1, p. 109] we have that

$$\overline{\mathrm{Br}}(X)' \cong H^1(F, \widehat{S}_{\mathrm{sep}})',$$

where $\overline{\mathrm{Br}}(X) := \mathrm{Br}(X)/\mathrm{Br}(F)$. By (the primed version of) [Col95, §3.4, Sequence (3.9)] we have that

$$\overline{\mathrm{Br}}(X)' \cong \overline{A}^0(X, H^2[\mathbb{Q}/\mathbb{Z}(1)]), \tag{3.3}$$

but since X/F is proper (and contains G as an open subset), Theorem 1.11 implies this is isomorphic to $\overline{H}^2_{\mathrm{nr}}(F(G), \mathbb{Q}/\mathbb{Z}(1))$.

Analogous to how we obtained Theorem 3.1, Proposition 1.16 and the Corollary to Theorem

2.2 imply that the two invariants groups satisfy

$$\begin{tikzcd} \mathrm{Inv}^2_{\mathrm{nr}}(T,\mathbb{Q}/\mathbb{Z}(1)) \arrow[d, hook] \arrow[r, phantom, "\cong"] & H^1(F,\widehat{N}_{\mathrm{sep}})' \\ \mathrm{Inv}^2(T,\mathbb{Q}/\mathbb{Z}(1)) \arrow[r, phantom, "\cong"] & H^1(F,\widehat{T}_{\mathrm{sep}})', \end{tikzcd}$$

where $\widehat{N}_{\mathrm{sep}}$ is a flasque module that fits into an exact sequence

$$0 \longrightarrow \widehat{P}_{\mathrm{sep}} \longrightarrow \widehat{N}_{\mathrm{sep}} \longrightarrow \widehat{T}_{\mathrm{sep}} \longrightarrow 0$$

with $\widehat{P}_{\mathrm{sep}}$ a permutation module, which explains the inclusion $H^1(F,\widehat{N}_{\mathrm{sep}}) \hookrightarrow H^1(F,\widehat{T}_{\mathrm{sep}})$ independent of the natural one for the corresponding invariants groups.

4 An Exact Sequence

In this section we construct an exact sequence that relates invariants of an algebraic group G to the partially unramified groups (*cf.* the introduction or §1.4) of auxiliary varieties. Applying this to tori in degree 2 (*cf.* Remark 4.1 below) recovers the Corollary to Theorem 2.2 and it will be the main tool for determining degree 3 invariants in the next section. We refer to it throughout as the *invariants sequence.*

Theorem 4.1. Let G/F be an algebraic group and C a Γ-module. Let $G \hookrightarrow S$ be an embedding into a universally special algebraic group S and let $X = S/G$ be the homogenous space. Then there exists an exact sequence

$$0 \longrightarrow \mathrm{Inv}^i(G,C) \xrightarrow{r} \overline{A}^0(X,H^i[C]) \xrightarrow{a^*-\pi^*} \overline{A}^0(S\times X,H^i[C]),$$

where r is the Rost embedding of Theorem 1.6 and a^* and π^* are the pullbacks induced by the action (*cf.* Remark 1.2) and projection morphisms $S\times X \rightrightarrows X$, respectively.

Proof. The proof of Lemma 1.4 implies that the natural map $\mu : S \longrightarrow X$ is a versal G-torsor hence exactness at $\mathrm{Inv}^i(G,C)$ follows from Theorem 1.6. Replacing $\mathbf{SL}_n$ by S, the arguments at the beginning of the proof of Theorem 2.1 (and the last paragraph) show that a given $\alpha \in \overline{A}^0(X,H^i[C])$ is in $\mathrm{Im}(r)$ if $a^*(\alpha) = \pi^*(\alpha)$, hence $\ker(a^*-\pi^*) \subseteq \mathrm{Im}(r)$.

Now let $\beta \in \operatorname{Inv}^i(G, C)$ so that $r(\beta) = \beta(S_\xi)$, where $\xi \in X$ is the generic point. Writing $\eta \in S \times X$ for the generic point, a and π induce maps on function fields and a commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(F(S \times X)) \arrow[r, shift left, "a'"] \arrow[r, shift right, "\pi'"'] \arrow[d, "\eta"'] & \operatorname{Spec}(F(X)) \arrow[d, "\xi"] \\
S \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "\pi"'] & X
\end{tikzcd} \tag{4.1}$$

hence they induce maps on torsors

$$\begin{tikzcd}
H^1(F(X), G) \arrow[r, shift left] \arrow[r, shift right] & H^1(F(S \times X), G)
\end{tikzcd}$$

given by pullback along a' and π', respectively. The key to comparing $a^*(\beta(S_\xi))$ and $\pi^*(\beta(S_\xi))$ is the fact that β is a natural transformation of functors hence one has a commutative diagram

$$\begin{tikzcd}
H^1(F(X), G) \arrow[r, shift left] \arrow[r, shift right] \arrow[d, "\beta"'] & H^1(F(S \times X), G) \arrow[d, "\beta"] \\
H^i(F(X), C) \arrow[r, shift left] \arrow[r, shift right] & H^i(F(S \times X), C) \\
\overline{A}^0(X, H^i[C]) \arrow[r, shift left, "a^*"] \arrow[r, shift right, "\pi^*"'] \arrow[u, hook] & \overline{A}^0(S \times X, H^i[C]). \arrow[u, hook]
\end{tikzcd}$$

Therefore

$$a^*(\beta(S_\xi)) = \beta(S_{\xi \circ a'})$$
$$\pi^*(\beta(S_\xi)) = \beta(S_{\xi \circ \pi'})$$

since S_ξ is itself the pullback of $\mu : S \longrightarrow X$ along $\xi : \operatorname{Spec}(F(X)) \longrightarrow X$. Writing $E_1 = S_{\xi \circ a'}$ and $E_2 = S_{\xi \circ \pi'}$ for clarity, it suffices to show that $E_1 \cong E_2$ in $H^1(F(S \times X), G)$. Moreover, by diagram (4.1) we have that $\xi \circ a' = a \circ \eta$ and analogously with π' and π. Therefore, one can realize E_1 and E_2 as pullbacks along η:

$$\begin{tikzcd}
E_1, E_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & \mathscr{E}_1, \mathscr{E}_2 \arrow[r, shift left] \arrow[r, shift right] \arrow[d, shift left] \arrow[d, shift right] & S \arrow[d, "\mu"] \\
\operatorname{Spec}(F(S \times X)) \arrow[r, "\eta"'] & S \times X \arrow[r, shift left, "a"] \arrow[r, shift right, "\pi"'] & X.
\end{tikzcd}$$

Hence we reduce to showing that $\mathscr{E}_1 \cong \mathscr{E}_2$ in $H^1(S \times X, G)$. In fact, we will see that both are isomorphic (as G-torsors) to

$$\begin{tikzcd} S \times S \arrow[d, "\mathrm{Id} \times \mu"] \\ S \times X. \end{tikzcd}$$

in which G acts trivially on the first component and by right multiplication on the second. It is clear we have a commutative diagram

$$\begin{tikzcd} S \times S \arrow[r, "\pi_2"] \arrow[d, "\mathrm{Id} \times \mu"'] & S \arrow[d, "\mu"] \\ S \times X \arrow[r, "\pi"'] & X, \end{tikzcd}$$

where m is the multiplication map. Moreover, for any scheme Y/X forming a commutative diagram

$$\begin{tikzcd} Y \arrow[drr, bend left, "\psi"] \arrow[ddr, bend right, "(\varphi_S, \varphi_X)"'] & & \\ & S \times S \arrow[r, "\pi_2"] \arrow[d, "\mathrm{Id} \times \mu"'] & S \arrow[d, "\mu"] \\ & S \times X \arrow[r, "\pi"'] & X, \end{tikzcd}$$

the map $(\varphi_S, \psi) : Y \longrightarrow S \times S$ completes the diagram uniquely, i.e., we have that

$$S \times S \cong \mathscr{E}_1 \in H^1(S \times X, G)$$

since $\mathscr{E}_1$ is the pullback of $\mu : S \longrightarrow X$ along $\pi : S \times X \longrightarrow X$ by definition and the G-torsor structures are induced by the pullback in both cases.

Analogously, by construction of the homogeneous space X as the "orbit space" of a fixed $w \in W$ for some representation $S \longrightarrow \mathbf{GL}(W)$ such that $G \hookrightarrow S$ is the stabilizer of w, we have a commutative diagram

$$\begin{tikzcd} S \times S \arrow[r, "m"] \arrow[d, "\mathrm{Id} \times \mu"'] & S \arrow[d, "\mu"] \\ S \times X \arrow[r, "a"'] & X. \end{tikzcd}$$

Moreover, for any scheme Y/X forming a commutative diagram

$$\begin{tikzcd}
Y \arrow[rrd, bend left, "\psi"] \arrow[rdd, bend right, "(\varphi_S{,}\varphi_X)"'] & & \\
& S \times S \arrow[r, "m"] \arrow[d, "\mathrm{Id} \times \mu"'] & S \arrow[d, "\mu"] \\
& S \times X \arrow[r, "a"'] & X,
\end{tikzcd}$$

the map $(\varphi_S, m \circ (i \circ \varphi_S, \psi)) : Y \longrightarrow S \times S$, with $i : S \longrightarrow S$ the inverse, completes the diagram uniquely, i.e., we have that

$$S \times S \cong \mathscr{E}_2 \in H^1(S \times X, G)$$

since $\mathscr{E}_2$ is the pullback of $\mu : S \longrightarrow X$ along $a : S \times X \longrightarrow X$ by definition and as before the G-torsor structures are induced by the pullback in both cases.

It remains only to see why the G-torsor structures on $\mathrm{Id} \times \mu : S \times S \longrightarrow S \times X$ induced by the two pullbacks are isomorphic. Since the structure morphism is the identity in the first component, we have that G must act trivially on the first component in both pullbacks. Since the G-torsor structure in both cases is induced from the G-torsor structure on the natural map $\mu : S \longrightarrow X$, which is given by right multiplication (*cf.* Remark 1.2), considering the two pullback diagrams we see that in fact G must also act by right multiplication on the second component in both cases. Thus indeed the two pullbacks are isomorphic as G-torsors over $S \times X$ since they are isomorphic as $S \times X$-varieties and have the same G-structure. ☐

Remark 4.1. We use the invariants sequence of the previous Theorem to recover the calculation of degree 2 invariants of tori obtained in the Corollary to Theorem 2.2 above. What makes the invariants sequence so useful for tori is the existence of toric resolutions: if

$$1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$$

is an exact sequence of tori with P quasitrivial (e.g., a coflasque resolution of T or a flasque resolution of Q) then the invariants sequence applies with $S = P$ and $X = Q$; in particular, toric resolutions allow one to use schemes X and $S \times X$ in the invariants sequence which are themselves tori. In degree 2 with $C = \mathbb{Q}/\mathbb{Z}(1)$, the invariants sequence becomes, after the identity $\overline{\mathrm{Br}}(Y)' \cong \overline{A}^0(Y, \mathbb{Q}/\mathbb{Z}(1))$ of Equation (3.3),

$$0 \longrightarrow \mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \xrightarrow{r} \overline{\mathrm{Br}}(Q)' \xrightarrow{a^* - \pi^*} \overline{\mathrm{Br}}(P \times Q)' .$$

Therefore, we are reduced to calculating Brauer groups of tori, which can be done by using the Hochschild-Serre spectral sequence, and $\ker(a^* - \pi^*)$. This is essentially the same approach we take in §5 to compute degree 3 invariants of tori, but in one degree higher (*cf.* Theorem 5.2). The spectral sequence is

$$E_2^{p,q} = H^p_{\text{ét}}(F, H^q(Q_{\text{sep}}, \mathbf{G}_m)) \implies H^{p+q}_{\text{ét}}(Q, \mathbf{G}_m)$$

and we have that

$$H^i_{\text{ét}}(Q_{\text{sep}}, \mathbf{G}_m) = \begin{cases} F_{\text{sep}}[Q]^\times = F_{\text{sep}}^\times \oplus \widehat{Q}_{\text{sep}} & i = 0 \\ \mathrm{Pic}(Q_{\text{sep}}) = 0 & i = 1 \\ \mathrm{Br}(Q_{\text{sep}}) & i = 2, \end{cases}$$

where the first equality is [Ros61, Thm. 3]. Considering the E_2-page, one obtains the exact sequence

$$0 \longrightarrow \mathrm{Br}(F) \oplus H^2(F, \widehat{Q}_{\text{sep}}) \longrightarrow \mathrm{Br}(Q) \xrightarrow{\theta} \mathrm{Br}(Q_{\text{sep}}).$$

But, the commutative diagram

$$\begin{tikzcd}
\overline{\mathrm{Br}}(Q) \arrow[r, "\overline{\theta}"] & \overline{\mathrm{Br}}(Q_{\text{sep}}) \\
\mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \arrow[u, hook] \arrow[r] & \mathrm{Inv}^2(T_{\text{sep}}, \mathbb{Q}/\mathbb{Z}(1)) = 0 \arrow[u, hook]
\end{tikzcd}$$

implies that $\mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \subseteq \ker(\overline{\theta}) = H^2(F, \widehat{Q}_{\text{sep}})$ so the invariants sequence becomes

$$0 \longrightarrow \mathrm{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \longrightarrow H^2(F, \widehat{Q}_{\text{sep}})' \xrightarrow{a^* - \pi^*} H^2(F, \widehat{P}_{\text{sep}})' \oplus H^2(F, \widehat{Q}_{\text{sep}})'.$$

To determine $\ker(a^* - \pi^*)$ we consider the short exact sequence of Γ-modules

$$0 \longrightarrow \widehat{Q}_{\text{sep}} \xrightarrow{\widehat{j}} \widehat{P}_{\text{sep}} \xrightarrow{\widehat{i}} \widehat{T}_{\text{sep}} \longrightarrow 0. \tag{4.2}$$

We have the commutative square

$$\begin{tikzcd}
P \times Q \arrow[r, "a"] & Q \\
P \times P \arrow[u, "(\mathrm{Id}, j)"] \arrow[r, "m"'] & P \arrow[u, "j"']
\end{tikzcd}$$

and its dual

$$\begin{tikzcd}
\widehat{P}_{\mathrm{sep}} \oplus \widehat{Q}_{\mathrm{sep}} \arrow[d, "(\mathrm{Id}, \widehat{j})"'] & \widehat{Q}_{\mathrm{sep}} \arrow[l, "\widehat{a}"'] \arrow[d, "\widehat{j}"] \\
\widehat{P}_{\mathrm{sep}} \oplus \widehat{P}_{\mathrm{sep}} & \widehat{P}_{\mathrm{sep}}, \arrow[l, "\Delta"]
\end{tikzcd}$$

where Δ is the diagonal map, hence we see that

$$\widehat{a} = (\widehat{j}, \mathrm{Id}).$$

Similarly, the dual of the projection $\pi : P \times Q \longrightarrow Q$ is $\widehat{\pi} = (0, \mathrm{Id})$ and so

$$a^* - \pi^* = f \oplus 0, \tag{4.3}$$

where $f : H^2(F, \widehat{Q}_{\mathrm{sep}}) \longrightarrow H^2(F, \widehat{P}_{\mathrm{sep}})$ is the map on cohology induced by $\widehat{j} : \widehat{Q}_{\mathrm{sep}} \longrightarrow \widehat{P}_{\mathrm{sep}}$. Since $\widehat{P}_{\mathrm{sep}}$ is a permutation Γ-module, the exact sequence (4.2) implies that

$$\ker(a^* - \pi^*) = \ker(f) \cong H^1(F, \widehat{T}_{\mathrm{sep}}).$$

5 Degree 3 Invariants

In this section we calculate $\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for T/F an algebraic torus and $p \neq 2, q$. We begin by connecting our work to an older construction which will allow us to make use of some auxiliary results. We then use the Hochschild-Serre spectral sequence to determine the necessary groups and maps in the relevant form of the invariants sequence of the previous section.

In [Lic87], Lichtenbaum defined a weight-two motivic complex $\Gamma(2)_X$ of étale sheaves on any regular noetherian scheme X. This complex is concentrated in degrees 1 and 2 and we write $\mathbb{H}^p(X, \Gamma(2))$ for the étale (hyper)cohomology groups. We let $\Gamma(0)$ be the constant sheaf $\mathbb{Z}$ concentrated in degree 0 and $\Gamma(1)$ the sheaf $\mathbf{G}_m$ concentrated in degree 1 and use the notation

$$\mathbb{H}^{p,q}(X) = \begin{cases} \mathbb{H}^p(X, \Gamma(q)) & q = 0, 1, 2 \\ 0 & \text{otherwise} \end{cases}$$

for the *motivic cohomology groups.* We write $\mathbb{H}^{p,q}(F)$ for $\mathbb{H}^{p,q}(\mathrm{Spec}(F))$. In particular, we have that

$$\mathbb{H}^{p,1}(X) = H^{p-1}_{\text{ét}}(X, \mathbf{G}_m),$$

hence

$$\mathbb{H}^{3,1}(F) \cong \mathrm{Br}(F) \cong H^2(F, \mathbb{Q}/\mathbb{Z}(1)) \oplus \mathrm{Br}(F)\{q\},$$

since the our (see below) Γ-module $\mathbb{Q}/\mathbb{Z}(1)$ has no characteristic component.

In the introduction to [Kah96], the author defines étale sheaves $\mathbb{Q}_p/\mathbb{Z}_p(2) = \mathbb{Q}/\mathbb{Z}(2)\{p\}$ whose sections over F_{sep} agree with the identically notated (see Remark 5.1 below) Γ-modules we defined in §1.4 for primes $p \neq q$, hence their cohomology over fields agrees canonically with our Galois cohomology groups. But, whereas our construction of the Γ-module

$$\mathbb{Q}/\mathbb{Z}(2) = \coprod_{p \text{ prime } \neq q} \mathbb{Q}_p/\mathbb{Z}_p(2) \tag{5.1}$$

avoids the characteristic part, in [Kah96], the author explicitly constructs a q-component $\mathbb{Q}_q/\mathbb{Z}_q(2)$. In [GMS03, Part 2, App. A], although the object $\mathbb{Q}_q/\mathbb{Z}_q(2)$ is not constructed, the author defines the "cohomology" groups $H^{i+1}(F, \mathbb{Q}_q/\mathbb{Z}_q(i))$ as $H^2(F, K_i(F_{\text{sep}}))\{q\}$ for all $i \geq 0$, which guarantees that $H^2(F, \mathbb{Q}/\mathbb{Z}(1)) \cong \mathrm{Br}(F)$ and not just $\mathrm{Br}(F)'$. The fact, that the two constructions of $H^3(F, \mathbb{Q}_q/\mathbb{Z}_q(2))$ in [Kah96] and [GMS03] agree is a highly nontrivial result related to the so-called Bloch-Kato conjecture in positive characteristic (*cf.* [BK86]).

Remark 5.1. Although in this work $\mathbb{Q}/\mathbb{Z}(i)$ will always refer to our "q-less" Γ-module construction (5.1), because we will appeal to results in [Kah96], one should keep in mind that the $\mathbb{Q}/\mathbb{Z}(2)$ which appears in that work is an étale sheaf and includes the characteristic component $\mathbb{Q}_q/\mathbb{Z}_q(2)$ constructed there. Thus, when referencing results from [Kah96] that include this object, we will write $\mathcal{Q}/\mathcal{Z}(2)$ for Kahn's characteristic-inclusive étale sheaf. We continue to write $\mathbb{Q}_p/\mathbb{Z}_p(2)$ for the components in both constructions because it will always be clear form context whether we mean the étale sheaf or the Γ-module. For clarity, we suppress subscripts in the cohomology groups when taking étale cohomology over fields. Moreover, we will have occasion (Theorem 5.2 below) to exploit the étale sheaf structure of $\mathcal{Q}/\mathcal{Z}(2)$ and its components $\mathbb{Q}_p/\mathbb{Z}_p(2)$ for $p \neq q$.

As in Remark 1.4, if $x \in X(F)$ one has that $\mathbb{H}^{p,q}(F)$ is a direct summand of $\mathbb{H}^{p,q}(X)$. Moreover, [Kah96, Thm. 1.1] implies that

$$\mathbb{H}^{i,2}(F) \cong H^{i-1}(F, \mathcal{Q}/\mathcal{Z}(2)) \cong H^{i-1}(F, \mathbb{Q}/\mathbb{Z}(2)) \oplus H^{i-1}(F, \mathbb{Q}_q/\mathbb{Z}_q(2)), \quad i \geq 4. \tag{5.2}$$

Avoiding the characteristic part, if $x \in X(F)$ we define the *normalized motivic cohomology groups*

$$\overline{\mathbb{H}}^{4,2}(X) := \mathbb{H}^{4,2}(X)\big/H^3(F, \mathbb{Q}/\mathbb{Z}(2)). \tag{5.3}$$

The reason for this (instead of the more natural $\mathbb{H}^{4,2}(X)/\mathbb{H}^{4,2}(F)$) will become apparent in the proof of Theorems 5.2 and 7.1 below.

For the invariants sequence, we need to determine (normalized) partially unramified groups $\overline{A}^0(Y, H^3[\mathbb{Q}/\mathbb{Z}(2)])$ for specific varieties Y/F (*cf.* Theorem 4.1). The connection to Lichtenbaum's complex is provided by a result of Kahn's and the cohomology of this complex is determined using the Hochschild-Serre spectral sequence. Moreover, this spectral sequence reveals how, when working with tori, one can shrink the partially unramified groups appearing in the invariants sequence and still obtain an exact sequence involving cohomological invariants. The resulting groups, and in particular maps between them, can then be analyzed using basic K-theory and Galois cohomology.

We begin with a lemma which will simplify the presentation of the next theorem.

Lemma 5.1. *If Q is a coflasque torus then $CH^2(Q) = 0$.*

Proof. In the Corollary to Theorem 2.2, we saw that $\mathrm{Pic}(Q) \cong H^1(F, \widehat{Q}_{\mathrm{sep}})$, and the latter is isomorphic to $H^1(G, \widehat{Q}_{\mathrm{sep}})$ by the inflation-restriction exact sequence ([Ser79, Ch. VII, §6, Prop. 4]), where $G = \mathrm{Gal}(L/F)$ is the Galois group of a finite splitting field L/F of Q. More generally, for all intermediate fields $F \subseteq E \subseteq L$, we have that $\mathrm{Pic}(Q_E) = H^1(H, \widehat{Q}_{\mathrm{sep}})$, where $H = \mathrm{Gal}(L/E)$. Since Q is coflasque, we obtain $\mathrm{Pic}(Q_E) = 0$ for all such intermediate fields.

By [MP97, Cor. 5.13], $K_0(Q)$ is generated by $(i_{E/F})_*(\mathrm{Pic}(Q_E))$ for all intermediate fields E as above, where $(i_{E/F}) : Q_E \longrightarrow Q$ is the natural map. By the above, $\mathrm{Pic}(Q_E)$ consists only of the structure sheaf $\mathcal{O}_{Q_E}$ and

$$(i_{E/F})_*(\mathcal{O}_{Q_E}) = \mathcal{O}_Q^{[E:F]},$$

hence $K_0(Q) \cong \mathbb{Z}$, generated by $\mathcal{O}_Q$. The second Chern class

$$c_2 : K_0(Q) \twoheadrightarrow CH^2(Q)$$

is surjective and although it is not a homomorphism, it is not too far off: by [Ful98, Thm. 3.2(e)], the composition

$$\begin{tikzcd} K_0(Q) \arrow[r, two heads, "c_2"] \arrow[rr, bend right, "\overline{c_2}"'] & CH^2(Q) \arrow[r, two heads] & CH^2(Q)/CH^1(Q).CH^1(Q) \end{tikzcd}$$

is a homomorphism. Since the generator $\mathcal{O}_Q$ corresponds to a line bundle in $K_0(Q)$, we have that $\overline{c_2} = 0$ and therefore $CH^2(Q) = CH^1(Q).CH^1(Q)$. But $CH^1(Q) \cong \mathrm{Pic}(Q) = 0$. ☐

Theorem 5.2. *Let T/F be an algebraic torus and let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be a coflasque resolution of T. Then for every prime $p \neq q$, there is an exact sequence*

$$0 \longrightarrow \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{r} H^2(F, A^0(Q_{\mathrm{sep}}, K_2))\{p\} \xrightarrow{a^* - \pi^*} H^2(F, A^0(P_{\mathrm{sep}} \times Q_{\mathrm{sep}}, K_2))\{p\},$$

where r is the Rost embedding of Theorem 1.6 and a^* and π^* are induced by the action and projection morphisms $P \times Q \rightrightarrows Q$, respectively.

Proof. In [Kah96, Thm. 1.1], the author constructs the short exact sequence

$$0 \longrightarrow CH^2(X) \longrightarrow \mathbb{H}^{4,2}(X) \longrightarrow H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathcal{Q}/\mathcal{Z})) \longrightarrow 0 \tag{5.4}$$

for any smooth, connected variety X/F, where $\mathcal{H}^3(\mathcal{Q}/\mathcal{Z}(2))$ is the Zariski sheaf associated to the the presheaf $U \longmapsto H^3_{\text{ét}}(U, \mathcal{Q}/\mathcal{Z}(2))$ (*cf.* Remark 5.1 and the paragraph proceeding it). Using the (étale cohomology) Gersten resolution of this sheaf (*cf.* [BO74]), one can see that

$$H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathcal{Q}/\mathcal{Z}))\{p\} \cong A^0(X, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]) \tag{5.5}$$

for primes $p \neq q$. As pointed out in Remark 4.1, the invariants sequence (Theorem 4.1) applies with $S = P$ and $X = Q$. Moreover, both Q and $P \times Q$ are coflasque tori hence the previous lemma combined with (5.4) and (5.5) imply that, for $p \neq q$,

$$\mathbb{H}^{4,2}(Q)\{p\} \cong A^0(Q, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]), \tag{5.6}$$

and analogously with $P \times Q$. Hence, after the definition in (5.3), the invariants exact sequence can be expressed

$$0 \longrightarrow \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow \overline{\mathbb{H}}^{4,2}(Q)\{p\} \longrightarrow \overline{\mathbb{H}}^{4,2}(P \times Q)\{p\}. \tag{5.7}$$

To determine the latter two groups, we use the Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(F, \mathbb{H}^{q,2}(X_{\mathrm{sep}})) \implies \mathbb{H}^{p+q,2}(X).$$

For smooth connected varieties X/F, [Kah96, Thm. 1.1] shows that

$$\mathbb{H}^{q,2}(X) = \begin{cases} 0 & q \leq 0 \\ K_3^{\mathrm{ind}}(F(X)) & q = 1 \\ A^0(X, K_2) & q = 2 \\ A^1(X, K_2) & q = 3, \end{cases} \tag{5.8}$$

where $K_3^{\mathrm{ind}}(F(X)) = \mathrm{coker}\left[K_3(F(X)) \longrightarrow K_3^Q(F(X))\right]$ and $K_3^Q(F(X))$ is Quillen's K-Theory group ([Qui73]). The group $K_3^{\mathrm{ind}}(F)$ is studied in [MS90].

Consider the case $X = Q$. Since $F_{\mathrm{sep}}(Q) \cong F_{\mathrm{sep}}(t_1, \ldots, t_n)$, by [MS90, Lemma 4.2], one has that $K_3^{\mathrm{ind}}(F_{\mathrm{sep}}(Q)) \cong K_3^{\mathrm{ind}}(F_{\mathrm{sep}})$, and by [MS90, Thm. 10.2] the latter is divisible. Moreover, [MS90, Prop. 11.1] shows that

$$K_3^{\mathrm{ind}}(F_{\mathrm{sep}})_{\mathrm{tor}} \cong \mathbb{Q}/\mathbb{Z}(2). \tag{5.9}$$

Note that the $\mathbb{Q}/\mathbb{Z}(2)$ used in [MS90] comes from [Tat76], but in fact agrees with our notation (*cf.* the discussion around (5.32)). Since $K_3^{\mathrm{ind}}(F_{\mathrm{sep}}(Q))/K_3^{\mathrm{ind}}(F_{\mathrm{sep}}(Q))_{\mathrm{tor}}$ is uniquely divisible, it has trivial Galois cohomology in degree ≥ 1 and therefore

$$H^p(F, K_3^{\mathrm{ind}}(F_{\mathrm{sep}}(Q))) \cong H^p(F, \mathbb{Q}/\mathbb{Z}(2)), \quad p > 1. \tag{5.10}$$

We have by [GMS03, Part 2, Cor. 5.6(2)] that

$$A^1(Q_{\mathrm{sep}}, K_2) = 0, \tag{5.11}$$

so combining the formulas in (5.8) with the computations in (5.10) and (5.11), a portion of the E_2-page of the Hochschild-Serre spectral sequence for $X = Q$ is

$$\begin{tikzcd}
q & & & & & \\
3 \arrow[u] & 0 \arrow[rrr] & & & {} & \\
2 & H^1(F, A^0(Q_{\mathrm{sep}}, K_2)) \arrow[drr, "d_2^{1,2}"'] & H^2(F, A^0(Q_{\mathrm{sep}}, K_2)) \arrow[drr, "d_2^{2,2}"] & & & \\
1 & & & H^3(F, \mathbb{Q}/\mathbb{Z}(2)) & H^4(F, \mathbb{Q}/\mathbb{Z}(2)) & \\
0 & 0 \arrow[rrr] & & & {} & \\
{} \arrow[rrrrr] & & & & & p \\
& 1 & 2 & 3 & 4 &
\end{tikzcd}$$

Considering the filtration

$$F^4 \subseteq F^3 \subseteq F^2 \subseteq F^1 \subseteq \mathbb{H}^{4,2}(Q)$$

we have that $F^4 \cong E_\infty^{4,0} = 0$ hence $E_\infty^{3,1} \cong F^3 \subseteq \mathbb{H}^{4,2}(Q)$. Using this, we can see that the differentials $d_2^{1,2}$ and $d_2^{2,2}$ are both trivial. Indeed, we have that the differentials $d_r^{3,1} = 0$ for $r \geq 2$ and the differentials mapping into $E_s^{3,1}$ are trivial for $s \geq 3$, hence $E_\infty^{3,1} = E_3^{3,1} = \operatorname{coker}(d_2^{1,2})$. Thus, we have the composition

$$\begin{tikzcd}
E_2^{3,1} = H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \arrow[r, two heads] \arrow[rr, bend right] & \operatorname{coker}(d_2^{1,2}) = E_\infty^{3,1} \arrow[r, hook] & \mathbb{H}^{4,2}(Q),
\end{tikzcd}$$

which is injective because the group identity in $Q(F)$ induces a splitting

$$\begin{tikzcd}
\mathbb{H}^{4,2}(F) \arrow[r] & \mathbb{H}^{4,2}(X) \arrow[r] & \mathbb{H}^{4,2}(F)
\end{tikzcd}$$

and $H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ is a direct summand of $\mathbb{H}^{4,2}(F)$ by (5.2), which is preserved by this mapping. Therefore, $H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \cong \operatorname{coker}(d_2^{1,2})$ hence $d_2^{1,2} = 0$. An entirely analogous argument, in one degree higher, applies to $d_2^{2,2}$ (except that, *a priori*, we only have a surjection $\operatorname{coker}(d_2^{2,2}) \twoheadrightarrow E_\infty^{4,1} \subseteq \mathbb{H}^{5,2}(Q)$ because although all the differentials $d_r^{4,1} = 0$ for $r \geq 2$, we do not bother investigating whether the differential mapping into $E_4^{4,1}$ is trivial).

Since $F^1/F^2 \cong E_\infty^{1,3} = 0$, the above implies that

$$\begin{aligned} F^3 &\cong E_\infty^{3,1} = \operatorname{coker}(d_2^{1,2}) = H^3(F, \mathbb{Q}/\mathbb{Z}(2)) \\ F^2\big/F^3 &\cong E_\infty^{2,2} = \ker(d_2^{2,2}) = H^2(F, A^0(Q_{\mathrm{sep}}, K_2)) \\ \mathbb{H}^{4,2}(Q)\big/F^2 &\cong \mathbb{H}^{4,2}(Q)\big/F^1 \cong E_\infty^{0,4}. \end{aligned}$$

Taking the quotient by $F^3 = H^3(F, \mathbb{Q}/\mathbb{Z}(2))$ everywhere in the filtration then gives the short exact sequence

$$0 \longrightarrow H^2(F, A^0(Q_{\mathrm{sep}}, K_2)) \longrightarrow \overline{\mathbb{H}}^{4,2}(Q) \longrightarrow E_\infty^{0,4} \longrightarrow 0,$$

justifying the definition in (5.3). Because all differentials into $E_r^{0,4}$ for $r \geq 2$ are necessarily 0, we have that

$$E_\infty^{0,4} \hookrightarrow E_2^{0,4} = \mathbb{H}^{4,2}(Q_{\mathrm{sep}})^\Gamma \subseteq \mathbb{H}^{4,2}(Q_{\mathrm{sep}}).$$

Combining this with the inclusion in (5.7), the previous exact sequence can be rewritten and augmented to

$$\begin{tikzcd}
0 \arrow[r] & H^2(F, A^0(Q_{\mathrm{sep}}, K_2)) \arrow[r] & \overline{\mathbb{H}}^{4,2}(Q) \arrow[r] & \overline{\mathbb{H}}^{4,2}(Q_{\mathrm{sep}}) \\
 & & \operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \arrow[u, hook] \arrow[r] & \operatorname{Inv}^3(T_{\mathrm{sep}}, \mathbb{Q}/\mathbb{Z}(2)) = 0, \arrow[u, hook]
\end{tikzcd}$$

hence the embedding $\operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2)) \hookrightarrow \overline{\mathbb{H}}^{4,2}(Q)$ has image in $H^2(F, A^0(Q_{\mathrm{sep}}, K_2))$. Since $P \times Q$ satisfies all the necessary hypotheses and the top row is functorial, we are done after the invariants exact sequence in (5.7). □

If X/F is a smooth variety then $X(F_{\mathrm{sep}}) \neq \emptyset$ and so $K_2(F_{\mathrm{sep}})$ is a direct summand of $A^0(X_{\mathrm{sep}}, K_2)$ by Remark 1.4(c). But, since the Γ-action on both is diagonal, the decomposition holds as Γ-modules, which is essential because we take Galois cohomology in the previous theorem. Writing

$$A^0(Q_{\mathrm{sep}}, K_2) = K_2(F_{\mathrm{sep}}) \oplus \overline{A}^0(Q_{\mathrm{sep}}, K_2),$$

and analogously with $P_{\mathrm{sep}} \times Q_{\mathrm{sep}}$, the following will allow us to ignore the first component and restate the previous theorem.

Lemma 5.3. *$H^i(F, K_2(F_{\mathrm{sep}}))\{p\} = 0$ for all $p \neq q$ and all $i \geq 1$.*

Proof. Write $A = K_2(F_{\mathrm{sep}})\{p\}$. Since $K_2(F_{\mathrm{sep}})$ is p-divisible, $K_2(F_{\mathrm{sep}})/A$ is uniquely p-divisible and so the p-primary component of its Galois cohomology is trivial in positive degree. Since localization is exact, it commutes with taking cohomology and so the exact sequence in cohomology gives a surjection

$$H^i(F, A)\{p\} \twoheadrightarrow H^i(F, K_2(F_{\mathrm{sep}}))\{p\}$$

for all $i \geq 1$. Therefore, it suffices to show that $A = 0$. By [MS82, Thm. 14.2], we have that a symbol in ${}_pK_2(F_{\mathrm{sep}})$ is necessarily of the form $\{\zeta_p, b\}$ for some p^{th}-root of unity ζ_p and $b \in F_{\mathrm{sep}}^{\times}$. Since $F_{\mathrm{sep}} = F_{\mathrm{sep}}^p$ for $p \neq q$, we have that $\{\zeta_p, b\} = \{\zeta_p, c^p\} = \{1, c\} = 0$. ☐

Remark 5.2. Note that the above result holds for any field L such that $\mu_p \subset L$ and $L = L^p$. Also, there is a more direct argument that does not rely on [MS82, Thm. 14.2] and which actually proves the stronger statement $K_2(L)\{p\} = 0$. One constructs an inverse to the multiplication by p map

$$K_2(L) \xrightarrow{p} K_2(L)$$

as follows. Begin with a map $L \otimes L \longrightarrow K_2(L)$ given by $a \otimes b \longmapsto \{a^{1/p}, b\}$ where $a^{1/p}$ is any p^{th}-root of a. This is well-defined because all p^{th}-roots of a differ by a ζ_p and we already saw that any $\{\zeta_p, b\} = 0$. Since the map is clearly linear, to descend this to $K_2(F)$ we need to show that it respects the Steinberg relation: $a \otimes (1-a) \longmapsto 0$. But, we can write

$$1 - a = \prod_{i=1}^{p}(1 - \zeta_p^i a^{1/p})$$

hence

$$a \otimes 1 - a \longmapsto \sum_{i=1}^{p} \{a^{1/p}, 1 - \zeta_p^i a^{1/p}\} = \sum_{i=1}^{p} \{\zeta_p^i a^{1/p}, 1 - \zeta_p^i a^{1/p}\} = 0.$$

Thus, we have a map

$$\begin{aligned} K_2(L) &\longrightarrow K_2(L) \\ \{a, b\} &\longmapsto \{a^{1/p}, b\} \end{aligned}$$

which is clearly inverse to the multiplication by p map.

After the lemma and the splitting above, we have that

$$H^i(F, A^0(Q_{\text{sep}}, K_2))\{p\} \cong H^i(F, \overline{A}^0(Q_{\text{sep}}, K_2))\{p\}, \quad i \geq 1,$$

for all $p \neq q$, and analogously with $P_{\text{sep}} \times Q_{\text{sep}}$. Hence, we can restate Theorem 5.2 as follows.

Corollary. *Let T/F be an algebraic torus and let $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ be a coflasque resolution of T. Then for every prime $p \neq q$, there is an exact sequence*

$$0 \longrightarrow \operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \xrightarrow{r} H^2(F, \overline{A}^0(Q_{\text{sep}}, K_2))\{p\} \xrightarrow{a^* - \pi^*} H^2(F, \overline{A}^0(P_{\text{sep}} \times Q_{\text{sep}}, K_2))\{p\}. \tag{5.12}$$

Therefore, we must investigate not only the group structure of $\overline{A}^0(Q_{\text{sep}}, K_2)$ and $\overline{A}^0(P_{\text{sep}} \times Q_{\text{sep}}, K_2)$ but also their Γ-module structure, and it is in doing so that we will impose the condition $p \neq 2$.

To begin with, [GMS03, Part 2, Cor. 5.6(1)] implies that we have a split short exact sequence *of abelian groups*

$$\begin{tikzcd} 0 \arrow[r] & F_{\text{sep}}^{\times} \otimes \widehat{Q}_{\text{sep}} \arrow[r, "\iota"] & \overline{A}^0(Q_{\text{sep}}, K_2) \arrow[r, "p"] & \Lambda^2(\widehat{Q}_{\text{sep}}) \arrow[r] \arrow[l, bend left, "\sigma"] & 0, \end{tikzcd} \tag{5.13}$$

with $\iota(a \otimes \eta) = \{a, \eta\}$ and $\sigma(f_i \wedge f_j) = \{f_i, f_j\}$ for $i \neq j$, where $\{f_1, \dots, f_n\}$ is a $\mathbb{Z}$-basis for $\widehat{Q}_{\text{sep}}$. Indeed, these maps make sense because

$$F_{\text{sep}}[Q]^{\times} = \ker \left[\, F_{\text{sep}}(Q)^{\times} \xrightarrow{\coprod v_x} \textstyle\coprod_{x \in Q_{\text{sep}}^{(1)}} \mathbb{Z} \, \right]$$

hence [Mil70, Lemma 2.1] implies that the symbols $\{g, h\} \in K_2(F_{\text{sep}}(Q))$ with $g, h \in F_{\text{sep}}[Q]^\times$ vanish under the residue homomorphisms ∂_x, i.e., all such $\{g, h\} \in A^0(Q_{\text{sep}}, K_2)$. By [Ros61, Thm. 3] we have that $F_{\text{sep}}[Q]^\times = F_{\text{sep}}^\times \oplus \widehat{Q}_{\text{sep}}$ hence we can view symbols of the form $\{a, \eta\}$ and $\{\eta, \chi\}$, with $a \in F_{\text{sep}}^\times$ and $\eta, \chi \in \widehat{Q}_{\text{sep}}$, in $A^0(Q_{\text{sep}}, K_2)$, and in fact in $\overline{A}^0(Q_{\text{sep}}, K_2)$.

Remark 5.3. The Γ-action on $F_{\text{sep}}^\times \otimes \widehat{Q}_{\text{sep}}$ and $\Lambda^2(\widehat{Q}_{\text{sep}})$ is the diagonal one and the action on $\overline{A}^0(Q_{\text{sep}}, K_2)$ is induced by the natural inclusion $\overline{A}^0(Q_{\text{sep}}, K_2) \subseteq K_2(F_{\text{sep}}(Q))$, hence is also diagonal. Therefore, although ι and p are Γ-equivariant, σ, *a priori*, is not. That is, the splitting is only as abelian groups, not as Γ-modules (yet). This is because the result [GMS03, Part 2, Cor. 5.6(1)] deals exclusively with split tori hence there the Γ-action is trivial and the character module is merely a free abelian group.

The cost of splitting the exact sequence (5.13) as Γ-modules will ultimately be the condition $p \neq 2$ in the exact sequence (5.12) of the previous corollary, but it will not be the only time we will need this assumption (*cf.* proof of Theorem 5.10 below). The main step towards a Γ-splitting is the following lemma.

Lemma 5.4. *The exact sequence (5.13) splits as a sequence of* Γ*-modules if for all* $\chi \in \widehat{Q}_{sep}$, $\{\chi, \chi\} = 0$ *in* $K_2(F_{\text{sep}}(Q))$.

Proof. Although a direct computation shows that σ becomes Γ-equivariant if $\{f_i, f_i\} = 0$ for all $i = 1, \dots, n$, a basis-free splitting is given by the map

$$\tau : \Lambda^2(\widehat{Q}_{\text{sep}}) \longrightarrow \overline{A}^0(Q_{\text{sep}}, K_2)$$
$$\eta \wedge \chi \longmapsto \{\eta, \chi\},$$

which is well-defined by the hypothesis and Γ-equivariant by construction. □

Since localization is exact, all of the above applies *mutatis mutandis* if the exact sequence (5.13) if localized at some prime p. Since $\{\chi, \chi\} = \{-1, \chi\} \in K_2(F_{\text{sep}}(Q))$ (which shows how the elements $\{f_i, f_i\}$ appear in the splitting (5.13)) and since -1 has order 2 in the group $F_{\text{sep}}(Q)^\times$, the symbol $\{\chi, \chi\}$ has order 2 in $K_2(F_{\text{sep}}(Q))$. In particular, the condition of the previous lemma in the localized setting is guaranteed if we localize at a prime $p \neq 2$

because this kills the 2-torsion in each group. Moreover, localization, being exact, commutes with taking cohomology. Since Galois cohomology groups are torsion in degree ≥ 1 and the localization of a torsion abelian group at a prime p can be identified with its p-primary component, localizing the exact sequence (5.13) at a prime $p \neq 2, q$ allows us to rewrite the previous corollary as follows.

Theorem 5.5. *Let T/F be an algebraic torus and let $1 \longrightarrow T \xrightarrow{i} P \xrightarrow{j} Q \longrightarrow 1$ be a coflasque resolution of T. Then for every prime $p \neq 2, q$ there is an exact sequence*

$$\begin{aligned} 0 \longrightarrow \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) &\xrightarrow{r} H^2(F, (F_{\mathrm{sep}}^\times \otimes \widehat{Q}_{\mathrm{sep}})_{(p)} \oplus \Lambda^2(\widehat{Q}_{\mathrm{sep}})_{(p)}) \\ &\xrightarrow{a^*-\pi^*} H^2(F, (F_{\mathrm{sep}}^\times \otimes (\widehat{P}_{\mathrm{sep}} \oplus \widehat{Q}_{\mathrm{sep}}))_{(p)}) \oplus \Lambda^2(\widehat{P}_{\mathrm{sep}} \oplus \widehat{Q}_{\mathrm{sep}})_{(p)}). \end{aligned}$$

Our goal then is to calculate $\ker(a^* - \pi^*)$. The maps a^* and π^* are, in this context, induced by $\widehat{a}, \widehat{\pi} : \widehat{Q}_{\mathrm{sep}} \longrightarrow \widehat{P}_{\mathrm{sep}} \oplus \widehat{Q}_{\mathrm{sep}}$, which are given by $q \longmapsto (\widehat{j}(q), q)$ and $q \longmapsto (0, q)$, respectively (*cf.* Remark 4.1) with $\widehat{j} : \widehat{Q}_{\mathrm{sep}} \hookrightarrow \widehat{P}_{\mathrm{sep}}$ dual to $j : P \longrightarrow Q$ above. Because Galois cohomology is additive and the maps $\overline{A}^0(Q_{\mathrm{sep}}, K_2) \longrightarrow \overline{A}^0(P_{\mathrm{sep}} \times Q_{\mathrm{sep}}, K_2)$ induced by a and π respect the splittings above, we can consider the two components separately and write

$$\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong \ker(a^* - \pi^*) = \ker(\phi) \oplus \ker(\psi), \tag{5.14}$$

where

$$H^2(F, (F_{\mathrm{sep}}^\times \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \xrightarrow{\phi} H^2(F, (F_{\mathrm{sep}}^\times \otimes \widehat{P}_{\mathrm{sep}})_{(p)})$$

and

$$H^2(F, \Lambda^2(\widehat{Q}_{\mathrm{sep}})_{(p)}) \xrightarrow{\psi} H^2(F, \Lambda^2(\widehat{P}_{\mathrm{sep}})_{(p)}) \oplus H^2(F, (\widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}),$$

the latter after the standard formula

$$\begin{aligned} \Lambda^2(\widehat{P}_{\mathrm{sep}} \oplus \widehat{Q}_{\mathrm{sep}}) &\xrightarrow{\cong} \Lambda^2(\widehat{P}_{\mathrm{sep}}) \oplus \Lambda^2(\widehat{Q}_{\mathrm{sep}}) \oplus (\widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}}) \\ (p, q) \wedge (p', q') &\longmapsto (p \wedge p', q \wedge q', p \otimes q' - p' \otimes q). \end{aligned}$$

Note that we have dropped the components $H^2(F, (F_{\mathrm{sep}}^\times \otimes \widehat{Q}_{\mathrm{sep}})_{(p)})$ and $H^2(F, (\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})$ in the codomain of ϕ and ψ, respectively, that depend on $\widehat{Q}_{\mathrm{sep}}$ because the $\widehat{Q}_{\mathrm{sep}}$ components of the images of $\widehat{a}$ and $\widehat{\pi}$ are the same, hence when subtracting the maps a^* and π^* that they induce in cohomology we get 0 in those components (*cf.* the following Remark and Equation (4.3) in Remark 4.1 for the analogous situation in the degree 2 case).

We write $\psi = (\kappa, \varepsilon)$ for its components.

Remark 5.4. It is important to note that, in general, the map $a^* - \pi^*$ is *not* induced by the map $\widehat{a} - \widehat{\pi}$. Rather, we must determine a^* as induced by $\widehat{a}$ and π^* as induced by $\widehat{\pi}$ separately and *then* subtract. This is because $\Lambda^2(-)$ is not an additive functor as demonstrated by the Künneth formula above and ignoring this one would lose the second, nontrivial, component ε of the image of ψ since $\widehat{a} - \widehat{\pi}$ reduces to $\widehat{j} : \widehat{Q}_{\text{sep}} \longrightarrow \widehat{P}_{\text{sep}}$. Indeed, the previous comment shows that this doesn't matter for ϕ because $F^{\times}_{\text{sep}} \otimes -$ is an additive functor nor does it matter for the "additive part" of $\Lambda^2(-)$; this is also what happened in Remark 4.1. In particular, ϕ is induced by $\widehat{a} - \widehat{\pi} = \widehat{j}$ as is the first component κ of ψ. On the other hand, the second component ε of ψ is the map on cohomology that is induced by the map

$$\begin{aligned} \widehat{\varepsilon} : \Lambda^2(\widehat{Q}_{\text{sep}}) &\longrightarrow \widehat{P}_{\text{sep}} \otimes \widehat{Q}_{\text{sep}} \\ q \wedge q' &\longmapsto \widehat{j}(q) \otimes q' - \widehat{j}(q') \otimes q \end{aligned} \tag{5.15}$$

coming from $\widehat{a}$ alone, because the corresponding map induced by $\widehat{\pi}$ happens to be the zero map.

Determining $\ker(\phi)$ is entirely analogous to the computation in Remark 4.1 and we will see that while the degree 2 invariants with coefficients in $\mathbb{Q}_p/\mathbb{Z}_p(1), p \neq q$, are given exactly by the p-primary component of the first cohomology group of the character module $\widehat{T}_{\text{sep}}$, in degree 3 (for $p \neq 2, q$) we obtain the p-primary component of the first cohomology group of the *twist* $F^{\times}_{\text{sep}} \otimes \widehat{T}_{\text{sep}}$, and moreover, this doesn't tell the whole story this time. For the final result, see Theorem 5.12.

Theorem 5.6. *We have that* $\ker(\phi) \cong H^1(F, F^{\times}_{\text{sep}} \otimes \widehat{T}_{\text{sep}})\{p\} \cong H^1(F, T^0)\{p\}$ *in Equation (5.14).*

Proof. Although $F^{\times}_{\text{sep}}$ is not a flat $\mathbb{Z}$-module (if $(n, q) = 1$ then $F^{\times}_{\text{sep}}$ is non-uniquely n-divisible and the short exact sequence $1 \longrightarrow \mu_n \longrightarrow F^{\times}_{\text{sep}} \xrightarrow{n} F^{\times}_{\text{sep}} \longrightarrow 1$ fails to be exact after tensoring with $F_{\mathrm{sep}}^{\times}$), the map

$$F_{\mathrm{sep}}^{\times} \otimes \widehat{Q}_{\mathrm{sep}} \xrightarrow{\mathrm{Id} \otimes \widehat{j}} F_{\mathrm{sep}}^{\times} \otimes \widehat{P}_{\mathrm{sep}}$$

is nonetheless injective because character modules of tori are torsion-free. Since tensoring is always right-exact and localization is exact, the short exact sequence

$$0 \longrightarrow \widehat{Q}_{\mathrm{sep}} \xrightarrow{\widehat{j}} \widehat{P}_{\mathrm{sep}} \xrightarrow{\widehat{i}} \widehat{T}_{\mathrm{sep}} \longrightarrow 0 \tag{5.16}$$

induces a short exact sequence

$$0 \longrightarrow (F_{\mathrm{sep}}^{\times} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)} \longrightarrow (F_{\mathrm{sep}}^{\times} \otimes \widehat{P}_{\mathrm{sep}})_{(p)} \longrightarrow (F_{\mathrm{sep}}^{\times} \otimes \widehat{T}_{\mathrm{sep}})_{(p)} \longrightarrow 0$$

and taking cohomology gives the exact sequence

$$\begin{aligned} H^1(F, (F_{\mathrm{sep}}^{\times} \otimes \widehat{P}_{\mathrm{sep}})_{(p)}) &\longrightarrow H^1(F, (F_{\mathrm{sep}}^{\times} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \\ &\longrightarrow H^2(F, (F_{\mathrm{sep}}^{\times} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \xrightarrow{\phi} H^2(F, (F_{\mathrm{sep}}^{\times} \otimes \widehat{P}_{\mathrm{sep}})_{(p)}), \end{aligned}$$

where we obtain ϕ because it is precisely the map on cohomology induced by $\widehat{j}$ as pointed out in the previous Remark. The result follows after the first statement of the next lemma and the (dual of the) observation that $H^1(F, T) \cong H^1(F, F_{\mathrm{sep}}^{\times} \otimes \widehat{T}_{\mathrm{sep}}^0)$, which follows because $T(F_{\mathrm{sep}}) \cong \mathrm{Hom}_{\mathrm{gp}}(\widehat{T}_{\mathrm{sep}}, F_{\mathrm{sep}}^{\times}) \cong F_{\mathrm{sep}}^{\times} \otimes \widehat{T}_{\mathrm{sep}}^0$. □

Lemma 5.7. *The groups $H^1(F, (F_{\mathrm{sep}}^{\times} \otimes \widehat{P}_{\mathrm{sep}})_{(p)})$ and $H^1(F, (\widehat{Q}_{\mathrm{sep}} \otimes \widehat{P}_{\mathrm{sep}})_{(p)})$ are both trivial.*

Proof. If $\Gamma_0 \leq \Gamma$ is a closed subgroup and N is a Γ_0-module, we call

$$\mathrm{Ind}_{\Gamma}^{\Gamma_0}(N) := \{\alpha : \Gamma \longrightarrow N \mid \alpha \text{ is continuous and } \alpha(\gamma_0\gamma) = \gamma_0\alpha(\gamma)\}$$

the induced module. It is a Γ-module whose action is given by ${}^{\gamma}\alpha(\gamma') = \alpha(\gamma'\gamma)$. We are interested in this construction because of the Faddeev-Shapiro Lemma [Ser97, Ch. I, §2.5, Prop. 10], which says that for all $i \geq 0$, the natural map

$$\begin{aligned} \mathrm{Ind}_{\Gamma}^{\Gamma_0}(N) &\longrightarrow N \\ f &\longmapsto f(1) \end{aligned} \tag{5.17}$$

induces an isomorphism

$$H^i(\Gamma, \mathrm{Ind}_\Gamma^{\Gamma_0}(N)) \cong H^i(\Gamma_0, N).$$

Let M be a Γ-module (viewed as Γ_0-module by restriction). Then we have an isomorphism of Γ-modules

$$\begin{gathered} M \otimes \mathrm{Ind}_\Gamma^{\Gamma_0}(N) \xrightarrow{\cong} \mathrm{Ind}_\Gamma^{\Gamma_0}(M \otimes N), \\ m \otimes f \longmapsto f_m \end{gathered} \tag{5.18}$$

where $f_m : \Gamma \longrightarrow M \otimes N$ is given by $f_m(\gamma) = \gamma m \otimes f(\gamma)$.

Now, assume that $\Gamma_0 \leq \Gamma$ is an open subgroup and choose representatives $g_1 = 1, \ldots, g_n$ for the set of *right* cosets Γ/Γ_0, which we endow with the *left* action $\gamma \cdot (\Gamma_0 g) = \Gamma_0 g \gamma^{-1}$. Then we have an isomorphism

$$\begin{gathered} \mathbb{Z}[\Gamma/\Gamma_0] \xrightarrow{\cong} \mathrm{Ind}_\Gamma^{\Gamma_0}(\mathbb{Z}) \\ \sum_{i=1}^n n_i(\Gamma_0 g_i) \longmapsto \alpha, \end{gathered} \tag{5.19}$$

where $\mathbb{Z}$ always has trivial (left) action, and $\alpha(\gamma) = n_i$ if $\Gamma_0\gamma = \Gamma_0 g_i$. Then indeed, $\alpha(\gamma_0\gamma) = \alpha(\gamma)$ and one can check this is a Γ-equivariant homomorphism with inverse given by $\alpha \longmapsto \sum_{i=1}^n \alpha(g_i)(\Gamma_0 g_i)$.

Since $\widehat{P}_{\mathrm{sep}}$ is a permutation Γ-module, we have that $\widehat{P}_{\mathrm{sep}} \cong \bigoplus_i \mathbb{Z}[\Gamma/\Gamma_i]$ for some finite collection of open subgroups $\Gamma_i \leq \Gamma$ (Example 1.2(b)). By the isomorphisms (5.18) and (5.19),

$$M \otimes \mathbb{Z}[\Gamma/\Gamma_i] \cong \mathrm{Ind}_\Gamma^{\Gamma_i}(M),$$

so we see that

$$\begin{aligned} H^i(F, M \otimes \widehat{P}_{\mathrm{sep}}) &\cong \bigoplus_i H^i(F, M \otimes \mathbb{Z}[\Gamma/\Gamma_i]) \\ &\cong \bigoplus_i H^i(F, \mathrm{Ind}_\Gamma^{\Gamma_i}(M)) \\ &\cong \bigoplus_i H^i(\Gamma_i, M). \end{aligned}$$

For $i = 1$ and $M = F_{\mathrm{sep}}^{\times}$, the last group vanishes by Hilbert's Theorem 90. If $M = \widehat{Q}_{\mathrm{sep}}$, it vanishes by definition of coflasque modules. The lemma then follows because localization is an exact functor hence commutes with taking cohomology. ☐

The next lemma will be useful in calculating the second component of $\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ in (5.14), $\ker(\psi)$.

Lemma 5.8. *For*

$$H^2(F, \Lambda^2(\widehat{Q}_{\mathrm{sep}})_{(p)}) \xrightarrow{\psi=(\kappa,\varepsilon)} H^2(F, \Lambda^2(\widehat{P}_{\mathrm{sep}})_{(p)}) \oplus H^2(F, (\widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)})$$

in (5.14), we have that $\ker(\psi) = \ker(\varepsilon)$.

Proof. Since $\psi = (\kappa, \varepsilon)$, we have that $\ker(\psi) = \ker(\kappa) \cap \ker(\varepsilon)$. After Remark 5.4 (esp. (5.15)), κ and ε are the maps in cohomology induced by the maps in the diagram

$$\begin{tikzcd}
 & \Lambda^2(\widehat{P}_{\mathrm{sep}}) & & \widehat{j}(q) \wedge \widehat{j}(q') \\
\Lambda^2(\widehat{Q}_{\mathrm{sep}}) \arrow[ur, "\widehat{\kappa}"] \arrow[dr, "\widehat{\varepsilon}"'] & & q \wedge q' \arrow[ur, mapsto] \arrow[dr, mapsto] & \\
 & \widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} & & \widehat{j}(q) \otimes q' - \widehat{j}(q') \otimes q.
\end{tikzcd} \tag{5.20}$$

If we let

$$\begin{tikzcd}
\Lambda^2(\widehat{P}_{\mathrm{sep}}) \\
\widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} \arrow[u, "\widehat{\theta}"']
\end{tikzcd}$$

be defined by the natural factorization through $\widehat{P}_{\mathrm{sep}} \otimes \widehat{P}_{\mathrm{sep}}$, i.e.,

$$\widehat{\theta}(p \otimes q) = p \wedge \widehat{j}(q),$$

then we have that

$$\begin{aligned}
(\widehat{\theta} \circ \widehat{\varepsilon})(q \wedge q') &= \widehat{\theta}\,(\widehat{j}(q) \otimes q' - \widehat{j}(q') \otimes q) \\
&= \widehat{j}(q) \wedge \widehat{j}(q') - \widehat{j}(q') \wedge \widehat{j}(q) \\
&= 2(\widehat{j}(q) \wedge \widehat{j}(q')) \\
&= 2 \cdot \widehat{\kappa}(q \wedge q').
\end{aligned}$$

Localizing and passing to cohomology, the following diagram therefore commutes *up to multiplication by* 2

$$\begin{tikzcd}
 & H^2(F, (\Lambda^2(\widehat{P}_{\text{sep}}))_{(p)}) \\
H^2(F, (\Lambda^2(\widehat{Q}_{\text{sep}}))_{(p)}) \arrow[ur, "\kappa"] \arrow[dr, "\varepsilon"'] & \\
 & H^2(F, (\widehat{P}_{\text{sep}} \otimes \widehat{Q}_{\text{sep}})_{(p)}). \arrow[uu, "\theta"']
\end{tikzcd}$$

Since we localized at $p \neq 2$, $\ker(\varepsilon) \subseteq \ker(\kappa)$ and so $\ker(\psi) = \ker(\varepsilon)$. □

To finish the calculation of $\ker(\psi)$, we introduce the following construction.

Construction 5.9. Let $\mathfrak{g}$ be a profinite group and M a finitely generated torsion-free $\mathfrak{g}$-module, i.e., a $\mathfrak{g}$-lattice. For every open subgroup $\mathfrak{h} \leq \mathfrak{g}$ choose representatives $\gamma_1 = 1, \ldots, \gamma_n$ for the left cosets of $\mathfrak{h}$ in $\mathfrak{g}$. We have the *trace map* $\mathrm{Tr} : M^{\mathfrak{h}} \longrightarrow M^{\mathfrak{g}}$ given by $\mathrm{Tr}(m) = \sum_{i=1}^{n} \gamma_i m$. We write $\mathrm{Dec}(M)$ for the subgroup of *decomposable elements* in $S^2(M)^{\mathfrak{g}}$ generated by the elements $\mathrm{Tr}(m \cdot m')$ for every open subgroup $\mathfrak{h} \leq \mathfrak{g}$ and all $m, m' \in M^{\mathfrak{h}}$. In particular, we have that the symmetric square $(M^{\mathfrak{g}})^2 \subset \mathrm{Dec}(M)$ by choosing $\mathfrak{h} = \mathfrak{g}$. We often write simply $S^2(M)^{\mathfrak{g}}/\operatorname{Dec}$ for $S^2(M)^{\mathfrak{g}}/\operatorname{Dec}(M)$ unless the emphasis is necessary for clarity.

If N is another $\mathfrak{g}$-lattice, write $\mathrm{Dec}(M, N)$ for the subgroup of $(M \otimes N)^{\mathfrak{g}}$ generated by all the traces $\mathrm{Tr}(m \otimes n)$ for every open subgroup $\mathfrak{h} \leq \mathfrak{g}$ and all $m \in M^{\mathfrak{h}}, n \in N^{\mathfrak{h}}$. The natural isomorphism

$$S^2(M \oplus N) \cong S^2(M) \oplus S^2(N) \oplus (M \otimes N) \tag{5.21}$$

given by the correspondence $(m + n) \cdot (m' + n') \longleftrightarrow m \cdot m' + n \cdot n' + m \otimes n' + m' \otimes n$ implies the decomposition

$$\mathrm{Dec}(M \oplus N) \cong \mathrm{Dec}(M) \oplus \mathrm{Dec}(N) \oplus \mathrm{Dec}(M, N). \tag{5.22}$$

Theorem 5.10. *We have that* $\ker(\psi) \cong \big(S^2(\widehat{T}_{\text{sep}})^{\Gamma}/\operatorname{Dec}\big)\{p\}$ *in (5.14).*

Proof. After the previous lemma, it suffices to compute $\ker(\varepsilon)$. As with $\ker(\phi)$ in Theorem 5.6, we determine $\ker(\varepsilon)$ by exploiting an exact sequence of cohomology groups induced by a specific short exact sequence of Γ-modules. Since a Γ-lattice N is torsion-free, tensoring it with the short exact sequence of character modules (5.16) gives the exact sequence

$$0 \longrightarrow N \otimes \widehat{Q}_{\text{sep}} \longrightarrow N \otimes \widehat{P}_{\text{sep}} \longrightarrow N \otimes \widehat{T}_{\text{sep}} \longrightarrow 0. \tag{5.23}$$

Recall that if A is a commutative ring and M is an A-module then the standard exact sequence

$$0 \longrightarrow \Lambda^2(M) \longrightarrow M \otimes M \longrightarrow S^2(M) \longrightarrow 0, \tag{5.24}$$

where the injection is given by $m \wedge m' \longmapsto m \otimes m' - m' \otimes m$, splits if $2 \in A^\times$: on the right by $m \cdot m' \longmapsto \frac{1}{2}(m \otimes m' + m' \otimes m)$ or on the left by $m \otimes m' \longmapsto \frac{1}{2}(m \wedge m')$. Moreover, if M is a Γ-module then the splitting respects the action because Γ acts diagonally everywhere hence, in particular, the splitting holds with Γ invariants and in cohomology. Localizing at $p \neq 2$ and applying this to $M = \widehat{Q}_{\text{sep}}$ we see that the short exact sequence (5.23) with $N = \widehat{Q}_{\text{sep}}$ induces the exact sequence in cohomology

$$\begin{aligned} 0 \longrightarrow H^1(F, (\widehat{Q}_{\text{sep}} \otimes \widehat{T}_{\text{sep}})_{(p)}) \xrightarrow{s} H^2(F, (\Lambda^2(\widehat{Q}_{\text{sep}}))_{(p)}) \oplus H^2(F, (S^2(\widehat{Q}_{\text{sep}}))_{(p)}) \\ \xrightarrow{u} H^2(F, (\widehat{Q}_{\text{sep}} \otimes \widehat{P}_{\text{sep}})_{(p)}) \longrightarrow 0 \end{aligned} \tag{5.25}$$

where $H^1(F, (\widehat{Q}_{\text{sep}} \otimes \widehat{P}_{\text{sep}})_{(p)}) = 0$ on the left because of the second statement in Lemma 5.7. If we let

$$\lambda : H^2(F, (\Lambda^2(\widehat{Q}_{\text{sep}}))_{(p)}) \longrightarrow H^2(F, (\widehat{Q}_{\text{sep}} \otimes \widehat{P}_{\text{sep}})_{(p)})$$

be the restriction of u to the factor $H^2(F, (\Lambda^2(\widehat{Q}_{\text{sep}}))_{(p)})$ then λ is induced by the composition

$$\begin{aligned} \Lambda^2(\widehat{Q}_{\text{sep}}) &\longrightarrow \widehat{Q}_{\text{sep}} \otimes \widehat{Q}_{\text{sep}} \longrightarrow \widehat{Q}_{\text{sep}} \otimes \widehat{P}_{\text{sep}} \\ q \wedge q' &\longmapsto q \otimes q' - q' \otimes q \longmapsto q \otimes \widehat{j}(q') - q' \otimes \widehat{j}(q). \end{aligned}$$

If we let $\widehat{\sigma}$ be the "flip" isomorphism of a tensor product and σ the isomorphism it induces on cohomology, then by diagram (5.20) (or (5.15)) we see that $\sigma \circ \lambda = -\varepsilon$ hence $\ker(\lambda) = \ker(\varepsilon)$.

Moreover, exactness of (5.25) implies that

$$\begin{aligned} \ker(\lambda) &\cong s^{-1}\big[H^2(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})\big] \\ &\cong s\big[H^1(F,(\widehat{Q}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}})_{(p)})\big]\cap H^2(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}) \end{aligned} \tag{5.26}$$

To determine this intersection we combine the various short exact sequences (5.23) for $N = \widehat{Q}_{\mathrm{sep}}, \widehat{T}_{\mathrm{sep}}$, and $\widehat{P}_{\mathrm{sep}}$, obtaining the diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & \widehat{Q}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{Q}_{\mathrm{sep}}\otimes\widehat{P}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{Q}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{P}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{P}_{\mathrm{sep}}\otimes\widehat{P}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{P}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{T}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{T}_{\mathrm{sep}}\otimes\widehat{P}_{\mathrm{sep}} \arrow[r] \arrow[d] & \widehat{T}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows and columns. Combining the induced cohomology exact sequences we have

$$\begin{tikzcd}
H^1(F,(\widehat{P}_{\mathrm{sep}}\otimes\widehat{P}_{\mathrm{sep}})_{(p)}) \arrow[r] & H^1(F,(\widehat{P}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "k"] & H^2(F,(\widehat{P}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}})_{(p)}) \\
H^1(F,(\widehat{Q}_{\mathrm{sep}}\otimes\widehat{P}_{\mathrm{sep}})_{(p)}) \arrow[r] & H^1(F,(\widehat{Q}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "s"] \arrow[u, "h"] & H^2(F,(\widehat{Q}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[u, "t"'] \\
 & H^0(F,(\widehat{T}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "l"'] \arrow[u, "g"] \arrow[ur, phantom, "-1"] & H^1(F,(\widehat{T}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[u, "r"'] \\
 & H^0(F,(\widehat{P}_{\mathrm{sep}}\otimes\widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r] \arrow[u, "f"] & H^1(F,(\widehat{P}_{\mathrm{sep}}\otimes\widehat{Q}_{\mathrm{sep}})_{(p)}), \arrow[u]
\end{tikzcd}$$

where the marked square anti-commutes by [CE99, §IV.2, Prop. 2.1]. After Lemma 5.7 and the decompositions of cohomology groups induced by the splitting of (5.24), this becomes

$$\begin{tikzcd}
0 \arrow[r] & H^1(F,(\widehat{P}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "k"] & H^2(F,(\widehat{P}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \\
0 \arrow[r] & H^1(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[u, "h"] \arrow[r, "s"] & H^2(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}) \oplus H^2(F,(S^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}) \arrow[u, "t"'] \\
 & (\Lambda^2(\widehat{T}_{\mathrm{sep}}))^{\Gamma}_{(p)} \oplus (S^2(\widehat{T}_{\mathrm{sep}}))^{\Gamma}_{(p)} \arrow[u, "g"] \arrow[r, "l"'] \arrow[ur, phantom, "-1"] & H^1(F,(\widehat{T}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[u, "r"'] \\
 & (\widehat{P}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \arrow[u, "f"] \arrow[r] & 0. \arrow[u]
\end{tikzcd} \tag{5.27}$$

By construction of the splitting (5.24), applying σ to the decomposition

$$H^i(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \cong H^i(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}) \oplus H^i(F,(S^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}) \tag{5.28}$$

fixes the symmetric part and multiplies the exterior part by -1, hence $-\sigma$ does the opposite. Moreover, because the flip isomorphism commutes with the maps in the short exact sequences (5.24) with $N = \widehat{Q}_{\mathrm{sep}}$ and $\widehat{T}_{\mathrm{sep}}$, i.e., we have the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} \arrow[r] \arrow[d, "\widehat{\sigma}"] & \widehat{Q}_{\mathrm{sep}} \otimes \widehat{P}_{\mathrm{sep}} \arrow[r] \arrow[d, "\widehat{\sigma}"] & \widehat{Q}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} \arrow[r] \arrow[d, "\widehat{\sigma}"] & 0 \\
0 \arrow[r] & \widehat{T}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}} \arrow[r] & \widehat{T}_{\mathrm{sep}} \otimes \widehat{P}_{\mathrm{sep}} \arrow[r] & \widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} \arrow[r] & 0,
\end{tikzcd}$$

we have that σ commutes with the connecting homomorphisms in cohomology, i.e., we have commutative diagrams

$$\begin{tikzcd}
H^1(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "s"] \arrow[d, "\sigma"'] & H^2(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[d, "\sigma"] \\
H^1(F,(\widehat{T}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[r, "r"'] & H^2(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}),
\end{tikzcd} \tag{5.29}$$

and

$$\begin{tikzcd}
H^0(F,(\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "l"] \arrow[d, "\sigma"'] & H^1(F,(\widehat{T}_{\mathrm{sep}} \otimes \widehat{Q}_{\mathrm{sep}})_{(p)}) \arrow[d, "\sigma"] \\
H^0(F,(\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \arrow[r, "g"'] & H^1(F,(\widehat{Q}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}).
\end{tikzcd} \tag{5.30}$$

Now choose

$$\alpha \in s^{-1}\big[H^2(F, (\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})\big] \cong \ker(\lambda).$$

Then the fact that $-\sigma$ fixes $s(\alpha)$ and (5.29) imply that

$$s(\alpha) = -\sigma(s(a)) = -r(\sigma(\alpha)),$$

so that $s(\alpha) \in \mathrm{Im}(r)$ and exactness of the second column in (5.27) implies $t(s(\alpha)) = 0$. Commutativity of the top square in (5.27) thus implies that $k(h(\alpha)) = 0$ hence $h(\alpha) = 0$ because k is injective, and therefore $\alpha \in \mathrm{Im}(g)$ by exactness of the first column. In particular, we have shown that

$$s^{-1}\big[H^2(F, (\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})\big] \subseteq \mathrm{Im}(g).$$

Claim. We have that

$$\begin{aligned} s\big[g\big(\Lambda^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}\big)\big] &\subseteq H^2(F, (S^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}), \\ s\big[g\big(S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}\big)\big] &\subseteq H^2(F, (\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}). \end{aligned}$$

This follows by the combining the two previous diagrams (5.29) and (5.30) with the anti-commutative square in (5.27). Let

$$\zeta \in \Lambda^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}.$$

Then $\sigma(\zeta) = -\zeta$ and thus

$$\sigma(s(g(\zeta))) \overset{(5.29)}{=} r(\sigma(g(\zeta))) \overset{(5.30)}{=} r(l(\sigma(\zeta))) \overset{(5.27)}{=} -s(g(\sigma(\zeta))) = s(g(\zeta)).$$

Since $s(g(\zeta))$ is fixed by σ, we must have that

$$s(g(\zeta)) \in H^2(F, (S^2(\widehat{Q}_{\mathrm{sep}}))_{(p)}).$$

If, on the other hand, we start with $\zeta \in S^2(\widehat{T})^{\Gamma}_{(p)}$ then we have $\sigma(\zeta) = \zeta$ and in the previous sequence of equalities, the last one gives $-s(g(\zeta))$ on the right so in this case σ maps $s(g(\zeta))$ to its negative and therefore $s(g(\zeta)) \in H^2(F, (\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})$, proving the claim.

Before the claim, we saw that

$$s^{-1}\big[H^2(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})\big] \subseteq \mathrm{Im}(g),$$

so write $\alpha = g(\gamma,\delta)$ for α on the left hand side with $\gamma \in \Lambda^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}$ and $\delta \in S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}$. If we write

$$s(\alpha) = s(g(\gamma,0)) + s(g(0,\delta))$$

then since $s(\alpha) = -\sigma(s(\alpha))$ by assumption, the claim implies that

$$\begin{aligned} s(g(\gamma,0)) + s(g(0,\delta)) &= -\sigma(s(g(\gamma,0))) - \sigma(s(g(0,\delta))) \\ &= -s(g(\gamma,0)) + s(g(0,\delta)) \end{aligned}$$

which means that $2 \cdot s(g(\gamma,0)) = 0$ and therefore $s(g(\gamma,0)) = 0$ because we have localized at $p \neq 2$. Since s is an embedding, $g(\gamma,0) = 0$ and thus $\alpha = g(0,\delta)$. This proves

$$s^{-1}\big[H^2(F,(\Lambda^2(\widehat{Q}_{\mathrm{sep}}))_{(p)})\big] \subseteq \mathrm{Im}\left[g|_{(S^2(\widehat{T}_{\mathrm{sep}}))^{\Gamma}_{(p)}}\right],$$

and since the other inclusion $\supseteq$ is just the second statement of the claim, we have equality. Since the group on the left is isomorphic to $\ker(\lambda)$ (*cf.* (5.26)), we have reduced to determining the image of the restriction of g to $(S^2(\widehat{T}_{\mathrm{sep}}))^{\Gamma}_{(p)}$. By exactness of the first column in (5.27), we have that

$$\mathrm{Im}\left[g|_{(S^2(\widehat{T}_{\mathrm{sep}}))^{\Gamma}_{(p)}}\right] \cong S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \big/ \big[\,\mathrm{Im}(f) \cap S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}\big].$$

The result then follows after the following lemma. ☐

Lemma 5.11. *We have that*

$$S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \big/ \big[\,\mathrm{Im}(f) \cap S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}\big] \cong \big(S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\mathrm{Dec}\big)\{p\}.$$

Proof. We begin by computing $\mathrm{Im}(f)$. We have that f is the map on Γ-invariants induced by

$$\begin{aligned} \widehat{P}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} &\longrightarrow \widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}} \\ p \otimes t &\longmapsto \widehat{i}(p) \otimes t, \end{aligned}$$

where $\widehat{i}$ is the map on character modules dual to $i : T \longrightarrow P$ from the coflasque resolution in Theorem 5.5. But, by Lemma 1.1, we know that if $i' : \widehat{P'} \longrightarrow \widehat{T}_{\mathrm{sep}}$ is any map from another permutation module $\widehat{P'}$ then we have a factorization

$$\begin{tikzcd}
0 \arrow[r] & \widehat{Q}_{\mathrm{sep}} \arrow[r, "\widehat{j}"] & \widehat{P}_{\mathrm{sep}} \arrow[r, "\widehat{i}"] & \widehat{T}_{\mathrm{sep}} \arrow[r] & 0, \\
 & & & \widehat{P'} \arrow[u, "i'"'] \arrow[ul, dashed] &
\end{tikzcd}$$

hence

$$\mathrm{Im}(\,\widehat{i}\,) = \sum \mathrm{Im}(i'),$$

the sum over all maps i' from some permutation module into $\widehat{T}_{\mathrm{sep}}$. By Example 1.2(b), an arbitrary permutation module is of the form $\widehat{P'} \cong \bigoplus_i \mathbb{Z}[\Gamma/\Gamma_i]$ for some open subgroups $\Gamma_i \leq \Gamma$, hence $\mathrm{Im}(f)$ is generated by the images of all maps

$$(\mathbb{Z}[\Gamma/\Gamma_0] \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \xrightarrow{f_0 = (i_0 \otimes \mathrm{Id})_{(p)}} (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}$$

induced by all Γ-equivariant maps

$$\mathbb{Z}[\Gamma/\Gamma_0] \xrightarrow{i_0} \widehat{T}_{\mathrm{sep}}$$

for every open subgroup $\Gamma_0 \leq \Gamma$. The following claims will allow us to describe $\mathrm{Im}(f)$ in terms of the group $\mathrm{Dec}(\widehat{T}_{\mathrm{sep}})$ defined in Construction 5.9 above.

Claim. $\mathrm{Hom}_{\Gamma}(\mathbb{Z}[\Gamma/\Gamma_0], \widehat{T}_{\mathrm{sep}}) \cong (\widehat{T}_{\mathrm{sep}})^{\Gamma_0} \cong (\mathbb{Z}[\Gamma/\Gamma_0] \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}$.

As in Lemma 5.7, we view Γ/Γ_0 as *right* cosets with *left* action given by $\gamma \cdot (\Gamma_0 g) = \Gamma_0 g \gamma^{-1}$ and fix representatives $g_1 = 1, \ldots, g_n$. Because Γ acts transitively on the cosets Γ/Γ_0, we see that every Γ-equivariant homomorphism

$$\mathbb{Z}[\Gamma/\Gamma_0] \xrightarrow{i_0} \widehat{T}_{\mathrm{sep}}$$

is determined by the image of the identity coset Γ_0. Moreover, if $\Gamma_0 \longmapsto t$ then Γ-equivariance implies that $\gamma_0 \cdot t = t$ for all $\gamma_0 \in \Gamma_0$ so $t \in (\widehat{T}_{\mathrm{sep}})^{\Gamma_0}$. For the second isomorphism, we need only appeal to the proof of Lemma 5.7, which shows that

$$\mathbb{Z}[\Gamma/\Gamma_0] \otimes \widehat{T}_{\mathrm{sep}} \cong \mathrm{Ind}_{\Gamma}^{\Gamma_0}(\mathbb{Z}) \otimes \widehat{T}_{\mathrm{sep}} \cong \mathrm{Ind}_{\Gamma}^{\Gamma_0}(\widehat{T}_{\mathrm{sep}})$$

hence

$$\begin{aligned}(\mathbb{Z}[\Gamma/\Gamma_0]\otimes \widehat{T}_{\text{sep}})^{\Gamma} &= H^0(F,\mathbb{Z}[\Gamma/\Gamma_0]\otimes \widehat{T}_{\text{sep}})\\ &\cong H^0(F,\operatorname{Ind}_{\Gamma}^{\Gamma_0}(\widehat{T}_{\text{sep}}))\\ &\cong H^0(\Gamma_0,\widehat{T}_{\text{sep}})\\ &= (\widehat{T}_{\text{sep}})^{\Gamma_0}.\end{aligned}$$

Next we determine $\operatorname{Im}(f_0)$ for a fixed $i_0 \in \operatorname{Hom}_{\Gamma}(\mathbb{Z}[\Gamma/\Gamma_0],\widehat{T}_{\text{sep}})$.

Claim. We have that $\operatorname{Im}(f_0) = \big\{\sum_{k=1}^{n} g_k^{-1}x\otimes g_k^{-1}y \mid y \in (\widehat{T}_{\text{sep}})^{\Gamma_0}\big\}_{(p)}$, where $x = i_0(\Gamma_0) \in (\widehat{T}_{\text{sep}})^{\Gamma_0}$.

By definition, if

$$z = \sum_{l=1}^{m}\left(\sum_{k=1}^{n}\Big[n_{lk}(\Gamma_0 g_k)\Big]\otimes t_l\right) \in (\mathbb{Z}[\Gamma/\Gamma_0]\otimes \widehat{T}_{\text{sep}})^{\Gamma}$$

then

$$\begin{aligned}(i_0\otimes \operatorname{Id})(z) &= \sum_{l=1}^{m}\left(i_0\Big(\sum_{k=1}^{n} n_{lk}(\Gamma_0 g_k)\Big)\otimes t_l\right)\\ &= \sum_{l=1}^{m}\sum_{k=1}^{n} g_k^{-1}x\otimes n_{lk}t_l.\end{aligned} \tag{5.31}$$

On the other hand, we can trace the element z through the second isomorphism of the previous claim (given explicitly in Lemma 5.7):

$$\begin{tikzcd}
(\mathbb{Z}[\Gamma/\Gamma_0]\otimes \widehat{T}_{\text{sep}})^{\Gamma} \arrow[r, phantom, "\cong"] & (\operatorname{Ind}_{\Gamma}^{\Gamma_0}(\mathbb{Z})\otimes \widehat{T}_{\text{sep}})^{\Gamma} \arrow[r, phantom, "\cong"] & (\operatorname{Ind}_{\Gamma}^{\Gamma_0}(\widehat{T}_{\text{sep}}))^{\Gamma} \arrow[r, phantom, "\cong"] & (\widehat{T}_{\text{sep}})^{\Gamma_0} \\
\sum_{l=1}^{m}\left(\sum_{k=1}^{n}\Big[n_{lk}(\Gamma_0 g_k)\Big]\otimes t_l\right) \arrow[r, leftrightarrow] & \sum_{l=1}^{m}\alpha_l\otimes t_l \arrow[r, leftrightarrow] & \sum_{l=1}^{m}\beta_l \arrow[r, leftrightarrow] & \sum_{l=1}^{m}\beta_l(1),
\end{tikzcd}$$

where $\alpha_l : \Gamma \longrightarrow \mathbb{Z}$ is given by $\alpha_l(\gamma) = n_{lk}$ if $\Gamma_0\gamma = \Gamma_0 g_k$ and $\beta_l : \Gamma \longrightarrow \widehat{T}_{\text{sep}}$ is given by $\beta_l(\gamma) = \alpha_l(\gamma)\gamma t_l$, hence $\beta_l(1) = n_{l1}t_l$ because $g_1 = 1$. The last map is the (degree 0) Faddeev-Shapiro isomorphism described by (5.17) in the proof of Lemma 5.7. Although we have that $\sum_{l=1}^{m}\beta_l(1) = \sum_{l=1}^{m} n_{l1}t_l$, the sum of the β_l is a Γ-invariant element of the induced module, hence by definition of the action (*cf.* the proof of Lemma 5.7), we have that

$$\sum_{l=1}^{m}\beta_l(\gamma) = \sum_{l=1}^{m}\beta_l(1)$$

for all $\gamma \in \Gamma$. In particular, we can choose $\gamma = g_k$ which gives

$$\sum_{l=1}^{m} n_{lk} g_k t_l = \sum_{l=1}^{m} \beta_l(1), \quad 1 \leq k \leq n.$$

Then,

$$\begin{aligned}\sum_{k=1}^{n} \left[g_k^{-1} x \otimes g_k^{-1} \Big(\sum_{l=1}^{m} \beta_l(1) \Big) \right] &= \sum_{k=1}^{n} \left[g_k^{-1} x \otimes g_k^{-1} \Big(\sum_{l=1}^{m} n_{lk} g_k t_l \Big) \right] \\ &= \sum_{l=1}^{m} \sum_{k=1}^{n} g_k^{-1} x \otimes n_{lk} t_l \\ &= (i_0 \otimes \mathrm{Id})(z),\end{aligned}$$

where the last equality follows from (5.31). Because the $z \in (\mathbb{Z}[\Gamma/\Gamma_0] \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}$ correspond isomorphically to the $y = \sum_{l=1}^{m} \beta_l(1) \in (\widehat{T}_{\mathrm{sep}})^{\Gamma_0}$, the claim follows since $(i_0 \otimes \mathrm{Id})_{(p)} = f_0$ and localization commutes with taking the image.

$\mathrm{Im}(f)$ is generated by the images of the f_0 for every $i_0 \in \mathrm{Hom}_{\Gamma}(\mathbb{Z}[\Gamma/\Gamma_0], \widehat{T}_{\mathrm{sep}})$ and all open subgroups $\Gamma_0 \leq \Gamma$, and we saw that the i_0 correspond isomorphically to the characters $x = i_0(\Gamma_0) \in (\widehat{T}_{\mathrm{sep}})^{\Gamma_0}$. Moreover, since the $g_1, \ldots, g_n$ are representatives for the *right* cosets of a subgroup $\Gamma_0 \leq \Gamma$, their inverses are representatives of the *left* cosets. Therefore, writing $\gamma_k = g_k^{-1}$ for these representatives and varying the x and the Γ_0, we obtain

$$\mathrm{Im}(f) = \left\langle \sum_{k=1}^{n} \gamma_k x \otimes \gamma_k y \;\middle|\; x, y \in (\widehat{T}_{\mathrm{sep}})^{\Gamma_0}, \Gamma_0 \leq \Gamma \text{ open} \right\rangle_{(p)}.$$

By definition, the intersection $\mathrm{Im}(f) \cap S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}$ is just the image of the group $\mathrm{Im}(f)$ under the natural projection

$$(\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \longrightarrow S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)}$$

since this map is the inverse to the split embedding

$$\begin{aligned} S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} &\hookrightarrow (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \\ \textstyle\sum_{i=1}^{l} \frac{x_i \cdot z_i}{n_i} &\longmapsto \textstyle\sum_{i=1}^{l} \frac{x_i \otimes z_i}{2n_i} + \sum_{i=1}^{l} \frac{z_i \otimes x_i}{2n_i}. \end{aligned}$$

Hence $\mathrm{Im}(f) \cap S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}$ is precisely $\mathrm{Dec}(\widehat{T}_{\mathrm{sep}})_{(p)}$ and therefore,

$$S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} / \big[\mathrm{Im}(f) \cap S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} \big] = S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}_{(p)} / \mathrm{Dec}_{(p)}.$$

Since localization commutes with taking quotients and this is a direct summand of the p-primary torsion group $\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$, the result follows. □

Remark 5.5. As in Construction 5.9 above, let $\mathfrak{h} \leq \mathfrak{g}$ be an open subgroup of a profinite group with left coset representatives $\gamma_1, \ldots, \gamma_n$ and M a $\mathfrak{g}$-lattice. In the upcoming paper [BM12], the authors extend the trace map $\mathrm{Tr} : M^{\mathfrak{h}} \longrightarrow M^{\mathfrak{g}}$ to the so-called *quadratic trace map* $\mathrm{Qtr} : M^{\mathfrak{h}} \longrightarrow S^2(M)^{\mathfrak{g}}$ given by $\mathrm{Qtr}(m) = \sum_{i<j} \gamma_i m \cdot \gamma_j m$ and instead define the decomposable elements as the subgroup generated by the square $(M^{\mathfrak{g}})^2$ and the elements $\mathrm{Qtr}(m)$ for every open subgroup $\mathfrak{h} \leq \mathfrak{g}$ and all $m \in M^{\mathfrak{h}}$. Note that the symmetric square is included separately since $\mathrm{Qtr}(m) = 0$ for all $m \in M^{\mathfrak{g}}$ by definition. Denoting this alternate subgroup of decomposable elements $\widetilde{\mathrm{Dec}}(M)$, the isomorphism (5.21) identifies

$$\mathrm{Qtr}(m+n) = \mathrm{Qtr}(m) + \mathrm{Qtr}(n) + [\mathrm{Tr}(m) \otimes \mathrm{Tr}(n) - \mathrm{Tr}(m \otimes n)]$$

and so implies a decomposition analogous to (5.22):

$$\widetilde{\mathrm{Dec}}(M \oplus N) \cong \widetilde{\mathrm{Dec}}(M) \oplus \widetilde{\mathrm{Dec}}(N) \oplus \mathrm{Dec}(M, N),$$

with the same $\mathrm{Dec}(M, N)$ we defined above. But, for $x \in M^{\mathfrak{h}}$, the formula

$$\begin{aligned}
2 \cdot \sum_{i<j} \gamma_i x \cdot \gamma_j x &= \sum_{i<j} \gamma_i x \cdot \gamma_j x + \sum_{j<i} \gamma_i x \cdot \gamma_j x \\
&= \sum_{i \neq j} \gamma_i x \cdot \gamma_j x \\
&= \left(\sum_{i=1}^{n} \gamma_i x \right) \cdot \left(\sum_{j=1}^{n} \gamma_j x \right) - \sum_{i=1}^{n} \gamma_i x \cdot \gamma_i x \\
&= \mathrm{Tr}(x) \cdot \mathrm{Tr}(x) - \mathrm{Tr}(x \cdot x)
\end{aligned}$$

shows that $2 \cdot \widetilde{\mathrm{Dec}}(M) \subseteq \mathrm{Dec}(M)$ since $\mathrm{Tr}(x) \in M^{\mathfrak{g}}$. Incidentally, this implies that $\mathrm{Tr}(x \cdot x) \in \widetilde{\mathrm{Dec}}(M)$ since both constructions of the decomposable elements contain $(M^{\mathfrak{g}})^2$. Similarly, if

$x, y \in M^{\mathfrak{h}}$ and $z = x + y$, this same formula shows

$$\begin{aligned} 2 \cdot \sum_{i<j} \gamma_i z \cdot \gamma_j z &= \operatorname{Tr}(z) \cdot \operatorname{Tr}(z) - \operatorname{Tr}(z \cdot z) \\ &= \operatorname{Tr}(z) \cdot \operatorname{Tr}(z) - \sum_{i=1}^{n} \gamma_i (x \cdot x + y \cdot y + 2x \cdot y) \\ &= \operatorname{Tr}(z) \cdot \operatorname{Tr}(z) - \operatorname{Tr}(x \cdot x) - \operatorname{Tr}(y \cdot y) - 2 \operatorname{Tr}(x \cdot y), \end{aligned}$$

which implies that $2 \cdot \operatorname{Dec}(M) \subseteq \widetilde{\operatorname{Dec}}(M)$. Therefore, localizing at a prime $p \neq 2$, we cannot tell the difference: $\operatorname{Dec}(M)_{(p)} = \widetilde{\operatorname{Dec}}(M)_{(p)}$ in $S^2(M)^{\mathfrak{g}}_{(p)}$.

In some sense, our decomposable elements $\operatorname{Dec}(M)$ are naive because they arise from the localized approach we used to compute $\operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for $p \neq 2, q$ (2 being the problematic prime as far as the decomposable elements are concerned since the characteristic restriction was imposed by our reliance on Rost's Chow groups with coefficients). But, this is precisely why we use them: they are the natural construction for our approach. This point of view is further supported by the fact that the decomposable elements $\widetilde{\operatorname{Dec}}(M)$ help capture the description of degree 3 invariants for all primes p, as demonstrated in the upcoming paper [BM12]. See the discussion after the next theorem for the relation between our work and [BM12].

Combining formula (5.14) with Theorems 5.6 and 5.10, we obtain the main result of this work.

Theorem 5.12. *Let T/F be an algebraic torus. Then for every prime $p \neq 2, q$ there is an isomorphism*

$$\operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong H^1(F, T^0)\{p\} \oplus \big(S^2(\widehat{T}_{\mathrm{sep}})^{\Gamma}/\operatorname{Dec}\big)\{p\}.$$

Remark 5.6. As in §3, when combined with Propositions 1.14 and 1.16, the previous theorem gives a description of the groups $\overline{H}^3_{\mathrm{nr}}(F(T), \mathbb{Q}_p/\mathbb{Z}_p(2))$ and $\operatorname{Inv}^3_{\mathrm{nr}}(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ for an arbitrary torus T/F and $p \neq 2, q$. By Proposition 1.14 we have that

$$\overline{H}^3_{\mathrm{nr}}(F(T), \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong H^1(F, S^0)\{p\} \oplus \big(S^2(\widehat{S}_{\mathrm{sep}})^{\Gamma}/\operatorname{Dec}\big)\{p\},$$

where $[\widehat{S}_{\text{sep}}] = p(\widehat{T}_{\text{sep}})$, which answers the motivating question in [Col95, p. 39] in the case $i = 3$ and n prime to $2, q$ (*cf.* the Introduction). On the other hand, Proposition 1.16 implies that

$$\operatorname{Inv}^3_{\text{nr}}(T, \mathbb{Q}_p/\mathbb{Z}_p(1)) \cong H^1(F, N^0)\{p\} \oplus \big(S^2(\widehat{N}_{\text{sep}})^{\Gamma}/\text{Dec}\big)\{p\},$$

where $\widehat{N}_{\text{sep}}$ is a flasque module that fits into an exact sequence

$$0 \longrightarrow \widehat{P}_{\text{sep}} \longrightarrow \widehat{N}_{\text{sep}} \longrightarrow \widehat{T}_{\text{sep}} \longrightarrow 0$$

with $\widehat{P}_{\text{sep}}$ a permutation module.

Unfortunately, the numerous technical calculations we used to derive the formula in the previous theorem hide the precise description of the isomorphism. Furthermore, our reliance on Rost's theory of cycle modules is ultimately responsible for the restriction $p \neq 2, q$ because of the necessity of splitting the exact sequence (5.13) (as Γ-modules) and the fact that the residue homomorphisms used to define the partially unramified groups $A^0(X, H^i[C])$ (and the unramified cohomology groups $H^i_{\text{nr}}(F, C)$) are not defined on the characteristic component (*cf.* Remarks 1.3(a) and 2.1(c)).

Defining the characteristic component $\mathbb{Q}_q/\mathbb{Z}_q(2)$ following [Kah96] and replacing Rost's cycle modules with other constructions, the upcoming paper [BM12] (developed from the ideas in this work) extends the formula of the previous theorem to the case $p = q$. Although the isomorphism does not hold in the case $p = 2$, one in fact obtains an exact sequence containing $\operatorname{Inv}^3(T, \mathbb{Q}/\mathbb{Z}(2) \oplus \mathbb{Q}_q/\mathbb{Z}_q(2))$ (recall we always write $\mathbb{Q}/\mathbb{Z}(i)$ for the characterstic-free module) which gives the above result on the p-primary components when $p \neq 2$. Using a more general definition of unramified cohomology that includes a characteristic component, the paper then extends the computation of degree 3 unramified cohomology of the function field of a torus in the previous remark to the case $p = q$, with the case $p = 2$ described via the aforementioned exact sequence for invariants of an auxiliary torus, namely the Picard torus (*cf.* Proposition 1.14).

Morevoer, the isomorphism of the previous theorem is constructed explicitly in [BM12] using advanced techniques in motivic cohomology and the results can be used to give a related description in our simplified setting of Galois cohomology as long as one stays away from the characteristic component, which suffices for our purposes. Although we do not verify this here to avoid a long digression into topics outside the scope of this work, the construction is worth describing insofar as it reveals an important difference between the degree 2 and degree 3 cases.

Following [Tat76], for $p \neq 2, q$, let $\mathbb{Z}_p = \varprojlim \mathbb{Z}/p^n\mathbb{Z}$ be the p-adic integers. Define the Γ-module

$$\mathbb{Z}_p(1) = \varprojlim \ \mu_{p^n},$$

where the inverse limit is taken over all non-negative integers n. Letting $\mathbb{Z}_p(0) = \mathbb{Z}_p$, define inductively

$$\mathbb{Z}_p(m+1) = \mathbb{Z}_p(m) \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1), \quad m \geq 0.$$

For any abelian group M which is both a $\mathbb{Z}_p$ and a Γ-module, set

$$M(m) = M \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(m),$$

so that in particular

$$\mathbb{Q}_p/\mathbb{Z}_p(1) = \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(1) = \varinjlim \ \mu_{p^n} = \bigcup_n \mu_{p^n} \subset F_{\mathrm{sep}}^{\times}, \tag{5.32}$$

hence $\mathbb{Q}/\mathbb{Z}(i)$ and $\mathbb{Q}_p/\mathbb{Z}_p(i)$ in this notation agree with the notation we have been using throughout. Moreover, we have the pairings

$$M(i) \times \mathbb{Z}_p(j) \longrightarrow M(i+j) \tag{5.33}$$

induced by the natural pairings $\mathbb{Z}_p(i) \times \mathbb{Z}_p(j) \longrightarrow \mathbb{Z}_p(i+j)$ given by the tensor product, since $\mathbb{Z}_p(i) = (\mathbb{Z}_p(1))^{\otimes i}$.

For each integer $n \geq 0$ and every field extension K/F we have the induced map $\Gamma_K \longrightarrow \Gamma$ and thus the following commutative diagram of Γ_K-modules

$$\begin{tikzcd}
0 \arrow[r] & \mu_{p^n} \arrow[r] & K_{\mathrm{sep}}^{\times} \arrow[r] & K_{\mathrm{sep}}^{\times} \arrow[r] & 0 \\
0 \arrow[r] & \mu_{p^{n+1}} \arrow[r] \arrow[u, "p"] & K_{\mathrm{sep}}^{\times} \arrow[r] \arrow[u, "p"] & K_{\mathrm{sep}}^{\times} \arrow[r] \arrow[u, "\mathrm{Id}"] & 0.
\end{tikzcd}$$

Passing to the inverse limit over n and tensoring (over $\mathbb{Z}$) with the torsion-free abelian group (and induced Γ_K-module) $\widehat{T}^0_{\text{sep}}$, we obtain the short exact sequence

$$0 \longrightarrow \mathbb{Z}_p(1) \otimes \widehat{T}^0_{\text{sep}} \longrightarrow \varprojlim K^\times_{\text{sep}} \otimes \widehat{T}^0_{\text{sep}} \longrightarrow K^\times_{\text{sep}} \otimes \widehat{T}^0_{\text{sep}} \longrightarrow 0.$$

Therefore, we have the connecting map in cohomology

$$H^1(K,T) = H^1(K, K^\times_{\text{sep}} \otimes \widehat{T}^0_{\text{sep}}) \xrightarrow{\nu} H^2(K, \mathbb{Z}_p(1) \otimes \widehat{T}^0_{\text{sep}}), \tag{5.34}$$

where the equality follows from the end of the proof of Theorem 5.6. After (5.32), we have the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(1) \longrightarrow K^\times_{\text{sep}} \longrightarrow \overline{K}^\times_{\text{sep}} \longrightarrow 0$$

hence also

$$0 \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \widehat{T}_{\text{sep}} \longrightarrow K^\times_{\text{sep}} \otimes \widehat{T}_{\text{sep}} \longrightarrow \overline{K}^\times_{\text{sep}} \otimes \widehat{T}_{\text{sep}} \longrightarrow 0.$$

Since $K^\times_{\text{sep}}$ is m-divisible for all m relatively prime to q and $\mathbb{Q}_p/\mathbb{Z}_p(1) = K^\times_{\text{sep}}\{p\}$ by (5.32), we see that the group $\overline{K}^\times_{\text{sep}} \otimes \widehat{T}_{\text{sep}}$ is uniquely p-divisible and thus the p-primary components of its Galois cohomology is trivial in positive degree. Hence, we have the surjection

$$H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \widehat{T}_{\text{sep}}) \overset{\omega}{\twoheadrightarrow} H^1(K, K^\times_{\text{sep}} \otimes \widehat{T}_{\text{sep}})\{p\} = H^1(K, T^0)\{p\}, \tag{5.35}$$

where as before, the last equality follows from the end of the proof of Theorem 5.6. Setting $M = \mathbb{Q}_p/\mathbb{Z}_p$ and $i = j = 1$ in (5.33) and considering the natural pairing between $\widehat{T}_{\text{sep}}$ and $\widehat{T}^0_{\text{sep}}$ into $\mathbb{Z}$, we obtain the cup-product

$$H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \widehat{T}_{\text{sep}}) \times H^2(K, \mathbb{Z}_p(1) \otimes \widehat{T}^0_{\text{sep}}) \xrightarrow{\cup} H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)) .$$

Using this, one can show that the first component of the isomorphism in Theorem 5.12 is described explicitly by the split embedding $H^1(F, T^0)\{p\} \hookrightarrow \operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ which takes a class $\alpha \in H^1(F, T^0)\{p\}$ to the invariant that maps a torsor $\mathscr{T} \in H^1(K, T)$ to the cohomology class $\omega^{-1}(\alpha_K) \cup \nu(\mathscr{T}) \in H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, where $\alpha_K \in H^1(K, T^0)$ is the image of α under the natural map $H^1(F, T^0) \longrightarrow H^1(K, T^0)$. By construction, two preimages of α_K

under ω differ by an element β coming from $H^0(K, \overline{K}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})$, hence β is p-divisible. Since Galois cohomology is torsion in positive degree, $\mathrm{Im}(\nu)$ is torsion. Moreover, multiplication by n relatively prime to p is an isomorphism of $\mathbb{Z}_p(1)$, thus $\mathrm{Im}(\nu)$ is in fact p-primary torsion. Therefore, the cup-product $\beta \cup \nu(\mathscr{T}) = 0$ and so the given map is well-defined. Moreover, because we take the cup-product with $\nu(\mathscr{T})$, it is irrelevant whether we view $\mathbb{Z}_p(i)$ with its natural topology (as is done in [Tat76]) or with the discreet topology when forming its cohomology groups.

Analogously, we claim that the second component of the isomorphism in Theorem 5.12 is described using a (different) cup-product. This time, the natural pairing between $\widehat{T}_{\mathrm{sep}}$ and $\widehat{T}^0_{\mathrm{sep}}$ is doubled:

$$(\mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \widehat{T}^0_{\mathrm{sep}}) \times (\mathbb{Z}_p(1) \otimes \widehat{T}^0_{\mathrm{sep}}) \times (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)} \longrightarrow \mathbb{Q}_p/\mathbb{Z}_p(2).$$

The induced cup-product is

$$H^1(K, \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes \widehat{T}^0_{\mathrm{sep}}) \times H^2(K, \mathbb{Z}_p(1) \otimes \widehat{T}^0_{\mathrm{sep}}) \times H^0(K, (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}) \\ \xrightarrow{\cup} H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2)),$$

and we claim that the second split injection $(S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec})\{p\} \hookrightarrow \operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ of Theorem 5.12 is given as follows. For a coset $\overline{\alpha} \in (S^2(\widehat{T}_{\mathrm{sep}})^\Gamma / \operatorname{Dec})\{p\}$, choose a representative $\alpha \in S^2(\widehat{T}_{\mathrm{sep}})^\Gamma_{(p)}$ and write α for its image under the natural embedding

$$S^2(\widehat{T}_{\mathrm{sep}})^\Gamma_{(p)} \hookrightarrow (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^\Gamma_{(p)}\ .$$

Since the Γ_K-module structure is given by the induced map $\Gamma_K \longrightarrow \Gamma$, we have that

$$\alpha \in (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})^{\Gamma_K}_{(p)} = H^0(K, (\widehat{T}_{\mathrm{sep}} \otimes \widehat{T}_{\mathrm{sep}})_{(p)}).$$

The invariant corresponding to $\overline{\alpha}$ maps a torsor $\mathscr{T} \in H^1(K, T)$ to the cohomology class $\omega_0^{-1}(\mathscr{T}_p) \cup \nu(\mathscr{T}) \cup \alpha \in H^3(K, \mathbb{Q}_p/\mathbb{Z}_p(2))$, where we write $\mathscr{T}_p$ for the projection of $\mathscr{T}$ onto the p-primary component of $H^1(K, T)$ and ω_0 for the dual of ω in (5.35) obtained by replacing T with T^0. As above, this is independent of the choice of preimage under ω_0 and one can check that it is also independent of the choice of representative α for $\overline{\alpha}$.

The fact that the pairing between $\widehat{T}_{\text{sep}}$ and its dual is doubled in the second cup-product is significant because the degree 3 invariants corresponding to the second component $(S^2(\widehat{T}_{\text{sep}})^{\Gamma}/\operatorname{Dec})\{p\}$ in Theorem 5.12 are of a new type, different from those corresponding to the first component $H^1(F, T^0)\{p\}$ and degree 2 invariants. This difference is captured by the following definitions. If G is a commutative algebraic group, e.g. a torus, then the (pointed) set of isomorphism classes of torsors $H^1(K, G)$ is in fact an abelian group. In this case, an invariant $i \in \operatorname{Inv}^i(G, C)$ is called *linear* if $i_K : H^1(K, G) \longrightarrow H^i(K, C)$ is a group homomorphism for every field K/F. An invariant $i \in \operatorname{Inv}^i(G, C)$ is called *quadratic* if the function $h(\mathscr{T}, \mathscr{T}') = i(\mathscr{T} + \mathscr{T}') - i(\mathscr{T}) - i(\mathscr{T}')$ is bilinear and $h(\mathscr{T}, \mathscr{T}) = 2 \cdot i(\mathscr{T})$ for all torsors $\mathscr{T}, \mathscr{T}' \in H^1(K, G)$ over all fields K/F. We write $\operatorname{Inv}^i_{\text{lin}}(G, C)$ and $\operatorname{Inv}^i_{\text{quad}}(G, C)$ for the linear and quadratic invariants, respectively.

In the Corollary to Theorem 2.2 we proved that $\operatorname{Inv}^2(T, \mathbb{Q}/\mathbb{Z}(1)) \cong H^1(F, \widehat{T}_{\text{sep}})'$. In [BM12], the authors describe this isomorphism (with a characteristic component, in fact) via a cup-product pairing in a manner similar to the description above for the degree 3 isomorphism in Theorem 5.12. It is induced by the natural pairing

$$H^1(K, \widehat{T}_{\text{sep}}) \times H^1(K, K_{\text{sep}}^{\times} \otimes \widehat{T}^0_{\text{sep}}) \xrightarrow{\cup} H^2(K, K_{\text{sep}}^{\times}),$$

i.e., identifying $H^1(K, T)$ with $H^1(K, K_{\text{sep}}^{\times} \otimes \widehat{T}^0_{\text{sep}})$ as in (5.34), the degree 2 invariant associated to $\alpha \in H^1(F, \widehat{T}_{\text{sep}})'$ maps a torsor $\mathscr{T} \in H^1(K, T)$ to the cohomology class $\alpha_K \cup \mathscr{T} \in H^2(K, \mathbb{Q}/\mathbb{Z}(1)) = H^2(K, K_{\text{sep}}^{\times})'$, where $\alpha_K \in H^1(K, \widehat{T}_{\text{sep}})'$ is the image of α under the natural map $H^1(F, \widehat{T}_{\text{sep}}) \longrightarrow H^1(K, \widehat{T}_{\text{sep}})$ and $\mathscr{T}'$ is the projection of $\mathscr{T}$ onto the characteristic-free part $H^1(K, T)'$. Because cup-products are bilinear, we see that all degree 2 invariants as well as the first component $H^1(F, \widehat{T}_{\text{sep}})\{p\}$ of degree 3 invariants are linear, while the second component $(S^2(\widehat{T}_{\text{sep}})^{\Gamma}/\operatorname{Dec})\{p\}$ of degree 3 invariants is quadratic. That is, Theorem 5.12 can be augmented to

$$\begin{aligned} \operatorname{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) &\cong H^1(F, T^0)\{p\} \oplus \big(S^2(\widehat{T}_{\text{sep}})^{\Gamma}/\text{Dec}\big)\{p\} \\ &= \operatorname{Inv}^3_{\text{lin}}(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \oplus \operatorname{Inv}^3_{\text{quad}}(T, \mathbb{Q}_p/\mathbb{Z}_p(2)). \end{aligned} \tag{5.36}$$

In Example 6.4 we give a construction of tori which exhibit nontrivial quadratic invariants.

6 An Equivalence for Algebraic Tori

In this section and the next, we present applications of our main result, Theorem 5.12. The first is an equivalence for tori which demonstrates a level of control that degree 3 invariants have over invariants of all other degrees.

We begin with some notation and a preliminary result. Let T/F be a torus and $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ an exact sequence of tori with P quasi-trivial (e.g., a coflasque resolution of T or a flasque resolution of Q). By Lemma 1.5, $P \longrightarrow Q$ is a versal T-torsor and we write P_ξ for the generic torsor in $H^1(F(Q), T)$ and e for its period (= order in this abelian group). Since $\operatorname{End}_\Gamma(\widehat{T}_{\text{sep}}) \cong (\widehat{T}_{\text{sep}} \otimes \widehat{T}^0_{\text{sep}})^\Gamma$ is a direct summand of $S^2(\widehat{T}_{\text{sep}} \oplus \widehat{T}^0_{\text{sep}})^\Gamma$, we can view the identity endomorphism Id in the symmetric square and we write $\overline{\text{Id}}$ for its image in the quotient $S^2(\widehat{T}_{\text{sep}} \oplus \widehat{T}^0_{\text{sep}})^\Gamma / \operatorname{Dec}$ (see Construction 5.9 for the definition of the subgroup Dec).

Proposition 6.1. *The following are equivalent:*

(1) *T is universally special, i.e., $H^1(K, T) = 0$ for all K/F.*

(2) $e = 1$.

(3) $\overline{\text{Id}} = 1$.

(4) *$\widehat{T}_{\text{sep}}$ is invertible.*

Proof. (1) $\implies$ (2) is obvious and (2) $\iff$ (3) follows from [Mer10, Thm. 2.2] (which in fact proves that $e = |\overline{\text{Id}}|$). Construction 5.9 shows that the projection of the subgroup of decomposable elements $\operatorname{Dec} \subseteq S^2(\widehat{T}_{\text{sep}} \oplus \widehat{T}^0_{\text{sep}})^\Gamma$ onto the factor $(\widehat{T}_{\text{sep}} \otimes \widehat{T}^0_{\text{sep}})^\Gamma$ is generated by the traces $\operatorname{Tr}(t \otimes s)$ for all open subgroups $\Gamma' \leq \Gamma$ and all elements $t \in (\widehat{T}_{\text{sep}})^{\Gamma'}, s \in (\widehat{T}^0_{\text{sep}})^{\Gamma'}$. Thus, if (3) holds, we can write

$$\text{Id} = \sum_{i=1}^{n} \operatorname{Tr}(t_i \otimes s_i)$$

for some open subgroups $\Gamma_i \leq \Gamma$ and $t_i \in (\widehat{T}_{\text{sep}})^{\Gamma_i}, s_i \in (\widehat{T}^0_{\text{sep}})^{\Gamma_i}$. Moreover, if we let $P_i = \mathbb{Z}[\Gamma/\Gamma_i]$ then we can view the t_i and s_i as Γ-equivariant homomorphisms $P_i \longrightarrow \widehat{T}_{\text{sep}}$ and

$\widehat{T}_{\mathrm{sep}} \longrightarrow P_i$, respectively (*cf.* the second claim in the proof of Lemma 5.11). Letting $P = \bigoplus_{i=1}^{n} P_i$, we then have a factorization

$$\begin{tikzcd}
\widehat{T}_{\mathrm{sep}} \arrow[r, "(s_i)"] \arrow[rr, bend right, "\mathrm{Id}"'] & P \arrow[r, "(t_i)"] & \widehat{T}_{\mathrm{sep}},
\end{tikzcd}$$

hence (4) holds since P is a permutation module. Since the implication (4) $\implies$ (1) is obvious, the result follows. ☐

The following Lemma will be useful in the next result.

Lemma 6.2. *If a commutative algebraic group G/F has no nontrivial degree i linear invariants with values in a Γ-module C universally, i.e.,* $\mathrm{Inv}^i_{\mathrm{lin}}(G_K, C) = 0$ *for all field extensions K/F, then it has no nontrivial degree i quadratic invariants with values in C universally, i.e.,* $\mathrm{Inv}^i_{\mathrm{quad}}(G_K, C) = 0$ *for all field extensions K/F.*

Proof. Let K/F be an extension and $i \in \mathrm{Inv}^i_{\mathrm{quad}}(G_K, C)$ be a quadratic invariant. If $\mathscr{T} \in H^1(K, G)$ is a fixed G_K-torsor, then the invariant which maps a torsor $\mathscr{T}' \in H^1(L, G)$ for some field extension L/K to the cohomology class $h(\mathscr{T}_L, \mathscr{T}') = i(\mathscr{T}_L + \mathscr{T}') - i(\mathscr{T}_L) - i(\mathscr{T}')$ is linear, i.e., $h(\mathscr{T}, -) \in \mathrm{Inv}^i_{\mathrm{lin}}(G_K, C)$, hence trivial by hypothesis. Choosing $L = K$, we obtain $i(\mathscr{T} + \mathscr{T}') = i(\mathscr{T}) + i(\mathscr{T}')$. Since K and the torsors $\mathscr{T}, \mathscr{T}'$ were arbitrary, we see that i is actually a linear invariant, hence trivial. ☐

Theorem 6.3. *For every prime $p \neq 2, q$, the following are equivalent:*

(1) $\mathrm{Inv}^3(T_K, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ *for every field extension K/F.*

(2) $H^1(K, T)\{p\} = 0$ *for every field extension K/F.*

(3) $\mathrm{Inv}^i_{\mathrm{lin}}(T_K, C)\{p\} = 0$ *for any field extension K/F, in any degree i, and for any Γ-module C.*

(4) $(e, p) = 1$.

(5) *There exists an integer n such that $(n, p) = 1$ and a factorization $n \cdot \mathrm{Id} : \widehat{T}_{\mathrm{sep}} \longrightarrow P \longrightarrow \widehat{T}_{\mathrm{sep}}$ for some permutation module P.*

(1^0) - (5^0), the same statements for the dual torus T^0.

Proof.

$\underline{(1) \implies (2^0),\ (1^0) \implies (2)}$: This follows directly from Theorem 5.12.

$\underline{(2) \implies (3),\ (2^0) \implies (3^0)}$: Let K/F be any field extension and $Y \longrightarrow X$ be any versal T_K-torsor and write Y_ξ for the generic torsor in $H^1(K(X), T)$. By the Rost embedding (Theorem 1.6), every invariant is determined by its value at the generic torsor so if $\alpha \in \mathrm{Inv}^i_{\mathrm{lin}}(T_K, C)$ for some Γ-module C then by definition of linear invariants, $|\alpha| \mid |Y_\xi|$. If (2) holds then $(|Y_\xi|, p) = 1$, hence $(|\alpha|, p) = 1$ and so $\mathrm{Inv}^i_{\mathrm{lin}}(T_K, C)\{p\} = 0$.

$\underline{(3) \implies (1),\ (3^0) \implies (1^0)}$: This follows from Lemma 6.2 after (5.36).

This establishes the equivalence of (1), (2), (3), and their duals.

$\underline{(2) \iff (4),\ (2^0) \iff (4^0)}$: This follows directly from [Mer10, Prop. 1.1] which shows that the period e of the generic torsor $P_\xi \in H^1(F(Q), T)$ coming from the specific versal T-torsor $P \longrightarrow Q$ arising from a resolution of T (see above) is divisible by the period of any T-torsor over any field extension of F.

$\underline{(4) \iff (5),\ (4^0) \iff (5^0)}$: [Mer10, Thm. 2.2] shows that $e = |\overline{\mathrm{Id}}|$ hence if (4) holds we have that $e \cdot \mathrm{Id} \in \mathrm{Dec}$ and we obtain the desired factorization with $n = e$ as in the proof of the previous Proposition. Conversely, the same proof shows that such a factorization implies $|\overline{\mathrm{Id}}| \mid n$ hence $(e, p) = 1$ and (4) holds. □

Using this result we give a construction of tori with nontrivial quadratic invariants in degree 3.

Example 6.4.

(a) If S/F is any torus with a nontrivial p-primary torsor $\mathscr{T} \in H^1(K,S)\{p\}$ for $p \neq 2, q$ over some field K/F then Theorem 6.3 implies that $p \mid e = |\overline{\mathrm{Id}}|$ and therefore the element $\frac{e}{p} \cdot \overline{\mathrm{Id}} \in (S^2(\widehat{S}_{\mathrm{sep}} \oplus \widehat{S}^0_{\mathrm{sep}})^\Gamma / \operatorname{Dec})\{p\}$ is nontrivial. Thus, the torus $T = S \times S^0$ has nontrivial quadratic invariants by (5.36). For example, if S is not invertible and split by a p-group $G = \mathrm{Gal}(L/F)$ for some prime $p \neq 2, q$ then $H^1(K,S) \neq 0$ for some $F \subset K \subset L$ by Proposition 6.1 and moreover is p-primary because composing the restriction and corestriction maps for the trivial subgroup implies that (finite) group cohomology is annihilated by the order of the group (*cf.* [Ser97, Ch. I, §2.5, Prop. 9]).

(b) Let L/F be a cyclic extension of prime order $p \neq 2, q$, i.e., $G = \mathrm{Gal}(L/F) = \mathbb{Z}/p\mathbb{Z}$. Consider the torus $T = R^{(1)}_{L/F}(\mathbf{G}_m) \times R^{(1)}_{L/F}(\mathbf{G}_m)$ and write $\widehat{T}_L = J \oplus J$ for the G-module of characters, shortening the notation introduced in Example 1.2(c) for clarity. Recall, J is defined by the exact sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{N} \mathbb{Z}[G] \longrightarrow J \longrightarrow 0, \tag{6.1}$$

with $N(1) = \sum_G g$. As we saw in Example 1.2(c), the G-invariants J^G are trivial. Since G is cyclic, $J \cong J^0$ so in fact T is of the type considered in (a) (and self-dual). But, in this case we can explicitly describe the entire factor containing the non-trivial element $\overline{\mathrm{Id}}$ considered above. Following the notation in Construction 5.9, we have the decomposition

$$S^2(\widehat{T}_L)^G/\mathrm{Dec} \cong \Big[S^2(J)/\mathrm{Dec}(J)\Big] \oplus \Big[S^2(J)/\mathrm{Dec}(J)\Big] \oplus \Big[(J \otimes J)^G/\mathrm{Dec}(J,J)\Big]. \tag{6.2}$$

Since G has prime order and $J^G = 0$, $\mathrm{Dec}(J,J)$ is generated only by the traces $\mathrm{Tr}(j \otimes k) = \sum_G gj \otimes gk$ for $j, k \in J$ and so

$$(J \otimes J)^G/\mathrm{Dec}(J,J) = (J \otimes J)^G/\big\langle \textstyle\sum_G gj \otimes gk \;\big|\; j,k \in J\big\rangle = \widehat{H}^0(G, J \otimes J), \tag{6.3}$$

where $\widehat{H}^i(G,-)$ refers to the Tate cohomology groups (*cf.* [Ser79, Ch. VIII, §1]). Moreover, by [Vos98, §4.8, p. 53, 56] there is an exact sequence

$$0 \longrightarrow J \longrightarrow \mathbb{Z}[G]^{p-1} \longrightarrow J \otimes J \longrightarrow 0, \tag{6.4}$$

hence we have the exact sequence in cohomology

$$\widehat{H}^0(G, \mathbb{Z}[G]^{p-1}) \longrightarrow \widehat{H}^0(G, J \otimes J) \longrightarrow H^1(G, J) \longrightarrow H^1(G, \mathbb{Z}[G]^{p-1}).$$

Since $\mathbb{Z}[G]^G = \langle \sum_G g \rangle$ we have that $\widehat{H}^0(G, \mathbb{Z}[G]^{p-1}) = 0$ and $H^1(G, \mathbb{Z}[G]^{p-1}) = 0$ as well because $\mathbb{Z}[G]$ is a permutation G-module. Therefore,

$$\widehat{H}^0(G, J \otimes J) \cong H^1(G, J). \tag{6.5}$$

But G is cyclic hence (Tate) cohomology is 2-periodic and so in the exact sequence

$$0 \longrightarrow H^1(G, J) \longrightarrow H^2(G, \mathbb{Z}) \longrightarrow H^2(G, \mathbb{Z}[G])$$

induced by (6.1) above, we have that $H^2(G, \mathbb{Z}[G]) \cong \widehat{H}^0(G, \mathbb{Z}[G])$, which we just saw is trivial. Therefore,

$$H^1(G, J) \cong H^2(G, \mathbb{Z}) \cong \widehat{H}^0(G, \mathbb{Z}) = \mathbb{Z}/|G|\mathbb{Z}, \tag{6.6}$$

where the last equality follows because G acts trivially on $\mathbb{Z}$. Combining (6.2), (6.3), (6.5), and (6.6) with (5.36), we see that the group $\mathbb{Z}/p\mathbb{Z}$ is (isomorphic to) a direct summand of $\mathrm{Inv}^3_{\mathrm{quad}}(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$.

7 Connections with Chow Groups

As a final application of our main result we obtain a new example of a recent result on Chow groups. Let X be a smooth projective variety over F and write $CH^i(X)$ for the Chow group of cycles of codimension i on X modulo rational equivalence. In [Pir11], the author answers the following question in the negative: in cases when the natural map $CH^1(X) \longrightarrow CH^1(X_{\mathrm{sep}})^\Gamma$ surjective, is the corresponding map $CH^2(X) \longrightarrow CH^2(X_{\mathrm{sep}})^\Gamma$ also surjective?

Using degree 3 cohomological invariants of a torus T, we produce a counterexample by constructing an embedding

$$\mathrm{coker}\left[H^1(F, T^0)\{p\} \longrightarrow \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))\right] \hookrightarrow \mathrm{coker}\left[CH^2(X)_{(p)} \longrightarrow CH^2(X_{\mathrm{sep}})^\Gamma_{(p)}\right]$$

for an auxiliary (smooth and projective) variety X/F, where the map on the left is the one given by Theorem 5.12 and the subscript denotes localization at a prime $p \neq 2, q$. The counterexample follows by exhibiting a torus T with nontrivial quadratic invariants so that the coker on the left is nontrivial (after (5.36)). Since localization is exact, this will imply that $CH^2(X) \longrightarrow CH^2(X_{\text{sep}})^{\Gamma}$ has nontrivial cokernel. In fact, our desire to have surjectivity on the level of the first Chow groups of X will lead us to an isomorphism of these cokernels (*cf.* Theorem 7.2 and Example 7.3). Moreover, we will see that the torus T is necessarily flasque, hence its invariants are all unramified (Proposition 1.15). Thus, we ultimately obtain our counterexample by showing that an unramified cohomology group (*cf.* Proposition 1.14) is large enough. Interestingly, Pirutka's result relied on demonstrating that a particular degree 3 unramified cohomology group was nontrivial.

In order to obtain the isomorphism above we form a cross diagram of two exact sequences. The first will be derived from the Hochschild-Serre spectral sequence in a manner analogous to the proof of Theorem 5.2 and the second will be a localization of (5.4). The connection between the two will be the two maps whose cokernels we consider above, which will determine precisely the relationship between the torus T and the smooth projective variety X/F, the latter of which will serve as our counterexample.

Theorem 7.1. *Let T/F be a flasque torus and $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ an exact sequence of tori with P quasitrivial (e.g., a coflasque resolution of T or a flasque resolution of Q). Then for every prime $p \neq q$ there exists a smooth projective variety X/F and an exact sequence*

$$0 \longrightarrow H^1(F, T^0)\{p\} \longrightarrow \overline{\mathbb{H}}^{4,2}(X)_{(p)} \longrightarrow CH^2(X_{\text{sep}})^{\Gamma}_{(p)} \longrightarrow H^2(F, \operatorname{Pic}(X_{\text{sep}}) \otimes F^{\times}_{\text{sep}})\{p\},$$

where the (normalized) motivic cohomology group $\overline{\mathbb{H}}^{4,2}(X)$ is introduced in §5 and the subscripts denote localization. If, moreover, the cohomological dimension of F is ≤ 1, then the last group vanishes and we obtain the short exact sequence

$$0 \longrightarrow H^1(F, T^0)\{p\} \longrightarrow \overline{\mathbb{H}}^{4,2}(X)_{(p)} \longrightarrow CH^2(X_{\text{sep}})^{\Gamma}_{(p)} \longrightarrow 0.$$

Proof. Although parts of the proof are analogous to the proof of Theorem 5.2, we include the details for completeness. Let X be a smooth projective toric model for Q ([CHS05]) and consider the *localized* Hochschild-Serre spectral sequence

$$E_2^{r,s} = H^r(F, \mathbb{H}^{q,2}(X_{\text{sep}}))_{(p)} \implies \mathbb{H}^{r+s,2}(X)_{(p)}.$$

Since Galois cohomology is torsion in positive degree and localization coincides with taking the p-primary component for a torsion abelian group, we will write the p-primary component for $E_2^{r,s}$ when $r \geq 1$. Since X is smooth and connected, [Kah96, Thm. 1.1] shows that

$$\mathbb{H}^{q,2}(X) = \begin{cases} 0 & q \leq 0 \\ K_3^{\text{ind}}(F(X)) & q = 1 \\ A^0(X, K_2) & q = 2 \\ A^1(X, K_2) & q = 3. \end{cases} \tag{7.1}$$

Since Q is an open subset of X, we obtain that

$$E_2^{r,1} = H^r(F, K_3^{\text{ind}}(F_{\text{sep}}(Q)))\{p\} \cong H^r(F, \mathbb{Q}_p/\mathbb{Z}_p(2)), \quad r > 1,$$

where the isomorphism follows from (5.10). By [Sus84, Thm. 25.5], we have that

$$A^0(X_{\text{sep}}, K_2) \cong K_2(F_{\text{sep}}). \tag{7.2}$$

Note that "rational" in [Sus84, Thm. 25.5] means "geometrically rational" and $\mathscr{K}$-cohomology (= Zariski cohomology for the sheaf associated to the presheaf $U \longmapsto K_*^Q(U)$) can be identified with Rost's Chow groups with values in K-Theory by Gersten's conjecture (proved in [Qui73]). After Lemma 5.3 we have that

$$E_2^{r,2} = H^r(F, K_2(F_{\text{sep}}))\{p\} = 0, \quad r \geq 1. \tag{7.3}$$

The vanishing of this row will be essential for obtaining our exact sequence from a filtration on the abutment , which is the reason for using the localized version of the Hochschild-Serre spectral sequence and ultimately why the result is the localized exact sequence away from the characteristic (*cf.* [BM12]).

Since X is a smooth toric model for Q, [Mer08, Cor. 2.2] implies that there is an isomorphism

$$A^1(X_{\text{sep}}, K_2) \cong \operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^{\times}.$$

Because X is cellular, the Γ-module $\operatorname{Pic}(X_{\text{sep}})$ is finitely generated and $\mathbb{Z}$-free (see also [Vos98, §4.5]). Moreover, $\operatorname{Pic}(X_{\text{sep}})$ is a flasque module and there is an exact sequence

$$0 \longrightarrow \widehat{Q}_{\text{sep}} \longrightarrow P' \longrightarrow \operatorname{Pic}(X_{\text{sep}}) \longrightarrow 0$$

for some permutation module P'. Since by hypothesis

$$0 \longrightarrow \widehat{Q}_{\text{sep}} \longrightarrow \widehat{P}_{\text{sep}} \longrightarrow \widehat{T}_{\text{sep}} \longrightarrow 0$$

is also flasque resolution of $\widehat{Q}_{\text{sep}}$, Lemma 1.1 implies that the Γ-modules $\widehat{T}_{\text{sep}}$ and $\operatorname{Pic}(X_{\text{sep}})$ are similar, i.e., become isomorphic after addition of permutation modules. By the proof of Lemma 5.7 and the end of the proof of Theorem 5.6, we see that

$$\begin{aligned} E_2^{1,3} &= H^1(F, A^1(X_{\text{sep}}, K_2))\{p\} \\ &\cong H^1(F, \operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^{\times})\{p\} \\ &\cong H^1(F, \widehat{T}_{\text{sep}} \otimes F_{\text{sep}}^{\times})\{p\} \\ &\cong H^1(F, T^0)\{p\}. \end{aligned}$$

Thus, a portion of our E_2-page is of the form

$$\begin{tikzcd}
q & & & & & \\
4 & \mathbb{H}^{4,2}(X_{\text{sep}})^{\Gamma}_{(p)} \arrow[drr, "d_2^{0,4}"] & & & & \\
3 & (\operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^{\times})^{\Gamma}_{(p)} & H^1(F, T^0)\{p\} & H^2(F, \operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^{\times})\{p\} & & \\
2 & & 0 \arrow[rr] & & {} & \\
1 & & & H^2(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) & H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) & \\
0 & 0 \arrow[rrr] & & & {} & \\
 & 0 & 1 & 2 & 3 & p
\end{tikzcd}$$

Because $E_2^{r,0} = 0 = E_2^{r,2}$ and the spectral sequence is first quadrant, we have the E_3-page differentials

$$\begin{aligned} (\mathrm{Pic}(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^\times)^\Gamma_{(p)} = E_3^{0,3} &\xrightarrow{d_3^{0,3}} E_3^{3,1} = H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ H^1(F, T^0)\{p\} = E_3^{1,3} &\xrightarrow{d_3^{1,3}} E_3^{4,1} = H^4(F, \mathbb{Q}_p/\mathbb{Z}_p(2)). \end{aligned} \tag{7.4}$$

The filtration

$$F^4 \subseteq F^3 \subseteq F^2 \subseteq F^1 \subseteq \mathbb{H}^{4,2}(X)_{(p)}$$

implies that $F^4 \cong E_\infty^{4,0} = 0$ hence $E_\infty^{3,1} \cong F^3 \subseteq \mathbb{H}^{4,2}(X)_{(p)}$. Moreover, the differentials $d_i^{3,1} = 0$ for $i \geq 2$ and the differentials mapping into $E_j^{3,1}$ are trivial for $j \geq 4$, hence $E_\infty^{3,1} = E_4^{3,1} = \mathrm{coker}(d_3^{0,3})$. The composition

$$\begin{tikzcd} E_3^{3,1} = H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r, two heads] \arrow[rr, bend right, hook] & \mathrm{coker}(d_3^{0,3}) = E_\infty^{3,1} \arrow[r, hook] & \mathbb{H}^{4,2}(X)_{(p)} \end{tikzcd}$$

is injective because the group identity of $Q(F)$ in $X(F)$ induces a splitting

$$\mathbb{H}^{4,2}(F) \longrightarrow \mathbb{H}^{4,2}(X) \longrightarrow \mathbb{H}^{4,2}(F) \tag{7.5}$$

and $H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is a direct summand of $\mathbb{H}^{4,2}(F)$ by (5.2), which is preserved by this mapping. Since the former is p-primary torsion, it is preserved in the localization $\mathbb{H}^{4,2}(X)_{(p)}$. Therefore, $H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong \mathrm{coker}(d_3^{0,3})$ hence $d_3^{0,3} = 0$. An entirely analogous argument, in one degree higher, applies to show that $d_3^{1,3} = 0$ (except that, *a priori*, we only have $\mathrm{coker}(d_3^{1,3}) \twoheadrightarrow E_\infty^{4,1} \subseteq \mathbb{H}^{5,2}(X)_{(p)}$ because although all the differentials $d_i^{4,1} = 0$ for $i \geq 2$, we do not bother investigating whether the differential mapping into $E_4^{4,1}$ is trivial).

Since $F^2/F^3 \cong E_\infty^{2,2} = 0$, the above implies that

$$\begin{gathered} F^3 \cong E_\infty^{3,1} = \mathrm{coker}(d_3^{0,3}) = H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ F^1/F^3 \cong E_\infty^{1,3} = \ker(d_3^{1,3}) = H^1(F, T^0)\{p\} \\ \mathbb{H}^{4,2}(X)_{(p)}/F^1 \cong E_\infty^{0,4}. \end{gathered}$$

Taking the quotient by $F^3 = H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2))$ everywhere in the filtration then gives the short exact sequence

$$0 \longrightarrow H^1(F, T^0)\{p\} \longrightarrow \overline{\mathbb{H}}^{4,2}(X)_{(p)} \longrightarrow E_\infty^{0,4} \longrightarrow 0, \tag{7.6}$$

where the second term follows from the definition in (5.3) and the fact that localization commutes with taking quotients. In fact, we have that

$$E_\infty^{0,4} \cong \ker\left[\, \mathbb{H}^{4,2}(X_{\text{sep}})^\Gamma_{(p)} \xrightarrow{d_2^{0,4}} H^2(F, \operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^\times)\{p\} \,\right]. \tag{7.7}$$

This follows because $d_3^{0,4} : \ker(d_2^{0,4}) = E_3^{0,4} \longrightarrow E_3^{3,2} = 0$ and thus

$$d_4^{0,4} : \ker(d_2^{0,4}) = E_4^{0,4} \longrightarrow E_4^{4,1} = H^4(F, \mathbb{Q}_p/\mathbb{Z}_p(2)).$$

One then shows that $d_4^{0,4} = 0$ in the same way we showed that $d_3^{0,3}$ and $d_3^{1,3}$ were trivial above and so $d_k^{0,4} = 0$ for all $k \geq 3$, which, along with the fact that all differentials mapping into $E_i^{0,4}$ for $i \geq 2$ are trivial, implies (7.7). Therefore, the short exact sequence (7.6) gives the exact sequence

$$0 \longrightarrow H^1(F, T^0)\{p\} \longrightarrow \overline{\mathbb{H}}^{4,2}(X)_{(p)} \longrightarrow \mathbb{H}^{4,2}(X_{\text{sep}})^\Gamma_{(p)} \xrightarrow{d_2^{0,4}} H^2(F, \operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^\times)\{p\}.$$

Localizing Kahn's short exact sequence (5.4) with X_{sep} and using (5.5) implies that

$$\mathbb{H}^{4,2}(X_{\text{sep}})_{(p)} / CH^2(X_{\text{sep}})_{(p)} \cong A^0(X_{\text{sep}}, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]).$$

But, X_{sep} is proper and rational over F_{sep}, hence Theorems 1.11 and 1.10 show that

$$\begin{aligned} A^0(X_{\text{sep}}, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]) &\cong H^3_{\text{nr}}(F_{\text{sep}}(X), \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &\cong H^3(F_{\text{sep}}, \mathbb{Q}_p/\mathbb{Z}_p(2)) \\ &= 0, \end{aligned}$$

and so $\mathbb{H}^{4,2}(X_{\text{sep}})_{(p)} \cong CH^2(X_{\text{sep}})_{(p)}$, and the result follows from the previous exact sequence. Our reliance on (5.5) to connect the terms in Kahn's exact sequence with Rost's Chow groups is another reason we work in the localized setting away from the characteristic.

Suppose now that $\operatorname{cd}(F) \leq 1$, which means that $H^r(F, A) = 0$ for all *torsion* Γ-modules A and $r \geq 2$. Although $\operatorname{Pic}(X_{\text{sep}}) \otimes F_{\text{sep}}^\times$ is not a torsion Γ-module, we can exploit the roots of unity inside $F_{\text{sep}}^\times$ to see that it has trivial cohomology in degree ≥ 2. Indeed, if $\mu = \bigcup_n \mu_n \subset F_{\text{sep}}^\times$ are all the roots of unity then $\mu = (F_{\text{sep}}^\times)_{\text{tor}}$. Since $F_{\text{sep}}^\times$ is divisible, the quotient $F_{\text{sep}}^\times/\mu$ is uniquely divisible and therefore has trivial Galois cohomology in positive degree. Since $\mathrm{Pic}(X_{\mathrm{sep}})$ is $\mathbb{Z}$-free, $\mathrm{Pic}(X_{\mathrm{sep}}) \otimes (F_{\mathrm{sep}}^{\times}/\mu)$ remains uniquely divisible and we have the short exact sequence of Γ-modules

$$0 \longrightarrow \mathrm{Pic}(X_{\mathrm{sep}}) \otimes \mu \longrightarrow \mathrm{Pic}(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^{\times} \longrightarrow \mathrm{Pic}(X_{\mathrm{sep}}) \otimes \left(F_{\mathrm{sep}}^{\times}/\mu\right) \longrightarrow 0.$$

Since $\mathrm{Pic}(X_{\mathrm{sep}}) \otimes \mu$ is torsion, we see that

$$H^r(F, \mathrm{Pic}(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^{\times}) \cong H^r(F, \mathrm{Pic}(X_{\mathrm{sep}}) \otimes \mu) = 0, \quad r \geq 2. \qquad \square$$

In order to relate invariants to coker $\left[CH^2(X)_{(p)} \longrightarrow CH^2(X_{\mathrm{sep}})^{\Gamma}_{(p)}\right]$, we combine the exact sequence from the previous theorem with the localization of Kahn's exact sequence (5.4)

$$0 \longrightarrow CH^2(X)_{(p)} \longrightarrow \mathbb{H}^{4,2}(X)_{(p)} \longrightarrow H^0_{\mathrm{Zar}}(X, \mathcal{H}^3(\mathcal{Q}/\mathcal{Z}))\{p\} \longrightarrow 0, \tag{7.8}$$

for the same X. By (5.5), we have that the last term is isomorphic to $A^0(X, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)])$. To make the connection with invariants we need X to be a classifying variety for T because then the Rost embedding (Theorem 1.6 and Remark 1.4) gives us the inclusion

$$\mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \hookrightarrow \overline{A}^0(X, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]).$$

If T is a flasque torus, then the proof of Proposition 1.15 shows that not only is the toric Q-model X a classifying variety for T, but all the invariants of T are unramified and we have isomorphisms

$$\mathrm{Inv}^3_{\mathrm{nr}}(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \cong \overline{A}^0(X, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)]). \tag{7.9}$$

Furthermore, since $X(F) \neq \emptyset$, the splitting (7.5) and the diagram

$$\begin{tikzcd}
 & \mathbb{H}^{4,2}(F) \arrow[d] \arrow[dd, bend left, "\mathrm{Id}"] \\
CH^2(X) \arrow[r, hook] \arrow[d] & \mathbb{H}^{4,2}(X) \arrow[d] \\
0 = CH^2(F) \arrow[r, hook] & \mathbb{H}^{4,2}(F)
\end{tikzcd}$$

show that $CH^2(X) \cap H^3(F, \mathbb{Q}_p/\mathbb{Z}_p(2)) = 0$ inside $\mathbb{H}^{4,2}(X)$ hence we can normalize (7.8) and obtain our second exact sequence

$$0 \longrightarrow CH^2(X)_{(p)} \longrightarrow \overline{\mathbb{H}}^{4,2}(X)_{(p)} \longrightarrow \mathrm{Inv}^3_{(\mathrm{nr})}(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \longrightarrow 0, \tag{7.10}$$

where the subscript on the invariants denotes the fact that all the invariants are unramified.

The following will allow us to construct our counterexample.

Theorem 7.2. *Let T/F be a flasque torus and $1 \longrightarrow T \longrightarrow P \longrightarrow Q \longrightarrow 1$ an exact sequence of tori with P quasitrivial (e.g., a coflasque resolution of T or a flasque resolution of Q). Let X be a toric model for Q (cf. [CHS05]). Then for every prime $p \neq 2, q$ we have an embedding*

$$\mathrm{coker}\left[H^1(F, T^0)\{p\} \xrightarrow{f} \mathrm{Inv}^3(T, \mathbb{Q}_p/\mathbb{Z}_p(2))\right] \hookrightarrow \mathrm{coker}\left[CH^2(X)_{(p)} \xrightarrow{g_p} CH^2(X_{\mathrm{sep}})^{\Gamma}_{(p)}\right],$$

where f is the map from Theorem 5.12 and g_p is the (localization of the) natural map. If, moreover, the cohomological dimension of F is ≤ 1 then this is an isomorphism.

Proof. Notice we must impose the condition $p \neq 2$ in addition to the characteristic restriction because of Theorem 5.12. Combining the exact sequence in the previous theorem and (7.10) we obtain the cross-diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & H^1(F, T^0)\{p\} \arrow[d] \arrow[drr, "f"] & & \\
0 \arrow[r] & CH^2(X)_{(p)} \arrow[r] \arrow[dr, "g_p"'] & \overline{\mathbb{H}}^{4,2}(X)_{(p)} \arrow[rr] \arrow[d] & & \mathrm{Inv}^3_{(\mathrm{nr})}(T, \mathbb{Q}_p/\mathbb{Z}_p(2)) \arrow[r] & 0 \\
 & & CH^2(X_{\mathrm{sep}})^{\Gamma}_{(p)} \arrow[d] & & \\
 & & H^2(F, \mathrm{Pic}(X_{\mathrm{sep}}) \otimes F^{\times}_{\mathrm{sep}})\{p\}, & &
\end{tikzcd}$$

and the result follows after the so-called "Lemma of the 700th" ([MT93, p. 142]), noting that the last term in the vertical sequence vanishes under the extra hpothesis $\mathrm{cd}(F) \leq 1$ by Theorem 7.1. □

After Theorem 7.2, we need only construct a flasque torus with non-trivial quadratic invariants to exhibit a smooth and projective variety X/F such that the natural map $CH^2(X) \longrightarrow CH^2(X_{\mathrm{sep}})^\Gamma$ is not surjective. But, we would also like an example for which surjectivity holds on the level of the first Chow groups. Since X is smooth, we have that $CH^1(X) \cong \mathrm{Pic}(X)$. The Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(F, H^q(X_{\mathrm{sep}}, \mathbf{G}_m)) \implies H^{p+q}(X, \mathbf{G}_m)$$

gives the exact sequence

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_{\mathrm{sep}})^\Gamma \longrightarrow \mathrm{Br}(F) \longrightarrow \mathrm{Br}(X),$$

hence the obstruction to surjectivity of the first Chow groups is the Brauer group of the ground field. If $\mathrm{cd}(F) \leq 1$ then the Brauer group vanishes by the same argument we used to show $H^2(F, \mathrm{Pic}(X_{\mathrm{sep}}) \otimes F_{\mathrm{sep}}^\times) = 0$ in the proof of Theorem 7.1. Although this is not necessary for $\mathrm{Br}(F)$ to vanish ($\mathrm{cd}(F) \leq 1$ is equivalent to the stronger statement "$\mathrm{Br}(K) = 0$ for all K/F" by [Ser97, Ch. II, §3.1, Prop. 5]), it is a convenient assumption and incidentally results in an isomorphism of the cokernels in Theorem 7.2 instead of merely an embedding.

Example 7.3. Let F be a field of cohomological dimension ≤ 1 and let L/F be a Galois extension with $G = \mathrm{Gal}(L/F)$ and $|G| = n$. Recall the G-module J introduced in Examples 1.2(c) and 6.4(b) above. It was defined by the short exact sequence (6.1)

$$0 \xrightarrow{N} \mathbb{Z} \longrightarrow \mathbb{Z}[G] \longrightarrow J \longrightarrow 0$$

with $N(1) = \sum_G g$ and we had the auxiliary exact sequence (6.4)

$$0 \longrightarrow J \longrightarrow \mathbb{Z}[G]^{n-1} \longrightarrow J \otimes J \longrightarrow 0.$$

If $H \leq G$ is a subgroup then $\widehat{H}^0(H, \mathbb{Z}[G]) = 0$ by direct calculation (i.e., free G-modules have trival Tate cohomology in degree $i = -1, 0$, and 1) and $H^1(H, \mathbb{Z}) = 0$ because G acts trivially on $\mathbb{Z}$ so this group can be identified with $\mathrm{Hom}_{\mathrm{cont}}(H, \mathbb{Z}) = \mathrm{Hom}(H, \mathbb{Z})$. In fact, this holds for profinite groups because the image of any such homomorphism is a compact subgroup of $\mathbb{Z}$, hence finite (*cf.* [Ser97, Ch. I, §2.2]). Therefore $\widehat{H}^0(H, J) = 0$. The second exact sequence then implies that $\widehat{H}^{-1}(H, J \otimes J) = 0$, hence $J \otimes J$ is flasque, i.e., this sequence is a flasque resolution of J. By [CS87, Lemma 0.6], there exists an exact sequence

$$0 \longrightarrow J \otimes J \longrightarrow M \longrightarrow P \longrightarrow 0$$

with M coflasque and P permutation G-modules. Since M is between two flasque modules, it is necessarily flasque. If G has a non-cyclic abelian subgroup then by [CS77, Cor. 1 to Prop. 1] we have that the Picard class $p(J) = J \otimes J$ is *not* coflasque, which implies that M cannot be invertible. If $S = D(M)$ is the torus with character module M then the torus $T = S \times S^0$ is flasque and not invertible. Example 6.4(a) implies that S has non-trivial quadratic invariants and so Theorem 7.2 produces a counterexample to the question at the beginning of the section.

For an explicit example, one can take $F = E(x)$ for some separably closed field $E = E_{\mathrm{sep}}$ and $L = F(x^{1/p}, (1+x)^{1/p})$ for some prime $p \neq 2, q$, so that $G = \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$.

Remark 7.1.

(a) It is worth noting that this counterexample can be constructed in arbitrary characteristic whereas Pirutka's original construction [Pir11] is over a finite field. In fact, our approach over a finite field $\mathbb{F}_q$ almost does the opposite because over a finite field every torus is split by a cyclic group hence by the Endo-Miyata Theorem (*cf.* [CS77, Prop. 2, p. 184]) all flasque G-modules are invertible and so have no invariants in any degree. Thus, $\mathrm{coker}(f) = 0$ in Theorem 7.2 and since $\mathrm{cd}(\mathbb{F}_q) \leq 1$, we see that $\mathrm{coker}(g_p) = 0$ for every $p \neq 2, q$.

(b) The reason we are interested in a counterexample with X projective is because the group $\mathrm{Inv}^3_{(\mathrm{nr})}(T, \mathbb{Q}_p/\mathbb{Z}_p(2))$ is identified with $\overline{A}^0(X, H^3[\mathbb{Q}_p/\mathbb{Z}_p(2)])$ by (7.9) and by construction this group can only become larger for an open subset $U \subset X$, which would make coker(f) in Theorem 7.2 larger. In the projective case, Rost's Chow group is as small as possible among all stably birationally equivalent spaces because it is isomorphic to the unramified cohomology group of the function field (Theorem 1.11), which is a birational invariant (Theorem 1.10). Thus, one is inclined to seek out a projective counterexample.

(c) The upcoming paper [BM12] obtains the cross-diagram above without the localizations, but the X there is not projective. Hence, although the approach taken here fails to capture the characteristic and 2-primary components of the invariants, it is able to produce a projective counterexample to the above question where [BM12] cannot. Nonetheless, from the cross-diagram there, one obtains a (non-projective) counterexample in the same way we have done here.

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