Title
Editorial introduction to the special issue “Plasma physics in the 20th century as told by players”

Journal
The European Physical Journal H, 43(4-5)

ISSN
2102-6459

Authors
Diamond, Patrick H
Frisch, Uriel
Pomeau, Yves

Publication Date
2018-12-01

DOI
10.1140/epjh/e2018-90061-5

Peer reviewed

Accepted manuscript
Published as:
Eur. Phys. J. H. (2018)
https://doi.org/10.1140/epjh/e2018-90061-5

Editorial introduction to the special issue "Plasma physics in the 20th century as told by players"

Patrick Diamond*
University of California San Diego, La Jolla, CA 92093-0319, USA

Uriel Frisch
Université Côte d'Azur, OCA, Lab. Lagrange, CS 34229, 06304 Nice Cedex 4, France

Yves Pomeau
Ladhyx, École Polytechnique, Palaiseau, France
(Dated: October 30, 2018)

Our ancestors lived in a world where — as far as they were aware — electrons and ions lived happily bound together. But with suitable conditions, e.g. low density and/or high temperature., electrons and ions will unbind and we obtain a plasma, a state of matter of which we became increasingly aware in the 20th century, and which pervades our universe near and far. There are also many applications of plasma physics: chip etching, TV screens, torches, propulsion, fusion (through magnetic or inertial confinement), astrophysics and space physics, and laser physics, to cite just a few. At the crossroads of electrodynamics, continuum physics, kinetic theory and nonlinear physics, plasma physics enjoys an abundance of riches. Actually, so much that we can only cover a fraction of its history in this limited volume.

The history of plasma physics presented here covers mostly the period from 1950 to 2000. The issue is focussed both on fundamentals and on applications in controlled fusion through magnetic confinement, which in turn raises a host of fundamental and interesting questions in various areas. However, it is not our intention to assess ongoing projects aiming for sustained energy production by controlled fusion.

Specifically, the activity in the Soviet Union/Russia and beyond is covered by a key player, Roald Sagdeev, interviewed by one of us (PD). The papers by Dominique Escande, by Guy Laval with Denis Pesme and Jean-Claude Adam, and by Akira Hasegawa with Kunioki Mima cover fundamentals and those by Paul-Henri Rebut, by Fritz Wagner and by Mitsuru Kikuchi discuss fusion-oriented applications.

There is also a story to tell of plasma physics and controlled fusion research in China and Korea (R.O.K.), which are both members of ITER (International Thermonuclear Experimental Reactor). By 1984, a medium-sized tokamak, HL-1, was built and in operation at the Southwest Institute of Physics in Chengdu. This program continued with the HL-1M, HL-2A and HL-2M devices, the last currently under construction, and has also involved an active theory and analysis component. An approximately contemporaneous research effort evolved at the Chinese Academy of Sciences Institute of Plasma Physics, in Hefei. This program ultimately designed, built and operated — since 2006 — the large superconducting tokamak EAST (Experimental Advanced Superconducting Tokamak), one of the two large long-pulse, superconducting machines in the world today. Theoretical and fundamental plasma physics in China have had a long tradition at Peking University, the University of Science and Technology of China (USTC) and, more recently, Zhejiang University, in Hangzhou.

In R.O.K., fusion research began in earnest in the mid-1990s, and led to the construction and operation of KSTAR (Korea Superconducting Tokamak Advanced Research) since 2008. KSTAR is also a large superconducting long-pulse tokamak. There is an established school of plasma theory and fundamental plasma physics at Seoul National University and KAIST (Korea Advanced Institute of Science and Technology).

As to the US contribution to plasma research, for which many references are available, it

*Electronic address: pdiamond@ucsd.edu

is covered in Section IV on supplemental reading material.

Of course, plasma physics was not born in a conceptual desert. The reader will thus also find below three short essays covering the early history of plasma physics in the first half of the 20th century. These essays, written under the responsibility of the guest editors, are about the ionosphere (Section I), gaz discharges and Langmuir (Section II) and the birth of nuclear fusion (Section III).

Finally, we provide a list of supplemental reading material (Section IV).

I. THE HEAVISIDE LAYER AND THE INCEPTION OF IONOSPHERIC PLASMA PHYSICS

(For further information: pdiamond@ucsd.edu)

What ambush lies beyond the heather
And behind the Standing Stones?
Beyond the Heaviside Layer
and behind the smiling moon?

T.S. Eliot, in *The Family Reunion*

A. Challenge

The age of wireless communication — in which we are ever so strongly enmeshed today — began on May 13, 1897, when Guglielmo Marconi transmitted the Morse code message "Are you ready?" over the open sea of the Bristol Channel from Flat Holm Island to Lavernock Point, in Penarth [1]. Earlier test transmissions had been over land, covering very short distances, only. Marconi soon sought to compete against undersea telegraph cables — in use at that time — by exploiting his wireless technology. Thus, the first transatlantic wireless transmission soon occurred on December 12, 1901, when J. A. Fleming — working with Marconi — propagated a wireless signal from Poldhu, in Cornwall, UK, to St. Johns, Newfoundland [2] now in Canada. The precise wavelength used for this first transatlantic transmission is unknown [3] but thought to be approximately 2000 meters. The challenge of accounting for the fact of propagation in the presence of a wall of seawater roughly 100 miles high immediately killed any putative explanation based on plane wave transmission. Lord Rayleigh [2] suggested the use of diffraction theory to explain how the wave might follow the curvature of the Earth. A flurry of analyses followed, and is briefly discussed below. These erudite diffraction calculations merely confirmed that most of the transmitted energy would radiate away into space.

B. Concept

Arthur E. Kennelly [4] and Oliver Heaviside [5] had other, newer ideas. They quickly responded to the challenge of explaining long-range transmission by independently proposing the existence of an electrically conducting layer in the upper atmosphere of the Earth. This layer, together with the ocean surface, was suggested to form an effective guide in which the transmitted wave propagated. The hypothesized conducting layer became known as "the Heaviside Layer", and later as the E-Layer of the ionosphere. It should be noted that some speculations concerning such a high-altitude conducting layer had been previously advanced in the course of study of anomalous geomagnetic variations [6].

Kennelly, an electrical engineer who, for a time, worked with Thomas Edison, exploited existing measurements of the conductivity of air at low pressures made by J. J. Thomson [3]. Kennelly noted [3]:

> On waves that are transmitted but a few miles the upper conducting strata of the atmosphere may have but little influence. On waves that are transmitted, however, to distances that are large by comparison with 50 miles, it seems likely

> that the waves may also find an upper reflecting surface in the conducting rarefied strata of the air. It seems reasonable to infer that electromagnetic disturbances emitted from a wireless sending antenna spread horizontally outwards, and also upwards, until the conducting strata of the atmosphere are encountered, after which the waves will move horizontally outward in a 50-mile layer between the reflecting surface of the ocean beneath, and an electrically reflecting surface, or successive series of surfaces, in the rarefied air above.

Kennelly also commented that propagation in this (effectively) cylindrical waveguide should significantly reduce signal attenuation, relative to the case of free space. He wrote: "If the attenuation is found to be nearly in simple proportion to the distance, it would seem that the existence of the upper reflecting surface could be regarded as demonstrated."

Also in 1902, in an article on the "Theory of Electric Telegraphy" in Encyclopedia Britannica, Heaviside — in the course of a discussion on how curved transmission line pairs can effectively guide a wave around a corner — observed [5]:

> There is something similar in "wireless" telegraphy. Sea water, though transparent to light, has quite enough conductivity to make it behave as a conductor for Hertzian waves, and the same is true in a more imperfect manner of the earth. Hence the waves accommodate themselves to the surface of the sea in the same way as waves follow wires. The irregularities make confusion, no doubt, but the main waves are pulled round by the curvature of the earth, and do not jump off. There is another consideration. There may possibly be a sufficiently conducting layer in the upper air. If so, the waves will, so to speak, catch on to it more or less. Then the guidance will be by the sea on one side and the upper layer on the other. But obstructions, on land especially, may not be conducting enough to make waves go round them fairly. The waves will go partly through them.

Heaviside, whose scientific career was devoted to the study of electromagnetics and telegraphy, evidently was motivated by analogy with transmission over a pair of wires. He saw that in the case of transatlantic wireless propagation, the ocean surface might correspond to one wire, while the hypothetical high-altitude conducting atmospheric layer might account for the second. That high-altitude layer came to bear his name. Oliver Heaviside (b. 1850–d. 1925) has been referred to — along with George Fitzgerald, Oliver Lodge and Heinrich Hertz — as one of the "Maxwellians" [7], a group of physicists who transformed Maxwell's difficult and unwieldy early theory of electromagnetism into the elegant and familiar form we use today. Heaviside's career spanned and was immersed in the era of transatlantic cable telegraphy, to which William Thomson (Lord Kelvin) and other notable physicists made major contributions. Heaviside developed what is known today as the "telegraph equation". He is sometimes referred to as "Heaviside the Telegrapher".

C. Confusion

The clever suggestions by Kennelly and Heaviside were at first ignored. Instead, the theory community of the time focused on attempting to explain the puzzle of wireless transmission by application of the theory of diffraction and on the validity of those analyses. There were attempts by Lord Rayleigh [8], H. M. MacDonald [9], Henri Poincaré [10], A. Sommerfeld [11] and others. None of these efforts succeeded, except to bury the proposal that diffraction was responsible for long-range transmission. Indeed, Poincaré later pointed out an error in McDonald's calculation and suggested reflection from the upper atmosphere as an explanation [10, 12]. Joseph Larmor succinctly stated the key point in his paper of 1924 [13], in which he wrote:

> In fact, in a single medium, propagating waves without absorption or dispersion, all features remain similar when the scales of space and time are altered in the same ratio: thus waves of length λ bending round the earth of radius a will behave similarly to waves of length $k\lambda$ bending round a smaller sphere of like quality of radius ka. To change from electric waves 10^4 cm long to waves of light 10^{-4} cm long, k would be 10^{-8}; hence the radius of the sphere corresponding to

> the earth would be $10^{-8} \times 10^9 / \left(\frac{\pi}{2}\right)$ or 6 cm; and familiar experience indicates that visible light could hardly creep to sensible degree round one-tenth of the circumference of a sphere of that radius.

It seems that the proponents of diffraction had not yet come to grips with the large scales and long wavelengths involved in transatlantic wireless communication. Also, as late as 1909 — on the occasion of his Nobel lecture — Marconi was still advocating ground connection (i.e. ground wave) effects as essential to the explanation [14]. Another reason for the slow acceptance of Heaviside's suggestion of an ionosphere is that for such a case, the phase speed of the electromagnetic wave undergoing internal reflection would exceed that of light. Of course, energy, etc., propagate at the group velocity, which is smaller than c. Understanding this point required a theory of electromagnetic propagation in plasma, which came only in 1924.

D. Clarification

The reader should observe that Kennelly and Heaviside advocated the existence of a conducting layer and not explicitly a plasma of free ions and electrons. The conducting layer notion encountered difficulties when spurious analogies with conduction in metals (with a short mean free path, in marked contrast to the long mean free paths in the ionosphere) were applied. These led to predictions of strong Ohmic dissipation of the wave, which the success of long-range transmission manifestly contradicted.

Joseph Larmor clarified the physics and introduced the idea of the Heaviside Layer-as-plasma in his 1924 paper, "Why Wireless Electric Rays Can Bend Round the Earth" [13]. Larmor first observed:

> The rays could not travel free, except in straight lines, in a medium of practically uniform speed of propagation, such as ordinary air, either they must be guided by being linked to the surface of the earth, much as cylindrical Hertz-ion waves follow a guiding central wire, or else the speed of propagation must for some cause increase notably upwards so as to bend them down.

Larmor also noted:

> The ultra-rarefied upper atmosphere with its long mean free path must be subject to strong ionization by the incident ultra-violet solar radiation, the effect persisting after daytime, and appeal has been generally made to such a conducting layer to provide the necessary increase in the velocity of the rays. But ordinary electric conduction operates by introducing a frictional term causing absorption into equations, and it is a familiar principle that when the absorption is of the first order the change of speed is only of the second order, so that if the latter change is to be adequate, the rays would be damped out immediately instead of being sensibly transmitted.

Thus, Larmor stated:

> To produce sensible bending of the rays without extinction, all action by conductive or other dissipation must thus be excluded. It would not suffice to say that the conductance has to become perfect, for then no waves could travel: the influence must be of the dielectric type. ... A sufficient cause for the increase of velocity, without dissipation, for waves travelling horizontally is, in fact, afforded by the free oscillations of ions even sparsely distributed in the very high regions of the atmosphere; though lower down their energy would be dissipated by collisions with the atoms and travelling waves would be gradually quenched.

He finally concluded:

> There must thus be a layer high almost beyond the sensible atmosphere, within the auroral domain, in which a sheaf of horizontal electric rays, provided they are long enough, will travel without loss by absorption or scattering, concentration

> in this stratum being due to the potent influence exerted on the velocity of long waves by the free ions of small inertia interacting with them. But we have to suppose that the energy transmitted in this high stratum is being shed down to sensible degree all along the path, for the signals to be everywhere received. Introducing optical imagery, we can contemplate a special component ray-sheaf connecting the transmitter to every receiver situated along the path of the beam; these rays all gather together into a sort of caustic curve or layer up aloft, and so long as they are in it the energy travels onward without loss except by sideways spreading.

We see that Larmor had grasped the essential point — namely that of a waveguide defined by a plasma of free charges, where dielectric response, not resistive dissipation, was the key element of the physics of transmission.

Larmor presented several important quantitative results. By elementary consideration of an ionized particle's response to the wave field, he deduced the refractive index μ due to a population density N_0 of charged particles:

$$\mu = \left(1 - \frac{N_0 e^2 \lambda^2}{\pi m}\right)^{1/2} \tag{1}$$

Equation (1) is as written by Larmor, and here μ is the index, N_0 the density of charged particles, λ the wavelength, e the charge and m the mass. Equation (1) is manifestly equivalent to the well-known result for the electromagnetic dispersion relation in a plasma, i.e. $k^2 = \omega^2 \varepsilon(\omega)/c^2$, with $\varepsilon(\omega) = 1 - \omega_p^2/\omega^2$. Here ω_p is the plasma frequency. It should be noted that Larmor's calculation preceded Langmuir's basic paper [15] on plasma oscillations. Clearly, Larmor had already grasped the significance of the plasma frequency $\frac{N_0 e^2}{m} \sim \omega_p^2$ and the physics of plasma oscillations. Noting that transmission requires the ray's radius of curvature to be comparable to (or smaller than) the Earth's radius, Larmor also deduced the required variation of charged particle density with amplitude h to be $\frac{dN_0}{\mathrm{dh}} \sim 3 \times 10^{-6}$. From this, and assuming $h \sim 10^5$ cm (too low!), he estimated that the Heaviside Layer must support a population of at least 5×10^2 hydrogen ions / cm^3. The actual value is much higher.

Larmor's paper of 1924 argued convincingly that the Heaviside Layer was fundamentally a plasma layer, and also set forth the fundamental physics of electromagnetic wave propagation in a plasma. It's a notable achievement in plasma and ionospheric physics, as well as the theory of electromagnetic wave propagation. The paper thus ended a long period of controversy which spanned the years 1902-1924, and established the central place of "magneto-ionic theory" — as the nascent study of plasma physics was then referred to — in the understanding of long-range wireless communication. At that point, only direct experimental verification of the Heaviside Layer remained to be accomplished.

E. Confirmation

The existence of the Heaviside Layer was at last directly confirmed by Edward Victor Appleton and M. A. F. Barnett in 1925 [16]. Appleton successfully exploited the phenomenon of "fading", in which short distance propagation of a shortwave signal degrades. Appleton correctly linked fading to the interference between ground waves and waves scattered downward from the Heaviside Layer. A loop antenna was more effective for detecting such downward indeed, scattered waves than a standard vertical antenna was. Attempts to detect downward propagating waves by using a linear antenna that pointed downward from the horizontal encountered difficulty, likely due to excess interference from the ground wave. Appleton thus detected downward propagating waves (scattered off the Heaviside Layer) thought to contribute to fading. He also observed that the scattered waves were elliptically polarized, in accord with predictions based on the magneto-ionic theory of atmospheric refraction. Appleton also deduced a lower bound of $\sim 10^5$ / cm^3 on the electron density in the Heaviside Layer. It should be noted that in the 1925 paper, Appleton warmly acknowledged his important interactions with Larmor.

Appleton received the 1947 Nobel Prize in Physics for his experimental studies of the ionosphere, as the Heaviside Layer came to be known. The Nobel citation lauded Appleton for his discovery of the F-Layer, at higher altitude than the E-Layer of Heaviside. Appleton's was arguably the first Nobel Prize in plasma physics.

F. Continuation

The theoretical work of Larmor and Appleton's clever experiments ended the early history of the Heaviside Layer — a story which began with the seminal contributions by Marconi, Kennelly and Heaviside. In 1926, Robert Watson–Watt coined the term "ionosphere" in a letter published only in 1969 [17]. This rapidly replaced the term Heaviside Layer.

Research in ionospheric plasma physics continues to the present day.

G. Culture

Paradoxically, though the use of the term "Heaviside Layer" has faded from scientific discourse, it remains firmly entrenched in popular culture through its prominence in Cats, the popular musical by Andrew Lloyd Webber [18], based on Old Possum's Book of Practical Cats [19] — a collection of whimsical poems by Thomas Stearns Eliot (b. 1888–d. 1965). At that time, the Two Cultures were not so distant, and the poet Eliot was aware of, and interested in, new trends and discoveries in science. The term "Heaviside Layer" does not appear in the published Old Possum's but is found in drafts left in Eliot's papers. This was noted by Webber, who used it in his script for Cats. The Heaviside Layer also appears in Eliot's verse play The Family Reunion, a fragment from which is the epigram for this essay.

While there is no precise understanding of Eliot's fascination with the Heaviside Layer, it appears to embody some concept of the unattainable (for mortals) the highest heaven, somewhat similar to Dante's Empyrean [20]. Indeed, in Cats, the story ends when the character Grizabella, who sings the signature hit song "Memory", is ultimately sent upward to the Heaviside Layer, there to be reborn. Perhaps this conveys some sense of the air of mystery which surrounded the early discoveries in wireless transmission?!

Acknowledgement

We thank Rick Salmon for bringing the excellent book — The Maxwellians — by Bruce J. Hunt to our attention.

II. LANGMUIR AND PLASMA PHYSICS

(For further information: yves.pomeau@lps.ens.fr)

The American scientist Irving Langmuir (1881–1957) was one of the great scientists of the Twentieth centurty. He is famous for many discoveries. On the theoretical side he was the first, with Lewis (1875–1946), to understand the shell like structure of electrons surrounding the central nucleus of atoms. He had also the idea, with Ms Blodgett (1898–1979) to deposit monoatomic layers floating on fluids on hard substrates for various purposes. His main contribution to plasma physics is the calculation of the frequency of oscillations of mobile charges (electrons) on a neutralizing background (practically, much heavier ions).

The discovery and understanding of waves in continuous media has a long and remarkable history in physical sciences. It started with Newton's Principia, where the author explained the physics of sound waves in air and showed that their speed is the square root of the compressibility (Newton took the isothermal compressibility, which was changed by Laplace into the — correct — adiabatic compressibility). Newton explained also the propagation of water waves on the surface of liquids. The next step in this march of giants was made by Maxwell who related the propagation of light to electromagnetic phenomena and showed that its speed is given by measurable electromagnetic properties of vacuum. Another step was made in the first half of the twentieth century by Alfvén for waves in magnetized plasmas and by Langmuir who explained the fast oscillations of a plasma of electrons neutralized by a background of ions. This discovery by Langmuir is discussed below.

When he published on plasma, Langmuir was an employee of the General Electric company in the US. A source of interest of this company for plasma physics came from the possible use of glowing gas discharges for lighting purposes. Langmuir had already contributed to the improvement of the efficiency of light bulbs by the idea of coiling Tungsten filaments to increase the temperature and also the stiffness of those filaments. Langmuir is also responsible for giving the name "plasma" to ionized gases: indeed, an ionized gas carries ions, electrons and atoms just as blood plasma carries hematocytes, germs, etc. Langmuir's calculation of the plasma frequency is one of the first thing a student in plasma physics learns. We present below this derivation and comment on it. The linearization is quite often poorly explained (because of the absence of dispersion it is hard to compare the wave amplitude to another length scale). It was also understood by Langmuir that the plasma frequency is a cut-off for the reflection of a wave incident on a conductor: at frequencies below it, the wave is totally reflected and it is not for frequencies above it. For instance the reflection of waves by the ionosphere occurs for frequencies low enough only to trap EM waves, as was understood in the early years of the 20th century when interpreting the successful transmission of radio waves across the Atlantic by Marconi (1874–1937) [12, 21]. Poincaré (1854–1912) having shown that the diffraction (without an upper conducting layer) gives too small amplitudes for long-range propagation across the Atlantic, the only explanation for Marconi's observations was that the waves are trapped between the sea surface and the upper ionized atmosphere (no ionosphere yet at this time) that was also conducting electricity and so reflects electromagnetic waves (see also essay I on the ionosphere).

Langmuir published two papers on plasmas, the first as a single author in the PNAS [15], another one more detailed with Tonks (1897–1971) as coauhor, where he repeats partly the PNAS paper and reports on an experiment where he puts in evidence plasma oscillations at the frequency he just computed, the latter being published in the Physical Review [22].

The Physical Review paper is rather complete and well organized and we shall base on it our discussion of Langmuir's contribution to plasma physics. At the beginning Tonks and Langmuir explain that, in gaseous discharges, the electrons tend to have a kinetic energy that is less than what they should gain from the potential difference between the two electrodes. The density is low enough to make the scattering by other particles negligible as a cooling mechanism. They thus conclude that the electrons leave part of their energy to collective oscillations, the plasma oscillations, and they are going to explain what they are. The calculation outlined is brilliant and very simple. It is done in Lagrangian coordinates, outside of the cumbersome Eulerian framework.

The derivation of Tonks and Langmuir deserves to be introduced, although some minor technical points may be questionable. The starting point is that, without plasma oscillations, the plasma is neutral because immobile ions cancel exactly the charges of the more mobile electrons. The fluctuation is such that the electrons are displaced by a quantity called $\xi(x, t)$, x being the undisturbed position. A non-uniform displacement field yields a modulated charge density

$$\delta n = -n \frac{\partial \xi}{\partial x},$$

where n is the background density. Our minus sign on the right-hand side of this equation is not in Tonks and Langmuir.

As often, when using Lagrangian coordinates in the mechanics of continuous media, care must be taken of the difference between the undeformed coordinates (the one without the disturbance) and the ones after it. It seems that this equation for the density change should have a negative sign on the right. Such a negative sign should be there both because it follows straightforwardly from the algebra and from the physics. The algebra is standard, the physics is that if the stretching $\partial \xi / \partial x$ is positive, then the density is lowered, whence the negative sign. This should be seen, we believe, with the eyes of Jean Cocteau (1889–1963), who claimed that "Chef -d'oeuvres" (translated a bit imperfectly as "masterpieces") are always imperfect because they are so far in advance of their time that they cannot incorporate all developments that occurred after their completion. The Poisson equation yields the electric field $E(x)$ due to this density fluctuation. This relation reads (with the minus sign restored) :

$$\frac{\partial E}{\partial x} = 4\pi e \delta n = -en\frac{\partial \xi}{\partial x},$$

where e is the electric charge of the electron. Integrating and assuming that there is no background charge because the plasma is neutral at rest, one finds

$$E = -en\,\xi(x).$$

The next step is the writing of Newton's equation of motion with the force eE acting on charges. It yields

$$m\frac{\partial^2 \xi}{\partial t^2} = -e^2 n\xi,$$

where m is the electron mass. Whence the equation for plasma oscillations:

$$\frac{\partial^2 \xi}{\partial t^2} + \omega^2 \xi = 0,$$

where $\omega = \sqrt{4\pi n e^2/m}$ is 2π times the plasma frequency. Note that, depending on the way one writes the Poisson equation (and ultimately on the choice of electrical units), one can get coefficients with the vacuum permittivity and without 4π in this expression.

It is worth noticing that the calculation by Tonks and Langmuir was very close to the one by Jeans, published in 1902 [23] on the stability of a self-gravitating fluid. There Jeans looked at the instability under modulation in space of a uniform density of masses in an infinite space. Formally, this problem looks a bit like the one of the plasma oscillations, because the long-range force of interaction decays as $1/r^2$. The main difference is in the integration of the equation for E, the force exerted on a mass by the other masses. In the case of a plasma, because of the assumed compensation of charges between ions and electrons, there is no average force without modulation. This is hardly so in the case of gravitational attraction: the cancelation of the average force due to point masses requires a fine-tuning of the mass distribution to exclude such a force. Assuming that this can be done, the derivation of the dynamics of the small amplitude fluctuations is very close to the Tonks–Langmuir calculation, the result being

$$\frac{\partial^2 \xi}{\partial t^2} - \omega_J^2 \xi = 0,$$

where $\omega_J = \sqrt{4\pi G n m^2}$, G being Newton's constant (taken as positive). This shows that there is an instability, instead of oscillations, fundamentally because masses attract each others gravitationally whereas identical electric charges repel each other. It is of interest to notice that, contrary to the plasma case, a scientifically consistent approach of the dynamics of a self-gravitating gas of stars requires an approach via kinetic theory: there, for an assembly of N stars the short-range collisions changes the velocity distribution by one order less with respect to the small parameter $1/N$ than the mean field effect described by the Vlasov kinetic equation [24]. This Vlasov theory is already non trivial when one looks at the steady states, which can be found in a number of cases thanks to mathematical tricks. The Jeans "frequency" ω_J is also the order of magnitude of the inverse of the period of the orbits of stars (in the case of a self-gravitating cluster of many stars) in the mean field of the others: the square root of the number density $n^{1/2}$ in the expression of ω_J is $N^{1/2}R^{-3/2}$, with R the radius of the self-gravitating cluster. The dependence of the period of the motion with respect to the 3/2 power is the same as the period vs radius dependence of the motion of planets in the third Kepler law (period squared goes like the cube of the orbit radius). The results derived from kinetic theory cannot be extended to the limit of an infinitely large number of stars at constant number density and so to infinitely large R, to obtain the Jeans instability in the fluid limit. In this limit the kinetic and gravitational energy of each star are of the same order of magnitude and so the dynamics of the cluster cannot be derived from a theory where the kinetic effects are neglected as was done, implicitly, by Jeans.

Tonks and Langmuir derive later the dispersion relation of transverse plasma oscillations due to the finite speed of light. They discuss also the effect of the thermal velocity of the electrons on the plasma waves. Their discussion is qualitative and predates Landau's calculation of the kinetic theory of plasma waves. However it brings the interesting remark that electron moving fast enough to go across several wavelength of the plasma oscillations during one period do not contribute to them because they have no net effect on the charge distribution. This a nice qualitative way of looking at the effect of a finite wave speed compared to the thermal velocity. The effect of finite temperature is taken into account, when this temperature is small (making the thermal velocity of electrons far less than the phase velocity of the plasma waves) by a correction to the plasma frequency. This changes the constant plasma frequency of Langmuir into the Bohm–Gross dispersion relation.

The same kinetic theory referred to in the case of gravitational interaction is also relevant for short plasma waves such that the thermal velocity is of the order of or much larger than the phase velocity of the plasma waves computed with Langmuir's frequency: as noticed by Tonks and Langmuir in the short-wave limit the polarization (in modern terms) of the plasma tends to zero. The general theory of oscillations of a collisionless plasma, taking fully into account the thermal velocity is due [25] to Landau (1908–1968) and is covered, including in its nonlinear part, in several papers of this topical issue. It is noticeable that by introducing a finite thermal velocity the frequency of the plasma waves gets an imaginary part, which represents either damping or instability, depending on the shape of the velocity distribution. When the oscillation become unstable this is the Landau instability, the wave energy is pumped out of the kinetic energy of the electrons, which is possible if the velocity distribution satisfies certain conditions. When the velocity distribution is an equilibrium Maxwellian, the plasma oscillations are damped by the Landau mechanism and there is stability. Other kind of waves exist in plasmas. At the lowest frequencies one finds ion acoustic waves which are non dispersive plasma oscillations of the ions where the electrons and the ions oscillate almost exactly together. A variety of waves can propagate in magnetized plasmas, like Alfvén waves and cyclotron waves. Drift waves exist in a plasma of non uniform density. This particularly rich physics has been a topic of study both because of its relevance for fusion devices and because of its interest for geophysics and astrophysics [26].

The rest of the paper of Tonks and Langmuir is about experiments intended to put in evidence oscillations at the plasma frequency. The oscillations observed were in the 100 MHz range. This was at the limit of technical observability at the time (end of the nineteen twenties), even for the Laboratory of the General Electric company where the two authors worked at the time.

III. THE BIRTH OF NUCLEAR FUSION

(For further information: uriel@oca.eu)

Nuclear fusion has a prehistory ranging from 1815 to 1905. In 1815 and 1816, a physician and chemist, William Prout (1785–1850), examining the tables of atomic weights available at the time, hypothesizes that the atomic weight of all elements is an integer multiple of that of Hydrogen, which might be their primordial building block [27, 28]. As the measurements of atomic weights become more precise, there are indications of small discrepancies from Prout's hypothesis.

Charles Darwin (1809–1882), in the first edition (1859) of "On the Origin of Species" takes argument of his geological estimate of 200 million years of age of The Weald (in South-East England) to state that the Earth must be much older than this. Thus, he believes there is enough time for natural selection to operate and produce very complex beings [29].

In a 1862 paper William Thomson (1824–1907), later known as Lord Kelvin, invoking the only mechanism for generating heat in the Sun, known at the time, namely gravitational contraction (proposed by Hermann Helmholtz in a 1854 popular conference), estimates the age of the Sun (and thus of Earth) to be around 20–50 million years [31]. Observe, however that in his 1854 conference, Helmholtz does not shy off estimating the age of the Earth to be in excess of 350 million years [30]. Darwin, who has great respect for Thomson, begins to doubt his own theory and even removes the mention of The Weald from the 3rd edition of his book. (For the Darwin–Kelvin controversy, see Bahcall [32].

As we shall see, some of the small discrepancies to Prout's hypothesis will eventually provide us with the explanation of how stars manage to remain hot for billions of years. But this requires Albert Einstein (1879–1955), who in one of his famous 1905 papers shows that relativity implies the equivalence of mass and energy, related by $E = mc^2$.

The idea of energy release by nuclear fusion goes back to 1913 when Paul Langevin (1872–1946) in a paper in French, titled "The inertia of energy and consequences", proposes an Einsteinian interpretation of the tiny discrepancies to Prout's hypothesis [33]. On page 587 he writes the following (our translation) "The explanation I propose follows immediately of all that precedes: the discrepancies would arise because the formation of atoms from primordial elements (by disintegration, as we see it in radioactivty, *or by an inverse process not yet observed that gives birth to heavy atoms*[1]), would be accompanied by variations of internal energy by emission or absorption of radiation." One of the few scientists to cite this idea of Langevin will be Perrin in an early 1920 paper (see below). Langevin's and Perrin's work is also discussed in detail by Wesemael [36].

In 1919, Jean Perrin (1870–1942) is probably the first to propose that nuclear reaction are responsible for maintaining the stars hot, well beyond what can be accounted for by the Helmholtz–Kelvin contraction theory. No quantitative estimates are given [34]. In February 1920, Jean Perrin returns to the subject he dealt with in his 1919 paper but, this time, he puts the matter in the context of Langevin's interpretation of discrepancies to Prout's law and in particular of the slight mass deficit of Helium compared to four Hydrogens. This allows Perrin to be somewhat more quentitative and he mentions the possibility to maintain the heat of the Sun for billions or even tens of billions of years [35]. In May 1920, Francis Aston (1877–1945) refines the measurement of the discrepancy: using a high-precision (10^{-3}) mass-spectrograph he determines the atomic weight of hydrogen to be 1.008 and that of helium to be 3.99, hence slightly less than one percent of four hydrogens [37]. In October 1920, Arthur Eddington (1882–1944) observes that Aston's mass discrepancy together with Einstein's equivalence of mass and energy leads to the idea that four hydrogens somehow manage to combine to form one helium with much energy release and in a way capable of keeping the Sun warm for 15 billion years [38] (Perrin's early 1920 work will be mentioned by Eddington in his 1926 treatise on the internal constitution of stars [39]). In the 1920 paper, Eddington also writes the following amazingly prescient statement: "If, indeed, the sub-atomic energy in the stars is being freely used to maintain their great furnaces, it seems to bring a little nearer to fulfilment our dream of controlling this latent power for the well-being of the human race — or for its suicide."

Before the release of energy by nuclear fusion — be it in a controlled or explosive way — can be seriously considered, a major hurdle is to be surmounted: atomic nuclei are positively charged and thus repel each other very strongly. How can this Coulomb repulsion be overcome, thereby allowing two nuclei to fuse? Classically, it cannot, but quantum-mechanically, it can: Indeed, in 1927 Friedrich Hund (1896–1997) observes that a potential barrier, which cannot be overcome classically, may nevertheless be overcome through quantum tunnelling, a mathematical analogue of evanescent waves in optics. In 1928 George Gamow (1904–1968) and, independently, Robert Guerney (1898–1953) and Edward Condon (1902–1974) discover that quantum tunnelling allows the escape of alpha-particles from an atomic nucleus, thereby explaining alpha decay. For this early history of quantum tunneling, see Merzbacher [40].

In 1929 Robert d'Escourt Atkinson (1898–1982), and Fritz Houtermans (1903–1966), use the same quantum tunnelling effect for the first attempt to build a quantitative theory of how a proton can enter a nucleus to form a heavier nucleus [41]. If there is a mass defect, it will be be radiated away into energy, just as conjectured by Langevin, Perrin and Eddington. With the help of Gamow, Atkinson and Houtermans succeed in getting the overall numbers essentially right. However, as stressed by Gamow in his 1970 posthumous memoirs, the quantitative estimate had two compensating errors of five orders of magnitude (regarding the correct collision cross section and a quadrupole vs dipole emission formula), which would

[1] Our emphasis.

be corrected only 6–7 years later by Gamow and Teller. Eventually, a somewhat similar mechanism, but way more complex and subtle, involving Carbon, Nitrogen and Oxygen will be discovered in 1939 by Hans Bethe (1906–2005) and will provide a full quantitative model of the heating of stars by nuclear fusion [43]. The whole story, around Gamow and his coworkers is told in detail in Kragh [44].

But back to Gamow, already in 1932 he gives some indication that production of energy by controlled fusion in the laboratory is a possibility, although it will require a plasma with a temperature of millions of degrees. This is proposed during a lecture at the Academy of science in Leningrad on thermonuclear reactions and their role in the energy production in the Sun and other stars. Nikolai Bukharin, a high-ranking Soviet official eventually executed by Stalin, attends the lecture and discusses with Gamow concrete steps to achieve controlled fusion experimentally. This is deemed premature by Gamow, perhaps because at the time he is already thinking about escaping from the Soviet Union. These matters are documented in Teller [45] and in Gamow's memoirs [42] and were pointed out to one of us (UF) by Roald Sagdeev (see also his interview by Patrick Diamond in this volume).

The first actual fusion experiment is then performed at the Cavendish Laboratory in 1934 by Mark Oliphant (1901–2000), Paul Harteck (1902–1985) and Ernest Rutherford (1871–1937)[46]. Using a 400,000 volt accelerator, they hurl deuterons (nuclei of Deuterium) at various targets. In the interaction of two deuterons, transmutation into helium is observed and much energy is released.

To summarize, between 1913 and 1920 the idea of nuclear fusion was born as an interesting speculation, capable among other things to explain how the stars can stay hot for a very long time. Then, in the twenties, a quantitative theory of nuclear fusion was developed making a crucial use of the quantum tunnelling effect. Finally, in 1934, nuclear fusion became an experimental field. Observe that nuclear fusion research owes much to relativity and quantum mechanics, but it has also benefitted from research in astrophysics, chemistry and natural selection.

It is interesting to point out that all this was done several years before nuclear fission was discovered by Otto Hahn and Fritz Strassmann in 1939 and then quickly explained by Lise Meitner and Otto Frisch. The Szilard/Einstein letter to Roosevelt in the context of WWII, allowed fission research to proceed at a gigantic pace: Enrico Fermi's Chicago Pile (a fission reactor) achieved criticality on December 2, 1942; the Manhattan Project, led by Robert Oppenheimer, performed the Trinity test (first fission bomb) on July 16, 1945. But during all this time, most nuclear physicists were fully aware of, let us call it the "suicidal wing" of Eddington's 1920 prediction, what was eventually called the Hydrogen bomb. Some of the physicists resisted the temptation and others took it as a major challenge. Eventually, the Teller–Ulam configuration led to a fusion bomb, ignited by a fission reaction, tested in the Enewetak Atoll on November 1, 1952.

When shall we see the other wing, the one benefitting mankind, now called controlled fusion? This is not a question within the scope of the present issue of EPJ H, devoted to the history of plasma physics. Nevertheless, we can observe that gravitational waves were detected directly only one hundred years after their prediction by Einstein. Sustained energy production by plasma fusion seems significantly more complex but, given enough international brain-power, time and money (in that order), it will be achieved, hopefully.

IV. SUPPLEMENTAL READING MATERIAL

This section presents supplementary reading material. This list is chosen with the general physicist reader in mind. We suggest general texts, some focused monographs and some reviews. Our preference is for works that have stood the test of time. Material on achievements of the US fusion program is found in Section C.

A) General Texts

Spitzer, L. 2006. Physics of fully ionized gases (2nd ed.). Dover Publications, Inc., Mineola.

Chandrasekhar, S. 1962. Plasma physics. University of Chicago Press, Chicago.

Krall, N. and A. Trivelpiece. 1973. Principles of plasma physics. McGraw–Hill, New York.

Kulsrud, R. M. 2004. Plasma physics for astrophysics. Princeton University Press, Princeton.

Sturrock, P. A. 1994. Plasma physics: an introduction to the theory of astrophysical, geophysical and laboratory plasmas. Cambridge University Press, Cambridge.

Kadomtsev, B. 2001. Collective effects in plasma. In: Reviews of plasma physics (vol. 22) (Shafranov, V. D., ed.). Kluwer Academic/Consultants Bureau, New York.

Lifshitz, E. and L. Pitaevski. 1981. Physical kinetics (vol. 10, Landau and Lifshitz course on theoretical physics; Sykes, J. B. and R. N. Franklin, transl.). Elsevier, Oxford.

B) Selected Focused Monographs

For MHD, dynamo theory, magnetic reconnection, with solar and astrophysical plasma applications:

Biskamp, D. 1993. Nonlinear magnetohydrodynamics. Cambridge University Press, Cambridge.

Moffatt, H. K. 1978. Magnetic field generation in electrically conducting fluids. Cambridge University Press, Cambridge.

Parker, E. N. 1979. Cosmic magnetic fields: their origin and their activity. Clarendon Press, Oxford.

For fundamentals of kinetic theory:

Montgomery, D. and D. Tidman. 1964. Plasma kinetic theory. McGraw–Hill, New York.

For confinement physics and tokamaks:

Kadomtsev, B. B. 1992. Tokamak plasma: a complex physical system. Institute of Physics, Bristol.

Wesson, J. 2011. Tokamaks (4^{th} ed.). Oxford University Press, New York.

For plasma turbulence and Hamiltonian chaos:

Sagdeev, R. Z. and A. A. Galeev. 1969. Nonlinear plasma theory. W. A. Benjamin, New York.

Kadomtsev, B. 1965. Plasma turbulence. Academic Press, London.

Zaslavsky, G. M. 2005. Hamiltonian chaos and fractional dynamics. Oxford University Press, Oxford.

Diamond, P. H., S.-I. Itoh, and K. Itoh. 2010. Modern plasma physics — vol. I: physical kinetics of turbulent plasmas. Cambridge University Pres, New York.

For a glimpse at the contributions of Marshall Rosenbluth and his school (see also the Sagdeev interview, this volume):

Van Dam, J. W. (ed.). 1989. From particles to plasmas: lectures honoring Marshall Rosenbluth. Addison–Wesley Publ. Co., Redwood City.

For an introduction to space plasma physics:

Kivelson, M. G. and C. T. Russell (eds.). 1995. Introduction to space physics. Cambridge University Press, New York.

C) Selected Reviews

For a look at some of the achievements of the US fusion program:

Hawryluk, R. G. 1998. Results from deuterium — tritium tokamak confinement experiments. Rev. Mod. Phys. **70**: 537-587. doi:10.1103/revmodphys.70.537.

Greenwald, M., A. Bader, S. Baek, M. Bakhtiari, H. Barnard, W. Beck, W. Bergerson, I. Bespamyatnov, P. Bonoli, D. Brunner, et al. 2014. 20 years of research on the Alcator C-Mod tokamak. Phys. Plasmas **21**: 110501. doi:10.1063/1.4901920.

Burrell, K. 1997. Effects of $E \times B$ velocity shear and magnetic shear on turbulence and transportation in magnetic confinement devices. Phys. Plasmas **4**: 1499-1518. doi:10.1063/1.872367.

For the plasma physics of cosmic ray acceleration:

Blandford, R. and D. Eichler. 1987. Particle acceleration at astrophysical shocks: a theory of cosmic ray origin. Phys. Rep. **154**: 1-75. doi:10.1016/0370-1573(87)90134-7.

For overviews of magnetic relaxation and reconnection:

Taylor, J. B. 1986. Relaxation and magnetic reconnection in plasmas. Rev. Mod. Phys. **58**: 741-763. doi:10.1103/revmodphys.58.741.

Yamada, M., R. Kulsrud and H. Ji. 2010. Magnetic reconnection. Rev. Mod. Phys. **82**: 603-664. doi:10.1103/revmodphys.82.603.

For a glimpse at the current thinking on drift–zonal flow turbulence:

Gürcan, D. and P. H. Diamond. 2015. Zonal flows and pattern formation. J. Phys. A-Math Theor. **48**: 293001. doi:10.1088/1751-8113/48/29/293001.

For a review of modern gyrokinetic theory (plasma with strong magnetic field):

Brizard, A. J. and T. S. Hahm. 2007. Foundations of nonlinear gyrokinetic theory. Rev. Mod. Phys. **79**: 421-467. doi:10.1103/revmodphys.79.421.

References

[1] BBC Wales. Marconi's waves [documentary film].

[2] Mimno, H. R. 1937. "The physics of the ionosphere", Rev. Mod. Phys. **9**: 1-45. doi:10.1103/RevModPhys.9.1

[3] Green, A. L. 1974. "Early history of the ionosphere", J. Atmos. Terr. Phys. **36**: 2159-2165. doi:10.1016/0021-9169(74)90146-9

[4] Kennelly, A. E. 1902. "On the elevation of the electrically-conducting strata of the Earth's atmosphere", Electr. World and Engr. **39**: 473.

[5] Heaviside, O. 1902. "Telegraphy", in: Encyclopedia Britannica (10^{th} edition), vol. 33; p. 215.

[6] Stewart, B. 1878. "Meteorology", in: Encyclopedia Britannica, 9th edition, (Baynes, T.S., ed.), vol. 16; p. 181. https://archive.org/details/encyclopediabrit16newyrich/page/180
[7] Hunt, B. J. 1991. The Maxwellians. Cornell University Press, Ithaca.
[8] Rayleigh, L. 1903. "On the bending of waves round a spherical obstacle", P. R. Soc. London **72**: 40-41. doi:10.1098/rspl.1903.0012
[9] MacDonald, H. M. 1903. "The bending of electric waves round a conducting obstacle", P. R. Soc. London **71**: 251-258.
[10] Poincaré, H. 1903. "Sur la diffraction des ondes électriques: à propos d'un article de M. Macdonald", P. R. Soc. London **72**: 42-52. doi:10.1098/rspl.1903.0013
[11] Sommerfeld, A. 1909. Über die Ausbreitung der Wellen in der drahtlosen Telegraphie", Ann. Phys. (Berl.) **333:** 665-736. doi:10.1002/andp.19093330402
[12] Pomeau, Y. Chapter 18 in "The scientific legacy of Poincaré", Eric Charpentier et al. editors, American Mathematical Society (2010) and references therein.
[13] Larmor, Sir Joseph 1924. XCVIII. Why wireless electric rays can bend round the earth. Philosophical Magazine Series 6, **48**: 1025-1036. doi:10.1080/14786442408634561
[14] Nobel Foundation. 1967. G. Marconi: Nobel lecture, 11 December 1909. In: Nobel lectures: physics 1901-1921; p. 206. Elsevier, Amsterdam.
[15] Langmuir, I. 1928. "Oscillations in ionized gases", Proc. Nat. Acad. Sci. **14**: 627-637. doi:10.1073/pnas.14.8.627
[16] Appleton, E. V. and M. A. F. Barnett. 1925. "On some direct evidence for downward atmospheric reflection of electric rays", P. R. Soc. London **109**: 621-641. doi:10.1098/rspa.1925.0149
[17] Wikipedia. 2018. "Ionosphere", in: Wikipedia: the free encyclopedia [Internet]. [last edited 2018 Sep. 13; accessed 2018 Oct. 18]. Wikipedia Foundation, San Francisco. https://en.wikipedia.org/w/index.php?title=Ionosphere&oldid=859372611
[18] Lloyd Webber, Andrew. 1981. Cats: the book of the musical (music by Andrew Lloyd Webber; based on Old Possum's book of practical cats by T. S. Eliot). Faber and Faber (Really Useful Company), London.
[19] Eliot, T. S. 1939. Old possum's book of practical cats. Harcourt, Brace and Co., New York.
[20] Alighieri, Dante. 1984. Paradiso (A. Mandelbaum, transl.). University of California Press, Berkeley.
[21] Yeang, C.P. "The study of long distance radio-wave propagation", Historical Studies in the Phys. and Bio. Sciences **33**(2003), 369.
[22] Tonks, L., Langmuir, I. "Oscillations in ionized gases", Phys. Rev. **33**, (1929), 195.
[23] Jeans, J.H. "The Stability of a Spherical Nebula", Philosophical Transactions of the Royal Society A. **199** (1902) p. 153
[24] Pomeau, Y. "Statistical mechanics of a gravitational plasma: the effect of close collisions", Proceedings of the Kazimiercz Summer School (2011), University of Warsaw press (Warsaw). and references therein.
[25] Landau, L.D. "On the vibrations of the electronic plasma", J. Phys. USSR **10** (1946) 496 and Sov. Phys. JETP **16** (1946) p. 574.
[26] Six, T.H. Waves in plasmas (1992) 2d edition, Springer (New-York).
[27] Prout, W. 1815. "On the Relation between the Specific Gravities of Bodies in their Gaseous State and the Weights of their Atoms", Annals of Philosophy. 6: 321–330 (at first, published anonymously).
[28] Prout, W. 1816. "Correction of a Mistake in the Essay on the Relation between the Specific Gravities of Bodies in their Gaseous State and the Weights of their Atoms", Annals of Philosophy. 7: 111–113 (at first, published anonymously).
[29] Darwin, C. 1859 On the Origin of Species . . . , John Murray (1859).
[30] Helmholtz, H. von 1854 "Ueber die Wechselwirkung der Naturkräfte und die darauf bezüglichen neuesten Ermittelungen der Physik. Ein populär-wissenschaftlicher Vortrag [gehalten am 7. Februar 1854]", Gräfe & Unzer (Königsberg), translated into English as "On the interaction of natural forces", Philosophical Magazine, Series 4, 11:75, 489–518 (1856) LXIV.
[31] Thomson, W. (Lord Kelvin) 1862, "On the Age of the Sun's Heat", Macmillan's Magazine, vol. 5: 388–393.
[32] Bahcall, J.N. 2000, "How the Sun Shines", Nobelprize.org. Nobel Media AB
[33] Langevin, P. 1913 "L'inertie de l'énergie et ses conséquences", J. Phys. Théor. Appl., 3 (1), pp. 553–591
[34] Perrin, J. 1919, "Matière et lumière. Essai de synthèse de la mécanique chimique", Ann. Phys., Vol. 9, N11 pp. 5108
[35] Perrin, J. 1920 "Atomes et lumière", La revue du Mois xxi, 113–166.
[36] Wesemael, F., 2009 "Harkins, Perrin and the alternative paths to the solution of the stellar energy problem", Journal for the History of Astronomy, 40, 277–296 (2009) JHA, xl (2009).
[37] Aston, F.W. 1920, "The mass-spectra of chemical elements", Phil. Mag. vol. 39, pp. 611–625.
[38] Eddington, A. 1920 "The internal constitution of the stars", The Observatory, vol. 43, pp. 341–

358.
[39] Eddington, 1926 “The internal constitution of the stars”, Cambridge University Press.
[40] Merzbacher, E. 2002 “The early history of quantum tunneling”, Physics Today (August 2002), pp. 44–49.
[41] Atkinson, R. d'E. and Houtermans, F. 1929 “Zur Frage der Aufbaumöglichkeit der Elemente in Sternen”, Zeits. f. Physik, vol. 54, pp. 656–665.
[42] Gamow, G. 1970 (posthumous) My World Line, Viking Press (New York).
[43] Bethe, H. 1939 “Energy production in stars”, Phys. Rev. Vol. 55, pp. 434–456.
[44] Kragh, H. 1996 “Cosmology and Controversy”, Princeton University Press.
[45] Teller, E. 1955 “The work of many people”, Science Vol. 121, issue 3139, pp. 267–275.
[46] Oliphant, M.L.E., Harteck, P. and Lord Rutherford 1934, “Transmutation effects observed with heavy hydrogen”, Proc. R. Soc. A vol. 144, pp. 692–703.