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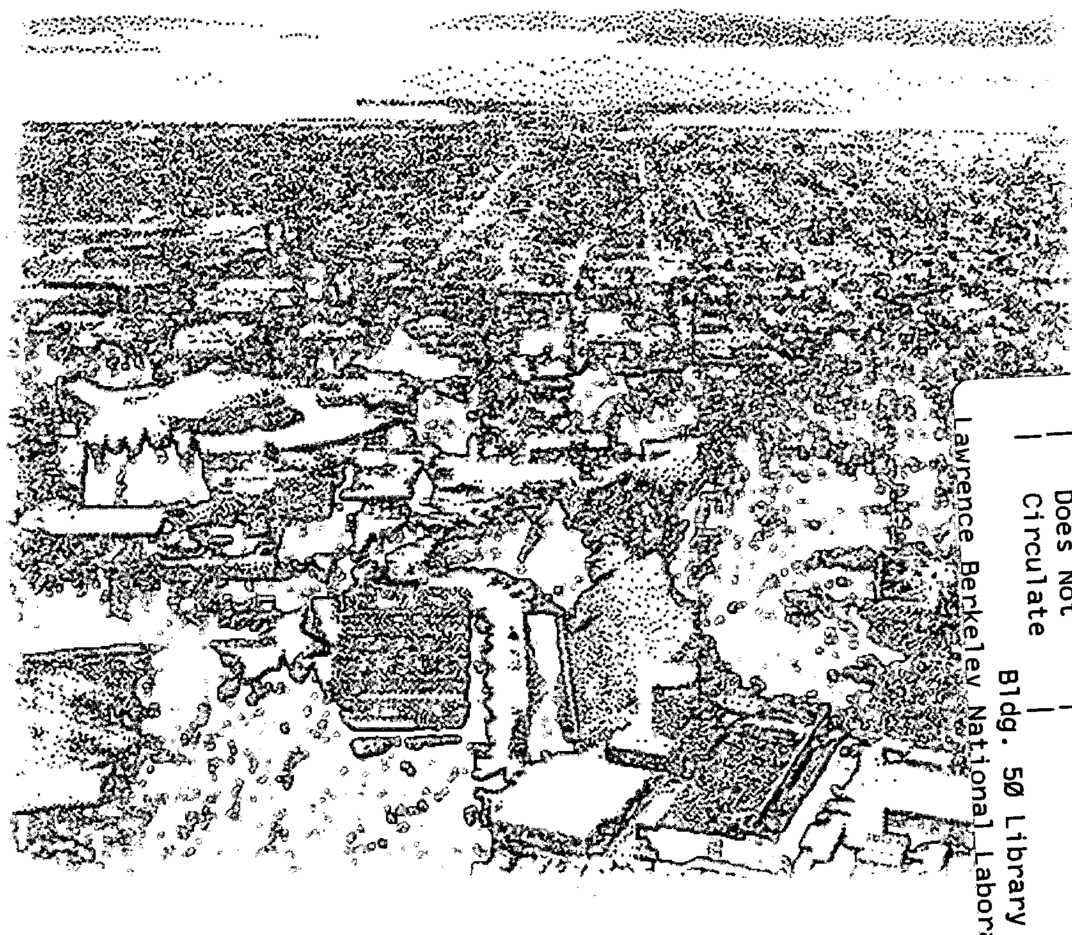

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Mirror Symmetry in Three-Dimensional Gauge Theories, Quivers and D-branes

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Abstract

We construct and analyze dual $N = 4$ supersymmetric gauge theories in three dimensions with unitary and symplectic gauge groups. The gauge groups and the field content of the theories are encoded in quiver diagrams. The duality exchanges the Coulomb and Higgs branches and the Fayet-Iliopoulos and mass parameters. We analyze the classical and the quantum moduli spaces of the theories and construct an explicit mirror map between the mass parameters and the the Fayet-Iliopoulos parameters of the dual. The results generalize the relation between ALE spaces and moduli spaces of $SU(n)$ and $SO(2n)$ instantons. We interpret some of these results from the string theory viewpoint, for $SU(n)$ by analyzing T-duality and extremal transitions in type II string compactifications, for $SO(2n)$ by using D-branes as probes. Finally, we make a proposal for the moduli space of vacua of these theories in the absence of matter.

1 Introduction

$N = 4$ supersymmetric gauge theories in three dimensions have been studied recently from string theory as well as field theory viewpoints [1–4]. In these theories both the Coulomb and the Higgs branches are hyperkähler manifolds. In [3] a duality between $N = 4$ supersymmetric gauge theories in three dimensions has been proposed under which the Higgs and Coulomb branches and the Fayet-Iliopoulos (FI) and mass terms are exchanged. The dual gauge theories have an ALE space as Higgs branch, and were based on Kronheimer's construction [5] of ALE spaces as an hyperkähler quotient.

In this paper we generalize the duality (mirror) proposal to other $N = 4$ supersymmetric gauge theories in three dimensions. A gauge theory and its conjectured dual will be called A-model and B-model respectively. The gauge groups and field content of the theories are encoded in quiver diagrams that correspond to Kronheimer-Nakajima's hyperkähler quotient construction of quiver varieties [6, 7], which will then automatically be the Higgs branch of the associated gauge theory. Specifically, we propose and study the duality between the following families of $N = 4$ supersymmetric gauge theories:

(1) The A-model has $U(k)$ gauge group, n hypermultiplets in the fundamental representation of the gauge group and one hypermultiplet in the adjoint representation. Its dual B-model has $U(k)^n$ gauge group and matter content specified by a quiver diagram corresponding to the Hilbert scheme of k points on an ALE space of A_{n-1} type [1]. By the Hilbert scheme of k points on a complex surface X we mean a smooth resolution of the k-symmetric product of X, $Sym^k X$. Concretely, there will be one hypermultiplet in the fundamental representation of one of the $U(k)$'s, and n hypermultiplets charged under a pair of $U(k)$'s.

(2) The A-model has $Sp(k)$ gauge group, n hypermultiplets in the fundamental representation of the gauge group and one hypermultiplet in the antisymmetric representation. Its dual B-model has $U(k)^4 U(2k)^{n-3}$ gauge group and matter content specified by a quiver diagram corresponding to the Hilbert scheme of k points on ALE space of D_n type.

(3) The A and B models have $U(k)^n$ and $U(k)^m$ gauge groups respectively, and matter content specified by quiver diagrams corresponding to the hyperkähler quotient construction of certain moduli spaces of instantons on vector bundles over an ALE space of A_{n-1} type. This is a generalization of (1).

The paper is organized as follows: Section 2 is a brief introduction to $N = 4$ su-

[1]The Hilbert schemes of k points on complex surfaces have recently appeared as the moduli spaces of D-branes [8, 9]

persymmetric gauge theories in three dimensions. In section 3 we define the dual gauge theories associated with quiver diagrams. We present the proposed dualities, the Higgs and Coulomb branches of the theories and the mirror map between the mass and FI parameters. In section 4 we study the first proposed family of dualities for $U(k)$ gauge theories. We start by providing the first evidence to this duality proposal by counting the dimensions of the Higgs and Coulomb branches as well as the number of mass and FI terms. We then study how the quantum corrections to the metric on the Coulomb branch fit into the mirror picture. We compute the one-loop corrections to the hyperkähler metric on the Coulomb branch of the A-model and compare to the exact metric on the Higgs branch of the B-model. The comparison yields strong support for the mirror map between the mass terms of the A-model and the FI terms of the B-model. In section 5 we analyze the structure of the Coulomb, Higgs and mixed branches for various mass and FI parameters. We observe a complete agreement of their dimensions which provide further evidence for the duality. In particular, we complete the proof of the mirror map by fixing the ambiguities left after the one-loop computation. We show how the proposed duality completely determines the quantum moduli space of vacua. In section 6 we examine type II string compactifications that in the field theory limit yield the A-model. The gauge symmetry and matter fields arise by wrapping D-branes around vanishing cycles and we use T-duality and extremal transitions to explain the gauge theory duality from a stringy viewpoint. In section 7 we study the second proposed family of dualities for $Sp(k)$ gauge theories. We provide the counting evidence for this duality proposal, study the quantum corrections, derive the mirror map and use D-brane probes and the Type I - M-theory duality to further support the gauge theory picture. In section 8 we study the third proposed family of dualities for $U(k)^n$ gauge theories. We provide the counting evidence for this duality proposal, study the Higgs, Coulomb and mixed branches of the dual theories, and give the mirror map. In section 9 we discuss the case of $U(k)$, $SU(k)$ and $Sp(k)$ gauge theories without matter, present a proposal for their moduli spaces, and conclude with open problems.

2 $N = 4$ supersymmetric gauge theories in three dimensions

We begin with a brief review of $N = 4$ supersymmetric gauge theories in three dimensions.

$N = 4$ supersymmetric gauge theories in three dimensions can be constructed by dimensional reduction of $N = 1$ supersymmetric gauge theories in six dimensions. The R-symmetry group is $SU(2)_L \times SU(2)_R$ with $SU(2)_L$ being the double cover of rotations in the three reduced coordinates and $SU(2)_R$ is the R symmetry group in six dimensions. The masses and FI parameters transform under $SU(2)_L \times SU(2)_R$ as $(\mathbf{3},\mathbf{1})$ and $(\mathbf{1},\mathbf{3})$ respectively. The mass terms deform the metric on the Coulomb branch and lift some of the Higgs branch, while the FI terms deform the metric on the Higgs branch and lift some of the Coulomb branch. The Higgs branch is constructed as a hyperkähler quotient with an $SU(2)_R$ action, and unlike the Coulomb branch is not modified by quantum corrections.

The $N=4$ vector multiplet in three dimensions contains three scalars $\phi^\alpha, \alpha = 1,2,3$ which transform as $(\mathbf{3},\mathbf{1})$ under the R-symmetry group. Their potential energy is

$$V = \frac{1}{e^2} \sum_{\alpha<\beta} Tr[\phi^\alpha, \phi^\beta]^2 \ , \tag{2.1}$$

where e is the gauge coupling. The potential energy vanishes if the ϕ^α commute and thus they take values in a common Cartan sub-algebra of the gauge group. For a generic vev in this Cartan subalgebra, the gauge group of rank r is broken to $U(1)^r$. Thus, in addition to the $3r$ scalars we have r massless photons which are dual to r scalars in three dimensions. The Coulomb branch is parametrized by the vevs of the $3r$ scalars and the r scalars dual to the photons and thus is of dimension $4r$. Due to the $N=4$ supersymmetry it is a hyperkähler manifold with an $SU(2)_L$ action. Its metric is corrected by loop and monopole corrections. The monopoles are instantons in three dimensions and they provide exponential corrections to the metric on the Coulomb branch.

The duality between $N = 4$ supersymmetric gauge theories in three dimensions exchanges the Higgs and Coulomb branches, the Fayet-Iliopoulos (FI) parameters and masses and the R-symmetry groups $SU(2)_L$ and $SU(2)_R$. The fact that the Higgs branch is not modified by quantum corrections while the Coulomb branch is, implies that like in mirror symmetry in string theory quantum corrections in one model are seen at the classical level of the dual and vice versa. Note that in general the duality between the A-models and B-models becomes exact only when the bare coupling constant e^2 is sent to infinity.

3 Mirror symmetric gauge theories and quivers

In this section we define the gauge theories associated with quiver diagrams. We present the proposed dualities, the Higgs and Coulomb branches of the theories and the mirror map between the mass and FI parameters. An object that will appear frequently in the discussion is the Hilbert scheme of k points on a complex surface X, $Hilb^{[k]}X$. As we

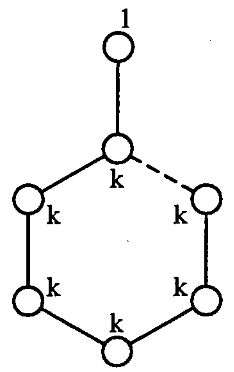

Figure 1: Quiver diagram for the B-model of $U(k)$ gauge theory

noted previously, this is a resolution of the the quotient singularities[1] of the k-symmetric product of X, $Sym^k X$. In the A model the Coulomb branch will be described by a Hilbert scheme and the parameter for the resolution of the quotient singularities will be found to be the adjoint hypermultiplet mass $\vec{m}_{adj}$ for $U(k)$ gauge theories and the mass of the antisymmetric hypermultiplet $\vec{m}_{as}$ for the $Sp(k)$ gauge theories. The parameters for the resolution of the singularities of the complex surface X will be shown to correspond to the masses of the fundamental hypermultiplets $\vec{m}_{fund}$ in both cases. In the B-model the Higgs branch will be described by a Hilbert scheme and the parameters for the resolution of all the singularities will be explicitly constructed from the FI parameters.

3.1 $U(k)$ Gauge Groups

The A-model has a $U(k)$ gauge group, n hypermultiplets in the fundamental representation and one hypermultiplet in the adjoint representation. This is precisely the field content needed for the hyperkähler quotient construction of the moduli space of $SU(n)$ $k-$instantons $\overline{\mathcal{M}}_k(SU(n))$ [10],[2] which is indeed the Higgs branch of the A-model.

The B-model is associated with the quiver diagram in figure 1.

We attach an index k_i at each node i. There are n nodes in the diagram with $k_i = k$ and one node with $k_i = 1$. The gauge group and the field content of the theory are encoded in the diagram in the following way: We associate to each node i with $k_i = k$

[1]We use the terminology quotient singularity to denote the singularities that arise in a symmetric product due to the action of the symmetric group.

[2]By $\overline{\mathcal{M}}_k(SU(n))$ we denote an enlarged moduli space which includes the small instantons. For more technical details see section 5.

a gauge group $U(k)_i$, to each link ${}_i\circ\!\!-\!\!-\!\!\circ_j$ with $k_i = k_j = k$ a hypermultiplet in the representation $(\mathbf{k}, \mathbf{k}^*)$ of $U(k)_i \times U(k)_j$, and to the link attached to the node with index 1 a hypermultiplet in the fundamental representation of the $U(k)$ gauge group associated with the other node of the link. This is the field content needed for the hyperkähler quotient construction of the Hilbert scheme of k points on ALE space of type A_{n-1}, which we will denote by $X_{A_{n-1}}$ [6, 7], and which is the Higgs branch of the B-model. The duality between the moduli spaces can be roughly summarized by the following table:

Model	$\mathcal{M}_V$	$\mathcal{M}_{\mathcal{H}}$
A	$Hilb^{[k]}X_{A_{n-1}}$	$\overline{\mathcal{M}}_k(SU(n))$
B	$\overline{\mathcal{M}}_k(SU(n))$	$Hilb^{[k]}X_{A_{n-1}}$

Table 1: The Coulomb and Higgs branches of A and B models

The precise structure is more detailed and depends on the mass and FI parameters. Consider the A-model: Without mass terms, the vector multiplet moduli space is the k-symmetric product $Sym^k X_{A_{n-1}}$ of the ALE space. It has singularities inherited from the simple singularity of A_{n-1} type of $X_{A_{n-1}}$, and also singularities coming from modding out by the action of the symmetric group. The masses for the fundamental hypermultiplets resolve the simple singularity of $X_{A_{n-1}}$. We denote the resolved ALE space as $\widetilde{X}_{A_{n-1}}$. The mass of the adjoint hypermultiplets resolves the quotient singularities of the symmetric product. In the following table, we show how the vector multiplet moduli space depends on the mass parameters[3].

Masses	$\mathcal{M}_V$
$\vec{m}_{fund} = 0,\ \vec{m}_{adj} = 0$	$Sym^k X_{A_{n-1}}$
$\vec{m}_{fund} \neq 0,\ \vec{m}_{adj} = 0$	$Sym^k \widetilde{X}_{A_{n-1}}$
$\vec{m}_{fund} = 0,\ \vec{m}_{adj} \neq 0$	$Hilb^{[k]}X_{A_{n-1}}$
$\vec{m}_{fund} \neq 0,\ \vec{m}_{adj} \neq 0$	$Hilb^{[k]}\widetilde{X}_{A_{n-1}}$

[3]In fact, there are two independent mass parameters $\vec{m}_{U(1)}$ and $\vec{m}_{SU(k)}$ for the adjoint hypermultiplet, associated to its trace and traceless part respectively. The metric on the A-model moduli space does not depend on $\vec{m}_{U(1)}$. Its only effect is to lift a trivial direction in the Higgs branch, corresponding to the center of mass of the instantons. Consequently, we do not count $\vec{m}_{U(1)}$ as an independent parameter, and there is no corresponding FI parameter in the B-model. Our duality applies to the cases where $\vec{m}_{SU(k)}$ and $\vec{m}_{U(1)}$ are either both vanishing or both non-vanishing.

Table 2: Mass parameters versus the vector multiplet moduli space (A-model)

The other effect of the mass terms is to lift some of the flat directions of the hypermultiplet moduli space. In section 5 we will analyze how this lifting is compatible with the resolution of the singularity.

In the B-model, the resolution of the singularity of the hypermultiplet moduli space and the lifting of some of the flat directions for the vector multiplets are caused by turning on FI terms. The way in which the moduli spaces are resolved or lifted matches exactly with the A-model when the vector multiplets and hypermultiplets are exchanged, provided that the FI parameters are related to the mass parameters of the A-model.

The mirror map between the mass parameters of the A-model and the FI parameters of the B-model takes the form

$$\vec{m}_i = \sum_{l=0}^{i} \vec{\zeta}_l, \qquad \vec{m}_{adj} = \sum_{l=0}^{n-1} \vec{\zeta}_l \ , \tag{3.1}$$

where $\vec{m}_i$ are the masses of the fundamental hypermultiplets, $\vec{m}_{adj}$ is the mass of the adjoint hypermultiplets and $\vec{\zeta}_l$ are the FI parameters. Note that a linear combination of masses can be eliminated for every $U(1)$ factor in the gauge group by shifting the origin of the Coulomb branch. In (3.1) we used this freedom to choose $\vec{m}_{n-1} = \vec{m}_{adj}$.

The first evidence that we will provide for the duality between the A and B models will be the matching of the dimensions of the Higgs and Coulomb branches and the number of FI and mass terms. We will then analyze the one-loop corrections and derive the mirror map (3.1). A detailed analysis of the moduli spaces will provide further evidence for the duality, which will in particular completely determine the mirror map, fixing all remaining ambiguities. Finally we will show how the duality structure arises from a stringy D-brane picture.

3.2 $Sp(k)$ Gauge Groups

We define the A-model to have $Sp(k)$ as its gauge group. The matter content consists of n hypermultiplets in the fundamental representation of $Sp(k)$ and one hypermultiplet in the antisymmetric representation of $Sp(k)$. The Higgs branch of the A-model is the moduli space of $SO(2n)$ $k-$instantons $\overline{\mathcal{M}}_k(SO(2n))$[11].

The B-model is associated with the quiver diagram in figure 2.

As described in the previous section we associate to each node a gauge group corresponding to its index. Diagram 2 has four nodes with index k and $n-3$ nodes with

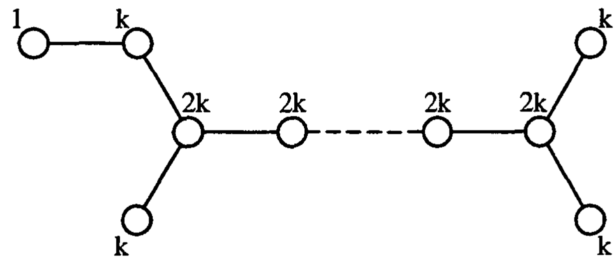

Figure 2: Quiver diagram for the B-model of $Sp(k)$ gauge theory

index $2k$, thus the gauge group of the B-model is $U(k)^4 U(2k)^{n-3}$. Again, the the matter content is $\oplus_{ij} a_{ij} (\mathbf{k_i}, \mathbf{k_j}^*)$ where a_{ij} is one if there is a link between the nodes i and j and zero otherwise. In addition, there is one fundamental hypermultiplet charged with respect to the $U(k)$ associated to the node that is connected to the exceptional one. The Higgs branch of the B-model is the Hilbert scheme of k points on an ALE space of D_n type [6].

The duality is roughly summarized in the following table:

Model	$\mathcal{M}_V$	$\mathcal{M}_{\mathcal{H}}$
A	$Hilb^{[k]} X_{D_n}$	$\overline{\mathcal{M}}_k(SO(2n))$
B	$\overline{\mathcal{M}}_k(SO(2n))$	$Hilb^{[k]} X_{D_n}$

Table 3: The Coulomb and Higgs branches of A and B models

As for the $U(k)$ case, the detailed structure depends on the mass and FI parameters. An illustrative table for the effect of the mass parameters[1] on the Coulomb branch is

[1]Here, $\vec{m}_{as}$ denotes the mass parameter for the hypermultiplet in the anti-symmetric representation. As in the $U(k)$ case, there are really two mass parameters for the anti-symmetric representation, one of which corresponds to the trivial representation, and the same statements made in the footnote for $U(k)$ apply here too.

Masses	$\mathcal{M}_V$
$\vec{m}_{fund} = 0$, $\vec{m}_{as} = 0$	$Sym^k X_{D_n}$
$\vec{m}_{fund} \neq 0$, $\vec{m}_{as} = 0$	$Sym^k \widetilde{X}_{D_n}$
$\vec{m}_{fund} = 0$, $\vec{m}_{as} \neq 0$	$Hilb^{[k]} X_{D_n}$
$\vec{m}_{fund} \neq 0$, $\vec{m}_{as} \neq 0$	$Hilb^{[k]} \widetilde{X}_{D_n}$

Table 4: Mass parameters versus the vector multiplet moduli space (A-model)

The structure of the moduli space of the B-model can be read of by exchanging the Higgs and Coulomb branches of the A-model. The masses of the the hypermultiplets in the fundamental representation $\vec{m}_i$ and the antisymmetric hypermultiplet mass $\vec{m}_{as}$ of the A-model are mapped under the duality to the FI parameters of the B-model

$$\begin{aligned} \vec{m}_i &= 2\sum_{l=1}^{i} \vec{\zeta}_l + \vec{\zeta}_{n-1} + \vec{\zeta}_n \qquad i < n, \qquad \vec{m}_n = \vec{\zeta}_n - \vec{\zeta}_{n-1} \ , \\ \vec{m}_{as} &= \vec{\zeta}_0 + \vec{\zeta}_1 + 2\sum_{l=2}^{n-2} \vec{\zeta}_l + \vec{\zeta}_{n-1} + \vec{\zeta}_n . \end{aligned} \tag{3.2}$$

Here, $\vec{\zeta}_0$ is associated to the node connected to the exceptional one, $\vec{\zeta}_1$ to the other leftmost node with index k, $\vec{\zeta}_l$ for $1 < l < n-1$ to the nodes with index $2k$ ordered from left to right, and $\vec{\zeta}_{n-1}$ and $\vec{\zeta}_n$ to the rightmost nodes with index k.

We will study this duality in section 7. We will provide the counting evidence, analyze the quantum corrections, derive the mirror map and support the duality by a D-brane picture based on the Type I - M-theory duality, and by the use of D-branes as probes.

3.3 $U(k)^n$ Gauge Groups

The gauge field and matter content of the A and B models are encoded in the quiver diagram in figure 3.

The A-model gauge group is $U(k)^n$, one $U(k)$ for each node of the extended Dynkin diagram. Notice that there is no gauge symmetry associated to the outside nodes with labels v_i. There are two kinds of matter. As before, for each pair of gauge groups whose nodes are connected by an edge there will be matter transforming as $(\mathbf{k}, \mathbf{k}^*)$ under $U(k) \times U(k)$. In addition, there will be v_i matter fields transforming in the fundamental representation of the ith $U(k)$ gauge group. We will denote the A-model as $(U(k)^n; \{v_i\})$.

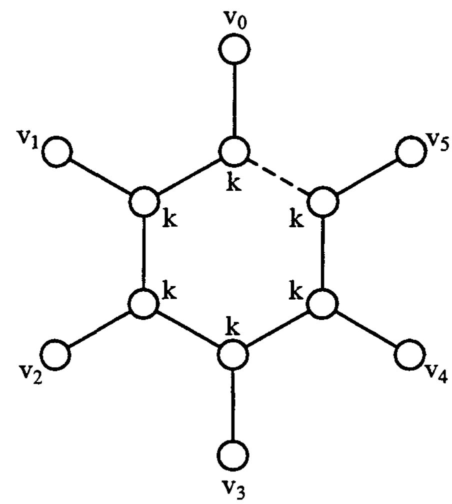

Figure 3: Quiver diagram for the A-model of $U(k)^n$ gauge theory

The Higgs branch of the A-model is the moduli space of instantons on a vector bundle V over an ALE space of type A_{n-1}. More precisely, it describes the moduli space $\mathcal{M}_k(V)$ of instantons of instanton number k on $V = \oplus\mathcal{R}_i^{\otimes v_i}$, with gauge group $U(V)$, where $\mathcal{R}_i$ are particular line bundles over the ALE space associated to the different representations of $\mathbf{Z}_n$ [6]. The B-model gauge theory is $(U(k)^m; \{w_i\})$, where

$$m = \sum_{i=0}^{n-1} v_i, \qquad n = \sum_{i=0}^{n-1} w_i \ . \tag{3.3}$$

The numbers v_i and w_i are related as follows: Consider a Young diagram whose rows have lengths $\sum_{i=0}^{p} v_i$, $p = 0, \ldots, n-1$. The lengths of the columns of this diagram can be parametrized as $\sum_{i=0}^{q} w_i$, $q = 0, \ldots m-1$, and the integers w_i are the ones appearing in the dual gauge theory. For example, $(U(k)^5, \{2,3,0,1,0\})$ is proposed as the dual of $(U(k)^6, \{2,2,0,0,1,0\})$. The $U(k)$ gauge theory we considered so far in this paper corresponds to a Young diagram which is a rectangle of size $n \times 1$.

The duality is summarized in the following table :

Model	$\mathcal{M}_V$	$\mathcal{M}_{\mathcal{H}}$
A	$\mathcal{M}_k(\oplus\mathcal{R}_i^{\otimes w_i})$	$\mathcal{M}_k(\oplus\mathcal{R}_i^{\otimes v_i})$
B	$\mathcal{M}_k(\oplus\mathcal{R}_i^{\otimes v_i})$	$\mathcal{M}_k(\oplus\mathcal{R}_i^{\otimes w_i})$

Table 5: The Coulomb and Higgs branches of A and B models

An important feature of this construction is that the dual of the dual theory is the original theory again, as one would expect, making duality a true involution in this set of theories. In section 8, we will provide counting evidence for the duality and analyse the structure of the moduli spaces, and give arguments for the following mirror map: Denote by $\vec{m}_i^{(B)}$, $\sum_{l=0}^{j-1} w_l \leq i < \sum_{l=0}^{j} w_l$ the masses of the hypermultiplets in the B-model charged only under the j^{th} $U(k)$. In addition, there are m masses of hypermultiplets charged under two $U(k)$'s. Using the freedom to shift the origin on the Coulomb branch, we can choose all these masses equal to same value which we denote by $\vec{m}_{2f}^{(B)}$. This leaves only the freedom to add a constant simultaneously to all $\vec{m}_i^{(B)}$, which we use to fix $\vec{m}_{n-1}^{(B)} = 0$. Then the relation between the FI parameters $\vec{\zeta}_i^{(A)}$ of the A-model and the masses of the B-model reads

$$\begin{aligned} \sum_{l=0}^{i} \vec{\zeta}_l^{(A)} &= \vec{m}_i^{(B)} + (\sum_{l=0}^{i} v_l)\vec{m}_{2f}^{(B)} \\ \sum_{l=0}^{n-1} \vec{\zeta}_l^{(A)} &= (\sum_{l=0}^{n-1} v_l)\vec{m}_{2f}^{(B)}. \end{aligned} \tag{3.4}$$

4 Duality for $U(k)$ Gauge Theories I: Quantum Corrections and Mirror Map

In this section we begin by providing the first preliminary counting evidence for the duality. We then turn to the computation of the one-loop corrections to the metric on the Coulomb branch of the $U(k)$ A model. We further compute the metric on the Higgs branch of the B model in the case where the sum of the Fayet-Iliopoulos terms vanishes. This corresponds in the A model to the case where the mass of the adjoint hypermultiplet vanishes. By comparing the two computations we derive the form of the mirror map between the fundamental hypermultiplets mass parameters of A model and the FI parameters of B model for $\vec{m}_{adj} = 0$. Finally we construct the mirror map with a non-vanishing adjoint mass.

4.1 Counting Evidence

As a first evidence for the duality between the A and B models we count in quaternionic units the dimensions of the Higgs and Coulomb branches and the number of independent FI and mass terms.

A-model: The dimension of the Coulomb branch is the rank of the gauge group $U(k)$

which is $d_V = k$. The Higgs branch is given by a hyperkähler quotient construction and accordingly, its dimension equals the dimension of the space of hypermultiplets minus the dimension of the gauge group. Therefore, $d_H = (nk + k^2) - k^2 = nk$. The number of FI terms is the number of $U(1)$ factors in the gauge group, $n_\zeta = 1$. In order to count the number of mass parameters note that a linear combination of masses can be eliminated for every $U(1)$ factor in the gauge group by shifting the origin of the Coulomb branch. Thus, in this case the number of mass parameters is $n_m = (n+1) - 1 = n$.

B-model: The dimension of the Coulomb branch is the rank of $U(k)^n$, thus $d_V = nk$. The dimension of the Higgs branch is the dimension of the space of hypermultiplets $(nk^2 + k)$ minus the dimension of the gauge group (nk^2), thus $d_H = k$. The number of FI terms is the number of $U(1)$ factors in $U(k)^n$ and therefore $n_\zeta = n$. The number of mass parameters is $n_m = (n+1) - n = 1$. The counting is summarized in the following table:

Model	d_V	d_H	n_ζ	n_m
A	k	nk	1	n
B	nk	k	n	1

Table 6: The dimension of the Coulomb and Higgs branches and the number of mass and FI parameters of A and B models

The counting shows that we indeed have a symmetry under A-model $\leftrightarrow$ B-model, $d_V \leftrightarrow d_H$ and $n_\zeta \leftrightarrow n_m$ which is a necessary condition for the duality to hold.

4.2 A model - One-loop Corrections

Consider the A model with gauge group $U(k)$, one hypermultiplet in the adjoint representation and n hypermultiplets in the fundamental representation. Let us parametrize the scalars that minimize the potential energy (2.1) by

$$\vec{\phi} = diag[\vec{r}_1, ..., \vec{r}_k] \ , \tag{4.1}$$

where $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$.

The one-loop corrected metric of the Coulomb branch of A model takes the form

$$ds^2 = g_{ab} d\vec{r}_a d\vec{r}_b + (g^{-1})_{ab}(d\sigma_a + \vec{\omega}_{ac} \cdot d\vec{r}_c)(d\sigma_b + \vec{\omega}_{bd} \cdot d\vec{r}_d) \ , \tag{4.2}$$

where

$$g_{aa} = \frac{1}{e^2} + \sum_{b\neq a}^{k}\left(\frac{-2}{|\vec{r}_a - \vec{r}_b|} + \frac{1}{|\vec{r}_a - \vec{r}_b + \vec{m}_{adj}|} + \frac{1}{|\vec{r}_a - \vec{r}_b - \vec{m}_{adj}|}\right) + \sum_{i=0}^{n-1}\frac{1}{|\vec{r}_a - \vec{m}_i|}$$

$$g_{ab} = \frac{2}{|\vec{r}_a - \vec{r}_b|} - \frac{1}{|\vec{r}_a - \vec{r}_b + \vec{m}_{adj}|} - \frac{1}{|\vec{r}_a - \vec{r}_b - \vec{m}_{adj}|} \qquad a \neq b\,, \tag{4.3}$$

where $a, b = 1...k$. The σ_a are variables dual to the photons that remain massless on the Coulomb branch. They are periodic with period 2π, and constant shifts of the σ_i are triholomorphic isometries of the hyperkähler metric (4.3). These isometries are unbroken in perturbation theory, and every hyperkähler metric of real dimension $4k$ with k commuting triholomorphic isometries can be written in the form (4.2), where g and ω satisfy [14–16]

$$\begin{aligned} \vec{\nabla}_a g_{bc} &= \vec{\nabla}_b g_{ac} \\ \frac{\partial}{\partial r_a^p}\omega_{bc}^q - \frac{\partial}{\partial r_b^q}\omega_{ac}^p &= \epsilon_{pqr}\frac{\partial}{\partial r_a^r} g_{bc}. \end{aligned} \tag{4.4}$$

This explains the form of the metric in (4.2), and can be used to express $\vec{\omega}_{ab}$ in terms of g_{ab}. Thus, in order to derive this form of the one-loop corrected metric we only need to look at the terms in the one-loop effective action coming from one-loop diagrams with two gauge fields on the external legs and the vector multiplet or hypermultiplet running in the loop. We then make use of the following limits:

(1) Reduction in color: Taking the limit $|\vec{r}_k| \to \infty$ is a reduction in the number of colors and we should recover the formula for the metric for the gauge group $U(k-1)$. This implies that the coefficients of the different terms are independent of the number of colors. Thus it is sufficient to consider the gauge group $U(2)$. The gauge group $U(1)$ is evidently not sufficient since the theory is free in the absence of matter.

(2) Reduction in flavor: Taking the limit $|\vec{m}_{n-1}| \to \infty$ is a reduction in the number of hypermultiplets in the fundamental representation of the gauge group, and we should recover the formula for the metric for $n-1$ flavors. This implies that the coefficients of the different terms are independent of the number of flavors.

(3) The first equation in (4.4) implies that the contributions of the vector multiplet and the adjoint hypermultiplet to the diagonal and off diagonal elements of the metric are of opposite sign and the same absolute value. It also implies that the hypermultiplets in the fundamental can contribute only to the diagonal terms of the metric. In order to see these it is sufficient to consider the $U(2)$ gauge group and use the equation for the metric $\partial_1 g_{21} = \partial_2 g_{11}$ implied by (4.4).

(4) Reduction of the gauge group to $U(1)$ and considering the case of n hypermultiplets

in the fundamental representation while taking the limit $|\vec{m}_{adj}| \to \infty$, should recover for massive hypermultiplets the Taub-NUT metric for a resolved A_{n-1} singularity [2]

$$ds^2 = g^{-2} d\vec{r} d\vec{r} + g^2 (d\sigma + \vec{\omega} \cdot d\vec{r})^2 \ , \tag{4.5}$$

where

$$g^{-2}(\vec{r}) = \frac{1}{e^2} + \sum_{i=0}^{n-1} \frac{1}{|\vec{r} - \vec{m}_i|} \ , \tag{4.6}$$

and

$$\vec{\nabla}(g^{-2}) = \vec{\nabla} \times \vec{\omega} \ . \tag{4.7}$$

This fixes the coefficient of the fundamental hypermultiplet contribution to the metric.
(5) Reduction of the gauge group to $SU(2)$ and considering the case of $n = 2$ hypermultiplets in the fundamental representation while taking the limit $|\vec{m}_{adj}| \to \infty$, should recover for massless hypermultiplets the classical metric since there are no quantum corrections in this case [2]. Using (4), this fixes the coefficient of the vector multiplet contribution to the metric. In order to see this explicitly consider the case of gauge group $U(2)$ with two massless hypermultiplets in the fundamental representation. For the metric g_{ab}, $a, b = 1, 2$ we take

$$\begin{aligned} g_{aa} &= \frac{1}{e^2} + \frac{\alpha}{|\vec{r}_a - \vec{r}_b|} + \frac{2}{|\vec{r}_a|} \\ g_{ab} &= \frac{-\alpha}{|\vec{r}_a - \vec{r}_b|} \qquad a \neq b \ . \end{aligned} \tag{4.8}$$

where α is the constant coefficient to be determined and the coefficient of the fundamental hypermultiplets has been determined in (4). Define

$$g_{ab} d\vec{r}_a d\vec{r}_b = g_{++} (d\vec{r}_+)^2 + 2 g_{+-} d\vec{r}_+ d\vec{r}_- + g_{--} (d\vec{r}_-)^2 \ , \tag{4.9}$$

where $\vec{r}_\pm = \frac{\vec{r}_1 \pm \vec{r}_2}{\sqrt{2}}$. g_{++} and g_{--} correspond to the $U(1)$ and $SU(2)$ parts of the metric respectively. Restricting to the $SU(2)$ part, $\vec{r}_1 = -\vec{r}_2$, and requiring that g_{--} does not get quantum corrections for two massless fundamentals we get the required result $\alpha = -2$.
(6) The coefficient of the adjoint hypermultiplet contribution is fixed by reading from the Lagrangian its relation to that of the fundamental hypermultiplets. Note that in the absence of hypermultiplets in the fundamental representation there are no one-loop corrections to the metric if there is no adjoint mass. This is consistent with the fact that in this case we have an $N = 8$ supersymmetry as a reduction of the $N = 4$ supersymmetry in four dimensions. In this case the complex structure of the hyperkähler manifold is, as expected [2], the same as that of the Jacobian corresponding to the $N = 4$ curve [12].

Consider the case with zero adjoint mass and n massive fundamentals, in the limit $e^2 \to \infty$. In this case the one-loop metric describes the k-symmetric product of resolved ALE spaces of A_{n-1} type $\widetilde{X}_{A_{n-1}}$ (the symmetric product arises because we still have to divide by the action of the Weyl group S_k of $U(k)$)

$$\mathcal{M}_V^{one-loop}(\mathrm{A-model}, \vec{m}_{adj} = 0, \vec{m}_{fund} \neq 0) = Sym^k \widetilde{X}_{A_{n-1}} \ , \tag{4.10}$$

where the masses of the hypermultiplets resolve the ALE singularities. We will argue in the next section that this result is in fact exact, namely

$$\mathcal{M}_V^{Exact}(\mathrm{A-model}, \vec{m}_{adj} = 0, \vec{m}_{fund} \neq 0) = Sym^k \widetilde{X}_{A_{n-1}} \ . \tag{4.11}$$

When the adjoint mass is nonzero, $\vec{m}_{adj} \neq 0$, the one-loop metric is not positive definite in the region $|\vec{r}_a - \vec{r}_b| \to 0$. We expect monopole corrections to contribute in this case, and that the metric will become positive definite upon including these corrections. A similar phenomenon happens in pure $SU(2)$ gauge theory with zero or one hypermultiplet in the fundamental representation [2], and also when considering monopole moduli spaces [17]. More specifically, in the region $|\vec{r}_a - \vec{r}_b| \ll |\vec{m}_{adj}|$ for some a, b, while keeping other pairs $\gg |\vec{m}_{adj}|$, the system can be well approximated by the $SU(2)$ gauge theory with one adjoint hypermultiplet with bare mass $\vec{m}_{adj}$. By a slight generalization of the results in [2] we see that there are no higher-loop corrections in this region, and we expect monopole corrections to restore the positivity of the metric. There is a close analogy between the quotient singularity $\vec{r}_a \leftrightarrow \vec{r}_b$ in our case and the $\mathbf{Z}_2$ singularity $\vec{r} \to -\vec{r}$ in the $SU(2)$ case, which is resolved by monopole corrections. Since we expect monopole corrections when $\vec{m}_{adj} \neq 0$, this suggests that the adjoint mass is a parameter for the resolution of the quotient singularities of the symmetric product. In the following sections we will provide further support to this picture.

4.3 B Model - Higgs Branch

In general, the Higgs branch of an $N = 4$ supersymmetric gauge theory in three dimensions is given by a hyperkähler quotient. Recall that a hyperkähler quotients are manifolds one constructs from a given hyperkähler manifold M with an action of group G that preserves the hyperkähler structure[14]. Associated to such a group action are three moment maps $\mu_i : M \to g^*$, one for each kähler form, where g^* is the dual of the Lie algebra g of G. The hyperkähler quotient is defined as the Riemannian quotient $\mu^{-1}(\vec{\zeta})/G$, where $\vec{\zeta}$ is a three vector with values in the center of g^*. In three-dimensional $N = 4$ gauge theories, one obtains a set of equations that determine the classical vacua by integrating out the auxiliary fields, and requiring the resulting potential to vanish. If we are interested in the Higgs branch we put the vevs of the scalars in the vector multiplet equal to zero, in the case of mixed branches we can take them equal to some other fixed value. In this case, we obtain a real equation from the D-terms in the lagrangian, and a complex equation from the F-terms. These together constitute the three equations $\vec{\mu}(x) = \vec{\zeta}$, that also appear in the hyperkähler quotient. The manifold M is spanned by the vector space of scalars in the hypermultiplets, which is hyperkähler in view of the $N = 4$ supersymmetry. The components of $\vec{\zeta}$ correspond to the Fayet-Iliopoulos parameters in the lagrangian. Finally, one has to divide by the action of the gauge group, to identify equivalent vacua, and one ends up with a Higgs branch which is precisely the hyperkähler quotient $\mu^{-1}(\vec{\zeta})/G$.

In the case at hand, the equations that govern the Higgs branch of the B-model are the same ones that appear in the hyperkähler quotient construction of the corresponding quiver variety [6] and read

$$\begin{aligned}
B_{01}B_{01}^\dagger - B_{10}^\dagger B_{10} + B_{0(n-1)}B_{0(n-1)}^\dagger - B_{(n-1)0}^\dagger B_{(n-1)0} + Q_0 Q_0^\dagger - \tilde{Q}_0^\dagger \tilde{Q}_0 &= 2\zeta_0^{\mathbf{R}}\mathbf{1} \\
B_{12}B_{12}^\dagger - B_{21}^\dagger B_{21} + B_{10}B_{10}^\dagger - B_{01}^\dagger B_{01} &= 2\zeta_1^{\mathbf{R}}\mathbf{1} \\
&\vdots \\
B_{(n-1)0}B_{(n-1)0}^\dagger - B_{0(n-1)}^\dagger B_{0(n-1)} + B_{(n-1)(n-2)}B_{(n-1)(n-2)}^\dagger & \\
-B_{(n-2)(n-1)}^\dagger B_{(n-2)(n-1)} &= 2\zeta_{n-1}^{\mathbf{R}}\mathbf{1}
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
B_{01}B_{10} - B_{0(n-1)}B_{(n-1)0} + Q_0\tilde{Q}_0 &= \xi_0^{\mathbf{C}}\mathbf{1} \\
B_{12}B_{21} - B_{10}B_{01} &= \zeta_1^{\mathbf{C}}\mathbf{1} \\
&\vdots \\
B_{(n-1)0}B_{0(n-1)} - B_{(n-1)(n-2)}B_{(n-2)(n-1)} &= \zeta_{n-1}^{\mathbf{C}}\mathbf{1}
\end{aligned} \tag{4.13}$$

where B_{ij} is a complex matrix of size $k \times k$, Q_0 and $\tilde{Q}_0$ are respectively a column and a row vector with k entries, and $\zeta_i^{\mathbf{R}}$ and $\zeta_i^{\mathbf{C}}$ are real and complex parameters which constitute the FI parameter associated to the ith diagonal $U(1) \subset U(k)$. The vector space V spanned by the components of B_{ij} and $Q_0, \tilde{Q}_0$ carries the standard metric

$$ds^2 = \sum_{i=0}^{n-1} \mathrm{Tr}(dB_{i(i+1)}dB_{i(i+1)}^\dagger) + dQ_0^\dagger dQ_0 + d\tilde{Q}_0 d\tilde{Q}_0^\dagger. \tag{4.14}$$

The gauge group $G = U(k)^n$ acts on V and on the space $\mathcal{M}'$ of solutions of (4.12) and (4.13), and the Higgs branch is the hyperkähler quotient of V with respect to G.

We will consider the case $\sum \zeta_i^{\mathbf{R}} = \sum \zeta_i^{\mathbf{C}} = 0$. In this case the hyperkähler quotient is the symmetric product of k ALE spaces of A_{n-1} type [6]. This implies that the manifold

$\mathcal{M}'$ is a submanifold of the set of G-orbits that intersect the vector space $V' \subset V$, where V' is constructed by taking all B_{ij} diagonal and $Q_0 = \tilde{Q}_0 = 0$. It is easy to see that $\mathcal{M}'/G$ is the same as $(\mathcal{M}' \cap V')/G'$, where G' is the subgroup of G that maps V' onto itself. The subgroup G' is given by the semidirect product of $U(1)^{k(n-1)}$ and the symmetric group S_k. The latter group acts by permuting the diagonals of all B_{ij} simultaneously. The equations (4.12) and (4.13) consist of k copies of the same set of equations, and are also permuted by S_k. Thus the Higgs branch is indeed given by the symmetric product of k copies of one and the same space. This space is determined by taking the B_{ij} in (4.12) and (4.13) to be equal to a complex number b_{ij}, and $Q_0 = \tilde{Q}_0 = 0$, and to divide by the group $U(1)^{n-1}$. The equations (4.12), (4.13) reduce to the hyperkähler quotient description of a single ALE space of type A_{n-1}, as given in [5], thus confirming that the Higgs branch is the symmetric product of k ALE spaces.

It remains to compute the metric on a single ALE space. For this it is convenient to replace each set of complex numbers $b_{i(i+1)}, b_{(i+1)i}$ by a three vector $\vec{r}_i$ and an angular variable σ_i, $0 \le \phi_i < 2\pi$ [13]. This change of variables is defined as follows: Given two complex numbers a, b, we can introduce the quaternion $q = a - bj$. Any quaternion can be written as $q = ce^{i\sigma}$, where c is a purely imaginary quaternion, $\bar{c} = -c$. The combination $qi\bar{q}$ does not depend on σ and is also purely imaginary, and we can define a vector $\vec{r}$ by [13]

$$\frac{1}{2}(qi\bar{q}) = r^x i + (r^y + ir^z)k \ . \tag{4.15}$$

The flat metric $ds^2 = da d\bar{a} + db d\bar{b}$ becomes in terms of σ and $\vec{r}$

$$ds^2 = \frac{1}{r} d\vec{r}^2 + r(d\sigma^2 + \vec{\omega} \cdot d\vec{r})^2 \tag{4.16}$$

where $\vec{\omega}$ has the form of a one-monopole gauge field and satisfies $\vec{\nabla} \times \vec{\omega} = \vec{\nabla}(\frac{1}{r})$, see (4.4).

The advantage of using variables $\vec{r}_i, \phi_i$ instead of b_{ij} is that they linearize the moment map equations (4.12) and (4.13), and that the metrics in these variables are similar to the ones we found from the one-loop computation (4.2). If we introduce a three-vector $\vec{\zeta}_i \equiv (\zeta_i^{\mathbf{R}}, \mathrm{Re}(\zeta_i^{\mathbf{C}}), \mathrm{Im}(\zeta_i^{\mathbf{C}}))$, then the moment map equations simply become

$$\vec{r}_i - \vec{r}_{i-1} = \vec{\zeta}_i \tag{4.17}$$

Thus, we can solve for all $\vec{r}_i$ in terms of $\vec{r}_0$,

$$\vec{r}_i = \vec{r}_0 + \sum_{l=1}^{i} \vec{\zeta}_l. \tag{4.18}$$

The general solution to (4.12), (4.13) is thus parametrized by $\vec{r}_0$ and the angular variables σ_i. The metric on the manifold of solutions is given by

$$ds^2 = \frac{1}{r_0} d\vec{r}_0^2 + r_0(d\sigma_0^2 + \vec{\omega}_0 \cdot d\vec{r}_0)^2 + \sum_{l=1}^{n-1} r_l d\sigma_l^2 \tag{4.19}$$

where the r_i are expressed in terms of $\vec{r}_0$ by means of (4.18). We next take the Riemannian quotient with respect to the group action of $U(1)^{n-1}$, which acts on the manifold of solutions by means of the vector fields $V_i = \frac{\partial}{\partial\sigma_i} - \frac{\partial}{\partial\sigma_{i+1}}$, $i = 0 \ldots n-2$. The $U(1)^{n-1}$ symmetry can be used to put $\sigma_1 = \ldots \sigma_{n-1} = 0$, leaving the four real coordinates $\vec{r}_0$ and σ_0. The vector field $\frac{\partial}{\partial\sigma_0}$ generates an isometry of (4.19) that commutes with the group action, and therefore also an isometry of the quotient. Any four dimensional hyperkähler manifold with a $U(1)$ isometry has a metric of the form [14–16]

$$ds^2 = g^{-2}(\vec{r}_0) d\vec{r}_0^2 + g^2(\vec{r}_0)(d\sigma_0 + \vec{\omega}(\vec{r}_0) \cdot d\vec{r}_0)^2 \tag{4.20}$$

where $\vec{\omega}$ is given in terms of g^{-2} by the equation $\vec{\nabla} \times \vec{\omega} = \vec{\nabla}(g^{-2}(\vec{r}_0))$, see (4.2) and (4.4). This means that we know the full metric once we know the inner product of the vector field $V = \frac{\partial}{\partial\sigma_0}$ with itself. This cannot be simply read off from (4.19), as we still have to take a quotient with respect to $U(1)^{n-1}$. If we denote by $(,)$ the metric (4.19) on the solution space and by $(,)_H$ the metric on the quotient, then à la Dirac the following relation holds

$$(V,V)_H = (V,V) - (V,V_i)(M^{-1})^{ij}(V_j,V) \tag{4.21}$$

where $M_{ij} = (V_i, V_j)$. The nonzero matrix elements of M are $M_{ii} = r_i + r_{i+1}$, $M_{ii+1} = -r_{i+1}$ and $M_{i+1i} = -r_{i+1}$. The determinant of M satisfies the recursion relation $M^{(n)} = (r_n + r_{n-1})M^{(n-1)} - r_{n-1}^2 M^{(n-2)}$ which is solved by $M^{(n-1)} = \prod_{i=0}^{n-1} r_i \left(\sum_{i=0}^{n-1} \frac{1}{r_i}\right)$. Using this result we obtain that the only non-vanishing matrix element of M^{-1} appearing in (4.21) is

$$(M^{-1})^{00} = \frac{\prod_{i=1}^{n-1} r_i (\sum_{i=1}^{n-1} \frac{1}{r_i})}{\prod_{i=0}^{n-1} r_i (\sum_{i=0}^{n-1} \frac{1}{r_i})} = \frac{1}{r_0} - \frac{1}{r_0^2 \sum_{i=0}^{n-1} \frac{1}{r_i}} \tag{4.22}$$

Putting everything together we obtain

$$g^2(\vec{r}_0) = r_0 - r_0^2 (M^{-1})^{00} = \frac{1}{\sum_{i=0}^{n-1} \frac{1}{r_i}} \tag{4.23}$$

and finally

$$g^{-2}(\vec{r}_0) = \sum_{i=0}^{n-1} \frac{1}{r_i}. \tag{4.24}$$

Using (4.18) and comparing with the one-loop result (4.3) with $m_{adj} = 0$ we have

$$g^{-2}(\vec{r}_0) = \sum_{i=0}^{n-1} \frac{1}{|\vec{r}_0 - \vec{m}_i|}. \tag{4.25}$$

where $\vec{m}_i = \sum_{l=1}^{i} \vec{\zeta}_l$. Up to a constant shift with $\vec{\zeta}_0$ this is precisely the mirror map (3.1). The fact that the one-loop metric on the Coulomb branch is positive definite and smooth for generic masses strongly suggests there are no monopole corrections to the metric on the Coulomb branch, and that the one-loop result is exact. In that case, both the exact Coulomb branch of the A model (in the infrared) and the exact Higgs branch of B model are given by a symmetric product of ALE spaces of type A_{n-1}, and the relation between the masses of the hypermultiplets in the fundamental representation in A model and the FI parameters in the B model is given by (3.1) with $\vec{m}_{adj} = 0$.

4.4 The Mirror Map

The above derivation of the mirror map was restricted to the case when the adjoint mass in the A-model and the sum of the FI terms in the B-model were set to zero. Consider now the case where the adjoint mass is different than zero. The mirror map for the adjoint mass can be generally written as

$$F(\vec{m}_{adj}, \vec{m}_{fund}) = \sum_{l=0}^{n-1} \vec{\zeta}_l \ . \tag{4.26}$$

If we assume that F is analytic at $\vec{m}_{adj} = \vec{m}_{fund} = 0$, then dimensional analysis, the requirement for the correct transformation under the global symmetry $SU(2)_L \times SU(2)_R$, and the requirement for a finite limit as $\vec{m}_{fund} \to 0$ force F to be linear. We also know that $F(0, \vec{m}_{fund}) = 0$, and this implies that F is proportional to $\vec{m}_{adj}$, in agreement with (3.1). In principle there is also a possibility that the mass of the adjoint will modify the mirror map for the fundamental hypermultiplets. This possibility will be excluded in the next section by a detailed study of the correspondence between the mass parameters of the A-model and the FI parameters of the B-model, and this will also fix the relative normalization of F with respect to the fundamental masses.

The fact that the relation between the mass and the FI parameters is linear is also expected by the following reasoning: The FI parameters $\vec{\zeta}_l$ of the B-model are given by the periods of the three covariantly constant two-forms $\vec{\omega}$ of the Higgs branch [18]

$$\vec{\zeta}_l = \int_{\Sigma_l} \vec{\omega} \ , \tag{4.27}$$

where $\{\Sigma_l\}$ is a basis for the second homology group of the Higgs branch. By duality it is the Coulomb branch of the A-model. It was argued in [2] [1] that the periods are linear

[1]The argument given in [2] was for the $SU(2)$ gauge group but it can be generalized at least to some of the higher rank groups such as $Sp(k)$. In fact our derivation of the mirror map shows that it is correct for $U(k)$ gauge groups too.

in the masses and thus we expect a linear relation between the mass parameters of the A-model and the FI parameters of the B-model and vice versa.

Finally, we note that there exists another viewpoint on the mirror map for the mass parameters of the hypermultiplets in the fundamental representation which will prove to be useful for other gauge groups. According to theorem 2.8 in [7], if $\sum \vec{\zeta}_l = 0$, the Higgs branch of the B-model develops a singularity if $\vec{\zeta}_k + \ldots + \vec{\zeta}_l = 0$, for $1 \leq k \leq l \leq n-1$, corresponding to the positive roots of A_{n-1} (The general case is given in (5.43).) On the other hand, by inspection of the one-loop metric (4.2) with $\vec{m}_{adj} = 0$ we see that we expect a singularity whenever $\vec{m}_i - \vec{m}_j = 0$. In order for these singularities to be in one-to-one correspondence with the singularities in the Higgs branch of the B-model, we need (up to an overall factor) the relations

$$\vec{m}_i - \vec{m}_{i-1} = \vec{\zeta}_i, \qquad i = 1, ..., n-1. \tag{4.28}$$

Equation (4.28) is equivalent to the mirror map (3.1) with $\vec{m}_{adj} = \sum \vec{\zeta} = 0$.

5 Duality for $U(k)$ Gauge Theories II: Structure of The Moduli Space of Vacua

In this section, we analyze the moduli spaces of vacua for various choices of mass and Fayet-Iliopoulos terms. In general, if mass terms are turned on, some of the Higgs branches are reduced. Conversely, some of the Coulomb branches are reduced by Fayet-Iliopoulos terms which, by turning on Higgs vevs, break part of the gauge symmetry. Here, we consider the case where we turn on masses of the A model and Fayet-Iliopoulos terms of the B model. We will observe a complete agreement between the dimensions of various Higgs branches of the A model and various Coulomb branches of the B model, provided that the masses and Fayet-Iliopoulos terms are related via the mirror map (3.1). This result provides strong evidence for the proposed duality and excludes possible corrections to the mirror map. Use of the proposed duality, in turn, makes it possible to determine how various branches touch each other.

5.1 Classical Moduli Space of Vacua of The A Model

In this subsection, we classify moduli spaces of hypermultiplet using classical arguments. Although there are possible quantum corrections to the way they intersect the moduli space of vector multiplet, the metric on them will not be corrected. Also, the structure of mixed branches will get corrected in the direction of vector multiplet but their dimensions will not, and we will count them.

The moduli space of hypermultiplet with its metric is obtained by a hyperkähler quotient based on classical data. Let $A_1 = ({A_1}^a_{\ b})$, $A_2 = ({A_2}^a_{\ b})$; $1 \leq a, b \leq k$ be a hypermultiplet in the adjoint representation of $U(k)$ and $Q = (Q^a_i)$, $\tilde{Q} = (\tilde{Q}^j_b)$; $1 \leq a, b \leq k$, $0 \leq i, j \leq n-1$ be n-hypermultiplets in the fundamental representation of $U(k)$. (Q and $\tilde{Q}$ transform under $U(k) \times SU(n)$ as $(\mathbf{k}, \mathbf{n}^*)$ and $(\mathbf{k}^*, \mathbf{n})$ respectively.) The classical equations determining the vacua are

$$[A_1, A_1^\dagger] + [A_2, A_2^\dagger] + QQ^\dagger - \tilde{Q}^\dagger \tilde{Q} = 0, \tag{5.1}$$

$$[A_1, A_2] + Q\tilde{Q} = 0, \tag{5.2}$$

$$\vec{\phi} Q - Q\vec{m} = 0, \qquad \tilde{Q}\vec{\phi} - \vec{m}\tilde{Q} = 0, \tag{5.3}$$

$$[\vec{\phi}, A_1] - \vec{m}_{adj} A_1 = 0, \quad [\vec{\phi}, A_2] + \vec{m}_{adj} A_2 = 0 \tag{5.4}$$

In the above expressions, $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$ denotes the scalers of the $U(k)$ vector multiplet and $\vec{m} = (m^1, m^2, m^3)$ is the mass matrix. By N=4 supersymmetry, they can be diagonalized

$$\vec{\phi} = \begin{pmatrix} \vec{r}_1 & & \\ & \ddots & \\ & & \vec{r}_k \end{pmatrix}, \qquad \vec{m} = \begin{pmatrix} \vec{m}_0 & & \\ & \ddots & \\ & & \vec{m}_{n-1} \end{pmatrix}. \tag{5.5}$$

Note that the trace part of $\vec{m}$ can be absorbed by a shift of $\vec{\phi}$. As we discussed before, the structure of vacua is substantially influenced by the bare mass $\vec{m}_{adj}$ of the adjoint hypermultiplet. When $\vec{m}_{adj} = 0$, the diagonal elements of A_1, A_2 are always massless, while there is no such flat direction if $\vec{m}_{adj} \neq 0$. We will treat the cases $\vec{m}_{adj} = 0$ and $\vec{m}_{adj} \neq 0$ separately.

5.1.1 Vanishing Adjoint Mass: $\vec{m}_{adj} = 0$

As a warm-up example, we start with the case of $n = 1$. In theories with a single flavor, the fundamental hypermultiplet cannot have non-zero vev, $Q = \tilde{Q} = 0$, which follows from the equations (5.1) and (5.2). The equations also imply that A_1 and A_2 can be diagonalized simultaneously with $\vec{\phi}$. Let $(\mathbf{C}^2_H)^k$ denote the set of eigenvalues of A_1 and A_2:

$$\left(\begin{pmatrix} z_1^{(1)} \\ z_2^{(1)} \end{pmatrix}, \ldots, \begin{pmatrix} z_1^{(k)} \\ z_2^{(k)} \end{pmatrix} \right) \longleftrightarrow A_1 = \begin{pmatrix} z_1^{(1)} & & \\ & \ddots & \\ & & z_1^{(k)} \end{pmatrix}, A_2 = \begin{pmatrix} z_2^{(1)} & & \\ & \ddots & \\ & & z_2^{(k)} \end{pmatrix}. \tag{5.6}$$

If $\vec{\phi}$ is generic, the moduli space of hypermultiplet is given by

$$\mathcal{M}_{\mathcal{H}} = (\mathbf{C}^2_H)^k. \tag{5.7}$$

If $\vec{\phi}$ is invariant under some subgroup, say G, of the Weyl group which acts by permuting the diagonal entries, then the moduli space of hypermultiplet is $(\mathbf{C}^2_H)^k/G$. Massless photons live in the subgroup of the gauge group which is unbroken by the vevs of scalar fields. Since the $U(1)^k$ subgroup is unbroken in the present case, there are k-flat directions for the vector multiplets.

From here on, we will consider the case with $n \geq 2$. Equations (5.1) and (5.2) are the same as the ADHM equations for the construction of $SU(n)$ instantons on $\mathbf{R}^4$ of instanton number k [10]. Thus, if the mass constraints (5.3) and (5.4) were absent, the moduli space of hypermultiplet would be the moduli space of k-$SU(n)$ instantons on $\mathbf{R}^4$. More precisely, a solution of (5.1) and (5.2) describes genuine k-instantons only if a condition on the rank of the matrices Q, $\widetilde{Q}$, A_1 and A_2 is satisfied[1]. However, we take into account all possible vacua including those which do not meet such a condition. A degenerate solution describes a configuration containing a number of small instantons, the so-called ideal instantons (see section 3.4 of [21]). Thus, the moduli space of hypermultiplet is in fact the moduli space $\overline{\mathcal{M}}_k(SU(n))$ of ideal instantons of instanton number k. This includes as subspaces the moduli spaces $\mathcal{M}_{k-\ell}(SU(n)) \times Sym^\ell \mathbf{R}^4$ where ℓ of the instantons are small. Their positions are labeled by $\mathbf{R}^4$. If we turn on $\vec{\phi}$ and the masses $\vec{m}$ (and also $\vec{m}_{adj}$), the mass constraints (5.3) and (5.4) reduce the moduli space of hypermultiplets to (a finite cover of) a certain subspace of $\overline{\mathcal{M}}_k(SU(n))$.

For generic values of $\vec{\phi}$, the gauge group $U(k)$ is broken to $U(1)^k$, and quarks and off-diagonal part of adjoint hypermultiplet acquire mass. Therefore the flat direction is $Q = \widetilde{Q} = 0$, $A_1 = A_2 =$ diagonal, and the moduli space of hypermultiplet is given by $(\mathbf{C}^2_H)^k$. As the gauge symmetry $U(1)^k$ is unbroken on such vacua, we have a mixed branch with $d_H = k$ and $d_V = k$ flat directions of hyper and vector multiplets.

Vanishing Quark Mass

We will consider first the case $\vec{m} = 0$ where the theory possesses global $SU(n)$ symmetry.

At the special point $\vec{\phi} = 0$, the mass constraint is trivial and the moduli space of hypermultiplets is the full moduli space of ideal instantons

$$\mathcal{M}_{\mathcal{H}} = \overline{\mathcal{M}}_k(SU(n)). \tag{5.8}$$

[1]The condition is: for any $\lambda, \mu \in \mathbf{C}$, both $(A_1 + \lambda, A_2 + \mu, {}^t\widetilde{Q})$ and $(\lambda - A_1, A_2 - \mu, Q)$ have maximal rank k (See [10]).

This has (quaternionic) dimension nk. The global $SU(n)$ symmetry is generically spontaneously broken but remains unbroken on the locus $Sym^k(\mathbf{C}^2_H) \subset \overline{\mathcal{M}}_k(SU(n))$ of vanishing squark vevs $Q = \tilde{Q} = 0$. The gauge group $U(k)$ is generically completely broken, and thus, the moduli space (5.8) is an isolated Higgs branch.

Let us consider a more general value

$$\vec{\phi} = \mathrm{diag}(0, \ldots, 0, \vec{r}_{k_0+1}, \ldots, \vec{r}_k). \tag{5.9}$$

If the non-zero entries are generic, the gauge symmetry is broken to $U(k_0) \times U(1)^{k-k_0}$, and A_1, A_2 and $Q, \tilde{Q}$ are constrained to be a $U(k_0) \times U(1)^{k-k_0}$ adjoint and $U(k_0)$ fundamental hypermultiplets with n flavors respectively. Thus, the moduli space of hypermultiplets is

$$\mathcal{M}_{\mathcal{H}} = \overline{\mathcal{M}}_{k_0}(SU(n)) \times (\mathbf{C}^2_H)^{k-k_0}, \tag{5.10}$$

which has dimension $d_H = nk_0 + k - k_0$. At generic point on this space, the gauge group $U(k_0) \times U(1)^{k-k_0}$ is broken to $U(1)^{k-k_0}$. Thus, the moduli space (5.10) extends to a mixed branch in the $d_V = k - k_0$ flat directions for the vector multiplets. At values of $\vec{\phi}$ whose non-zero entries are invariant under a group G of permutations, the factor $(\mathbf{C}^2_H)^{k-k_0}$ is replaced by the quotient $(\mathbf{C}^2_H)^{k-k_0}/G$.

To summarize, we list the dimensions d_H and d_V of the mixed branches:

d_H	k	$n+k-1$	$\cdots$	$nk-n+1$	nk
d_V	k	$k-1$	$\cdots$	1	0

Table 7: Mixed branches for $\vec{m}_{adj} = 0, \vec{m}_{fund} = 0$

Non-Vanishing Quark Mass

We consider the case

$$\vec{m} = \mathrm{diag}(\underbrace{\vec{m}_1, \ldots, \vec{m}_1}_{n_1}, \ldots, \underbrace{\vec{m}_s, \ldots, \vec{m}_s}_{n_s}), \quad \vec{m}_i \neq \vec{m}_j \quad i \neq j \tag{5.11}$$

in which the global symmetry $SU(n)$ is broken to $SU(n_1) \times \cdots \times SU(n_s)$. We assume here $n_i \geq 2$ but other cases can also be worked out.

The most general choice of $\vec{\phi}$ is

$$\vec{\phi} = \mathrm{diag}(\overbrace{\vec{m}_1, \ldots, \vec{m}_1}^{k_1}, \ldots, \overbrace{\vec{m}_s, \ldots, \vec{m}_s}^{k_s}, \vec{r}_{k_1+\cdots+k_s+1}, \ldots, \vec{r}_k)\ . \tag{5.12}$$

If $\vec{r}_{k_1+\cdots+k_s+1}, \ldots, \vec{r}_k$ are generic, the gauge group $U(k)$ is broken to the subgroup $U(k_1) \times \cdots \times U(k_s) \times U(1)^{k-k_1-\cdots-k_s}$. Due to equations (5.3) and (5.4), the hypermultiplets A_1, A_2 and $Q, \tilde{Q}$ are constrained respectively to be the hypermultiplet in the adjoint representation of this subgroup and $U(k_i)$ fundamental hypermultiplets with flavors n_i, $i = 1, \ldots, s$. The moduli space of hypermultiplets at this point is thus

$$\mathcal{M}_{\mathcal{H}} = \overline{\mathcal{M}}_{k_1}(SU_{n_1}) \times \cdots \times \overline{\mathcal{M}}_{k_s}(SU_{n_s}) \times (\mathbf{C}_H^2)^{k-k_1-\cdots-k_s}, \tag{5.13}$$

which has dimension $d_H = \sum n_i k_i + k - \sum k_i$. Generically on this moduli space, the gauge group is broken to $U(1)^{k-k_1-\cdots-k_s}$. Thus, the moduli space (5.13) extends to a mixed branch which has dimensions

$$\begin{aligned} d_H &= n_1 k_1 + \cdots + n_s k_s + k - k_1 - \cdots - k_s \\ d_V &= k - k_1 - \cdots - k_s \end{aligned} \tag{5.14}$$

in the directions of hyper and vector multiplets respectively.

5.1.2 Non-Vanishing Adjoint Mass: $\vec{m}_{adj} \neq 0$

Consider next the case where $\vec{m}_{adj} \neq 0$. We start again with the single flavor case $n = 1$. It follows from the ADHM equations (5.1) and (5.2) that $Q = \tilde{Q} = 0$ and that A_1 and A_2 are diagonalizable. On the other hand, the mass constraint (5.4) shows that A_1 and A_2 are nilpotent for any choice of $\vec{\phi}$; we conclude that $A_1 = A_2 = 0$. Thus, hypermultiplets do not have a flat direction for any value of $\vec{\phi}$ and there is only a Coulomb branch of dimension k.

For $n \geq 2$ one can also turn on the quark mass $\vec{m}$. However, we will mainly treat the case with $\vec{m} = 0$ where the theory has global $SU(n)$ symmetry. Later we make a few comments on the case $\vec{m} \neq 0$.

Coulomb Branch

For generic values of $\vec{\phi}$, quarks get mass and decouple $Q = \tilde{Q} = 0$. We can also show $A_1 = A_2 = 0$ by repeating the above argument. Thus, we see that there is no flat direction for the hypermultiplets. Since $U(1)^k$ is unbroken, we have a Coulomb branch of dimension k.

Higgs and Mixed Branches with $A_1 = A_2 = 0$

At the special point $\vec{\phi} = 0$, massive adjoint hypermultiplet decouples $A_1 = A_2 = 0$ but the quarks do not. This is the case of QCD with $U(k)$ gauge group and n flavors[2]. Using the $U(k) \times SU(n)$ rotations, a solution to the vacuum equations $QQ^\dagger - \tilde{Q}^\dagger\tilde{Q} = 0$ and $Q\tilde{Q} = 0$ can be expressed as

$$Q = \begin{pmatrix} q_1 & & 0 & & 0 & \\ & \ddots & & \ddots & & \ddots \\ & & q_r & & 0 & \\ & & & & & \end{pmatrix}, \quad {}^t\tilde{Q} = \begin{pmatrix} 0 & & q_1 & & 0 & \\ & \ddots & & \ddots & & \ddots \\ & & 0 & & q_r & \\ & & & & & \end{pmatrix} \tag{5.15}$$

for some r, where $q_1, \ldots, q_r$ are real non-negative numbers. Note that the maximum number that r can take is k if $n \geq 2k$ and $[\frac{n}{2}]$ if $n < 2k$. Let $\mathcal{H}_r$ be the moduli space of hypermultiplets consisting of vacua with rank $\leq r$ squark vevs. The global symmetry $SU(n)$ is broken to $SU(n-2r) \times U(1)^r$ and there are $4nr - 4r^2 - r$ Nambu-Goldstone bosons. Since there are r real parameters, the moduli space $\mathcal{H}_r$ has (quaternionic) dimension $nr - r^2$. Remark that $\mathcal{H}_r$ is obtained by hyperkähler quotient of a nr dimensional vector space by the completely broken subgroup $U(r)$, and the dimension is given by the naive counting: $\dim \mathcal{H}_r = nr - \dim U(r)$. This turns out to be a useful method to count the dimension in complicated situations which we will encounter in the following. Since the gauge group is broken to $U(k-r)$, $\mathcal{H}_r$ extends to a mixed branch in the $k-r$ flat directions of vector multiplet. An isolated Higgs branch $\mathcal{H}_k$ exists only when the flavor n is not less than $2k$.

Higgs and Mixed Branches with $A_1 \neq 0$, $A_2 \neq 0$

We can find other type of Higgs or mixed branches at some values of $\vec{\phi}$. For example let us consider

$$\vec{\phi} = \mathrm{diag}(\overbrace{0, \ldots, 0}^{\ell_0}, \overbrace{\vec{m}_{adj}, \ldots, \vec{m}_{adj}}^{\ell_1}). \tag{5.16}$$

At this point, gauge group is broken to $U(\ell_0) \times U(\ell_1)$ and some of the adjoint and fundamental hypermultiplets remain massless. The mass constraints (5.3) and (5.4) impose the vevs to be of the following form

$$A_1 = \left(\begin{array}{c|c} 0 & 0 \\ \hline \alpha & 0 \end{array}\right), \quad A_2 = \left(\begin{array}{c|c} 0 & \tilde{\alpha} \\ \hline 0 & 0 \end{array}\right), \quad Q = \left(\begin{array}{c} q \\ \hline 0 \end{array}\right), \quad {}^t\tilde{Q} = \left(\begin{array}{c} {}^t\tilde{q} \\ \hline 0 \end{array}\right), \tag{5.17}$$

where the k columns (rows) are decomposed into blocks of size ℓ_0 and ℓ_1. Under the local and global symmetry $U(\ell_0) \times U(\ell_1) \times SU(n)$, α and $\tilde{\alpha}$ transform as $(\ell_0{}^*, \ell_1, \mathbf{1})$, $(\ell_0, \ell_1{}^*, \mathbf{1})$

[2] The moduli space of hypermultiplet of $N = 2$ $SU(N_c)$ QCD in four dimension was analyzed in [19].

while q and $\tilde{q}$ transform as $(\ell_0, \mathbf{1}, \mathbf{n}^*)$, $(\ell_0{}^*, \mathbf{1}, \mathbf{n})$ respectively. The D-term equations with respect to $U(\ell_0)$ and $U(\ell_1)$ gauge symmetry read

$$qq^\dagger - \tilde{q}^\dagger\tilde{q} = \alpha^\dagger\alpha - \tilde{\alpha}\tilde{\alpha}^\dagger, \quad q\tilde{q} = \tilde{\alpha}\alpha, \tag{5.18}$$

$$\alpha\alpha^\dagger - \tilde{\alpha}^\dagger\tilde{\alpha} = 0, \quad \alpha\tilde{\alpha} = 0. \tag{5.19}$$

Equations (5.19) admit solutions as (5.15) and in particular require α and $\tilde{\alpha}$ to be of the same rank, say k_1. If we insert such a solution, equations (5.18) also requires q and $\tilde{q}$ to be of the same rank, say k_0. In addition to the obvious bound $k_1 \leq \ell_1$ and $k_0 \leq \ell_0$, the ranks must satisfy the relation

$$k_0 \geq 2k_1, \qquad n \geq 2k_0 - k_1. \tag{5.20}$$

Let $\mathcal{H}_{k_0,k_1}$ be the moduli space of such vacua with lower rank cases being included. It is the hyperkähler quotient of a vector space of dimension $nk_0 + k_0k_1$ by the completely broken subgroup $U(k_0) \times U(k_1)$. Thus, according to the previous remark, its dimension is $nk_0 + k_0k_1 - k_0^2 - k_1^2$. Generically on this space, the gauge group is broken to $U(\ell_0 - k_0) \times U(\ell_1 - k_1)$. Thus $\mathcal{H}_{k_0,k_1}$ extends to a mixed branch in the $k - k_0 - k_1$ flat directions of vector multiplet. Note that it exists when $k_0 + k_1 \leq k$, $n \geq 2k_0 - k_1$ and $k_0 \geq 2k_1$ (irrespectively of ℓ_0, ℓ_1). It is an isolated Higgs branch if $k_0 + k_1 = k$ which is possible only when $n \geq k$.

In general, flat directions of hypermultiplet can be found for values of $\vec{\phi}$ whose entries are integer multiples of $\vec{m}_{adj}$. Let us consider the case in which k_j entries are $j\vec{m}_{adj}$ where j runs over integers from $-p \leq 0$ to $q \geq 0$. There exists a non-trivial moduli space of hypermultiplet $\mathcal{H}_{\{k_i\}}$ when the k_j satisfy the following conditions

$$\sum_{i=-p}^{q} k_i \leq k, \qquad 2k_0 - k_{-1} - k_1 \leq n \tag{5.21}$$

$$k_{i-1} - k_i \geq k_i - k_{i+1}, \quad i \neq 0. \tag{5.22}$$

It extends to a mixed branch which has dimensions

$$d_H = nk_0 + \sum_{i=-p}^{q-1} k_ik_{i+1} - \sum_{i=-p}^{q} k_i^2, \qquad d_V = k - \sum_{i=-p}^{q} k_i. \tag{5.23}$$

Note that the condition (5.22) which is a generalization of (5.20) means that the plot of k_j against the horizontal j axis is concave in the regions $j > 0$ and $j < 0$. This concave property will become more important in the next subsection.

Generic Quark Mass

When two or more quark masses are coincident, quarks have a flat direction. Otherwise, a flat direction for the hypermultiplets is possible only when the mass constraints (5.3) and (5.4) allow some components of the quarks and adjoint hypermultiplet that are charged under common subgroups to be massless. However, this cannot happen at any value of $\vec{\phi}$ if the masses are generic in the following sense

$$\vec{m}_{adj} \neq 0 \quad \text{and} \quad \vec{m}_i - \vec{m}_j \neq \ell \vec{m}_{adj} \quad \text{for any } 0 \leq j < i \leq n-1 \text{ and } -k < \ell < k. \quad (5.24)$$

Conversely, when this condition is broken, a flat direction for the hypermultiplets does exist for some value of $\vec{\phi}$.

5.2 Classical Moduli Space of Vacua of The B Model

We look at the moduli space of vacua of the B model in such a way that various Coulomb or mixed branches are emanating from the underlying moduli space $\mathcal{M}_H$ of hypermultiplet. As FI terms are turned on, the moduli space $\mathcal{M}_H$ is deformed and the Coulomb branches get reduced. The dimension of the moduli space of vector multiplet emanating from a point of $\mathcal{M}_H$ is given by the rank of the unbroken gauge group. In this subsection, we characterize and classify points of $\mathcal{M}_H$ with respect to the unbroken gauge group.

Recall that the B model has gauge group $U(k)^n = \prod_{i=0}^{n-1} U(k)_i$ and matter hypermultiplets $B_{i(i+1)}, B_{(i+1)i}$ in the "bifundamental" representation of $U(k)_i \times U(k)_{i+1}$ and $Q_0, \tilde{Q}_0$ in the fundamental representation of $U(k)_0$. ($B_{i(i+1)}$ and $B_{(i+1)i}$ transform as $(\mathbf{k}, \mathbf{k}^*)$ and $(\mathbf{k}^*, \mathbf{k})$ under $U(k)_i \times U(k)_{i+1}$ respectively.) The moduli space $\mathcal{M}_H$ of hypermultiplet is determined at the classical level as the set of solutions of the classical equations (4.12), (4.13) modulo the $U(k)^n$ gauge group action. We note that this is the same as the hyperkähler quotient construction of Hilbert Scheme of points on an ALE space by Kronheimer and Nakajima [6, 20, 7]. We do not impose mass constraints like (5.3) and (5.4) on hypermultiplets. Instead, we use them to force the flat directions of the vector multiplet to lie in the direction of the unbroken gauge group.

As we will see, the structure of vacua is greatly affected by the trace part $\sum \vec{\zeta}_i$ of the FI parameters $\vec{\zeta} = (\vec{\zeta}_0, \vec{\zeta}_1, \ldots, \vec{\zeta}_{n-1})$.

5.2.1 Tracefree FI Parameters: $\sum_i \vec{\zeta}_i = 0$

In the case where the trace part of the FI parameters vanishes, one can show that $Q_0 = \tilde{Q}_0 = 0$ and B_{ij} are diagonalizable at the same time

$$B_{ij} = \begin{pmatrix} b_{ij}^{(1)} & & \\ & \ddots & \\ & & b_{ij}^{(k)} \end{pmatrix}. \tag{5.25}$$

The l^{th} diagonal entries $b_l = (b_{ij}^{(l)})$ satisfy a system of equations. Namely, that of the $k = 1$ model:

$$|b_{i(i-1)}|^2 - |b_{(i-1)i}|^2 + |b_{i(i+1)}|^2 - |b_{(i+1)i}|^2 = \zeta_i^{\mathbf{R}} \tag{5.26}$$

$$b_{i(i-1)}b_{(i-1)i} - b_{i(i+1)}b_{(i+1)i} = \zeta_i^{\mathbf{C}} \tag{5.27}$$

where $\zeta^{\mathbf{R}} = \zeta^1, \zeta^{\mathbf{C}} = \zeta^2 + i\zeta^3$ for $\vec{\zeta} = (\zeta^1, \zeta^2, \zeta^3)$. The moduli space of hypermultiplet for $k = 1$ model is the quotient by $U(1)^{n-1}$ of the set of solutions of these equations. (Note that the diagonal $U(1)$ subgroup of $U(1)^n$ is always unbroken, and can be forgotten upon quotient.) This is the same as the Kronheimer's hyperkähler quotient construction [5] of the ALE space $X_{\vec{\zeta}}$ of type A_{n-1}. For $k \geq 1$, we have k copies of this space and dividing by the residual permutation symmetry S_k we obtain

$$\mathcal{M}_H = Sym^k(X_{\vec{\zeta}}). \tag{5.28}$$

At generic points of $\mathcal{M}_H$, all $b_{ij}^{(l)}$ are non-zero and the gauge group $U(k)^n$ is broken to the diagonal subgroup $U(1)^k$ of the maximal torus $(U(1)^k)^n$. Thus, the vector multiplet generically has k-flat directions and there is no pure Higgs branch. As we will see in the following, for a non-generic choice of the FI parameters $\vec{\zeta}$ there are special points in $\mathcal{M}_H$ at which the unbroken gauge group has higher rank. This enhancement of unbroken gauge group corresponds to the singularity of the space $X_{\vec{\zeta}}$.

Turning Off FI Parameters $\vec{\zeta} = 0$

We first consider the case with FI terms turned off. Let us look at the moduli space $\mathcal{M}_H = X_{\vec{0}}$ for the $k = 1$ model. It follows from the equations (5.26) and (5.27) that $|b_{i(i+1)}|$, $|b_{(i+1)i}|$ and $b_{i(i+1)}b_{(i+1)i}$ are independent of i. Then, we can define z_1 and z_2 by

$$z_1 z_2 = b_{i(i+1)}b_{(i+1)i} \tag{5.29}$$

$$z_1^n = b_{01}b_{12}\cdots b_{(n-1)0} \tag{5.30}$$

$$z_2^n = b_{0(n-1)}\cdots b_{21}b_{10} \tag{5.31}$$

up to $\mathbf{Z}_n$ ambiguity $(z_1, z_2) \sim (e^{\frac{2\pi i}{n}} z_1, e^{-\frac{2\pi i}{n}} z_2)$. Thus, we see that the $k = 1$ moduli space is just the $\mathbf{Z}_n$ orbifold $\mathbf{C}^2/\mathbf{Z}_n$. Note also that by introducing gauge invariant variables $x = z_1 z_2$, $y = z_1^n$, and $z = z_2^n$, we obtain the standard relation $x^n = yz$. The A_{n-1} simple singularity at the origin corresponds to the solution $b_{ij} \equiv 0$ on which the gauge group $U(1)^n$ is totally unbroken. At other points, one or both of $b_{i(i+1)}$ and $b_{(i+1)i}$ is non-vanishing for each i and hence the gauge group is broken to the diagonal $U(1)$. Thus, we have a Coulomb branch of dimension n and a mixed branch with a single flat direction for each of the hyper- and vectormultiplets.

For general k, the moduli space of hyper multiplet is the k^{th} symmetric product

$$\mathcal{M}_H = Sym^k(\mathbf{C}^2/\mathbf{Z}_n). \tag{5.32}$$

For each k_0, $0 \leq k_0 \leq k$, let $\mathcal{N}_{k_0} \subset Sym^k(\mathbf{C}^2/\mathbf{Z}_n)$ be the submanifold of dimension $k - k_0$ corresponding to the set of points in $(\mathbf{C}^2/\mathbf{Z}_n)^k$ whose k_0 entries are the A_{n-1} singularity. A generic point in $\mathcal{N}_{k_0}$ corresponds to a vacuum with $b^{(1)} = \cdots = b^{(k_0)} = 0$ on which the gauge symmetry $U(k)^n$ is broken to the subgroup $U(k_0)^n \times U(1)^{k-k_0}$ of rank $nk_0 + k - k_0$. If the non-zero entries $b^{(k_0+1)}, \ldots, b^{(k)}$ are invariant under a group of permutations, the factor $U(1)^{k-k_0}$ is replaced by a larger group but the rank is still $k - k_0$. Thus, along the submanifold $\mathcal{N}_{k_0}$ of the moduli space of hypermultiplet, the vector multiplet has $nk_0 + k - k_0$ flat directions. To summarize, we list the dimensions of mixed branches where d_H, d_V denotes the number of flat directions of hyper and vector multiplets:

d_H	k	$k-1$	$\cdots$	1	0
d_V	k	$n+k-1$	$\cdots$	$nk-n+1$	nk

Table 8: Mixed branches for $\vec{\zeta}_0 = \cdots = \vec{\zeta}_{n-1} = 0$

Turning On Tracefree FI Parameters

We consider the case

$$\vec{\zeta} = (\underbrace{\vec{\zeta}_1, 0, \ldots, 0}_{n_1}, \underbrace{\vec{\zeta}_2, 0, \ldots, 0}_{n_2}, \ldots, \underbrace{\vec{\zeta}_s, 0, \ldots, 0}_{n_s}) \tag{5.33}$$

in which

$$\vec{\zeta}_1 + \cdots + \vec{\zeta}_s = 0 \quad \text{but} \quad \vec{\zeta}_{i+1} + \cdots + \vec{\zeta}_j \neq 0, \quad 0 < i < j \leq s. \tag{5.34}$$

This corresponds to the choice of mass (5.11) under the mirror map (3.1) where $\vec{m}_i = \vec{\zeta}_1 + \cdots + \vec{\zeta}_i$.

The moduli space $\mathcal{M}_H = X_{\vec{\zeta}}$ for the $k = 1$ model is an orbifold with singularities of types $A_{n_1-1}, A_{n_2-1}, \dots, A_{n_s-1}$ at s distinct points. This can be seen by an argument as in the $\vec{\zeta} = 0$ case. For instance, the point with A_{n_1-1} singularity corresponds to the vacuum with $b_{i,i+1} = b_{i+1,i} = 0$ for $i = 0, 1, \dots, n_1 - 1$, which is invariant under the subgroup $U(1)^{n_1}$ of the gauge group $U(1)^n$. In general, at the point with A_{n_i-1} singularity the unbroken gauge group is $U(1)^{n_i}$, while it is the diagonal $U(1)$ at other points. Thus, we have s-Coulomb branches of dimensions $n_1, \dots, n_s$ and one mixed branch of dimensions $\widetilde{d}_H = \widetilde{d}_V = 1$.

For general k, let $\mathcal{N}_{k_1,\dots,k_s}$ be the submanifold of $\mathcal{M}_H = Sym^k(X_{\vec{\zeta}})$ corresponding to the points in $(X_{\vec{\zeta}})^k$ whose k_i entries are the A_{n_i-1} singularity. On this submanifold, the gauge group is broken to $U(k_1)^{n_1-1} \times \cdots \times U(k_s)^{n_s-1} \times U(1)^k$. Thus, we have a mixed branch of dimensions

$$\begin{aligned} d_H &= k - k_1 - \cdots - k_s \\ d_V &= n_1 k_1 + \cdots + n_s k_s + k - k_1 - \cdots - k_s \end{aligned} \tag{5.35}$$

along the submanifold $\mathcal{N}_{k_1,\dots,k_s}$.

5.2.2 FI Parameters With Non-Vanishing Trace: $\sum_i \vec{\zeta}_i \neq 0$

When the trace $\sum \vec{\zeta}_i$ of the FI parameters is non-vanishing, things drastically change. By summing up the equations (4.12) and (4.13) and taking the trace, we obtain $||Q_0||^2 - ||\widetilde{Q}_0||^2 = 2k \sum \zeta_i^{\mathbf{R}}$ and $\widetilde{Q}_0 Q_0 = k \sum \zeta_i^{\mathbf{C}}$, and thus Q_0 or $\widetilde{Q}_0$ cannot be zero. In addition, the $B_{i,j}$ cannot be simultaneously diagonalizable. Namely, we have lost the structure of the symmetric product of the moduli space $X_{\vec{\zeta}}$ of the $k = 1$ model. Instead, our moduli space is the hyperkähler quotient construction of the Hilbert scheme of k-points on $X_{\vec{\zeta}}$:

$$\mathcal{M}_H = Hilb^{[k]} X_{\vec{\zeta}}. \tag{5.36}$$

When $\vec{\zeta}$ is generic and $X_{\vec{\zeta}}$ is smooth, it is known that $Hilb^{[k]} X_{\vec{\zeta}}$ is a resolution of the diagonal or quotient singularities of $Sym^k X_{\vec{\zeta}}$ and is in particular smooth. This means that the gauge group $U(k)^n$ is completely broken at every point of $\mathcal{M}_H$ and there is no flat direction for the vector multiplets.

For some special values of $\vec{\zeta}$ such that $X_{\vec{\zeta}}$ is singular, $Hilb^{[k]} X_{\vec{\zeta}}$ inherits the singularity of $X_{\vec{\zeta}}$. At a singular point, some subgroup of $U(k)^n$ remains unbroken and flat directions of vector multiplet appear. Here we classify such unbroken subgroups for the special value

$$\vec{\zeta} = (\vec{\zeta}_0, 0, \dots, 0) \tag{5.37}$$

for which the $k=1$ moduli space $X_{\vec{\zeta}}$ is the orbifold $\mathbf{C}^2/\mathbf{Z}_n$ with an A_{n-1} simple singularity. Using an $SU(2)_L$ rotation, we may put $\vec{\zeta}_0=(c,0,0)$ with $c>0$.

Convex Graphs and Mixed Branches

By looking at the first equation of (4.12), we see that $(B_{0,n-1},B_{0,1},Q_0)$ has rank k and hence $U(k)_0$ is always completely broken. However, the groups $U(k)_i$ at other sites may contain unbroken pieces. Let us consider configurations such that $U(k)_i$ is broken to $U(\ell_i)_i$ and its centralizer $U(k-\ell_i)_i$ is completely broken. Such configurations exist only when the ℓ_i satisfy a certain condition. Let us consider making a plot of ℓ_i against the horizontal i axis where i runs from 0 to $n\equiv 0$. As we noted in subsection 5.1.2, the plot of the rank $k-\ell_i$ of the completely broken gauge groups must be concave. In other words, the plot of ℓ_i is convex. Thus, for each convex integral graph $\{\ell_i\}_{i=0}^n$ with $\ell_0=\ell_n=0$, $\ell_i\leq k$, we have a submanifold of $\mathcal{M}_H$ with unbroken gauge group $\prod_i U(\ell_i)$. Its dimension is $d_H=k+\sum_{i=0}^{n-1}(k-\ell_i)(k-\ell_{i+1})-\sum_{i=0}^{n-1}(k-\ell_i)^2$ and it extends to a mixed branch with dimension $d_V=\sum\ell_i$ in the direction of vector multiplet.

This result can be rephrased in the following way. Suppose that the steepest ascending slope of the plot of ℓ_i is $q+1$, and the steepest descending slope is $-p-1$. For $-p-1\leq i\leq q+1$, let e_i be the number of steps with slope i. Since the plot starts with $\ell_0=0$ and ends with $\ell_n=0$, these numbers satisfy $\sum ie_i=0$. Let us introduce numbers k_i; $-p\leq i\leq q$ by

$$\begin{array}{ll}
e_{q+1}=k_q & e_{-p-1}=k_{-p} \\
e_q=k_{q-1}-2k_q & e_{-p}=k_{-p+1}-2k_{-p} \\
e_{q-1}=k_{q-2}-2k_{q-1}+k_q & e_{-p+1}=k_{-p+2}-2k_{-p+1}+k_{-p} \\
\cdots\cdots & \cdots\cdots \\
e_2=k_1-2k_2+k_3 & e_{-2}=k_{-1}-2k_{-2}+k_{-3} \\
e_1=k_0-2k_1+k_2 & e_{-1}=k_0-2k_{-1}+k_{-2}.
\end{array} \tag{5.38}$$

It appears that k_0 has two solutions $\sum_{i>0}ie_i$ and $\sum_{i>0}ie_{-i}$ but they coincide due to the relation $\sum ie_i=0$. In fact, it is the highest value of ℓ_i. In terms of $\{k_i\}$, the dimensions d_H and d_V of the mixed branch can be expressed as

$$d_H=k-\sum k_i,\qquad d_V=nk_0+\sum k_ik_{i+1}-\sum k_i^2. \tag{5.39}$$

In particular $\sum k_i\leq k$. Since e_i are non-negative integers, k_i satisfy the concave property

$$k_{i-1}-k_i\geq k_i-k_{i+1},\qquad i\neq 0. \tag{5.40}$$

Since the total number of steps is n, we have $n \geq \sum_{i\neq 0} e_i$, i.e.

$$n \geq 2k_0 - k_{-1} - k_1. \tag{5.41}$$

It is easy to see that each sequence $\{k_i\}_{-p\leq i\leq q}$ satisfying (5.41), (5.40) and $\sum k_i \leq k$ determines a convex graph $\{\ell_i\}_{i=0}^{n}$ having e_i steps of slope i where e_i is given by (5.38).

Adjacency Relations

Let us denote by $\mathcal{N}_{\{k_i\}}$ the submanifold of $Hilb^{[k]}(\mathbf{C}^2/\mathbf{Z}_n) = \mathcal{M}_H$ corresponding to a convex graph determined by the sequence $\{k_i\}$. As we move around the moduli space $\mathcal{M}_H$, unbroken gauge group can suddenly be enhanced but the converse will never occur. This property tells us some information on how the submanifolds $\mathcal{N}_{\{k_i\}}$ are related with each other. Let us consider two graphs $\{\ell_i\}$ and $\{\ell'_i\}$ determined by $\{k_i\}$ and $\{k'_i\}$ respectively. Then, $\mathcal{N}_{\{k'_i\}}$ intersects with a boundary of $\mathcal{N}_{\{k_i\}}$ only if $\ell'_i \geq \ell_i$ for any i. It is easy to see that the latter condition holds if and only if $k'_i \geq k_i$ for any i which we represent by $\{k'_i\} \geq \{k_i\}$. Thus, we have seen that

$$\overline{\mathcal{N}_{\{k_i\}}} \subset \bigcup_{\{k'_i\}\geq\{k_i\}} \mathcal{N}_{\{k'_i\}}. \tag{5.42}$$

Generic Values of FI Parameters

According to [7] theorem 2.8, $Hilb^{[k]}X_{\vec{\zeta}}$ is smooth when the FI parameter $\vec{\zeta}$ satisfies a certain condition. In our language it reads as

$$\sum_h \vec{\zeta}_h \neq 0 \quad \text{and} \quad \vec{\zeta}_i + \cdots + \vec{\zeta}_j \neq \ell \sum_h \vec{\zeta}_h \quad \text{for any } 1 \leq j \leq i \leq n-1,\ -k < \ell < k. \tag{5.43}$$

When this condition is satisfied, gauge group is completely broken everywhere and there is only a Higgs branch of dimension k.

5.3 The Mirror Map Revisited

In subsection 5.1, we determined and classified the various moduli spaces of hypermultiplet emanating from the classical moduli space of vector multiplet. In subsection 5.2, we classified submanifolds of the moduli space $\mathcal{M}_H$ of hypermultiplet with respect to the rank of the unbroken gauge group. If we compare the results, we can see an agreement of dimensions of mixed branches

$$(d_H, d_V)_{\text{A-model}} = (d_V, d_H)_{\text{B-model}} \tag{5.44}$$

provided that masses and FI parameters are related under the mirror map (3.1). For example, compare

- Table 7 for $\vec{m}_{adj} = \vec{m} \equiv 0$ and Table 8 for $\vec{\zeta}_i \equiv 0$
- Dimensions (5.14) for the mass (5.11) and (5.35) for the FI parameter (5.33)
- Dimensions (5.23) with (5.21), (5.22) for $\vec{m}_{adj} \neq 0, \vec{m} = 0$ and Dimensions (5.39) with (5.41), (5.40) for $\vec{\zeta}_0 \neq 0, \vec{\zeta}_{i>0} = 0$
- Condition (5.24) for the mass to be generic and Condition (5.43) for the FI parameters to be generic.

This agreement gives strong evidence of our duality proposal. In particular, the third one excludes the possibility of non-trivial dependence of the trace $\sum \vec{\zeta}_j$ in $\vec{m}_i$ like $\vec{m}_i = \sum_{l=0}^{i} \vec{\zeta}_l + c_i \sum_{j=0}^{n-1} \vec{\zeta}_j$. Also, the last one shows that absence of a flat direction for the hypermultiplets corresponds to the smoothness of $Hilb^{[k]}X_{\vec{\zeta}}$ only when the mirror map is normalized as in (3.1). Thus, we have excluded all possible corrections to the mirror map (3.1) and completed the proof of it.

5.4 Quantum Moduli Space of Vacua

In this subsection, which is mostly a summary of the results we obtained so far, we give a description what the quantum moduli space of vacua of the A model looks like if our duality conjecture is assumed to be correct. In particular, we locate the moduli spaces of hypermultiplet on the *quantum* moduli space of vector multiplet $\mathcal{M}_V$ by identifying the latter with the moduli space of hypermultiplet $\mathcal{M}_H$ of the B model.

The Self-Dual Model

When there is only a single flavor $n = 1$, the A model coincides with the B model and therefore is expected to be self-dual. The model has two parameters: the bare mass $\vec{m}_{adj}$ of the adjoint hypermultiplet and the FI parameter $\vec{\zeta}$ for the unique $U(1)$ factor of the gauge group.

$\underline{\vec{m}_{adj} = 0, \vec{\zeta} = 0}$

In this case, quantum moduli space of vector multiplet is $\mathcal{M}_V = Sym^k(\mathbf{C}^2_V)$ where $\mathbf{C}^2_V = X_{\vec{0}}$ is the quantum moduli space for the $k = 1$ model. At the generic point of $\mathcal{M}_V$ represented by a point of $(\mathbf{C}^2_V)^k$ whose entries are distinct with each other, we have $(\mathbf{C}^2_H)^k$ as the moduli space of hypermultiplet. When the representative in $(\mathbf{C}^2_V)^k$ is invariant under a group G of permutations, the moduli space of hypermultiplet collapses to $(\mathbf{C}_H^2)^k/G$. Thus, the quantum moduli space is given by

$$\mathcal{M}_{\rm total} = Sym^k(\mathbf{C}_V^2 \times \mathbf{C}_H^2). \tag{5.45}$$

$\vec{m}_{adj} \neq 0, \vec{\zeta} = 0$

When $\vec{m}_{adj} \neq 0$, there is a monopole correction that smooths out the singularity due to S_k quotient, and the quantum moduli space of vector multiplet is the Hilbert scheme of k-points on $\mathbf{C}_V^2$: $\mathcal{M}_V = Hilb^{[k]}(\mathbf{C}_V^2)$. Since $\vec{m}_{adj} \neq 0$ meets the condition (5.24) to be generic, the flat directions of hypermultiplet are completely lifted.

$\vec{m}_{adj} = 0, \vec{\zeta} \neq 0$

Since $\vec{\zeta} \neq 0$ meets the condition (5.43) to be generic, the flat directions of vector multiplet are completely lifted and we have a single smooth Higgs branch which is again the Hilbert scheme of points $Hilb^{[k]}(\mathbf{C}_H^2)$.

In summary, we list the quantum moduli space of vacua:

	Moduli Space	(d_V, d_H)
$\vec{m}_{adj} = 0,\ \vec{\zeta} = 0$	$Sym^k(\mathbf{C}_V^2 \times \mathbf{C}_H^2)$	(k, k)
$\vec{m}_{adj} \neq 0,\ \vec{\zeta} = 0$	$Hilb^{[k]}\mathbf{C}_V^2$	$(k, 0)$
$\vec{m}_{adj} = 0,\ \vec{\zeta} \neq 0$	$Hilb^{[k]}\mathbf{C}_H^2$	$(0, k)$

Table 9: Quantum Moduli Space of Vacua of the $n = 1$ Model

Multi Flavor Case

$\vec{m}_{adj} = 0,\ \vec{m} = 0$

When all the mass terms are turned off, the moduli space of vector multiplet is given by $\mathcal{M}_V = Sym^k(\mathbf{C}^2/\mathbf{Z}_n)$ which decomposes into $k+1$ submanifolds $\mathcal{N}_{k_0}$, $0 \le k_0 \le k$. Recall that $\mathcal{N}_{k_0}$ corresponds to the set of points in $(\mathbf{C}_V^2/\mathbf{Z}_n)^k$ whose k_0 entries are the A_{n-1} singularity. The moduli space of hypermultiplet emanating from a generic point of $\mathcal{N}_{k_0}$ is $(\mathbf{C}_H^2)^{k-k_0} \times \overline{\mathcal{M}}_{k_0}(SU(n))$. At the point whose representative in $(\mathbf{C}^2/\mathbf{Z}_n)^k$ is invariant under a group $G \times S_{k_0}$ of permutations, the moduli space of hypermultiplet collapses to $(\mathbf{C}_H^2)^{k-k_0}/G \times \overline{\mathcal{M}}_{k_0}(SU(n))$. Thus, we have located the moduli spaces of hypermultiplet on the submanifold $\mathcal{N}_{k_0}$. The resulting mixed branch, including its boundary, is given by

$$\mathcal{M}_{k_0} = Sym^{k-k_0}(\mathbf{C}^2/\mathbf{Z}_n \times \mathbf{C}^2_H) \times \overline{\mathcal{M}}_{k_0}(SU(n)). \tag{5.46}$$

It has dimensions $d_V = k - k_0$ and $d_H = nk_0 + k - k_0$. The quantum moduli space is now represented as a union of these branches:

$$\mathcal{M}_{\text{total}} = \bigcup_{0 \leq k_0 \leq k} \mathcal{M}_{k_0}. \tag{5.47}$$

Note that $\mathcal{M}_k = \overline{\mathcal{M}}_k(SU(n))$ is a unique Higgs branch of dimension nk. The "basic branch"

$$\mathcal{M}_0 = Sym^k(\mathbf{C}^2/\mathbf{Z}_n \times \mathbf{C}^2_H) \tag{5.48}$$

is a mixed branch of dimension $2k$ which has no non-trivial $SU(n)$ action. On any other branch $\mathcal{M}_{k_0}$, the global $SU(n)$ symmetry is generically spontaneously broken due to squark vevs. It touches the basic branch $\mathcal{M}_0$ along the submanifold of dimension $2k - k_0$ of $SU(n)$-fixed points. The theories in that submanifold have unbroken $SU(n)$ symmetry. The branches $\mathcal{M}_{k_0}$ with $k_0 \geq 1$ also touch each other; a boundary of $\mathcal{M}_{k_0}$ is embedded in $\mathcal{M}_{k_0+\ell}$ according to the embedding of $Sym^\ell(\mathbf{C}^2_H) \times \overline{\mathcal{M}}_{k_0}(SU(n))$ in $\overline{\mathcal{M}}_{k_0+\ell}(SU(n))$.

$\vec{m}_{adj} = 0$, $\vec{m} \neq 0$

We consider the case with the bare mass $\vec{m}$ being given by (5.11) in which the theory has global symmetry $SU(n_1) \times \cdots \times SU(n_s)$. The moduli space of vector multiplet is $\mathcal{M}_V = Sym^k(X_{\vec{\zeta}(m)})$ where $\vec{\zeta}(m)$ is mirror image of $\vec{m}$.

The quantum moduli space is represented as:

$$\mathcal{M}_{\text{total}} = \bigcup_{\substack{k_i \geq 0 \\ k_1 + \cdots + k_s \leq k}} \mathcal{M}_{k_1,\ldots,k_s}. \tag{5.49}$$

where

$$\mathcal{M}_{k_1,\ldots,k_s} = Sym^{k-\sum k_i}(X_{\vec{\zeta}(m)} \times \mathbf{C}^2_H) \times \overline{\mathcal{M}}_{k_1}(SU_{n_1}) \times \cdots \times \overline{\mathcal{M}}_{k_s}(SU_{n_s}) \tag{5.50}$$

is a mixed branch of dimensions $d_V = k - \sum k_i$ and $d_H = \sum n_i k_i + k - \sum k_i$. The basic branch

$$\mathcal{M}_{0,\ldots,0} = Sym^k(X_{\vec{\zeta}(m)} \times \mathbf{C}^2_H) \tag{5.51}$$

has no non-trivial $SU(n_1) \times \cdots \times SU(n_s)$ action. Any other branch $\mathcal{M}_{k_1,\ldots,k_s}$ has non-trivial action of this group and touches the basic branch $\mathcal{M}_{0,\ldots,0}$ along the submanifolds of fixed points. Theories in the fixed point submanifold have unbroken $SU(n_1) \times \cdots \times SU(n_s)$ global symmetry.

$\vec{m}_{adj} \neq 0$, $\vec{m} = 0$

Finally, let us consider the case $\vec{m}_{adj} \neq 0$ and $\vec{m} = 0$ in which the theory possesses global $SU(n)$ symmetry. This choice of mass is mapped to the FI parameter $\vec{\zeta} = (\vec{m}_{adj}, 0, \ldots, 0)$. The moduli space of vector multiplet is $\mathcal{M}_V = Hilb^{[k]}(\mathbf{C}^2/\mathbf{Z}_n)$, the Hilbert scheme of k-points on $\mathbf{C}^2/\mathbf{Z}_n$.

The quantum moduli space is represented as follows:

$$\mathcal{M}_{\text{total}} = \bigcup_{\{k_i\}} \mathcal{M}_{\{k_i\}}. \tag{5.52}$$

Here $\{k_i\}$ runs over sequences of integers satisfying the conditions (5.21) and (5.22), and $\mathcal{M}_{\{k_i\}}$ is roughly a direct product $\mathcal{N}_{\{k_i\}} \times \mathcal{H}_{\{k_i\}}$ with its boundary being included. The space

$$\mathcal{M}_{\{0\}} = Hilb^{[k]}(\mathbf{C}^2/\mathbf{Z}_n) \tag{5.53}$$

is the unique Coulomb branch of dimension k, on which the global $SU(n)$ symmetry acts trivially. Any other branch $\mathcal{M}_{\{k_i\}}$ has a non-trivial action of $SU(n)$ and touches the Coulomb branch along the submanifold $\overline{\mathcal{N}_{\{k_i\}}}$. If $\{k_i\} \leq \{k_i'\}$, $\mathcal{H}_{\{k_i\}}$ is embedded in $\mathcal{H}_{\{k_i'\}}$ and it is possible that a boundary of $\mathcal{N}_{\{k_i\}}$ intersects with $\mathcal{N}_{\{k_i'\}}$. When this happens to be the case, a boundary of $\mathcal{M}_{\{k_i\}}$ is embedded in $\mathcal{M}_{\{k_i'\}}$ as a submanifold. To know whether this really happens or not, we need more information on the adjacency relations of the $\mathcal{N}_{\{k_i\}}$'s in $Hilb^{[k]}(\mathbf{C}^2/\mathbf{Z}_n)$.

6 Duality for $U(k)$ Gauge Theories III: T-Duality and Extremal Transition Picture

In this section, we discuss how to understand the mirror symmetry between the A and B-models from the string theory view point. It has been suggested in [3], [22] that the mirror symmetry in three dimensions should be a consequence of the T-duality between IIA and IIB strings. The type IIA string compactified on a Calabi-Yau 3-fold M times S^1 is, by the T-duality, equivalent to the type IIB string on the same geometry except for the change of the radius of S^1. Under the T-duality, the vector and the hypermultiplet moduli spaces of the two theories are interchanged. This is exactly the situation of the mirror symmetry in three dimensions. Here we will examine how this suggestion can be implemented explicitly in our case.

There is a particular Calabi-Yau 3-fold M on which the type IIA string gives the field content of the A-model [23–26]. In order to turn off gravity, we take the Planck mass to infinity after the compactification. At the same time, we would like to have finite masses for relevant charged particles coming from D-branes wrapping cycles in M. Thus we have to consider a singular limit of M, where we scale the relevant Kähler moduli of M to zero simultaneously. In fact a local description of the singularity of M is sufficient in order to understand the field theory limit of the compactified IIA string [27]. To realize the A-model in three dimensions, we send the radius R_A of S^1 to zero also while keeping $R_A m_{string}$ finite.

This compactification of the IIA string is related, by the T-duality, to the type IIB string on $M \times S^1$ with the radius of S^1 being $R_B = (R_A m_{string}^2)^{-1}$. Since R_B scales as $1/m_{string}$, the T-dual of the A-model should also give a three-dimensional field theory with rigid $N = 4$ supersymmetry. In fact, in the case of $k = 1$ with n being arbitrary, we will show that the type IIB string on M reproduces the field content of the B-model. This means that, in this case, the mirror symmetry of the A and B-models can indeed be interpreted as a consequence of the T-duality of the type IIA and IIB string theories. We also present some evidences for the $k > 1$ case.

6.1 A-Model

The A-model of the gauge theory arises from the type IIA string on a Calabi-Yau 3-fold M constructed as a family of K3 fibered over a complex one-dimensional torus C [23, 24]. In order to reproduce the field content in the A-model, namely:

- vector multiplet with $U(k)$ gauge group,
- one hypermultiplet $(A, \tilde{A})$ in the adjoint representation,
- n hypermultiplets $(Q_i, \tilde{Q}_i)$ $(i = 1, ..., n)$ in the fundamental representation,

we consider a case when K3 has singularities of type A_k at n isolated points $w = w_1, ..., w_n$ on C which are resolved to type A_{k-1} over a generic point [25, 26]. The geometry of the Calabi-Yau manifold M near the singularities can be modeled by the equation,

$$z^k(z + P_n(w)) + x^2 + y^2 = 0, \tag{6.1}$$

where $P_n(w)$ is a polynomial of degree n with n zeroes at $w_1, ..., w_n$, and (x, y, z) parametrize the K3 fiber. We can see that the fiber develops an A_k singularity at n points on C where $P_n(w) = 0$. The A_{k-1} singularity on a generic fiber can be resolved as

$$\prod_{a=1}^{k} (z + \mu_a) \cdot (z + P_n(w)) + x^2 + y^2 = 0. \tag{6.2}$$

However, for each $a = 1, ..., k$, the fiber still has A_1 singularity at n points on $\mathcal{C}$ solving $P_n(w) = \mu_a$.

Let us demonstrate that this geometry indeed generates the field content of the A-model. Due to the A_k singularity, there are $\frac{1}{2}k(k+1)$ 2-cycles S_{ab} $(a, b = 1, ..., k+1; a < b)$ on each fiber. The cycle S_{ab} with $a, b \leq k$ vanishes when we choose the complex moduli $\mu_a = \mu_b$. On the other hand, $S_{a(k+1)}$ always vanishes at n points on $\mathcal{C}$ satisfying $P_n(w) = \mu_a$. Among these cycles, k of them are homologically independent. We can choose S_{12}, S_{23}, ..., $S_{k(k+1)}$ as primitive cycles and correspondingly there are k Kähler moduli of M. For later convenience, we denote the one associated to $S_{a(a+1)}$ by $(t_a - t_{a+1})$ with $t_{k+1} = 0$ (alternatively one may choose $S_{1(k+1)}, ..., S_{k(k+1)}$ as primitive cycles and t^a as the Kähler moduli associated to them). Each of the Kähler moduli can be identified as a vev of charge neutral scalar component of the vector multiplet. Since the base of M is a torus $\mathcal{C}$, there is also a generator η_a of $H^{2,1}(M)$ associated to each $S_{a(k+1)}$. To see this explicitly, one may take the $(1,1)$ form on the fiber corresponding to $S_{a(k+1)}$ and tensor it with the holomorphic 1-form on the base $\mathcal{C}$. A 3-cycle dual to η_a is S^1 of the base $\mathcal{C}$ times $S_{a(k+1)}$ of the fiber[1]. In fact these $\eta_a \in H^{2,1}(M)$ correspond to the complex modulus μ_a in the resolved space (6.2). The complex modulus μ_a together with a vev of the RR 3-form $B^{(3)}$ on $S^1 \times S_{a(k+1)}$ make charge neutral scalar components of the adjoint hypermultiplet $(A, \tilde{A})$. The charged components of the vector and the adjoint hypermultiplets correspond to wrapping D2-branes on the 2-cycles S_{ab} $(a, b = 1, ..., k)$. Among them, the cycles $S_{a(a+1)}$ correspond to simple roots of $U(k)$ while others correspond to non-simple roots.

As we mentioned in the above, for generic values of μ_a, S_{ab} with $a, b \leq k$ are non-vanishing, but each $S_{a(k+1)}$ vanishes at n special points satisfying $P_n(w) = \mu_a$. The cycle $S_{a(k+1)}$ is homologous to the sum of the primitive cycles $S_{a(a+1)} \cup S_{(a+1)(a+2)} \cup \cdots \cup S_{k(k+1)}$. Thus, by wrapping a D2-brane on $S_{a(k+1)}$, we find one hypermultiplet carrying charges $q_a = 1$ and $q_b = 0$ $(b \neq a)$. Here q_a means the charge for the $U(1)$ vector associated to the Kähler moduli t^a. Of course, if we choose $\mu_a = 0$ for all a, the vanishing of $S_{a(k+1)}$ takes place at the zeroes of $P_n(w)$. Thus we find that, from each of the n exceptional fibers, we obtain one hypermultiplet in the fundamental representation of $U(k)$. This completes the field content of the A-model.

Now let us examine the structure of the Coulomb branch of the A-model using this string theory construction. In four dimensions, the RR 3-form $B^{(3)}$ associated to the 2-cycle $S_{a(k+1)}$ and the RR 5-form $B^{(5)}$ associated to its dual 4-cycle ${}^*S_{a(k+1)}$ give a vector field v_μ and its dual ${}^*v_\mu$. Upon compactification to three dimensions, their Wilson line

[1] In general, if the base is a genus-g curve, each $S_{a(k+1)}$ should give g elements of $H^{2,1}(M)$ since there are g holomorphic 1-forms.

expectation values on S^1 make a complex scalar field, which we denote by u_a. It is then paired with the scalar component of the vector multiplet corresponding to the Kähler moduli t_a, and the vector multiplet moduli space also become a hyperkähler space. Since the RR charges are quantized, the RR fields u_a are periodically identified. Thus the vector multiplet moduli space can be viewed as a family of tori of the RR field u_a fibered over the Kähler moduli space of t_a.

Let us examine the conifold singularity in the moduli space which is generated when the quantum corrected size of one of $S_{a(k+1)}$'s vanishes, paying attention to the 2-dimensional subspace of (t_a, u_a) taking all other moduli to be of generic value[2]. In fact there are n homologous 2-cycles (one at each of the n special points) whose quantum volumes vanish simultaneously in this limit. This is a situation in which the extremal transition is possible [28]. Traveling around the singular point, the RR fields experience the monodromy transformation $u_a \to u_a + n$ [29]. This means that the moduli space near the conifold point has an orbifold singularity $\mathbf{C}^2/\mathbf{Z}_n$ in the subspace of (t_a, u_a).

When the two complex moduli coincide, $\mu_a = \mu_b$, there appears an additional symmetry which exchanges (t_a, u_a) and (t_b, u_b). Thus, in particular when all the complex moduli coincide, the vector multiplet moduli space becomes $Sym^k(\mathbf{C}^2/\mathbf{Z}_n)$. This is exactly the structure of the Coulomb branch of the A-model which we found previously from the mirror symmetry and the one-loop test.

After the extremal transition, the n homologous 2-cycles are replaced by n 3-cycles with 1 homology relation. Thus the origin of the 2-dimensional subspace (t_a, u_a) of the vector multiplet moduli space is connected to $2(n-1)$-dimensional subspace of the hypermultiplet moduli space, $(n-1)$ of which correspond to the complex moduli of M. We can repeat this procedure k times to completely Higgs the vector multiplet. This gives us $k + k(n-1) = kn$ hypermultiplets (the first k are the complex moduli μ_a which are there before the extremal transition). This correctly reproduces the Coulomb-Higgs transition discussed in the previous sections.

6.2 B-Model

Here we will consider the same Calabi-Yau manifold M, but put the IIB string on it. As we mentioned before, if the field content of the B-model is reproduced in this way, the mirror symmetry may be considered as a consequence of the T-duality of the IIA and IIB

[2]We are considering the quantum corrected size (the one which takes into account worldsheet instanton effects) since it is the one that is proportional to the BPS mass of a D-brane wrapped around a cycle homologous to $S_{a(k+1)}$.

string theories.

To begin with, let us consider a case of $k = 1$ with n being arbitrary. In this case, we have a family of K3 fibered over $\mathcal{C}$, with n special points on which K3 develops the A_1 singularity. Elsewhere K3 is regular in this case. We would like to show that the type IIB string theory on this geometry gives the field content of the B-model:

- n vector multiplets v_i $(i = 1, ..., n)$ of $U(1)$ gauge group,
- n hypermultiplets $(B_{i(i+1)}, B_{(i+1)i})$ with charges $q_i = \pm 1$, $q_{i+1} = \mp 1$ and $q_{j \neq i, i+1} = 0$,
- a hypermultiplet $(Q_0, \tilde{Q}_0)$ with charges $q_1 = 1$, $q_2 = \cdots = q_n = 0$

The geometry as it is has one complex modulus μ and the one Kähler modulus t corresponding to the vanishing S^2 on the fiber. As we have seen in the previous subsection, the type IIA string on this geometry gives the Coulomb branch of the A-model. Since the T-duality exchanges the vector and the hypermultiplet moduli spaces, the type IIB string theory on the same geometry should be in the Higgs branch. To identify the field content of the B-model, however, it seems easier to work in the Coulomb branch. This means that we have to perform an extremal transition of the geometry.

Since there are n vanishing 2-cycles at the n special points and since they are all homologically equivalent, the extremal transition changes them into n 3-cycles $S^{(i)}$ $(i = 1, ..., n)$ with 1 homology relation $\sum_{i=1}^{n} S^{(i)} = 0$. This gives us $(n-1)$ complex moduli, in addition to one complex modulus μ which had been there before the extremal transition. Let us denote complex moduli associated to $S^{(i)}$ by $(\mu^{(i)} - \mu^{(i+1)})$ with $\mu^{(n+1)} = \mu^{(1)}$. There is a redundancy in this parametrization corresponding to simultaneous shift of $\mu^{(i)}$'s, and we fix it by choosing $\mu^{(1)}$ to be equal to the complex modulus μ.

Now we can identify the field content of the B-model. The $U(1)^n$ vector multiplet comes from the n complex moduli and the n hypermultiplets $(B_{i(i+1)}, B_{(i+1)i})$ are obtained by wrapping D3-branes on $S^{(i)}$'s. Since the dimensions of $H_{2,1}$ is n, there must be one more 3-cycle which is not homologous to $S^{(i)}$'s. In fact it is not difficult to identify one. Before the extremal transition, there is a unique homology 3-cycle which is the 2-cycle on the fiber times S^1 of $\mathcal{C}$. Since the extremal transition is a local operation near the n special points, this 3-cycle should remain after the transition as far as we choose S^1 to be away from these points. In fact, in our notation, the complex moduli $\mu^{(1)}$ corresponds to this 3-cycle. By wrapping a D3-brane on this cycle, we obtain one hypermultiplet $(Q_0, \tilde{Q}_0)$ with charges $q_1 = 1$, $q_2 = \cdots q_n = 0$. This completes the field content of the B-model. As one can see, the n hypermultiplets $(B_{i(i+1)}, B_{(i+1)i})$ are massless at the conifold point where all $S^{(i)}$ are vanishing. On the other hand, $(Q_0, \tilde{Q}_0)$ is massive even at the conifold point. This is also consistent with what we know about the B-model, i.e. the vev of $(Q_0, \tilde{Q}_0)$ is zero in both Coulomb and Higgs branches. Thus we have found that, in this case, the B-model arises from the type IIB string on $M \times S^1 \times \mathbf{R}^3$. This shows the mirror symmetry of the A and B-models is in fact a consequence of the T-duality of the IIA and IIB string theories.

Let us turn to general case when both k and n are arbitrary. We would like to identity

- n vector multiplets v_i $(i = 1, ..., n)$ of $U(k)$ gauge group,
- n hypermultiplets $(B_{i(i+1)}, B_{(i+1)i})$ where $B_{i(i+1)}$ is in $(\mathbf{k}_i, \mathbf{k}^*_{i+1})$ of $U(k)_i \times U(k)_{i+1}$,
- one hypermultiplet $(Q_0, \tilde{Q}_0)$ in the fundamental representation of the first $U(k)$.

Before the extremal transition, the number of holomogy 3-cycles is k, and this corresponds to the number of unbroken $U(1)$ gauge symmetries in the fully Higgsed branch of the B-model. After the extremal transition, the k 2-cycles at each of the n special points are replaced by k 3-cycles $S_a^{(i)}$ $(a = 1, ..., k;\ i = 1, ..., n)$ with k homology relations. Thus the number of homology 3-cycles becomes $k + (n-1)k = nk$ after the series of transitions, and this also agrees with the number of unbroken $U(1)$ symmetries in the Coulomb branch of the B-model. By counting charges with respect to these $U(1)$'s, we can identity wrappings of D3-branes on $S_a^{(i)}$'s as diagonal elements of $(B_{i(i+1)}, B_{(i+1)i})$, and wrappings of D3-brane on the original k 3-cycles as $(Q_0, \tilde{Q}_0)$.

We have not yet identified the roots of $U(k)^n$ and the off-diagonal elements of $(B_{i(i+1)}, B_{(i+1)i})$. Before the extremal transitions, in additions to the vanishing 2-cycles at the n special fibers, there are $\frac{1}{2}k(k-1)$ 2-cycles S_{ab}. After the extremal transitions, they should also transform into 3-cycles. In fact, they appear to carry appropriate $U(1)$ charges to be identified with these fields. It would be very interesting to work out the relevant homology relations among the 3-cycles after the extremal transition and to fully identify the fields in the B-model.

7 Duality for $Sp(k)$ Gauge Theories

In this section we study the second proposed family of dualities for $Sp(k)$ gauge theories. We provide the counting evidence for this duality proposal, study the quantum corrections, derive the mirror map and use D-brane probes and the Type I - M-theory duality to further support the gauge theory picture.

7.1 Counting Evidence

Again, as a first necessary evidence for the duality between the A and B models we count in quaternionic units the dimensions of the Higgs and Coulomb branches and the number of FI and mass terms.

A-model: The dimension of the Coulomb branch is the rank of the gauge group which is $d_V = k$. The dimension of the Higgs branch is the dimension of the hypermultiplet content $(2nk + 2k^2 - k)$ minus the dimension of the gauge group $(2k^2 + k)$. Thus, $d_H = 2k(n-1)$. The number of FI terms is zero since there are no $U(1)$ factors in the gauge group, and the number of mass parameters equals $n + 1$.

B-model: The dimension of the Coulomb branch is the rank of $U(k)^4 U(2k)^{n-3}$, thus $d_V = 2k(n-1)$. The dimension of the Higgs branch is the dimension of the hypermultiplet content $(k+4(2k^2)+(n-4)(4k^2))$ minus the dimension of the gauge group $(4k^2+(n-3)4k^2)$, thus $d_H = k$. The number of FI terms is $n + 1$, while the number of mass parameters $n_m = (n + 1) - (n + 1) = 0$. Altogether, we have the following table:

Model	d_V	d_H	n_ζ	n_m
A	k	$2k(n-1)$	0	$n+1$
B	$2k(n-1)$	k	$n+1$	0

Table 10: The dimension of the Coulomb and Higgs branches and the number of mass and FI parameters of A and B models

The counting shows that we have the required symmetry under A-model $\leftrightarrow$ B-model, $d_V \leftrightarrow d_H$ and $n_\zeta \leftrightarrow n_m$.

7.2 A model - One-loop Corrections

In this section we compute the one-loop corrections to the metric on the Coulomb branch of the A model with $Sp(k)$ gauge group, one hypermultiplet in the antisymmetric representation and n hypermultiplets in the fundamental representation.

Let us parametrize the scalars that minimize the potential energy (2.1) of A model by

$$\vec{\phi} = diag[\vec{r}_1, -\vec{r}_1, ..., \vec{r}_k, -\vec{r}_k] \;, \tag{7.1}$$

where as before $\vec{\phi} = (\phi^1, \phi^2, \phi^3)$.

The one-loop corrected metric of the Coulomb branch of A model takes the form

$$
\begin{aligned}
g_{aa} &= \frac{1}{e^2} - \sum_{b\neq a}^{k}\left(\frac{1}{|\vec{r}_a-\vec{r}_b|}+\frac{1}{|\vec{r}_a+\vec{r}_b|}\right) - \frac{2}{|\vec{r}_a|} + \sum_{i=1}^{n}\left(\frac{1}{2|\vec{r}_a-\vec{m}_i|}+\frac{1}{2|\vec{r}_a+\vec{m}_i|}\right) \\
&+ \sum_{b\neq a}^{k}\left(\frac{1}{|\vec{r}_a-\vec{r}_b+\vec{m}_{as}|}+\frac{1}{|\vec{r}_a+\vec{r}_b+\vec{m}_{as}|}+\frac{1}{|\vec{r}_a-\vec{r}_b-\vec{m}_{as}|}+\frac{1}{|\vec{r}_a+\vec{r}_b-\vec{m}_{as}|}\right) \\
g_{ab} &= \frac{1}{|\vec{r}_a-\vec{r}_b|}+\frac{1}{|\vec{r}_a+\vec{r}_b|}+\frac{2}{|\vec{r}_a|}+\frac{2}{|\vec{r}_b|} \qquad\qquad a\neq b \\
&- \left(\frac{1}{|\vec{r}_a-\vec{r}_b+\vec{m}_{as}|}+\frac{1}{|\vec{r}_a+\vec{r}_b+\vec{m}_{as}|}+\frac{1}{|\vec{r}_a-\vec{r}_b-\vec{m}_{as}|}+\frac{1}{|\vec{r}_a+\vec{r}_b-\vec{m}_{as}|}\right) \quad (7.2)
\end{aligned}
$$

As for the $U(k)$ gauge group case, in order to compute the one-loop correction one need only consider all possible one-loop diagrams with two gauge fields on the external legs and a vector multiplet or hypermultiplet running in the loop. Reduction in the number of colors k and flavors n imply that the all coefficients of the different diagrams are independent of k, n. The hyperkähler properties of the metric (4.4) implies that the contributions of the vector multiplet and the antisymmetric hypermultiplet to the diagonal and off diagonal elements of the metric are of opposite sign and the same absolute value, and that the hypermultiplets in the fundamental can contribute only to the diagonal terms of the metric. We then make use of the fact that $Sp(1)$ yields the $SU(2)$ case. For $SU(2)$ the antisymmetric representation is trivial. Taking the number of fundamentals to be zero fixes the coefficient of the vector multiplet contribution, while the case of two massless fundamentals fixes the coefficient of the contribution of the fundamental hypermultiplets. Finally, the coefficient of the antisymmetric hypermultiplet contribution is fixed by reading from the Lagrangian its relation to that of the fundamental hypermultiplets.

Consider the case where the mass of the antisymmetric hypermultiplet vanishes and we have $n > 1$ massless fundamentals. In this case the one-loop metric describes the k-symmetric product of non-resolved ALE spaces of D_n type X_{D_n} [1]

$$\mathcal{M}_V^{One-loop}(\mathrm{A-model}, \vec{m}_{as}=0, \vec{m}_{fund}=0) = Sym^k X_{D_n} \ . \quad (7.3)$$

The one-loop result is expected to be exact in these cases since the metric corresponds to a product of k copies of the moduli space for $SU(2)$ where there are no higher loop or monopole corrections for $n > 1$ [2]. Thus we conclude that

$$\mathcal{M}_V^{Exact}(\mathrm{A-model}, \vec{m}_{as}=0, \vec{m}_{fund}=0) = Sym^k X_{D_n} \ . \quad (7.4)$$

[1]For $n = 0$ and $n = 1$ we get, after including the one loop and the monopole corrections, the k-symmetric product of an Atiyah-Hitchin space and its simply connected double cover respectively.

This is exactly the Higgs branch of the B-model when all the FI parameters are set to zero [6]. Consider now the inclusion of masses for the fundamental hypermultiplets while still setting the mass of the antisymmetric hypermultiplet to zero. This case still corresponds a product of k copies of the $SU(2)$ case. However when the fundamental hypermultiplets are massive the metric is no longer positive definite and we expect that there will be monopole corrections. The masses of the hypermultiplets resolve the ALE singularities and we expect

$$\mathcal{M}_V^{Exact}(\mathrm{A-model}, \vec{m}_{as} = 0, \vec{m}_{fund} \neq 0) = Sym^k \widetilde{X}_{D_n} , \tag{7.5}$$

which is in agreement with the Higgs branch of the B-model [6] when the weighted sum (trace) of the FI parameters vanishes. As we said above, the one-loop metric (7.2) for massless antisymmetric and massive fundamentals the metric in not positive definite, which indicates that indeed there are monopole corrections that contribute and make the metric positive definite. In this case the metric is similar to (4.3) and the mechanism of resolving the quotient singularities by adjoint mass there is like the mechanism of resolving the D_n singularities by fundamental masses: In both cases there are monopole corrections.

7.3 The Mirror Map

The mass of the antisymmetric multiplet is expected to correspond to the resolution of the quotient singularities of the symmetric product in (7.4) and (7.5). The weighted sum of the FI parameters in the B-model (3.2) resolves these singularities [6]. The reason for the weights can be traced to the equations defining the hyperkähler quotient construction of the Higgs branch of the B-model [6]. The analogue of equations (4.12) and (4.13) contain in our case two types of matrix equations: Those of size $k \times k$ that correspond to the $U(k)$ nodes of the quiver diagram in figure 2 and those of size $2k \times 2k$ that correspond to the $U(2k)$ nodes. The parameter for the resolution of the symmetric product is [6]

$$Tr[\vec{\zeta}_0 \mathbf{1}_{k\times k} + \vec{\zeta}_1 \mathbf{1}_{k\times k} + \sum_{l=2}^{n-2} \vec{\zeta}_l \mathbf{1}_{2k\times 2k} + \vec{\zeta}_{n-1} \mathbf{1}_{k\times k} + \vec{\zeta}_n \mathbf{1}_{k\times k}] , \tag{7.6}$$

which, using similar argument as in the $U(k)$ case, we identify up to an overall constant with $\vec{m}_{as}$ in (3.2).

In order to derive the mirror map for the masses of the hypermultiplets in the fundamental representation we use the same reasoning that led to (4.28). In the Higgs branch of the B model we expect singularities whenever a linear combination of the FI parameters corresponding to a positive root of D_n vanishes. In the Coulomb branch of the A-model we expect, from the one-loop metric (7.2), such singularities to appear when $\vec{m}_i = \pm\vec{m}_j$[1]. Requiring that these singularities match yield the following identification

$$\vec{m}_i - \vec{m}_{i+1} = 2\vec{\zeta}_i, \qquad i = 1, ..., n \qquad \vec{m}_{n-1} + \vec{m}_n = 2\vec{\zeta}_n \ , \tag{7.7}$$

where $\vec{m}_1 = \vec{\zeta}_1$. Equations (7.7) are consistent with the mirror map (3.2). One can extend the singularity analysis to include the mass of the antisymmetric hypermultiplet, and recover the complete mirror map (3.2) that way.

As a further consistency check we repeated part of the analysis of section 5, namely verifying that the change of the dimension of the hypermultiplet moduli space when turning masses in the A-model matches the change of the dimension of the vector multiplet moduli space of the B-model when turning on the corresponding FI parameters.

7.4 D-brane Picture

We have argued in the previous section that in the A-model the masses of the n fundamental hypermultiplets resolve the D_n singularities in the Coulomb branch while the mass of the antisymmetric hypermultiplet resolves the quotient singularities associated with the symmetric product (7.4). In this section we show that this scenario is expected from string theory viewpoint.

It has been suggested that D-branes can be used to probe the space-time geometry and the background gauge fields [31–38]. In particular, enhanced gauge symmetry in the space-time theory is reflected in the D-brane world volume theory by enhanced global symmetry.

Consider a type I string theory on $R^7 \times T^3$, and k D5-branes wrapping the T^3 and yielding k 2-branes in R^7. When the k branes coincide the world volume theory has an $Sp(k)$ gauge group [39]. The matter fields consist of 16 hypermultiplets in the fundamental representation of the gauge group arising from the DN sector and one hypermultiplet in the antisymmetric representation from the DD sector. This is precisely the field content of the A-model of the previous section. The mass terms for the fundamental hypermultiplets arise from the Wilson lines around T^3. Thus, breaking the $SO(32)$ space-time gauge group by the Wilson lines corresponds to breaking the $SO(32)$ global symmetry on the

[1] It is worth to note that we do not expect a singularity when a single mass $\vec{m}_i \to 0$: The role of a single mass parameter is to deform but not to resolve a singularity. For instance, for the gauge group $SU(2)$ with one massless hypermultiplet the Coulomb branch is the double cover of the Atiyah-Hitchin manifold which is smooth. When turning a mass term for the hypermultiplet we get a deformation to the Dancer manifold [30].

world volume of the brane by masses of the fundamental hypermultiplets. For massless hypermultiplets the Higgs branch of the world volume theory is the moduli space of $SO(32)$ k-instantons which is the Higgs branch of the A-model.

Consider now the antisymmetric hypermultiplet ϕ_{as}. When its mass is zero it can get a generic vev of the form

$$\langle \phi_{as} \rangle = \begin{pmatrix} 0 & s_1 & & & \\ -s_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & s_k \\ & & & -s_k & 0 \end{pmatrix}_{2k \times 2k} , \tag{7.8}$$

where s_i consists of four real components. This vev breaks the gauge group to $Sp(1)^k$ and thus separates the k coinciding branes. The Coulomb branch of each brane can be determined using the duality between M-theory on $R^7 \times K_3$ and Type I or heterotic string on $R^7 \times T^3$ [1]. Under this duality the the type I five brane wrapping the T^3 is mapped to the M-theory 2-brane whose world volume is $R^3 \times \{pt \in K_3\}$, which implies that its Coulomb branch is K_3. The precise Coulomb branch in our case is an ALE space of D_{16} type. In order to derive that in this context one has to keep track of the precise duality map. The Coulomb branch for k separated branes is the product of the Coulomb branches for each brane modded by the action of the Weyl group which permutes them. Consequently, we get the k-symmetric product of the Coulomb branch of a single brane. This is consistent with the field theory picture for $\vec{m}_{as} = 0$ (7.4) and (7.5).

In order to have a massive antisymmetric hypermultiplet we need to modify the stringy scenario, so that $\vec{m}_{as}$ will arise as a parameter of the string theory picture. If such a stringy picture exists, and if the mass of the antisymmetric hypermultiplet is different from zero, it cannot get a vev. Thus we see that the k branes cannot be separated and we expect that the Coulomb branch will become the Hilbert scheme of k points on an ALE space of D_{16} type. It would be interesting to verify this explicitly in string theory.

8 Duality for $U(k)^n$ Gauge Theories

In this section we study the third proposed family of dualities for $U(k)^n$ gauge theories. We provide the counting evidence for this duality proposal and study the Higgs and mixed branches of the dual theories. Finally, we briefly discuss the mirror map.

8.1 Counting Evidence

First, we can count the dimensions of the Coulomb and Higgs branches, as well as the number of masses and Fayet-Iliopoulos parameters, as we did previously. The moduli space of vacua of the theory $(U(k)^n; \{v_i\})$ contains a Coulomb branch, with unbroken gauge group $U(1)^{nk}$. In contrast to the case $n = 1$, for $n > 1$ this is a pure Coulomb branch, and not a mixed branch. The moduli space of vacua also contains a pure Higgs branch (unless $\sum v_i = 1$), that is described by a hyperkähler quotient. The quaternionic dimension of this hyperkähler quotient equals $(nk^2 + \sum kv_i - nk^2) = k \sum v_i = km$.

The number of mass parameters equals the number of irreducible representations of the gauge group $(n + \sum v_i)$ minus the number of $U(1)$'s in the gauge group (n), leading to a total of $\sum v_i = m$. Finally, the number of FI parameters is equal to the number of $U(1)$ factors in the gauge group, which is n. These results can be summarized in the following table

Model	d_V	d_H	n_ζ	n_m
$(U(k)^n; \{v_i\})$	nk	mk	n	m
$(U(k)^m; \{w_i\})$	mk	nk	m	n

Table 11: The dimension of the Coulomb and Higgs branches and the number of mass and FI parameters of A and B models

where $\sum v_i = m$ and $\sum w_i = n$.

Again, the counting shows that we have a symmetry under A-model $\leftrightarrow$ B-model, $d_V \leftrightarrow d_H$ and $n_\zeta \leftrightarrow n_m$, in accordance with the duality proposal.

8.2 Mixed Branches

As a further check of the conjecture we will now consider some of the mixed Coulomb/Higgs branches that both theories posses in their moduli space of vacua, restricting our attention to the case where the masses and FI parameters vanish. Such mixed branches appear when we restrict the vev's of the scalars that parametrize the Coulomb branch in such a way that some of the matter fields become massless, and can acquire a nonzero expectation value. Their expectation values parametrize a hyperkähler quotient, the group being that piece of the unbroken gauge group under which the massless matter fields are charged.

The global geometry of such mixed branches can be quite complicated, as the Coulomb branch can receive quantum corrections, but we expect in general that the mixed branches have the structure of a fiber bundle whose fiber is described by a hyperkähler quotient. In any case we will here only count the dimensions of some of the mixed branches, and not consider their global structure.

When analyzing such mixed branches, it may happen the the hyperkähler quotient corresponds to a case where the group does not act properly on the hyperkähler manifold, and the quotient is singular. We will be mainly interested in the case where G acts nowhere properly, so that part of the gauge group is unbroken and we are dealing with a mixed branch. Consider such a case and denote the hyperkähler manifold by M and the group by G. Since G does not act properly, at every $p \in M$ there is a nontrivial subgroup G_p of G that leaves p invariant. The submanifold M_{G_p} of points $q \in M$ such that $G_q = G_p$ is properly acted upon by the centralizer $Z_{G_p}(G)$ of G_p in G (which is the broken part of the gauge group), and in addition M_{G_p} is hyperkähler. Therefore we can take the hyperkähler quotient of M_{G_p} with respect to $Z_{G_p}(G)$, and the result is one of the smooth strata of the hyperkähler quotient M/G of M with respect to G. By varying G_p, we obtain in this way all the strata of M/G, and two G_p's related by conjugation give rise to the same stratum.

Let us now analyze the mixed branches of the theory $(U(k)^n; \{v_i\})$. We impose constraints on the vev's of the scalars that parametrize the Coulomb branch in such a way that the vev a appears $n^i(a)$ times in the scalars coming from the ith gauge group in $U(k)^n$. This puts us on a submanifold of dimension $\sum_{i,a} \delta_{n^i(a)\neq 0}$ of the Coulomb branch, with unbroken gauge group $\otimes_{i,a} U(n^i(a))$. The number of massless matter fields on this submanifold equals $\sum_i v_i n^i(0) + \sum_{i,a} n^i(a) n^{i+1}(a)$. The Higgs branch $\mathcal{M}_H$ over this submanifold of the Coulomb branch is given by a direct product of hyperkähler quotients. If we denote each hyperkähler quotient by its corresponding quiver diagram, we have explicitly

$$\mathcal{M}_H = (\otimes_i U(n^i(0)), \{v_i\}) \times \prod_{a\neq 0} (\otimes_i U(n^i(a)), \{0,\ldots,0\}) \tag{8.1}$$

Now it will in general happen that the groups in (8.1) do not act properly. Then we are in the situation of the previous paragraph, and we have to specify a broken gauge group to describe a stratum of the hyperkähler quotient. Although more exotic possibilities are possible, a typical broken gauge group could be $\otimes_{i,a} U(t^i(a))$, and we will restrict our attention to this type. The centralizer of this subgroup is $\otimes_{i,a} U(n^i(a) - t^i(a))$, and one easily sees that the stratum associated to it is a dense submanifold of the hyperkähler quotient

$$\mathcal{M}'_H = (\otimes_i U(n^i(0) - t^i(0)), \{v_i\}) \times \prod_{a \neq 0} (\otimes_i U(n^i(a) - t^i(a)), \{0, \ldots, 0\}) \tag{8.2}$$

Each of the factors $(\otimes_i U(n^i(a) - t^i(a)), \{0, \ldots, 0\})$ has the property that the quaternionic dimension of the manifold is smaller than or equal to then dimension of the group, and therefore the group cannot act properly, unless the quotient has zero dimension. This implies that for $a \neq 0$ the only consistent choice is $n^i(a) = t^i(a)$, and that we can forget this part of $\mathcal{M}'_H$, leaving us with

$$\mathcal{M}'_H = (\otimes_i U(n^i(0) - t^i(0)), \{v_i\}) \tag{8.3}$$

By assumption, the group $\otimes_i U(n^i(0) - t^i(0))$ acts properly almost everywhere. But this Higgs branch already emerges over a bigger submanifold of the Coulomb branch: if we choose $n^i(0)' = n^i(0) - t^i(0)$ and all other $n^i(a)$ arbitrary, we will still encounter $\mathcal{M}'_H$ as Higgs branch.

Ignoring the possibility of different types of broken gauge groups, this leads to the following picture. If we choose integers k_i in such a way that $\otimes_i U(k_i)$ acts properly in $(\otimes_i U(k_i); \{v_i\})$, then associated to $\{k_i\}$ is a mixed branch in the moduli space of vacua, where we restrict the vevs of k_i of the k scalars coming from the ith $U(k)$ to vanish, and keep the others arbitrary. The dimensions of these mixed branches are

$$(d_V, d_H) = (\sum_{i=0}^{n-1} (k - k_i), \sum_{i=0}^{n-1} k_i v_i - \frac{1}{2} \sum_{i=0}^{n-1} (k_{i+1} - k_i)^2). \tag{8.4}$$

If our duality conjecture is to be correct, we should be able to find similar mixed branches, with d_V and d_H interchanged, in the dual theory $(U(k)^m; \{w_i\})$. We do not know precisely which sets of integers $\{k_i\}$ appear in (8.4), but the results of section 5.1.2 and section 5.2.2 suggest that the integers have to obey the "convexity" condition

$$2k_i - k_{i-1} - k_{i+1} \leq v_i. \tag{8.5}$$

We have not yet completely solved the problem of finding a mixed branch in the B-model for each solution of (8.5), but luckily we can show a correspondence in a large class of examples which is already remarkable in itself.

As our example, we take the theory $(U(k)^n; \{v_i\})$, with $v_i > 0$ for each i, and we impose a requirement on the integers k_i that is stronger than (8.5), namely that $2k_i \leq v_i$ for each i. For each such choice we can indeed find a corresponding mixed branch in $(U(k)^m; \{w_i\})$, as we now describe. The w_i satisfy $w_{v_0} = w_{v_0+v_1} = \ldots = 1$, and all other

w_i vanish. Mixed branches in the B-model are given by integers l_i, $i = 0, \ldots, m-1$. Let us choose them as follows

$$l_{v_0+v_1+\ldots+v_{i-1}+p} = k - \min(k_{i-1} + p, k_i, k_{i+1} + v_i - p), \qquad p = 0, \ldots, v_i. \tag{8.6}$$

Note that the integers l_i also satisfy the convexity condition (8.5). One can compute that

$$d_V = \sum_{i=0}^{m-1} (k - l_i) = \sum_i (k_i v_i - \frac{1}{2}(k_{i+1} - k_i)^2) \tag{8.7}$$

and that

$$d_H = \sum_{i=0}^{m-1} (w_i l_i - \frac{1}{2}(l_{i+1} - l_i)^2) = \sum_{i=0}^{n-1} (k - k_i). \tag{8.8}$$

These numbers are indeed respectively equal to d_H and d_V of the dual theory. The mixed branches that may be relevant to study this duality from the point of view of T-duality and extremal transitions in string theory, correspond to taking all k_i equal to a fixed k_0, and $l_i = k - k_0$. To summarize the results for the mixed branches:

Model	mixed branch	d_V	d_H
$(U(k)^n; \{v_i\})$	$\{k_i\}$	$\sum(k - k_i)$	$\sum(k_i v_i - \frac{1}{2}(k_{i+1} - k_i)^2)$
$(U(k)^m; \{w_i\})$	$\{l_i\}$	$\sum(k_i v_i - \frac{1}{2}(k_{i+1} - k_i)^2)$	$\sum(k - k_i)$

Table 12: The dimensions of the Coulomb and Higgs mixed branches

Let us give one explicit example: The theory $(U(k)^3, \{2,6,2\})$ has a mixed branch with $k_1 = 1, k_2 = 2, k_3 = 1$, with dimension $(d_V, d_H) = (3k - 4, 15)$. The dual theory is $(U(k)^{10}, \{1,0,1,0,0,0,0,0,1,0\})$, and according to (8.6) the corresponding mixed branch in the dual theory should have $\{l_i\} = \{k-1, k-1, k-1, k-2, k-2, k-2, k-2, k-2, k-1, k-1\}$. Indeed, the dimensions of this mixed branch are $d_V = \sum(k - l_i) = 15$ and $d_H = \sum_i (l_i w_i - \frac{1}{2}(l_{i+1} - l_i)^2) = 3k - 4$, in accordance with the duality conjecture.

The analysis of the mixed branches provides a highly nontrivial check on the consistency of the proposed duality. The check might be improved even further if one could demonstrate a one-to-one correspondence between k_i satisfying (8.5) and l_i satisfying a similar condition. Also, it would be very interesting to incorporate masses and FI parameters in the discussion, and to try to derive the mirror map as in section 5. Right now, the evidence we have for the mirror map (3.4) is based on an analysis of the singularities. According to theorem 2.8 in [7], singularities in the Higgs branch of the A-model will appear if $\sum n_l \vec{\zeta}_l = 0$, where the n_l are nonnegative integers that satisfy $\sum_i n_i(n_i - n_{i-1}) \leq 2$

and $n_i \leq v_i$. The singularities in the Higgs branch appear whenever there is an unbroken gauge group. In the B-model, one can analyze for which masses one expects a singularity in the Coulomb branch and the appearance of flat directions for the hypermultiplets. The result is a generalization of (5.24). Requiring that there is a one-to-one correspondence between these sets of singularities in the A-model and B-model, one ends up with the mirror map given in (3.4).

9 Discussion

Quiver diagrams provide a natural framework for the study of the mirror phenomena in three dimensional gauge theories. In this paper we studied three families of mirror gauge theories based on unitary and symplectic gauge groups. All these theories contain matter hypermultiplets in various representations. In the absence of matter there is no Higgs branch but nevertheless it is still natural to ask whether our results can shed light on the Coulomb branches of the pure gauge theories, and in fact they do.

Consider first a $U(k)$ gauge theory without hypermultiplets in the fundamental representation but with one massless adjoint hypermultiplet. Being an $N = 8$ supersymmetric gauge theory, the Coulomb branch does not receive quantum corrections and is therefore the hyperkähler manifold $Sym^k(\mathbf{C}^* \times \mathbf{C})$ [1]. Upon adding a mass to the adjoint the Higgs branch is lifted and we expect the quotient singularities to be resolved and the moduli space to become the Hilbert scheme of k points on $\mathbf{C}^* \times \mathbf{C}$. However we should be cautious since the fact that the non-compact space $\mathbf{C}^* \times \mathbf{C}$ is hyperkähler does not guarantee that $Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})$ is hyperkähler too. Keeping this point in mind we decouple the adjoint by sending its mass to infinity. The structure of the moduli space metric, for which we can gain some understanding from the one-loop calculation (4.3), suggests that the limit $\vec{m}_{adj} \to \infty$ scales the metric in such a way that we probe only a small open subset of the Hilbert scheme. This may well still be the same algebraic variety as $Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})$ if it has a hyperkähler quotient construction, similar to the one fundamental hypermultiplet case, since we expect such a construction to be scale invariant[2]. Thus we propose that the moduli space is either $Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})$ or some subset of it.

The moduli space for an $SU(k)$ gauge theory without matter hypermultiplets follows from the above since the $U(1)$ and the $SU(k)$ parts of the $U(k)$ gauge theory decouple in the absence of matter. This indicates that the moduli space of the $SU(k)$ gauge theory

[1]Recall that $\mathbf{R}^3 \times \mathbf{S}^1 \cong \mathbf{C}^* \times \mathbf{C}$.

[2]There will be only one parameter in such a hyperkähler quotient construction which is the mass of the adjoint hypermultiplet $\vec{m}_{adj}$ and we can scale the algebraic equation and absorb the scale of $\vec{m}_{adj}$.

is $Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})$ modded by $\mathbf{C}^* \times \mathbf{C}$, or some subset of it. It is curious to note that moduli space of pure $SU(2)$ k-monopoles, which have been proposed in [4] as the moduli space of pure $SU(k)$ gauge theory is an open subset of this space [40]. It is clear however, that in order to correctly identify the moduli space we need a better understanding of the quantum and monopole corrections.

Consider now an $Sp(k)$ gauge theory without hypermultiplets in the fundamental representation and with one massless antisymmetric hypermultiplet. The Coulomb branch is the symmetric product of Atiyah-Hitchin spaces, each of which we denote by X_{AH}. Upon adding a mass term for the antisymmetric hypermultiplet, the Higgs branch is lifted, and we expect to resolve the quotient singularities of the moduli space and get the Hilbert scheme of k points on the Atiyah-Hitchin space. Again, it is not guaranteed that this space is hyperkähler and we may need a suitable subset of it. Decoupling the antisymmetric hypermultiplet will scale the metric in a similar manner as in the $U(k)$ case.

The following table summarizes this discussion:

Gauge Group	$\mathcal{M}_V$
$U(k)$	$Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})$
$SU(k)$	$Hilb^{[k]}(\mathbf{C}^* \times \mathbf{C})/(\mathbf{C}^* \times \mathbf{C})$
$Sp(k)$	$Hilb^{[k]} X_{AH}$

Table 13: The proposed moduli spaces in the absence of matter

There are several natural directions for future studies. From a field theory viewpoint it is important to understand the role of the monopole corrections to the metric on the Coulomb branch and in particular the mechanism by which it resolves singularities. It is also interesting to explore the D-brane wrapping mechanism that corresponds to the monopole corrections. From a string theory viewpoint it would be important to further explore the stringy origin of the mirror phenomena and the mirror map. In particular it would be interesting to uncover the role played by the moduli space of D-branes that exists in the wrapping picture.

The detailed study of the moduli space of vacua exhibits a rich structure of mixed branches and possibly non-trivial RG fixed points which is worth exploring. We expect that other dual quiver diagrams exist which encode the field data for other families of mirror gauge theories and it would be interesting to find them.

Besides being a remarkable rich structure in its own right, we believe our results will be useful in obtaining a better understanding of non-perturbative effects in type II string compactifications, the physics of small instantons, monopole corrections in three dimensions and (possibly non-trivial) IR fixed points.

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