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Mock-Modular Forms of Weight One

A dissertation submitted in partial satisfaction
of the requirements for the degree
Doctor of Philosophy in Mathematics

by

Yingkun Li

2013

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Abstract of the Dissertation

Mock-Modular Forms of Weight One

by

Yingkun Li

Doctor of Philosophy in Mathematics

University of California, Los Angeles, 2013

Professor William Duke, Chair

In this thesis, we will study mock-modular forms of weight one and their Fourier coefficients. In particular, we will concentrate on the mock-modular forms whose shadows are dihedral newforms arising from ray class group characters of imaginary quadratic fields. We will show that certain linear combinations of their Fourier coefficients are logarithms of CM values of the modular j-function We will also make a conjecture about the algebraicity of the individual Fourier coefficients.

The dissertation of Yingkun Li is approved.

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University of California, Los Angeles

2013

Table of Contents

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Publications

K. Bringmann, Y. Li, R. Rhoades, *Asymptotics for the number of row-Fishburn matrices*, submitted (2013)

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CHAPTER 1

Introduction

1.1 Introduction and Setup

The main object we are interested in studying, mock-modular forms, started with the last letter of Ramanujan to Hardy, dated January 1920. In this letter, Ramanujan described a class of q-series, which he called "Mock ϑ function", by writing down several examples and stating interesting combinatorial and asymptotic properties. In contrast to other q-series, such as the generating function of the partition function, these examples cannot be made holomorphic and modular, hence the name "mock ϑ function". Nevertheless, their properties were quite similar to those of the modular theta functions. Unfortunately, Ramanujan passed away before he could give the definition of his mock ϑ function, and left the world 17 such examples and a big mystery. Over the next eighty years, many people, including G. E. Andrews, L. Dragonette, A. Selberg and Watson, have studied these special examples in the absence of a uniform theory of mock-modular forms [1, 20, 47, 55, 56].

In 2003, Dutch mathematician Sander Zwegers gave a defining property of mock-modular forms in his thesis [63], by realizing them as the holomorphic part of a non-holomorphic modular form $\hat{f}(z)$. Furthermore, these non-holomorphic modular forms have poles and are annihilated by the weight k Laplacian operator

$$\Delta_k = \xi_{2-k} \circ \xi_k, \quad \xi_k := 2iy^k \overline{\partial_{\bar{z}}}. \tag{1.1.1}$$

Fittingly, these non-holomorphic modular objects are called harmonic weak Maass forms. The differential operator ξ_k is anti-holomorphic and commutes with the slash operator by changing the weight from k to $2 - k$ and conjugating the nebentypus character. So the function $\xi_k \hat{f}$ is a holomorphic modular form of weight $2-k$ and is called the *shadow* of the mock-modular form.

The case of weight $k=1$ has always been mysterious and important in the theory of modular forms. One of the mysteries is the number of weight one modular forms. Unlike in the cases when $k \geq 2$, the dimension of the space of modular forms over the complex numbers is unknown as the Riemann-Roch theorem yields trivial information. Using analytic techniques, various people have obtained asymptotic bounds on the dimensions of these spaces (see for example [4, 21, 41]). Over finite field, there are more weight one modular forms, some of which does not even come from reduction of modular forms over $\overline{\mathbb{Q}}$ (see [45]).

Another important features of weight one modular forms is their connection to Galois representations. By the Deligne-Serre's theorem, one could attach to each weight one newform f an odd, irreducible Artin representation ρ_f of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$

$$\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

Since ρ_f is continuous, it has finite image and the field fixed by $\ker \rho_f$ is an algebraic number field over $\mathbb{Q}$. Let $\tilde{\rho}_f$ be the composition of ρ_f and the surjection $\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$. Then the image of $\tilde{\rho}_f$, which is finite, is isomorphic to one of the following groups

- Dihedral, or D_{2n},
- Tetrahedral, or A_4,
- Octahedral, or S_4,
- Icosahedral, or A_5.

We call modular forms with the last three types of projective images *exotic* and use this classification to denote the types of weight one newforms and mock-modular forms. Following Langlands' philosophy and works by Deligne-Serre, Langlands, Tunnell and Khare-Wintenberger (see [34, 35, 39, 54]), this correspondence is functorial and bijective, and provides a bridge between weight one modular forms, which are a priori complex analytic, with algebraic number fields. This correspondence also enables one to check Stark's conjecture on L-series attached to weight one modular forms [51].

In this thesis, we will study the Fourier coefficients of weight one mock-modular forms, whose shadows are newforms. The case where the newform arises from a class group character of an imaginary quadratic field has been treated in [24, 26, 58]. Here, we will generalize the techniques in [24] to treat the case when the newform arises from a ray class group character of an imaginary quadratic field. The main goal is to relate the linear combinations of these Fourier coefficients to logarithms of CM values of modular functions

To be precise, let $D \equiv 1 \pmod 4$ be an odd, negative fundamental discriminant and $\chi_D = \left(\frac{D}{\cdot}\right)$ the associated Dirichlet character. Let p be an odd prime with $\chi_D(p) = 1$. It splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in the imaginary quadratic field $K = \mathbb{Q}(\sqrt{D})$. Let $\phi : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathbb{C}^\times$ be a ray class group character with modulus $\mathfrak{p}$ such that the induced representation

$$\rho_\phi := \mathrm{Ind}_K^{\mathbb{Q}}(\phi) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \longrightarrow \mathbb{C}^\times$$

is odd and irreducible. Then $\det(\rho_\phi) = \chi_D \phi_1$ with ϕ_1 a character of conductor p defined by

$$\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow I_\mathfrak{p}/P_{\mathfrak{p},1} \xrightarrow{\phi} \mathbb{C}^\times.$$

Denote the weight one newform associated to ρ_ϕ by $f_\phi(z)$. Then it is a newform of level $|D|p$ and character $\chi_{-7}\phi_1$. For example, suppose $D = -7, p = 11$ and $\phi_1(2) = \zeta_5$. Then there exists a newform in $f_\phi(z) \in S_1(77, \chi_D\phi_1)$ with the Fourier expansion

$$\begin{aligned} f_\phi(z) = q &+ \left(-\zeta_5^3 - \zeta_5 - 1\right) q^2 + \left(-\zeta_5^2 - 1\right) q^4 + \zeta_5 q^7 + \left(\zeta_5^3 + \zeta_5^2 + \zeta_5 + 1\right) q^8 + \zeta_5^3 q^9 + \zeta_5^2 q^{11} \\ &+ \left(\zeta_5^3 + 1\right) q^{14} + \left(\zeta_5^2 + 1\right) q^{18} + \left(-\zeta_5^3 - \zeta_5^2 - 1\right) q^{22} + \left(-\zeta_5^3 - \zeta_5^2 - 1\right) q^{23} + O(q^{25}). \end{aligned}$$

In general, the dimension of $S_1(|D|p, \chi_D\phi_1)$ is expected to be the class number of D plus the number of exotic forms in this space.

For a discriminant $D' < 0$, let $\mathcal{C}(D')$ be the set of positive definite binary quadratic forms $Q = [A, B, C]$ with discriminant D'. To each Q, one could associate a point $\tau_Q \in \mathcal{H}$. The group $\mathrm{SL}_2(\mathbb{Z})$ has an action on $\mathcal{C}(D')$, which translates into linear fractional transformation on τ_Q. We use w_Q to denote the size of the stabilizer of this action on $Q \in \mathcal{C}(D')$. A binary quadratic form is called *primitive* if $\gcd(A, B, C) = 1$. This property is preserved by the action of $\mathrm{SL}_2(\mathbb{Z})$. Let $C(D')$ be the group of equivalence classes of primitive binary quadratic forms of discriminant D'. One could evaluate modular functions, such as $j(z)$, at these τ_Q and obtain algebraic values by the theory of complex multiplication. Furthermore, this value only depends on the equivalence class of Q. We could now state a simple case of the main result.

Theorem 1.1.1. *Suppose $D < -5$ is a prime, fundamental discriminant such that the space $S_1(|D|p, \chi_D\phi_1)$ is one dimensional. Then there exists a unique mock-modular form $\tilde{f}_\phi(z)$ with shadow $f_\phi(z) \in S_1(|D|p, \chi_D\phi_1)$ and Fourier expansion*

$$\tilde{f}_\phi(z) = \tilde{c}_\phi(-1)q^{-1} + \sum_{\substack{n>1 \\ \chi_D(n)=-1}} \tilde{c}_\phi(n)q^n \tag{1.1.2}$$

at the cusp infinity. Furthermore, for any fundamental discriminant $D' < 0$ satisfying $\chi_D(D') = -1$, we have

$$\sum_{k\in\mathbb{Z}} \tilde{c}_\phi\left(\frac{p^2DD' - k^2}{4}\right)\phi_1\left(\frac{k}{2}\right)\delta_D(k) = -4\sum_{\substack{Q\in C(Dp^2) \\ Q'\in C(D')}} \psi^2(Q)\log|j(\tau_Q) - j(\tau_{Q'})|^{2/w_{Q'}}, \tag{1.1.3}$$

where $\psi : \mathrm{Pic}(\mathcal{O}_p) \longrightarrow \mathbb{C}^\times$ is the ring class group character associated to ϕ as in Prop. (3.3.4).

Remark 1.1.2. *The case that* $\dim S_1(|D|p, \chi_D\phi_1) = 1$ *happens when the class number of D is 1, ϕ_1 has order greater than two and there is no exotic form in $S_1(|D|p, \chi_D, \phi_1)$. For the general version of the main result, see Theorem 4.3.4.*

This is result is the analogue of Theorem 1.2 in [24]. Its generalization will be helpful for the future when we study mock-modular forms with shadows arising from ray class group characters of real quadratic field.

Suppose Q has discriminant Df^2 with D an odd, negative fundamental discriminant. By the theory of complex multiplication, the quantity

$$\prod_{Q'\in C(D')} (j(\tau_Q) - j(\tau_{Q'}))$$

is an algebraic number lying in the ring class field of K of conductor f, denoted by H_f. In [28], Gross and Zagier gave a factorization of the rational norm of this quantity when $f = 1$ and $\gcd(D, D') = 1$. They gave two proofs of this factorization, one algebraic, one analytic. The algebraic proof in fact gives the valuation of this quantity at various primes in the Hilbert class field of K. Later, people have given various generalizations of this factorization, both to non-fundamental discriminants (see [19, 40]) and to Hilbert modular functions (see [15]). These factorizations prompt us to make the following conjecture regarding the individual coefficient $\tilde{c}_\phi(n)$.

Conjecture 1.1.3. *Let D be an odd, negative fundamental discriminant, $K = \mathbb{Q}(\sqrt{D})$ and $\phi : \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathbb{C}^\times$ a character modulo $\mathfrak{p}$ such that $\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ has order greater than two. Let $f_{\phi,1}(z) \in S_1(|D|p, \chi_D\phi_1)$ be the cusp form associated to $f_\phi(z)$ as in Eq.* (3.3.13). *Then there exists $\kappa \in \mathbb{Z}, u(n, \mathcal{A}') \in \mathcal{O}_{H_p}$ and a mock-modular form $\tilde{f}_{\phi,1}(z) = \sum_{n \geq -n_0} \tilde{c}_\phi(n) q^n$ with shadow $f_\phi(z)$ such that*

$$\tilde{c}_\phi(n)\phi_1(\sqrt{n}) = \frac{1}{\kappa} \sum_{\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_p)} \psi^2(\mathcal{A}') \log |u(n, \mathcal{A}')|,$$

and $\sigma_{\mathcal{C}'}(u(n, \mathcal{A}')) = u(n, \mathcal{A}'\mathcal{C}'^{-1})$, where $\sigma_{\mathcal{C}'} \in \mathrm{Gal}(H_p/K)$ is associated to $\mathcal{C}' \in \mathrm{Pic}(\mathcal{O}_p)$ via class field theory.

The structure of the thesis is as follows. In Chapter 1, we will give a brief introduction to mock-modular forms. In Chapter 2, some preliminary results on existence of mock-modular forms and modular form transformations are given. In Chapter 3, we give the facts on weight one newforms, such as their associated Galois representation and Petersson inner products. In Chapter 4, we will give the proof of Theorem 4.3.4, from which Theorem 1.1.1 can be deduced.

1.2 Modular Forms

In this section, we will give some basic background on modular forms, following the reference [36].

1.2.1 Integral Weight Modular Forms

For any commutative ring R, denote the 2×2 matrices with entries in R by $M_2(R)$. The groups $\mathrm{GL}_2(\mathbb{R}), \mathrm{GL}_2^+(\mathbb{R})$ and $\mathrm{SL}_2(\mathbb{Z})$ are defined by

$$\begin{aligned} \mathrm{GL}_2(\mathbb{R}) &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{R}) : ad - bc \in \mathbb{R}^\times \right\}, \\ \mathrm{GL}_2^+(\mathbb{R}) &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{R}) : ad - bc > 0 \right\}, \\ \mathrm{SL}_2(\mathbb{Z}) &:= \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in M_2(\mathbb{Z}) : ad - bc = 1 \right\}. \end{aligned}$$

When $R = \mathbb{Z}$ and $M \in \mathbb{N}$, we can define the congruence subgroups $\Gamma_0(M)$ by

$$\Gamma_0(M) := \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{M} \right\}.$$

Let $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the upper half plane. The group $\mathrm{GL}_2^+(\mathbb{R})$ acts on $\mathcal{H}$ via linear fractional transformation, i.e. for $\gamma \in \mathrm{GL}_2(\mathbb{R})$ and $z \in \mathcal{H}$, we have

$$\gamma z := \frac{az + b}{cz + d}.$$

Similarly, $\Gamma_0(M)$ also act on $\mathcal{H}$ by linear fractional transformation. Modulo this action, the upper half plane becomes an open Riemann surface possibly with pinched points. It can be compactified by adding $\mathbb{P}^1(\mathbb{Q})$ modulo the action of $\Gamma_0(M)$, which is a finite set. We call points in this set *cusps*.

Given a function $f(z)$ on $\mathcal{H}$ and a integer k, the group $\mathrm{GL}_2(\mathbb{R})$ acts on $f(z)$ by the weight k slash operator defined by

$$(f \mid_k \gamma) := (\det \gamma)^{k/2} (cz + d)^{-k} f(\gamma z),$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$. For a Dirichlet character $\chi : (\mathbb{Z}/M\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$, one could view it as a character of $\Gamma_0(M)$ via $\chi(\gamma) := \chi(d)$. Now, we can define modular forms of integral weight.

Definition 1.2.1. *Let $k \geq 0$ be an integer, M a positive integer and $\chi : (\mathbb{Z}/M\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$ a character such that $\chi(-1) = (-1)^k$. Then a function $f(z)$ on the upper half plane is called a* modular form *of level M and nebetypus character χ if it satisfies the following three conditions*

(1) *$f(z)$ is holomorphic.*

(2) *The following equation holds for all $\gamma = \left(\begin{smallmatrix} a & b \\ Mc & d \end{smallmatrix}\right) \in \Gamma_0(M)$*

$$(f \mid_k \gamma)(z) = \chi(d) f(z).$$

(3) *$f(z)$ does not have poles on $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$.*

We denote the $\mathbb{C}$-vector space of function by $M_k(M, \chi)$. Suppose $f(z)$ satisfies the following stronger version of (3)

(3)' $f(z)$ does not have poles on $\mathcal{H}$ and <u>vanishes</u> at all cusps,

then $f(z)$ is called a *cuspform* and the $\mathbb{C}$-vector space of all such functions is denoted by $S_k(M, \chi)$.

1.2.2 Half-Integral Weight Modular Forms

When the weight k above is a half-integer, then the weight k slash operator is not well-defined for $\mathrm{GL}_2^+(\mathbb{R})$. Different branch choices are the reasons for this problem. To overcome it, we define the group $\tilde{G}$, which is a four-sheeted cover of $\mathrm{GL}_2^+(\mathbb{R})$, by

$$\tilde{G} := \left\{ (\alpha, \phi(z)) \mid \alpha = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R}), \phi(z) : \mathcal{H} \longrightarrow \mathbb{C} \text{ holomorphic }, \phi(z)^2 = \pm \tfrac{cz+d}{\sqrt{\det \alpha}} \right\}.$$

The group law is defined by

$$(\alpha, \phi(z)) \cdot (\beta, \psi(z)) = (\alpha\beta, \phi(\beta z)\psi(z)).$$

For any element $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(4)$, define the automorphy factor $j(\gamma, z)$ by

$$j(\gamma, z) := \left(\tfrac{c}{d}\right) \epsilon_d^{-1} \sqrt{cz + d},$$

where $\epsilon_d = 1$, resp. i, if d is congruent to 1, resp. 3, modulo 4. Then there is a natural copy of $\Gamma_0(4)$ in $\tilde{G}$ via $\gamma \mapsto \tilde{\gamma} = (\gamma, j(\gamma, z))$. We will denote the image of any congruence subgroup $\Gamma_0(4M)$ in $\tilde{G}$ by $\tilde{\Gamma}_0(4M)$.

Let k be an integer. For a function $f(z)$ defined on $\mathcal{H}$, $(\alpha, \phi(z)) \in \tilde{G}$ acts on it with weight $\frac{k}{2}$ by

$$f\mid_{k/2}(\alpha, \phi(z)) := \phi(z)^{-k} f(\alpha z).$$

With the half-integral weight slash operator defined, we can give the definition of modular forms of half-integral weight.

Definition 1.2.2. *Let $k \geq 1$ be an integer, M an integer and $\chi : (\mathbb{Z}/4\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ a character. A function $f(z)$ on $\mathcal{H}$ is called a modular form of weight $\frac{k}{2}$, level $4M$ and nebentypus character χ if the following conditions are satisfied.*

(1) *$f(z)$ is holomorphic.*

(2) *The following equation holds for all $(\alpha, \phi(z)) \in \tilde{\Gamma}_0(4M)$*

$$(f\mid_{k/2}(\alpha, \phi(z)))(z) = \chi(\alpha) f(z).$$

(3) *$f(z)$ does not have poles on $\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$.*

We denote the $\mathbb{C}$ vector space of such functions by $M_{k/2}(4M, \chi)$. Similarly, we call $f(z)$ a cuspform *of weight $\frac{k}{2}$, level $4M$ and character χ if condition (3) is replaced by condition (3)' above, and denote the space of cuspforms by $S_{k/2}(4M, \chi)$.*

1.2.3 Mock-modular forms

In this section, we will give some background information on mock-modular forms. Let $k \in \mathbb{Z}$, $M \in \mathbb{Z}^+$ and $\chi : (\mathbb{Z}/M\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be as before. Denote by $\mathcal{F}_k(M, \chi)$ the space of smooth functions $f : \mathcal{H} \to \mathbb{C}$ such that

$$(f\mid_k \gamma)(z) = \chi(\gamma) f(z)$$

for all $\gamma \in \Gamma_0(M)$. Recall from Eq. (1.1.1) the differential operator ξ_k and the weight k hyperbolic Laplacian Δ_k. If $z = x + iy$, then Δ_k can be written as

$$\Delta_k = y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) - iky \left(\frac{\partial}{\partial x} + i \frac{\partial}{\partial y} \right).$$

We say that $f(z) \in \mathcal{F}_k(M, \chi)$ is a *harmonic weak Maass form* of weight k, level M and character χ (or more briefly, a weakly harmonic form) if it satisfies the following properties.

(i) $f(z)$ is real-analytic.

(ii) $\Delta_k(f) = 0$.

(iii) The function $f(z)$ has at most linear exponential growth at all cusps of $\Gamma_0(M)$.

Let $H_k(M, \chi)$ be the space of weakly harmonic forms of weight k, level M and character χ, whose image under ξ_k is a holomorphic modular form. Denote by $M_k^!(M, \chi)$ the usual subspaces of weakly holomorphic modular forms. It also contains $M_k(M, \chi)$ and $S_k(M, \chi)$ as subspaces as well. A *mock-modular form* is a formal Laurent series in q,

$$\tilde{g}(z) = \sum_{n \gg -\infty} c^+(n) q^n,$$

such that there exists $g(z) = \sum_{n \geq 0} c(n) q^n \in M_{2-k}(M, \overline{\chi})$ satisfying

$$\sum_{n \gg -\infty} c^+(n) q^n - \sum_{n \geq 0} \overline{c(n)} \beta_k(n, y) q^{-n} \in H_k(M, \chi).$$

The form $g(z)$ is called the *shadow* of $\tilde{g}(z)$. The expression $\sum_{n<0} c^+(n) q^n$ is called the *principal part* of $\tilde{g}(z)$. Let $\mathbb{M}_k(M, \chi)$ be the subspace of mock-modular forms whose shadows are in $M_{2-k}(M, \overline{\chi})$. Since every weakly harmonic form can be written uniquely as the sum of a holomorphic part and a non-holomorphic part, the spaces $H_k(M, \chi)$ and $\mathbb{M}_k(M, \chi)$ are canonically isomorphic to each other.

With some computations, one could verify that ξ_k commutes with the slash operator as follows

$$\xi_k \left(f \mid_k \gamma \right) = \left(\xi_k f \right) \mid_{2-k} \gamma \tag{1.2.1}$$

for all $\gamma \in \mathrm{GL}_2(\mathbb{R})$. Property (ii) and Eq. (1.1.1) then gives the following map

$$\xi_k : H_k(M, \chi) \longrightarrow M_{2-k}(M, \overline{\chi}),$$

whose kernel is exactly $M_k^!(M, \chi)$. When $k \neq 1$, the map above is surjective as shown in [10] and [12]. When $k = 1$, one can still prove surjectivity by analytically continuing the weight one Poincaré series, the same family as in [10] for $k = 1$, via spectral expansion. We will carry this out in §2.1.

CHAPTER 2

Preliminary Results

2.1 Existence of Mock-Modular Forms of Weight One

Given a cusp form of weight one, we will show the existence of a weight one mock-modular forms with it as shadow in this section. There are several different approaches to proving this result, such as a geometric approach in [12], or another approach found by Zwegers using the holomorphic projection trick. For completeness and since it might have some independent interest, we prove the existence by analytically continuing weight one Poincaré series via the spectral expansion.

From now on we fix $k = 1$ and write $|$ for the slash operator $|_1$. The notations $H_1(M, \chi)$, $M_1^!(M, \chi)$, $M_1(M, \chi)$ and $S_1(M, \chi)$ are the same as in §1.2. Let $\mathbb{M}_1(M, \chi)$ be the space of mock-modular forms, which is canonically isomorphic to $H_1(M, \chi)$.

To proceed, we will construct two families of Poincaré series $P_m(z, s), Q_m(z, s)$, where the first family is similar to the one used in [10]. They are a priori defined for $\mathrm{Re}(s) > 1$ and will be analytically continued to $\mathrm{Re}(s) > 0$ through their spectral expansions. Unlike the cases $k \geq 2$, this will only be a statement about existence, and not a formula that could be used to calculate the Fourier coefficients of the preimage explicitly. To prove the analytic continuation, we will refer to results in [42] and [44].

Given any positive integer m, define

$$\phi_m^*(z, s) := e^{2\pi i m x}(4\pi|m|y)^{-1/2} M_{\frac{\mathrm{sgn}(m)}{2}, s-\frac{1}{2}}(4\pi|m|y), \tag{2.1.1}$$

$$\varphi_m^*(z, s) := e^{2\pi i m x}(4\pi|m|y)^{s-1/2} e^{-2\pi|m|y}. \tag{2.1.2}$$

Here $M_{\mu,\nu}(y)$ is the M-Whittaker function defined by

$$M_{\mu,\nu}(z) := z^{\nu+1/2}e^{z/2}{}_1F_1\left(\tfrac{1}{2}+\mu+\nu; 1+2\nu; -z\right), \tag{2.1.3}$$

with ${}_1F_1(\alpha;\beta;z)$ being the generalized hypergeometric function. Averaging them over the coset representatives of $\Gamma_\infty\backslash\Gamma_0(M)$, we can define the following Poincaré series

$$\begin{aligned} P_m(z,s,\chi) &:= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)} \overline{\chi(\gamma)}(\phi_m^* \mid \gamma)(z,s), \\ Q_m(z,s,\chi) &:= \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(M)} \overline{\chi(\gamma)}(\varphi_m^* \mid \gamma)(z,s). \end{aligned}$$

For convenience, we shall omit χ and write $P_m(z,s)$ and $Q_m(z,s)$ instead. Both families are absolutely convergent for $\mathrm{Re}(s)>1$ and define a holomorphic function in s. Also, $P_m(z,s)$ is an eigenfunction of $-\Delta_1$ with eigenvalue $(s-1/2)((1-s)-1/2)$ and belongs to $\mathcal{F}_1(M,\chi)$.

Let Δ_k' be the differential operator defined by

$$-\Delta_k' := -y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)+iky\frac{\partial}{\partial x}.$$

It is related to Δ_k by the following equation

$$\Delta_k' + \frac{k}{2}\left(1-\frac{k}{2}\right) = y^{k/2}\Delta_k y^{-k/2}.$$

Define the space $\mathcal{D}_1(M,\chi)$ by

$$\mathcal{D}_1(M,\chi) := \{y^{1/2}f(z) : f(z)\in\mathcal{F}_1(M,\chi) \text{ is smooth with compact support on } \mathcal{H}\} \tag{2.1.4}$$

Let $\tilde{\mathcal{D}}_1(M,\chi)$ be the completion of $\mathcal{D}_1(M,\chi)$ with respect to the Petersson norm

$$||g||^2 = \langle g,g\rangle := \int_{\Gamma_0(M)\backslash\mathcal{H}} |g(z)|^2\frac{dxdy}{y^2}.$$

Satz 3.2 in [44] implies that $-\Delta_k'$ has a self-adjoint extension $-\tilde{\Delta}_k'$ from $\mathcal{D}_1(M,\chi)$ to $\tilde{\mathcal{D}}_1(M,\chi)$. Also, $-\tilde{\Delta}_k'$ has a countable system of Maass cusp forms $\{e_n(z)\}_{n\in\mathbb{N}}\subset\tilde{\mathcal{D}}_1(M,\chi)$ forming the discrete spectrum. Each Maass cusp form $e_n(z)$ has eigenvalue λ_n and each eigenvalue has finite multiplicity. For any $f\in\tilde{\mathcal{D}}_1(M,\chi)$, the discrete spectrum contributes the following sum to its spectral expansion

$$\sum_{n=1}^{\infty}\langle f,e_n\rangle e_n(z),$$

which converges absolutely and uniformly for all $z \in \mathcal{H}$ [44, Satz 8.1].

Let ι be a cusp of $\Gamma_0(M)$, $\sigma_\iota \in \mathrm{GL}_2(\mathbb{R})$ the scaling matrix sending ∞ to ι, and $\Gamma_\iota \subset \Gamma_0(M)$ the subgroup fixing ι. One can define the weight one real analytic Eisenstein series $E_\iota(z,s)$ by

$$E_\iota(z,s) := \tfrac{y^{1/2}}{2} \sum_{\gamma \in \Gamma_\iota \backslash \Gamma_0(M)} \tfrac{\chi(\gamma)}{cz+d} (\mathrm{Im}(\sigma_\iota^{-1}\gamma z))^{s-1/2}.$$

Selberg showed that the Eisenstein series has meromorphic continuation to the whole complex plane in s. For ι ranging over the cusps of $\Gamma_0(M)$, the Eisenstein series $\{E_\iota(z,s)\}_\iota$ make up the continuous spectrum of $-\tilde{\Delta}_1'$. So for any $f \in \tilde{\mathcal{D}}_1(M,\chi)$, the contribution of the Eisenstein series to the spectral expansion of f is

$$\frac{1}{4\pi} \sum_\iota \int_{-\infty}^{\infty} \langle f(\cdot), E_\iota(\cdot, \tfrac{1}{2}+ir)\rangle E_\iota(z, \tfrac{1}{2}+ir) dr.$$

If f is smooth, then the integral in r converges absolutely and uniformly for z in any fixed compact subset of $\mathcal{H}$ [44, Satz 12.3]. Applying the completeness theorem [44, Satz 7.2], we have the spectral expansion for any smooth $f \in \tilde{\mathcal{D}}_1(M,\chi)$ in the form

$$f(z) = \sum_{n=1}^{\infty} \langle f, e_n\rangle e_n(z) + \sum_\iota \frac{1}{4\pi} \int_{-\infty}^{\infty} \langle f(\cdot), E_\iota(\cdot, \tfrac{1}{2}+ir)\rangle E_\iota(z, \tfrac{1}{2}+ir) dr, \tag{2.1.5}$$

where the sum over $n \in \mathbb{N}$ and the integral in r both converge uniformly and absolutely for z in any compact set of $\mathcal{H}$.

By setting $f(z) = y^{1/2} Q_m(z,s)$ and comparing $P_m(z,s)$ to $Q_m(z,s)$ as in [42], we can deduce the following proposition.

Proposition 2.1.1. *For any positive integer m, the Poincaré series $Q_m(z,s)$ and $P_m(z,s)$ have analytic continuation to Re$(s) > 0$. At $s = 1/2$, $P_m(z,s)$ has at most a simple pole. Furthermore, the residues of $P_m(z,s)$ at $s = 1/2$ generate $S_1(M,\chi)$.*

Proof. First, we will prove the analytic continuation $Q_m(z,s)$. For any $m > 0$, the function $\tilde{Q}_m(z,s) := y^{1/2} Q_m(z,s)$ is square integrable for $\mathrm{Re}(s) > 1$, hence contained in $\tilde{\mathcal{D}}_1(M,\chi)$.

So we can write out its spectral expansion

$$\begin{aligned}\tilde{Q}_m(z,s) &= D(z,s)+C(z,s), \qquad (2.1.6)\\ D(z,s) &:= \sum_{n=1}^{\infty}\langle \tilde{Q}_m(\cdot,s), e_n(\cdot)\rangle e_n(z),\\ C(z,s) &:= \tfrac{1}{4\pi}\sum_{\iota}\int_{-\infty}^{\infty}\langle \tilde{Q}_m(\cdot,s), E_\iota(\cdot,\tfrac{1}{2}+ir)\rangle E_\iota(z,\tfrac{1}{2}+ir)dr,\end{aligned}$$

where $D(z,s)$ and $C(z,s)$ are the contributions from the dicrete and continuous spectrum of $-\tilde{\Delta}_1'$ respectively.

By Satz 5.2 and Satz 5.5 in [44], $e_n(z)$ has eigenvalue $\lambda_n\in[1/4,\infty)$ under $-\tilde{\Delta}_1'$. If $\lambda_n=1/4$, then $y^{-1/2}e_n(z)$ is in $S_1(M,\chi)$. Since each eigenvalue has finite multiplicity, we can let $N_0\geq 0$ such that $\{y^{-1/2}e_n(z): 1\leq n\leq N_0\}$ is an orthonormal basis of $S_1(M,\chi)$. Note that $S_1(M,\chi)$ could be empty, in which case $N_0=0$.

For $n\in\mathbb{N}$, let $t_n=\sqrt{\lambda_n-1/4}$, $s_n=1/2+it_n$. We can use Eq. (66) in [27], the asymptotic $M_{\mu,\nu}(y)=O_{\mu,\nu}(e^y)$ as $y\to\infty$ and the vanishing property of cusp forms at all cusps to write

$$e_n(z)=\sum_{u=-\infty,u\neq 0}^{\infty} c_{n,u}W_{\frac{\mathrm{sgn}(u)}{2},s_n-\frac{1}{2}}(4\pi|u|y)e^{2\pi iux},$$

where $W_{\mu,\nu}(z)$ is the W-Whittaker function and $c_{n,u}$ are constants. Note if $t_n=0$, i.e. $y^{-1/2}e_n(z)$ is a holomorphic cusp form, then $W_{\mathrm{sgn}(u)/2,s_n-1/2}(4\pi|u|y)=(4\pi|u|y)^{1/2}e^{-2\pi|u|y}$ and $c_{n,u}=0$ for $u\leq 0$. Now we can use the Rankin-Selberg unfolding trick to calculate $\langle\tilde{Q}_m(\cdot,s),e_n(\cdot)\rangle$

$$\begin{aligned}\langle\tilde{Q}_m(\cdot,s),e_n(\cdot)\rangle &= \int_{\Gamma_0(M)\backslash\mathcal{H}} y^{1/2}Q_m(z,s)\overline{e_n(z)}\frac{dxdy}{y^2}\\ &= \int_0^\infty\int_0^1 y^{1/2}\varphi_m^*(z,s)\overline{e_n(z)}\frac{dxdy}{y^2}\\ &= (4\pi|m|)^{1/2}\int_0^\infty e^{-2\pi|m|y}(4\pi|m|y)^{s-1}\overline{c_{n,m}W_{\frac{\mathrm{sgn}(m)}{2},s_n-\frac{1}{2}}(4\pi|m|y)}\frac{dy}{y}\\ &= (4\pi|m|)^{1/2}\overline{c_{n,m}}\frac{\Gamma\left(s-\frac{1}{2}-it_n\right)\Gamma\left(s-\frac{1}{2}+it_n\right)}{\Gamma\left(s-\frac{\mathrm{sgn}(m)}{2}\right)} \qquad (2.1.7)\end{aligned}$$

The last step uses the Mellin transform of the W-Whittaker function [5, Eq. (8b)] and the substitution $s_n=1/2+it_n$. When $\mathrm{Re}(s)>1$, the sum defining $D(z,s)$ is absolutely convergent [44, Satz 8.1], since $y^{1/2}Q_m(z,s) \in \tilde{\mathcal{D}}_1(M,\chi)$ for $\mathrm{Re}(s)>1$. When $0<\mathrm{Re}(s)\le 1$, we can write $D(z,s)$ as

$$D(z,s) = (4\pi|m|)^{1/2}\sum_{n\in\mathbb{N}} \overline{c_{n,m}} \frac{\Gamma\left(s+1-\frac{1}{2}-it_n\right)\Gamma\left(s+1-\frac{1}{2}+it_n\right)}{\Gamma\left(s+1-\frac{\mathrm{sgn}(m)}{2}\right)} e_n(z)\cdot \frac{s-\frac{\mathrm{sgn}(m)}{2}}{(s-\frac{1}{2})^2+t_n^2}$$

Since $t_n^2 = \lambda_n - 1/4$ and $\sum_{n>M}\lambda_n^{-2}$ converges [44, Satz 8.1], we can apply Cauchy-Schwarz inequality to see that the sum on the right hand side above converges absolutely on compact subsets of $\{s\in\mathbb{C} : \mathrm{Re}(s)>0, s\neq 1/2\}$. At $s=1/2$, the first N_0 terms in the sum produce a simple pole since $t_n=0$ for all $1\le n\le N_0$. The rest of the sum still converges absolutely. So the right hand side above gives the analytic continuation of $D(z,s)$ to $\mathrm{Re}(s)>0$.

For the continuous spectrum, the contribution from the Eisenstein series on the right hand side of (2.1.6) can be treated similarly. For any cusp ι, we can write the Fourier expansion of $E_\iota(z,s)$ at infinity in the following form (for $\iota=\infty$, see [27, eq. (76)']])

$$E_\iota(z,s) = y^s + \psi_\iota(s)y^{1-s} + \sum_{m\neq 0}\psi_{\iota,m}(s) W_{\frac{\mathrm{sgn}(m)}{2},s-\frac{1}{2}}(4\pi|m|y)e^{2\pi imx}, \tag{2.1.8}$$

where $\psi_\iota(s)$ and $\psi_{\iota,m}(s)$ are products of gamma factors and Selberg-Kloosterman zeta functions. It is well-known that The Eisenstein series can be analytically continued in s to the whole complex plane. When $\mathrm{Re}(s)>1/2$, the poles of $E_\iota(z,s)$ are in the interval $s\in(1/2,1]$ [44, Satz 10.3]. On the line $\mathrm{Re}(s)=1/2$, $E_\iota(z_0,s)$ is holomorphic in s for any fixed $z_0\in\mathcal{H}$ [44, Satz 10.4]. So both $\psi_\iota(s)$ and $\psi_{\iota,m}(s)$ admit analytic continuation to $\mathrm{Re}(s)>0$ and are holomorphic on $\mathrm{Re}(s)=1/2$. Using the same unfolding trick above, we can evaluate

$$\langle \tilde{Q}_m(\cdot,s), E_\iota(\cdot,\tfrac{1}{2}+ir)\rangle = \overline{\psi_{\iota,m}\left(\tfrac{1}{2}+ir\right)}(4\pi|m|)^{1/2}\frac{\Gamma\left(s-\frac{1}{2}-ir\right)\Gamma\left(s-\frac{1}{2}+ir\right)}{\Gamma\left(s-\frac{\mathrm{sgn}(m)}{2}\right)}.$$

Then $C(z,s)$ can be written as

$$\begin{aligned} C(z,s) &= \sum_\iota \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\psi_{\iota,m}\left(\tfrac{1}{2}+ir\right)}(4\pi|m|)^{1/2}\frac{\Gamma\left(s-\frac{1}{2}-ir\right)\Gamma\left(s-\frac{1}{2}+ir\right)}{\Gamma\left(s-\frac{\mathrm{sgn}(m)}{2}\right)}E_\iota(z,\tfrac{1}{2}+ir)dr \\ &= \sum_\iota \frac{1}{4\pi}\int_{-\infty}^{\infty} \overline{\psi_{\iota,m}\left(\tfrac{1}{2}+ir\right)}(4\pi|m|)^{1/2}\frac{\Gamma\left(s+1-\frac{1}{2}-ir\right)\Gamma\left(s+1-\frac{1}{2}+ir\right)}{\Gamma\left(s+1-\frac{\mathrm{sgn}(m)}{2}\right)} \\ &\qquad \cdot\frac{(s-\frac{\mathrm{sgn}(m)}{2})}{(s-\frac{1}{2}-ir)(s-\frac{1}{2}+ir)}\cdot E_\iota(z,\tfrac{1}{2}+ir)dr. \end{aligned} \tag{2.1.9}$$

When $\mathrm{Re}(s) > 0$, $C(z, s+1)$ is absolutely convergent. So when $\mathrm{Re}(s) \neq 1/2$, we can apply Cauchy-Schwarz to bound expression (2.1.9) by

$$\int_{-\infty}^{\infty} \left| \frac{s - \frac{\mathrm{sgn}(m)}{2}}{(s-\frac{1}{2})^2 + r^2} \right| dr. \tag{2.1.10}$$

When $\mathrm{Re}(s) > 1/2$, let $\tilde{C}(z, s)$ be the expression (2.1.9), which gives the analytic continuation of $C(z, s)$ in this region. When $\mathrm{Re}(s) < 1/2$, we can define $\tilde{C}(z, s)$ in a similar fashion as in §6 of [42]. We start with (2.1.9) and $\mathrm{Re}(s) \in (1/2, 1/2 + \epsilon)$ for some small $\epsilon > 0$, then deform the contour so that it goes above $\frac{1/2-s}{i}$ and below $\frac{s-1/2}{i}$. In the process, the following residue is picked up

$$\frac{(\pi|m|)^{1/2}\Gamma(2s-1)}{\Gamma\left(s - \frac{\mathrm{sgn}(m)}{2}\right)} \sum_{\iota} \overline{\psi_{\iota,m}(s)} E_{\iota}(z, s) + \overline{\psi_{\iota,m}(1-s)} E_{\iota}(z, 1-s). \tag{2.1.11}$$

Finally we can reduce the real part of s to less than $1/2$ and change the contour back to the real line. So when $\mathrm{Re}(s) < 1/2$, let $\tilde{C}(z, s)$ be the sum of (2.1.11) and (2.1.9), which are holomorphic. The bound (2.1.10) and $m \geq 1$ guarantees the existence of the limits of $\tilde{C}(z, s)$ as $\mathrm{Re}(s)$ approaches $1/2$ from the left and from the right. The procedure defining $\tilde{C}(z, s)$ shows that the limits agree. So we define $\tilde{C}(z, s)$ on $\mathrm{Re}(s) = 1/2$ to be this limit. Then $\tilde{C}(z, s)$ gives the analytic continuation of $C(z, s)$ to $\mathrm{Re}(s) > 0$. Putting these together with the analytic continuation of $D(z, s)$, we have the analytic continuation of $Q_m(z, s)$ to $\mathrm{Re}(s) > 0$.

Now to analytically continue $P_m(z, s)$, we can simply compare it to $Q_m(z, s)$. Applying the power series expansions of ${}_1F_1(\alpha; \beta; z)$ and the exponential map to (2.1.1) and (2.1.2), we see that for y small and $0 < \mathrm{Re}(s) < 2$,

$$|(4\pi|m|y)^{-s+1/2}(\phi_m^*(z, s) - \varphi_m^*(z, s))| = O_m(y).$$

So the difference $P_m(z, s) - Q_m(z, s)$ defined by termwise subtraction is a holomorphic function in s for $\mathrm{Re}(s) > 0$. That means the analytic continuation of $P_m(z, s)$ to $\mathrm{Re}(s) > 0$ follows from that of $Q_m(z, s)$. Furthermore, they have the same poles and residues in the region $\mathrm{Re}(s) > 0$. Thus, to prove the second half of the proposition, it suffices to analyze the poles and residues of $Q_m(z, s)$ at $s = 1/2$.

Since $m > 0$, expression (2.1.10) is bounded by an absolute constant for all $s \in (1/2, 2]$. So $C(z,s)$ does not contribute to the pole at $s = 1/2$. For a Maass cusp form $e_n(z)$, the right hand side of (2.1.7) vanishes at $s = 1/2$ if $t_n \neq 0$. Otherwise, $y^{-1/2}e_n(z) \in S_1(M,\chi)$ and $\langle \tilde{Q}_m(\cdot, s), e_n(\cdot)\rangle$ has a simple pole of residue $(4\pi|m|)^{1/2}\overline{c_{n,m}}$. Thus, we have

$$\mathrm{Res}_{s=1/2} P_m(z,s) = \mathrm{Res}_{s=1/2} Q_m(z,s) = (4\pi|m|)^{1/2} \sum_{n=1}^{N_0} \overline{c_{n,m}} y^{-1/2} e_n(z). \tag{2.1.12}$$

Since $\{y^{-1/2}e_n(z) : n = 1, \ldots, N_0\}$ is a basis of $S_1(M,\chi)$, the matrix

$$\{\overline{c_{n,m}} : 1 \leq n \leq N_0, 1 \leq m \leq K\}$$

has rank equals to M for K sufficiently large. Therefore the residues at $s = 1/2$ of $P_m(z,s)$ generate $S_1(M,\chi)$. □

The following theorem is an immediate consequence of the proposition above.

Theorem 2.1.2. *Using the notations above, the following map is a surjection*

$$\xi_1 : H_1(M, \overline{\chi}) \to S_1(M,\chi),$$

i.e. for any cusp form $h(z) \in S_1(M,\chi)$, there exists $\tilde{h}(z) \in \mathbb{M}_1(M,\overline{\chi})$ with shadow $h(z)$.

Proof. When $k = 1$, Eq. (1.1.1) becomes $\Delta_1 = \xi_1 \circ \xi_1$. So the Poincaré series $P_m(z,s)$ satisfies

$$\Delta_1(P_m(z,s)) = \left(s - \tfrac{1}{2}\right)^2 P_m(z,s) \tag{2.1.13}$$

when $\mathrm{Re}(s) > 1$. Since the difference between both sides is holomorphic in s and $P_m(z,s)$ can be analytically continued to $\mathrm{Re}(s) > 0$ as in the proposition, Eq. (2.1.13) is valid for $\mathrm{Re}(s) > 0$. At $s = 1/2$, suppose $P_m(z,s)$ has the following Taylor series expansion in s

$$P_m(z,s) = g_{-1}(z)\left(s - \tfrac{1}{2}\right)^{-1} + g_0(z) + g_1(z)\left(s - \tfrac{1}{2}\right) + O_z\left(\left(s - \tfrac{1}{2}\right)^2\right), \tag{2.1.14}$$

with $g_j(z) \in \mathcal{F}_1(M,\chi)$ real-analytic for $j = -1, 0, 1$. Since ξ_1 commutes with the slash operator and $\Delta_1(g_1(z)) = \xi_1^2(g_1(z)) = g_{-1}(z)$, we have $\xi_1(g_1(z)) \in \mathcal{F}_1(M,\overline{\chi})$ is real-analytic and a preimage of $g_{-1}(z)$ under ξ_1.

By considering the Fourier expansion of $g_1(z)$, we know that $g_1(z)$ has at most linear exponential growth near the cusps implies $\xi_1(g_1(z))$ has the same property. Suppose $Q_m(z,s)$ has the Laurent expansion

$$g_{-1}(z)\left(s-\tfrac{1}{2}\right)^{-1}+f_0(z)+f_1(z)\left(s-\tfrac{1}{2}\right)+O_z\left(\left(s-\tfrac{1}{2}\right)^2\right)$$

near $s=1/2$. Then from the spectral expansion of $Q_m(z,s)$, i.e. $y^{-1/2}(D(z,s)+C(z,s))$ as in (2.1.6), it is not hard to see that $f_1(z)$ has at most linear exponential growth near the cusps. The difference $P_m(z,s)-Q_m(z,s)$ is a Poincaré series defined for $\mathrm{Re}(s)>0$. So the coefficient of $(s-1/2)$ of its Laurent series expansion around $s=1/2$, say $h_1(z)$, has at most linear exponential growth near the cusps. That means the sum of $f_1(z)$ and $h_1(z)$, which is $g_1(z)$ by analytic continuation, also has this property.

Thus, $\xi_1(g_1(z))$ is in $H_1(M,\overline{\chi})$ and a preimage of $g_{-1}(z)\in S_1(M,\chi)$ under ξ_1. Since the functions $\{\mathrm{Res}_{s=1/2}P_m(z,s) : m\geq 1\}$ span the space $S_1(M,\chi)$, the map $\xi_1 : H_1(M,\overline{\chi})\to S_1(M,\chi)$ is surjective. □

2.2 Principal Part Coefficients of Harmonic Maass Forms

Let M be an odd, square-free positive integer and $\chi:(\mathbb{Z}/M\mathbb{Z})^\times\longrightarrow\mathbb{C}^\times$ a character. In this section, we will relate the regularized inner products between $g(z)\in S_1(M,\chi)$ and $f(z)\in M_1^!(M,\chi)$ to linear combinations of coefficients of a mock-modular form $\tilde{g}(z)$, whose shadow is $g(z)$, via Stokes' theorem. The regularization technique is standard and has been used in many places before (see for example [8, 11, 12, 14, 23]).

The usual Petersson inner product $\langle f,g\rangle$ can be regularized as follows. Since M is square-free, $\Gamma_0(M)$ has $2^{\omega(M)}$ inequivalent cusps

$$\{\iota_d : d\mid M\},$$

with ι_1 equivalent to the cusp infinity. Here, $\omega(M)$ is the number of distinct prime divisors of M. The cusp ι_d is related to the cusp infinity by the matrix $\sigma_d=\left(\begin{smallmatrix}\sqrt{d}\alpha_d & \beta_d/\sqrt{d}\\ M/\sqrt{d} & \sqrt{d}\end{smallmatrix}\right)$ in $\mathrm{SL}_2(\mathbb{R})$. Take a fundamental domain of $\Gamma_0(M)\backslash\mathcal{H}$, cut off the portion with $\mathrm{Im}(z)>Y$ for a large

Y and intersect it with its translate under σ_d for all $d \mid M$. We will call this the truncated fundamental domain $\mathcal{F}(Y)$. Now, define the regularized inner product by

$$\langle f, g \rangle_{\text{reg}} := \lim_{Y \to \infty} \int_{\mathcal{F}(Y)} f(z)\overline{g(z)} y \, \frac{dxdy}{y^2}. \tag{2.2.1}$$

If $f(z) \in M_1(M, \chi)$, then this is the usual Petersson inner product. Now let $\hat{g} \in H_1(M, \overline{\chi})$ be a preimage of $g(z)$ under ξ_1 with the following Fourier expansions at each cusp ι_d

$$(\hat{g}|_1\sigma_d)(z) = \sum_{n \in \mathbb{Z}} c_d^+(n) q^n - \sum_{n \geq 1} \overline{c(g|_1\sigma_d, n)} \beta_1(n, y) q^{-n},$$

The expression for $(\hat{g}|_1\sigma_d)(z)$ follows from the commutativity between ξ_1 and the slash operator. Note that $\sum_{n \in \mathbb{Z}} c_d^+(n) q^n$ is a mock-modular forms with shadows $(g|_1\sigma_d)(z)$. As a special case of Prop. 3.5 in [12], we can express $\langle f, g \rangle_{\text{reg}}$ in terms of these Fourier coefficients.

Lemma 2.2.1 (See Prop. 3.5 in [12]). *Let $f(z) \in M_1^!(M, \chi)$ and $g(z) \in S_1(M, \chi)$. In the notations above, we have*

$$\langle f, g \rangle_{\text{reg}} = \sum_{n \in \mathbb{Z}} \sum_{d \mid M} c_d^+(n) c_d(f, -n). \tag{2.2.2}$$

Notice that the right hand side of equation (2.2.2) depends on the choice of $\hat{g}$, whereas the left hand side only depends on $g(z)$. So if we replace $\hat{g}$ with $h(z) \in M_1^!(M, \overline{\chi})$, then Lemma 2.2.1 still holds and we obtain

$$0 = \sum_{n \in \mathbb{Z}} \sum_{d \mid M} c_d(h, n) c_d(f, -n)$$

where $h(z)$ has Fourier expansions $\sum_{n \in \mathbb{Z}} c_d(h, n) q^n$ at the cusp ι_d. So the right hand side of Eq. (2.2.2) gives a pairing between $f \in M_1^!(M, \chi)$ and $G \in H_1(M, \overline{\chi})/M_1^!(M, \overline{\chi})$ defined by

$$\{f, G\} := \langle f, \xi_1(G) \rangle_{\text{reg}} = \sum_{d \mid M} \text{Const}((f \mid_1 \sigma_d) \cdot (G \mid_1 \sigma_d)). \tag{2.2.3}$$

This is in fact a perfect pairing when one restricts to $f \in S_1(M, \chi)$ as a consequence of Serre duality (see [9, §3]). In that case, the first sum is only over $n < 0$ and we obtain relations among the principal part coefficients of $h(z)$ at various cusps. We remark that this holds for other weights as well. So given some Fourier coefficients, we know that they are the principal part coefficients of a weakly holomorphic modular form in $M_k^!(M, \chi)$ if and only if its pairing with cusp forms in $S_{2-k}(M, \overline{\chi})$ vanishes.

Proposition 2.2.2. *Let $\{c_d(-n) \in \mathbb{C} : d \mid M,\ 1 \le n \le n_0\}$ be a set of complex numbers. Then there exists $f \in M_1^!(M, \chi)$ such that*

$$f \mid_1 \sigma_d = \sum_{n=1}^{n_0} c_d(-n) q^{-n} + O(1)$$

for each $d \mid M$ if and only if

$$\sum_{d \mid M} \sum_{n=1}^{n_0} c_d(-n) c_d(h, n) = 0$$

for all $h(z) \in S_1(M, \overline{\chi})$ with $h \mid_1 \sigma_d = \sum_{n \ge 1} c_d(h, n) q^n$.

2.3 Transformation Calculations

In this section, we will give some results on transformations of modular forms under different operators. These will be useful in calculating the Fourier expansion of certain modular forms in Chapter 3.

2.3.1 Commutation Lemmas

First, we will state some general results about the commutations between the U-operator and different Atkin-Lehner involutions. In this section, $M > 0$ will be an odd integer and $\Phi(z)$ will be a real-analytic modular form of level $4M$, weight $k/2$ and character χ for some $k \in \mathbb{Z}$ and character $\chi : (\mathbb{Z}/4M)^\times \longrightarrow \mathbb{C}^\times$. For $p \mid M$ a prime, let $\tilde{U}_p$ be the U-operator defined by

$$\tilde{U}_p := \sum_{\lambda=0}^{p-1} \left[\left(\begin{smallmatrix} 1 & \lambda \\ & p \end{smallmatrix} \right), p^{1/4} \right]. \tag{2.3.1}$$

For $d \mid 4M$ a positive integer such that $\gcd(d, M/d) = 1$, write $\chi = \chi_d \chi_{4M/d}$ and let $\tilde{W}_d$ be the Atkin-Lehner involution defined by

$$\begin{aligned} \tilde{W}_d &:= \left[\left(\begin{smallmatrix} dr & s \\ 4Mt & du \end{smallmatrix} \right), d^{-1/4} \left(\frac{Mt/d}{u} \right) \sqrt{4Mtz + du} \right] \\ &= \widetilde{\left(\begin{smallmatrix} r & s \\ 4Mt/d & du \end{smallmatrix} \right)} \left[\left(\begin{smallmatrix} d & \\ & 1 \end{smallmatrix} \right), d^{-1/4} \left(\frac{Mt/d}{d} \right) \epsilon_{du} \right] \\ &= \left[\left(\begin{smallmatrix} 1 & \\ & d \end{smallmatrix} \right), d^{1/4} \epsilon_u \right] \widetilde{\left(\begin{smallmatrix} dr & s \\ 4Mt/d & u \end{smallmatrix} \right)}. \end{aligned} \tag{2.3.2}$$

The following lemma shows the effect of the Atkin-Lehner involution on the character.

Lemma 2.3.1. *In the notations above, let*

$$\Phi'(z) := (\Phi \mid_{k/2} \tilde{W}_d)(z).$$

Then $\Phi'(z)$ is a real-analytic modular form of level $4M$, weight $k/2$ and character χ', where

$$\chi' : (\mathbb{Z}/4M\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$$
$$\alpha \mapsto \left(\tfrac{d}{\alpha}\right)^k \overline{\chi_d(\alpha)}\chi_{4M/d}(\alpha).$$

Proof. It is not hard to see that

$$\tilde{W}_d^{-1} = \left[\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right), d^{1/4}\left(\tfrac{Mt/d}{d}\right)\epsilon_{du}^{-1}\right]\widetilde{\begin{pmatrix} du & -s \\ -4Mt/d & r\end{pmatrix}}.$$

Suppose $\gamma = \left(\begin{smallmatrix}A & B \\ 4MC & D\end{smallmatrix}\right) \in \Gamma_0(4M)$. Then we have

$$\begin{aligned}
\Phi \mid_{k/2} \tilde{W}_d\gamma\tilde{W}_d^{-1} &= \Phi \mid_{k/2} \widetilde{\left(\begin{smallmatrix}r & s \\ 4Mt/d & du\end{smallmatrix}\right)}\left[\left(\begin{smallmatrix}d & \\ & 1\end{smallmatrix}\right), d^{-1/4}\right]\widetilde{\left(\begin{smallmatrix}A & B \\ 4MC & D\end{smallmatrix}\right)}\left[\left(\begin{smallmatrix}1 & \\ & d\end{smallmatrix}\right), d^{1/4}\right]\widetilde{\begin{pmatrix} du & -s \\ -4Mt/d & r\end{pmatrix}} \\
&= \left(\tfrac{d}{D}\right)^k \Phi \mid_{k/2} \widetilde{\left(\begin{smallmatrix}r & s \\ 4Mt/d & du\end{smallmatrix}\right)}\widetilde{\left(\begin{smallmatrix}A & Bd \\ 4MC/d & D\end{smallmatrix}\right)}\widetilde{\begin{pmatrix} du & -s \\ -4Mt/d & r\end{pmatrix}} \\
&= \left(\tfrac{d}{D}\right)^k \Phi \mid_{k/2} \widetilde{\left(\begin{smallmatrix}* & * \\ 4M* & \delta\end{smallmatrix}\right)} \\
&= \left(\tfrac{d}{D}\right)^k \chi(\delta)\Phi \mid_{k/2},
\end{aligned}$$

where $\delta \equiv -4MtsA/d + durD \pmod{4M}$. Since $dur \equiv 1 \pmod{4M/d}$ and $-4Mts/d \equiv 1 \pmod{d}$, we have

$$\chi(\delta) = \chi_{4M/d}(durD)\chi_d(-4MtsA/d) = \chi_{4M/d}(D)\chi_d(A).$$

This implies the lemma as $AD \equiv 1 \pmod{4M}$. ☐

Lemma 2.3.2. *The following quantity is independent of the choice of r, s, t, u for $\tilde{W}_d$*

$$\epsilon_{du}^k \left(\tfrac{t}{d}\right)^k \overline{\chi_d}(t)\overline{\chi_{4M/d}}(u)\Phi \mid_{k/2} \tilde{W}_d. \tag{2.3.3}$$

Proof. For a different choice

$$\tilde{W}_d' = \widetilde{\left(\begin{smallmatrix}r' & s' \\ 4Mt'/d & du'\end{smallmatrix}\right)}\left[\left(\begin{smallmatrix}d & \\ & 1\end{smallmatrix}\right), d^{-1/4}\left(\tfrac{Mt'/d}{d}\right)\epsilon_{du'}\right],$$

we have the following calculations

$$\begin{aligned}\Phi\mid_{k/2}\tilde{W}'_d(\tilde{W}_d)^{-1}&=\left(\frac{\epsilon_{du}}{\epsilon_{du'}}\right)^k\left(\left(\frac{Mt'/d}{d}\right)\left(\frac{Mt/d}{d}\right)\right)^k\Phi\mid_{k/2}\widetilde{\begin{pmatrix} r' & s' \\ 4Mt'/d & du'\end{pmatrix}}\widetilde{\begin{pmatrix} du & -s \\ -4Mt/d & r\end{pmatrix}}\\&=\left(\frac{\epsilon_{du}}{\epsilon_{du'}}\right)^k\left(\frac{t't}{d}\right)^k\chi_{4M/d}(du'r)\chi_d(-4Mt's/d)\Phi\\&=\left(\frac{\epsilon_{du}}{\epsilon_{du'}}\right)^k\left(\frac{t't}{d}\right)^k\overline{\chi_{4M/d}}(u)\overline{\chi_d}(t)\chi_{4M/d}(u')\chi_d(t')\Phi,\end{aligned}$$

which implies Eq. (2.3.3). □

For simplicity, we will take the Atkin-Lehner involution with $t=u=1$ and $r=\alpha, s=\beta$. So from now on,

$$\tilde{W}_d:=\left[\begin{pmatrix} d\alpha & \beta \\ 4M & d\end{pmatrix}, d^{-1/4}\sqrt{4Mz+d}\right]. \tag{2.3.4}$$

Lemma 2.3.3. *In the notations above, suppose $d,d'\mid M$ and $\gcd(d,d')=1$. Then*

$$\Phi\mid_{k/2}\tilde{W}_d\tilde{W}_{d'}=\left(\frac{d}{d'}\right)^k\chi_{d'}(d)\Phi\mid_{k/2}\tilde{W}_{dd'}. \tag{2.3.5}$$

Proof. Since $\gcd(d,d')=1$, some matrix calculations tell us that

$$\begin{aligned}\begin{pmatrix} d\alpha & \beta \\ 4M & d\end{pmatrix}\begin{pmatrix} d'\alpha' & \beta' \\ 4M & d'\end{pmatrix}&=\begin{pmatrix} * & * \\ 4M/d & d\end{pmatrix}\begin{pmatrix} \alpha' & \beta'd \\ 4M/(dd') & d'\end{pmatrix}\begin{pmatrix} dd' & \\ & 1\end{pmatrix}\\&=\begin{pmatrix} 1 & \\ & dd'\end{pmatrix}\begin{pmatrix} * & * \\ 4M/(dd') & 1\end{pmatrix}\begin{pmatrix} d\alpha' & \beta' \\ 4M/d' & 1\end{pmatrix}.\end{aligned}$$

Since $\epsilon_d^{-1}\epsilon_{d'}^{-1}\left(\frac{M/d}{d}\right)\left(\frac{M/(dd')}{dd'}\right)=\epsilon_{dd'}^{-1}\left(\frac{4M\alpha'/d+4M/d'}{4M\beta'+dd'}\right)$, we could set

$$\tilde{W}'_{dd'}:=\widetilde{\begin{pmatrix} * & * \\ 4M/d & d\end{pmatrix}}\widetilde{\begin{pmatrix} \alpha' & \beta'd \\ 4M/(dd') & d'\end{pmatrix}}\left[\begin{pmatrix} dd' & \\ & 1\end{pmatrix}, (dd')^{-1/4}\left(\frac{\epsilon_{dd'}}{\epsilon_d\epsilon_{d'}}\right)\left(\frac{M/d}{d}\right)\left(\frac{M/(dd')}{dd'}\right)\epsilon_{dd'}^{-1}\right],$$

and obtain Eq. (2.3.5) as follows

$$\begin{aligned}\Phi\mid_{k/2}\tilde{W}_d\tilde{W}_{d'}=&\Phi\mid_{k/2}\tilde{W}'_{dd'}\\&=\left(\frac{d'\alpha'+d}{dd'}\right)^k\chi_{dd'}(d'\alpha'+d)\Phi\mid_{k/2}\tilde{W}_{dd'}\\&=\left(\frac{d}{d'}\right)^k\chi_{d'}(d)\Phi\mid_{k/2}\tilde{W}_{dd'}.\end{aligned}$$

This also implies that

$$\Phi\mid_{k/2}\tilde{W}_d\tilde{W}_{d'}=\left(\frac{d}{d'}\right)^k\left(\frac{d'}{d}\right)^k\chi_{d'}(d)\chi_d(d')\Phi\mid_{k/2}\tilde{W}_{d'}\tilde{W}_d. \tag{2.3.6}$$

□

Lemma 2.3.4. *In the notations above, if* $\gcd(p,d)=1$*, then*

$$\Phi \mid_{k/2} \tilde{U}_p \tilde{W}_d = \left(\tfrac{p}{d}\right)^k \chi_d(p) \Phi \mid_{k/2} \tilde{W}_d \tilde{U}_p. \tag{2.3.7}$$

Proof. This again follows from the straightforward calculation

$$\left(\begin{smallmatrix} 1 & \lambda \\ & p \end{smallmatrix}\right) \left(\begin{smallmatrix} d\alpha & \beta \\ 4M & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} d\alpha+4M\lambda & * \\ 4Mp & d\delta_\lambda \end{smallmatrix}\right) \left(\begin{smallmatrix} 1 & (\beta+d\lambda)\beta' \\ & p \end{smallmatrix}\right),$$

with $d\alpha\beta' \equiv 1 \pmod p$ and

$$\delta_\lambda = 1 - \tfrac{4M(\beta+d\lambda)\beta'}{d} \equiv 1 \pmod{4M/d}$$

for all $0 \le \lambda \le p-1$. Set

$$\tilde{W}_{d,\lambda} = [\left(\begin{smallmatrix} d\alpha+4M\lambda & * \\ 4Mp & d\delta_\lambda \end{smallmatrix}\right), d^{-1/4} \left(\frac{Mp/d}{\delta_\lambda}\right) \sqrt{4Mpz + d\delta_\lambda}].$$

Then by Lemma 2.3.2, we have

$$\begin{aligned} \Phi \mid_{k/2} \tilde{U}_p \tilde{W}_d &= \sum_{\lambda=0}^{p-1} \left(\frac{Mp/d}{\delta_\lambda}\right)^k \Phi \mid_{k/2} \tilde{W}_{d,\lambda} \left[\left(\begin{smallmatrix} 1 & (\beta+d\lambda \\ & \beta' \end{smallmatrix}\right), p^{1/4}\right] \\ &= \sum_{\lambda=0}^{p-1} \left(\tfrac{p}{d}\right)^k \chi_d(p) \chi_{4M/d}(\delta_\lambda) \Phi \mid_{k/2} \tilde{W}_d \left[\left(\begin{smallmatrix} 1 & (\beta+d\lambda \\ & \beta' \end{smallmatrix}\right), p^{1/4}\right] \\ &= \left(\tfrac{p}{d}\right)^k \chi_d(p) \Phi \mid_{k/2} \tilde{W}_d \tilde{U}_p. \end{aligned}$$

□

2.3.2 Trace Down Lemmas

Now, we will prove two level-lowering lemmas for modular forms.

Lemma 2.3.5. *Let* $\Phi(z)$ *be a real-analytic function on* $\mathcal{H}$ *such that it has at most linear exponential growth at the cusps and*

$$(\Phi \mid_{k/2} \tilde{\gamma})(z) = \chi(d)\Phi(z)$$

for all $\tilde{\gamma} \in \tilde{\Gamma}_0(4Np^{r+1})$ *with* $r \ge 1$, $\gcd(N,p)=1$ *and* $\chi : (\mathbb{Z}/4Np^r\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$ *a character. Define* $\tilde{\Phi}(z)$ *to be*

$$\tilde{\Phi}(z) := \Phi \mid_{k/2} \tilde{U}_p.$$

Then $\tilde{\Phi}(z)$ *satisfies*

$$\left(\tilde{\Phi}\mid_{k/2}\tilde{\gamma}\right)(z)=\chi(d)\left(\tfrac{p}{d}\right)^k\tilde{\Phi}(z)$$

for all $\tilde{\gamma}\in\tilde{\Gamma}_0(4Np^r)$.

Proof. This is a purely group theoretic lemma and quite similar to Lemma 7 in [2], its the integral weight counterpart. For completeness, we will include the argument here.

Let $\gamma=\left(\begin{smallmatrix}a & b\\ 4Np^rc & d\end{smallmatrix}\right)\in\Gamma_0(4Np^r)$. Suppose $r\geq 1$, then

$$\begin{aligned}
\tilde{\Phi}\mid_{k/2}\tilde{\gamma} &= \sum_{\lambda=0}^{p-1}\Phi\mid_{k/2}\left[\left(\begin{smallmatrix}1 & \lambda\\ & p\end{smallmatrix}\right),p^{1/4}\right]\left[\left(\begin{smallmatrix}a & b\\ 4Np^rc & d\end{smallmatrix}\right),j(\gamma,z)\right]\\
&= \sum_{\lambda=0}^{p-1}\Phi\mid_{k/2}\left[\left(\begin{smallmatrix}a+4Np^rc\lambda & \beta_\lambda\\ 4Np^{r+1}c & \delta_\lambda\end{smallmatrix}\right)\left(\begin{smallmatrix}1 & \overline{a}(b+d\lambda)\\ & p\end{smallmatrix}\right),p^{1/4}\left(\tfrac{Np^rc}{d}\right)\epsilon_d^{-1}\sqrt{4Np^rcz+d}\right]\\
&= \sum_{\lambda=0}^{p-1}\Phi\mid_{k/2}\widetilde{\left(\begin{smallmatrix}a+4Np^rc\lambda & \beta_\lambda\\ 4Np^{r+1}c & \delta_\lambda\end{smallmatrix}\right)}\left[\left(\begin{smallmatrix}1 & \overline{a}(b+d\lambda)\\ & p\end{smallmatrix}\right),p^{1/4}\left(\tfrac{p}{d}\right)\right]\\
&= \sum_{\lambda=0}^{p-1}\chi(\delta_\lambda)\Phi\mid_{k/2}\left[\left(\begin{smallmatrix}1 & \overline{a}(b+d\lambda)\\ & p\end{smallmatrix}\right),p^{1/4}\right]\\
&= \sum_{\lambda=0}^{p-1}\chi(d)\left(\tfrac{p}{d}\right)^k\Phi\mid_{k/2}\left[\left(\begin{smallmatrix}1 & \overline{a}(b+d\lambda)\\ & p\end{smallmatrix}\right),p^{1/4}\right]=\chi(d)\left(\tfrac{p}{d}\right)^k\tilde{\Phi}.
\end{aligned}$$

□

When $r=0$ in the above lemma, one needs both the U-operator and the Atkin-Lehner involution to lower the level.

Lemma 2.3.6. *Let* $\Phi(z)$ *be a real-analytic function on* $\mathcal{H}$ *such that it has at most linear exponential growth at the cusps and*

$$(\Phi\mid_{k/2}\tilde{\gamma})(z)=\chi(d)\left(\tfrac{p}{d}\right)^k\Phi(z)$$

for all $\tilde{\gamma}\in\tilde{\Gamma}_0(4Np)$ *with* $\gcd(N,p)=1$ *and* $\chi:(\mathbb{Z}/4N)^*\longrightarrow\mathbb{C}^\times$ *a character. Define* $\tilde{\Phi}(z)$ *to be*

$$\tilde{\Phi}(z):=\Phi\mid_{k/2}\tilde{U}_p+\Phi\mid_{k/2}\tilde{W}_p \tag{2.3.8}$$

where $\tilde{W}_p$ *is defined as in Eq.* (2.3.4). *Then* $\tilde{\Phi}(z)$ *satisfies*

$$\left(\tilde{\Phi}\mid_{k/2}\tilde{\gamma}\right)(z)=\chi(d)\tilde{\Phi}(z)$$

for all $\tilde{\gamma} \in \tilde{\Gamma}_0(4N)$.

Remark 2.3.7. *For different choices*

$$\tilde{W}_p' := \left[\begin{pmatrix} pa' & b' \\ 4Npc' & pd' \end{pmatrix}, p^{-1/4}\left(\frac{Nc'}{d'}\right)\sqrt{4Npc'z+pd'}\right],$$
$$\tilde{W}_p'' := \left[\begin{pmatrix} pa'' & b'' \\ 4Npc'' & pd'' \end{pmatrix}, p^{-1/4}\left(\frac{Nc''}{d''}\right)\sqrt{4Npc''z+pd''}\right],$$

Lemma 2.3.2 implies that

$$\epsilon_{pd'}^k\overline{\chi(d')}\Phi\mid_{k/2}\tilde{W}_p' = \epsilon_{pd''}^k\overline{\chi(d'')}\Phi\mid_{k/2}\tilde{W}_p''.$$

So for general $\tilde{W}_p'$, *one has* $\tilde{\Phi} = \Phi\mid_{k/2}\tilde{U}_p + \left(\frac{\epsilon_{pd'}}{\epsilon_p}\right)^k\overline{\chi(d')}\Phi\mid_{k/2}\tilde{W}_p'$.

Proof. The proof is almost the same as the proof of Lemma 2.3.5. Let $\gamma = \left(\begin{smallmatrix} a & b \\ 4Nc & d \end{smallmatrix}\right) \in \Gamma_0(4N)$. Notice that the condition satisfied by $\Phi(z)$ is equivalent to

$$\left(\Phi\mid_{k/2}\left[\gamma, \left(\frac{p}{d}\right)j(\gamma,z)\right]\right)(z) = \chi(d)\Phi(z).$$

By quadratic reciprocity, the character of Φ can be written as

$$\chi(\cdot)\left(\frac{(-1)^{(p-1)/2}}{\cdot}\right)^k\left(\frac{\cdot}{p}\right)^k.$$

The only difference now in evaluating $\tilde{\Phi}\mid_{k/2}\tilde{\gamma}$ is that one of the λ in the sum of $\tilde{U}_p$ will switch with $\tilde{W}_p$. For completeness, we record the calculations here. When $p \mid a+4Nc\lambda$, we have by the remark above

$$\begin{aligned}\Phi\mid_{k/2}\left[\left(\begin{smallmatrix} 1 & \lambda \\ & p \end{smallmatrix}\right), p^{1/4}\right]\tilde{\gamma} &= \epsilon_d^k\Phi\mid_{k/2}\tilde{W}_p', \\ &= \left(\frac{\epsilon_d\epsilon_p}{\epsilon_{pd}}\right)^k\left(\frac{c}{p}\right)^k\left(\frac{c}{p}\right)^k\chi(d)\left(\frac{(-1)^{(p-1)/2}}{d}\right)^k\Phi\mid_{k/2}\tilde{W}_p \\ &= \chi(d)\Phi\mid_{k/2}\tilde{W}_p.\end{aligned}$$

where $\tilde{W}_p' = \left[\begin{pmatrix} p\frac{a+4Nc\lambda}{p} & b+d\lambda \\ 4Npc & pd \end{pmatrix}, p^{-1/4}\left(\frac{Nc}{d}\right)\sqrt{4Npcz+pd}\right]$. On the other hand, we have

$$\begin{aligned}\Phi\mid_{k/2}\tilde{W}_p\tilde{\gamma} &= \Phi\mid_{k/2}\left[\left(\begin{smallmatrix} p\alpha & \beta \\ 4Np & p \end{smallmatrix}\right), p^{-1/4}\sqrt{4Npz+p}\right]\left[\left(\begin{smallmatrix} a & b \\ 4Nc & d \end{smallmatrix}\right), j(\gamma,z)\right] \\ &= \Phi\mid_{k/2}\left[\left(\begin{smallmatrix} 1 & \\ & p \end{smallmatrix}\right), p^{1/4}\right]\widetilde{\left(\begin{smallmatrix} p\alpha & \beta \\ 4N & 1 \end{smallmatrix}\right)}\widetilde{\left(\begin{smallmatrix} a & b' \\ 4Nc & pd' \end{smallmatrix}\right)}\left[\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), p^{-1/4}\right]\left[\left(\begin{smallmatrix} 1 & \overline{4Nc} \\ & p \end{smallmatrix}\right), p^{1/4}\right] \\ &= \Phi\mid_{k/2}\left[\gamma, \left(\frac{p}{4Nb'+pd'}\right)j(\gamma,z)\right]\left[\left(\begin{smallmatrix} 1 & \overline{4Nc} \\ & p \end{smallmatrix}\right), p^{1/4}\right] \\ &= \chi(d)\Phi\mid_{k/2}\left[\left(\begin{smallmatrix} 1 & \overline{4Nc} \\ & p \end{smallmatrix}\right), p^{1/4}\right],\end{aligned}$$

where $\gamma = \begin{pmatrix} p\alpha a+4Nc\beta & \alpha b'+\beta d' \\ 4Np(a+c) & 4Nb'+pd' \end{pmatrix}$ and $pd' \equiv d \pmod{4Nc}$. □

Corollary 2.3.8. *In the notation of Lemma 2.3.6, define*

$$\hat{\Phi}(z) := \Phi \mid_{k/2} \left(\epsilon_p^k \left(\tfrac{-N}{p}\right)^k \overline{\chi(p)} \tilde{U}_p \tilde{W}_p + 1 \right).$$

Then we have

$$\hat{\Phi} \mid_{k/2} \tilde{U}_p = \hat{\Phi} \mid_{k/2} \tilde{W}_p.$$

Proof. Applying $\tilde{W}_p$ to the definition of $\tilde{\Phi}(z)$ in Eq. (2.3.8) gives us

$$\begin{aligned}
\tilde{\Phi} \mid_{k/2} \tilde{W}_p &= \Phi \mid_{k/2} \tilde{U}_p \tilde{W}_p + \Phi \mid_{k/2} \tilde{W}_p^2 \\
&= \Phi \mid_{k/2} \tilde{U}_p \tilde{W}_p + \epsilon_p^{-k} \left(\tfrac{N}{p}\right)^k \Phi \mid_{k/2} \widetilde{\left(\begin{smallmatrix} \alpha & \beta \\ 4N & p \end{smallmatrix}\right)} \widetilde{\left(\begin{smallmatrix} p\alpha & \beta \\ 4N & 1 \end{smallmatrix}\right)} \\
&= \Phi \mid_{k/2} \tilde{U}_p \tilde{W}_p + \epsilon_p^{-k} \left(\tfrac{-N}{p}\right)^k \chi(p) \Phi \\
&= \epsilon_p^{-k} \left(\tfrac{-N}{p}\right)^k \chi(p) \hat{\Phi}.
\end{aligned}$$

On the other hand, since $\tilde{\Phi}$ transforms with level $4N$, we can write

$$\begin{aligned}
\tilde{\Phi} \mid_{k/2} \tilde{W}_p &= \epsilon_p^{-k} \left(\tfrac{N}{p}\right)^k \tilde{\Phi} \mid_{k/2} \widetilde{\left(\begin{smallmatrix} \alpha & \beta \\ 4N & p \end{smallmatrix}\right)} \left[\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), p^{-1/4}\right] \\
&= \epsilon_p^{-k} \left(\tfrac{N}{p}\right)^k \chi(p) \tilde{\Phi} \mid_{k/2} \left[\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), p^{-1/4}\right] \\
\tilde{\Phi} \mid_{k/2} \tilde{W}_p^2 &= \epsilon_p^{-k} \left(\tfrac{N}{p}\right)^k \tilde{\Phi} \mid_{k/2} \widetilde{\left(\begin{smallmatrix} \alpha & \beta \\ 4N & p \end{smallmatrix}\right)} \widetilde{\left(\begin{smallmatrix} p\alpha & \beta \\ 4N & 1 \end{smallmatrix}\right)} \\
&= \epsilon_p^{k} \left(\tfrac{-N}{p}\right)^k \chi(p) \tilde{\Phi}
\end{aligned}$$

Combining the two produces

$$\begin{aligned}
\hat{\Phi} \mid \tilde{W}_p &= \epsilon_p^k \left(\tfrac{-N}{p}\right)^k \overline{\chi(p)} \tilde{\Phi} \mid_{k/2} \tilde{W}_p^2 = \left(\tfrac{-1}{p}\right)^k \tilde{\Phi}, \\
\hat{\Phi} \mid_{k/2} \tilde{U}_p &= \left(\tfrac{-1}{p}\right)^k \tilde{\Phi} \mid_{k/2} \left[\left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right), p^{-1/4}\right] \tilde{U}_p = \left(\tfrac{-1}{p}\right)^k \tilde{\Phi}.
\end{aligned}$$

This completes the proof. □

2.4 Weight One Space Decomposition

2.4.1 Projection Operators

Let M be a positive, odd integer and Φ be a real-analytic function on $\mathcal{H}$ with at most linear exponential growth at all cusps and $k \in \mathbb{Z}$. Suppose it satisfies the transformation property

$$(\Phi\mid_k \gamma)(z) = \chi_M(d)\Phi(z),$$

for all $\gamma \in \Gamma_0(M)$ and $\chi_M : (\mathbb{Z}/M\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$ a character satisfying $\chi_M(-1) = (-1)^k$. Recall that the space of such functions is denoted by $\mathcal{F}_k(M, \chi_M)$. Let $\ell \mid M$ be a prime such that $\gcd(\ell, M/\ell) = 1$ and $\ell > 0$. Write

$$M = M'\ell, \quad \chi_M = \chi_{M'}\chi_\ell$$

with χ_* a character having conductor $* = M', \ell$.

Notice that any such $\Phi \in \mathcal{F}_k(M, \chi_M)$ has a Fourier expansion at the cusp $1/M'$ with variable q. If χ_ℓ is non-trivial, then we *cannot* trace Φ down to level M'. Fortunately, it is still possible to obtain results similar to Corollary 2.3.8 when $\chi_\ell(\cdot) = \overline{\chi_\ell}(\cdot) = \left(\frac{\cdot}{\ell}\right)$ is quadratic.

For $\varepsilon = \pm 1$, define the operator $\mathrm{pr}^\varepsilon_{\chi_\ell}$ by

$$\mathrm{pr}^\varepsilon_{\chi_\ell}(\Phi) := \tfrac{1}{2}\left(\varepsilon \tfrac{\overline{\chi_{M'}(\ell)\chi_\ell(-M')}}{G(\chi_\ell)} \Phi\mid_k U_\ell W_\ell + \Phi\right) \tag{2.4.1}$$

where $G(\chi_\ell)$ is the Gauss sum and

$$U_\ell = \sum_{\lambda=0}^{\ell-1} \left(\begin{smallmatrix} 1 & \lambda \\ & \ell \end{smallmatrix}\right), W_\ell = \left(\begin{smallmatrix} \ell\alpha & \beta \\ M & \ell \end{smallmatrix}\right).$$

It is not difficult to check that $\mathrm{pr}^\varepsilon_{\chi_\ell}(\Phi)$ is in $\mathcal{F}_k(M, \chi_{M'}\overline{\chi_\ell})$. The following lemma describes the kernel of these operators in this case.

Proposition 2.4.1. *Let $\varepsilon \in \{\pm 1\}$, $\Phi(z) \in \mathcal{F}_k(M, \chi_{M'}\chi_\ell)$ with $\chi_\ell = \left(\frac{\cdot}{\ell}\right)$ and Fourier expansion*

$$\Phi(z) = \sum_{n\in\mathbb{Z}} a(\Phi, n, y)q^n$$

at infinity. Then the followings are equivalent.

(1) *For all* $n \in \mathbb{Z}$ *relatively prime to* ℓ,

$$\chi_\ell(n)a(\Phi, n, y) = -\varepsilon a(\Phi, n, y). \tag{2.4.2}$$

(2) $\mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi) = 0$.

(2)′ $\mathrm{pr}^{-\varepsilon}_{\chi_\ell}(\Phi) = \Phi$.

(3)

$$\Phi \mid_k W_\ell = -\varepsilon \tfrac{\overline{\chi_\ell(M')}}{G(\chi_\ell)} \Phi \mid_k U_\ell. \tag{2.4.3}$$

Proof. For any $\Psi \in \mathcal{F}_k(M, \chi_{M'}\chi_p)$, write its Fourier expansions at infinity as

$$\Psi(z) = \sum_{n \in \mathbb{Z}} a(\Psi, n, y) q^n.$$

Some calculations show that

$$\begin{aligned}
\Psi \mid_k U_\ell W_\ell &= \sum_{\lambda=0}^{\ell-1} \Psi \mid_k \left(\begin{smallmatrix} \alpha+M'\lambda & \beta+\ell\lambda \\ M & \ell^2 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) \\
&= \sum_{\lambda=0, \ell\nmid\alpha+M'\lambda}^{\ell-1} \Psi \mid_k \left(\begin{smallmatrix} \alpha+M'\lambda & * \\ M & * \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell & \overline{(\alpha+M'\lambda)}\beta \\ & \ell \end{smallmatrix}\right) + \Psi \mid_k W'_\ell \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) \\
&= \chi_{M'}(\ell)\chi_\ell(-M') \sum_{\mu=1}^{\ell-1} \chi_\ell(\mu) \Psi \mid_k \left(\begin{smallmatrix} \ell & \mu \\ & \ell \end{smallmatrix}\right) + \Psi \mid_k W'_\ell \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) \\
&= \chi_{M'}(\ell)\chi_\ell(-M') G(\chi_\ell) \sum_{n \in \mathbb{Z}} \overline{\chi_\ell(n)} a(\Psi, n, y) q^n + \chi_{M'}(\ell) \Psi \mid_k W_\ell \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right),
\end{aligned}$$

where $W'_\ell = \left(\begin{smallmatrix} \ell \frac{\alpha+M'\lambda_0}{\ell} & \beta+\ell\lambda \\ M & \ell^2 \end{smallmatrix}\right)$ with $\ell \mid \alpha + M'\lambda_0$. That means

$$\mathrm{pr}^{\epsilon}_{\chi_\ell}(\Psi) = \frac{1}{2} \left(\sum_{n \in \mathbb{Z}} (\epsilon \overline{\chi_\ell(n)} + 1) a(\Psi, n, y) q^n + \frac{\epsilon \overline{\chi_\ell(-M')}}{G(\chi_\ell)} \Psi \mid_k W_\ell \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) \right). \tag{2.4.4}$$

(1) ⇔ (2)

Taking $\Psi = \Phi$ in Eq. (2.4.4) shows that ⇐ is clear. For ⇒, we know that there exists $\Phi_\varepsilon(z)$ such that

$$\Phi_\varepsilon(z+1) = \Phi_\varepsilon(z),$$

$$\Phi_\varepsilon \mid_k \left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right) = \mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi).$$

Since $\mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi) \in \mathcal{F}_k(M, \chi_{M'}\chi_\ell)$, the second condition implies that

$$\Phi_\varepsilon \mid_k \left(\begin{smallmatrix} a & \ell b \\ M'c & d \end{smallmatrix}\right) = \chi_{M'}(d)\chi_\ell(d)\Phi_\varepsilon$$

for all $\left(\begin{smallmatrix} a & \ell b \\ M'c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ with $a, b, c, d \in \mathbb{Z}$. Given any $\left(\begin{smallmatrix} * & * \\ M'* & d \end{smallmatrix}\right) \in \Gamma_0(M')$ with $\gcd(\ell, d) = 1$, we can write it as $\left(\begin{smallmatrix} 1 & t \\ & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} * & \ell * \\ M'* & d \end{smallmatrix}\right)$. Thus,

$$\Phi_\varepsilon \mid_k \left(\begin{smallmatrix} * & * \\ M'* & d \end{smallmatrix}\right) = \chi_{M'}(d)\chi_\ell(d)\Phi_\varepsilon.$$

On the other hand, since χ_ℓ is non-trivial, one can find $b', c, d, d' \in \mathbb{Z}$ such that $\gcd(d, Mc') = \gcd(d', Mb') = 1$ and $\chi_\ell(dd') \neq \chi_\ell(M'b'c + dd')$. That means

$$\begin{aligned} \chi_{M'}(dd')\chi_\ell(dd')\Phi_\varepsilon &= \Phi_\varepsilon \mid_k \left(\begin{smallmatrix} * & * \\ M'c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} * & b' \\ M'* & d' \end{smallmatrix}\right) \\ &= \Phi_\varepsilon \mid_k \left(\begin{smallmatrix} * & * \\ M'* & M'b'c+dd' \end{smallmatrix}\right) \\ &= \chi_{M'}(dd')\chi_\ell(M'b'c + dd')\Phi_\varepsilon. \end{aligned}$$

By the choice of b', c, d and d', we know that $\chi_{M'}(dd') \neq 0$. Thus, Φ_ε must vanish and $\mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi) = 0$.

(2) $\Leftrightarrow$ (2)$'$

This follows easily from

$$\Phi = \mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi) + \mathrm{pr}^{-\varepsilon}_{\chi_\ell}(\Phi) = \mathrm{pr}^{-\varepsilon}_{\chi_\ell}(\Phi). \tag{2.4.5}$$

(2) $\Leftrightarrow$ (3)

Notice that (2) and Eq. (2.4.1) is the same as

$$\Phi = -\varepsilon \overline{\tfrac{\chi_{M'}(\ell)\chi_\ell(-M')}{G(\chi_\ell)}} \Phi \mid_k U_\ell W_\ell$$

The equivalence then from applying W_ℓ to both sides and using the relations

$$\Phi \mid_k U_\ell W_\ell^2 = \chi_\ell(-1)\chi_{M'}(\ell)\Phi \mid_k U_\ell.$$

$\square$

If $\chi_{M'}$ is quadratic and χ_ℓ is an arbitrary character, then the proposition above can be modified to yield similar results. In this case, set

$$M' = N, \ell = p, \chi_M = \chi_N \chi_p, \chi_N(\cdot) = \left(\frac{\cdot}{N}\right).$$

Let χ_p be an arbitrary character of conductor p. Define the operator $\mathrm{pr}^{\varepsilon}_{\chi_p}$ for $\varepsilon = \pm 1$ by

$$\mathrm{pr}^{\varepsilon}_{\chi_p}(\Phi) := \tfrac{1}{2}\left(\varepsilon \tfrac{\overline{\chi_N(p)\chi_p(-N)}}{G(\chi_p)} \Phi \mid_k U_p W_p + \Phi^c\right) \tag{2.4.6}$$

where $\Phi^c(z) = \overline{\Phi(\overline{z})}$. Since W_p sends $\mathcal{F}(M, \chi_N\chi_p)$ to $\mathcal{F}(M, \chi_N\overline{\chi_p})$ and $\Phi^c \in \mathcal{F}(M, \chi_N\overline{\chi_p})$, we use Φ^c instead of Φ in defining the projection operator here. The following proposition is the analogue of Prop. 2.4.1.

Proposition 2.4.2. *Let $\varepsilon \in \{\pm 1\}$, $\Phi(z) \in \mathcal{F}_k(Np, \chi_N\chi_p)$ with $\chi_N = \left(\frac{\cdot}{N}\right)$ quadratic, χ_p non-trivial and the Fourier expansion*

$$\Phi(z) = \sum_{n \in \mathbb{Z}} a(\Phi, n, y) q^n$$

at infinity. Then the followings are equivalent

(1) *For all $n \in \mathbb{Z}$ relatively prime to p,*

$$\chi_p(n)\overline{a(\Phi, n, y)} = -\varepsilon a(\Phi, n, y) \tag{2.4.7}$$

(2) $\mathrm{pr}^{\varepsilon}_{\chi_p}(\Phi) = 0$.

(2)′ $\mathrm{pr}^{-\varepsilon}_{\chi_p}(\Phi) = \Phi^c$.

(2)″ $\mathrm{pr}^{-\varepsilon}_{\chi_p}(i\Phi) = 0$.

(3)

$$\Phi^c \mid_k W_p = -\varepsilon \tfrac{\overline{\chi_p(N)}}{G(\chi_p)} \Phi \mid_k U_p. \tag{2.4.8}$$

Remark 2.4.3. *The subspace of $\mathcal{F}_k(Np, \chi_N\chi_p)$ satisfying any of the four conditions above is a **real** vector subspace of $\mathcal{F}_k(Np, \chi_N\chi_p)$.*

Proof. Let $\Psi \in \mathcal{F}_k(Np, \chi_N\chi_p)$ with Fourier expansions

$$\Psi(z) = \sum_{n\in\mathbb{Z}} a(\Psi, n, y)q^n$$

at infinity. Then $\Psi^c(z) = \sum_{n\in\mathbb{Z}} \overline{a(\Psi, n, y)}q^n$ and the same calculations in Prop. 2.4.1 gives us the following analogue of Eq. (2.4.4).

$$\mathrm{pr}^{\epsilon}_{\chi_p}(\Psi) = \frac{1}{2}\left(\sum_{n\in\mathbb{Z}}(\epsilon\overline{\chi_p(n)}a(\Psi, n, y) + \overline{a(\Psi, n, y)})q^n + \frac{\epsilon\overline{\chi_p(-M')}}{G(\chi_p)}\Psi \mid_k W_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)\right).$$

The rest of the proof follows, mutatis mutandis, from that of Prop. 2.4.1. The equivalence to $(2)''$ follows from substituting $i\Phi$ for Φ in condition (1) and replacing ε with $-\varepsilon$. ☐

Proposition 2.4.4. *For distinct $\ell, \ell' \mid M$ and $\varepsilon, \varepsilon' \in \{\pm 1\}$, the projection operators satisfy the following properties*

$$\begin{aligned} \mathrm{pr}^{-\varepsilon}_{\overline{\chi_\ell}} \circ \mathrm{pr}^{\varepsilon}_{\chi_\ell} &= 0, \\ \mathrm{pr}^{\varepsilon}_{\chi_\ell} \circ \mathrm{pr}^{\varepsilon'}_{\chi_{\ell'}} &= \mathrm{pr}^{\varepsilon'}_{\chi_{\ell'}} \circ \mathrm{pr}^{\varepsilon}_{\chi_\ell}, \\ \mathrm{pr}^{\varepsilon}_{\chi_\ell} \circ \xi_k &= \xi_k \circ \mathrm{pr}^{\varepsilon\chi_\ell(-1)}_{\chi_\ell}. \end{aligned} \tag{2.4.9}$$

Proof. For $\Phi \in \mathcal{F}(M, \chi_M)$, let $a(\Phi, n, y), a(\Phi^c, n, y)$ and $b_\ell(\Phi, n, y)$ be the Fourier coefficients of $\Phi, \Phi^c, \mathrm{pr}^{\varepsilon}_{\ell}(\Phi)$ and $\mathrm{pr}^{\varepsilon}_{\chi_\ell}(\Phi)$ respectively. For $\ell \neq p$ in the first equation, the calculations in Prop. 2.4.1 tells us that whenever $\gcd(\ell, n) = 1$,

$$b_\ell(\Phi, n, y) = \tfrac{1}{2}\left(\varepsilon\overline{\chi_\ell(n)}a(\Phi, n, y) + a(\Phi, n, y)\right).$$

When $\ell = p$, we could write

$$b_p(\Phi, n, y) = \tfrac{1}{2}\left(\varepsilon\overline{\chi_p(n)}a(\Phi^c, n, y) + a(\Phi, n, y)\right).$$

Since $\varepsilon^2 = 1$, we have in both cases.

$$\chi_\ell(n)b_\ell(\Phi, n, y) = \varepsilon b_\ell(\Phi, n, y) \text{ whenever } \gcd(\ell, n) = 1.$$

By Prop. 2.4.1, the first two equations hold.

For the second equation, notice that we can write

$$\mathrm{pr}_\ell^\varepsilon \circ \mathrm{pr}_{\ell'}^{\varepsilon'}(\Phi) = A((\varepsilon,\ell),(\varepsilon',\ell'))\Phi \mid_k U_\ell W_\ell U_{\ell'} W_{\ell'} + B((\varepsilon,\ell),(\varepsilon',\ell')),$$

where $A(\varepsilon,\varepsilon',\ell,\ell')$ and $B(\varepsilon,\varepsilon',\ell,\ell')$ are stable under switching (ε,ℓ) and (ε',ℓ'). By Lemma 2.3.4 and 2.3.3, we have

$$\Phi \mid_k U_\ell W_\ell U_{\ell'} W_{\ell'} = \overline{\chi_{\ell'}(\ell)}\Phi \mid_k U_\ell U_{\ell'} W_\ell W_{\ell'} = \Phi \mid_k U_\ell U_{\ell'} W_{\ell\ell'},$$

which is also stable under switching (ε,ℓ) and (ε',ℓ'). So the third equation holds. By the same procedure, one could verify that the fourth equation holds as well.

The last equation follows from the definitions of the projection operators and the fact that ξ_k commutes with slash operator by changing the weight from k to $2-k$, character from χ_M to $\overline{\chi_M}$, and the coefficients to their complex conjugates. □

2.4.2 Applications of the Projection Operators

Now, we will apply the projection operator to the space $H_1(M,\chi_M)$, where $M = Np$ is odd, square-free and $\chi_M = \chi_N\chi_p$ with $\chi_N(\cdot) = \left(\frac{\cdot}{N}\right)$ quadratic and χ_p an arbitrary, non-trivial character. All the results in §2.4.1 still holds since the subspace $H_1(M,\chi_M) \subset \mathcal{F}_1(M,\chi_M)$ is defined by the differential operator ξ_1, which commutes with the slash operator. For $\varepsilon \in \{\pm 1\}$ and each prime $\ell \mid M$, define the space $H_{1,\ell}^\varepsilon(M,\chi_M)$ by

$$H_{1,\ell}^\varepsilon(M,\chi_M) := \left\{\hat{f} \in H_1(M,\chi_M) : \mathrm{pr}_{\chi_\ell}^\varepsilon(\hat{f}) = 0\right\}. \tag{2.4.10}$$

The space $H_{1,\ell}^\varepsilon(M,\chi_M)$ is a complex vector space unless $\ell = p$, in which case it is a real vector space. For each $d \mid M$, we could define the real vector space $H_{1,d}(M,\chi_M)$ by

$$H_{1,d}(M,\chi_M) := \bigcap_{\ell \mid d \text{ positive prime}} H_{1,\ell}^+(M,\chi_M) \bigcap_{\ell \mid \frac{M}{d} \text{ positive prime}} H_{1,\ell}^-(M,\chi_M). \tag{2.4.11}$$

By the definitions of the projeciton operators and Props. 2.4.4, we could write

$$H_1(M,\chi_M) = H_{1,\ell}^+(M,\chi_M) \oplus H_{1,\ell}^-(M,\chi_M)$$

for all $\ell \mid M$. By Props. 2.4.1 and 2.4.2, we know that $\hat{f} \in H_{1,d}(M, \chi_M)$ if and only if the following conditions are satisfied for all $\ell \mid N$

$$\hat{f} \mid_1 W_\ell = -\varepsilon \frac{\chi_\ell(Np/\ell)}{\epsilon_\ell \sqrt{\ell}} \hat{f} \mid_1 U_\ell,$$
$$\hat{f} \mid_1 W_p = -\varepsilon \frac{\chi_p(N)}{G(\overline{\chi_p})} \hat{f}^c \mid_1 U_p.$$

Furthermore, Prop. 2.4.4 tells us that projection operators of different ℓ commute with each other. This implies that

$$H_1(M, \chi_M) = \bigoplus_{d \mid M} H_{1,d}(M, \chi_M) \tag{2.4.12}$$

as **real** vector spaces. Remark 2.4.3 also implies that for any $d \mid N$, $\hat{f} \in H_{1,d}(M, \chi_M)$ if and only if $i\hat{f} \in H_{1,dp}(M, \chi_M)$. Since the decomposition 2.4.12 is defined entirely using the slash operator, it could be naturally defined for subspaces $S_1(M, \chi_M) \subset M_1(M, \chi_M) \subset M_1^!(M, \chi_M)$ of $H_1(M, \chi_M)$ and $\mathbb{M}_1(M, \chi_M) \cong H_1(M, \chi_M)$.

For any positive $\ell \mid M$, define $\ell^* \mid M$ to be

$$\ell^* := \chi_\ell(-1)\ell. \tag{2.4.13}$$

Then we could write $M = M_+ \cdot M_-$, where

$$M_\varepsilon := \prod_{\substack{\ell \mid M \\ \ell^* = \varepsilon \ell}} \ell \tag{2.4.14}$$

for $\varepsilon \in \{\pm 1\}$. Since $\chi_\ell(\cdot) = \left(\frac{\cdot}{\ell}\right)$ when $\ell \mid N$ is prime, M_ε contains all the primes dividing N which are congruent to ε modulo 4. Define the quantity $\hat{d} \mid M$ by

$$\hat{d} := \gcd(d, M_+) \cdot \gcd(M/d, M_-). \tag{2.4.15}$$

The next proposition shows that decomposition in Eq. (2.4.12) behaves nicely with respect to the differential operator ξ_1.

Proposition 2.4.5. *The following sequence is exact*

$$M^!_{1,\hat{d}}(M, \chi_M) \hookrightarrow H_{1,\hat{d}}(M, \chi_M) \xrightarrow{\xi_1} S_{1,d}(M, \overline{\chi_M}). \tag{2.4.16}$$

Proof. By the last two equations in (2.4.9), the image of $H_{1,\hat{d}}(M,\chi_M)$ lies in $S_{1,d}(M,\overline{\chi_M})$. For any $f \in S_{1,d}(M,\overline{\chi_M})$, let $\hat{f} \in H_1(M,\chi_M)$ be its preimage under ξ, whose existence is given by Theorem 2.1.2. Define $\hat{f}_{\hat{d}}$ by

$$\hat{f}_{\hat{d}} := \left(\prod_{\ell\mid\hat{d}} \mathrm{pr}^-_{\chi_\ell} \prod_{\ell\mid(M/\hat{d})} \mathrm{pr}^+_{\chi_\ell}\right)\hat{f}.$$

Then the Eqs. (2.4.9) tell us that

$$\begin{aligned}\xi_1(\hat{f}_{\hat{d}}) &= \xi_1\left(\prod_{\ell\mid\hat{d}} \mathrm{pr}^-_{\chi_\ell} \prod_{\ell\mid(M/\hat{d})} \mathrm{pr}^+_{\chi_\ell}\right)\hat{f} = \left(\prod_{\ell\mid d} \mathrm{pr}^-_{\chi_\ell} \prod_{\ell\mid\frac{M}{d}} \mathrm{pr}^+_{\chi_\ell}\right)\xi_1\hat{f} \\ &= \left(\prod_{\ell\mid d} \mathrm{pr}^-_{\chi_\ell} \prod_{\ell\mid\frac{M}{d}} \mathrm{pr}^+_{\chi_\ell}\right) f = f.\end{aligned}$$

The last step follows from (2) $\Leftrightarrow$ (2)$'$ in Props. 2.4.1 and 2.4.2. By the first two equations in Eq. (2.4.9), we know that

$$\begin{aligned}\mathrm{pr}^+_{\chi_\ell}(\hat{f}_{\hat{d}}) &= 0 \text{ for all } \ell \mid \hat{d}, \\ \mathrm{pr}^-_{\chi_\ell}(\hat{f}_{\hat{d}}) &= 0 \text{ for all } \ell \mid \frac{M}{\hat{d}}.\end{aligned}$$

So $\hat{f}_{\hat{d}} \in H_{1,\hat{d}}(M,\chi_M)$ by definition and $\xi_1 : H_{1,\hat{d}}(M,\chi_M) \longrightarrow S_{1,d}(M,\overline{\chi_M})$ is surjective. The kernel is the holomorphic subspace of $H_{1,\hat{d}}(M,\chi_M)$, which is exactly $M^!_{1,\hat{d}}(M,\chi_M)$. $\square$

Combining the proposition above with Lemma 2.2.1, we could deduce the following result on regularized inner product.

Proposition 2.4.6. *Let $d_1, d_2 \mid N$, $h \in M^!_{1,d_1}(M,\overline{\chi_M})$, $f \in S_{1,d_2}(M,\overline{\chi_M})$ with Fourier expansions*

$$h = \sum_{n\in\mathbb{Z}} c(h,n)q^n,\ f = \sum_{n\geq 1} c(f,n)q^n$$

at infinity. Let $\hat{f} \in H_{1,\hat{d}_2}(M,\chi_M)$ be any preimage of f under ξ_1 as in Prop. 2.4.5 with Fourier expansion

$$\hat{f} = \sum_{n\in\mathbb{Z}} c(\hat{f},n)q^n - \sum_{n\geq 1} \overline{c(f,n)}\beta_1(n,y)q^{-n}$$

at infinity. If $d_1 = d_2$, then

$$\langle h, f\rangle_{\text{reg}} = \sum_{n\in\mathbb{Z}} \left(c(\hat{f}, -n)c(h,n) + \overline{c(\hat{f}, -pn)c(h,pn)}\right) \delta_N(n), \tag{2.4.17}$$

where $\delta_N(n) := 2^{\omega(\gcd(N,n))}$ is the number of divisors of $\gcd(N,n)$. Otherwise, the regularized inner product is 0.

Proof. By Lemma 2.2.1, we could rewrite $\langle h, f\rangle_{\text{reg}}$ as

$$\langle h, f\rangle_{\text{reg}} = \sum_{d\mid M} \text{Const}\left((h \mid_1 W_d)(\hat{f} \mid_1 W_d)\right).$$

Applying Lemmas 2.3.3, 2.3.4 and Props. 2.4.1 and 2.4.2 to the right hand side above gives us

$$\begin{aligned}\langle h, f\rangle_{\text{reg}} &= \sum_{d\mid N} \frac{\delta_{d_1,d_2}(d)}{d} \text{Const}\left((h \mid_1 U_d)(\hat{f} \mid_1 U_d) + \frac{1}{p}(h^c \mid_1 U_{dp})(\hat{f}^c \mid_1 U_{dp})\right), \\ &= \sum_{n\in\mathbb{Z}} \left(c(h,n)c(\hat{f}, -n) + \overline{c(h,pn)c(\hat{f}, -pn)}\right) \prod_{\ell\mid\gcd(n,N)\text{ prime}} (\delta_{d_1,d_2}(\ell) + 1) \end{aligned} \tag{2.4.18}$$

where $\delta_{d_1,d_2}(\ell)$ is defined by

$$\delta_{d_1,d_2}(\ell) := \begin{cases} 1, & \ell \mid \gcd(d_1, d_2) \cdot \gcd(M/d_1, M/d_2), \\ -1, & \ell \mid \gcd(d_1, M/d_2) \cdot \gcd(M/d_1, d_2). \end{cases}$$

and $\delta_{d_1,d_2}(d) := \prod_{\ell\mid d \text{ prime}} \delta_{d_1,d_2}(\ell)$.

If $\ell \mid d_2$, then Prop. 2.4.4 tells us that

$$\chi_\ell(n)c(\hat{f}, -n) = -c(\hat{f}, -n),$$

for all $n \in \mathbb{Z}$ not divisible by ℓ. So if $d_1 \neq d_2$, then there exists ℓ' such that

$$\chi_{\ell'}(n)c(\hat{f}, -n) = -\varepsilon c(\hat{f}, -n), \chi_{\ell'}(n)c(h,n) = \varepsilon c(h,n)$$

with $\varepsilon \in \{\pm 1\}$ for all $n \in \mathbb{Z}$ satisfying $\ell' \nmid n$ by Prop. 2.4.1. That means for these n, we have

$$c(\hat{f}, -n)c(h,n) = \left(\chi_{\ell'}(n)c(\hat{f}, -n)\right)(\chi_{\ell'}(n)c(h,n)) = -c(\hat{f}, -n)c(h,n),$$

and $c(\hat{f}, -n)c(h, n) = 0$. Furthermore, $\delta_{d_1,d_2}(\ell') = -1$ and the factor $\prod_{\ell | \gcd(n,N)}(\delta_{d_1,d_2}(\ell)+1)$ vanishes. Thus, the sum in Eq. (2.4.18) vanishes identically and $\langle h, f\rangle_{\text{reg}} = 0$.

If $d_1 = d_2$, then $\delta_{d_1,d_2}(\ell) = 1$ for all $\ell \mid N$ and the factor

$$\prod_{\ell | \gcd(n,N) \text{ prime}} (\delta_{d_1,d_2}(\ell) + 1) = 2^{\omega(\gcd(n,N))}$$

is the number of divisors of $\gcd(n, N)$. Substituting this into Eq. (2.4.18) gives us Eq. (2.4.17). ☐

For each $d \mid D$, let $n_d, r_d \in \mathbb{N}$ be the quantities

$$\begin{aligned} n_d &:= \max\{\text{ord}_\infty f : f \in S_{1,d}(Np, \chi_D\chi_p)\}, \\ r_d &:= \dim_{\mathbb{R}} S_{1,d}(Np, \chi_D\chi_p). \end{aligned} \tag{2.4.19}$$

Props. 2.2.2 and 2.4.6 together gives us the following result about the order at infinity of the preimage in 2.4.5.

Proposition 2.4.7. *For any $d \mid N$ and $f \in S_{1,d}(Np, \chi_N\chi_p)$, there exists $\hat{f} \in H_{1,\hat{d}}(Np, \chi_N\overline{\chi_p})$ such that $\xi_1(\hat{f}) = f$ and*

$$\text{ord}_\infty(\hat{f}) \geq -n_d.$$

Proof. Let $\hat{f}_1 \in H_{1,}(Np, \chi_N\overline{\chi_p})$ be a preimage of f under ξ_1 as in Prop. 2.4.5 with $\text{ord}_\infty(\hat{f}_1) = -n_0$ and principal part

$$\sum_{n=1}^{n_0} c(\hat{f}_1, -n)q^{-n}.$$

If $n_0 > n_d$, then one could find complex numbers $\{c(-n) : 1 \leq n \leq n_0\}$ such that $c(-n_0) \neq 0$ and

$$\begin{aligned}
&\text{for all } \ell \mid \hat{d},\ \chi_\ell(-n)c(-n) = -c(-n),\ \text{for all } n \in \mathbb{Z} \text{ relatively prime to } \ell, \\
&\text{for all } \ell \mid \tfrac{N}{\hat{d}},\ \chi_\ell(-n)c(-n) = c(-n),\ \text{for all } n \in \mathbb{Z} \text{ relatively prime to } \ell, \\
&\chi_p(-n)c(-n) = -\overline{c(-n)},\ \text{for all } n \in \mathbb{Z} \text{ relatively prime to } p, \\
&\sum_{n=1}^{n_0} \left(c(-n)c(h,n) + \overline{c(-pn)c(h,pn)}\right)\delta_N(n) = 0,\ \text{for all } h \in S_{1,d}(Np, \chi_D\chi_p).
\end{aligned}$$

By Prop. 2.2.2, there exists $f_2 \in M^!_{1,\hat{d}}(Np, \chi_D\overline{\chi_p})$ with the principal part

$$c(-n_0)q^{-n_0} + \sum_{n=1}^{n_0-1} c(-n)q^{-n}$$

if $p \nmid n_0$. Since $\chi_p(-n_0)c(\hat{f}, -n_0) = -\overline{c(\hat{f}, -n_0)}$ and $c(-n_0) \neq 0$, the ratio $c(\hat{f}, -n_0)/c(-n_0)$ is a real number and $\hat{f}_1 - c(\hat{f}, -n_0)/c(-n_0)f_2 \in H_{1,\hat{d}}(Np, \chi_D\overline{\chi_p})$ is a harmonic Maass form with image $f_{\phi,\mathcal{A}}$ under ξ_1 and smaller order of pole at infinity.

If $p \mid n_0$, then Prop. 2.2.2 also implies that there exist $f_3, f_4 \in M^!_{1,\hat{d}}(Np, \chi_D\overline{\phi})$ with the following principal parts at infinity respectively

$$q^{-n_0} + \sum_{n=1}^{n_0-1} c(-n)q^{-n}, iq^{-n_0} + \sum_{n=1}^{n_0-1} c(-n)q^{-n}.$$

By the same idea as before, we could still subtract $\mathbb{R}$ multiples of f_3 and f_4 from $\hat{f}_1$ to reduce n_0. An induction on the size of n_0 shows that we could take it to be n_d. $\square$

CHAPTER 3

Weight One Newforms of Imaginary Dihedral Type

Let $D < 0$ be an **odd, fundamental** discriminant and $K = \mathbb{Q}(\sqrt{D})$ an imaginary quadratic field with ring of integers $\mathcal{O}_K$. The weight one newforms we want to study arise from non-trivial characters of certain ray class groups of $\mathcal{O}_K$, which becomes the newform associated to characters of $\mathrm{Pic}(\mathcal{O})$ for some order $\mathcal{O} \subset \mathcal{O}_k$ after suitable twisting. The structure of the chapter is as follows. In §3.1, we will give some background information on weight one newforms and Galois representations following [48]. Then in §3.2, we will gather some facts about $\mathrm{Pic}(\mathcal{O})$ and use its characters to construct newforms. In §3.3, we will look at two dimensional, odd, complex Galois representations arising from ray class group characters of imaginary quadratic fields, and the weight one newforms associated to them via Theorem 3.1.1. Finally in §3.4.1, we will look at the inner product between newforms constructed in §3.3.

3.1 Newforms of Weight One

In this section, we will describe the results connecting complex, odd, two-dimensional, irreducible representations of $G_{\mathbb{Q}}$ and weight one newforms.

Let $\overline{\mathbb{Q}}/\mathbb{Q}$ be an algebraic closure of $\mathbb{Q}$ and $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ be the absolute Galois group of $\mathbb{Q}$, with the profinite topology. Because of the difference in topologies, a continuous representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ will have finite image. Its projective image under the projection $\mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathrm{PGL}_2(\mathbb{C})$ is a *finite* subgroup of $\mathrm{PGL}_2(\mathbb{C})$. The finite subgroups of $\mathrm{GL}_n(\mathbb{C})$ is well-known to be classified as either cyclic, dihedral, tetrahedral, octahedral, or icosahedral, when the projective image is C_n, D_{2n}, A_4, S_4 or A_5. If the representation is irreducible, then the projective image can only be one of the last four types. We say a continuous representations $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ is odd if $\rho(c) = -1$, where $c \in G_{\mathbb{Q}}$ is complex conjugation. The conductor of ρ can be defined using the data $\rho|_{I_p}$ with $I_p \subset G_{\mathbb{Q}}$ the inertia subgroup at rational prime p.

In [18], Deligne and Serre attached odd, irreducible Galois representations $\rho_f : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$ to weight one newforms $f = \sum_{n \geq 1} a(f,n)q^n$ of level N and nebentypus χ such that the conductor and determinant of ρ_f are N and χ respectively, and for all primes $p \nmid N$

$$\mathrm{tr}\rho_f(\mathrm{Frob}_p) = a(f,p).$$

In the other direction, Langlands used solvable base change and proved the modularity of Galois representations with solvable image [39]. In particular, he attached weight one newforms to odd, continuous, irreducible representation $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ with projective image A_4 and Tunnell did the case with projective image S_4 [54]. Finally, the A_5 case, which is not solvable, was proved as a consequence of the resolution of Serre's Conjecture by Khare-Wintenberger [34, 35]. As a result, we have the following bijection between Galois representations and weight one newforms.

Theorem 3.1.1 (Deligne-Serre, Langlands-Tunnell, Khare-Wintenberger)**.** *There is bijection between the set of odd, irreducible, continuous representations $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ of conductor N and $\det(\rho) = \chi$ and weight one newforms of level N and nebentypus χ.*

From now on, we will say a weight one newform is of dihedral, resp. tetrahedral, octahedral, or icosahedral type if its associated Galois representation has dihedral, resp. tetrahedral, octahedral, icosahedral projective images. In the dihedral case, the Galois representation is induced from a Hecke character of either a real or imaginary quadratic field. We will say that it is of imaginary or real dihedral type depending on the nature of the quadratic field.

3.2 Structure of $\mathrm{Pic}(\mathcal{O})$

Let $\mathcal{O} \subset \mathcal{O}_K \subset K$ be a subring. Its index in $\mathcal{O}_K$ is f^2 for some positive integer f. We call f the *conductor* of $\mathcal{O}$ and denote $\mathcal{O}$ by $\mathcal{O}_f$. Explicitly, one can write

$$\mathcal{O}_f = \{\tfrac{m+nf\sqrt{D}}{2} : m, n \in \mathbb{Z}, m \equiv nfD \pmod 2\}.$$

When $f > 1$, the order $\mathcal{O}_f$ is not a Dedekind domain. So we do not expect every fractional ideal to factor uniquely into prime ideals. On the other hand, every fractional ideal *relatively prime* to the conductor f has unique factorization into primes ideals in $\mathcal{O}_f$. Let $I(\mathcal{O}_f)$, resp. $P(\mathcal{O}_f)$, be the group of invertible, resp. principal ideals of $\mathcal{O}_f$. Define the Picard group of $\mathcal{O}_f$ to be

$$\mathrm{Pic}(\mathcal{O}_f) := I(\mathcal{O}_f)/P(\mathcal{O}_f).$$

When $f = 1$, $\mathrm{Pic}(\mathcal{O}_f)$ is just the class group of K, which is canonically isomorphic to $C(D)$, the group of equivalence classes of primitive binary quadratic forms of discriminant $D < 0$ (see §4.1.1 for details). Via the following map, this relationship holds more generally between $\mathrm{Pic}(\mathcal{O}_f)$ and $C(Df^2)$ (Theorem 7.7 [17]).

$$\begin{aligned} C(Df^2) &\xrightarrow{\cong} \mathrm{Pic}(\mathcal{O}_f) \\ [A, B, C] &\longmapsto [A, \tfrac{-B+f\sqrt{D}}{2}]\mathcal{O}_f \end{aligned} \tag{3.2.1}$$

In the notation of §4.1.1, we will denote $\mathrm{Pic}_0(\mathcal{O}_f)$ and $\mathrm{Pic}^2(\mathcal{O}_f)$ the kernel and image of $\mathrm{Pic}(\mathcal{O}_f)$ under the squaring map.

By Proposition 7.20 in [17], there is an isomorphism between $I(\mathcal{O}_f)$ and $I_K(f)$, the group of fractional ideals of $\mathcal{O}_K$ relatively prime to f. So there is another way to describe $\mathrm{Pic}(\mathcal{O}_f)$ (Prop 7.22 [17])

$$\mathrm{Pic}(\mathcal{O}_f) \cong I_K(f)/P_{K,\mathbb{Z}}(f), \tag{3.2.2}$$

where $P_{K,\mathbb{Z}}(f) \leq I_K(f)$ is the subgroup of principal ideals (α) of K with $\alpha \equiv a \pmod f$ for some $a \in \mathbb{Z}, \gcd(a, f) = 1$. Let $I_K = I(\mathcal{O}_K)$ and $P_K = P(\mathcal{O}_K)$. By Chebotarev density theorem, the natural map $I_K(f) \to I_K/P_K \cong \mathrm{Pic}(\mathcal{O}_K)$ is surjective with kernel $I_K(f) \cap P_K$ containing $P_{K,\mathbb{Z}}(f)$. So we have the following exact sequence (Eq. (7.25) in [17])

$$1 \longrightarrow I_K(f) \cap P_K/P_{K,\mathbb{Z}}(f) \longrightarrow I_K(f)/P_{K,\mathbb{Z}}(f) \longrightarrow I_K/P_K \longrightarrow 1 \tag{3.2.3}$$

The kernel is well-understood from the following exact sequence (Eq. (7.27) & Ex. 7.30 in [17])

$$\begin{aligned} 1 \longrightarrow \{\pm 1\} \longrightarrow (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{O}_K^\times \longrightarrow (\mathcal{O}_K/f\mathcal{O}_K)^\times& \\ \longrightarrow I_K(f) \cap P_K/P_{K,\mathbb{Z}}(f) \longrightarrow 1.& \end{aligned} \tag{3.2.4}$$

From this description, we can determine $h(\mathcal{O}_f)$, the size of $\operatorname{Pic}(\mathcal{O}_f)$. It is given by the following formula (Theorem 7.24 [17])

$$h(\mathcal{O}_f) = \frac{h(\mathcal{O}_K)f}{[\mathcal{O}_K^\times : \mathcal{O}_f^\times]} \prod_{p \mid f} \left(1 - \left(\frac{D}{p}\right)\frac{1}{p}\right). \tag{3.2.5}$$

Here $\left(\frac{D}{p}\right)$ is the Kronecker symbol and $\mathcal{O}^\times$ denotes the units in the ring $\mathcal{O}$. In particular, $h(\mathcal{O}_f)$ is always an integral multiple of $h(\mathcal{O}_K)$. When $f \mid f'$, we have $\mathcal{O}_{f'} \subset \mathcal{O}_f$ and hence a canonical map

$$\begin{aligned} \pi : \operatorname{Pic}(\mathcal{O}_{f'}) &\twoheadrightarrow \operatorname{Pic}(\mathcal{O}_f) \\ \mathfrak{a}\mathcal{O}_{f'} &\mapsto \mathfrak{a}\mathcal{O}_f. \end{aligned} \tag{3.2.6}$$

For a class $\mathcal{A} \in \operatorname{Pic}(\mathcal{O}_f)$, define the theta series $\vartheta_{\mathcal{A}}(z)$ by

$$\vartheta_{\mathcal{A}}(z) := \frac{1}{\#\mathcal{O}_f^\times} + \sum_{\mathfrak{a} \in I_K(f), [\mathfrak{a}] = \mathcal{A}} q^{\mathrm{Nm}(\mathfrak{a})} = \sum_{n \geq 0} r_{\mathcal{A}}(n) q^n. \tag{3.2.7}$$

Let $\psi : \operatorname{Pic}(\mathcal{O}_f) \to \mathbb{C}^\times$ be a character and define

$$g_\psi(z) := \sum_{\mathcal{A} \in \operatorname{Pic}(\mathcal{O}_f)} \psi(\mathcal{A}) \vartheta_{\mathcal{A}}(z) = \sum_{n \geq 1} c_\psi(n) q^n. \tag{3.2.8}$$

By Hecke's work [31], we know that $g_\psi \in M_1(\Gamma_0(|D|f^2), \chi_D \mathbb{1}_f)$ is an eigenform, where $\chi_D = \left(\frac{D}{\cdot}\right)$ is the Dirichlet character associated to K and $\mathbb{1}_f$ is the trivial character of conductor f. Furthermore, if ψ is not a genus character of $C(Df^2) \cong \operatorname{Pic}(\mathcal{O}_f)$, then $g_\psi(z)$ is a cuspform.

When $f = p$ is a prime that splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_K$, the ring $\mathcal{O}_K/p\mathcal{O}_K$ is isomorphic to $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ via

$$\begin{aligned} \mathcal{O}_K/p\mathcal{O}_K &\longrightarrow \mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z} \\ \alpha &\mapsto (\alpha \pmod{\mathfrak{p}}, \alpha \pmod{\overline{\mathfrak{p}}}). \end{aligned} \tag{3.2.9}$$

So the group $(\mathcal{O}_K/p\mathcal{O})^\times/(\mathbb{Z}/p\mathbb{Z})^\times$, which surjects onto $I_K(p) \cap P_K/P_{K,\mathbb{Z}}(p)$ with kernel $\mathcal{O}_K^\times/\{\pm 1\}$, is isomorphic to $(\mathbb{Z}/p\mathbb{Z})^\times$ via

$$\begin{aligned}(\mathbb{Z}/p\mathbb{Z})^\times \times (\mathbb{Z}/p\mathbb{Z})^\times &\longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \\ (u,v) &\mapsto uv^{-1}.\end{aligned}$$

Then we could restrict ψ to $(\mathcal{O}_K/p\mathcal{O})^\times/(\mathbb{Z}/p\mathbb{Z})^\times$ and view it as a character of $(\mathbb{Z}/p\mathbb{Z})^\times$.

3.3 Modular Forms of Imaginary Dihedral Type

This section will describe the weight one newforms we are interested in. First, we will study the structure of the ray class group and its characters. They give rise to holomorphic modular forms of imaginary dihedral type. Then we will analyze the relationship between the characters of ray class group and ring class group. Finally, we will discuss mock-modular forms with imaginary dihedral shadows.

3.3.1 Ray Class Group and Its Characters

Let $K = \mathbb{Q}(\sqrt{D})$ as before and $p \geq 3$ a rational prime ideal that splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_K$. Let $I_\mathfrak{p}$, resp. $P_\mathfrak{p}$, be the group of fractional, resp. principal fractional, ideals of $\mathcal{O}_K$ relatively prime to $\mathfrak{p}$ and $P_{\mathfrak{p},1}$ be the group of principal fractional ideals $\mathfrak{a}$ of K such that there exists $\alpha \in K$ satisfying $\mathfrak{a} = (\alpha)$ and $v_\mathfrak{p}(\alpha - 1) > 0$. Suppose $H_\mathfrak{p}$ is the ray class field of modulus $\mathfrak{p}$ with $M/\mathbb{Q}$ the Galois closure over $\mathbb{Q}$, then $H_\mathfrak{p}/K$ is abelian and contains H, the Hilbert class field of K. Furthermore, $\mathrm{Gal}(H_\mathfrak{p}/K)$ is canonically isomorphic to $I_\mathfrak{p}/P_{\mathfrak{p},1}$, which fits into exact sequences similar to (3.2.3) and (3.2.4)

$$\begin{gathered}1 \longrightarrow P_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow I_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow \mathrm{Pic}(\mathcal{O}_K) \longrightarrow 1 \\ 1 \longrightarrow \mathcal{O}_K^\times \longrightarrow (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow P_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow 1.\end{gathered} \tag{3.3.1}$$

The map $(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow P_\mathfrak{p}/P_{\mathfrak{p},1}$ comes from the embedding $\mathbb{Z} \hookrightarrow \mathcal{O}_K$.

Let $\phi : I_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ be a non-trivial character and ϕ_1 the composition

$$\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times/\mathcal{O}_K^\times \longrightarrow I_\mathfrak{p}/P_{\mathfrak{p},1} \xrightarrow{\phi} \mathbb{C}^\times$$

Since $-1 \in \mathcal{O}_K^\times$, ϕ_1 as a character of $(\mathbb{Z}/p\mathbb{Z})^\times$ has even order and there exists $\phi_2 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ satisfying

$$\phi_1\phi_2^2 = \mathbb{1}_p.$$

Let $\phi_2'(\cdot) := \phi_2(\cdot)\left(\frac{\cdot}{p}\right)$ be another character of conductor p. Then it also satisfies $\phi_1(\phi_2')^2 = \mathbb{1}_p$. For another $\phi' : I_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ giving rise to the same ϕ_1, we know that $\overline{\phi}\phi'$ factors through $(\mathbb{Z}/p\mathbb{Z})^\times/\mathcal{O}_K^\times$ and is a character of $\mathrm{Pic}(\mathcal{O}_K)$.

Let $\rho_\phi : G_\mathbb{Q} \longrightarrow \mathrm{GL}_2(\mathbb{C})$ be the induced representation from ϕ. Then the kernel of ρ_ϕ is contained in $\mathrm{Gal}(\overline{M}/M)$ and the projective image of $\rho(\phi)$ is dihedral. Since ϕ is non-trivial, ρ_ϕ is irreducible and

$$\det(\rho_\phi) = \chi_D\phi_1,$$

where $\chi_D = \left(\frac{D}{\cdot}\right)$. Let $c \in G_\mathbb{Q}$ be complex conjugation, then

$$\det(\rho_\phi(c)) = (\chi_D\phi_1)(-1) = -1.$$

So one can attach a weight one newform $f_\phi(z) \in S_1(|D|p, \chi_D\phi_1)$ of imaginary dihedral type to ρ_ϕ. It can be written out explicitly as

$$f_\phi(z) = \sum_{\mathfrak{a}\in I_\mathfrak{p}, \mathfrak{a}\subset\mathcal{O}_K} \phi(\mathfrak{a})q^{\mathrm{Nm}(\mathfrak{a})} = \sum_{n\geq 1} c_\phi(n)q^n. \tag{3.3.2}$$

From this expression, one can express the Fourier coefficients $c_\phi(n)$ explicitly in terms of the Hecke character ϕ.

Proposition 3.3.1. *The Fourier coefficients $c_\phi(n)$ is multiplicative with respect to n. For ℓ prime and $r \in \mathbb{N}$, we have*

$$c_\phi(\ell^r) = \begin{cases} \frac{\phi(\mathfrak{l})^{r+1}-\phi(\bar{\mathfrak{l}})^{r+1}}{\phi(\mathfrak{l})-\phi(\bar{\mathfrak{l}})} & (\ell) = \mathfrak{l}\bar{\mathfrak{l}} \text{ in } \mathcal{O}_K \\ \phi(\ell)^{r/2} & (\ell) \text{ inert in } \mathcal{O}_K \text{ and } 2 \mid r \\ 0 & (\ell) \text{ inert in } \mathcal{O}_K \text{ and } 2 \nmid r \\ (r+1)\phi(\mathfrak{l})^r & (\ell) = \mathfrak{l}^2 \text{ in } \mathcal{O}_K \\ \phi(\overline{\mathfrak{p}})^r & (\ell) = p \text{ in } \mathcal{O}_K. \end{cases} \tag{3.3.3}$$

Let T_ℓ be the Hecke operator defined by

$$T_\ell(f_\phi)(z) := \sum_{n\geq 1} c_\phi(\ell n)q^n + \sum_{n\geq 1}(\chi_D\phi_1)(\ell)c_\phi(n)q^{\ell n}. \tag{3.3.4}$$

Then $T_\ell(f_\phi) = c_\phi(\ell) f_\phi$ for all prime ℓ.

Proof. The multiplicativity of $c_\phi(n)$ follows from the Euler product of the L-series associated to f_ϕ, which is the same L-series associated to ρ_ϕ. Eq. (3.3.3) and $T_\ell(f_\phi) = c_\phi(\ell) f_\phi$ are both direct consequences of Eq. (3.3.2). □

Proposition 3.3.2. *In the notation above, suppose $f_\phi \in S_1(|D|p, \chi_D \phi_1)$. Then for all characters $\varphi : \mathrm{Pic}(\mathcal{O}_K) \longrightarrow \mathbb{C}^\times$,*

$$f_{\phi\varphi}(z) = \sum_{\mathfrak{a} \in I_\mathfrak{p}, \mathfrak{a} \subset \mathcal{O}_K} \phi(\mathfrak{a}) \varphi(\mathfrak{a}) q^{\mathrm{Nm}(\mathfrak{a})} \in S_1(|D|p, \chi_D \phi_1).$$

Proof. By Eq. (3.3.1), $\phi\varphi$ is a character of $I_\mathfrak{p}/P_{\mathfrak{p},1}$ and its composition with $(\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow I_\mathfrak{p}/P_{\mathfrak{p},1}$ is the same as $\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow I_\mathfrak{p}/P_{\mathfrak{p},1} \xrightarrow{\phi} \mathbb{C}^\times$ since $(\mathbb{Z}/p\mathbb{Z})^\times$ is in the kernel of φ. □

3.3.2 Genus Theory

In this section, we will decompose the form f_ϕ into the sum of forms from different genera. Define the set

$$\Sigma_D := \{d : d \equiv 1 \pmod 4, d \mid D\}, \ \tilde{\Sigma}_D := \Sigma/\sim, \tag{3.3.5}$$

where $d \sim d'$ if $dd' = D$. For each $d \in \Sigma_D$, write $d\mathcal{O}_K = \mathfrak{d}^2$ and let φ_d be the genus character of $\mathrm{Pic}(\mathcal{O}_K)$ determined by d. Notice that this is well-defined with respect to $\sim$. Since D is an odd, fundamental discriminant, $\tilde{\Sigma}_D$ has size $2^{\omega(D)-1}$ and is isomorphic as sets to the subgroup of $\mathrm{Pic}(\mathcal{O}_K)$ consisting of elements of order at most 2, which gives it a group structure.

For an arbitrary class $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, we could define the form $f_{\phi,\mathcal{A}}$ by

$$f_{\phi,\mathcal{A}}(z) := \frac{1}{H(D)} \sum_{\varphi : \mathrm{Pic}(\mathcal{O}_K) \to \mathbb{C}^\times} \varphi(\mathcal{A}^{-1}) f_{\phi\varphi}(z) = \sum_{n \geq 1} c_{\phi,\mathcal{A}}(n) q^n, \tag{3.3.6}$$

where $H(D)$ is the class number of $K = \mathbb{Q}(\sqrt{D})$. By the orthogonality of the characters φ, we could write the Fourier coefficient $c_{\phi,\mathcal{A}}(n)$ in the following form

$$c_{\phi,\mathcal{A}}(n) = \frac{1}{H(D)} \sum_{\varphi:\mathrm{Pic}(\mathcal{O}_K)\to\mathbb{C}^\times} \varphi(\mathcal{A}^{-1}) c_{\phi\varphi}(n) = \sum_{\substack{\mathfrak{a}\subset\mathcal{O}_K,\mathfrak{a}\in I_\mathfrak{p}\\ \mathrm{Nm}(\mathfrak{a})=n\\ [\mathfrak{a}]=\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)}} \phi(\mathfrak{a}). \tag{3.3.7}$$

Then we have the following decomposition

$$f_\phi(z) = \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)} f_{\phi,\mathcal{A}}(z), \tag{3.3.8}$$

which follows from the orthogonality of the characters.

Let $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ be a split prime in $\mathcal{O}_K$ and $\mathcal{L} = [\mathfrak{l}] \in \mathrm{Pic}(\mathcal{O}_K)$. As an immediate consequence of Prop. 3.3.1, we have

$$\begin{aligned} T_\ell(f_{\phi,\mathcal{A}}) &= \frac{1}{H(D)} \sum_{\varphi:\mathrm{Pic}(\mathcal{O}_K)\to\mathbb{C}^\times} \varphi(\mathcal{A}^{-1}) T_\ell(f_{\phi\varphi}) \\ &= \frac{1}{H(D)} \sum_{\varphi} \varphi(\mathcal{A}^{-1}) \left((\phi\varphi)(\mathfrak{l}) + (\phi\varphi)(\bar{\mathfrak{l}})\right) f_{\phi\varphi} \\ &= \phi(\mathfrak{l}) f_{\phi,\mathcal{A}\mathcal{L}} + \phi(\bar{\mathfrak{l}}) f_{\phi,\mathcal{A}\mathcal{L}^{-1}}. \end{aligned} \tag{3.3.9}$$

Now, define $d_\mathcal{A} \in \Sigma_D$ by

$$d_\mathcal{A} := \prod_{\ell|D,\ \varphi_{\ell^*}(\mathcal{A})=-1} \ell^*, \tag{3.3.10}$$

where ℓ^* is given in Eq. (2.4.13). The following result tells us about the vanishing of the coefficients $c_{\phi,\mathcal{A}}(n)$.

Proposition 3.3.3. *The Fourier coefficient $c_{\phi,\mathcal{A}}(n)$ vanishes whenever $\left(\frac{d}{n}\right) \neq \varphi_d(\mathcal{A})$ for some $d \in \Sigma_D$. Equivalently, $f_{\phi,\mathcal{A}} \in S_{1,d_\mathcal{A}}(|D|p, \chi_D\phi_1)$.*

Proof. In the last sum in Eq. (3.3.7), the ideal $\mathfrak{a} \subset \mathcal{O}_K$ is in the same genus class as $\mathcal{A}$, which implies that $\varphi_d(\mathcal{A}) = \varphi_d(\mathfrak{a}) = \left(\frac{d}{\mathrm{Nm}(\mathfrak{a})}\right)$ for all $d \in \Sigma_D$. Thus, this sum is empty, hence zero, if $\left(\frac{d}{n}\right) \neq \varphi_d(\mathcal{A})$ for some $d \in \Sigma_D$. This is equivalent to

$$\left(\tfrac{d}{n}\right) c_{\phi,\mathcal{A}}(n) = \varphi_d(\mathcal{A}) c_{\phi,\mathcal{A}}(n) \text{ whenever } \gcd(n,d) = 1. \tag{3.3.11}$$

By taking $d = \ell^*$ for $\ell \mid D$ a positive prime and applying Prop. 2.4.1, we see that

$$\begin{aligned}\mathrm{pr}^+_{\chi_\ell} f_{\phi,\mathcal{A}} &= 0 \text{ for all } \ell \mid d_{\mathcal{A}},\\ \mathrm{pr}^-_{\chi_\ell} f_{\phi,\mathcal{A}} &= 0 \text{ for all } \ell \mid \frac{|D|}{d_{\mathcal{A}}}.\end{aligned}$$

Using Prop. 3.3.1, we could deduce that

$$\overline{\phi_1(n)} c_{\phi,\mathcal{A}}(n) = \overline{c_{\phi,\mathcal{A}}(n)}$$

whenever $\gcd(n,p) = 1$ for all $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$. Prop. 2.4.2 then implies that $\mathrm{pr}^-_{\phi_1}(f_{\phi,\mathcal{A}}) = 0$. By Definition 2.4.11, $f_{\phi,\mathcal{A}} \in S_{1,d_{\mathcal{A}}}(|D|p, \chi_D\phi_1)$. □

For each $d \in \Sigma_D$, we could now define $f_{\phi,d} \in S_{1,d}(|D|p, \chi_D\phi_1)$ by

$$f_{\phi,d}(z) := \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K),\, d_{\mathcal{A}}=d} f_{\phi,\mathcal{A}}(z) = \sum_{n\geq 1} c_{\phi,d}(n) q^n. \tag{3.3.12}$$

Then $f_\phi = \sum_{d\in\Sigma_D} f_{\phi,d}$ with each summand containing the information in different genera of $\mathrm{Pic}(\mathcal{O}_K)$. For example, $d_{\mathcal{A}} = 1$ if and only if $\mathcal{A}$ is in the principal genus of $\mathrm{Pic}(\mathcal{O}_K)$, also a subgroup of $\mathrm{Pic}(\mathcal{O}_K)$. Since there are $2^{\omega(D)-1}$ genera and every genus is represented by a class in $\mathrm{Pic}(\mathcal{O}_K)$, the size of the principal genus is $\frac{H(D)}{2^{\omega(D)-1}}$. The principal genus contains the subgroup $\mathrm{Pic}^2(\mathcal{O}_K)$ of $\mathrm{Pic}(\mathcal{O}_K)$, consisting of squared classes. By Lemma 4.1.5, the size of $\mathrm{Pic}_0(\mathcal{O}_K)$ is the same as the number of genera. Thus, $\mathrm{Pic}^2(\mathcal{O}_K)$ has size $\frac{H(D)}{2^{\omega(D)-1}}$ and is exactly the principal genus. For each $d \in \Sigma_D$, pick any $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ such that $d_{\mathcal{A}} = d$, if it exists. Then we could write

$$f_{\phi,d} = \sum_{\mathcal{C}\in\mathrm{Pic}^2(\mathcal{O}_K)} f_{\phi,\mathcal{AC}} = \frac{1}{2^{\omega(D)-1}} \sum_{\mathcal{B}\in\mathrm{Pic}(\mathcal{O}_K)} f_{\phi,\mathcal{AB}^2}. \tag{3.3.13}$$

Let $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ be a split prime ideal in $\mathcal{O}_K$ and $\mathcal{L} = [\mathfrak{l}] \in \mathrm{Pic}(\mathcal{O}_K)$. Applying Eq. (3.3.9) produces

$$\begin{aligned} T_\ell f_{\phi,d} &= \frac{1}{2^{\omega(D)-1}} \sum_{\mathcal{B}\in\mathrm{Pic}(\mathcal{O}_K)} \left(\phi(\mathfrak{l}) f_{\phi,\mathcal{AB}^2\mathcal{L}} + \phi(\bar{\mathfrak{l}}) f_{\phi,\mathcal{AB}^2\mathcal{L}^{-1}}\right)\\ &= \left(\phi(\mathfrak{l}) + \phi(\bar{\mathfrak{l}})\right) f_{\phi,d'}, \end{aligned} \tag{3.3.14}$$

where $d' = d_{\mathcal{AL}} = d_{\mathcal{AL}^{-1}}$.

3.3.3 Relationship to Characters of $\mathrm{Pic}(\mathcal{O}_p)$

After understanding the relationship between f_ϕ and $\mathrm{Pic}(\mathcal{O}_K)$, we will now consider its relationship to $\mathrm{Pic}(\mathcal{O}_p)$. The following result tells us that the twist of $f_\phi = \sum_{n\geq 1} c_\phi(n)q^n$ by ϕ_2, defined as

$$f_\phi \otimes \phi_2 := \frac{1}{G(\overline{\phi_2})} \sum_{\mu=1}^{p} \overline{\phi_2}(\mu) f_\phi \mid_1 \left(\begin{smallmatrix} p & \mu \\ & p \end{smallmatrix}\right) = \sum_{n\geq 1} \phi_2(n) c_\phi(n) q^n, \tag{3.3.15}$$

is a newform of the shape (3.2.8) for some character ψ of $\mathrm{Pic}(\mathcal{O}_p)$.

Proposition 3.3.4. *In the notation above, let $\psi : \mathrm{Pic}(\mathcal{O}_p) \longrightarrow \mathbb{C}^\times$ be a character defined by*

$$\begin{aligned}\psi : \mathrm{Pic}(\mathcal{O}_p) &\longrightarrow \mathbb{C}^\times \\ &\qquad \mathfrak{a} \mapsto \phi(\mathfrak{a})\phi_2(\mathrm{Nm}(\mathfrak{a})).\end{aligned}$$

Suppose $\phi : I_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ is non-trivial. Then $f_\phi \otimes \phi_2$ satisfies

$$(f_\phi \otimes \phi_2)(z) = \sum_{\mathcal{A}' \in \mathrm{Pic}(\mathcal{O}_p)} \psi(\mathcal{A}')\vartheta_{\mathcal{A}'}(z) \in S_1(|D|p^2, \chi_{Dp^2}).$$

Proof. The level and nebentypus of $f_\phi \otimes \phi_2$ is given by Prop. 3.1 in [3]. Note that $\chi_{Dp^2} = \chi_D \mathbb{1}_p$ since $p \nmid D$. To prove the rest of the proposition, we first need to check that $\psi : I_K(p) \longrightarrow \mathbb{C}^\times$ is a well-defined character of $\mathrm{Pic}(\mathcal{O}_p) \cong I_K(p)/P_{K,\mathbb{Z}}(p)$. This is clear from the following calculations

$$\begin{aligned}\psi(\mathfrak{a}) =& \phi(\mathfrak{a})\phi_2\left(\tfrac{a^2+b^2p^2D}{r^2}\right) = \phi_1\left(\tfrac{a+bp\sqrt{D}}{r}\right)\phi_2\left(\tfrac{a^2}{r^2}\right) \\ =& (\phi_1\phi_2^2)(a^2/r^2) = 1\end{aligned}$$

for all $\mathfrak{a} = \left(\frac{a+bp\sqrt{D}}{r}\right) \in P_{K,\mathbb{Z}}(p)$.

Now since $f_\phi \otimes \phi_2$ and $g_\psi(z) := \sum_{\mathcal{A}' \in \mathrm{Pic}(\mathcal{O}_p)} \psi(\mathcal{A}')\vartheta_{\mathcal{A}'}(z) = \sum_{n\geq 1} c_\psi(n)q^n$ are both eigenforms in $S_1(|D|p^2, \chi_{Dp^2})$, it is enough to compare their ℓ^{th} coefficients, for almost all primes ℓ, to show that they are the same. Suppose $(\ell) = \mathfrak{l}\bar{\mathfrak{l}}$ splits in $\mathcal{O}_K$, then

$$\begin{aligned}c_\phi(n)\phi_2(n) &= (\phi(\mathfrak{l}) + \phi(\bar{\mathfrak{l}}))\phi_2(\ell) \\ &= \phi(\mathfrak{l})\phi_2(\mathrm{Nm}(\mathfrak{l})) + \phi(\bar{\mathfrak{l}})\phi_2(\mathrm{Nm}(\bar{\mathfrak{l}})) \\ &= \psi(\mathfrak{l}) + \psi(\bar{\mathfrak{l}}) = c_\psi(\ell).\end{aligned}$$

When ℓ is inert in $\mathcal{O}_K$, $c_\phi(\ell)\phi_2(\ell) = c_\psi(\ell) = 0$. Finally the $(pn)^{\text{th}}$ coefficient can be written as

$$\begin{aligned} c_\psi(pn) &= \sum_{\mathcal{A}' \in \operatorname{Pic}(\mathcal{O}_p)} \psi(\mathcal{A}') r_{\mathcal{A}'}(pn) = \sum_{\mathcal{A}' \in \operatorname{Pic}(\mathcal{O}_p)} \psi(\mathcal{A}') r_{\pi(\mathcal{A}')}(n/p) \\ &= \sum_{\mathcal{B} \in \operatorname{Pic}(\mathcal{O}_K)} \psi(\mathcal{B}) r_{\mathcal{B}}(n/p) \sum_{\substack{\mathcal{A}' \in \operatorname{Pic}(\mathcal{O}_p) \\ \mathcal{A}' \in \ker(\pi)}} \psi(\mathcal{A}'). \end{aligned}$$

Here $\pi : \operatorname{Pic}(\mathcal{O}_p) \longrightarrow \operatorname{Pic}(\mathcal{O}_K)$ is the projection in Eq. (3.2.6) with $f' = p$ and $f = 1$. Since ϕ_1 is non-trivial, ψ does not factor through $\pi : \operatorname{Pic}(\mathcal{O}_p) \longrightarrow \operatorname{Pic}(\mathcal{O}_K)$ and $c_\psi(pn) = 0 = c_\phi(pn)\phi_2(pn)$.

Thus, the difference $f_\phi \otimes \phi_2 - g_\psi$ is of the form $h(dz)$ for some $d \mid D$ and modular form $h(z) \in \overline{\mathbb{Q}}[\![q]\!]$ of level dividing D. Since this difference has level Dp^2 and D is square-free, we must have $h(z) = 0$. □

Let ψ be defined as above. Consider $m + n\sqrt{D} \in \mathcal{O}_K$ in the following subgroup of $\operatorname{Pic}(\mathcal{O}_p)$

$$\begin{aligned} \ker(\pi : \operatorname{Pic}(\mathcal{O}_p) \longrightarrow \operatorname{Pic}(\mathcal{O}_K)) &= (\mathcal{O}_K/p\mathcal{O}_K)^\times / \big((\mathbb{Z}/p\mathbb{Z}) \times \mathcal{O}_K^\times/\{\pm 1\}\big) \\ &\cong (\mathbb{Z}/p\mathbb{Z})^\times / (\mathcal{O}_K^\times/\{\pm 1\}), \end{aligned}$$

which is defined in the exact sequence (3.2.4). If $v_{\mathfrak{p}}(m + n\sqrt{D} - a) > 0$ with $a \in \mathbb{Z}$, then the value of ψ on $m + n\sqrt{D}$ becomes

$$\psi(m + n\sqrt{D}) = \phi_1(m + n\sqrt{D})\phi_2(m^2 - n^2 D) = \phi_1(a)\phi_2(2am - a^2) = \phi_2\left(\tfrac{2m}{a} - 1\right).$$

By changing $m + n\sqrt{D}$ to $m + \lambda + n\sqrt{D}$ for $\lambda \in \mathbb{Z}$, the quantity $\frac{2(m+\lambda)}{a+\lambda} - 1$ runs through all residue classes modulo p. Thus, the restriction of ψ to the kernel of $\pi : \operatorname{Pic}(\mathcal{O}_p) \longrightarrow \operatorname{Pic}(\mathcal{O}_K)$ is ϕ_2.

For each $\mathcal{A} \in \operatorname{Pic}(\mathcal{O}_K)$, define

$$g_{\psi,\mathcal{A}} := f_{\phi,\mathcal{A}} \otimes \phi_2 = \sum_{n \geq 1} c_{\psi,\mathcal{A}}(n), \tag{3.3.16}$$

$$c_{\psi,\mathcal{A}}(n) = \sum_{\substack{\mathfrak{a} \subset \mathcal{O}_K, \mathfrak{a} \in I_K(p), \\ \operatorname{Nm}(\mathfrak{a}) = n, \\ \pi([\mathfrak{a}]) = \mathcal{A} \in \operatorname{Pic}(\mathcal{O}_K)}} \psi(\mathfrak{a}) = \sum_{\substack{\mathcal{A}' \in \operatorname{Pic}(\mathcal{O}_p) \\ \pi(\mathcal{A}') = \mathcal{A}}} \psi(\mathcal{A}') r_{\mathcal{A}'}(n), \tag{3.3.17}$$

where $\pi : \mathrm{Pic}(\mathcal{O}_p) \longrightarrow \mathrm{Pic}(\mathcal{O}_K)$ is the projection maps as in (3.2.6). Note that if $n = pn'$, then $r_{\mathcal{A}'}(pn') = r_{\mathcal{A}}\left(\frac{n'}{p}\right)$ and

$$\sum_{\substack{\mathcal{A}'\in\mathrm{Pic}(\mathcal{O}_p)\\ \pi(\mathcal{A}')=\mathcal{A}}} \psi(\mathcal{A}') = \psi(\mathcal{A}'') \sum_{\mathcal{B}'\in\ker(\pi)} \phi_2(\mathcal{B}') = 0 \tag{3.3.18}$$

since ϕ_2 is non-trivial. Here $\mathcal{A}'' \in \mathrm{Pic}(\mathcal{O}_p)$ is any class whose image under π is $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$. Thus, $c_{\psi,\mathcal{A}}(pn') = 0$.

As immediate consequences of Props. 3.3.3 and 3.3.4, we have

$$g_\psi(z) = \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)} g_{\psi,\mathcal{A}}(z), \tag{3.3.19}$$

$$\left(\tfrac{d}{n}\right) c_{\psi,\mathcal{A}}(n) = \varphi_d(\mathcal{A}) c_{\psi,\mathcal{A}}(n) \text{ whenever } \gcd(n,d) = 1, \tag{3.3.20}$$

for all $d \in \Sigma_D$. When $\mathcal{A} = \mathcal{A}_0 \in \mathrm{Pic}(\mathcal{O}_K)$ is the principal class, the Fourier coefficients $c_{\phi,\mathcal{A}_0}(n)$ have the following properties.

Proposition 3.3.5. *When $n < \frac{|D|}{4}$, the Fourier coefficient $c_{\psi,\mathcal{A}_0}(n)$ has the form*

$$c_{\psi,\mathcal{A}_0}(n) = \begin{cases} 1, & \text{when } n = k^2, p \nmid k \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.21}$$

Proof. Let $1 \le n < \frac{|D|}{4}$ be a positive integer and $\mathfrak{a} \subset \mathcal{O}_K$ an ideal in the set

$$I_{K,p}(\mathcal{A}_0, n) := \{\mathfrak{a} \subset \mathcal{O} : \mathfrak{a} \in I_K(p), \mathrm{Nm}(\mathfrak{a}) = n, \pi([\mathfrak{a}]) = \mathcal{A}_0\}.$$

By the exact sequences (3.2.3) and (3.2.4), we know that $\ker(\pi)$ consists of principal ideals $\mathfrak{a} = (\alpha)$, $\alpha \in K$, such that

$$\mathrm{ord}_{\mathfrak{p}}(\alpha) = \mathrm{ord}_{\bar{\mathfrak{p}}}(\alpha) = 0.$$

When $\alpha \in \mathcal{O}_K$, we could write $\alpha = \frac{a+b\sqrt{D}}{2}$ with $a, b \in \mathbb{Z}$. Since $\mathrm{Nm}(\mathfrak{a}) = \mathrm{Nm}(\alpha) = n < \frac{|D|}{4}$, we have $b = 0$. In that case, the ideal $\mathfrak{a} = (\alpha)$ is in $P_{K,\mathbb{Z}}(p)$, hence principal in $\mathrm{Pic}(\mathcal{O}_p)$. So when $n < \frac{|D|}{4}$, the set $I_{K,p}(\mathcal{A}_0, n)$ is non-empty only for n a square not divisible by p, in which case it has only one element. So whenever $n < \frac{|D|}{4}$ is a perfect square not divisible by p, Eq. (3.3.17) tells us that

$$c_{\psi,\mathcal{A}_0}(n) = \sum_{\mathfrak{a}\in I_{K,p}(\mathcal{A}_0,n)} \psi(\mathfrak{a}) = 1.$$

□

Corollary 3.3.6. *When $n < \frac{|D|}{4}$, the Fourier coefficient $c_{\phi,\mathcal{A}_0}(n)$ has the form*

$$c_{\phi,\mathcal{A}_0}(n) = \begin{cases} \phi_1(k), & \text{when } n = k^2, p \nmid k, \\ \phi(\overline{\mathfrak{p}})c_{\phi,[\overline{\mathfrak{p}}]^{-1}}(m), & \text{when } n = pm, \\ 0, & \text{otherwise.} \end{cases} \tag{3.3.22}$$

Proof. Using the relation $c_{\psi,\mathcal{A}}(n) = c_{\phi,\mathcal{A}}(n)\phi_2(n)$, it is easy to deduce Eq. (3.3.22) from Prop. 3.3.5 when $p \nmid n$. Otherwise for $n = pm$, Prop. 3.3.1 and Eq. (3.3.7) implies that

$$c_{\phi,\mathcal{A}_0}(pm) = \frac{1}{H(D)}\sum_{\varphi} c_{\phi\varphi}(pm) = \frac{1}{H(D)}\sum_{\varphi}(\phi\varphi)(\overline{\mathfrak{p}})c_{\phi\varphi}(m) = \phi(\overline{\mathfrak{p}})c_{\phi,[\overline{\mathfrak{p}}]^{-1}}(m).$$

□

3.4 Mock-Modular Forms with Imaginary Dihedral Shadow

3.4.1 Petersson Inner Product of Newforms

For an integer M, let $E_M(z,s)$ be the non-holomorphic Eisenstein series of weight zero, level M defined by

$$E_M(z,s) := \sum_{\gamma \in \tilde{\Gamma}_\infty \backslash \Gamma_0(M)} (\mathrm{Im}(\gamma z))^s, \tag{3.4.1}$$

where $\tilde{\Gamma}_\infty = \{\left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})\}$. In particular, we can write

$$E_1(z,s) = \frac{1}{2}\sum_{\substack{c,d\in\mathbb{Z} \\ \gcd(c,d)=1, \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}} = \frac{1}{2\zeta(2s)}\sum_{\substack{c,d\in\mathbb{Z} \\ (c,d)\neq(0,0)}} \frac{y^s}{|cz+d|^{2s}}.$$

For convenience, we denote $E_1(z,s)$ by $E(z,s)$. It has a simple pole at $s = 1$ and the well-known expansion

$$E(z,s) = y^s + \varphi(s)y^{1-s} + O(e^{-y})$$

as $y \to \infty$, where $z = x + iy$ and

$$\varphi(s) = \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\frac{\zeta(2s-1)}{\zeta(2s)}.$$

Kronecker's first limit formula states that

$$2\zeta(2s)E(z,s) = \frac{\pi}{s-1} + 2\pi\left(\gamma - \log(2) - \log(\sqrt{y}|\eta(z)|^2)\right) + O(s-1), \tag{3.4.2}$$

where γ is the Euler constant. The factor of 2 comes from $\pm I \in \tilde{\Gamma}_\infty$.

By Eq. (II 2.16) in [29], the Eisenstein series $E_M(z,s)$ can be expressed in terms of $E(z,s)$ as

$$E_M(z,s) = M^{-s}\prod_{\ell|M}(1-\ell^{-2s})^{-1}\sum_{d|M}\frac{\mu(d)}{d^s}E\left(\frac{M}{d}z, s\right), \tag{3.4.3}$$

where $\mu(d)$ is the Möbius function. Using (3.4.2) and Rankin-Selberg unfolding trick, we can relate the inner product between dihedral newforms to values of modular functions.

Proposition 3.4.1. *Let $\phi, \phi' : I_{\mathfrak{p}}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ be ray class group characters and $\phi_1, \phi_1' : (\mathbb{Z}/p\mathbb{Z})^\times/\mathcal{O}_K^\times \longrightarrow \mathbb{C}^\times$ their associated characters via* (3.3.1)*. Suppose that $\phi_1 = \phi_1'$ are not quadratic. Then inner product $\langle f_\phi, f_{\phi'}\rangle$ vanishes unless $\phi' = \phi$, in which cases we have*

$$\begin{aligned} \langle f_\phi, f_\phi\rangle &= -\frac{4H(D)}{\#\mathcal{O}_K^\times \#\mathcal{O}_p^\times} I_{\psi^2}, \\ I_{\psi^2} &:= \sum_{[Q]\in C(Dp^2)} \psi^2(Q)\log(\sqrt{y_Q}|\eta(\tau_Q)|^2), \end{aligned} \tag{3.4.4}$$

where ψ is the character of $\mathrm{Pic}(\mathcal{O}_p) \cong C(Dp^2)$ associated to ϕ by Prop. 3.3.4 and $\tau_Q = x_Q + iy_Q = \frac{B+p\sqrt{D}}{2}$ is the CM point associated to the binary quadratic form $Q = [A,B,C]$.

Proof. Let $\phi_2 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ be a character such that $\phi_1\phi_2^2 = \mathbb{1}_p$. Since $\phi_1 = \phi_1'$, the map $(\overline{\phi}\phi')\circ \mathrm{Nm}$ is trivial and we could write

$$\phi = \phi'\varphi$$

for some character $\varphi : \mathrm{Pic}(\mathcal{O}_K) \longrightarrow \mathbb{C}^\times$. By Prop. 3.3.2, we could associate ring class group character ψ and ψ' to ϕ and ϕ' respectively. Defined a character ψ_2 of $\mathrm{Pic}(\mathcal{O}_p)$ by

$$\psi_2(\mathfrak{a}) := \phi(\mathfrak{a})\overline{\phi(\overline{\mathfrak{a}})} = \psi(\mathfrak{a})\overline{\psi'(\overline{\mathfrak{a}})}. \tag{3.4.5}$$

When $\phi = \phi'$, we have

$$\psi_2(\mathfrak{a}) = \phi(\mathfrak{a})\overline{\phi(\overline{\mathfrak{a}})} = \phi^2(\mathfrak{a})\overline{\phi(\mathrm{Nm}(\mathfrak{a}))} = \psi^2(\mathfrak{a}).$$

Since $\phi_1 = \phi_1'$ is not quadratic, we have

$$\phi(a) = \phi_1(a) \neq \phi_1'(\overline{a}) = \phi'(\overline{a})$$

for some $a \in (\mathbb{Z}/p\mathbb{Z})^\times/\mathcal{O}_K^\times$ and ψ_2 is **not** the trivial character. By class field theory, the characters φ, ψ, ψ_2 are also characters of $\mathrm{Gal}(\overline{K}/K)$. Denote their induced two-dimensional representations of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ by ρ_φ, ρ_ψ and ρ_{ψ_2} respectively.

Let $M = Np$ be a square-free integer. Since the residue of $E(z,s)$ at $s = 1$ is $\frac{3}{\pi}$ and independent of z, the residue of $E_M(z,s)$ at $s = 1$ is

$$\begin{aligned}\mathrm{Res}_{s=1} E_M(z,s) &= \frac{3}{\pi} M^{-1} \prod_{\ell | M} (1-\ell^{-2})^{-1} \sum_{d|M} \frac{\mu(d)}{d} \\ &= \frac{3}{\pi} \prod_{\ell|M} (1+\ell)^{-1}.\end{aligned}$$

This gives us the relationship

$$\frac{3}{\pi} \prod_{\ell|M} (1+\ell)^{-1} \cdot \langle f_\phi, f_{\phi'} \rangle = \mathrm{Res}_{s=1} \int_{\Gamma_0(M)\backslash\mathcal{H}} f_\phi(z) \overline{f_{\phi'}(z)} E_M(z,s) y \frac{dxdy}{y^2}. \tag{3.4.6}$$

Now, we can use the Rankin-Selberg method to unfold the right hand side and obtain

$$\begin{aligned}\int_{\Gamma_0(M)\backslash\mathcal{H}} f_\phi(z)\overline{f_{\phi'}(z)} E_M(z,s) y \frac{dxdy}{y^2} &= \frac{\Gamma(s)}{(4\pi)^s} L(s, \phi, \overline{\phi'}), \\ L(s,\phi,\overline{\phi'}) &:= \frac{c_\phi(n)\overline{c_{\phi'}(n)}}{n^s}.\end{aligned} \tag{3.4.7}$$

Up to Euler factors at primes dividing M, the function $L(s,\phi,\phi')$ equals to $L(s, \rho_\phi \otimes \overline{\rho_{\phi'}})$, the L-function of the tensor product of the representations ρ_ϕ and $\rho_{\phi'}$. Alternatively, we could explicitly compute the Euler factors at all the places using Prop. 3.3.1. For a prime ℓ, we want to evaluate the sum

$$\sum_{r\geq 0} \frac{c_\phi(\ell^r)\overline{c_{\phi'}(\ell^r)}}{(\ell^s)^r}. \tag{3.4.8}$$

If (ℓ) splits into $\mathfrak{l}\bar{\mathfrak{l}}$ in $\mathcal{O}_K$ and $\ell \neq p$, then Eq. (3.4.8) becomes

$$\sum_{r\geq 0} \frac{\left(\sum_{t=0}^{r} \phi(\mathfrak{l})^t \phi(\bar{\mathfrak{l}})^{r-t}\right)\left(\sum_{t'=0}^{r} \overline{\phi'(\mathfrak{l})^{t'} \phi'(\bar{\mathfrak{l}})^{r-t'}}\right)}{(\ell^s)^r} = \sum_{r\geq 0} \frac{\sum_{t,t'=0}^{r} \phi(\mathfrak{l})^t \phi(\bar{\mathfrak{l}})^{r-t} \overline{\phi'(\mathfrak{l})^{t'} \phi'(\bar{\mathfrak{l}})^{r-t'}}}{(\ell^s)^r}$$

$$= (1-\ell^{-2s})\left(1-\phi(\mathfrak{l})\overline{\phi'(\mathfrak{l})}\ell^{-s}\right)^{-1}\left(1-\phi(\bar{\mathfrak{l}})\overline{\phi'(\bar{\mathfrak{l}})}\ell^{-s}\right)^{-1}$$
$$\left(1-\phi(\bar{\mathfrak{l}})\overline{\phi'(\mathfrak{l})}\ell^{-s}\right)^{-1}\left(1-\phi(\mathfrak{l})\overline{\phi'(\bar{\mathfrak{l}})}\ell^{-s}\right)^{-1}$$
$$= (1-\ell^{-2s})\left(1-\varphi(\mathfrak{l})\ell^{-s}\right)^{-1}\left(1-\varphi(\bar{\mathfrak{l}})\ell^{-s}\right)^{-1}\left(1-\psi_2(\mathfrak{l})\ell^{-s}\right)^{-1}\left(1-\psi_2(\mathfrak{l})\ell^{-s}\right)^{-1}.$$

If (ℓ) is inert in $\mathcal{O}_K$, then $\psi_2(\ell) = 1$ and Eq. (3.4.8) becomes

$$\sum_{r\geq 0} \frac{\phi(\ell)^r \overline{\phi'(\ell)^r}}{(\ell^s)^{2r}} = (1-\phi(\ell)\overline{\phi'(\ell)}\ell^{-2s})^{-1} = (1-\ell^{-2s})^{-1}.$$

If $(\ell) = (p) = \mathfrak{p}\bar{\mathfrak{p}}$ or $(\ell) = \mathfrak{l}^2$ is ramified in $\mathcal{O}_K$, then $\psi_2(\mathfrak{l}) = 0$ or

$$\psi_2(\mathfrak{l}) = \phi(\mathfrak{l})\phi'(\mathfrak{l})\overline{\phi'(\mathfrak{l})\phi'(\bar{\mathfrak{l}})} = \phi(\mathfrak{l}^2)\varphi(\mathfrak{l})\overline{\phi'(\ell)} = \varphi(\mathfrak{l}).$$

respectively and Eq. (3.4.8) becomes

$$\sum_{r\geq 0} \frac{\phi(\mathfrak{l})^r \overline{\phi'(\mathfrak{l})^r}}{(\ell^s)^r} = (1-\varphi(\mathfrak{l})\ell^{-s})^{-1}.$$

Multiplying these together, we find that

$$L(s,\phi,\overline{\phi'}) = \frac{L(s,\rho_{\psi_2})L(s,\rho_\varphi)(1-\varphi(\mathfrak{p})p^{-s})}{\zeta(2s)(1-p^{-2s})} \prod_{\ell | N}(1+\varphi(\mathfrak{l})\ell^{-s})^{-1}, \tag{3.4.9}$$

By definition, we could write

$$L(s,\rho_{\psi_2}) = \sum_{\mathcal{A}\in \mathrm{Pic}(\mathcal{O}_p)} \psi_2(\mathcal{A})L(s,\mathcal{A}),$$
$$L(s,\mathcal{A}) := \sum_{\substack{\mathfrak{a}\in\mathcal{O}_p \\ [\mathfrak{a}]=\mathcal{A}}} \mathrm{Nm}(\mathfrak{a})^{-s} = \frac{1}{\#\mathcal{O}_p^\times} \sum_{\substack{\alpha\in\mathfrak{b} \\ [\mathfrak{b}]=\mathcal{A}^{-1}}} \left(\frac{\mathrm{Nm}(\alpha)}{\mathrm{Nm}(\mathfrak{b})}\right)^{-s}.$$

where $\mathfrak{b} = [A, \frac{B+p\sqrt{D}}{2}] \subset \mathcal{O}_p$ be an ideal such that $[\mathfrak{b}] = \mathcal{A}^{-1}$ and $\mathrm{Nm}(\mathfrak{b}) = A$. Set $C = (B^2 - p^2 D)/4A$ and we have the following bijections

$$\{(m,n)\in\mathbb{Z}^2\} \leftrightarrow \{\mathrm{Nm}(\alpha) : \alpha\in\mathfrak{b}\}$$
$$(m,n) \leftrightarrow \mathrm{Nm}\left(Am + \frac{B+p\sqrt{D}}{2}n\right) = \mathrm{Nm}(\mathfrak{b})(Am^2+Bmn+Cn^2).$$

Then we have

$$\begin{aligned} L(s,\mathcal{A}) =& \frac{1}{\#\mathcal{O}_p^\times} \sum\nolimits'_{m,n\in\mathbb{Z}} \left(Am^2+Bmn+Cn^2\right)^{-s} \\ =& \frac{(p\sqrt{|D|}/2)^{-s}}{\#\mathcal{O}_p^\times} \sum\nolimits'_{m,n\in\mathbb{Z}} \frac{y_Q^s}{|m+n\tau_Q|^{2s}} \\ =& \frac{2^{1+s}\zeta(2s)}{(p\sqrt{|D|})^s\#\mathcal{O}_p^\times} E(\tau_Q,s) \end{aligned}$$

where $Q=[A,B,C]$ and $\tau_Q = x_Q+iy_Q = \frac{-B+p\sqrt{D}}{2A}$. By the isomorphism (3.2.1), the function $L(s,\psi_2)$ becomes

$$L(s,\rho_{\psi_2}) = \frac{2^{1+s}\zeta(2s)}{(p\sqrt{|D|})^s\#\mathcal{O}_p^\times} \sum_{[Q]\in C(Dp^2)} \psi_2(\tau_Q)E(\tau_Q,s). \tag{3.4.10}$$

Similarly, we could deduce that

$$L(s,\rho_\varphi) = \frac{2^{1+s}\zeta(2s)}{(\sqrt{|D|})^s\#\mathcal{O}_K^\times} \sum_{[Q]\in C(D)} \varphi(\tau_Q)E(\tau_Q,s). \tag{3.4.11}$$

From Eqs. (3.4.10), (3.4.11), (3.4.9) and the fact that ψ_2 is not trivial, we see that $L(s,\phi,\phi')$ has a pole at $s=1$ when φ is trivial. In that case,

$$\psi_2(\mathfrak{l}) = \phi(\mathfrak{l})\overline{\phi(\bar{\mathfrak{l}})}.$$

Now putting together Eqs. (3.4.2), (3.4.6), (3.4.7), (3.4.10) and (3.4.11), we obtain Eq. (3.4.4). □

Corollary 3.4.2. *Let $\mathcal{A}_1,\mathcal{A}_2\in \mathrm{Pic}(\mathcal{O}_K)$ and ϕ be a ray class group character such that ϕ_1 is non-quadratic. Then we have*

$$\begin{aligned} \langle f_{\phi,\mathcal{A}_1}, f_{\phi,\mathcal{A}_2}\rangle &= -\frac{4}{\#\mathcal{O}_K^\times\#\mathcal{O}_p^\times} I_{\psi^2,\mathcal{A}_1(\mathcal{A}_2)^{-1}}, \\ I_{\psi^2,\mathcal{B}^2} &:= \sum_{\substack{[Q]\in C(Dp^2)\\ \pi(Q)^2=\mathcal{B}^2}} \psi^2([Q])\log(\sqrt{y_Q}|\eta(\tau_Q)|^2). \end{aligned} \tag{3.4.12}$$

Remark 3.4.3. *Notice that if $\mathcal{A}_1(\mathcal{A}_2)^{-1}\notin \mathrm{Pic}^2(\mathcal{O}_K)$, then the inner product above vanishes.*

Proof. By Eq. (3.3.6) and Prop. 3.4.1, we have

$$\begin{aligned}
\langle f_{\phi,\mathcal{A}_1}, f_{\phi,\mathcal{A}_2}\rangle &= \frac{1}{H(D)^2}\sum_{\varphi_1,\varphi_2}\varphi_1({\mathcal{A}_1}^{-1})\overline{\varphi_2((\mathcal{A}_2)^{-1})}\langle f_{\phi\varphi_1}, f_{\phi\varphi_2}\rangle \\
&= \frac{1}{H(D)^2}\sum_{\varphi_1}\varphi_1({\mathcal{A}_1}^{-1})\varphi_1(\mathcal{A}_2)\langle f_{\phi\varphi_1}, f_{\phi\varphi_1}\rangle \\
&= -\frac{4}{H(D)\#\mathcal{O}_K^\times\#\mathcal{O}_p^\times}\sum_{\varphi_1}\varphi_1({\mathcal{A}_1}^{-1}\mathcal{A}_2)\sum_{[Q]\in C(Dp^2)}\psi^2(Q)\varphi_1^2(Q)\log(\sqrt{y_Q}|\eta(z_Q)|^2) \\
&= -\frac{4}{H(D)\#\mathcal{O}_K^\times\#\mathcal{O}_p^\times}\sum_{[Q]\in C(Dp^2)}\psi^2(Q)\log\left(\sqrt{y_Q}|\eta(z_Q)|^2\right)\sum_{\varphi_1}\varphi_1((\mathcal{A}_1)^{-1}\mathcal{A}_2Q^2) \\
&= -\frac{4}{\#\mathcal{O}_K^\times\#\mathcal{O}_p^\times}\sum_{\substack{[Q]\in C(Dp^2)\\ \pi(Q)^2=\mathcal{A}_1(\mathcal{A}_2)^{-1}}}\psi^2(Q)\log(\sqrt{y_Q}|\eta(z_Q)|^2).
\end{aligned}$$

□

Corollary 3.4.4. *Suppose $d\in\Sigma_D$, $\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)$ and ϕ is a ray class group character such that $\phi_1=\phi_1'$ is non-quadratic. Then we have*

$$\langle f_{\phi,d}, f_{\phi,\mathcal{A}}\rangle = \begin{cases} -\frac{4}{\#\mathcal{O}_K^\times\#\mathcal{O}_p^\times}I_{\psi^2}, & d_{\mathcal{A}}=d, \\ 0, & \text{Otherwise.}\end{cases} \tag{3.4.13}$$

Proof. Since $f_{\phi,d}\in S_{1,d}(|D|p,\chi_D\phi_1)$ and $f_{\phi,\mathcal{A}}\in S_{1,d_{\mathcal{A}}}(|D|p,\chi_D\phi_1)$, Prop. 2.4.6 implies that $\langle f_{\phi,d}, f_{\phi,\mathcal{A}}\rangle=0$ if $d_{\mathcal{A}}\neq d$. Otherwise, Eqs. (3.3.13) and (3.4.12) give us

$$\begin{aligned}
\langle f_{\phi,d}, f_{\phi,\mathcal{A}}\rangle &= \frac{1}{2^{\omega(D)-1}}\sum_{\mathcal{B}\in\mathrm{Pic}(\mathcal{O}_K)}\langle f_{\phi,\mathcal{A}^{-1}\mathcal{B}^2}, f_{\phi,\mathcal{A}}\rangle. \\
&= -\frac{4}{\#\mathcal{O}_K\#\mathcal{O}_p}\frac{1}{2^{\omega(D)-1}}\sum_{\mathcal{B}\in\mathrm{Pic}(\mathcal{O}_K)}\sum_{\substack{[Q]\in C(Dp^2)\\ \pi(Q)^2=\mathcal{B}^2}}\psi^2(Q)\log\left(\sqrt{y_Q}|\eta(\tau_Q)|^2\right) \\
&= -\frac{4}{\#\mathcal{O}_K\#\mathcal{O}_p}I_{\psi^2}.
\end{aligned}$$

□

3.4.2 Principal Part Coefficients of Mock-Modular Forms

For each $d\in\Sigma_D$, recall that n_d and r_d are defined in (2.4.19). Let $\{f_t : 1\le t\le r_d\}$ be a q-echelon basis of $S_{1,d}(|D|p,\chi_D\phi_1)$ over $\mathbb{R}$ and $m_t := \mathrm{ord}_\infty(f_t)$. Then $1\le m_t\le n_d$ for all

$1 \leq t \leq r_d$. Now we could apply Prop. 2.4.7 and Cor. 3.4.2 to $f_{\phi,d} \in S_{1,d}(|D|p, \chi_D\phi_1)$ to obtain the following result about mock-modular forms with special principal part coefficients.

Proposition 3.4.5. *There exists a mock-modular form*

$$\tilde{f}_{\phi,d} = \sum_{t=1}^{r_d} \tilde{c}_{\phi,d}(-m_t)q^{-m_t} + \sum_{n\geq 0} \tilde{c}_{\phi,d}(n)q^n \in \mathbb{M}_{1,\hat{d}}(|D|p, \chi_D\overline{\phi_1})$$

with shadow $f_{\phi,d}$ such that when $n \leq 0$,

$$\tilde{c}_{\phi,d}(n) = \alpha_n I_{\psi^2} \tag{3.4.14}$$

for some $\alpha_n \in \mathbb{Q}(\phi)$, where $\mathbb{Q}(\phi)$ is the number field obtained from $\mathbb{Q}$ by adjoining the values of ϕ and ϕ_2.

Proof. By Prop. 3.3.3 and Eq. (3.3.12), $f_{\phi,d} \in S_{1,d}(|D|p, \chi_D\phi_1)$. So the existence of $\tilde{f}_{\phi,d} \in \mathbb{M}_{1,\hat{d}}(|D|p, \chi_D\overline{\chi_p})$ with $\mathrm{ord}_\infty(\tilde{f}_{\phi,d}) < n_d$ with such principal part is given by Prop. 2.4.7. This, along with Prop. 2.4.6 and Cor. 3.4.4, gives us the equation

$$\sum_{t=1}^{r_d} \tilde{c}_{\phi,d}(-m_t)c_{\phi,\mathcal{A}}(m_t)\delta_{|D|}(m_t) + \sum_{\substack{1\leq t\leq r_d \\ p|m_t}} \overline{\tilde{c}_{\phi,d}(-m_t)c_{\phi,\mathcal{A}}(m_t)}\delta_{|D|}(m_t) = \langle f_{\phi,d}, f_{\phi,\mathcal{A}}\rangle = -\frac{4I_{\psi^2}}{\#\mathcal{O}_K^\times \#\mathcal{O}_p^\times}$$

for any $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ satisfying $d_\mathcal{A} = d$.

If there exists any subspace of $S_{1,d}$ orthogonal to $f_{\phi\varphi,d}$ for any character φ of $\mathrm{Pic}(\mathcal{O}_K)$, then we could obtain equations similar to the one above with the right hand side replaced by 0. These r_d equations are sufficient to determine $\tilde{c}_{\phi,d}(-m_t)$ for all $1 \leq t \leq r_d$ by Prop. 2.4.7. Since all the entries in this $r_d \times r_d$ matrix are in $\mathbb{Q}(\phi)$ and the right hand side $r_d \times 1$ matrix has entries either 0 or $\frac{-4}{\#\mathcal{O}_K^\times \#\mathcal{O}_p^\times} I_{\psi^2}$, we know that each $\tilde{c}_{\phi,d}(-m_t)$ can be written in the form of Eq. (3.4.15). Applying the same procedure to the inner product between $f_{\phi,d}$ and the Eisenstein series proves Eq. (3.4.15) for $n = 0$. □

There is also a refined result regarding the principal part coefficients of the mock-modular form $\tilde{f}_{\phi,\mathcal{A}} \in \mathbb{M}_{1,\widehat{d_\mathcal{A}}}$, which we state here. The proof is the same as the one for Prop. 2.4.7.

Proposition 3.4.6. *For every $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, there exists a mock-modular form*

$$\tilde{f}_{\phi,\mathcal{A}} = \sum_{t=1}^{r_{d_\mathcal{A}}} \tilde{c}_{\phi,\mathcal{A}}(-m_t)q^{-m_t} + \sum_{n\geq 0} \tilde{c}_{\phi,\mathcal{A}}(n)q^n \in \mathbb{M}_{1,\widehat{d_\mathcal{A}}}(|D|p, \chi_D\overline{\phi_1})$$

with shadow $f_{\phi,\mathcal{A}}$ *such that when* $n \leq 0$,

$$\tilde{c}_{\phi,\mathcal{A}}(n) = \sum_{\mathcal{B}^2 \in \mathrm{Pic}^2(\mathcal{O}_K)} \alpha_{n,\mathcal{A},\mathcal{B}^2} I_{\psi^2,\mathcal{B}^2} \tag{3.4.15}$$

for some $\alpha_{n,\mathcal{A},\mathcal{B}^2} \in \mathbb{Q}(\phi)$.

CHAPTER 4

Proof of Main Theorem

4.1 Counting Arguments

In this section, we will prove a counting argument crucial to our results.

4.1.1 Background

In this section, we will go over the background on positive definite binary quadratic forms following the treatments in [6] and [17]. The proofs of the results directly from these sources are omitted.

Let $D_0 = Df^2 < 0$ be a discriminant with $D < 0$ a fundamental discriminant and f an integer. Denote a positive definite binary quadratic form with discriminant D_0 by $[A, B, C], A > 0$ and the set of all such forms by $\mathcal{C}(D_0)$. If $\gcd(A, B, C) = 1$, then the form is call *primitive*. From now on, forms $[A, B, C]$ can be taken to be positive definite and primitive.

The group $\mathrm{SL}_2(\mathbb{Z})$ has a right action on $[A, B, C]$ via

$$[A, B, C] \cdot \gamma := [A', B', C'],$$

where $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z})$ and

$$\begin{aligned} A' &= Aa^2 + Bac + Cc^2, \\ C' &= Ab^2 + Bbd + Cd^2, \\ B' &= 2Aab + 2Ccd + B(ad + bc). \end{aligned} \tag{4.1.1}$$

It is not hard to see that the action preserves primitivity and positive definiteness. Notice γ

and $-\gamma = \left(\begin{smallmatrix} -1 & \\ & -1 \end{smallmatrix}\right) \cdot \gamma$ have the same action. So the group

$$\Gamma := \mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{\pm \left(\begin{smallmatrix} 1 & \\ & 1 \end{smallmatrix}\right)\}$$

acts on Q. Two forms $[A, B, C]$ and $[A', B', C']$ are *equivalent* [1] if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q \cdot \gamma = Q'$. A form $[A, B, C]$ is called *reduced* (see Eq. (2.7) in [17]) if

$$|B| \leq A \leq C, \text{ and } B \geq 0 \text{ if either } |B| = A \text{ or } A = C. \tag{4.1.2}$$

This is a convenient notion due to this theorem.

Theorem 4.1.1 (Theorem 2.8 [17]). *Every primitive positve definite form is equivalent to a* ***unique*** *reduced form.*

After Dirichlet, we say two forms $[A_1, B_1, C_1]$ and $[A_2, B_2, C_2]$ of the same discriminants D_0 are *united forms* if

$$\gcd(A_1, A_2, \tfrac{B_1+B_2}{2}) = 1.$$

As a consequence of some simple calculations, we can find representatives of united forms that are convenient for defining a composition law.

Proposition 4.1.2 (Proposition 4.5 [6]). *If* $[A_1, B_1, C_1]$ *and* $[A_2, B_2, C_2]$ *are united forms, then there exist forms* $[A_1, B, A_2C]$ *and* $[A_2, B, A_1C]$ *such that*

$$[A_1, B_1, C_1] \sim [A_1, B, A_2C],$$
$$[A_2, B_2, C_2] \sim [A_2, B, A_1C].$$

We define the composition of two united forms $[A_1, B, A_2C]$ and $[A_2, B, A_1C]$ by

$$[A_1, B, A_2C] \circ [A_2, B, A_1C] := [A_1A_2, B, C]. \tag{4.1.3}$$

The composition law is well-defined with respect to the equivalence relationship (Theorem 4.7 [6]). In fact, it makes the equivalence classes of primitive binary quadratic forms into a group, which we denote by $C(D_0)$. As a corollary, we have

[1] This is proper equivalence in the work of Gauss. Lagrange used the action of $\mathrm{GL}_2(\mathbb{Z})$ instead of $\mathrm{SL}_2(\mathbb{Z})$ in his notion of equivalence.

Corollary 4.1.3. *Let $Q = [A, B, AC]$ be a binary quadratic form. Then Q^2 is represented by the binary quadratic form $[A^2, B, C]$.*

Remark 4.1.4. *To be more precise, if $[A, B, AC] \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = [A', B', A'C']$, then $[A^2, B, C] \cdot \left(\begin{smallmatrix} X & W \\ Y & Z \end{smallmatrix}\right) = [A'^2, B', C']$ with*

$$X = a^2 - Cc^2, Y = 2Aac + Bc^2.$$

For $T \in \mathbb{R}$, let τ_T be the translation defined by

$$\tau_T := \left(\begin{smallmatrix} 1 & T \\ & 1 \end{smallmatrix}\right). \tag{4.1.4}$$

If $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \tau_{At}$ for some $t \in \mathbb{Z}$, then Eq. (4.1.1) implies $\left(\begin{smallmatrix} X & W \\ Y & Z \end{smallmatrix}\right) = \tau_t$, where

It is clear from the definition that $C(D_0)$ is abelian. So the image of $C(D_0)$ under the squaring map is again an abelian group, which we denote by $C^2(D_0)$. The kernel, denoted by $C_0(D_0)$, contains forms of order at most 2 and we have the following short exact sequence

$$0 \longrightarrow C_0(D_0) \longrightarrow C(D_0) \xrightarrow{\cdot 2} C^2(D_0) \to 0.$$

The subgroup $C_0(D_0)$ is well-understood. When D_0 is **odd**, let $\mathcal{Q}_m \in C(D_0)$ be the class represented by the forms Q_m, which is defined as

$$\begin{aligned} Q_m &:= [A_m, B_m, A_m], \\ A_m &= \frac{1}{4}\left(m - \frac{D_0}{m}\right), B_m = \frac{1}{2}\left(m + \frac{D_0}{m}\right), \end{aligned} \tag{4.1.5}$$

where $m \mid D_0$ and $\gcd(m, \frac{D_0}{m}) = 1$. It is clear that $\mathcal{Q}_m \in C_0(D_0)$ as $Q_m^2 \cdot \gamma_m = [A_m^2, B_m, 1] \cdot \gamma_m = Q_0$ where

$$\gamma_m = \begin{pmatrix} & -1 \\ 1 & \frac{B_m+1}{2} \end{pmatrix}.$$

Thus, $Q_m \sim Q_{D_0/m}$ and the set

$$C_0'(D_0) := \{\mathcal{Q}_m : m \mid D_0, m^2 < -D_0, \gcd(m, \tfrac{D_0}{m}) = 1\} \subseteq C_0(D_0)$$

has size $2^{\omega(D_0)-1}$, where for any $N \in \mathbb{Z}$

$$\omega(N) := \text{ number of distinct prime divisors of } N. \tag{4.1.6}$$

On the other hand, Prop. 3.11 in [17] tells us that there are exactly $2^{\omega(D_0)-1}$ elements in $C_0(D_0)$. It is then natural to expect the following lemma to hold.

Lemma 4.1.5. *In the notations above, $C_0'(D_0) = C_0(D_0)$.*

Proof. It suffices to show that for two $m, m' \mid D_0$ satisfying

$$\gcd(m, D_0/m) = \gcd(m', D_0/m') = 1,$$

the classes $\mathcal{Q}_m$ and $\mathcal{Q}_{m'}$ are the same if and only if $m = m'$ or $mm' = D_0$. By Theorem 4.1.1, it is enough to look at the reduced forms equivalent to the Q_m's. Since $Q_m \sim Q_{D_0/m}$, it is enough to cnosider $m \mid D_0$ such that $\frac{-D_0}{m^2} > 1$. Suppose $\frac{-D_0}{m^2} \geq 3$, then $0 < m \leq A_m$ and Q_m is equivalent to the reduced form $[m, m, A_m]$. Otherwise if $1 < \frac{-D_0}{m^2} \leq 3$, then $|B_m| \leq A_m$ and Q_m is equivalent to the reduced form $[A_m, |B_m|, A_m]$. Thus, for any given $m, m' \mid D_0$ such that $m^2 < -D_0, (m')^2 < -D_0$ and $Q_m \sim Q_{m'}$, we know that $A_m = A_{m'}$, which implies $m = m'$. $\square$

The composition law on $C_0(D_0)$ can now be easily described in terms of $\mathcal{Q}_m$.

Lemma 4.1.6. *For $m, m' \mid D_0$ satisfying $\gcd(m, D_0/m) = \gcd(m', D_0/m') = 1$, define $M \mid D_0$ by*

$$M := \frac{mm'}{\gcd(m, m')^2}. \tag{4.1.7}$$

Then $\gcd(M, D_0/M) = 1$ and $\mathcal{Q}_m \circ \mathcal{Q}_{m'} = \mathcal{Q}_M$.

Proof. Since $\gcd(m, D_0/m) = \gcd(m', D_0/m') = 1$, it is not hard to see that

$$M = \gcd(m, \tfrac{D_0}{m'}) \cdot \gcd(m', \tfrac{D_0}{m}),$$
$$\tfrac{D_0}{M} = \gcd(m, m') \cdot \gcd(\tfrac{D_0}{m}, \tfrac{D_0}{m'}).$$

Thus, $M \mid D_0$ and $\gcd(M, D_0/M) = 1$.

For the second part of the claim, we will first prove the case $m \mid m'$. Let $m' = mn$ with $n \in \mathbb{Z}$. We have seen before that $Q_m \sim [m, m, A_m], Q_{m'} \sim [m', m', A_{m'}]$. Using Arndt's composition algorithm (Theorem 4.10 [6]), it is easy to compute that

$$[m, m, A_m] \circ [mn, mn, A_{mn}] = [n, mn, *] \sim [n, n, A_n].$$

The equivalence step follows from applying $\tau_{(1-m)/2}$ and $m \mid D_0$ is odd. Thus, $\mathcal{Q}_m \circ \mathcal{Q}_{mn} = \mathcal{Q}_n$.

Next, suppose $\gcd(m, m') = 1$, then $M = mm'$, $m \mid D_0/m'$ and we have

$$\begin{aligned}\mathcal{Q}_m \circ \mathcal{Q}_{m'} &= \mathcal{Q}_m \circ \mathcal{Q}_{D_0/m'} \\ &= \mathcal{Q}_{D_0/(mm')} = \mathcal{Q}_{mm'} = \mathcal{Q}_M.\end{aligned}$$

Finally, suppose $\gcd(m, m') = n$, then $M = \frac{mm'}{n^2}$ and

$$\begin{aligned}\mathcal{Q}_m \circ \mathcal{Q}_{m'} &= \mathcal{Q}_m \circ \mathcal{Q}_n \circ \mathcal{Q}_n \mathcal{Q}_{m'} \\ &= \mathcal{Q}_{m/n} \circ \mathcal{Q}_{m'/n} = \mathcal{Q}_{\frac{mm'}{n^2}} = \mathcal{Q}_M.\end{aligned}$$

□

When D_0 is even, similar results hold and details can be found in §3.B in [17].

4.1.2 Counting Theorem

In this section, we will prove Lemma 4.1.9, which is crucial to the counting result, Theorem 4.1.10. Throughout, $D_0 = Df^2 < 0$ will be an odd discriminant with D fundamental. For any primitive, positive definite binary quadratic form $P = [A^2, B, C]$ of discriminant D_0, $A > 0$ and $x, y \in \mathbb{Z}$, define the quantity

$$I(P, [x, y]) := \tfrac{2A^2x + By}{A} \in \mathbb{Q}. \tag{4.1.8}$$

If $\gcd(A, B) = 1$, then $Q = [A, B, AC]$ is primitive and $P = Q^2 = [A^2, B, C]$ by Corollary 4.1.3. Since Q is primitive, $\gcd(A, D_0) = 1$ and $I_p(x, y)$ is well-defined modulo D_0.

In general, the function $I(Q^2, [x, y])$ is not well-defined as a function on the class of Q or Q^2. Fortunately, Lemma 4.1.7 below shows that $I(Q^2, [x, y]) \pmod{D_0}$ is well-defined as a function on the class of Q for many choices of $x, y \in \mathbb{Z}$.

Lemma 4.1.7. *Let $Q = [A, B, AC]$ be primitive and $x, y \in \mathbb{Z}$ such that*

$$\gcd(Q^2(x, y), f) = 1.$$

Suppose $Q \cdot \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = Q' := [A', B', A'C']$ and $Q^2 \cdot \gamma = (Q')^2$ with $\gamma = \left(\begin{smallmatrix} X & W \\ Y & Z \end{smallmatrix}\right)$ defined as in Remark 4.1.4, then

$$I(Q^2, [x, y]) \equiv I((Q')^2, [x', y']) \pmod{D_0}, \tag{4.1.9}$$

where $[x', y'] := [x, y] \cdot (\gamma^{-1})^t$.

Remark 4.1.8. *There will be a negative sign in the congruence if we choose* $-\gamma$ *instead of* γ.

Proof. For simplicity, write

$$I_Q = I(Q^2, [x, y]), I_{Q'} = I((Q')^2, [x', y']).$$

First, we have the following important observation from completing the square

$$4A^2Q^2(x, y) = (2A^2x + By)^2 + D_0y^2. \quad (4.1.10)$$

After unfolding the definitions, it is not hard to see that

$$\begin{aligned} Q^2(x, y) &= (Q')^2(x', y'), \\ I_Q^2 &= 4Q^2(x, y) + \tfrac{D_0}{A^2}y^2, \\ I_{Q'}^2 &= 4(Q')^2(x', y') + \tfrac{D_0}{(A')^2}(y')^2. \end{aligned}$$

This implies that

$$I_Q^2 \equiv (I_{Q'})^2 \equiv 4Q^2(x, y) \pmod{D_0}. \quad (4.1.11)$$

Let $p \mid D_0$ be a prime. If $r := \mathrm{ord}_p(D_0) \geq 2$, then $p \mid f$ and $p \nmid Q^2(x, y)$ since $\gcd(f, Q^2(x, y)) = 1$. That means only one of $I_Q + I_{Q'}$ and $I_Q - I_{Q'}$ is divisible by p, hence

$$I_Q \equiv \pm I_{Q'} \pmod{p^r} \iff I_Q \equiv \pm I_{Q'} \pmod{p} \quad (4.1.12)$$

So to prove Eq. (4.1.9), it suffices to show

$$I_Q \equiv I_{Q'} \pmod{D'},$$

where $D' \mid D_0$ is square-free and defined by

$$D' := \prod_{p \mid D_0, p \nmid Q^2(x,y)} p. \quad (4.1.13)$$

To simplify the proof, we will consider two cases depending on γ modulo D'. Bear in mind that $B^2 \equiv 4A^2C \pmod{D_0}$ in both cases.

Case (1) $\gamma \equiv \left(\begin{smallmatrix} 1 & W \\ 0 & 1 \end{smallmatrix}\right) \pmod{D_0'}$

Remark 4.1.4 implies that

$$a^2 - Cc^2 = X \equiv 1 \pmod{D'},$$
$$2Aac + Bc^2 = Y \equiv 0 \pmod{D'}.$$

Applying the congruences above and equation (4.1.1) to $Q \sim Q'$ gives us

$$\begin{aligned} 4AA' &= 4A(Aa^2 + Bac + ACc^2) \\ &\equiv 4A^2a^2 + 4ABac + B^2c^2 \\ &\equiv 4A^2(1 + Cc^2) - 2B^2c^2 + B^2c^2 \\ &\equiv 4A^2 \pmod{D'} \end{aligned}$$

Since $\gcd(A, D_0) = 1$, we conclude that $A \equiv A' \pmod{D'}$. Now applying equation (4.1.1) to $Q^2 \cdot \gamma = (Q')^2$ gives us

$$\begin{aligned} B' = 2A^2XW + 2CYZ + B(XZ + WY) &\equiv 2A^2W + B \pmod{D'}, \\ x' = Zx - Wy &\equiv x - Wy \pmod{D'}, \\ y' = -Yx + Xy &\equiv y \pmod{D'}. \end{aligned}$$

Putting these together, we see that

$$\begin{aligned} I' &\equiv \overline{A'}(2(A')^2x' + B'y') \\ &\equiv \overline{A}(2A^2(x - Wy) + (2A^2W + B)y) \\ &\equiv I \pmod{D'} \end{aligned}$$

Case (2) $\gamma \equiv \left(\begin{smallmatrix} \alpha & 0 \\ \beta & \alpha \end{smallmatrix}\right) \pmod{D'}$

Similar to before, we have

$$
\begin{aligned}
(A')^2 &\equiv A^2\alpha^2 + B\alpha\beta + C\beta^2, \\
x' &\equiv \overline{\alpha}x \pmod{D'}, \\
y' &\equiv -\beta x + \alpha y \pmod{D'}, \\
B' &\equiv B + 2C\beta\overline{\alpha} \pmod{D'}, \\
a^2 - Cc^2 = X &\equiv \alpha \pmod{D'}, \\
2Aac + Bc^2 = Y &\equiv \beta \pmod{D'}, \\
A' &= Aa^2 + Bac + ACc^2.
\end{aligned}
$$

Substituting these into I' gives

$$
\begin{aligned}
I' &\equiv \overline{A'}(2(A^2\alpha^2 + B\alpha\beta + C\beta^2)(\overline{\alpha}x) + (B + 2C\beta\overline{\alpha})(-\beta x + \alpha y)) \\
&\equiv \overline{2A'A^2}(4A^4\alpha x + 2A^2B\beta x + B^2\beta y + A^2B\alpha y) \\
&\equiv \overline{2A'A}(2A^2\alpha + B\beta)I \\
&\equiv \overline{4A'A}(4A^2(a^2 - Cc^2) + 2B(2Aac + Bc^2))I \\
&\equiv \overline{4A'A}(2Aa + Bc)^2 I \\
&\equiv \overline{(4A^2a^2 + 4ABac + 4A^2Cc^2)}(2Aa + Bc)^2 I \\
&\equiv I \pmod{D'}
\end{aligned}
$$

Notice the analysis in both cases works fine if D' is replaced with any of its divisor. Now in the general case $\gamma = \left(\begin{smallmatrix} X & W \\ Y & Z \end{smallmatrix}\right)$, let $p \mid D'$ be a prime. We say that $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{F}_p)$ are *translation equivalent* if there exists $t, t' \in \mathbb{F}_p$ such that

$$
\tau_t \gamma_1 \tau_{t'} = \gamma_2,
$$

where τ_T is translation by T as in Eq. (4.1.4). It is not difficult to see that any $\gamma \in \mathrm{SL}_2(\mathbb{F}_p)$ is translation equivalent to a matrix γ' of the shape $\left(\begin{smallmatrix} * & 0 \\ * & * \end{smallmatrix}\right)$. [2]

[2]Indeed, $\tau_t \gamma \tau_{t'} = \left(\begin{smallmatrix} * & t'(X+tY)+W+tZ \\ * & * \end{smallmatrix}\right)$ and $p \nmid (X + tY)$ for some t.

Let $\gamma' = \tau_{-t_1}\gamma\tau_{t_2}$ such that $\gamma' \equiv \left(\begin{smallmatrix} * & \\ * & * \end{smallmatrix}\right) \pmod p$. Note the choice of t_1, t_2 depends on p. Set $Q_1 := Q \cdot \tau_{At_1}, Q_2 := Q' \cdot \tau_{A't_2}$. Then Remark 4.1.4 implies

$$Q_1 \cdot (\tau_{-At_1} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \tau_{A't_2}) = Q_2,$$
$$(Q_1)^2 \cdot \gamma' = (Q_2)^2$$

Then the analysis of the two cases above shows that

$$I_Q \equiv I_{Q_1} \equiv I_{Q_2} \equiv I_{Q'} \pmod p,$$

where the I_{Q_j}'s are defined in the same way as $I_Q, I_{Q'}$. Since this congruence holds for all $p \mid D'$, which is square-free by definition, we have $I_Q \equiv I_{Q'} \pmod{D_0}$ for any γ.

□

From this Lemma, we see that the value $I(Q^2, [x, y])$ modulo D_0 only depends on the class of Q if $Q^2(x, y)$ is relatively prime to f.

One can now ask whether the congruence in Eq. (4.1.9) could distinguish the classes of Q and Q'. There will be a minus sign in it since replacing γ by $-\gamma$ has this effect on Eq. (4.1.9). If $f = 1$ and $D_0 = D$ is composite and divides k, then the answer is certainly false since Eq. (4.1.9) will be true for any form Q'' with $(Q'')^2 = Q^2$. On the other hand, if $\gcd(k, D_0) = 1$, then the answer is positive, as we will see.

The discussion in §4.1.1 then tells us that the ambiguity comes from $C_2(D_0)$ and should be related to $\gcd(D_0, k)$ intuitively. The following lemma gives a precise statement, which strengthens and completes Lemma 4.1.7.

Lemma 4.1.9. *Let $Q = [A, B, AC]$ be primitive of discriminant D_0 and $x, y \in \mathbb{Z}$ such that* $\gcd(Q^2(x, y), f) = 1$. *Suppose $Q' = [A', B', A'C']$ satisfies $(Q')^2 = Q^2 \cdot \gamma$. Then*

$$I(Q^2, [x, y]) \equiv \pm I((Q')^2, [x, y](\gamma^{-1})^t) \pmod{D_0}$$

if and only if $Q'Q^{-1} \sim Q_M$ for some Q_M as in Eq. (4.1.5) *such that $M \mid Q^2(x, y)$.*

Proof. Notice that Lemma 4.1.7 allows us to choose convenient representatives of Q and Q' here in the proof. We will use the same notation as in Lemma 4.1.7 and consider two cases.

Case (i): $Q^2 \sim Q_0 = [1, 1, \frac{1-D_0}{4}]$

Choose $Q = Q_m$ and $Q' = Q_{m'}$ with $m, m' \mid D_0$ as in Eq. (4.1.5). Then $Q^2 \cdot \gamma = (Q')^2$ where

$$\gamma = \gamma_m \gamma_{m'}^{-1} = \begin{pmatrix} 1 & \\ \frac{B_{m'}-B_m}{2} & 1 \end{pmatrix}.$$

Substitute these and $B_m^2 \equiv 4A_m^2 \pmod{D_0}$ into I_{Q_m} gives us

$$I_{Q_m} \equiv \overline{2A_m} B_m (B_m x + 2y) \pmod{D_0}.$$

Since $\gcd(m, \frac{D_0}{m}) = 1$, we know that $\gcd(B_m, D_0) = 1$. This implies

$$\begin{aligned}\gcd(Q^2(x,y), D_0) &= \gcd(I_Q, D_0) = \gcd((2A_m^2 x + B_m y), D_0)\\ &= \gcd((B_m^2 x + 2B_m y), D_0) = \gcd((B_m x + 2y), D_0).\end{aligned}$$

Now substitute the $[x', y'] = [x, y](\gamma^{-1})^t$ and $B_{m'}^2 - 4A_{m'}^2 \equiv 0 \pmod{D_0}$ into $I_{Q_{m'}}$ yields

$$I_{Q_{m'}} \equiv \overline{2A_{m'}} B_{m'} (B_m x + 2y) \pmod{D_0}.$$

Let $p \mid D_0$ be a prime. Suppose $p \mid Q^2(x,y)$, then $p \mid \gcd(Q^2(x,y), D_0) = \gcd((B_m x + 2y), D_0)$ and $I_{Q_m} \equiv I_{Q_{m'}} \equiv 0 \pmod{p}$. Otherwise, if $p \nmid Q^2(x,y)$ and $p \mid m$, then

$$I_{Q_m} \equiv -(B_m x + 2y) \equiv \begin{cases} I_{Q_{m'}} \pmod{p} & p \mid m' \\ -I_{Q_{m'}} \pmod{p} & p \nmid m'. \end{cases}$$

Similarly, when $p \nmid Q^2(x,y)$ and $p \mid \frac{D_0}{m}$, we have

$$I_{Q_m} \equiv (B_m x + 2y) \equiv \begin{cases} I_{Q_{m'}} \pmod{p} & p \mid \frac{D_0}{m'} \\ -I_{Q_{m'}} \pmod{p} & p \nmid \frac{D_0}{m'}. \end{cases}$$

Let M be defined as in Eq. (4.1.7). We can then summarize the results as follows

$$I_{Q_m} \equiv \begin{cases} \pm I_{Q_{m'}} \pmod{p}, & p \mid Q_m^2(x,y) \\ I_{Q_{m'}} \pmod{p}, & p \nmid Q_m^2(x,y) \text{ and } p \mid \frac{D_0}{M} \\ -I_{Q_{m'}} \pmod{p}, & p \nmid Q_m^2(x,y) \text{ and } p \mid M. \end{cases} \tag{4.1.14}$$

Since $\gcd(Q^2(x,y), f) = 1$, the same argument in Lemma 4.1.7 implies

$$I_Q \equiv \pm I_{Q'} \pmod{D_0} \Longleftrightarrow I_Q \equiv \pm I_{Q'} \pmod{\sigma(D_0)}, \tag{4.1.15}$$

where $\sigma(N)$ is the largest square-free integer dividing $N \in \mathbb{Z}$ and defined by

$$\sigma(N) := \prod_{p|N} p.$$

Combining this general fact with the analysis above in the specific case $Q^2 \sim Q_0$, we have

$$\begin{aligned} I_{Q_m} \equiv \pm I_{Q_{m'}} \pmod{D_0} &\iff I_{Q_m} \equiv \pm I_{Q_{m'}} \pmod{\sigma(D_0)} \\ &\iff \sigma(M) \mid Q_m^2(x,y) \text{ or } \sigma(\tfrac{D_0}{M}) \mid Q_m^2(x,y) \\ &\iff M \mid Q_m^2(x,y) \text{ or } \tfrac{D_0}{M} \mid Q_m^2(x,y). \end{aligned}$$

If $\sigma(M) \mid Q_m^2(x,y)$, then $\gcd(M,f) = 1$ since $\gcd(Q_m^2(x,y), f) = 1$. Since $M \mid D_0 = Df^2$, we know that $M \mid D$ is square-free and the third equivalence above follows. Now Lemma 4.1.6, we have $Q_m Q_{m'}^{-1} \sim Q_M \sim Q_{D_0/M}$ and finished proving this case. Notice that since $\gcd(M, \frac{D_0}{M}) = 1$, $M \mid Q_m^2(x,y)$ and $\frac{D_0}{M} \mid Q_m^2(x,y)$ happen simultaneously only when $f = 1$ and $D_0 \mid Q_m^2(x,y)$.

Case (ii): $Q^2 \not\sim Q_0$

Let ℓ be a prime represented by $Q_M \sim Q^{-1}Q'$ such that $\gcd(\ell, D_0) = 1$. This is equivalent to finding a prime ideal $\mathfrak{l}$ of the ring of integers of the imaginary quadratic field $\mathbb{Q}(\sqrt{D_0})$ with norm ℓ and contained in the ideal class corresponding to Q_M, which is possible by Chebotarev density theorem.

For the representative of Q, choose $[A, B, AC]$ such that $\ell \mid A$. Since ℓ is prime and represented by Q_M, any form representing ℓ is equivalent to Q_M or $Q_M^{-1} = Q_M$, in particular

$$Q'_M := [\ell, B, \tfrac{A^2C}{\ell}] \sim Q_M.$$

We can then choose $Q' = [A', B', A'C']$, where

$$A' = \tfrac{A}{\ell}, B' = B, C' = \ell^2 C.$$

By the definition of composition in Eq. (4.1.3), it is easy to check that $Q' \circ Q'_M = Q$. Also, we have

$$Q^2 = [A^2, B, C], (Q')^2 = [(A')^2, B, C'].$$

Since $[\ell^2, B, (A')^2C] = (Q'_M)^2 \sim Q_M^2 \sim [1, 1, \frac{1-D_0}{4}]$, there exist $a, c \in \mathbb{Z}$ such that

$$\ell^2 a^2 + Bac + (A')^2 Cc^2 = 1,$$

which also implies $\gcd(a, (A')^2 c) = 1$ and

$$A^2(a)^2 + Ba((A')^2c) + C((A')^2c)^2 = (A')^2.$$

That means $Q^2 \cdot \gamma = (Q')^2$ with

$$\gamma = \left(\begin{smallmatrix} a & b \\ (A')^2c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{Z}).$$

Set $\tilde{\gamma} := \left(\begin{smallmatrix} a & (A')^2b \\ c & d \end{smallmatrix}\right)$, $X := A'x, X' := A'x'$. It is easy to check that

$$[X', y'] = [X, y](\tilde{\gamma}^{-1})^t, (Q'_M)^2 \cdot \tilde{\gamma} = [1, B, A^2C].$$

Substituting these into Eq. (4.1.9) gives us

$$\overline{\ell}(2\ell^2 X + By) \equiv (2X' + By') \pmod{D_0},$$

and equivalence (4.1.15) becomes

$$\begin{aligned} I_Q \equiv \pm I_{Q'} \pmod{D_0} &\iff I_Q \equiv \pm I_{Q'} \pmod{\sigma(D_0)} \\ &\iff \overline{\ell}(2\ell^2 X + By) \equiv \pm(2X' + By') \pmod{\sigma(D_0)} \\ &\iff I((Q'_M)^2, [X, y]) \equiv \pm I([1, B, A^2C]^2, [X', y']) \pmod{\sigma(D_0)} \\ &\iff M \mid \gcd((Q'_M)^2(X, y), D_0) \text{ or } \tfrac{D_0}{M} \mid \gcd((Q'_M)^2(X, y), D_0) \\ &\iff M \mid \gcd((2\ell^2 X + By), D_0) \text{ or } \tfrac{D_0}{M} \mid \gcd((2\ell^2 X + By), D_0) \\ &\iff M \mid \gcd((2A^2 x + By), D_0) \text{ or } \tfrac{D_0}{M} \mid \gcd((2A^2 x + By), D_0) \\ &\iff M \mid Q^2(x, y) \text{ or } \tfrac{D_0}{M} \mid Q^2(x, y). \end{aligned}$$

Here, the fourth "iff" follows from case (i) proved above. The fifth and seventh "iff" both follows from Eq. (4.1.10)

□

Now, we are ready to prove the main counting theorem. For a binary quadratic for $Q = [A, B, C]$ of discriminant D_0, let $\tau_Q := \frac{B+\sqrt{D_0}}{2A} \in \mathcal{H}$ be the CM point associated to it. Let $d < 0$ be another discriminant, $q = [a, b, c]$ a binary quadratic form with discriminant d, and $d(\tau_Q, \tau_q)$ the hyperbolic distance between τ_Q and τ_q Its hyperbolic cosine has the following convenient expression

$$\cosh(d(\tau_Q, \tau_q)) = \tfrac{2Ac+2Ca-Bb}{\sqrt{D_0 d}}. \tag{4.1.16}$$

The group $\mathrm{SL}_2(\mathbb{Z})$ acts isometrically on $\mathcal{H}$ via linear fractional transformation. It is easy to check that for $\gamma \in \mathrm{SL}_2(\mathbb{Z})$,

$$\gamma \cdot \tau_Q = \tau_{Q\cdot\left(\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)\gamma^{-1}\left(\begin{smallmatrix}1 & \\ & -1\end{smallmatrix}\right)\right)}. \tag{4.1.17}$$

Let $\mathcal{Q} \in C(D_0)$ and $k \in \mathbb{Z}$. Define the sets and their sizes by

$$S_Q(k, d) := \{q = [a, b, c] \in \mathbb{Z}^3 : a > 0, \mathrm{disc}(q) = d, \cosh(d(\tau_Q, \tau_q)) = \tfrac{k}{\sqrt{D_0 d}}\}, \tag{4.1.18}$$

$$R_{\mathcal{Q}}(n) := \{(Q, \pm(x, y)) : x, y \in \mathbb{Z}, [Q] = \mathcal{Q}, Q(x, y) = n\}/\sim, \tag{4.1.19}$$

$$\rho_Q(k, d) = \#S_Q(k, d), \tag{4.1.20}$$

$$r_{\mathcal{Q}}(n) = \#R_{\mathcal{Q}}(n). \tag{4.1.21}$$

Here $(Q, \pm(x, y)) \sim (Q', \pm(x', y'))$ if there exists $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ such that $Q' = Q' \cdot \gamma$ and $[x', y'] = [x, y](\gamma^{-1})^t$.

The set $S_Q(k, d)$ counts the number of CM points of a fixed discriminant and at a fixed hyperbolic distance from a given CM point τ_Q. By Eq. (4.1.17), the quantity $\rho_Q(k, d)$ depends only on the class of Q. The main counting theorem will tell us that this number is closely related to $r_{\mathcal{Q}^2}(n)$.

Theorem 4.1.10. *Let $\mathcal{P} \in C^2(D_0)$ and $k \in \mathbb{Z}$ such that* $\gcd(k, f) = 1$*. Then*

$$\sum_{\mathcal{Q} \in C(D_0), \mathcal{Q}^2 = \mathcal{P}} \rho_{\mathcal{Q}}(k, d) = r_{\mathcal{P}}\left(\tfrac{k^2 - D_0 d}{4}\right) \cdot 2^{\omega(\gcd(D_0, k))}. \tag{4.1.22}$$

Proof. Let $Q = [A, B, AC]$ such that $[Q^2] = \mathcal{P}$. By Eq. (4.1.16), a form $q \in S_Q(k, d)$ is the same as a triple $[a, b, c]$ satisfying

$$b^2 - 4ac = -d,$$
$$2Ac + 2ACa - Bb = k.$$

Now there is a map ϕ_Q between $S_Q(k, d)$ and $R_{Q^2}\left(\frac{k^2 - D_0 d}{4}\right)$ defined by

$$\phi_Q : S_Q(k, d) \longrightarrow R_{Q^2}\left(\frac{k^2 - D_0 d}{4}\right)$$
$$[a, b, c] \mapsto (Q^2, \pm(c - Ca, Ba - Ab)).$$

It is more convenient to describe ϕ_Q by the linear map

$$\begin{pmatrix} 2AC & -B & 2A \\ B & -A & 0 \\ -C & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} k \\ \pm y \\ \pm x \end{pmatrix}.$$

Here the $\pm$ sign in front of x and y are the same. The determinant of the 3×3 matrix above is D_0, so is an injective linear map, and ϕ_Q is a function.

Now given $(Q^2, \pm(x, y)) \in R_{Q^2}\left(\frac{k^2 - D_0 d}{4}\right)$, it is in the image of ϕ_Q if and only if

$$\frac{1}{D_0}\begin{pmatrix} -A & B & 2A^2 \\ -B & 4AC & 2AB \\ AC & BC & B^2 - 2A^2C \end{pmatrix} \cdot \begin{pmatrix} k \\ \pm y \\ \pm x \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in \mathbb{Z}^3.$$

If $a = (-Ak + By + 2A^2x)/D_0 \in \mathbb{Z}$, then $c = Ca + x \in \mathbb{Z}$ and $b = (Ba - y)/A = \sqrt{d + 4ac} \in \mathbb{Z}$. Thus, we have

$$\begin{aligned} (Q^2, \pm(x, y)) \in \operatorname{im}(\phi_Q) &\Longleftrightarrow (2A^2x + By) \equiv \pm Ak \pmod{D_0}, \\ &\Longleftrightarrow I(Q^2, [x, y]) \equiv \pm k \pmod{D_0} \end{aligned} \tag{4.1.23}$$

Now only one of $\pm$ holds unless $D_0 \mid k$, in which case both signs can occur. That means ϕ_Q is an injective map when $D_0 \nmid k$ and a 2 to 1 map when $D_0 \mid k$.

By Eq. (4.1.10), we have

$$I(Q^2, [x, y])^2 \equiv k^2 \pmod{D_0}.$$

That means there exists $\epsilon(p, Q, [x,y]) \in \{\pm 1\}$ for each $p \mid D_0$ such that

$$I(Q^2, [x,y]) \equiv \epsilon(p, Q, [x,y])k \pmod{p}.$$

So Eq. (4.1.23) is satisfied if and only if $\epsilon(p, Q, [x,y]) = \pm 1$ is *independent* of $p \mid D_0$.

Using the congruence summary (4.1.14), we can find $((Q')^2, \pm(x', y')) \sim (Q^2, \pm(x,y))$ such that

$$I(Q^2, [x,y]) \equiv \epsilon(p, Q, [x,y]) I((Q')^2, [x', y']) \pmod{p}.$$

So equivalence (4.1.23) implies that $((Q')^2, \pm(x', y')) \in \mathrm{im}(\phi_{Q'})$. Furthermore, Lemma 4.1.9 also tell us that

$$\mathrm{im}(\phi_{Q'}) = \mathrm{im}(\phi_Q) \iff Q^{-1}Q' \sim Q_M \text{ for some } M \mid k.$$

Now if $D_0 \nmid k$, then every element in $R_{Q^2}\left(\frac{k^2 - D_0 d}{4}\right)$ lies $\mathrm{im}(\phi_{QQ_m})$ for exactly $2^{\omega(\gcd(k, D_0))}$ of $m \mid D_0$. If $D_0 \mid k$, then ϕ_{QQ_m} is a 2 to 1 surjection for every $m \mid D_0, \gcd(m, D_0/m) = 1$. By Lemma 4.1.5, the set $\{Q \circ Q_m : m \mid D_0, m^2 < -D_0\}$ contains exactly all the representatives of the classes $\mathcal{Q} \in C(D_0)$ satisfying $\mathcal{Q}^2 = \mathcal{P}$. By counting $\cup_{m \mid D_0, m^2 < -D_0} \mathrm{im}(\phi_{QQ_m})$ with repetition, we obtain Eq. (4.1.22). □

When $\gcd(k, f) > 1$, the function $\rho_Q(k, d)$ is more difficult to evaluate. In the special case when $\gcd(k, f) = f_1$ and d is a non-square residue modulo some prime dividing f_1, we have the following result.

Proposition 4.1.11. *Write $f = f_1 f_2$ and let $k \in \mathbb{Z}$ be an integer such that $\gcd(k, f_2) = 1$. If $d < 0$ is a discriminant such that $\left(\frac{d}{\ell}\right) = -1$ for some $\ell \mid f_1$, then $\rho_{\mathcal{Q}}(f_1 k, d) = 0$ for all $\mathcal{Q} \in C(D_0)$.*

Proof. First, we could choose a representative $Q = [A, fB, f^2AC]$ of $\mathcal{Q}$. Then $\gcd(A, f) = 1$ since Q is primitive. Using this Q, we see that if $q = [a, b, c] \in S_{\mathcal{Q}}(f_1 k, d)$, then

$$2Ac + 2f^2ACa - fBb = \sqrt{D_0 d} \cosh(d(\tau_Q, \tau_q)) = f_1 k.$$

Since $\gcd(2A, f) = 1$, there exists $c' \in \mathbb{Z}$ such that $c = f_1 c'$. So the set $S_{\mathcal{Q}}(f_1 k, d)$ becomes

$$\begin{aligned} S_{\mathcal{Q}}(f_1k, d) &= \{q = [a, b, f_1c'] \in \mathbb{Z}^3 : a > 0, b^2 - 4af_1c' = d, 2Ac' + 2f_1f_2^2ACa - f_2Bb = k\} \\ &= \{q = [f_1a, b, c'] \in \mathbb{Z}^3 : f_1a > 0, b^2 - 4(f_1a)c' = d, 2Ac' + 2f_2^2AC(f_1a) - f_2Bb = k\} \\ &= \left\{q = [a', b, c'] \in \mathbb{Z}^3 : a' > 0, f_1 \mid a', \operatorname{disc}(q) = d, \cosh(\tau_{Q'}, \tau_q) = \frac{k}{\sqrt{Df_2^2 d}}\right\}, \end{aligned}$$

where $a' = f_1 a$, $Q' = [A, B', AC'] = [A, f_2B, ACf_2^2] \in C(Df_2^2)$. For any $[a', b, c'] \in S_{\mathcal{Q}}(f_1k, d)$, define $x, y \in \mathbb{Z}$ by

$$\begin{pmatrix} 2AC' & B' & 2A \\ B' & -A & 0 \\ -C' & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} a' \\ b \\ c' \end{pmatrix} = \begin{pmatrix} k \\ y \\ x \end{pmatrix}.$$

It is easy to check that $4(A^2x^2 + B'xy + Cy^2) = (k^2 - Df_2^2 d)$. After multiplying on both sides by the inverses of the two 3×3 matrices, we obtain

$$\begin{pmatrix} a' \\ b \\ c' \end{pmatrix} = \frac{1}{Df_2^2} \begin{pmatrix} -A & B' & 2A^2 \\ -B' & 4AC' & 2AB' \\ AC' & B'C' & (B')^2 - 2A^2C' \end{pmatrix} \cdot \begin{pmatrix} k \\ y \\ x \end{pmatrix}.$$

Since $f_1 \mid a'$, we have $\frac{-Ak + B'y + 2A^2x}{Df_2^2} \in \mathbb{Z}$ and

$$\frac{-Ak + B'y + 2A^2x}{Df_2^2} \equiv 0 \pmod{f_1}.$$

Some calculations show that

$$\begin{aligned} (B'y + 2A^2x - Ak)(B'y + 2A^2x + Ak) &= 4A^2(A^2x^2 + B'xy + C'y^2) + ((B')^2 - 4A^2C')y^2 - A^2k^2 \\ &= A^2(k^2 - Df_2^2 d) + Df_2^2 y^2 - A^2k^2 \\ &= Df_2^2(y^2 - A^2 d). \end{aligned}$$

So $A^2 d \equiv y^2 \pmod{f_1}$, which implies that $\left(\frac{d}{\ell}\right) \neq -1$ for all $\ell \mid f_1$ since $\gcd(A, f) = 1$. This contradicts our condition on d in the statement of the proposition. So the set $S_{\mathcal{Q}}(f_1k, d)$ is empty for all $\mathcal{Q} \in C(D_0)$. $\square$

4.1.2.1 Character Sum Identities

In this section, we will give some character sum identities necessary for our results.

Let $p > 2$ be a prime, $\chi_p = \left(\frac{\cdot}{p}\right)$ the Dirichlet character of conductor p and $\chi : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$. For $\lambda \in \mathbb{Z}/p\mathbb{Z}$, define the character sum $S(\lambda, \chi)$ by

$$S_1(\lambda, \chi) := \tfrac{G(\chi)}{G(\chi\chi_p)G(\chi_p)} \sum_{r=0}^{p-1} \overline{\chi}(r+1)\overline{\chi}(r+\lambda)\chi(r)\chi_p(r), \tag{4.1.24}$$

where $G(\psi) = \sum_{k\in(\mathbb{Z}/p\mathbb{Z})^\times} \psi(k)e_p(k)$ is the Gauss sum associated to ψ for any character ψ. Via a substitution, $S_1(\lambda, \chi)$ is closely related to another character sum defined for $a, b \in \mathbb{Z}/p\mathbb{Z}$ by

$$S_2(a, b, \chi) := \tfrac{1}{G(\chi)G(\chi_p)} \sum_{u,v=0}^{p-1} \chi(uv)(\overline{\chi}\chi_p)(u+v)\, e_p(au+bv), \tag{4.1.25}$$

where $e_p(z) := e^{2\pi iz/p}$. The following lemma gives an explicit evaluation of $S_1(\lambda, \chi)$.

Lemma 4.1.12. *In the notations above, if $\chi \neq \chi_p$, then*

$$S_1(\lambda, \chi) = \tfrac{1}{2}(1 + \chi_p(\lambda)) \cdot \left(\overline{\chi}(1 + \sqrt{\lambda})^2 + \overline{\chi}(1 - \sqrt{\lambda})^2\right). \tag{4.1.26}$$

Remark 4.1.13. *When $\chi_p(\lambda) \neq -1$, the quantity $\sqrt{\lambda}$ makes sense in $\mathbb{F}_p$. Otherwise, the factor $(1 + \chi_p(\lambda))$ vanishes and it is not necessary to evaluate the second factor.*

Proof. When $\chi_p(\lambda) = -1$, the substitution $r \mapsto \frac{\lambda}{r}$ yields

$$S_1(\lambda, \chi) = \chi_p(\lambda)S_1(\lambda, \chi).$$

So $S_1(\lambda, \chi) = 0$ in this case.

When $\chi_p(\lambda) = 0$, Eq. (4.1.24) becomes

$$\begin{aligned}
\tfrac{G(\chi\chi_p)G(\chi_p)}{G(\chi)} S_1(\lambda, \chi) &= \sum_{r=0}^{p-1} \overline{\chi}(r+1)e_p(r) \\
&= \tfrac{1}{G(\chi)} \sum_{r,s=0}^{p-1} \chi(s)\chi_p(r)e_p(s(r+1)) \\
&= \tfrac{1}{G(\chi)} \sum_{s=0}^{p-1} (\chi\chi_p)(s)e_p(s) \sum_{rs=0}^{p-1} \chi_p(rs)e_p(rs) \\
&= \tfrac{G(\chi\chi_p)G(\chi_p)}{G(\chi)}.
\end{aligned}$$

Here in the second step, we used the Gauss sum substitution for

$$\overline{\chi}(r+1)G(\chi) = \sum_{s\in\mathbb{Z}/p\mathbb{Z}} \chi(s)e_p(s(r+1)).$$

When $\chi_p(\lambda) = 1$, apply Gauss sum substitution for $\overline{\chi}((r+1)(r+\lambda))$ gives us

$$\begin{aligned} S_1(\lambda,\chi) &= \tfrac{G(\chi)}{G(\chi\chi_p)G(\chi_p)} \sum_{s,r=0}^{p-1} \chi(rs)\chi_p(r)e_p(s(r+1)(r+\lambda)) \\ &= \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{s,r=0}^{p-1} (\chi\chi_p)(rs)\chi_p(s)e_p(s(r+\sqrt{\lambda})^2 + sr(1-\sqrt{\lambda})^2). \end{aligned}$$

Define a map $(u,v) : (\mathbb{Z}/p\mathbb{Z})^2 \longrightarrow (\mathbb{Z}/p\mathbb{Z})^2$ by

$$u(s,r) = sr, v(s,r) = s(r+\sqrt{\lambda})^2.$$

For each $(u,v) \in (\mathbb{Z}/p\mathbb{Z})^2$, the number of preimages under this map depends on the number of solutions in $(r,s) \in (\mathbb{Z}/p\mathbb{Z})^2$ to

$$rs = u, rv = u(r+\sqrt{\lambda})^2.$$

When $v \neq 0$, this quantity is either 0, 1, or 2 and can be expressed as

$$\delta(u,v) := (1 + \chi_p(1 - 4\sqrt{\lambda}u\overline{v})).$$

Thus, we have

$$\begin{aligned} S_1(\lambda,\chi) =& \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u=0}^{p-1} (\chi\chi_p)(u)\chi_p\left(-u\overline{\sqrt{\lambda}}\right) e_p(u(1-\sqrt{\lambda})^2) \\ &+ \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u,v=0}^{p-1} (\chi\chi_p)(u)\chi_p(v)e_p(v+u(1-\sqrt{\lambda})^2)\delta(u,v) \\ =& \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u=0}^{p-1} (\chi\chi_p)(u)\chi_p\left(-u\overline{\sqrt{\lambda}}\right) e_p(u(1-\sqrt{\lambda})^2) \\ &+ \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u,v=0}^{p-1} (\chi\chi_p)(u)\chi_p(v)e_p(v+u(1-\sqrt{\lambda})^2) \\ &+ \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u=0}^{p-1}\sum_{v=1}^{p-1} (\chi\chi_p)(u)e_p(v+u(1-\sqrt{\lambda})^2)\chi_p(v-4\sqrt{\lambda}u). \end{aligned}$$

Notice that $\chi_p(-4\sqrt{\lambda}u) = \chi_p(-u\overline{\sqrt{\lambda}})$ for any $u \in \mathbb{Z}/p\mathbb{Z}$ since χ_p is a quadratic character. So the first term and third term combines to change the index of v to $v \in \mathbb{Z}/p\mathbb{Z}$. The summation in the second term can also be evaluated easily as the product of two Gauss sums. This gives us

$$\begin{aligned} S_1(\lambda,\chi) =& \tfrac{1}{G(\chi\chi_p)G(\chi_p)} \sum_{u,v=0}^{p-1} (\chi\chi_p)(u)\chi_p(v-4\sqrt{\lambda}u)e_p(v-4\sqrt{\lambda}u+(1+\sqrt{\lambda})^2u) \\ &+ \overline{(\chi\chi_p)}(1-\sqrt{\lambda})^2 \\ =& \overline{\chi}(1+\sqrt{\lambda})^2 + \overline{\chi}(1-\sqrt{\lambda})^2. \end{aligned}$$

□

Corollary 4.1.14. *Suppose $\chi \neq \chi_p$, then*

$$S_2(a,b,\chi) = \tfrac{1}{2}\left(\chi_p(a)+\chi_p(b)\right) \cdot \left(\overline{\chi}(a+2\sqrt{ab}+b) + \overline{\chi}(a-2\sqrt{ab}+b)\right) \tag{4.1.27}$$

Proof. The Gauss sum substitution for $(\overline{\chi}\chi_p)(u+v)$ gives us

$$\begin{aligned} S_2(a,b,\chi) &= \tfrac{1}{G(\chi)G(\chi_p)G(\chi\chi_p)} \sum_{u,v,r=0}^{p-1} \chi(uvr)\chi_p(r)e_p((r+a)u+(r+b)v) \\ &= \tfrac{G(\chi)}{G(\chi_p)G(\chi\chi_p)} \sum_{r=0}^{p-1} \overline{\chi}(r+a)\overline{\chi}(r+b)\chi(r)\chi_p(r). \end{aligned}$$

This is clearly 0 when $a = b = 0$ since $\chi \neq \chi_p$. Otherwise, since it is easy to see that $S_2(a,b,\chi) = S_2(b,a,\chi)$ from its definition, we can suppose $a \neq 0$ without loss of generality. Then

$$S_2(a,b,\chi) = \overline{\chi}(a)\chi_p(a)S_1(\tfrac{b}{a},\chi).$$

So Eq. (4.1.27) follows directly from Eq. (4.1.26). □

4.2 Fourier Expansions

The notation in this section will be consistent with those in §3.3. Let $D < 0$ be an odd fundamental discriminant and $p \nmid D$ be an odd prime that splits into $\mathfrak{p}\overline{\mathfrak{p}}$ in $\mathcal{O}_K$, where $K = \mathbb{Q}(\sqrt{D})$. Let $\phi : I_{\mathfrak{p}}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ be a ray class group character with $\phi_1 : (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow$

$I_\mathfrak{p}/P_{\mathfrak{p},1} \xrightarrow{\phi} \mathbb{C}^\times$ non-quadratic and $\phi_2 : (\mathbb{Z}/p\mathbb{Z})^* \longrightarrow \mathbb{C}^\times$ a character satisfying $\phi_1\phi_2^2 = \mathbb{1}_p$. Denote the absolute value of D by

$$N := |D|. \tag{4.2.1}$$

Let $f_\phi = \sum_{n\geq 1} a(\phi, n)q^n \in S_1(Np, \chi_D\phi_1)$ be the weight one newform of imaginary dihedral type associated to ϕ and $g_\psi = f_\phi \otimes \phi_2 \in S_1(Np^2, \chi_{Dp^2})$ as in Prop. 3.3.4. We could write

$$f_\phi = \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)} f_{\phi,\mathcal{A}}, \quad g_\psi = \sum_{\mathcal{A}\in\mathrm{Pic}(\mathcal{O}_K)} g_{\psi,\mathcal{A}}$$

as in Eq. (3.3.8) and (3.3.19). Then we could find mock-modular forms with shadow $g_{\psi,\mathcal{A}}$ by twisting mock-modular forms $\tilde{f}_{\phi,\mathcal{A}}$, whose shadow is $f_{\phi,\mathcal{A}}$.

Proposition 4.2.1. *Let $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ be any class and $\tilde{f}_{\phi,\mathcal{A}} \in \mathbb{M}_{1,\widehat{d_\mathcal{A}}}(|D|p, \chi_D\phi_1)$ be any mock-modular form with shadow $f_{\phi,\mathcal{A}} \in S_{1,d_\mathcal{A}}(|D|p, \chi_D\phi_1)$ as in Prop. 2.4.5. Then the mock-modular form*

$$\tilde{g}_{\psi,\mathcal{A}}(z) := \phi_2(-1)(\tilde{f}_{\phi,\mathcal{A}} \otimes \overline{\phi_2})(z) \in \mathbb{M}_1(|D|p^2, \chi_D) \tag{4.2.2}$$

has shadow $g_{\psi,\mathcal{A}}(z)$ and the associated harmonic Maass form $\hat{g}_{\psi,\mathcal{A}}(z)$ satisfies

$$\begin{aligned} \hat{g}_{\psi,\mathcal{A}} \mid_1 W_\ell &= \frac{\varphi_{\ell^*}(\mathcal{A})\left(\frac{D/\ell}{\ell}\right)}{\epsilon_\ell\sqrt{\ell}}\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell, \\ \hat{g}_{\psi,\mathcal{A}} \mid_1 W_{p^2} &= \phi_2(-1)\hat{g}_{\psi,\mathcal{A}} \end{aligned} \tag{4.2.3}$$

for all primes $\ell \mid D$, $\ell > 0$, where $\ell^* = (-1)^{(\ell-1)/2}\ell$, $W_p = \begin{pmatrix} p^2\alpha & \beta \\ |D|p & p \end{pmatrix}$, $W_{p^2} = \begin{pmatrix} p^2\alpha & \beta \\ |D|p^2 & p^2 \end{pmatrix}$, W_ℓ and U_ℓ are defined as in §2.4.1.

Proof. Let $\hat{f}_{\phi,\mathcal{A}}(z) \in H_{1,\widehat{d_\mathcal{A}}}$ be harmonic Maass form associated to $\tilde{f}_{\phi,\mathcal{A}}(z)$. Then the function $\tilde{g}_{\psi,\mathcal{A}}(z)$ is the holomorphic part of the harmonic Maass form $\phi_2(-1)\left(\hat{f}_{\phi,\mathcal{A}} \otimes \overline{\phi_2}\right)$, which is sent to $g_{\psi,\mathcal{A}}$ under ξ_1.

Write the Fourier expansion of $\hat{f}_{\phi,\mathcal{A}}$ and $\hat{g}_{\psi,\mathcal{A}}$ at infinity as

$$\hat{f}_{\phi,\mathcal{A}} = \sum_{n\in\mathbb{Z}} \hat{c}_{\phi,\mathcal{A}}(n,y)q^n, \quad \hat{g}_{\psi,\mathcal{A}} = \sum_{n\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}(n,y)q^n.$$

Since $\hat{f}_{\phi,\mathcal{A}} \in H_{1,\widehat{d_{\mathcal{A}}}}(|D|p, \chi_D\overline{\phi_1})$ and $\hat{c}_{\psi,\mathcal{A}}(n,y) = \overline{\phi_2(n)}\hat{c}_{\phi,\mathcal{A}}(n,y)$, the Fourier coefficients $\hat{c}_{\psi,\mathcal{A}}(n,y)$ satisfy

$$\chi_\ell(n)\hat{c}_{\psi,\mathcal{A}}(n,y) = \varphi_{\ell^*}(\mathcal{A})\hat{c}_{\psi,\mathcal{A}}(n,y) \text{ for all } n \in \mathbb{Z} \text{ relatively prime to } \ell$$

for all $\ell \mid D$. The first equation in (4.2.3) is then implied by Prop. 2.4.1.

The second equation in (4.2.3) follows from Eq. (3.3.15) and the calculations below.

$$\begin{aligned}
\phi_2(-1)\hat{g}_{\psi,d} \mid_1 W_{p^2} &= \frac{1}{G(\phi_2)} \sum_{\mu=1}^{p-1} \phi_2(\mu)\hat{f}_{\phi,d} \mid_1 \left(\begin{smallmatrix} p & \mu \\ & p \end{smallmatrix}\right) W_{p^2} \\
&= \frac{1}{G(\phi_2)} \sum_{\mu=1}^{p-1} \phi_2(\mu)\hat{f}_{\phi,d} \mid_1 \left(\begin{smallmatrix} p\alpha+|D|\mu & * \\ |D|p & d_\mu \end{smallmatrix}\right) \left(\begin{smallmatrix} p & \beta\overline{(|D|\mu)} \\ & p \end{smallmatrix}\right) \\
&= \frac{1}{G(\phi_2)} \sum_{\mu=1}^{p-1} \phi_2(\mu)\chi_D(p)\phi_1(|D|\mu)\hat{f}_{\phi,d} \mid_1 \left(\begin{smallmatrix} p & \beta\overline{(|D|\mu)} \\ & p \end{smallmatrix}\right) \\
&= \frac{\overline{\phi_2(|D|\beta)}}{G(\phi_2)} \sum_{\mu'=1}^{p-1} \phi_2(\mu')\hat{f}_{\phi,d} \mid_1 \left(\begin{smallmatrix} p & \mu' \\ & p \end{smallmatrix}\right) = \hat{g}_{\psi,d}.
\end{aligned}$$

Here, $(p\alpha + |D|\mu)d_\mu \equiv 1 \pmod{|D|p}$, $|D|\beta \equiv -1 \pmod{p}$ and $\chi_D(p) = 1$. $\square$

Since $\hat{g}_{\psi,\mathcal{A}}$ is obtained from $\hat{f}_{\phi,\mathcal{A}}$ by twisting, its transformation under $\Gamma_0(Np)$ can be described by the following lemma.

Lemma 4.2.2. *For any* $\gamma = \left(\begin{smallmatrix} r & b \\ Npc & u \end{smallmatrix}\right) \in \Gamma_0(Np)$, *we have*

$$\begin{aligned}
\phi_2(-1)\hat{g}_{\psi,\mathcal{A}} \mid_1 \gamma = &\frac{\phi_2(-Nrc)\chi_N(u)}{G(\phi_2)G(\phi_1)}\hat{f}^c_{\phi,\mathcal{A}} \mid_1 U_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \\
&+\frac{\chi_N(u)}{G(\phi_2)} \sum_{\substack{\mu=1 \\ p\nmid r+Nc\mu}}^{p-1} \phi_2\left(\mu\overline{(r+Nc\mu)}^2\right) \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} p & \overline{(r+Nc\mu)}u\mu \\ & p \end{smallmatrix}\right).
\end{aligned} \tag{4.2.4}$$

Proof. Notice that if $p \mid c$, Eq. (4.2.4) is just $\hat{g}_{\psi,\mathcal{A}} \mid_1 \gamma = \chi_N(u)\hat{g}_{\psi,\mathcal{A}}$. So we suppose $p \nmid c$.

The rest of the proof follows from Eq. (3.3.15) and (4.2.2) and the calculations below.

$$\begin{aligned}
\phi_2(-1)\hat{g}_{\psi,\mathcal{A}} \mid_1 \gamma &= \frac{1}{G(\phi_2)} \sum_{\mu=1}^{p} \phi_2(\mu) \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} p & \mu \\ & p \end{smallmatrix}\right) \left(\begin{smallmatrix} r & b \\ Npc & u \end{smallmatrix}\right) \\
&= \frac{1}{G(\phi_2)} \sum_{\substack{\mu=1 \\ p \nmid r+Nc\mu}}^{p} \phi_2(\mu) \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} r+Nc\mu & b_\mu \\ Npc & u_\mu \end{smallmatrix}\right) \left(\begin{smallmatrix} p & \overline{(r+Nc\mu)}u\mu \\ & p \end{smallmatrix}\right) \\
&\quad + \frac{\phi_2(\mu_0)}{G(\phi_2)} \hat{f}_{\phi,\mathcal{A}} \mid_1 \begin{pmatrix} p\frac{r+Nc\mu_0}{p} & bp+u\mu_0 \\ Npc & pu \end{pmatrix} \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \\
&= \frac{1}{G(\phi_2)} \sum_{\substack{\mu=1 \\ p \nmid r+Nc\mu}}^{p} \phi_2(\mu) \chi_N(u_\mu) \overline{\phi_1(u_\mu)} \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} p & \overline{(r+Nc\mu)}u\mu \\ & p \end{smallmatrix}\right) \\
&\quad + \frac{\phi_2(-r\overline{Nc})\chi_N(u)\overline{\phi_1(c)\phi_1(N)}}{G(\phi_2)G(\phi_1)} \hat{f}^c_{\phi,\mathcal{A}} \mid_1 U_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right) \\
&= \frac{\chi_N(u)}{G(\phi_2)} \sum_{\substack{\mu=1 \\ p \nmid r+Nc\mu}}^{p} \phi_2(\mu) \phi_1(r+Nc\mu) \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} r+Nc\mu & b_\mu \\ Npc & u_\mu \end{smallmatrix}\right) \left(\begin{smallmatrix} p & \overline{(r+Nc\mu)}u\mu \\ & p \end{smallmatrix}\right) \\
&\quad + \frac{\phi_2(-Nrc)\chi_N(u)}{G(\phi_2)G(\phi_1)} \hat{f}^c_{\phi,\mathcal{A}} \mid_1 U_p \left(\begin{smallmatrix} p & \\ & 1 \end{smallmatrix}\right)
\end{aligned}$$

□

Let $\hat{f}_{\phi,\mathcal{A}}(z)$ and $\hat{g}_{\psi,\mathcal{A}}(z)$ be the harmonic Maass forms as in Prop. 4.2.1 with the following Fourier expansions at infinity

$$\hat{f}_{\phi,\mathcal{A}}(z) = \sum_{n\in\mathbb{Z}} \hat{c}_{\phi,\mathcal{A}}(n) q^n, \ \hat{g}_{\psi,\mathcal{A}}(z) = \sum_{n\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}(n) q^n.$$

For each $\delta \mid N$, $\delta > 0$ and p, let $\tilde{W}_\delta$ and $\tilde{W}_p$ be the Atkin-Lehner involutions as in Eq. (2.3.4)

$$\tilde{W}_\delta = \left[\left(\begin{smallmatrix} \delta\alpha_\delta & \beta_\delta \\ 4Np^2 & \delta \end{smallmatrix}\right), \delta^{-1/4}\sqrt{4Np^2 z + \delta}\right], \ \tilde{W}_p = \left[\left(\begin{smallmatrix} p^2\alpha & \beta \\ 4Np & p \end{smallmatrix}\right), p^{-1/4}\sqrt{4Npz + p}\right],$$

and $\tilde{U}_\delta, \tilde{U}_p$ be the U-operator as in Eq. (2.3.1). For each $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$, define the following functions

$$\begin{aligned}
\Phi_{\mathcal{A}}(z) &:= \hat{g}_{\psi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \theta(z), \\
\Psi_{\mathcal{A}}(z) &:= (\Phi_{\mathcal{A}}(z)) \mid_{3/2} \left(\prod_{\ell \mid N, \ell \text{ prime}} (\tilde{U}_\ell + \tilde{W}_\ell) \right) \tilde{U}_p(\tilde{U}_p + \tilde{W}_p),
\end{aligned} \tag{4.2.5}$$

where $\theta = \sum_{n\in\mathbb{Z}} q^{n^2}$ is the weight $\frac{1}{2}$ theta function. It is easy to check that for any $\gamma = \left(\begin{smallmatrix} a & b \\ 4Np^2 & d \end{smallmatrix}\right) \in \Gamma_0(Np^2)$, we have

$$(\Phi_{\mathcal{A}} \mid_{3/2} \gamma)(z) = \left(\tfrac{N}{d}\right) \Phi_{\mathcal{A}}(z).$$

Then by Lemma 2.3.5 and 2.3.6, the function $\Psi_{\mathcal{A}}(z)$ satisfies

$$(\Psi_{\mathcal{A}} \mid_{3/2} \tilde{\gamma})(z) = \Psi_{\mathcal{A}}(z)$$

for all $\gamma \in \Gamma_0(4)$. After subtracting off appropriate poles from $\Phi_{\mathcal{A}}(z)$, we could apply holomorphic projection to it and obtain an identity between finite linear combinations of $c_\psi^+(n)$ and an infinite sum from the Fourier expansion of $\Psi_{\mathcal{A}}(z)$. This infinite sum will become the special values of modular function after applying the appropriate counting argument. Before calculating the Fourier coefficients of $\Psi_{\mathcal{A}}$, we need the following lemma.

Lemma 4.2.3. *For any $\delta \mid N, \delta > 0$, we have*

$$\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta = \frac{\varphi_{\delta^*}(\mathcal{A})}{\delta} \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta. \tag{4.2.6}$$

Proof. This lemma follows from an induction on the number of prime divisors of δ. When $\delta = \ell$ is prime, we have

$$\begin{aligned}
\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\ell &= \hat{g}_{\psi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \left(\begin{smallmatrix} \ell\alpha_\ell & \beta_\ell \\ 4Np^2 & \ell \end{smallmatrix}\right) \cdot \theta \mid_{1/2} \widetilde{\left(\begin{smallmatrix} \alpha_\ell & \beta_\ell \\ 4Np^2/\ell & \ell \end{smallmatrix}\right)} \left[\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right), \ell^{-1/4} \left(\tfrac{N/\ell}{\ell}\right) \epsilon_\ell\right] \\
&= \hat{g}_{\psi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix} \ell\alpha_\ell & 4\beta_\ell \\ Np^2 & \ell \end{smallmatrix}\right) \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \cdot \theta \mid_{1/2} \left[\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right), \ell^{-1/4} \left(\tfrac{N/\ell}{\ell}\right) \epsilon_\ell\right] \\
&= \frac{\varphi_{\ell^*}(\mathcal{A}) \left(\frac{-N/\ell}{\ell}\right)}{\epsilon_\ell \sqrt{\ell}} \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \cdot \left(\tfrac{N/\ell}{\ell}\right) \epsilon_\ell^{-1} \theta \mid_{1/2} \left[\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right), \ell^{-1/4}\right] \\
&= \frac{\varphi_{\ell^*}(\mathcal{A})}{\sqrt{\ell}} \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \cdot \theta \mid_{1/2} \left[\left(\begin{smallmatrix} \ell & \\ & 1 \end{smallmatrix}\right), \ell^{-1/4}\right] \\
&= \frac{\varphi_{\ell^*}(\mathcal{A})}{\ell} \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell \left(\begin{smallmatrix} 4 & \\ & 1 \end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\ell.
\end{aligned}$$

When $\delta = \delta'\ell$, we could use Lemma 2.3.3 and 2.3.4 to find that

$$\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta = \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_{\delta'} \tilde{W}_\ell$$

$$\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell W_{\delta'} = \left(\tfrac{\ell}{\delta'}\right) \hat{g}_{\psi,\mathcal{A}} \mid_1 W_{\delta'} U_\ell,$$

$$\theta \mid_{1/2} \tilde{U}_\ell \tilde{W}_{\delta'} = \left(\tfrac{\ell}{\delta'}\right) \theta \mid_{1/2} \tilde{W}_{\delta'} \tilde{U}_\ell.$$

Using this, we could obtain

$$\begin{aligned}\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta =&\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_{\delta'}\tilde{W}_\ell \\ =&\frac{\varphi_{(\delta')^*}(\mathcal{A})}{\delta}\hat{g}_{\psi,\mathcal{A}} \mid_1 U_{\delta'} \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right) \begin{pmatrix} \ell\alpha_\ell & \beta_\ell \\ 4Np^2 & \ell \end{pmatrix} \cdot \theta \mid_{1/2} \tilde{U}_{\delta'}\tilde{W}_\ell \\ =&\frac{\varphi_{(\delta')^*}(\mathcal{A})}{\delta} \left(\tfrac{\ell}{\delta'}\right) \hat{g}_{\psi,\mathcal{A}} \mid_1 \begin{pmatrix} \ell\alpha_\ell & 4\beta_\ell \\ Np^2 & \ell \end{pmatrix} U_{\delta'} \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right) \cdot \left(\tfrac{\ell}{\delta'}\right) \theta \mid_{1/2} \tilde{W}_\ell \tilde{U}_{\delta'} \\ =&\frac{\varphi_{(\delta')^*}(\mathcal{A})\varphi_\ell(\mathcal{A})}{\delta'\ell}\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\ell U_{\delta'} \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\ell \tilde{U}_{\delta'} \\ =&\frac{\varphi_{\delta^*}(\mathcal{A})}{\delta}\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta.\end{aligned}$$

By induction, the proof is complete. □

Now, we are ready to calculate the Fourier expansion of $\Psi_{\mathcal{A}}$ at infinity.

Proposition 4.2.4. *The m^{th} Fourier coefficient of $\Psi_{\mathcal{A}}(z)$ at infinity, $\hat{c}(\Psi_{\mathcal{A}}, m, y)$, has the form*

$$\hat{c}(\Psi_{\mathcal{A}}, m, y) = 2\sqrt{p} \sum_{\delta \mid N, \delta>0} \varphi_{\delta^*}(\mathcal{A})(S_{\mathcal{A},\delta}(m,y) + S'_{\mathcal{A},\delta}(m,y)), \tag{4.2.7}$$

where

$$\begin{aligned} S_{\mathcal{A},\delta}(m,y) =& \sum_{k\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}\left(\frac{Np^2m - \delta^2k^2}{4}, \frac{4y}{p^2N}\right), \\ S'_{\mathcal{A},\delta}(m,y) =& \phi_2(-1)\sum_{k\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}\left(\frac{Nm - p^2\delta^2k^2}{4}, \frac{4y}{N}\right) \\ &+ \frac{\overline{\phi_2(-4Nm)}\epsilon_p\sqrt{p}G(\phi_2')}{G(\phi_1)G(\phi_2)} \sum_{\substack{k\in\mathbb{Z} \\ \delta^2k^2 \equiv Nm \pmod p}} \overline{\hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-\delta^2k^2}{4}, \frac{4y}{pN}\right)} \\ &+ \phi_2(4)\sum_{k\in\mathbb{Z}} S_2(Nm, \delta^2k^2, \phi_2)\hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-\delta^2k^2}{4}, \frac{4y}{N}\right), \\ \phi_2'(\cdot) =& \phi_2(\cdot)\left(\frac{\cdot}{p}\right). \end{aligned}$$

When $m = pm'$ is divisible by p, the expression $S'_{\mathcal{A},\delta}(pm)$ above simplifies to

$$\sum_{k\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}\left(\frac{Npm' - \delta^2k^2}{4}, \frac{4y}{N}\right).$$

When $\left(\frac{-m}{p}\right) = -1$, the expression $S'_{\mathcal{A},\delta}(m)$ vanishes identically.

Proof. By Lemma 2.3.3 and 2.3.4, we know that for any positive $\ell, \delta \mid N$ satisfying $\gcd(\ell, \delta) = 1$

$$\Phi_{\mathcal{A}} \mid_{3/2} \tilde{U}_\ell \tilde{W}_\delta = \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_\ell,$$
$$\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{W}_\ell = \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_{\delta\ell}.$$

So we could write

$$\Phi_{\mathcal{A}} \mid_{3/2} \prod_{\ell \mid N, \ell \text{ prime}} (\tilde{U}_\ell + \tilde{W}_\ell) = \sum_{\delta \mid N, \delta > 0} \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_{N/\delta}.$$

Applying Lemma 2.3.4 yields

$$\begin{aligned}
\Psi_{\mathcal{A}} &= \sum_{\delta \mid N, \delta > 0} \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_{N/\delta} \tilde{U}_p (\tilde{U}_p + \tilde{W}_p) \\
&= \sum_{\delta \mid N, \delta > 0} \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_{Np^2/\delta} + \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_{N/\delta} \tilde{U}_p \tilde{W}_p \\
&= \sum_{\delta \mid N, \delta > 0} \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_{Np^2/\delta} + \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_p \tilde{W}_p \tilde{U}_{N/\delta}.
\end{aligned}$$

The first term in the summand can be handled easily using Lemma 4.2.3. The main technical complications arise in the calculations of the second term, which we will carry out. Here, we will choose

$$\tilde{W}_p = \left[\left(\begin{smallmatrix} p^2\alpha & \beta \\ 4Np & p \end{smallmatrix} \right), p^{-1/4}\sqrt{4Npz + p} \right]$$

This is possible since $\gcd(N, p) = 1$. It is not essential to the result, but simplifies the calculations. For each $\delta \mid N, \delta > 0$, we denote

$$\Phi_{\mathcal{A},\delta} := \Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta.$$

Then we have

$$\begin{aligned}
\Phi_{\mathcal{A},\delta} \mid_{3/2} \tilde{U}_p \tilde{W}_p =& \Phi_{\mathcal{A},\delta} \mid_{3/2} \tilde{W}_{p^2} + \sum_{\lambda=1}^{p-1} \left(\tfrac{Np}{d_\lambda} \right) \epsilon_{d_\lambda}^3 \Phi_{\mathcal{A},\delta} \mid_{3/2} \tilde{\gamma}_\lambda \cdot \left[\left(\begin{smallmatrix} p & \beta\overline{(4N\lambda)} \\ & p \end{smallmatrix} \right), 1 \right], \\
\tilde{W}_{p^2} =& \left[\left(\begin{smallmatrix} p^2\alpha & \beta \\ 4Np^2 & p^2 \end{smallmatrix} \right), p^{-1/2}\sqrt{4Np^2 z + p^2} \right], \\
\gamma_\lambda =& \left(\begin{smallmatrix} p\alpha + 4N\lambda & b_\lambda \\ 4Np & d_\lambda \end{smallmatrix} \right) \in \Gamma_0(4Np), \\
d_\lambda \equiv& \overline{p\alpha + 4N\lambda} \pmod{4p}, d_\lambda \equiv p \pmod{4N}.
\end{aligned}$$

Set $C_{\mathcal{A},\delta} := \frac{\varphi_{\delta*}(\mathcal{A})}{\delta}$, then we can apply Lemma 4.2.3 to substitute $C_{\mathcal{A},\delta}(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta)$ for $\Phi_{\mathcal{A},\delta}$ and obtain

$$\begin{aligned}\Phi_{\mathcal{A},\delta} \mid_{3/2} \tilde{U}_p\tilde{W}_p &= C_{\mathcal{A},\delta}\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta\right) \mid_{3/2} \tilde{W}_{p^2} \\ &+ C_{\mathcal{A},\delta}\epsilon_p^3\left(\tfrac{N}{p}\right)\sum_{\lambda=1}^{p-1}\left(\tfrac{p}{d_\lambda}\right)\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta\right) \mid_{3/2} \tilde{\gamma}_\lambda \cdot \left[\left(\begin{smallmatrix}p & \beta\overline{(4N\lambda)}\\ & p\end{smallmatrix}\right), 1\right].\end{aligned} \tag{4.2.8}$$

The first term on the right hand side of Eq. (4.2.8) can be evaluated using Eq. (4.2.2) as follows.

$$\begin{aligned}\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta\right) \mid_{3/2} \tilde{W}_{p^2} &= \left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) W_{p^2}\right) \cdot \left(\theta \mid_{1/2} \tilde{U}_\delta\tilde{W}_{p^2}\right) \\ \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) W_{p^2} &= \hat{g}_{\psi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix}p^2\alpha & 4\beta\\ Np^2 & p^2\end{smallmatrix}\right) U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \\ &= \phi_2(-1)\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \\ \theta \mid_{1/2} \tilde{U}_\delta\tilde{W}_{p^2} = \theta \mid_{1/2} \tilde{W}_{p^2}\tilde{U}_\delta &= \theta \mid_{1/2} \left[\left(\begin{smallmatrix}p^2&\\&1\end{smallmatrix}\right), p^{-1}\right]\tilde{U}_\delta.\end{aligned}$$

The sum over λ in the second term on the right hand side of Eq. (4.2.8) could be evaluated as follows.

$$\begin{aligned}\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right) \cdot \theta \mid_{1/2} \tilde{U}_\delta\right) \mid_{3/2} \tilde{\gamma}_\lambda &= \epsilon_p^2\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right)\gamma_\lambda\right) \cdot \left(\theta \mid_{1/2} \tilde{U}_\delta\tilde{\gamma}_\lambda\right) \\ \hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right)\gamma_\lambda\left(\begin{smallmatrix}p & \beta\overline{(4N\lambda)}\\ & p\end{smallmatrix}\right) &= \hat{g}_{\psi,\mathcal{A}} \mid_1 \gamma'_\lambda\left(\begin{smallmatrix}p & \beta\overline{(\delta N\lambda)}\\ & p\end{smallmatrix}\right) U_\delta \left(\begin{smallmatrix}4&\\&1\end{smallmatrix}\right),\end{aligned} \tag{4.2.9}$$

where $\gamma'_\lambda = \left(\begin{smallmatrix}p\alpha+4N\lambda & 4b'_\lambda\\ \delta Np & d'_\lambda\end{smallmatrix}\right) \in \Gamma_0(Np)$ and $d'_\lambda \equiv d_\lambda \pmod{Np}$. By Lemma 4.2.2, we have

$$\begin{aligned}\phi_2(-1)\hat{g}_{\psi,\mathcal{A}} \mid_1 \gamma'_\lambda\left(\begin{smallmatrix}p & \beta\overline{(\delta N\lambda)}\\ & p\end{smallmatrix}\right) &= \frac{\phi_2(-4N^2\lambda\delta)\chi_N(p)}{G(\phi_2)G(\phi_1)}\hat{f}^c_{\phi,\mathcal{A}} \mid_1 U_p\left(\begin{smallmatrix}p&\\&1\end{smallmatrix}\right) \\ &+ \frac{\chi_N(p)}{G(\phi_2)}\sum_{\substack{\mu=1\\ p\nmid 4\lambda+\delta\mu}}^{p-1}\phi_2(\mu)\phi_1(N(4\lambda+\delta\mu))\hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix}p & \beta\overline{(\delta N\lambda)}(1-\overline{(4\lambda+\delta\mu)}\delta\mu)\\ & p\end{smallmatrix}\right).\end{aligned}$$

Also, we have

$$\begin{aligned}\theta \mid_{1/2} \tilde{U}_\delta\tilde{\gamma}_\lambda\left[\left(\begin{smallmatrix}p & \beta\overline{(4N\lambda)}\\ & p\end{smallmatrix}\right), 1\right] &= \left(\tfrac{\delta}{p}\right)\theta \mid_{1/2} \tilde{\gamma}''_\lambda\left[\left(\begin{smallmatrix}p & \beta\overline{(4\delta N\lambda)}\\ & p\end{smallmatrix}\right), 1\right]\tilde{U}_\delta \\ &= \left(\tfrac{\delta}{p}\right)\theta \mid_{1/2} \left[\left(\begin{smallmatrix}p & \beta\overline{(4\delta N\lambda)}\\ & p\end{smallmatrix}\right), 1\right]\tilde{U}_\delta,\end{aligned}$$

where $\gamma''_\lambda = \left(\begin{smallmatrix}p\alpha+4N\lambda & b''_\lambda\\ 4\delta Np & d''_\lambda\end{smallmatrix}\right)$ and $d''_\lambda \equiv d_\lambda \pmod{4Np}$.

Substituting these terms into Eq. (4.2.8), we have

$$\Phi_{\mathcal{A}} \mid_{3/2} \tilde{W}_\delta \tilde{U}_p \tilde{W}_p \tilde{U}_{N/\delta} = C_{\mathcal{A},\delta}(P_{\mathcal{A},\delta,1} + P_{\mathcal{A},\delta,2} + P_{\mathcal{A},\delta,3}) \mid_{3/2} \tilde{U}_{N/\delta}, \tag{4.2.10}$$

where

$$\begin{aligned}
P_{\mathcal{A},\delta,1} =& \phi_2(-1)\left(\hat{g}_{\psi,\mathcal{A}} \mid_1 U_\delta \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right)\right)\left(\theta \mid_{1/2} \left[\left(\begin{smallmatrix}p^2 & \\ & 1\end{smallmatrix}\right), p^{-1}\right] \tilde{U}_\delta\right),\\
P_{\mathcal{A},\delta,2} =& \frac{\epsilon_p \chi_N(p)\overline{\phi_2(-4)}G(\phi_2')}{G(\phi_1)G(\phi_2)} \hat{f}^c_{\phi,\mathcal{A}} \mid_1 U_{p\delta}\left(\begin{smallmatrix}4p & \\ & 1\end{smallmatrix}\right) \cdot (\theta \otimes \overline{\phi_2'}) \mid_{1/2} \tilde{U}_\delta,\\
P_{\mathcal{A},\delta,3} =& \frac{\epsilon_p \chi_N(p)\phi_2(-\delta)}{G(\phi_2)} \sum_{\substack{\lambda,\mu'=1\\ p\nmid 4\lambda+\mu'}}^{p-1} \left(\tfrac{\lambda\delta}{p}\right) \phi_2(\mu')\overline{\phi_2^2(\delta N(4\lambda+\mu'))} \hat{f}_{\phi,\mathcal{A}} \mid_1 \left(\begin{smallmatrix}p & -\frac{\overline{\delta N^2(4\lambda+\mu')}}{p}\\ & \end{smallmatrix}\right) U_\delta \left(\begin{smallmatrix}4 & \\ & 1\end{smallmatrix}\right)\\
&\cdot \theta \mid_{1/2} \left[\left(\begin{smallmatrix}p & \frac{\beta\overline{(4\delta N\lambda)}}{p}\\ & \end{smallmatrix}\right), 1\right] \tilde{U}_\delta,\\
\phi_2'(\cdot) =& \phi_2(\cdot)\left(\tfrac{\cdot}{p}\right),\\
\mu' =& \mu\delta.
\end{aligned}$$

The m^{th} Fourier coefficient of $\frac{1}{2}\delta^{-5/4}P_{\mathcal{A},\delta,1}$ and $\frac{1}{2}\delta^{-5/4}P_{\mathcal{A},\delta,2}$ are

$$\sqrt{p}\phi_2(-1)\sum_{k\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}\left(\delta\frac{m-p^4\delta k^2}{4}, \frac{4y}{\delta}\right) \tag{4.2.11}$$

$$p\frac{\epsilon_p \chi_N(p)\overline{\phi_2(-4)}G(\phi_2')}{G(\phi_1)G(\phi_2)} \sum_{\substack{k\in\mathbb{Z}\\ \delta k^2\equiv m \pmod p}} \overline{\hat{c}_{\phi,\mathcal{A}}\left(\delta\frac{m-\delta k^2}{4}, \frac{4y}{p\delta}\right)}\phi_1(\delta k) \tag{4.2.12}$$

The m^{th} Fourier coefficient of $\frac{1}{2}\delta^{-5/4}P_{\mathcal{A},\delta,3}$ is

$$\begin{aligned}
&\frac{\epsilon_p \chi_N(p)\phi_2(-\delta)}{G(\phi_2)} \sum_{\substack{\lambda,\mu'=1\\ p\nmid 4\lambda+\mu'\\ k\in\mathbb{Z}}}^{p-1} \left(\tfrac{\lambda\delta}{p}\right) \phi_2(\mu')\overline{\phi_2^2(\delta N(4\lambda+\mu'))}\hat{c}_{\phi,\mathcal{A}}\left(\delta\frac{m-\delta k^2}{4}, \frac{y}{\delta}\right)\\
&\qquad \cdot e_p\left(\beta\overline{N(4\lambda+\mu')}(m-\delta k^2) + \beta\overline{(4N\lambda)}\delta k^2\right),
\end{aligned}$$

where $e_p(z) = e^{2\pi i z/p}$. Set

$$u = \overline{N(4\lambda+\mu')},\ v = \overline{4N\lambda} - \overline{N(4\lambda+\mu')},$$

then $\mu' = v\overline{(N(u+v)u)}$ and the change of variable (u, v) is injective. So we could rewrite the m^{th} Fourier coefficient of $P_{\mathcal{A},\delta,3}$ as

$$\chi_N(p)\overline{\phi_2(-N\delta)}\left(\tfrac{-N\delta}{p}\right)\sqrt{p}\sum_{k\in\mathbb{Z}} S_2(\beta m, \beta\delta k^2, \phi_2)\hat{c}_{\phi,\mathcal{A}}\left(\delta\frac{m-\delta k^2}{4}, \frac{y}{\delta}\right), \tag{4.2.13}$$

where $S_2(a,b,\chi)$ is defined in Eq. (4.1.25) and given by

$$S_2(\beta m, \beta\delta k^2, \phi_2) = \frac{1}{\epsilon_p\sqrt{p}G(\phi_2)}\sum_{u,v=1}^{p-1}\left(\tfrac{u+v}{p}\right)\overline{\phi_2(u+v)}\phi_2(uv)\cdot e_p\left(\beta(mu+\delta k^2 v)\right).$$

After Substituting Using the fact that $\chi_N(p) = \left(\frac{-N}{p}\right) = \left(\frac{p}{N}\right) = 1$ and adding together Eq. (4.2.11), (4.2.12) and (4.2.13), we could write the m^{th} Fourier coefficient of $(2C_{\mathcal{A},\delta}\delta\sqrt{p})^{-1}N^{1/4}\Phi_{\mathcal{A}}\mid_{3/2} \tilde{W}_\delta\tilde{U}_p\tilde{W}_p\tilde{U}_{N/\delta}$ as follows.

$$\begin{aligned}
&\phi_2(-1)\sum_{k\in\mathbb{Z}}\hat{c}_{\psi,\mathcal{A}}\left(\frac{Nm-p^2\delta^2k^2}{4},\frac{y}{N}\right)\\
&+\frac{\overline{\phi_2(-4Nm)}\epsilon_p\sqrt{p}G(\phi_2')}{G(\phi_1)G(\phi_2)}\sum_{\substack{k\in\mathbb{Z}\\ \delta^2k^2\equiv Nm\ (\mathrm{mod}\ p)}}\overline{\hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-\delta^2k^2}{4},\frac{y}{pN}\right)}\\
&+\overline{\phi_2(-N\delta)}\left(\tfrac{\delta}{p}\right)\sum_{k\in\mathbb{Z}}S_2(\beta Nm/\delta,\beta\delta k^2,\phi_2)\hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-\delta^2k^2}{4},\frac{y}{N}\right),
\end{aligned}\tag{4.2.14}$$

The sum $S_2(a,b,\phi_2)$ can be evaluated using Eq. (4.1.27). After simplifying the expression, we get $S'_{\mathcal{A},\delta}(m,y)$.

When $m = pm'$, the first two terms in expression (4.2.14) both vanishes since $\hat{c}_{\psi,\mathcal{A}}(pn,y) = 0$, and the third term becomes

$$\sum_{k\in\mathbb{Z}}\hat{c}_{\psi,\mathcal{A}}\left(\frac{Npm'-\delta^2k^2}{4},\frac{4y}{N}\right).$$

Thus, the $(pm')^{\text{th}}$ Fourier coefficient of $\Phi_{\mathcal{A}}\mid_{3/2}\tilde{W}_\delta\tilde{U}_p\tilde{W}_p\tilde{U}_{N/\delta}$ is

$$2\sqrt{p}\varphi_{\delta^*}(\mathcal{A})\sum_{k\in\mathbb{Z}}\hat{c}_{\psi,\mathcal{A}}\left(\frac{Npm'-\delta^2k^2}{4},\frac{4y}{N}\right).\tag{4.2.15}$$

By Lemma 4.2.3, we could find the $(pm')^{\text{th}}$ Fourier coefficient of $\Phi_{\mathcal{A}}\mid_{3/2}\tilde{W}_\delta\tilde{U}_{Np^2/\delta}$ to be

$$2\sqrt{p}\varphi_{\delta^*}(\mathcal{A})\sum_{k\in\mathbb{Z}}\hat{c}_{\psi,\mathcal{A}}\left(\frac{Np^3m'-\delta^2k^2}{4},\frac{4y}{p^2N}\right).\tag{4.2.16}$$

Summing these together over $\delta\mid N$, we find that the $(pm')^{\text{th}}$ Fourier coefficient of $\Psi_{\mathcal{A}}$ to be the expression

$$2\sqrt{p}\sum_{\substack{\delta\mid N,\delta>0\\ k\in\mathbb{Z}}}\varphi_{\delta^*}(\mathcal{A})\left(\hat{c}_{\psi,\mathcal{A}}\left(\frac{Npm'-\delta^2k^2}{4},\frac{4y}{N}\right)+\hat{c}_{\psi,\mathcal{A}}\left(\frac{Np^3m'-\delta^2k^2}{4},\frac{4y}{p^2N}\right)\right).\tag{4.2.17}$$

When $\left(\frac{-m}{p}\right) = -1$, the second term in (4.2.14) vanishes since the summation is empty. The third term has contribution only from $k = pk'$ since $S_2(\beta Nm/\delta, \beta\delta k^2, \phi_2) = 0$ by Eq. (4.1.27) otherwise. So this term becomes

$$\begin{aligned}
&\overline{\phi_2(-N\delta)}\left(\tfrac{\delta}{p}\right)\sum_{k\in\mathbb{Z}} S_2(\beta Nm/\delta, \beta\delta k^2, \phi_2)\hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-\delta^2k^2}{4}, \frac{4y}{N}\right)\\
&= \overline{\phi_2(4N\delta\beta/\delta)}\left(\tfrac{\delta\beta Nm/\delta}{p}\right)\sum_{k'\in\mathbb{Z}} \hat{c}_{\phi,\mathcal{A}}\left(\frac{Nm-p^2\delta^2(k')^2}{4}, \frac{4y}{N}\right)\overline{\phi_2(Nm/4)}\phi_2(-1)\\
&= -\phi_2(-1)\sum_{k'\in\mathbb{Z}} \hat{c}_{\psi,\mathcal{A}}\left(\frac{Nm-p^2\delta^2(k')^2}{4}, \frac{4y}{N}\right),
\end{aligned}$$

which cancels the first term exactly. So $S'_{\mathcal{A},\delta}(m,y) = 0$ in this case. □

A special case of this proposition is when $\mathcal{A} = \mathcal{B}^2 \in \mathrm{Pic}^2(\mathcal{O}_K)$ and $\left(\frac{-m}{p}\right) = -1$. Here, $\varphi_{\delta^*}(\mathcal{B}^2) = 1$ for all $\delta \mid N$ and the m^{th} Fourier coefficient of $\Psi_{\mathcal{B}^2}(z)$ is simply

$$2\sqrt{p}\sum_{k\in\mathbb{Z}} 2^{\omega(\gcd(D,k))}\left(\begin{array}{l}\tilde{c}_{\phi,\mathcal{B}^2}\left(\frac{Np^2m-k^2}{4}\right)\phi_1\left(\frac{k}{2}\right) - \\ \overline{c_{\psi,\mathcal{B}^2}\left(\frac{k^2-Np^2m}{4}\right)}\beta_1\left(\frac{(k^2-Np^2m)}{4}, \frac{4y}{p^2N}\right)\end{array}\right) \tag{4.2.18}$$

where $\omega(\cdot)$ is the function defined in Eq. (4.1.6). Summing over such classes $\mathcal{A} \in \mathrm{Pic}^2(\mathcal{O}_K)$ gives us the m^{th} Fourier coefficient of $\Psi_1 := \sum_{\mathcal{A}\in\mathrm{Pic}^2(\mathcal{O}_K)} \Psi_{\mathcal{A}}$, which is

$$2\sqrt{p}\sum_{k\in\mathbb{Z}} 2^{\omega(\gcd(D,k))}\left(\tilde{c}_{\psi,1}\left(\frac{Np^2m-k^2}{4}\right) - \overline{c_{\psi,1}\left(\frac{k^2-Np^2m}{4}\right)}\beta_1\left(\frac{(k^2-Np^2m)}{4}, \frac{4y}{p^2N}\right)\right) \tag{4.2.19}$$

when $\left(\frac{m}{p}\right) = -1$.

4.3 Fourier Coefficients and Values of Modular Functions

4.3.1 Borcherds Lift

Let $M^!_{1/2}$ be the space of weakly holomorphic modular forms of weight 1/2 and level 4 satisfying Kohnen's plus space condition. It has a canonical basis $\{f_{-d}\}_{d\leq 0}$ with $d \equiv 0, 1 \pmod 4$ and Fourier expansions

$$f_{-d}(z) = q^d + \sum_{n\geq 1} c(f_{-d}, n)q^n.$$

Let $f(z) \in M^!_{1/2}$ be a weakly holomorphic form with integral Fourier coefficients $c(f,n)$. In [7], Borcherds constructed an infinite product $\Psi_f(z)$ using $c(f,n)$ as exponents, and showed that it is a modular form of weight $c(f,0)$ and some character. The divisors of $\Psi_f(z)$ are supported on cusps and imaginary quadratic irrationals. In particular, if τ is a quadratic irrational of discriminant $D < 0$, then its multiplicity in $\Psi_f(z)$ is

$$\mathrm{ord}_\tau(\Psi_f) = \sum_{k>0} c(f, Dk^2).$$

For example, when $f(z) = f_{-d}(z)$ with $d < 0$, the Borcherds product $\Psi_{-d}(z) := \Psi_{f_d}(z)$ equals to

$$\prod_{q \in \mathcal{C}(d)/\Gamma} (j(z) - j(\tau_q))^{1/w_q}, \tag{4.3.1}$$

where $\mathcal{C}(d)$ is the set of all positive definite binary quadratic forms $[a,b,c]$ of discriminant d satisfying $a > 0$. Note that when d is fundamental, w_d, the number of roots of unity in $\mathbb{Q}(\sqrt{d})$, is equal to $2w_q$ for all $q \in \mathcal{C}(d)$.

4.3.2 Automorphic Green's Function

In this section, we will follow the construction in [28, §5] to express the automorphic Green's function as an infinite sum. For two distinct points $z_j = x_j + iy_j \in \mathcal{H}$, the invariant hyperbolic distance $d(z_1,z_2)$ between them is defined by

$$\begin{aligned} \cosh d(z_1,z_2) &:= \frac{|z_1 - z_2|^2}{2y_1y_2} + 1 \\ &= \frac{(x_1-x_2)^2 + y_1^2 + y_2^2}{2y_1y_2}. \end{aligned} \tag{4.3.2}$$

Note $d(z_1,z_2) = d(\gamma z_1, \gamma z_2)$ for all $\gamma \in \mathrm{PSL}_2(\mathbb{R})$. The Legendre function of the second kind $Q_{s-1}(t)$ is defined by

$$\begin{aligned} Q_{s-1}(t) &= \int_0^\infty (t + \sqrt{t^2-1}\cosh u)^{-s} du, \ \mathrm{Re}(s) > 1, t > 1, \\ Q_0(t) &= \tfrac{1}{2}\log\left(1 + \tfrac{2}{t-1}\right). \end{aligned} \tag{4.3.3}$$

Let $\Gamma = \mathrm{PSL}_2(\mathbb{Z})$. For two distinct points $z_1, z_2 \in \Gamma\backslash\mathcal{H}$, the following convergent series defines the automorphic Green's function

$$G_s(z_1,z_2) := \sum_{\gamma \in \Gamma} g_s(z_1, \gamma z_2), \ \mathrm{Re}(s) > 1, \tag{4.3.4}$$

where

$$g_s(z_1, z_2) := -2Q_{s-1}(\cosh d(z_1, z_2)). \tag{4.3.5}$$

Recall that $E(\tau, s)$ is defined in (3.4.1) and $\varphi_1(s)$ is the coefficient of y^{1-s} in the Fourier expansion of $E(\tau, s)$. Proposition 5.1 in [28] tells us that for distinct $z_1, z_2 \in \Gamma\backslash\mathcal{H}$, the values of the j-function are related to the values of the automorphic Green's function by

$$\log |j(z_1) - j(z_2)|^2 = \lim_{s\to 1}(G_s(z_1, z_2) + 4\pi E(z_1, s) + 4\pi E(z_2, s) - 4\pi\varphi_1(s)) - 24. \tag{4.3.6}$$

For a fixed $z_1 \in \mathcal{H}$, one could evaluate z_2 at CM points arising from binary quadratic forms in $\mathcal{C}(d)$. The number of such CM points is give by the Hurwitz class number $H(-d)$. Adding up these values gives us the following proposition.

Proposition 4.3.1. *Let $d, D_0 < 0$ be congruent to 0 or 1 modulo 4 and $Q \in C(D_0)$. If $\tau_Q \neq \tau_q$ for any $q \in \mathcal{C}(d)$, then*

$$\log |\Psi_{-d}(\tau_Q)|^2 = \lim_{s\to 1}\left(\sum_{k>\sqrt{dD_0}} \rho_Q(k, d)(-2)Q_{s-1}\left(\frac{k}{\sqrt{dD_0}}\right) + H(-d)4\pi E(\tau_Q, s) + R(d, s)\right), \tag{4.3.7}$$

where $R(d, s) = \sum_{q\in\mathcal{C}(d)/\Gamma}(4\pi E(\tau_q, s) - 4\pi\varphi_1(s) - 24)$ and $\rho_Q(k, d)$ is the counting function defined by Eq. (4.1.20)*.*

Proof. Let $Q = [A, B, C] \in C(D_0), q = [a, b, c] \in \mathcal{C}(d)$, then $\tau_Q = \frac{-B+\sqrt{D_0}}{2A}, \tau_q = \frac{-b+\sqrt{d}}{2a}$. Some computations verify that

$$k := \sqrt{dD_0}\cosh d(\tau_Q, \tau_q) = 2Ac + 2Ca - Bb \in \mathbb{Z}.$$

Thus, the set $S_Q(k, d)$ defined by Eq. (4.1.18) can be rewritten as

$$S_Q(k, d) = \left\{q \in \mathcal{C}(d) : \cosh d(\tau_q, \tau_Q) = \tfrac{k}{\sqrt{dD_0}}\right\}.$$

Now let $z_1 = \tau_Q, z_2 = \tau_q$ in Eq. (4.3.6) and sum over $q \in \mathcal{C}(d)/\Gamma$. With the following observation

$$\begin{aligned}\sum_{q\in\mathcal{C}(d)/\Gamma}\frac{1}{w_q}G_s(\tau_Q,\tau_q) &= \sum_{q\in\mathcal{C}(d)/\Gamma}\sum_{\gamma\in\Gamma}\frac{1}{w_q}(-2)Q_{s-1}(\cosh d(\tau_Q,\gamma\tau_q))\\ &= \sum_{q\in\mathcal{C}(d)}(-2)Q_{s-1}(\cosh d(\tau_Q,\tau_q))\\ &= \sum_{k>\sqrt{dD_0}}\rho_Q(k,d)(-2)Q_{s-1}\left(\frac{k}{\sqrt{dD_0}}\right),\end{aligned}$$

we have Eq. (4.3.7). The sum is over $k > \sqrt{dD_0}$ since $\cosh d(\tau_Q,\tau_q) = 1$ precisely when $\tau_Q = \tau_q$ and $\tau_Q \neq \tau_q$ for any $q \in C(d)$. □

4.3.3 Holomorphic Projection

In this section, we will use holomorphic projection to express a finite linear combination of the Fourier coefficients of a mock-modular form as an infinite sum similar to the right hand side of Eq. (4.3.7).

Recall that $\Psi_{\mathcal{A}}(z)$ is defined by Eq. (4.2.5) for each $\mathcal{A} \in \mathrm{Pic}(\mathcal{O}_K)$ and has Fourier expansion

$$\Psi_{\mathcal{A}}(z) = \sum_{m\in\mathbb{Z}}\hat{c}(\Psi_{\mathcal{A}},m,y)q^m.$$

We have calculated the Fourier coefficients $\hat{c}(\Psi_{\mathcal{A}},m,y)$ explicitly in Prop. 4.2.4. Using the following facts

$$\begin{aligned}\hat{c}_{\phi,\mathcal{A}}(n) &= \tilde{c}_{\phi,\mathcal{A}}(n) - \overline{c_{\phi,\mathcal{A}}(-n)}\beta_1(-4\pi n,y),\\ \hat{c}_{\psi,\mathcal{A}}(n) &= \overline{\phi_2(-n)}\hat{c}_{\phi,\mathcal{A}}(n),\\ \beta_1(\alpha_1,\alpha_2 y) &= \beta_1(\alpha_1\alpha_2,y), \quad \alpha_1,\alpha_2>0\end{aligned}$$

we could write $\hat{c}(\Psi_{\mathcal{A}},m,y)q^m$ into the sum of a holomorphic part, $\left(2\sqrt{p}\right)a_{\phi,\mathcal{A}}(m)$, and non-holomorphic part $\left(2\sqrt{p}\right) b_{\phi,\mathcal{A}}(m,y)$, where for all $m \in \mathbb{Z}$

$$a_{\phi,\mathcal{A}}(m) = \sum_{\delta|N,\, k\in\mathbb{Z}} \varphi_{\delta^*}(\mathcal{A}) \left(\tilde{c}_{\psi,\mathcal{A}} \left(\frac{Np^2m - \delta^2k^2}{4} \right) + \phi_2(-1)\tilde{c}_{\psi,\mathcal{A}} \left(\frac{Nm - p^2\delta^2k^2}{4} \right) \right)$$
$$+ \frac{\overline{\phi_2(-4Nm)}\epsilon_p\sqrt{p}G(\phi_2')}{G(\phi_1)G(\phi_2)} \sum_{\substack{\delta|N,\, k\in\mathbb{Z} \\ \delta^2k^2\equiv Nm \pmod p}} \overline{\tilde{c}_{\phi,\mathcal{A}} \left(\frac{Nm - \delta^2k^2}{4} \right)} \tag{4.3.8}$$
$$+ \phi_2(4) \sum_{\delta|N,\, k\in\mathbb{Z}} S_2(Nm, \delta^2k^2, \phi_2)\tilde{c}_{\phi,\mathcal{A}} \left(\frac{Nm - \delta^2k^2}{4} \right),$$

and when $\left(\frac{-m}{p}\right) = -1$

$$b_{\phi,\mathcal{A}}(m,y) = -\sum_{\delta|N,\, k\in\mathbb{Z}} \varphi_{\delta^*}(\mathcal{A}) \overline{c_{\psi,\mathcal{A}} \left(\frac{\delta^2k^2 - Np^2m}{4} \right)} \beta_1 \left(\frac{\delta^2k^2 - Np^2m}{Np^2}, y \right). \tag{4.3.9}$$

Because $\beta_1(4\pi n, y)q^{-n}$ decays exponentially when $n \geq 1$, the pole and constant term of $\Psi_{\mathcal{A}}$ at infinity has the form

$$2\sqrt{p} \sum_{m\geq 0} a_{\phi,\mathcal{A}}(-m)q^{-m}.$$

To apply holomorphic projection to $\Psi_{\mathcal{A}}$, one needs to first subtract the pole and constant term.

For an integer $n \geq 1$ congruent to $0, 3$ modulo 4, let

$$g_n(z) = q^{-n} + \sum_{m\geq 1} c(g_n, m)q^m$$

be the unique weakly holomorphic modular form of level 4, weight 3/2 in the Kohnen plus space. They have integral Fourier coefficients and could be constructed explicitly (see [60]). Let $\mathcal{F}(z)$ be the weight 3/2 Eisenstein series studied in [33], which has the following Fourier expansion

$$\mathcal{F}(z) = \sum_{m=0}^{\infty} H(m)q^m + y^{-1/2} \sum_{m=-\infty}^{\infty} \frac{1}{16\pi} \beta_{3/2}(m^2, y)q^{-m^2},$$

and satisfies Kohnen's plus space condition. Here $H(m)$ is the Hurwitz class number when $m \geq 1$ and $H(0) = -\frac{1}{12}$. Define the function $\Psi^*_{\mathcal{A}}(z)$ to be

$$\Psi^*_{\mathcal{A}}(z) := \frac{1}{2\sqrt{p}}\Psi_{\mathcal{A}}(z) - \frac{a_{\phi,\mathcal{A}}(0)}{H(0)}\mathcal{F}(z) - \sum_{n\geq 1} a_{\phi,\mathcal{A}}(-n)g_n(z). \tag{4.3.10}$$

Denote its m^{th} Fourier coefficient by $\hat{c}(\Psi^*_{\mathcal{A}}, m, y)$. Then its holomorphic part, denoted by $a^*_{\phi,\mathcal{A}}(m)$, is

$$a^*_{\phi,\mathcal{A}}(m) = a_{\phi,\mathcal{A}}(m) - \frac{H(m)a_{\phi,\mathcal{A}}(0)}{H(0)} - \sum_{n\geq 1} a_{\phi,\mathcal{A}}(-n)c(g_n, m). \tag{4.3.11}$$

The function $\Psi^*_{\mathcal{A}}(z)$ has order $O(y^{-1/2})$ at the cusp infinity. The same decaying property holds at the other two cusps of $\Gamma_0(4)$ as well, since $\Psi^*_{\mathcal{A}}(z)$ satisfies the Kohnen plus space condition. So we can consider its holomorphic projection to the Kohnen plus space $S^+_{3/2}(\Gamma_0(4))$. This will produce the following identities between $a^*_{\phi,\mathcal{A}}(m)$ and an infinite sum similar to the one on the right hand side of Eq. (4.3.7).

Proposition 4.3.2. *Let $m \geq 1$ be a positive integer such that $\left(\frac{-m}{p}\right) = -1$. Then*

$$\overline{a^*_{\phi,\mathcal{A}}(m)} = 2\lim_{s\to 1}\left(\sum_{\delta|N}\varphi_{\delta^*}(\mathcal{A})\sum_{k>p\sqrt{Nm}/\delta} c_{\psi,\mathcal{A}}\left(\frac{\delta^2k^2 - Np^2m}{4}\right)2Q_{s-1}\left(\frac{\delta k}{p\sqrt{Nm}}\right)\right). \tag{4.3.12}$$

Proof. To execute the holomorphic projection, we first need to define the weight 3/2 Poincaré series for $m \geq 1$ by

$$\mathcal{P}_m(z,s) := \sum_{\gamma\in\Gamma_\infty\backslash\Gamma_0(4)} j(\gamma,z)^{-3}e^{2m\pi i\gamma z}\operatorname{Im}(\gamma z)^{s/2},$$

where for $\gamma \in \Gamma_0(4)$

$$j(\gamma,z) := \frac{\theta(\gamma z)}{\theta(z)}.$$

This series converges absolutely for $\operatorname{Re}(s) > 1$ and can be analytically continued to $\operatorname{Re}(s) \geq 0$. As $s \to 0$, the inner product $\langle \mathcal{P}_m(z,s), \Psi^*_{\mathcal{A}}(z)\rangle$ is the m^{th} Fourier coefficient of a cusp form in $S^+_{3/2}(\Gamma_0(4))$, since $\Psi^*_{\mathcal{A}}(z)$ is already in the plus space. Given $S^+_{3/2}(\Gamma_0(4)) = \{0\}$, we know the limit is zero and obtain the following equation after applying Rankin-Selberg unfolding,

$$\lim_{s\to 0}\left(\frac{\Gamma(\frac{1+s}{2})}{(4\pi m)^{1/2+s/2}}\overline{a^*_{\phi,\mathcal{A}}(m)} + \int_0^\infty \overline{b_{\phi,\mathcal{A}}(m,y)}e^{-4\pi my}y^{1/2+s/2}\frac{dy}{y}\right) = 0. \tag{4.3.13}$$

After some manipulations, we have

$$\int_0^\infty \beta_1(m,\mu y)e^{-4\pi my}y^{1/2+s/2}\frac{dy}{y} = \frac{\Gamma(\frac{1+s}{2})}{(4\pi m)^{1/2+s/2}}\varrho_s(\mu),$$

where the function $\varrho_s(\mu)$ is defined by

$$\varrho_s(\mu) := \int_1^\infty \frac{du}{(\mu u+1)^{\frac{1+s}{2}} u}, \quad \mu > 0. \tag{4.3.14}$$

After substituting Eq. (4.3.9) and $\mu = \frac{\delta^2 k^2}{Np^2 m} - 1$ into Eq. (4.3.13), we arrive at the following equation

$$-\int_0^\infty \overline{b_{\phi,\mathcal{A}}(m,y)} e^{-4\pi m y} y^{1/2+s/2} \frac{dy}{y} = \frac{\Gamma(\frac{1+s}{2})}{(4\pi m)^{\frac{1+s}{2}}} 2 \sum_{\delta|N} \varphi_{\delta^*}(\mathcal{A}) \sum_{k > p\sqrt{Nm}/\delta} c_{\psi,\mathcal{A}}\left(\frac{\delta^2 k^2 - Np^2 m}{4}\right) \varrho_s\left(\frac{\delta^2 k^2}{Np^2 m} - 1\right). \tag{4.3.15}$$

Since $c_{\psi,\mathcal{A}}(n) = 0$ whenever $n \leq 0$, the sum changed from $k \in \mathbb{Z}$ to $k > p\sqrt{N}$ and produced a factor of 2. Now substituting (4.3.15) into (4.3.13) gives us

$$\overline{a^*_{\phi,\mathcal{A}}(m)} = 2 \lim_{s\to 0} \left(\sum_{\delta|N} \varphi_{\delta^*}(\mathcal{A}) \sum_{k > p\sqrt{Nm}/\delta} c_{\psi,\mathcal{A}}\left(\frac{\delta^2 k^2 - Np^2 m}{4}\right) \varrho_s\left(\frac{\delta^2 k^2}{Np^2 m} - 1\right)\right).$$

With the following comparisons (see [28, §7] for similar arguments).

$$\varrho_0(\mu) = 2Q_0(\sqrt{\mu+1}),$$
$$Q_{s-1}(\sqrt{\mu+1}) - \frac{s\Gamma(s)^2}{2^{2-s}\Gamma(2s)} \varrho_{s-1}(\mu) = O(\mu^{-1/2-s/2}),$$

we could substitute $\varrho_s\left(\frac{\delta^2 k^2}{Np^2 m} - 1\right)$ with $2Q_{s-1}\left(\frac{\delta k}{p\sqrt{Nm}}\right)$ in the limit and obtain Eq. (4.3.12). □

4.3.4 Proof of Main Theorem

In this section, we will prove Theorem 1.1.1 stated in the introduction by proving a more general equality. As before, $D < 0$ is an odd, fundamental discriminant, $(p) = \mathfrak{p}\overline{\mathfrak{p}}$ a prime that splits in $K = \mathbb{Q}(\sqrt{D})$ and ϕ is a non-trivial ray class group character modulo $\mathfrak{p}$ such that $\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is non-quadratic.

Theorem 4.3.3. *Let $m \geq 1$ be a positive integer such that $\left(\frac{-m}{p}\right) = -1$ and $\mathcal{A} = \mathcal{B}^2 \in$ $\mathrm{Pic}(\mathcal{O}_K)$. Then we have*

$$a^*_{\phi,\mathcal{A}}(m) = -2 \sum_{\substack{[Q]\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}^{-1}}]}} \psi^2(Q)\left(\log|\Psi_m(\tau_Q)|^2 - \tfrac{2H(m)}{H(0)}\log\left(y_Q|\eta(\tau_Q)|^2\right)\right). \tag{4.3.16}$$

Proof. The right hand of Eq. (4.3.16) can be rewritten as

$$\sum_{\substack{[Q]\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}}]}} \psi^2(Q)\log|\Psi_m(\tau_Q)|^2 = \sum_{\substack{[P]\in C^2(Dp^2)\\ \pi([P])=[Q_{\mathcal{A}}]}} \psi(P) \sum_{\substack{[Q]\in C(Dp^2)\\ [Q]^2=[P]}} \log|\Psi_m(\tau_Q)|^2. \tag{4.3.17}$$

Applying Theorem 4.1.10 and Props. 4.1.11 and 4.3.1 with $d = -m, D_0 = -Np^2$ then gives us

$$\sum_{\substack{Q\in C(Dp^2)\\ [Q^2]=[P]}} \log|\Psi_m(\tau_Q)|^2 = \lim_{s\to 1}\left(\begin{array}{l}\displaystyle\sum_{k>p\sqrt{Nm}} 2^{\omega(\gcd(Np,k))} r_P\left(\tfrac{k^2-Np^2m}{4}\right)(-2)Q_{s-1}\left(\tfrac{k}{p\sqrt{Nm}}\right)\\ \displaystyle+\sum_{\substack{Q\in C(Dp^2)\\ [Q^2]=[P]}} (H(m)4\pi E(\tau_Q,s) + R(-m,s))\end{array}\right).$$

Notice that this substitution is valid even when $p \mid k$ by Prop. 4.1.11. Substituting this into the right hand side of Eq. (4.3.17) and applying Eqs. (3.3.17) and (3.3.18) then gives us

$$\sum_{\substack{[Q]\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}}]}} \psi^2(Q)\log|\Psi_m(\tau_Q)|^2 = \lim_{s\to 1}\sum_{k>p\sqrt{Nm}} 2^{\omega(\gcd(N,k))} c_{\psi,\mathcal{A}}\left(\tfrac{k^2-Np^2m}{4}\right)(-2)Q_{s-1}\left(\tfrac{k}{p\sqrt{Nm}}\right)$$
$$+ 4\pi H(m)\lim_{s\to 1}\sum_{\substack{Q\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}}]}} \psi^2(Q)E(\tau_Q,s).$$

The p disappears from $\gcd(Np,k)$ in the exponent since $c_{\psi,\mathcal{A}}(n) = 0$ whenever $p \mid n$. Eq. (3.3.18) implies that the term $R(-m,s)$ also vanishes since it is independent of $[Q] \in C(Dp^2)$. By Kronecker's first limit formula (Eq. (3.4.2)) and Eq. (3.3.18), we have

$$4\pi\lim_{s\to 1}\sum_{\substack{Q\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}}]}} \psi^2(Q)E(\tau_Q,s) = \frac{2}{H(0)}\sum_{\substack{Q\in C(Dp^2)\\ \pi([Q])^2=[Q_{\mathcal{A}}]}} \psi^2(Q)\log\left(y_Q|\eta(\tau_Q)|^2\right).$$

Since $\mathcal{A} \in \mathrm{Pic}^2(\mathcal{O}_K)$, Eq. (4.3.12) becomes

$$\overline{a^*_{\phi,\mathcal{A}}(m)} = 2\lim_{s\to 1}\left(\sum_{k>p\sqrt{Nm}} 2^{\omega(\gcd(N,k))} c_{\psi,\mathcal{A}}\left(\frac{k^2-Np^2m}{4}\right) 2Q_{s-1}\left(\frac{\delta k}{p\sqrt{Nm}}\right)\right).$$

Putting together the last three equations, we obtain

$$\overline{a^*_{\phi,\mathcal{A}}(m)} = -2 \sum_{\substack{[Q]\in C(Dp^2)\\ \pi([Q])^2=[Q_\mathcal{A}]}} \psi^2(Q) \left(\log|\Psi_m(\tau_Q)|^2 - \tfrac{2H(m)}{H(0)} \log\left(y_Q|\eta(\tau_Q)|^2\right)\right).$$

Conjugating both sides and using the fact that $\overline{\psi^2(Q)} = \psi^2(Q^{-1}), \tau_{Q^{-1}} = -\overline{\tau_Q}$ give us Eq. (4.3.16). □

From Eq. (3.3.13), we know that the shadow of

$$\tilde{f}_{\phi,1}(z) = \sum_{n\geq -n_1} \tilde{c}_{\phi,1}(n)q^n := \sum_{\mathcal{A}\in \mathrm{Pic}^2(\mathcal{O}_K)} \tilde{f}_{\phi,\mathcal{A}}(z)$$

is $f_{\phi,1}$. Here $n_1 \in \mathbb{Z}$ is defined in Eq. (2.4.19). In the same notations as before, we could now state the theorem relating finite linear combinations of $\tilde{c}_{\phi,1}(n)$ with the values of Borcherds lift.

Theorem 4.3.4. *Let $\phi : I_\mathfrak{p}/P_{\mathfrak{p},1} \longrightarrow \mathbb{C}^\times$ be a non-trivial character such that $\phi_1 : (\mathbb{Z}/p\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$ is non-quadratic and $m \geq 1$ be a positive integer such that $\left(\frac{-m}{p}\right) = -1$. Then for any $\tilde{f}_{\phi,1} = \sum_{n\geq -n_1} \tilde{c}_{\phi,1}(n)q^n \in \mathbb{M}_{1,1}(Np, \chi_D\overline{\phi_1})$, we have*

$$\frac{\sum_{k\in\mathbb{Z}} \tilde{c}_{\phi,1}\left(\frac{Np^2m-k^2}{4}\right)\phi_1\left(\frac{k}{2}\right)\delta_N(k) + 4\sum_{Q\in C(Dp^2)} \psi^2(Q)\log\left(|\Psi_m(\tau_Q)|\right)}{I_{\psi^2}} \in \mathbb{Q}(\phi). \quad (4.3.18)$$

Furthermore, if $n_1 < \min\left\{\frac{N}{4}, p\right\}$, then we have the equality

$$\sum_{k\in\mathbb{Z}} \tilde{c}_{\phi,1}\left(\frac{Np^2m-k^2}{4}\right)\phi_1\left(\frac{k}{2}\right)\delta_N(k) = -4 \sum_{Q\in C(Dp^2)} \psi^2(Q)\log\left(y_Q|\Psi_m(\tau_Q)|\right). \quad (4.3.19)$$

Proof. So we could define

$$a_{\phi,1}(m) := \sum_{\mathcal{A}\in \mathrm{Pic}^2(\mathcal{O}_K)} a_{\phi,\mathcal{A}}(m), a^*_{\phi,1}(m) := \sum_{\mathcal{A}\in \mathrm{Pic}^2(\mathcal{O}_K)} a^*_{\phi,\mathcal{A}}(m).$$

Summing Eq. (4.3.16) over $\mathcal{A} \in \mathrm{Pic}^2(\mathcal{O}_K)$ gives us

$$a^*_{\phi,1}(m) + 4 \sum_{Q\in C(Dp^2)} \psi^2(Q)\log|\Psi_m(\tau_Q)| = 4\frac{H(m)}{H(0)} I_{\psi^2}.$$

Combining with Eq. (4.3.11), we could write

$$a_{\phi,1}(m) + 4 \sum_{Q \in C(Dp^2)} \psi^2(Q) \log|\Psi_m(\tau_Q)| = \frac{H(m)}{H(0)} \left(4I_{\psi^2} + a_{\phi,1}(0)\right) + \sum_{n' \geq 1} a_{\phi,1}(-n')c(g_n, m). \tag{4.3.20}$$

Since $\left(\frac{-m}{p}\right) = -1$, we could simplify Eq. (4.3.8) and use $\tilde{c}_{\psi,\mathcal{A}}(n) = \overline{\phi_2(-n)}\tilde{c}_{\phi,\mathcal{A}}(n), \phi_1\phi_2^2 = \mathbb{1}_p$ to obtain

$$a_{\phi,1}(m) = \sum_{k \in \mathbb{Z}} \tilde{c}_{\phi,1}\left(\frac{Np^2m - k^2}{4}\right) \phi_1\left(\frac{k}{2}\right) \delta_N(k).$$

Now on the right hand side of Eq. (4.3.20), $c(g_n, m) \in \mathbb{Z}$ and $a_{\phi,1}(-n')$ is some linear combination of $\tilde{c}_{\phi,1}(n)$ with $n \leq 0$ with coefficients in $\mathbb{Q}(\phi)$ by Eq. (4.3.8). Prop. 3.4.5 then gives us (4.3.18).

If $n_1 < \frac{N}{4}$, then the term $a_{\phi,1}(-n')$ vanishes for all $n' \geq 1$ since the sums in Eq. (4.3.8) are all empty. Eq. (4.3.20) then becomes

$$a_{\phi,1}(m) + 4 \sum_{Q \in C(Dp^2)} \psi^2(Q) \log|\Psi_m(\tau_Q)| = \frac{H(m)}{H(0)} \left(4I_{\psi^2} + a_{\phi,1}(0)\right). \tag{4.3.21}$$

By Prop. 4.2.4, the term $a_{\phi,1}(0)$ can be written as

$$a_{\phi,1}(0) = 2 \sum_{k \in \mathbb{Z}} \tilde{c}_{\phi,1}(-k^2)\phi_1(k)\delta_N(k) = 4 \sum_{k=1}^{\lfloor\sqrt{n_1}\rfloor} \tilde{c}_{\phi,1}(-k^2)\phi_1(k)\delta_N(k). \tag{4.3.22}$$

Combinig Prop. 2.4.6 and Cor. 3.4.4 gives us

$$\sum_{n=1}^{n_1} \left(\tilde{c}_{\phi,1}(-n)c_{\phi,\mathcal{A}_0}(n) + \overline{\tilde{c}_{\phi,1}(-pn)c_{\phi,\mathcal{A}_0}(pn)}\right) \delta_N(n) = \langle f_{\phi,1}, f_{\phi,\mathcal{A}_0}\rangle = -\frac{4}{\#\mathcal{O}_K^\times \#\mathcal{O}_p^\times} I_{\psi^2}.$$

If $n_1 < \min\left\{\frac{N}{4}, p\right\}$, $\#\mathcal{O}_K^\times = \#\mathcal{O}_p^\times = 2$ and the sum above simplifies

$$\sum_{n=1}^{n_1} \tilde{c}_{\phi,1}(-n)c_{\phi,\mathcal{A}_0}(n)\delta_N(n) = -I_{\psi^2}.$$

Cor. 3.3.6 reduces the equation above further to

$$\sum_{k=1}^{\lfloor\sqrt{n_1}\rfloor} \tilde{c}_{\phi,1}(-k^2)\phi_1(k)\delta_N(k) = -I_{\psi^2}.$$

Substituting this into Eq. (4.3.22) yields $a_{\phi,1}(0) = -4I_{\psi^2}$. Then Eq. (4.3.21) becomes Eq. (4.3.19). □

Theorem 1.1.1 is now a consequence of Theorem 4.3.4. Since $S_1(|D|p, \chi_D\phi_1)$ is one dimensional and $|D| > 5$, $n_1 = 1$ and the condition $n_1 < \min\left\{\frac{N}{4}, p\right\}$ is satisfied. Since $|D|$ is prime, we know that $f_{\phi,1} = f_\phi$ and there are two Eisenstein series in $M_1(|D|p, \chi_D\phi_1)$. Because of the vanishing conditions we imposed on the Fourier coefficients $\tilde{c}(n)$, the mock-modular form $\tilde{f}_\phi(z)$ in Eq. (1.1.2) is unique. When $D' = -m$ is a fundamental discriminant, the function $\Psi_m(z)$ becomes

$$\Psi_m(z) = \prod_{Q' \in C(D')} (j(z) - j(\tau_{Q'}))^{2/w_{Q'}}.$$

So Eq. (4.3.19) becomes Eq. (1.1.3).

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