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THE RADON TRANSFORM:
SOME REMARKS AND FORMULAS FOR TWO DIMENSIONS

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Abstract

The importance of Radon transform inversion is beginning to be recognized in several areas of the physical and biological sciences. Various methods for inversion in two dimensions are discussed jointly with the interrelationships among some important formulas which naturally arise in the inversion process. These results serve as a foundation for the development of numerical inversion techniques, and for generalizations to higher dimensions.

1. INTRODUCTION

The Radon transform of a function F defined on Euclidean n-space $\mathbb{R}^n$ establishes a relation between F and integrals of F over all hyperplanes contained in $\mathbb{R}^n$. Following the early work by Radon [1] in 1917 the theory of the Radon transform has been developed by several authors [2-6]; however, it is only within the past few years that major applications have emerged. In most of these applications there is an attempt to obtain detailed information about some aspect of the internal structure of an object by studying the effect the object has upon some probe, such as x-rays when they pass through the object. For some specific applications we refer the reader to [7-9] and additional references contained therein.

The unification of all of these applications where there is an attempt to *reconstruct* certain internal structure *from* experimental knowledge about *projections* of the internal structure information is to be found in the theory of the Radon transform. The success of a given application ultimately depends upon one's ability to invert the Radon transform relative to the specific object and specific probe. It is our purpose here to investigate the Radon transform in two dimensions and its inversion which yields information about a *plane* cross section of the object. Specifically, we shall be concerned with the Radon transform and its inversion for a function $F \in \mathbb{D}$ where F represents the desired information about the internal structure and $\mathbb{D}$ is the space of C^∞ functions with compact support [10].

We shall present many of our results in considerable detail for several reasons. (i) It is important to point out certain interrelationships among various formulas which naturally appear in the inversion process. (ii) Various methods for inversion are suggested. (iii) There is a need to correct errors and simplify certain formulas which have appeared in the literature. (iv) This approach is especially relevant at this time since some of these results are of importance in current work on the development of computer codes for Radon transform inversion.

2. REDUCTION TO TWO DIMENSIONS

The n-dimensional Radon transform of the function $F(x) = F(x_1, x_2, \cdots, x_n)$ may be expressed as [2]

$$f(p,\xi) = R\{F\} = \int F(x)\ \delta(p - \xi \cdot x)\ dx \ , \qquad (2.1)$$

where $x \in \mathbb{R}^n$ and the integral is over all x-space, ξ is an arbitrary fixed unit vector, and p is a real number. Observe that since $F \in \mathbb{D}$ the function f is also a C^∞ function with compact support [6] and the presence of the Dirac δ function causes the integration to be performed over all hyperplanes where

$$p = \xi \cdot x = \xi_1 x_1 + \xi_2 x_2 + \cdots + \xi_n x_n \ .$$

Note that since ξ is a unit vector p represents the distance from the origin to the hyperplane. Moreover, if F is a function of n independent variables then f is also a function of n independent variables and f satisfies the symmetry property

$$f(-p,-\xi) = f(p,\xi).$$

We now assume that $F \in \mathbb{D}$ is defined on $\mathbb{R}^2$ and that the support of F is bounded by the unit circle. If at this point there is concern about the severe and perhaps unphysical restrictions which have been placed upon F it is appropriate to recall the Approximation Theorem [10].

THEOREM. (Schwartz) For any $\varepsilon>0$, any continuous function f, with a bounded support K, can be uniformly approximated to within a distance ε by some function $\phi \in \mathbb{D}$, and ϕ can be required to have its support contained within an arbitrary neighbourhood of the support K of f.

The main reason for working with functions from $\mathbb{D}$ is to insure that the results will be rigorous, since there are numerous changes in the order of integration and generalized functions appear all through the development.

With the stated assumptions (2.1) may be written as

$$f(p,\theta) = \int_0^{\pi} \int_{-1}^{1} F(r,\phi)\ \delta[p - r\cos(\phi-\theta)]\ |r|\, dr\, d\phi \quad , \quad -1 \le p \le 1 \ , \qquad (2.2)$$

where we have replaced $f(p,\xi)$ and $F(x)$ by $f(p,\theta)$ and $F(r,\phi)$, respectively, in going over to a modified form of polar coordinates. Explicitly,

$$\xi_1 = \cos\theta \ , \quad \xi_2 = \sin\theta$$
$$x_1 = r\cos\phi \ , \quad x_2 = r\sin\phi$$

and the symmetry condition becomes

$$f(-p,\theta+\pi) = f(p,\theta) \ .$$

By use of standard Fourier techniques [11,12] it is possible to obtain a deceptively simple expression for F which shows a striking resemblance to (2.2),

$$F(r,\phi) = \int_0^{\pi} \int_{-1}^{1} g(p,\theta)\ \delta[p - r\cos(\phi-\theta)]\ dp\, d\theta \quad , \ -1 \le r \le 1 \ , \ 0 \le \phi < \pi \ , \quad (2.3)$$

where $g(p,\theta)$ is related to $f(p,\theta)$ by

$$g(p,\theta) = \int_{-1}^{1} dt\ f(t,\theta) \int_{-\infty}^{\infty} dy\ |y|\ e^{2\pi i y(t-p)} \quad . \qquad (2.4)$$

Note that F satisfies the symmetry condition

$$F(r,\phi) = F(-r,\phi+\pi) \ , \quad -1 \le r \le 1 \ , \quad 0 \le \phi < \pi \ .$$

By consideration of the argument of the δ function and the symmetry satisfied by F (2.3) may be modified to read (for $r \neq 0$)

$$F(r,\phi) = \int_{0}^{\pi} \int_{-|r|}^{|r|} g(p,\theta)\ \delta[p - r\cos(\phi-\theta)]\ dp\, d\theta \ ;$$

however, for reasons which will be apparent as we proceed it is very useful to rewrite (2.3) with $r > 0$. (The $r = 0$ case is treated separately in Sec. 7) This modification can be made if we agree to calculate $F(-r,\phi)$ from the symmetry condition with r replaced by $-r$,

$$F(-r,\phi) = F(r,\phi+\pi) \ , \quad 0 < r \le 1 \ , \quad 0 \le \phi < \pi \ .$$

Then, F is calculated from

$$F(r,\phi) = \int_{0}^{\pi} \int_{-r}^{r} g(p,\theta)\ \delta[p - r\cos(\phi-\theta)]\ dp\, d\theta \ , \quad 0 < r \le 1 \ , \quad 0 \le \phi < 2\pi . \qquad (2.5)$$

The y integration in (2.4) can be done by considering the integrand as a generalized function [13],

$$g(p,\theta) = \frac{-1}{2\pi^2} \int_{-1}^{1} \frac{f(t,\theta)\ dt}{(t-p)^2} \quad .$$

An integration by parts serves to cast this result into the desired form,

$$g(p,\theta) = \frac{-1}{2\pi^2} \int_{-1}^{1} \frac{\partial f}{\partial t}(t,\theta)\ \frac{dt}{t-p} = \frac{-1}{2\pi} \mathsf{H}\{\frac{\partial f}{\partial t}\} \ , \qquad (2.6)$$

where H stands for the Hilbert transform [14].

Upon substituting (2.6) into (2.5) and making a change in the order of integration we obtain ($0 < r \leq 1$, $0 \leq \phi < 2\pi$)

$$F(r,\phi) = \frac{-1}{2\pi^2} \int_0^\pi d\theta \int_{-1}^{1} dt \int_{-r}^{r} \frac{dp}{t-p} \frac{\partial f(t,\theta)}{\partial t} \delta[p - r\cos(\phi-\theta)] . \qquad (2.7)$$

By doing the p integration we obtain the often publicized but seldom derived result,

$$F(r,\phi) = \frac{-1}{2\pi^2} \int_0^\pi \int_{-1}^{1} \frac{\partial f(t,\theta)}{\partial t} \frac{dt\, d\theta}{t - r\cos(\phi-\theta)} . \qquad (2.8)$$

It is useful to observe that this same result follows directly from the $n = 2$ special case of the general Radon inversion formula [6].

3. DECOMPOSITION OF F AND f

Our purpose in the previous section was not to obtain (2.8), but to indicate the steps one may follow in going from (2.3) to (2.8). Indeed, there are very good reasons for using (2.3) as a starting point for the inversion process [12]; however, for most of the interesting physical problems F is not in $\mathbb{D}$ and the convergence is usually in the space of distributions $\mathbb{D}'$ or in the space of tempered distributions. Convergence in these spaces is known as *weak convergence* and cannot be compared in a simple way to the usual convergence of functions. The Approximation Theorem would still apply but it is usually ignored and one might justifiably ask why many of the current methods work at all. Although a careful answer may be slightly different for each successful case the basic reason can be traced to the fact that from a numerical point of view all δ-convergent sequences are very similar in appearance, and in many situations a lack of uniqueness may not be of great concern if the final result "looks" close.

The main concern here is with the intermediate result (2.7). Suppose we do not do the p integration which yielded (2.8); instead, let us assume that $f(t,\theta)$ may be decomposed into the product

$$f(t,\theta) = a_\ell(t) \cos \ell\theta , \qquad \ell = 0,1,2,\cdots \tag{3.1a}$$

or

$$f(t,\theta) = b_\ell(t) \sin \ell\theta , \qquad \ell = 1,2,3,\cdots \tag{3.1b}$$

where

$$a_\ell(t) = (-1)^\ell a_\ell(-t) , \quad b_\ell(t) = (-1)^\ell b_\ell(-t) ,$$

a condition which follows directly from the symmetry condition satisfied by f. This decomposition is the most general one which preserves the symmetry, and by linearity it is easy to extend this to an expansion of the form

$$f(t,\theta) = \sum_{\ell=0}^{\infty} a_\ell(t) \cos \ell\theta + \sum_{\ell=1}^{\infty} b_\ell(t) \sin \ell\theta . \tag{3.2}$$

Note that the same form is obtained if one assumes that the *real* function f can be expanded as

$$f(t,\theta) = \sum_{\ell=-\infty}^{\infty} z_\ell(t) e^{i\ell\theta} ,$$

where $z_\ell(t)$ is complex, $z_\ell = x_\ell + iy_\ell$.

We now substitute the forms (3.1) into (2.7) and pick out the θ integration. This yields integrals J which depend upon r and ϕ,

$$J_a = \int_0^\pi \cos \ell\theta \ \delta[p - r\cos(\theta - \phi)]\, d\theta , \tag{3.3a}$$

$$J_b = \int_0^\pi \sin \ell\theta \ \delta[p - r\cos(\theta - \phi)]\, d\theta . \tag{3.3b}$$

These integrals are easier to evaluate after a change of variable, $x = \cos(\theta - \phi)$. Full details of the calculation for (3.3b) will be shown. The other case is very similar. After the change of variables,

$$J_b = \int_{-\cos\phi}^{\cos\phi} \frac{\sin(\ell\phi + \ell\cos^{-1}x)}{(1-x^2)^{\frac{1}{2}}} \delta(p - rx)\, dx \,.$$

Upon making use of the property $\delta(p - rx) = \frac{1}{r}\delta(\frac{p}{r} - x)$ with $r > 0$ it follows that

$$J_b = (r^2 - p^2)^{-\frac{1}{2}} \sin \ell\phi \; T_\ell(\tfrac{p}{r}) \; + \; r^{-1} \cos \ell\phi \; U_{\ell-1}(\tfrac{p}{r}) \,,$$

where T_ℓ and U_ℓ are Tchebycheff polynomials of the first and second kinds, respectively. For convenience, some properties of these functions have been included in the Appendix.

It turns out that only one of the terms in the expression for J_b will actually contribute to F; however, that may not be obvious at this point so we keep both terms and substitute this result into (2.7),

$$F(r,\phi) = \frac{-1}{2\pi^2} \int_{-1}^{1} dt\; b'_\ell(t) \int_{-r}^{r} \frac{dp}{t-p} J_b \,,$$

where $b'_\ell(t) = \frac{db_\ell}{dt}$. By another change of variables $p = rx$ we obtain

$$F(r,\phi) = \frac{1}{2\pi^2 r} \int_{-1}^{1} dt\;\; b'_\ell(t) \int_{-1}^{1} dx\;\; (x - \tfrac{t}{r})^{-1} \;\{(1-x^2)^{-\frac{1}{2}} \sin \ell\phi \;\; T_\ell(x) + \cos \ell\phi \;\; U_{\ell-1}(x)\}$$

The term involving $\cos \ell\phi$ may be dropped, since by a change of variables $x \to -x$, $t \to -t$ it is easy to show that

$$\int_{-1}^{1} \int_{-1}^{1} (x - \frac{t}{r})^{-1} \, b_\ell'(t) \, U_{\ell-1}(x) \, dx \, dt$$

vanishes. This leaves

$$F(r,\phi) = \frac{\sin \ell\phi}{2\pi^2 r} \int_{-1}^{1} dt \, b_\ell'(t) \int_{-1}^{1} dx \, (x - \frac{t}{r})^{-1} \, (1 - x^2)^{-\frac{1}{2}} \, T_\ell(x). \qquad (3.4)$$

The case involving J_a may be worked out in a similar fashion. The result is

$$F(r,\phi) = \frac{\cos \ell\phi}{2\pi^2 r} \int_{-1}^{1} dt \, a_\ell'(t) \int_{-1}^{1} dx \, (x - \frac{t}{r})^{-1} \, (1 - x^2)^{-\frac{1}{2}} \, T_\ell(x). \qquad (3.5)$$

Consequently, we see that the decomposition (3.1) implies that it is possible to also decompose $F(r,\phi)$ in the form

$$F(r,\phi) = A_\ell(r) \cos \ell\phi \, , \quad \ell = 0, 1, 2, \cdots \qquad (3.6a)$$

or

$$F(r,\phi) = B_\ell(r) \sin \ell\phi \, , \quad \ell = 1, 2, 3, \cdots \qquad (3.6b)$$

with the understanding that both $A_\ell(-r)$ and $B_\ell(-r)$ are to be calculated from an equation of the form $F_\ell(-r) = (-1)^\ell F_\ell(r)$, where

$$F_\ell(r) = \frac{1}{2\pi^2 r} \int_{-1}^{1} dt \, f_\ell'(t) \int_{-1}^{1} dx \, (x - \frac{t}{r})^{-1} \, (1 - x^2)^{-\frac{1}{2}} \, T_\ell(x). \qquad (3.7)$$

For brevity we have written F_ℓ to represent either A_ℓ or B_ℓ and f_ℓ to represent either a_ℓ or b_ℓ.

For future reference it is convenient to select the x integration from (3.7) and define

$$I_\ell(\frac{t}{r}) = \int_{-1}^{1} (x - \frac{t}{r})^{-1} \, (1 - x^2)^{-\frac{1}{2}} \, T_\ell(x) \, dx \, . \qquad (3.8)$$

The evaluation of this integral will be discussed in Sec. 5.

4. THE FORWARD INTEGRAL EQUATION

Once we know that $F(r,\phi)$ may be decomposed into the form (3.6), it is immediately possible to obtain another equation relating F_ℓ and f_ℓ which in effect solves (3.7) for $f_\ell(t)$. The desired result may be obtained by replacing f and F in (2.2) by their respective decompositions (3.1) and (3.6), and observing that the presence of the δ function allows a modification of the limits of integration, $\int_0^\pi \int_{-1}^1 \rightarrow \int_{-\pi}^\pi \int_p^1$, $0 \le p \le 1$. Then, by proceeding in a fashion very similar to the approach used in Sec. 3 one obtains

$$f_\ell(p) = 2 \int_p^1 [1 - (\tfrac{p}{r})^2]^{-\frac{1}{2}} \; F_\ell(r) \; T_\ell(\tfrac{p}{r}) \; dr, \qquad 0 \le p \le 1 \; , \tag{4.1}$$

with the understanding that $f_\ell(-p) = (-1)^\ell f_\ell(p)$. Here, the $p = 0$ case can be correctly calculated without difficulty from (4.1),

$$f_\ell(0) = 2 \cos(\tfrac{\ell\pi}{2}) \int_0^1 F_\ell(r) \; dr \; . \tag{4.2}$$

5. THE INVERSION FORMULA

Equation (4.1) has a rather simple appearance while its inversion (3.7) is considerably more complicated. It is possible to simplify (3.7) but care must be taken since the evaluation of $I_\ell(\frac{t}{r})$ defined in (3.8) depends upon whether $|\frac{t}{r}| < 1$ or $|\frac{t}{r}| > 1$. To properly take this into account we write $(0 < r \le 1)$

$$F_\ell(r) = \frac{1}{2\pi^2 r} \Big[\int_{-1}^{-r} f'_\ell(t) \; I_\ell(\tfrac{t}{r}) \; dt \; + \int_{-r}^{r} f'_\ell(t) \; I_\ell(\tfrac{t}{r}) \; dt$$

$$+ \int_r^1 f'_\ell(t) \; I_\ell(\tfrac{t}{r}) \; dt \; \Big] \; .$$

By making use of the $x \to -x$, $t \to -t$ change of variables again, we obtain

$$F_\ell(r) = \frac{1}{\pi r}\left[\int_0^r f_\ell'(t)\, I_\ell\left(\frac{t}{r}\right) dt + \int_r^1 f_\ell'(t)\, I_\ell\left(\frac{t}{r}\right) dt\right], \tag{5.1}$$

where $0 < r \leq 1$ and $F_\ell(-r) = (-1)^\ell F_\ell(r)$.

It is now possible to evaluate $I_\ell(\frac{t}{r})$ by use of tabulated results [15], since in the first integral of (5.1) $t/r \leq 1$ and in the second integral $t/r \geq 1$. The result is

$$F_\ell(r) = \frac{1}{\pi r}\int_0^r f_\ell'(t)\, U_{\ell-1}\left(\frac{t}{r}\right) dt - \frac{1}{\pi r}\int_r^1 \left[\left(\frac{t}{r}\right)^2 - 1\right]^{-\frac{1}{2}} \left[\frac{t}{r} + \sqrt{(t/r)^2 - 1}\right]^{-\ell} f_\ell'(t)\, dt \tag{5.2}$$

This inversion formula has been obtained by other authors [16,17] using different methods.

6. SIMPLIFICATION OF THE INVERSION FORMULA

The inversion formula (5.2) can be simplified by making use of the identities in the Appendix. The desired result is obtained by applying (A-10). This immediately yields

$$F_\ell(r) = \frac{1}{\pi r}\int_0^1 f_\ell'(t)\, U_{\ell-1}\left(\frac{t}{r}\right) dt - \frac{1}{\pi r}\int_r^1 \left[\left(\frac{t}{r}\right)^2 - 1\right]^{-\frac{1}{2}} f_\ell'(t)\, T_\ell\left(\frac{t}{r}\right) dt. \tag{6.1}$$

We now show that the first integral on the right gives zero. Consider the integral

$$K = \int_0^1 f_\ell'(t)\, U_{\ell-1}\left(\frac{t}{r}\right) dt \quad . \tag{6.2}$$

An integration by parts yields

$$K = -\frac{1}{r}\int_0^1 f_\ell(t)\; U'_{\ell-1}(\tfrac{t}{r})\; dt \;,$$

where the prime means derivative with respect to the argument. The integrated part does not appear since it vanishes by use of symmetry combined with the assumption that $f_\ell(1) = 0$. Now, by application of (4.1), K becomes

$$K = -\frac{2}{r}\int_0^1 dt\; U'_{\ell-1}(\tfrac{t}{r}) \int_t^1 dx\,[1-(t/x)^2]^{-\frac{1}{2}}\, F_\ell(x)\; T_\ell(\tfrac{t}{x}) \;,$$

and by interchanging the order of integration,

$$K = -\frac{2}{r}\int_0^1 dx\; F_\ell(x) \int_0^x dt\,[1-(t/x)^2]^{-\frac{1}{2}}\, U'_{\ell-1}(\tfrac{t}{r})\; T_\ell(\tfrac{t}{x}) \;.$$

(If there is any question about the limits set up a horizontal x axis and a vertical t axis and observe that the integration is over a triangular region in the first quadrant of the xt plane.) Next, by changing the t variable through $t = xy$,

$$K = -\frac{2}{r}\int_0^1 dx\;\; x\,F_\ell(x) \int_0^1 dy\,(1-y^2)^{-\frac{1}{2}}\, U'_{\ell-1}(\tfrac{xy}{r})\; T_\ell(y) \;.$$

At this point we focus attention on the y integration which we designate by L,

$$L = 2\int_0^1 U'_{\ell-1}(\tfrac{xy}{r})\; T_\ell(y)\;(1-y^2)^{-\frac{1}{2}}\, dy = \int_{-1}^1 U'_{\ell-1}(\tfrac{xy}{r})\; T_\ell(y)\;(1-y^2)^{-\frac{1}{2}}\, dy$$

or from (A-9)

$$L = \frac{1}{\ell} \int_{-1}^{1} T_\ell''\left(\frac{xy}{r}\right) T_\ell(y) \left(1 - y^2\right)^{-\frac{1}{2}} dy \quad . \tag{6.3}$$

Observe that $T_\ell''\left(\frac{xy}{r}\right)$ is a polynomial of degree $\ell - 2$ in the argument xy/r, where x and r are to be considered fixed and y as variable. Hence, in general we can write

$$T_\ell''\left(\frac{xy}{r}\right) = \sum_{m=0}^{\ell-2} C_m \left(\frac{x}{r}\right)^m y^m \; . \tag{6.4}$$

The constants C_m can be determined explicitly, but need not be for our current purposes. Upon substitution of (6.4) into (6.3) we see that we must evaluate integrals of the form

$$\int_{-1}^{1} y^m \, T_\ell(y) \left(1 - y^2\right)^{-\frac{1}{2}} dy \quad , \quad (m = 0,\ 1,\ 2, \cdots, \ell - 2) \; .$$

But this integral vanishes for all $m < \ell$, since over the interval $[-1,+1]$ $T_\ell(y)$ is orthogonal to any polynomial of degree less than ℓ. This is easy to verify by expanding y^m in a series of the Tchebycheff polynomials of the first kind and making use of the orthogonality relations (A-11).

Consequently, we have the result that L vanishes. Thus, K vanishes and $F_\ell(r)$ is given by the deceptively simple result

$$F_\ell(r) = \frac{-1}{\pi r} \int_r^1 f_\ell'(t) \, T_\ell\left(\frac{t}{r}\right) \left[\left(\frac{t}{r}\right)^2 - 1\right]^{-\frac{1}{2}} dt \; , \quad 0 < r \leq 1 \quad . \tag{6.5}$$

With the exception of an unfortunate extra factor of r in the numerator this result was obtained by Cormack [18] by a different line of reasoning.

7. THE $r = 0$ CASE

In deriving (6.5) it was necessary to exclude $r = 0$ and work over the interval $0 < r \leq 1$. Here, we consider the $r = 0$ case and work first with $\ell = 0$ and then with $\ell \neq 0$.

If $\ell = 0$, we obtain Abel transform pairs [19]

$$F_0(r) = -\frac{1}{\pi} \int_r^1 f_0'(t)\ (t^2 - r^2)^{-\frac{1}{2}}\, dt \tag{7.1a}$$

$$f_0(t) = 2 \int_t^1 r\, F_0(r)\, (r^2 - t^2)^{-\frac{1}{2}}\, dr \quad . \tag{7.1b}$$

In this case $F_0(r)$ may be evaluated at $r = 0$ with no difficulty.

On the other hand, for $\ell \neq 0$ the symmetry condition $F_\ell(-r) = (-1)^\ell F_\ell(r)$ immediately yields

$$F_\ell(0) = 0 \quad , \quad (\ell = 1,\ 3,\ 5,\ \cdots) \quad . \tag{7.2}$$

This leaves only the even ℓ values $\ell = 2, 4, 6, \cdots$. We may proceed (i) by direct evaluation of (2.3) or (ii) by examining the limiting case of (6.5). Either way the result is the same,

$$F_\ell(r) \to 0 \quad \text{as} \quad r \to 0 \quad , \quad (\ell = 2,\ 4,\ 6,\ \cdots) \ . \tag{7.3}$$

Method (i) is fairly straightforward but in method (ii) one must make use of arguments similar to those in Sec. 6. Such a result is perhaps most easily seen to follow from (6.1) since for arbitrarily small r

$$\frac{1}{r}\, U_{\ell-1}\left(\frac{t}{r}\right) \to (t^2 - r^2)^{-\frac{1}{2}}\, T_\ell\left(\frac{t}{r}\right) \to r^{-\ell}\, t^{\ell-1}$$

and the two integrals simply add to zero.

8. TRANSFORM METHOD

Given (4.1) it is possible to obtain the inversion formula (6.5) by use of the Mellin transform. By using Sneddon's [14] approach we write

$$\mathsf{f}_\ell(p) = f_\ell(p)\, H(1-p) ,$$

$$\mathsf{F}_\ell(r) = 2r\, F_\ell(r)\, H(1-r) ,$$

$$\mathsf{K}(\tfrac{p}{r}) = [1-(\tfrac{p}{r})^2]^{-\frac{1}{2}}\, T_\ell(\tfrac{p}{r})\, H(1-\tfrac{p}{r}) ,$$

where $H(x)$ is the Heaviside step function, zero if $x < 0$ and unity if $x > 0$. With these definitions (4.1) assumes the standard form

$$\mathsf{f}_\ell(p) = \int_0^\infty \mathsf{K}(\tfrac{p}{r})\, \mathsf{F}_\ell(r)\, \frac{dr}{r} . \tag{8.1}$$

The Mellin transform of both sides of (8.1) yields

$$\mathsf{f}^*_\ell(s) = \mathsf{K}^*(s)\, \mathsf{F}^*_\ell(s) , \tag{8.2}$$

where the star indicates the Mellin transform,

$$\mathsf{G}^*(s) = \int_0^\infty \mathsf{G}(x)\, x^{s-1}\, dx .$$

To cast (8.2) into the appropriate form for solution we rewrite it as

$$\mathsf{F}^*_\ell(s-1) = \frac{1}{(s-1)\,\mathsf{K}^*(s-1)} \cdot (s-1)\, \mathsf{f}^*_\ell(s-1)$$

$$= \frac{1}{(s-1)\,\mathsf{K}^*(s-1)} \cdot \frac{\Gamma(s)\, \mathsf{f}^*_\ell(s-1)}{\Gamma(s-1)} ,$$

where $\Gamma(s)$ is the Gamma function. In this form it is clear that Sneddon's general Mellin inversion formula applies, page 279 in [14]. The result is

$$2r\,F_\ell(r)\;H(1-r) = -\frac{2r}{\pi}\int_0^\infty [1-(\frac{r}{p})^2]^{-\frac{1}{2}}\;T_\ell(\frac{p}{r})\;H(1-\frac{r}{p})\frac{d}{dp}[f_\ell(p)\;H(1-p)]\frac{dp}{p}\;,$$

and upon simplification,

$$F_\ell(r) = -\frac{1}{\pi}\int_r^1 (p^2-r^2)^{-\frac{1}{2}}\;f'_\ell(p)\;T_\ell(\frac{p}{r})\;dp\;,$$

which is the same as (6.5).

9. SUMMARY AND CONCLUSIONS

We have demonstrated several techniques and methods for inversion of the Radon transform of a function $F \in \mathbb{D}$ defined on a plane, and have obtained the Tchebycheff transform pair given by (4.1) and (6.5). This lays the foundation for work involving the numerical inversion of the Radon transform with (6.5) used as the starting point rather than (2.3). This is not a trivial problem for although the inversion formula (6.5) is exact it is not in a good form for certain numerical calculations since there are large cancellations which may occur in the integrand for some values of r. Finally, the results here form the basis for generalizations to a study of the Radon transform of a function of n variables where n is even. New results in this area making use of Gegenbauer functions of the second kind will appear soon [22].

APPENDIX

In this appendix we collect some formulas involving Tchebycheff polynomials T_ℓ and U_ℓ of the first and second kinds, respectively. None of these results are new and can be obtained from standard sources [15,20,21]. They are included here for convenience.

For any nonnegative integer ℓ, the Tchebycheff polynomials of the first kind are given by

$$T_\ell(x) = \cos(\ell \operatorname{arc\,cos} x)\ , \qquad 0 < x < 1\ , \tag{A-1}$$

$$T_\ell(x) = \cosh(\ell \operatorname{arc\,cosh} x)\ , \quad 1 < x < \infty\ , \tag{A-2}$$

or

$$T_\ell(x) = \tfrac{1}{2}[(x+\sqrt{x^2-1})^\ell + (x-\sqrt{x^2-1})^\ell]\ , \quad 0 < x < \infty\ , \tag{A-3}$$

and

$$T_\ell(-x) = (-1)^\ell\, T_\ell(x)\ , \quad T_\ell(1) = 1\ , \quad T_\ell(0) = \cos \tfrac{1}{2}\ell\pi\ . \tag{A-4}$$

The Tchebycheff polynomials of the second kind are given by $(\ell \geq 1)$

$$U_{\ell-1}(x) = \frac{1}{\ell}\, T_\ell'(x) = \frac{\sin(\ell \operatorname{arc\,cos} x)}{(1-x^2)^{\frac{1}{2}}}\ , \quad 0 < x < 1\ , \tag{A-5}$$

$$U_{\ell-1}(x) = \frac{\sinh(\ell \operatorname{arc\,cosh} x)}{(x^2-1)^{\frac{1}{2}}}\ , \quad 1 < x < \infty \tag{A-6}$$

or

$$U_{\ell-1}(x) = \frac{(x+\sqrt{x^2-1})^\ell - (x-\sqrt{x^2-1})^\ell}{2(x^2-1)^{\frac{1}{2}}}\ , \quad 0 < x < \infty\ , \quad x \neq 1\ , \tag{A-7}$$

and

$$U_\ell(-x) = (-1)^\ell\, U_\ell(x)\ , \quad U_\ell(1) = \ell+1\ , \quad U_\ell(0) = \cos \tfrac{1}{2}\ell\pi\ . \tag{A-8}$$

From the above expressions we observe that

$$U'_{\ell-1}(x) = \frac{1}{\ell} T''_{\ell}(x) , \tag{A-9}$$

and for $x \neq 1$

$$\frac{T_{\ell}(x)}{(x^2-1)^{\frac{1}{2}}} - U_{\ell-1}(x) = \frac{(x - \sqrt{x^2-1})^{\ell}}{(x^2-1)^{\frac{1}{2}}} = \frac{(x + \sqrt{x^2-1})^{-\ell}}{(x^2-1)^{\frac{1}{2}}} . \tag{A-10}$$

The orthogonality relations hold over the interval $[-1,+1]$,

$$\int_{-1}^{1} T_{\ell}(x)\; T_m(x)\; (1-x^2)^{-\frac{1}{2}}\, dx = \begin{cases} 0 , & \ell \neq m \\ \frac{1}{2}\pi , & \ell = m \neq 0 \\ \pi , & \ell = m = 0 \end{cases} \tag{A-11}$$

$$\int_{-1}^{1} U_{\ell}(x)\; U_m(x)\; (1-x^2)^{\frac{1}{2}}\, dx = \frac{1}{2}\pi\, \delta_{\ell,m} . \tag{A-12}$$

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