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Arithmetic Sums of Nearly Affine Cantor Sets

DISSERTATION

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for the degree of

DOCTOR OF PHILOSOPHY

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by

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DEDICATION

To Cynthia and Max.

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ABSTRACT OF THE DISSERTATION

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Associate Professor Anton Gorodetski, Chair

For a compact set $K \subset \mathbb{R}^1$ and a family $\{C_\lambda\}_{\lambda \in J}$ of dynamically defined Cantor sets sufficiently close to affine with $\dim_H K + \dim_H C_\lambda > 1$ for all $\lambda \in J$, we prove that the sum $K + C_\lambda$ has positive Lebesgue measure for almost all values of the parameter λ. As a corollary, we show that generically the sum of two affine Cantor sets has positive Lebesgue measure provided the sum of their Hausdorff dimensions is greater than one.

Introduction

A dynamically defined Cantor set is a set which is given by a family of $C^{1+\varepsilon}$ contractions $\{\phi_i\}_{i=1}^m$ with the following properties:

1. $\phi_i : I_0 \to I_0$ for a compact interval I_0, and $i = 1, \ldots, m$
2. $\phi_i(I_0) \cap \phi_j(I_0) = \emptyset$ for $i \neq j$

If $I_{n+1} = \cup_{i=1}^m \phi_i(I_n)$, then $C = \cap_{n=0}^\infty I_n$ is a *dynamically defined Cantor set*. This class also includes the "famous" middle$-\alpha$ Cantor sets; in this case, $m = 2$, $I_0 = [0, 1]$, $\phi_1(x) = \lambda x$ and $\phi_2(x) = \lambda x + (1 - \lambda)$, where $\lambda = \frac{1}{2}(1 - \alpha)$.

Questions regarding the structure of arithmetic sums of Cantor sets, that is, sets of the form $\{x+y \,|\, x \in C_1, y \in C_2\}$ where C_1 and C_2 are Cantor sets, arise naturally in the study of fields such as number theory [7, 31, 38], spectral theory [17, 18, 19, 71], and dynamical systems [42, 43, 44, 48].

In number theory, arithmetic sums of Cantor sets have been used to answer questions about Lagrange and Markov spectra. By Dirichlet's theorem, given an irrational number α, the inequality

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \tag{1}$$

has infinitely many rational solutions. If we define the function

$$k(\alpha) = \sup\{k : |\alpha - p/q| < 1/kq^2 \text{ has infinitely many rational solutions}\} \tag{2}$$

then we can consider the *Lagrange spectrum*

$$L = \{k(\alpha) : \alpha \in \mathbb{R} \setminus \mathbb{Q} \text{ and } k(\alpha) < \infty\} \tag{3}$$

Hall was able to prove that L contained the interval $[6, \infty)$ by proving that $C+C$ contained an interval, where $C = F(4) \cap [0, 1]$, and $F(m) = \{[t, a_0, a_1, \dots] : t \in \mathbb{Z}, 1 \leq a_i \leq m \text{ for } m \geq 1\}$. Here $[a_0, a_1, a_2, \dots]$ denotes the continued fraction expansion

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\dots}}}$$

In fact, Hall proved that $F(4) + F(4) = \mathbb{R}$, which is equivalent to $C + C$ containing $[0, 1]$. More recently, Astels proved that for m, n integers, $F(m)+F(n) = \mathbb{R}$ and $F(m) - F(n) = \mathbb{R}$ provided (m, n) is $(2, 5)$ or $(3, 4)$, and these equations do not hold if (m, n) is $(2, 4)$. Additionally, $F(3) + F(3) \neq \mathbb{R}$ and $F(3) - F(3) = \mathbb{R}$ (see [23, 21, 35, 1, 2] for more on the subject).

Regarding spectral theory, consider the *Fibonacci Hamiltonian* which acts on square-summable two-sided sequences of complex numbers, $\ell^2(\mathbb{Z})$.

$$(H_\lambda \phi)_n = \phi_{n+1} + \phi_{n-1} + \lambda \omega_n \phi_n \tag{4}$$

for *coupling constant* $\lambda > 0$, and $\{\omega_n\}_{n \in \mathbb{Z}}$ is given by

$$\omega_n = 1_{[1-\alpha,1)}(n\alpha + \omega_0 \mod 1) \tag{5}$$

where 1_E is the indicatrix function of E, $\alpha = \frac{\sqrt{5}-1}{2}$, the reciprocal of the golden ratio, and ω_0 is in $\mathbb{R}/\mathbb{Z}$. The Fibonnaci Hamiltonian has been used to model the electron transfer properties of quasicrystals (see [9, 15], for instance). Its spectrum Σ_λ is independent of ω_0 and is a dynamically defined Cantor set (again, see [15]) , whose fractal properties and structure have implications regarding quantum dynamics (see [16, 28]). While the properties of these spectra are mostly understood, less is known about the *square Fibonacci Hamiltonian* which acts on $\ell^2(\mathbb{Z}^2)$

$$(H^2_\lambda \phi)_{n,m} = \phi_{n+1,m} + \phi_{n-1,m} + \phi_{n,m+1} + \phi_{n,m-1} + \lambda(\omega_n + \omega_m)\phi_{n,m}. \tag{6}$$

It is easily shown that for such operators, the spectrum $\Sigma^2_\lambda = \Sigma_\lambda + \Sigma_\lambda$, and so is an arithmetic sum of dynamically defined Cantor sets.

Arithmetic sums of Cantor sets have an important role in hyperbolic dynamics. Suppose M is a 2-dimensional manifold and $\varphi : M \to M$ a smooth diffeomorphism and p a hyperbolic fixed point (that is $D\varphi(p)$ has one eigenvalue with norm smaller than one and another with norm larger than one). Define $W^s(p) = \{x \in M : \lim_{n\to\infty} \varphi^n(x) = p\}$ and $W^u(p) = \{x \in M : \lim_{n\to-\infty} \varphi^n(x)\}$, the stable and unstable manifolds, respectively; it is well known that they are immersed 1-dimensional submanifolds of M. We call points $q \neq p$ homoclinic to p if $\lim_{n\to\pm\infty} \varphi^n(q) = p$; these points can be *transversal homoclinic points*, i.e.

$$T_q(M) = T_q(W^s(p)) \oplus T_q(W^u(p)) \tag{7}$$

or they can be *homoclinic points of tangency.*

A set Λ is *hyperbolic* if the tangent bundle $T_\Lambda(M)$ splits into stable and unstable $D\varphi$-invariant sub-bundles E^s and E^u on which $D\varphi$ is uniformly contracting and expanding, respectively. For instance, if we consider the horseshoe map, the set of points invariant under the map form a hyperbolic set (often called Smale's horseshoe, or just a horseshoe). For a point x in a hyperbolic set Λ, we define the stable manifold of x as $W^s(x) = \{y \in M : \lim_{n\to\infty} d(\varphi^n(x), \varphi^n(y)) = 0\}$, and the unstable manifold of x similarly. From these, we can construct the stable foliations $\mathcal{F}^s$ by taking the union of stable manifolds over points of Λ. Likewise, we can also construct the unstable foliations $\mathcal{F}^u$ in a similar manner. These foliations can be extended to a neighborhood of Λ. The *non-wandering set*, $\Omega(\varphi)$, is all $x \in M$ such that for any neighborhood U of x, there exists n so that $\varphi^n(U) \cap U \neq \emptyset$, and say that φ is hyperbolic if its non-wandering set is a hyperbolic set.

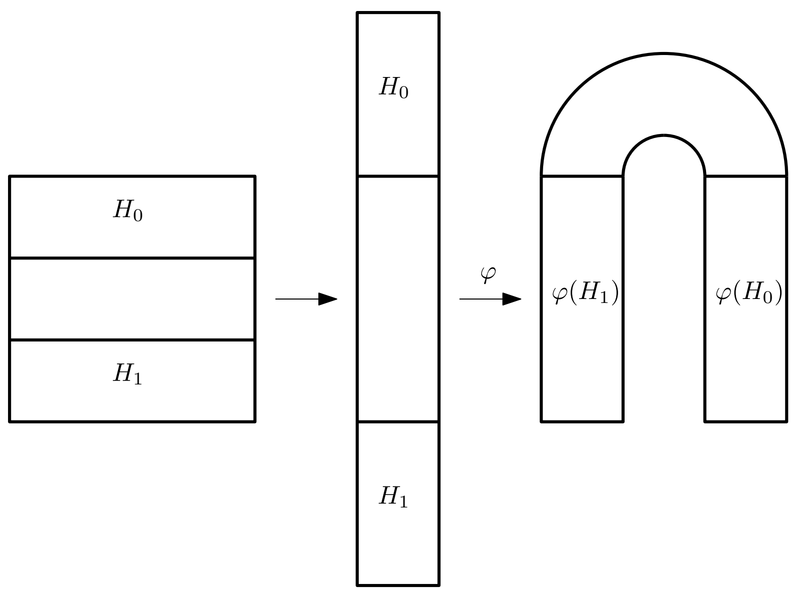

Figure 1: The horseshoe map; φ contracts the square horizontally and expands it vertically, before folding it back onto itself. The set of points invariant under this map will form a hyperbolic set, called *Smale's Horseshoe*.

In [44] Newhouse considered a C^2 diffeomorphism φ with a homoclinic tangency associated with a hyperbolic fixed point p. If Λ is an invariant hyperbolic set associated with p, then we can consider the dynamically defined Cantor sets $C^s = W^s(p) \cap \Lambda$ and $C^u = W^s(p) \cap \Lambda$. Let's suppose that the products of the thicknesses (see section 2.2 or chapter 4 of [48]) of C^s and C^u are greater than one. Now, if we consider the extensions of $\mathcal{F}^s$ and $\mathcal{F}^u$ to a neighborhood of Λ, we can take ℓ to be the line of their tangent intersections. By intersecting ℓ with C^s and C^u, we have dynamically defined Cantor sets on a line, call them C_1 and C_2. By the Gap Lemma (again, see section 2.2 or chapter 4 of [48]), C_1 and C_2 will have non-empty intersection, and in fact, the existence of homoclinic tangencies coincides with non-empty intersection.

If φ has these properties, then it is in the closure of an open set $U \subset \text{Diff}^2(M)$ which has persistent tangencies, that is to say, U has a dense subset which exhibit homoclinic tangencies. Since thicknesses depend continuously on the Cantor sets, Cantor sets C_1' and C_2' that are near C_1 and C_2 will still intersect. Notice that the existence of persistent tangencies discussed here implies that the difference $C_1 - C_2$ contains an interval.

In [58] Palis and Takens considered a C^2 family of diffeomorphisms φ_μ, $\mu \in (-1, 1)$, such that φ_μ is persistently hyperbolic (that is, any $\varphi_{\mu'}$ which is C^r-close to φ_μ is also hyperbolic, for $r \geq 1$) and φ_0 has a hyperbolic fixed point p_0 with associated homoclinic tangency. They showed that if Λ_0 is a basic set (see section 2.1) associated with p_0 and if $\dim_H(W^s(p_0) \cap \Lambda_0) + \dim_H(W^u(p_0) \cap \Lambda_0) < 1$ then

$$\lim_{\mu \to 0^+} \frac{Leb(B \cap [0, \mu])}{\mu} = 0 \tag{8}$$

where B is the set of parameters $\mu > 0$ for which φ_μ is not persistently hyperbolic.

A crucial point of this proof is that $W^s(p_0) \cap \Lambda_0$ and $W^u(p_0) \cap \Lambda_0$ are dynamically defined Cantor sets, and when $\dim_H(W^s(p_0) \cap \Lambda_0) + \dim_H(W^u(p_0) \cap \Lambda_0) < 1$ (where $\dim_H$ denotes the Hausdorff dimension) then the difference of these two Cantor sets has Lebesgue measure zero.

Later in [46, 47], while attempting to find a converse to the above statement, J. Palis conjectured that if the sums of the Hausdorff dimensions of two Cantor sets is larger than one, then their arithmetic sum (generically) has measure zero, or else contains an interval. This is known as the "Palis Conjecture". The conjecture was answered in the affirmative– in fact the sumset will contain an interval, generically– by Moreira and Yoccoz in [39] for dynamically defined Cantor sets. However, the genericity assumptions used by Moreira and Yoccoz are difficult to check in many examples of one parameter families of Cantor sets that appear in dynamics. Additionally, the conjecture has not been answered for affine Cantor sets.

Chapter 1

Outline

1.1 Statement of the Results

In this paper we provide a partial answer to the Palis Conjecture. Through dimensional arguments (see, for instance, [48]), one can show that given two Cantor sets, if the sum of their Hausdorff dimensions is less than one, then their arithmetic sum must have Lebesgue measure zero.

When the sum of the Hausdorff dimensions is greater than one, the answer is less clear. In [67], Solomyak provided an answer in the case of the middle$-\alpha$ Cantor sets, and then later he and Peres would answer the question for homogeneous self-similar Cantor sets (when the contractions outlined in the introduction are all linear with the same contraction coefficient) in [53]. As mentioned in the introduction, Moreira and Yoccoz would answer the conjecture for generic dynamically defined Cantor sets in the affirmative in [39]; however, their genericity assumptions would make it nearly impossible to check for a particular pair of Cantor sets.

Here we provide a partial answer for *nearly* affine Cantor sets; that is, whose contractions are almost affine.

Theorem 1.1. *Suppose $J \subseteq \mathbb{R}$ is an interval and $\{C_\lambda\}_{\lambda \in J}$ is a family of dynamically defined Cantor sets generated by contracting maps*

$$\{f_{i,\lambda}(x) = c_i(\lambda)x + b_i(\lambda) + g_i(x,\lambda)\}_{i=1}^m \tag{1.1}$$

such that the following holds:

$$c_i(\lambda), b_i(\lambda) \text{ are } C^1\text{-functions of } \lambda; \tag{1.2}$$

$$\frac{dc_i}{d\lambda} \leq -\delta < 0 \text{ for all } \lambda \in J \text{ and some uniform } \delta > 0; \tag{1.3}$$

$$g_i(x,\lambda) \in C^2, \text{ with small (based on } \{c_i(\lambda)\}, \{b_i(\lambda)\}\text{) } C^2\text{-norm.} \tag{1.4}$$

Then for any compact $K \subset \mathbb{R}$ with

$$\dim_H(K) + \dim_H(C_\lambda) > 1 \quad \text{for all } \lambda \in J, \tag{1.5}$$

the sumset $K + C_\lambda$ has positive Lebesgue measure for a.e. $\lambda \in J$.

Theorem 1.1 has implications for affine Cantor sets as well. By affine, we mean dynamically defined Cantor sets whose generating contractions $\{\phi_k\}_{k=1}^m$ are of the form $\phi_k(x) = \lambda_k x + d_k$, where $\lambda_k \in J_k \subset (-1,0) \cup (0,1)$, and $d_k = d_k(\lambda_1 \ldots, \lambda_m)$ is C^1, for $k = 1, \ldots, m$. If we define $\Lambda = (\lambda_1, \ldots, \lambda_m)$, let C_Λ be such a set. Then we have the following corollary.

Corollary 1.1. *Suppose K is a compact subset of the real line, and a family $\{C_\Lambda\}$ of affine Cantor sets as above is given such that*

$$\dim_H K + \dim_H C_\Lambda > 1 \quad \text{for all } (\lambda_1, \ldots, \lambda_m) \in J_1 \times \cdots \times J_m.$$

Then for a.e. $(\lambda_1, \ldots, \lambda_m) \in J_1 \times \cdots \times J_m$ the set $K + C_\Lambda$ has positive Lebesgue measure.

Thus in this this setting, any affine Cantor set is determined by $2m$ parameters, namely $\lambda_1, \ldots, \lambda_m$ and $d_1, \ldots, d_m$, such that $\phi_k = \lambda_k x + d_k$ satisfies the necessary requirements for

$k = 1, \ldots, m$. These parameters form a subset of $\mathbb{R}^{2m}$, and as a consequence of Corollary 1.1, we have the following corollary.

Corollary 1.2. *Generically (for Lebesgue almost all admissible tuples of the parameters) the sum of two affine Cantor sets has positive Lebesgue measure provided the sum of their Hausdorff dimensions is greater than one.*

We remark that this Corollary provides an interesting comparison with Theorem E of [63], which claims that for any two affine Cantor sets C_1 and C_2 with sum of Hausdorff dimensions greater than one, $\dim_H\{u \in \mathbb{R} \mid Leb(C_1 + uC_2) = 0\} = 0$.

1.2 Organization of the Thesis

We will now outline the structure of the paper. In chapter 2 we first discuss some preliminary definitions and ideas. In section 2.1 we define the idea of a dynamically defined Cantor set, including using the definition put forth by Palis and Takens in [48] which includes some of the flavor that motivated them to study dynamically defined Cantor sets. In section 2.2 we discuss ideas in dimensional theory, defining Hausdorff dimension, box counting dimension, as laid out by Falconer in [20]. We also define the thickness of a Cantor set, and lay out several theorems which will help us later on.

Starting in section 2.3 we discuss a result of Palis and Takens from [57, 58]. An attempt to provide a converse to these results would inspire the Palis conjecture (see [46, 47]). In section 2.4 we discuss Solomyak's results from [67] where he provides a partial answer to the Palis conjecture for middle-α Cantor sets, as well as some results of Mendes and Oliveira related to the structure of sums of Cantor sets from [37]. In section 2.5 we discuss results of Solomyak and Peres which generalize Solomyak's work to homogeneous self-similar Cantor sets from [53]. In section 2.6 we discuss results of Moreira and Yoccoz in [39] which positively answers the Palis conjecture, though with some hard to verify genericity conditions. In section 2.7 we discuss the known results for when $\dim_H(C_1+C_2) = \dim_H C_1 + \dim_H C_2$ for dynamically defined Cantor sets C_1 and C_2, including results from [51, 24]

In chapter 3 we prove the results outlined in section 1.1, notably the proof of Theorem 1.1. We start in section 3.1 by stating Proposition 3.1, which appears in [14], and which will we will use to help us prove Theorem 1.1; the proof of this Proposition can be found in Appendix A. Then in section 3.3 we start building a setting in which the conditions of Proposition 3.1 can be verified, and then verify those conditions in section 3.4. In section 3.5 we state some corollaries which will provide a partial answer to the Palis conjecture with respect to affine Cantor sets.

Finally, in chapter 4 we discuss a few questions which we have not answered but for which we have some ideas on how to proceed.

Chapter 2

Cantor Sets

In this chapter we will review the definition of a dynamically defined Cantor set, and discuss previous results related to the Palis conjecture.

Recall that a Cantor set in $\mathbb{R}$ a non-empty, totally disconnected, perfect (containing no isolated points), and compact set.

2.1 Dynamically Defined Cantor Sets

First, we restate the definition of a dynamically defined Cantor set.

Definition 2.1. *Consider a family of $C^{1+\varepsilon}$ contractions $\{\phi_i : I \to I\}_{i=1}^{m}$, where I is a compact interval, such that*

$$\phi_i(I) \cap \phi_j(I) = \emptyset \text{ for } i \neq j. \tag{2.1}$$

We say C is a dynamically defined Cantor set if

$$C = \cap_{n=0}^{\infty} I_n, \text{ where } I_0 = I \text{ and } I_{n+1} = \cup_{i=1}^{m} \phi_i(I_n) \tag{2.2}$$

Note that we often call a finite family of contractions on $\mathbb{R}^d$ an *iterated function system*, or IFS, for short.

For new results discussed in this paper, we will use Definition 2.1. However, let us now discuss a more general definition stemming from hyperbolic dynamics, and used by Palis and Takens in [48].

Suppose M is a 2-dimensional differentiable manifold, and let $\varphi : M \to M$ be a C^3 diffeomorphism. Suppose that p is a hyperbolic fixed point of φ of saddle-type, part of a non-trivial basic set Λ.

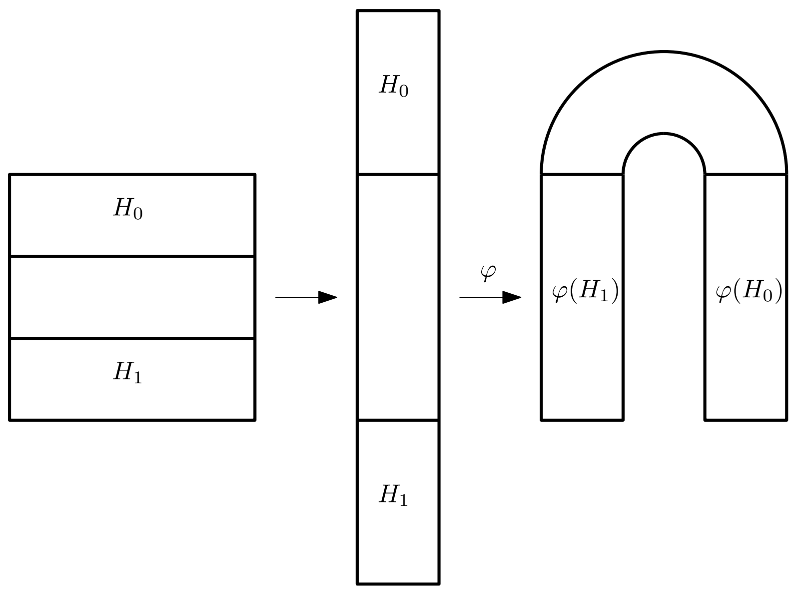

Figure 2.1: The Horseshoe map, again

By basic set, we mean a compact, hyperbolic, invariant set with a dense orbit and which is the maxical invariant set in a neighborhood of it. By non-trivial, we mean that it does not consist of finitely many periodic orbits.

For $p \in \Lambda$, $\Lambda \cap W^s(p)$ is what we want to call a dynamically defined Cantor set, where $W^s(p)$ is the stable manifold of p. To be precise, suppose that $\alpha : \mathbb{R} \to W^s(p)$ is a smooth identification such that $\alpha^{-1} \circ \varphi|_{W^s(p)} \circ \alpha$ is a linear contraction. Let C be an open and compact neighborhood of 0 in $\alpha^{-1}(W^s(p) \cap \Lambda)$. C is called a dynamically defined Cantor set.

Generally, we assume C is obtained by intersecting $\alpha^{-1}(W^s(p) \cap \Lambda)$ with an interval $C_0 \subset \mathbb{R}$ containing 0 and whose boundary points are not contained in $\alpha^{-1}(W^s(p) \cap \Lambda)$.

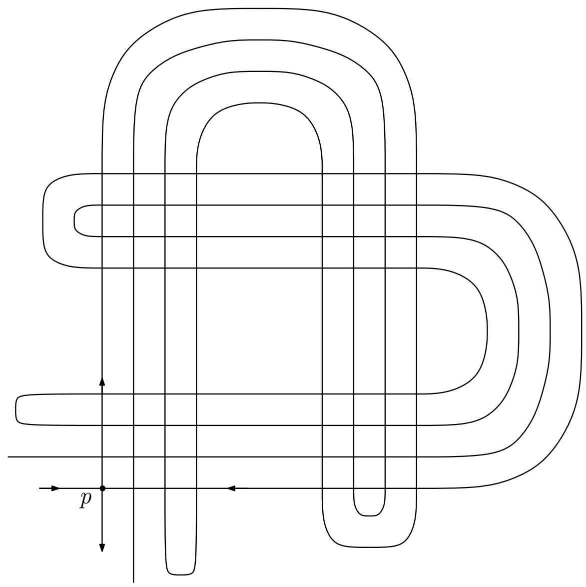

Figure 2.2: A few iterations of the Stable Manifold

A dynamically defined Cantor set exhibit the following three properties:

1. Scaling

   Since $\alpha^{-1} \circ \varphi|_{W^s(p)} \circ \alpha$ is a linear contraction by $\lambda > 0$, then $\lambda C = C \cap (\lambda C_0)$. Thus our choice of C_0 is arbitrary.

2. Expanding Structure

   Choose an unstable foliation $\mathcal{F}^u$ defined on a neighborhood U of Λ. If φ is C^3, then the foliation will be $C^{1+\varepsilon}$. If C_0 is sufficiently big, we have a projection π along the leaves of $\mathcal{F}^u$ of a neighborhood U' of Λ to $\alpha(C_0)$. That is, points of U' are projected onto the intersection of its leaf of $\mathcal{F}^u$ with $\alpha(C_0)$. Then $\pi(\Lambda) = \alpha(C)$. Note that π is not necessarily unique; a leaf of $\mathcal{F}^u$ may have more than one intersection with $\alpha(C_0)$.

Since Λ is totally disconnected, one can still take π on a small neighborhood of Λ to be continuous and hence $C^{1+\varepsilon}$. The derivative of $\pi|_{W^s(p)\cap U'}$ is bounded and bounded away from zero since the components of $W^s(p) \cap U'$ are leaves of the stable foliation which is transverse to $\mathcal{F}^u$.

For N sufficiently large, $\psi = \alpha^{-1} \circ \pi \circ \varphi^{-N} \circ \alpha : C_0 \to C_0$ is, where defined, expanding in the sense that its derivative has norm larger than 1. By construction, it follows that $\psi(C) = C$ and ψ is $C^{1+\varepsilon}$ in a neighborhood of C.

3. Markov Partition

   If we have a Cantor Set C with an expanding map ψ, then a Markov Partition is a finite set of disjoint intervals $C_1, \ldots, C_k \subset C_0$ such that

   (a) ψ is defined in a neighborhood of C_i, for $i \geq 1$.

   (b) $C \subset \cup_{i=1}^k C_i$ and $\partial C_i \subset C$ for each i.

   (c) For $1 \leq i \leq k$, $\psi(C_i)$ is an interval which is the convex hull of a finite collection of intervals of the Markov Partition.

   (d) For $1 \leq i \leq k$ and n sufficiently large, $\psi^n(C \cap C_i) = C$, i.e. $\psi|_C$ is topologically mixing.

Definition 2.2. *A dynamically defined Cantor set is a Cantor set $C \subset \mathbb{R}$ together with*

- *a scaling factor $\lambda \in \mathbb{R}$, such that $0 < |\lambda| < 1$ and λC is a neighborhood of 0 in C,*
- *a map $\psi : C \to C$ having a $C^{1+\varepsilon}$ expansive extension to a neighborhood of C,*
- *a Markov Partition $\{C_1, \ldots, C_k\}$.*

We can actually define C by ψ and a Markov partition $C_1, \ldots, C_k$. That is, if ψ is expanding, then we can take C to be $C = \cap_{i=1}^{\infty} \psi^{-i}(C_1 \cup \cdots \cup C_k)$.

This construction of a dynamically defined Cantor set yields a definition more general than Definition 2.1, but still includes those constructed in the manner of Defintion 2.1. If C is a dynamically defined Cantor set with expanding map ψ and Markov partition $C_1, \ldots, C_k$, then C will be generated by the contractions $\phi_i = (\psi|_{C_i})^{-1}$, for $i = 1, \ldots, k$. It will occasionally be useful in the proof of Theorem 1.1 to use both definitions.

Finally, of particular significance to this paper will be the *affine Cantor sets*. By an affine Cantor set C, we mean a dynamically defined Cantor set whose generating contractions $\{\phi_i\}_{i=1}^m$ are of the form $\phi_i(x) = \lambda_i x + d_i$.

2.2 Dimension Theory

We have previously discussed results on Cantor sets related to their Hausdorff dimensions, which we shall define here. We also define the box counting dimension and thickness of a Cantor set. For Hausdorff and box counting dimensions, we will use the definitions as laid out by Falconer in [20], and start with box counting dimension.

Suppose $E \subset \mathbb{R}^n$, and let $N_r(E)$ be the smallest number of sets of diameter r that can cover E. The *lower* and *upper box counting dimensions* of E are

$$\underline{\dim}_B E = \liminf_{r \to 0} \frac{\log N_r(E)}{-\log r} \tag{2.3}$$

and

$$\overline{\dim}_B E = \limsup_{r \to 0} \frac{\log N_r(E)}{-\log r}, \tag{2.4}$$

respectively. When the limit in 2.3 and 2.4 exists, then these two values are equal. We define the box counting dimension of E to be this common value, written

$$\dim_B E = \lim_{r \to 0} \frac{\log N_r(E)}{-\log r}. \tag{2.5}$$

Let a countable collection of subsets $\{U_i\}$ of $\mathbb{R}^n$ be a δ-cover of E; that is the diameter of any U_i is at most δ, and $E \subset \cup_{i=1}^{\infty} U_i$. Let $s \geq 0$; for all $\delta > 0$, define

$$\mathcal{H}_{\delta}^{s}(E) = \inf \left\{ \sum_{i=1}^{\infty} diam(U_i)^s \ : \ \{U_i\} \text{ is a } \delta\text{-cover of } E \right\}. \tag{2.6}$$

As δ decreases, so will the number of δ-covers of E, and so the infimum increases monotonically as δ decreases to zero. We can define the s-dimensional Hausdorff measure as

$$\mathcal{H}^{s}(E) = \lim_{\delta \to 0^+} \mathcal{H}_{\delta}^{s}(E) \tag{2.7}$$

We are now ready to define the *Hausdorff dimension*, as

$$\dim_H E = \inf\{s : \mathcal{H}^s(E) = 0\} = \sup\{s : \mathcal{H}^s(E) = \infty\} \tag{2.8}$$

which can be shown to exist for all subsets $E \subset \mathbb{R}^n$.

The upper and lower box counting dimensions and Hausdorff dimension have the following relationship (see [20] again):

$$\dim_H \leq \underline{\dim}_B E \leq \overline{\dim}_B E$$

Combined with Theorem 3 from [48] (see also [34]), we have the following helpful result:

Theorem 2.1. *If $C \subset \mathbb{R}$ is a dynamically defined Cantor set, then $\dim_H C = \dim_B C$.*

Among other things, this result makes it simple to calculate the Hausdorff dimension of self-similar Cantor sets. Consider the middle-third Cantor set $C_{1/3}$, constructed by removing the open interval comprising the middle third of the interval [0,1], and then removing the middle-third again of the remaining two intervals, and so on.

At each step of the construction we have 2^n intervals of length $(1/3)^n$ which will cover $C_{1/3}$. Thus,

$$\dim_H C_{1/3} = \dim_B C_{1/3} = \lim_{n \to \infty} \frac{\log 2^n}{\log 3^n} = \frac{\log 2}{\log 3}$$

Figure 2.3: The First three steps of construction for the middle-third Cantor set

Now to define the *thickness* of a Cantor set C. We call a *gap* of C a connected component of $\mathbb{R} \setminus C$. If U is a bounded gap of C and u a boundary point of U, a *bridge* of B is any interval in $\mathbb{R}$ such that u is also a boundary point of B and doesn't intersect any gap whose length is longer than the length of U.

The thickness of C at u is $\tau(C, u) = |B|/|U|$, where $|B|$ and $|U|$ are the lengths of B and U, respectively. We define the *thickness* of C to be the infimum of $\tau(C, u)$ over all u which are boundary points of bounded gaps of C, and write it as $\tau(C)$.

If we again consider the middle-third Cantor set, it is straightforward to calculate the thickness. Consider step n in the construction of $C_{1/3}$. For the endpoint of any interval (which will be a boundary point of a bounded gap in $C_{1/3}$), the length of any bridge will be $(1/3)^n$, as will the length of any bounded gap. Thus, the thickness must be 1.

We have the following useful theorem from [48]

Theorem 2.2. *The Hausdorff dimension (and so box counting dimension) and thickness of a dynamically defined Cantor set C depend continuously on C.*

Finally, the Newhouse Gap Lemma (see [42] or [48]) will be used to establish a relationship between the thicknesses of two Cantor sets and the structure of their arithmetic sum.

Lemma 2.1 (Newhouse Gap Lemma)**.** *Let C_1, $C_2 \subset \mathbb{R}$ be Cantor sets with thicknesses τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$ then one of the following is true: C_1 is contained in a gap of C_2, C_2 is contained in a gap of C_1, or $C_1 \cap C_2 \neq \emptyset$.*

The Newhouse Gap Lemma implies that if the product of thicknesses of two Cantor sets is greater than one, then the sum of the two Cantor sets must contain an interval; since gaps are open sets, if two such Cantor sets intersect, then small translations of either set will still have non-empty intersection. This gives us the following corollary.

Corollary 2.1. *Let C_1, $C_2 \subset \mathbb{R}$ be Cantor sets with thicknesses τ_1 and τ_2. If $\tau_1 \cdot \tau_2 > 1$, then $C_1 + C_2$ contains an interval.*

Proof. Consider the Cantor sets C_1 and $-C_2 = \{-x : x \in C_2\}$; we have that $\tau(-C_2) = \tau_2$, trivially. If C_1 is contained in a gap of $-C_2$ or vice versa, we can translate $-C_2$ by some t until $C_1 \cap (t - C_2) \neq \emptyset$, by the Gap Lemma. Now, gaps are open sets, and so for t in some interval, we will still have $C_1 \cap (t - C_2) \neq \emptyset$.

However, $C_1 + C_2 = \{t = x_1 + x_2 : x_1 \in C_1, x_2 \in C_2\} = \{t : C_1 \cap (t - C_2) \neq \emptyset\}$, and so $C_1 + C_2$ must contain an interval. $\square$

2.3 The Palis Conjecture

While trying to prove a theorem related to smooth dynamics (see the introduction) in [57] and [58], Palis and Takens proved the following result

Theorem 2.3. *If C_1 and C_2 are dynamically defined Cantor sets such that $\dim_H C_1 + \dim_H C_2 < 1$ then $C_1 - C_2$ has Lebesgue measure zero.*

Later while attempting to find a converse (see [46] and [47]), Palis conjectured the following

Conjecture 2.1 (Palis Conjecture)**.** *If C_1 and C_2 are dynamically defined Cantor sets such that $\dim_H C_1 + \dim_H C_2 > 1$ then generically $C_1 - C_2$ either has Lebesgue measure zero or else contains an interval.*

It should be noted that the Palis conjecture can only be true generically; in [61], Sannami constructs an example of a dynamically defined Cantor set C such that $C - C$ has positive Lebesgue measure but empty interior.

This is addition to a theorem by Marstrand (see [30], or see again [48]).

Theorem 2.4. *For C_1 and C_2 Cantor sets with $\dim_H C_1 + \dim_H C_2 > 1$, then $C_1 - \lambda C_2$ has positive Lebesgue measure for Lebesgue-a.e. $\lambda \in \mathbb{R}$.*

Note that $C_1 - \lambda C_2$ is simply the projection of $C_1 \times C_2$ onto the line with slope λ; since $\dim_H(C_1 \times C_2) = \dim_H C_1 + \dim_H C_2$ for dynamically defined Cantor sets C_1 and C_2 (see [48] again), we have the following inequality

$$\dim_H(C_1 + C_2) \leq \min\{1, \dim_H C_1 + \dim_H C_2\} \tag{2.9}$$

While we know that $C_1 + C_2$ will have Lebesgue measure zero if $\dim_H C_1 + \dim_H C_2 < 1$, in this case we are also interested in knowing when $\dim_H(C_1 + C_2) = \dim_H C_1 + \dim_H C_2$.

2.4 Middle-α Cantor Sets

The Palis conjecture is non-trivial, even when we restrict our view to one of the simplest class of Cantor sets, the middle-α Cantor sets. As mentioned in the introduction, a middle-α Cantor set C_λ is a dynamically defined Cantor set with contractions $\phi_1, \phi_2 : [0,1] \to [0,1]$, with $\phi_1(x) = \lambda x$ and $\phi_2(x) = \lambda x + (1 - \lambda)$, where $\lambda = \frac{1}{2}(1 - \alpha)$ (called the similarity ratio). As an example, the middle third Cantor set is in this class.

We know via geometric arguments that $\dim_H C_\lambda = \frac{\log 2}{\log 1/\lambda}$, and that $\tau(C_\lambda) = \frac{\lambda}{1-2\lambda}$; the proof of either is nearly identical to the middle-third case. If $\dim_H C_\lambda + \dim_H C_\gamma < 1$, then $C_\lambda + C_\gamma$ will be a Cantor set. On the other hand, if $\tau(C_\lambda)\tau(C_\gamma) > 1$, then the Newhouse Gap Lemma (see Corollary 2.1) states that $C_\lambda + C_\gamma$ will contain an interval.

In [67], Solomyak studied behavior of arithmetic sums of middle-α Cantor sets in what he called "The Mysterious Region R", that is the region with

$$\frac{\log 2}{\log 1/\lambda} + \frac{\log 2}{\log 1/\gamma} > 1 \tag{2.10}$$

and

$$\frac{\lambda}{1-2\lambda}\frac{\gamma}{1-2\gamma} < 1 \tag{2.11}$$

i.e. where the sums of Hausdorff dimensions of C_λ and C_γ are greater than one but the product of their thicknesses is less than one.

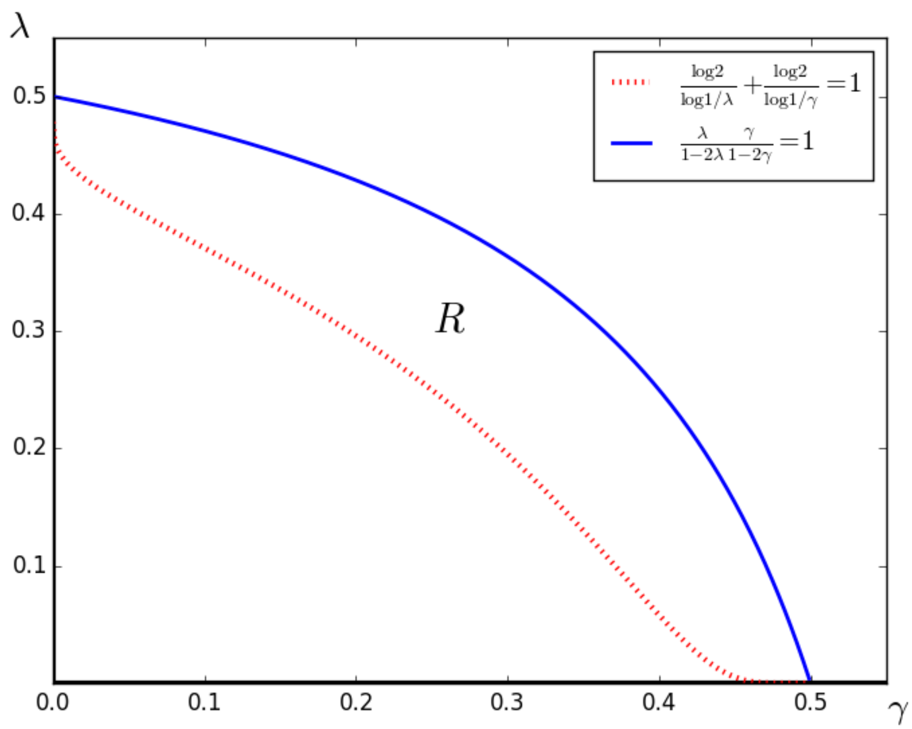

Figure 2.4: Solomyak's 'Mysterious Region' R.

In his paper, Solomyak proved the following theorem, which says that the set $C_\lambda + C_\gamma$ has positive Lebesgue measure for a.e. set of parameters (γ, λ) in R.

Theorem 2.5 (Theorem 1 from [67]). *For any* $\gamma \in (0, 1/2)$ *there exists a set* $E_\gamma \subset (0, 1/2)$ *of zero Lebesgue measure such that*

$$\left(\lambda \notin E_\gamma, \frac{\log 2}{\log 1/\lambda} + \frac{\log 2}{\log 1/\gamma} > 1\right) \implies Leb(C_\lambda + C_\gamma) > 0 \tag{2.12}$$

Additionally, in his paper, Solomyak discusses the known exceptional parameters, i.e. the sets of parameters for which $C_\lambda + C_\gamma$ has Lebesgue measure zero. It turns out that these exceptional sets are of the form $\lambda = \gamma^\theta$, for θ rational. Furthermore, the results related to the topological structure of $C_\lambda + C_\gamma$ are are outlined.

In the region where $\frac{\lambda}{1-2\lambda}\frac{\gamma}{1-2\gamma} > 1$ (which Solomyak calls the 'large' region), Mendes and Oliveira demonstrated in [37] that $C_\lambda + C_\gamma = [0, 2]$. In R, they also demonstrated that $C_\lambda + C_\gamma$ can have one of three different structures: it can be the interval $[0, 2]$, a Cantor set, or an M-Cantorval, a perfect set in $\mathbb{R}$, where any gap is accumulated on both sides by infinitely many intervals and gaps (for instance, by blowing up the right and left endpoints of a Cantor set). They also gave criterion which can determine when $C_\lambda + C_\gamma = [0, 2]$.

1. If $\theta = \frac{\log \lambda}{\log \gamma}$ is irrational, then $C_\lambda + C_\gamma \neq [0, 2]$.

2. If $\theta = n$ or $\theta = \frac{n+1}{n}$, then $C_\lambda + C_\gamma$ will be an interval if and only if $\lambda \geq 1 - 2\gamma$.

However, beyond these results, it is unknown whether $C_\lambda + C_\gamma$ will have nonempty interior, even generically.

2.5 Homogeneous Self-Similar Cantor Sets

By a homogeneous self-similar Cantor set C_λ we mean a dynamically defined Cantor set where the family of generating contractions $\{\phi_i\}_{i=1}^m$ are affine maps with identical contraction constants; that is to say, $\phi_i(x) = \lambda x + d_i(\lambda)$, such that

$$\{d_i(\lambda) + \lambda C_\lambda\} \cap \{d_j(\lambda) + \lambda C_\lambda\} = \emptyset \text{ for } i \neq j \tag{2.13}$$

which is equivalent to (2.1) in Definition 2.1 and is often called the strong separation condition. As with the middle$-\alpha$ Cantor sets (with similarity ratio λ), the strong separation condition implies that $\dim_H C_\lambda = \frac{\log m}{\log 1/\lambda}$.

Theorem 2.6 (Theorem 1.1 in [53]). *Suppose that K is a compact set on the real line and $J \subset (0,1)$ is an interval such that the family $\{C_\lambda\}$ are homogeneous self-similar Cantor sets for all $\lambda \in J$. Then*

(a) for a.e. $\lambda \in J$ with $\dim_H K + \dim_H C_\lambda < 1$ we have $\dim_H(K + C_\lambda) = \dim_H K + \dim_H C_\lambda$;

(b) for a.e. $\lambda \in J$ with $\dim_H K + \dim_H C_\lambda > 1$, the set $K + C_\lambda$ has positive Lebesgue measure.

The proof of the main result of this paper, Theorem 1.1 is similar to that of part (b) above (not surprisingly, since the proof of Theorem 1.1 uses a Proposition from [14] for which Boris Solomyak is a co-author); a Frostman measure for K is convolved with a self-similar measure for C_λ; this convolution is supported on $K + C_\lambda$. It can be shown that for a.e. $\lambda \in J$ which satisfy some conditions related to the sum of the Hausdorff dimenions of K and C_λ, this convolution will be absolutely continuous with respect to the Lebesgue measure, and so $K + C_\lambda$ will have positive Lebesgue measure for such λ.

2.6 A Positive Answer to the Palis Conjecture

In [36], Moreira established the stable intersection property for a pair of dynamically defined Cantor sets C_1 and C_2. We say that the pair (C_1, C_2) have the stable intersection property if there is a neighborhood V of (C_1, C_2) in the space of pairs of $C^{1+\varepsilon}$ dynamically defined Cantor sets such that for any (C_1', C_2') in V we have $C_1' \cap C_2' \neq \emptyset$.

Moreira posed the following question: does there exist an open and dense subset $U \subset \Omega^\infty = \{(C_1, C_2) : C_1, C_2$ dynamically defined Cantor sets such that $\dim_H C_1 + \dim_H C_2 > 1\}$ so that if (C_1, C_2) are in U, then will there exist $t \in \mathbb{R}$ such that $(C_1, C_2 - t)$ has the stable intersection property?

An affirmative answer to this question implies a strong version of the Palis conjecture; if $\dim_H C_1 + \dim_H C_2 > 1$ and $(C_1, C_2 - t)$ has stable intersection then $C_1 \cap (C_2 - t) \neq \emptyset$ for some neighborhood of t and so $C_1 - C_2$ must contain an interval.

In [39], Moreira and Yoccoz would provide the affirmative answer to this question and thus to the Palis conjecture with the following theorem

Theorem 2.7. *There is an open and dense subset $U \subset \Omega^\infty$ such that if $(C_1, C_2) \in U$ then $I_s(C_1, C_2) := \{t \in \mathbb{R} : C_1$ and $(C_2 + t)$ have stable intersection$\}$ is dense in $C_1 - C_2$ and*

$$\dim_H((C_1 - C_2) \setminus I_s(C_1, C_2)) < 1. \tag{2.14}$$

This proves the strong form of the Palis conjecture. However, the genericity assumptions are difficult to verify, even for a one parameter family of dynamically defined Cantor sets, as often arises during applications. Neither does it confirm the Palis conjecture for affine Cantor sets.

2.7 The $\dim_H C_1 + \dim_H C_2 < 1$ Case

As we mentioned in section 2.3, when $\dim_H C_1 + \dim_H C_2 < 1$, we are interested in knowing precisely when $\dim_H(C_1 + C_2) = \dim_H C_1 + \dim_H C_2$ for dynamically defined Cantor sets C_1

and C_2.

We've already mentioned the work of Peres and Solomyak regarding homogeneous Cantor sets in section 2.5. Part (a) of Theorem 2.6 states that for C_λ a homogeneous Cantor set, then for a compact set K such that $\dim_H K + \dim_H C_\lambda < 1$ for all λ in an interval J, then we will have that $\dim_H(K + C_\lambda) = \dim_H K + \dim_H C_\lambda$ for almost all choices of parameter.

In [51], Peres and Shmerkin were able to further elaborate on this result, providing a description to the possible exceptions for affine Cantor sets (including middle-α and homogeneous Cantor sets)

Theorem 2.8 (Theorem 2 in [51]). *Suppose $\{\lambda_k x + d_k\}_{k=1}^{m_\lambda}$ and $\{\gamma_k x + d'_k\}_{k=1}^{m_\gamma}$ are finite sets of affine contractions which generate dynamically defined (affine) Cantor sets C_λ and C_γ, respectively. If there exist j and j' such that $\log|\lambda_j|/\log|\gamma_{j'}|$ is irrational, then*

$$\dim_H(C_\lambda + C_\gamma) = \min\{1, \dim_H C_\lambda + \dim_H C_\gamma\} \tag{2.15}$$

We have the homogeneous case when $\lambda_k = \lambda$ and $\gamma_k = \gamma$ for all possible choices of k.

In [24], Hochman and Shmerkin lay out conditions for which a pair of dynamically defined Cantor sets C_1 and C_2 will have $\dim_H(C_1 + C_2) = \min\{1, \dim_H C_1 + \dim_H C_2\}$

If a function f is a smooth contraction of $[0,1]$ then it has a unique fixed point x and we can define

$$\lambda(f) = -\log(f'(x)) \tag{2.16}$$

If $\mathcal{I} = \{f_i\}_{i=1}^m$ is an IFS generating a dynamically defined Cantor set, define

$$L(\mathcal{I}) = \{\lambda(f_{x_1} \circ \cdots \circ f_{x_n}) : n \in \mathbb{N},\, x_1, \ldots, x_n \in \{1, \ldots, m\}\} \tag{2.17}$$

Theorem 2.9 (Theorem 1.4 from [24]). *Let $\mathcal{I}^{(i)} = \{f_j^{(i)}\}_{j=1}^{m^{(i)}}$, $i = 1, \ldots, d$ be IFSs generating dynamically defined Cantor sets C_i. Suppose the following minimality assumption holds: the* *set* $L(\mathcal{I}^{(1)}) \times \cdots \times L(\mathcal{I}^{(d)})$ *is dense in the quotient space* $(\mathbb{R}^d, +)/\Delta$*, where* Δ *is the diagonal subgroup of* $\mathbb{R}^d$*. Then if* $\pi(x) = \sum_{i=1}^{d} t_i x_i$ *with all* t_i *non-zero,*

$$\dim_H \pi(C_1 \times \cdots \times C_d) = \min\{1, \dim_H C_1 + \cdots + \dim_H C_d\}.$$

Chapter 3

Results for Nearly Affine Cantor Sets

3.1 Absolute Continuity of Convolutions

Let $\mathcal{A}$ be an alphabet of size $m \geq 2$ and let $\Omega = \mathcal{A}^{\mathbb{Z}_+}$ be the standard symbolic space, equipped with the product topology. Let μ be a Borel probability measure on Ω.

Let J be a compact interval and assume we are given a family of continuous maps $\Pi_\lambda : \Omega \to \mathbb{R}$, for $\lambda \in J$, such that $C_\lambda = \Pi_\lambda(\Omega)$ are the Cantor sets, and let $\nu_\lambda = \Pi_\lambda(\mu) := \mu \circ \Pi_\lambda^{-1}$.

For a word $u \in \mathcal{A}^n$, $n \geq 0$, denote by $|u| = n$ its length and by $[u]$ the cylinder set of elements of Ω that have u as a prefix. For $\omega, \tau \in \Omega$ we write $\omega \wedge \tau$ for the maximum common subword in the beginning of ω and τ (empty if $\omega_0 \neq \tau_0$; we set the length of the empty word to be zero). For $\omega, \tau \in \Omega$, let $\phi_{\omega,\tau}(\lambda) := \Pi_\lambda(\omega) - \Pi_\lambda(\tau)$.

We will need the following statement, which is Proposition 2.3 from [14].

Proposition 3.1. *Let η be a compactly supported Borel probability measure on $\mathbb{R}$ of exact local dimension d_η. Suppose that for any $\varepsilon > 0$ there exists a subset $\Omega_\varepsilon \subset \Omega$ such that $\mu(\Omega_\varepsilon) > 1 - \varepsilon$ and the following holds; there exist constants $C_1, C_2, C_3, \alpha, \beta, \gamma > 0$ and*

$k_0 \in \mathbb{Z}_+$ *such that*

(a) $d_\eta + \dfrac{\gamma}{\beta} > 1$ *and* $d_\eta > \dfrac{\beta - \gamma}{\alpha}$,

(b) $\max_{\lambda \in J} |\phi_{\omega,\tau}(\lambda)| \le C_1 m^{-\alpha|\omega\wedge\tau|}$ *for all* $\omega, \tau \in \Omega_\varepsilon, \omega \neq \tau$,

(c) $\sup_{v\in\mathbb{R}} Leb(\{\lambda \in J : |v + \phi_{\omega,\tau}(\lambda)| \le r\}) \le C_2 m^{|\omega\wedge\tau|\beta} r$ *for all* $\omega, \tau \in \Omega_\varepsilon, \omega \neq \tau$ *such that* $|\omega \wedge \tau| \ge k_0$, *and*

(d) $\max_{u\in\mathcal{A}^n, [u]\cap\Omega_\varepsilon\neq\emptyset} \mu([u]) \le C_3 m^{-\gamma n}$ *for all* $n \ge 1$.

Then the convolution $\eta * \nu_\lambda$ *is absolutely continuous with respect to the Lebesgue measure for a.e.* $\lambda \in J$.

Remark 3.1. *In fact, in Proposition 3.1 the condition on exact dimensionality of the measure* η *can be replaced by the following condition (and this is the only consequence of exact dimensionality of* η *that was used in the proof of Proposition 3.1 in [14]):* η *is a compactly supported Borel probability measure on the real line, then there exists* $D > 0$ *such that*

$$\eta[B_r(x)] \le D r^{d_\eta}, \text{ for all } x \in \mathbb{R} \text{ and } r > 0. \tag{3.1}$$

Remark 3.2. *Condition (c) is the hardest to check in typical applications. It follows from*

$$\inf_{\lambda_1,\lambda_2\in J,\, \lambda_1\neq\lambda_2} \frac{|\phi_{\omega,\tau}(\lambda_1) - \phi_{\omega,\tau}(\lambda_2)|}{|\lambda_1 - \lambda_2|} \ge C_2' m^{-|\omega\wedge\tau|\beta} \quad \text{for all} \ \ \omega, \tau \in \Omega_\varepsilon, \ \omega \neq \tau.$$

An estimate of this kind in turn follows from a suitable lower bound on $|\frac{d}{d\lambda}\phi_{\omega,\tau}(\lambda)|$.

For the interested reader, the proof of Proposition 3.1 can be found in Appendix A.

3.2 Theorem 1.1

We are now ready to prove Theorem 1.1. We repeat it here for convenience.

Theorem 1.1. *Suppose $J \subseteq \mathbb{R}$ is an interval and $\{C_\lambda\}_{\lambda \in J}$ is a family of dynamically defined Cantor sets generated by contracting maps*

$$\{f_{i,\lambda}(x) = c_i(\lambda)x + b_i(\lambda) + g_i(x,\lambda)\}_{i=1}^m \tag{3.2}$$

such that the following holds:

$$c_i(\lambda), b_i(\lambda) \text{ are } C^1\text{-functions of } \lambda; \tag{3.3}$$

$$\frac{dc_i}{d\lambda} \leq -\delta < 0 \text{ for all } \lambda \in J \text{ and some uniform } \delta > 0; \tag{3.4}$$

$$g_i(x,\lambda) \in C^2, \text{ with small (based on } \{c_i(\lambda)\}, \{b_i(\lambda)\}\text{) } C^2\text{-norm.} \tag{3.5}$$

Then for any compact $K \subset \mathbb{R}$ with

$$\dim_H(K) + \dim_H(C_\lambda) > 1 \quad \text{for all} \quad \lambda \in J, \tag{3.6}$$

the sumset $K + C_\lambda$ has positive Lebesgue measure for a.e. $\lambda \in J$.

Though we have written our family of Cantor sets in Theorem 1.1 using definition 2.1, we will assume that they are also dynamically defined in the sense of definition 2.2; that is, there is an expansive map Φ_λ which is defined on $I_{1,\lambda} \cup \cdots \cup I_{m,\lambda}$, where $I_{k,\lambda} = f_{k,\lambda}([0,1])$, and $f_{k,\lambda} = (\Phi_\lambda|_{I_{k,\lambda}})^{-1}$. $I_{1,\lambda}, \ldots, I_{m,\lambda}$ form the Markov partition for C_λ

To prove Theorem 1.1 we will construct particular measures for K and C_λ and apply Proposition 3.1.

3.3 Construction of Measures for K and C_λ

Here we will construct the measure η supported on K and a family of measures $\{\nu_\lambda\}$ with $supp\, \nu_\lambda = C_\lambda$ such that Proposition 3.1 can be applied. Since absolute continuity of the convolution $\eta * \nu_\lambda$ implies that $Leb(C_\lambda + K) > 0$, this will prove Theorem 1.1.

Let us start with construction of the measure η. The compact set $K \subset \mathbb{R}$ satisfies the condition (3.6), i.e. $\dim_H(K)+\dim_H(C_\lambda) > 1$ for all $\lambda \in J$. Take any constant $d \in (0, \dim_H(K))$ that is sufficiently close to $\dim_H(K)$ to guarantee that $d + \dim_H(C_\lambda) > 1$ for all $\lambda \in J$. We will need Frostman's lemma (see, e.g., [33, Theorem 8.8])

Lemma 3.1 (Frostman's Lemma)**.** *Let E be a Borel subset of $\mathbb{R}^n$, and let $s > 0$. Then the following are equivalent:*

- $\mathcal{H}^s(E) > 0$ *where $\mathcal{H}^s(E)$ is the s-dimensional Hausdorff measure*
- *There is an (unsigned) Borel measure η satisfying $\eta(E) > 0$, and such that $\eta(B(x,r)) \leq r^s$ for all $x \in \mathbb{R}^n$ and $r > 0$.*

Since $d < \dim_H K = \inf\{s : \mathcal{H}^s(K) = 0\}$, we have that $\mathcal{H}^d(K) > 0$. Thus, by Frostman's Lemma, there exists a Borel measure η supported on K such that (3.1) holds with $d_\eta = d$.

To construct measures for $\{C_\lambda\}$, first let us define the Hausdorff dimension of a measure. For a measure μ, define

$$\dim_H(\mu) = \inf\{\dim_H E : \mu(E) = 1\} \tag{3.7}$$

In general, for measures μ supported on a set E we have that $\dim_H \mu \leq dim_H E$. However, from Theorem 13.1 from [52] (see also [35, 34]), if C_λ is a dynamically defined Cantor set with expansive map Φ_λ, then there exists a Φ_λ-invariant ergodic measure μ_λ, supported on C_λ, such that

$$\dim_H C_\lambda = \dim_H \mu_\lambda = \frac{h_{\mu_\lambda}(\Phi_\lambda)}{Lyap^u(\mu_\lambda)} \tag{3.8}$$

where $h_{\mu_\lambda}(\Phi_\lambda)$ is the measure theoretic entropy of Φ_λ, and $Lyap^u(\mu_\lambda)$ is the Lyapunov exponent of Φ_λ, both with respect to the ergodic measure μ_λ. We call this the *measure of* *maximal dimension.*

In the setting of Proposition 3.1, we have one measure on the symbolic space Ω which defines the family of measures ν_λ on C_λ. However, we would like to use the particular properties of the measure of maximal dimension. To construct the measure μ on the symbolic space, we will pick some λ_0 and then consider the measure $\mu := \Pi_\lambda^{-1}(\mu_{\lambda_0})$. Then the measures $\nu_\lambda = \Pi_\lambda(\mu)$ will have the desired properties for λ close enough to λ_0. Since J is assumed to be compact, we only need to partition J into finitely many subintervals and apply Proposition 3.1 on each one. We demonstrate how to do this now.

For each $\omega \in \Omega$ let $F_\lambda^n(\omega) = f_{\omega_0,\lambda} \circ \cdots \circ f_{\omega_n,\lambda}(x)$, for a fixed $x \in [0,1]$. Then we can define the map $\Pi_\lambda : \Omega \to \mathbb{R}$ given by

$$\Pi_\lambda(\omega) = \lim_{n\to\infty} F_\lambda^n(\omega), \tag{3.9}$$

The limit does not depend on the initial point $x \in [0,1]$; this is a result of the Cantor Intersection Theorem and because the functions $f_{\omega_i,\lambda}$ are contractions. For any $\lambda_1, \lambda_2 \in J$ the map $h_{\lambda_1,\lambda_2} : C_{\lambda_2} \to C_{\lambda_1}$ defined by $h_{\lambda_1,\lambda_2} = \Pi_{\lambda_1} \circ \Pi_{\lambda_2}^{-1}$ is a homeomorphism. It is well known (e.g. see Section 19 in [27]) that this homeomorphism must be Hölder continuous. Moreover, due to [49] the following statement holds.

Lemma 3.2. *For any $\lambda_0 \in J$ and any $\tau \in (0,1)$ there exists a neighborhood $V \subseteq J$, $\lambda_0 \in V$, such that for any $\lambda \in V$ the conjugacy $h_{\lambda,\lambda_0} : C_{\lambda_0} \to C_\lambda$ as well as its inverse $h_{\lambda_0,\lambda} : C_\lambda \to C_{\lambda_0}$ are Hölder continuous with Hölder exponent τ.*

We also need the following easy lemma

Lemma 3.3. *If $f : \mathbb{R}^d \to \mathbb{R}^d$ is Hölder continuous with Hölder exponent τ then for any $E \subset \mathbb{R}^d$*

$$\dim_H f(E) \le \frac{1}{\tau} \dim_H E \tag{3.10}$$

Proof. If $\{U_i\}$ is a δ-cover of E, then $diam(U_i) < \delta$ and so $diam(f(U_i)) \le C \cdot diam(U_i)^\tau$.

Thus, $\{f(U_i)\}$ is a $C\delta^\tau$-cover of $f(E)$ and for any $s \geq 0$, we have that

$$\sum_i diam(f(U_i))^s \leq C^s \sum_i diam(U_i)^{\tau s} \tag{3.11}$$

Therefore, we have that $\mathcal{H}^s_{C\delta^\tau}(f(E)) \leq C^s \mathcal{H}^{\tau s}_\delta(E)$. Taking $\delta \to 0$, we have $\mathcal{H}^s(f(E)) \leq C^s \mathcal{H}^{\tau s}(E)$.

Now, $\frac{1}{\tau} \dim_H E = \inf\{s : \mathcal{H}^{\tau s}(E) = 0\}$. Since $\mathcal{H}^s(f(E)) \leq C^s \mathcal{H}^{\tau s}(E)$ for all s, we must have that

$$\dim_H f(E) \leq \frac{1}{\tau} \dim_H E \tag{3.12}$$

□

Define the measure μ_0 on Ω by $\mu_0 := \Pi_{\lambda_0}^{-1}(\mu_{\lambda_0})$, and set

$$\nu_\lambda := \Pi_\lambda(\mu_0) = \Pi_\lambda(\Pi_{\lambda_0}^{-1}(\mu_{\lambda_0})) = h_{\lambda,\lambda_0}(\mu_{\lambda_0}) = h_{\lambda,\lambda_0}(\nu_{\lambda_0}).$$

If both h_{λ,λ_0} and $h_{\lambda_0,\lambda}$ are Hölder continuous with Hölder exponent τ then by Lemma 3.3

$$\tau \dim_H C_{\lambda_0} = \tau \dim_H \nu_{\lambda_0} \leq \dim_H \nu_\lambda \leq \frac{1}{\tau} \dim_H \nu_{\lambda_0} = \frac{1}{\tau} \dim_H C_{\lambda_0}.$$

Since in Lemma 3.2 the value of τ can be taken arbitrarily close to one, we get the following statement.

Lemma 3.4. *For any $\lambda_0 \in J$ there exists a neighborhood $W \subseteq J$, $\lambda_0 \in W$, such that for any $\lambda \in W$ we have*

$$d_\eta + \dim_H \nu_\lambda > 1.$$

It is clear that in order to prove Theorem 1.1 it is enough to prove that for each $\lambda_0 \in J$ there exists a neighborhood $W, \lambda_0 \in W$, such that the sum $K + C_\lambda$ has positive Lebesgue measure for a.e. λ from W. Decreasing if needed the neighborhood W given by Lemma 3.4 we can choose positive $\varepsilon, \alpha, \beta$, and γ in such a way that for all $\lambda \in W$ the following properties hold:

$$
\begin{gathered}
d_\eta + \dim_H \nu_\lambda > 1 + \varepsilon, \\
\alpha < \frac{Lyap^u(\nu_\lambda)}{\log m} < \beta, \\
\gamma < \frac{h_{\nu_\lambda}(\Phi_\lambda)}{\log m}, \\
\frac{\beta - \alpha}{\gamma} < 1, \\
\frac{\beta}{\alpha} < 1 + \frac{\varepsilon}{2}, \\
\frac{h_{\nu_\lambda}(\Phi_\lambda)}{Lyap^u(\nu_\lambda)} - \frac{\gamma}{\alpha} < \frac{\varepsilon}{2}.
\end{gathered}
$$

Therefore

$$d_\eta > 1 + \varepsilon - \dim_H(\nu_\lambda) > \frac{\beta}{\alpha} - \frac{\gamma}{\alpha}$$

for $\lambda \in W$, which implies the conditions in part (a) of Proposition 3.1, namely,

$$d_\eta + \frac{\gamma}{\beta} > 1 \text{ and } d_\eta > \frac{\beta - \gamma}{\alpha}.$$

3.4 Verifying the Conditions of Proposition 3.1

We are now ready to verify the rest of the conditions of Proposition 3.1, we will try to mimic the proof of Theorem 3.7 from [14]. The plan will be to fix points $p, q \in C_\lambda$. We can iterate p and q by Φ_λ until their distance apart is of order 1. To simplify notation, it will be helpful to parameterize $C_\lambda, \Phi_\lambda(C_\lambda), \dots, \Phi_\lambda^n(C_\lambda)$ so that $\Phi_\lambda^i(p)$ is the origin for each iterate of C_λ. Then we can find a word $u \in \mathcal{A}^{n+1}$ that represents the "itinerary" of q. That is to say, if $u = u_0 u_1 \dots u_n$, then $\Phi_\lambda^i(q) \in I_{u_i,\lambda}$ where $I_{1,\lambda}, \dots, I_{m,\lambda}$ are the elements of the Markov partition.

Now, take $k_\lambda^{(t)} = f_{u_t,\lambda}$, and take $a^{(n)}$ to be the distance between $\Phi_\lambda^n(p)$ and $\Phi_\lambda^n(q)$ in $\Phi_\lambda^n(C_\lambda)$.

Define $P_n(\lambda) = \Phi^n_\lambda(p)$ and $Q_n = \Phi^n_\lambda(q)$. For $m < n$ Define

$$P^n_m(\lambda) = k^{(m)}_\lambda \circ k^{(m+1)}_\lambda \circ \cdots \circ k^{(n)}_\lambda(P_n(\lambda))$$

$$Q^n_m(\lambda) = k^{(m)}_\lambda \circ k^{(m+1)}_\lambda \circ \cdots \circ k^{(n)}_\lambda(Q_n(\lambda)),$$

$P^n_n(\lambda) = k^{(n)}_\lambda(P_n(\lambda))$ and $P^n_{n+1}(\lambda) = P_n(\lambda)$ (similarly for Q^n_n and Q^n_{n+1}). Let $K^n_m(\lambda) = P^n_m(\lambda) - Q^n_m(\lambda)$.

In order to verify the first conditions of Proposition 3.1 we need to find a bound on $\frac{d}{d\lambda}(q(\lambda) - p(\lambda)) = \frac{d}{d\lambda}(K^n_1(\lambda))$.

$$\frac{d}{d\lambda}(P^n_1(\lambda) - Q^n_1(\lambda)) =$$

$$\begin{aligned}
&= \sum_{i=1}^n \left(\prod_{s=1}^{i-1} \frac{\partial k^{(s)}_\lambda}{\partial x}(P^n_{s+1}(\lambda))\right) \frac{\partial k^{(i)}_\lambda}{\partial \lambda}(P^n_{i+1}(\lambda)) + \left(\prod_{s=1}^{n} \frac{\partial k^{(s)}_\lambda}{\partial x}(P^n_{s+1}(\lambda))\right) \frac{\partial P_n}{\partial \lambda}(\lambda) \\
&- \sum_{i=1}^n \left(\prod_{s=1}^{i-1} \frac{\partial k^{(s)}_\lambda}{\partial x}(Q^n_{s+1}(\lambda))\right) \frac{\partial k^{(i)}_\lambda}{\partial \lambda}(Q^n_{i+1}(\lambda)) - \left(\prod_{s=1}^{n} \frac{\partial k^{(s)}_\lambda}{\partial x}(Q^n_{s+1}(\lambda))\right) \frac{\partial Q_n(\lambda)}{\partial \lambda} \\
&= \sum_{i=1}^n \left(\prod_{s=1}^{i-1} \frac{\partial k^{(s)}_\lambda}{\partial x}(P^n_{s+1}(\lambda))\right) \left(\frac{\partial k^{(i)}_\lambda}{\partial \lambda}(P^n_{i+1}(\lambda)) - \frac{\partial k^{(i)}_\lambda}{\partial \lambda}(Q^n_{i+1}(\lambda))\right) \\
&+ \sum_{i=1}^n \frac{\partial k^{(i)}_\lambda}{\partial \lambda}(Q^n_{i+1}(\lambda)) \left(\prod_{s=1}^{i-1} \frac{\partial k^{(s)}_\lambda}{\partial x}(P^n_{s+1}(\lambda)) - \prod_{s=1}^{i-1} \frac{\partial k^{(s)}_\lambda}{\partial x}(Q^n_{s+1}(\lambda))\right) \\
&+ \left(\left(\prod_{s=1}^{n} \frac{\partial k^{(s)}_\lambda}{\partial x}(P^n_{s+1}(\lambda))\right) \frac{\partial P_n}{\partial \lambda}(\lambda) - \left(\prod_{s=1}^{n} \frac{\partial k^{(s)}_\lambda}{\partial x}(Q^n_{s+1}(\lambda))\right) \frac{\partial Q_n(\lambda)}{\partial \lambda}\right) \\
&= S_1 + S_2 + S_3
\end{aligned}$$

First, define $\frac{\partial k^{(t)}_\lambda}{\partial x}(P^n_{t+1}(\lambda)) = l^{(t)}(\lambda)$. Then we have that

$$k^{(t)}_\lambda(x) = k^{(t)}_\lambda(P^n_{t+1}(\lambda)) + l^{(t)}(x - P^n_{t+1}(\lambda)) + O((x - P^n_{t+1}(\lambda))^2)$$

and

$$\frac{\partial k^{(t)}_\lambda}{\partial x}(x) = l^{(t)} + O(x - P^n_{t+1}(\lambda))$$

It's clear that $S_3 = O(\prod_{s=1}^n l^{(s)})$. Let's first consider S_1:

$$S_1 = \sum_{i=1}^n \left(\prod_{s=1}^{i-1} \frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda)) \right) \left(\frac{\partial k_\lambda^{(i)}}{\partial \lambda}(P_{i+1}^n(\lambda)) - \frac{\partial k_\lambda^{(i)}}{\partial \lambda}(Q_{i+1}^n(\lambda)) \right)$$
$$= \sum_{i=1}^n \left(\prod_{s=1}^{i-1} l^{(s)} \right) \frac{\partial^2 k_\lambda^{(s)}}{\partial x \partial \lambda}(W_{i+1}^n(\lambda))(P_{i+1}^n(\lambda) - Q_{i+1}^n(\lambda))$$

where $W_{i+1}^n(\lambda)$ is between $P_{i+1}^n(\lambda)$ and $Q_{i+1}^n(\lambda)$. We will need the following lemma.

Lemma 3.5 (from [14])**.** *There is a constant $C > 0$ such that if $m, n \in \mathbb{Z}_+$ and $m \leq n+1$ then*

$$\frac{1}{C}\left(\prod_{s=m}^n l^{(s)} \right) \leq K_m^n(\lambda) \leq C \left(\prod_{s=m}^n l^{(s)} \right)$$

Moreover, for any $\varepsilon > 0$ there is $n_0 \in \mathbb{Z}_+$ such that for all $n > n_0$, there exists $A_n > 0$ such that for all $I \in \mathbb{Z}_+$ with $I \leq n - n_0 + 1 < n$, we have

$$\left| \frac{K_{I+1}^n(\lambda)}{\prod_{s=I+1}^n l^{(s)}} - A_n \right| \leq \varepsilon.$$

The sequence A_n is uniformly bounded above and away from zero.

Proof.

$$P_m^n(\lambda) - Q_m^n(\lambda) =$$
$$= k_\lambda^{(m)}(P_{m+1}^n) - k_\lambda^{(m)}(Q_{m+1}^n) =$$
$$= k_\lambda^{(m)}(P_{m+1}^n) - (k_\lambda^{(m)}(P_{m+1}^n) + l^{(m)}(-K_{m+1}^n) + O((-K_{m+1}^n)^2)$$
$$= K_{m+1}^n(l^{(m)} + O(K_{m+1}^n)$$
$$= \dots$$
$$= (P_n(\lambda) - Q_n(\lambda)) \prod_{s=m}^n (l^{(s)} + O(K_{s+1}^n))$$
$$= K_{n+1}^n(\lambda) \left(\prod_{s=m}^n l^{(s)} \right) \left(\prod_{s=m}^n 1 + \frac{O(K_{s+1}^n))}{l^{(s)}} \right)$$

Since $\{K_m^n(\lambda)\}$ is bounded by a geometric progression, the product $\prod_{s=m}^n 1 + \frac{O(K_{s+1}^n))}{l^{(s)}}$ is bounded above and below, and since $P_n(\lambda) - Q_n(\lambda)$ is bounded above and below, we have the first inequality.

For $n_0 \in \mathbb{Z}_+$ sufficiently large, and $I \le n - n_0 + 1 < n$, we have

$$\frac{K_{I+1}^n(\lambda)}{\prod_{s=I+1}^n l^{(s)}} = \prod_{s=I+1}^{n} \left(1 + \frac{O(K_{s+1}^n(\lambda))}{l^{(s)}}\right)(P_n(\lambda) - Q_n(\lambda))$$

If we have

$$A_n = \prod_{s=n-n_0+2}^{n} \left(1 + \frac{O(K_{s+1}^n(\lambda))}{l^{(s)}}\right)(P_n(\lambda) - Q_n(\lambda))$$

Then

$$\left|\frac{K_{I+1}^n(\lambda)}{\prod_{s=I+1}^n l^{(s)}} - A_n\right| = A_n \left|\prod_{s=I+1}^{n-n_0+1} \left(1 + \frac{O(K_{s+1}^n(\lambda))}{l^{(s)}}\right) - 1\right|$$

Since $\{K_{s+1}^n(\lambda)\}_{s=I+1}^{n-n_0+1}$ is bounded by a geometric progression, the choice of n_0 can guarantee that its terms are sufficiently small and hence

$$\left|\prod_{s=I+1}^{n-n_0+1} \left(1 + \frac{O(K_{s+1}^n(\lambda))}{l^{(s)}}\right) - 1\right| < \varepsilon.$$

Which gives the second estimate and completes the proof. $\square$

Using this lemma,

$$S_1 = \sum_{i=1}^{n-n_0} \left(\prod_{s=1}^{i-1} l^{(s)}\right) \frac{\partial^2 k_\lambda^{(s)}}{\partial x \partial \lambda}(W_{i+1}^n(\lambda))K_{i+1}^n(\lambda)$$
$$+ \sum_{i=n-n_0+1}^{n} \left(\prod_{s=1}^{i-1} l^{(s)}\right) \frac{\partial^2 k_\lambda^{(s)}}{\partial x \partial \lambda}(W_{i+1}^n(\lambda))K_{i+1}^n(\lambda)$$

The second sum is $O(\prod_{s=1}^{n} l^{(s)})$. As for the first sum

$$\begin{aligned}
&\sum_{i=1}^{n-n_0}\left(\prod_{s=1}^{i-1} l^{(s)}\right)\frac{\partial^2 k_\lambda^{(s)}}{\partial x\partial\lambda}(W_{i+1}^n(\lambda))K_{i+1}^n(\lambda) = \\
&= \sum_{i=1}^{n-n_0}\left(\prod_{s=1}^{n} l^{(s)}\right)\left(\frac{\partial^2 k_\lambda^{(s)}}{\partial x\partial\lambda}(W_{i+1}^n(\lambda))\right)\frac{K_{i+1}^n(\lambda)}{\prod_{i+1}^n l^{(s)}}\frac{1}{l^{(i)}} \\
&\leq \sum_{i=1}^{n-n_0}\left(\prod_{s=1}^{n} l^{(s)}\right)(-B)(A_n-\varepsilon) = -(n-n_0)B\left(\prod_{s=1}^{n} l^{(s)}\right)(A_n-\varepsilon)
\end{aligned}$$

Since $\frac{\partial^2 k_\lambda^{(s)}}{\partial x\partial\lambda} \leq -\delta < 0$ for all x and λ by assumption.

As far as S_2, note that we're assuming that each C_λ is a perturbation of an affine Cantor set, that is each contraction $f_{k,\lambda}(x) = c_k(\lambda)x + b_k(\lambda) + g_k(x,\lambda)$ where the C^2-norm of $g_k(x,\lambda)$ is small.

$$\begin{aligned}
|S_2| &= \\
&= \left|\sum_{i=1}^{n}\frac{\partial k_\lambda^{(i)}}{\partial\lambda}(Q_{i+1}^n(\lambda))\left(\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda)) - \prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))\right)\right| \\
&\leq \sum_{i=1}^{n}\left|\frac{\partial k_\lambda^{(i)}}{\partial\lambda}(Q_{i+1}^n(\lambda))\right|\left|\left(\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda)) - \prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))\right)\right| \\
&\leq \sum_{i=1}^{n} C\left|\left(\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda)) - \prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))\right)\right| \\
&= C\sum_{i=1}^{n}\left(\prod_{s=1}^{i-1} l^{(s)}\right)\left|1 - \frac{\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))}{\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))}\right|
\end{aligned}$$

Lemma 3.6.

$$\left|1 - \frac{\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))}{\prod_{s=1}^{i-1}\frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))}\right| \leq C'''\left(\prod_{s=i}^{n} l^{(s)}\right)\max_{t=1,\dots,m}\|g_t\|_{C^2}$$

for some $C''' > 0$.

Proof of Lemma 3.6. Note that if A is near 1 and B is much smaller than 1, we have that

$$|\log A| < B \text{ implies } |A-1| \leq 2B$$

for some $c > 0$. We have

$$
\begin{aligned}
|\log A| < B &\Rightarrow e^{-B} - 1 < A - 1 < e^{B} - 1 \\
&\Rightarrow -B + O(B^2) < A - 1 < B + O(B^2) \\
&\Rightarrow |A - 1| < 2B
\end{aligned}
$$

for small B.

To prove Lemma 3.6, we will show that $\left|\log \frac{\prod_{s=1}^{i-1} \frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))}{\prod_{s=1}^{i-1} \frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))}\right|$ is small. By the mean value theorem we get

$$
\begin{aligned}
&\left|\log \frac{\prod_{s=1}^{i-1} \frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda))}{\prod_{s=1}^{i-1} \frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))}\right| = \\
&= \left|\sum_{s=1}^{i-1} \log \frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda)) - \sum_{s=1}^{i-1} \log \frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))\right| \\
&\leq C \sum_{s=1}^{i-1} \left|\frac{\partial k_\lambda^{(s)}}{\partial x}(Q_{s+1}^n(\lambda)) - \frac{\partial k_\lambda^{(s)}}{\partial x}(P_{s+1}^n(\lambda))\right| \\
&= C \sum_{s=1}^{i-1} \left|\frac{\partial g_s}{\partial x}(Q_{s+1}^n(\lambda)) - \frac{\partial g_s}{\partial x}(P_{s+1}^n(\lambda))\right| \\
&= C \sum_{s=1}^{i-1} \left|\frac{\partial^2 g_s}{\partial x^2}(V_{s+1})\right| \left|Q_{s+1}^n(\lambda) - P_{s+1}^n(\lambda)\right| \\
&\leq \max_{t=1,\dots,m} \|g_t\|_{C^2} C' \sum_{s=1}^{i-1} \left(\prod_{j=s+1}^{n} l^{(s)}\right) \\
&\leq \max_{t=1,\dots,m} \|g_t\|_{C^2} C'' \prod_{s=i}^{n} l^{(s)}
\end{aligned}
$$

since the terms of the last sum are bounded by a geometric progression. This proves the estimate Lemma 3.6. □

Therefore we have

$$
|S_2| \leq n\tilde{C} \left(\prod_{i=1}^{n} l^{(s)}\right) \max_{t=1,\dots,m} \|g_t\|_{C^2}
$$

Hence we have

$$
\begin{aligned}
&S_1 + S_2 + S_3 \leq \\
&\leq -nB\left(\prod_{s=1}^{n} l^{(s)}\right)(A_n - \varepsilon) + n \max_{t=1,\dots,m} \|g_t\|_{C^2} C''' \left(\prod_{s=1}^{n} l^{(s)}\right) + O\left(\prod_{s=1}^{n} l^{(s)}\right) < \\
&< -n\delta\left(\prod_{s=1}^{n} l^{(s)}\right)
\end{aligned}
$$

if $\max_{t=1,\dots,m} \|g_t\|_{C^2}$ is sufficiently small and $\delta > 0$.

Lemma 3.7. *Given $\epsilon > 0$, there exists a set $\Omega_\epsilon \subset \Omega$ with $\mu_0(\Omega_\epsilon) > 1 - \epsilon$ and $N \in \mathbb{N}$ so that*

$$\alpha \log m < -\frac{1}{N} \sum_{s=1}^{N} \log l^{(s)}(\lambda)$$

for every $\lambda \in J$ and all $p \in \Pi_\lambda(\Omega_\epsilon)$.

Proof of Lemma 3.7. First we will show that for a fixed $\lambda \in J$ and a given $\epsilon' > 0$, there exists Ω' with $\mu_0(\Omega') > 1 - \epsilon'$ and $n' \in \mathbb{N}$ such that

$$\alpha + \xi < -\frac{1}{n'} \sum_{s=1}^{n'} \log l^{(s)}(\lambda),$$

where $0 < \xi < Lyap^u(\nu_\lambda) - \alpha$, for all $p \in \Pi_\lambda(\Omega')$.

By the Birkhoff Ergodic Theorem (see [70]),

$$
\begin{aligned}
Lyap^u(\nu_\lambda) &= \\
&= \int \log \|D\Phi_\lambda(\Pi_\lambda(\omega))\| \, d\mu_0(\omega) \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{s=1}^{n} \log \|D\Phi_\lambda(\Phi_\lambda^s(\Pi_\lambda(\omega))\| \\
&= \lim_{n\to\infty} -\frac{1}{n} \sum_{s=1}^{n} \log l_\omega^{(s)}(\lambda)
\end{aligned}
$$

for μ_0-a.e. $\omega \in \Omega$. Thus by Egorov's theorem (see [59], for instance), there exists $\Omega' \subset \Omega$ with $\mu_0(\Omega') > 1 - \epsilon'$ such that the convergence is uniform on Ω'. Thus there exists $n' \in \mathbb{N}$

such that $\alpha + \xi < -\frac{1}{n'}\sum_{s=1}^{n'} l^{(s)}(\lambda)$ for all $p \in \Pi_\lambda(\Omega')$.

Next we will show that N can be chosen for every $\lambda \in J$. Let $\epsilon > 0$ be given. Consider the family of functions

$$L_\omega(\lambda) = -\log \|D\Phi_\lambda(\Pi_\lambda(\omega)\|$$

We can treat this family as a functions of λ with parameter ω. Then $\{L_\omega(\lambda)\}_{\omega\in\Omega}$ is an equicontinuous family of functions and there exists $t > 0$ such that if $|\lambda_1 - \lambda_2| \leq t$, then $|L_\omega(\lambda_1) - L_\omega(\lambda_2)| < \frac{\xi}{100}$ for any $\omega \in \Omega$. Consider a finite t-net $\{y_1, \ldots, y_M\}$ in J, containing $M = M(J,t)$ points. For each point y_j we can find a set $\Omega_j \subset \Omega$, $\mu_0(\Omega_j) > 1 - \frac{\epsilon}{M}$, and $N_j \in \mathbb{N}$ such that for every $N \geq N_j$ and every $\omega \in \Omega_J$, we have

$$\frac{1}{N}\sum_{s=1}^{N} L_{\sigma^s(\omega)}(y_j) = -\frac{1}{N}\sum_{s=1}^{N} \log l_\omega^{(s)}(y_j) > \alpha + \xi,$$

where $\sigma : \Omega \to \Omega$ is the shift map, that is $\sigma(\omega_0\omega_1\ldots) = \omega_1\omega_2\ldots$. Take $\Omega_\epsilon = \cap_{j=1}^{M}\Omega_j$. We have

$$\mu_0(\Omega_\epsilon) > 1 - M\frac{\epsilon}{M} = 1 - \epsilon,$$

and for every $\lambda \in J$ there exists y_j with $|y_j - \lambda| \leq t$. So for every $\omega \in \Omega_\epsilon \subset \Omega_j$ and every $N > N_0 = \max\{N_1, \ldots, N_M\}$, we have

$$\begin{aligned}
-\frac{1}{N}\sum_{s=1}^{N} \log l_\omega^{(s)}(\lambda) &= \frac{1}{N}\sum_{s=1}^{N} L_{\sigma^s(\omega)}(\lambda) \\
&\geq \frac{1}{N}\sum_{s=1}^{N} L_{\sigma^s(\omega)}(y_j) - \left|\frac{1}{N}\sum_{s=1}^{N} L_{\sigma^s(\omega)}(y_j) - \frac{1}{N}\sum_{s=1}^{N} L_{\sigma^s(\omega)}(\lambda)\right| \\
&\geq \alpha + \xi - \frac{\xi}{100} \\
&> \alpha + \frac{\xi}{2} \\
&> \alpha
\end{aligned}$$

which concludes the proof of Lemma 3.7. $\square$

So by Lemma 3.7, for any $\epsilon > 0$ we can find Ω_1 with $\mu_0(\Omega_1) > 1 - \frac{\epsilon}{2}$. Thus for all $p \in \Pi_\lambda(\Omega_1)$, we have that

$$\alpha < \frac{Lyap^u(\nu_\lambda)}{\log m} = \frac{1}{\log m} \lim_{n\to\infty} \left(-\frac{1}{n} \sum_{s=1}^{n} \log l^{(s)} \right),$$

and so for sufficiently large $n \in \mathbb{N}$, we have that $n\alpha \log m < -\log \left(\prod_{s=1}^{n} l^{(s)} \right)$. Thus we have

$$m^{-\alpha|\omega\wedge\tau|} = m^{-\alpha n} > \left(\prod_{s=1}^{n} l^{(s)} \right) > C|\phi_{\omega,\tau}(\lambda)|$$

for n large enough.

Since $\frac{d}{d\lambda}\phi_{\omega,\tau}(\lambda) < -n\delta \left(\prod_{s=1}^{n} l^{(s)} \right)$ and $\beta > \frac{1}{\log m} Lyap^u(\nu_\lambda)$, we have

$$m^{-\beta|\omega\wedge\tau|} = m^{-n\beta} < C \left(\prod_{s=1}^{n} l^{(s)} \right) < \left| \frac{d}{d\lambda}\phi_{\omega,\tau}(\lambda) \right|,$$

for n large enough.

Now from the Shannon-McMillan-Breiman theorem (see [6]) we have that

$$-\frac{1}{n} \log \mu_0([\omega]_n) \to h_{\mu_0}(\sigma)$$

for μ_0-a.e. $\omega \in \Omega$. By Egorov's theorem, there exists a set $\Omega_2 \subset \Omega$ with $\mu_0(\Omega_2) > 1 - \epsilon/2$ such that this convergence is uniform in $\omega \in \Omega_2$. Thus we have

$$-\frac{1}{n} \log \mu_0([\omega]_n) \to h_{\mu_0}(\sigma) > \gamma \log m$$

uniformly for $\omega \in \Omega_\epsilon$. So for n sufficiently large, we have that

$$\mu_0([\omega]_n) < m^{-n\gamma}$$

or choosing a sufficient constant $C > 0$, then for all $n \geq 1$ we have

$$\mu_0([\omega]_n) < Cm^{-n\gamma}.$$

Now let $\Omega_\epsilon = \Omega_1 \cap \Omega_2$, then $\mu_0(\Omega_\epsilon) > 1 - \epsilon$ and all conditions of Proposition 3.1 hold on Ω_ϵ, concluding the proof.

3.5 Corollaries

Theorem 1.1 gives us the following corollaries, which provide partial answers to the Palis conjecture regarding affine Cantor sets.

Consider a family of affine Cantor sets C_Λ generated by contractions $\{\phi_k\}_{k=1}^m$ such that $\phi_k(x) = \lambda_k x + d_k$, where $\lambda_k \in J_k \subset (-1,0) \cup (0,1)$, and $d_k = d_k(\lambda_1 \ldots, \lambda_m)$ is C^1, for $k = 1, \ldots, m$. If we define $\Lambda = (\lambda_1, \ldots, \lambda_m)$, let C_Λ be such a set. Then by combining Fubini's theorem with Theorem 1.1 we have the following corollary.

Corollary 1.1. *Suppose K is a compact subset of the real line, and a family $\{C_\Lambda\}$ of affine Cantor sets as above is given such that*

$$\dim_H K + \dim_H C_\Lambda > 1 \quad \textit{for all } (\lambda_1, \ldots, \lambda_m) \in J_1 \times \cdots \times J_m.$$

Then for a.e. $(\lambda_1, \ldots, \lambda_m) \in J_1 \times \cdots \times J_m$ the set $K + C_\Lambda$ has positive Lebesgue measure.

Thus in this setting, any affine Cantor set is determined by $2m$ parameters, namely $\lambda_1, \ldots, \lambda_m$ and $d_1, \ldots, d_m$, such that $\phi_k = \lambda_k x + d_k$ satisfies the necessary requirements for $k = 1, \ldots, m$. These parameters form a subset of $\mathbb{R}^{2m}$, and as a consequence of Corollary 1.1, we have the following corollary.

Corollary 1.2. *Generically (for Lebesgue almost all admissible tuples of the parameters) the sum of two affine Cantor sets has positive Lebesgue measure provided the sum of their Hausdorff dimensions is greater than one.*

Corollary 1.2 continues the work of Peres and Solomyak and provides a partial answer to the Palis conjecture with respect to arithmetic sums of affine Cantor sets.

Chapter 4

Open Questions

Our result raises several questions which need to be addressed.

1. The assumption of C^2 smoothness of the perturbations is is necessary to complete the proof. We believe this is merely an artifact of the proof and can be improved upon. Typically for the contractions associated with a dynamically defined Cantor set, we assume they are $C^{1+\alpha}$ for $\alpha > 0$.

2. Can one prove the result in general for a family of dynamically defined Cantor sets given by C^2 (or $C^{1+\alpha}$) contractions? In our result, the near-linearity of our contractions was crucial to the proof, but we conjecture that this is also an artifact of the proof.

3. Unfortunately, the proof of this result says nothing about the topological structure of the arithmetic sums of Cantor sets fitting the hypothesis; merely that the sum will have positive Lebesgue measure. The results of Moreira and Yoccoz [39] and Mendes and Oliveira ([37]) strongly suggest that the arithmetic sums of Cantor sets generically contain an interval when the sum of their Hausdorff dimensions is larger than one. When does positive Lebesgue measure of the sum of Cantor sets (in the setting of this

paper) imply that that this sum will contain an interval. When will this sum be an interval or an M-Cantorval?

4. The monotonicity of the contraction rate with respect to the parameter seems to be an important part of the proof. This monotonicity with respect to the parameter is also important in other similar results, such as in [14]. We would like to formulate another conjecture in the same spirit.

   Conjecture 4.1. *Suppose that K is a compact set and $J \subset (0,1)$ is an interval, and $\{C_\lambda\}$ a family of dynamically defined Cantor sets such that $\frac{d}{d\lambda}(\dim_H C_\lambda) \geq \delta > 0$ for all $\lambda \in J$. Then if $\dim_H(K) + \dim_H(C_\lambda) > 1$ for all $\lambda \in J$, then for a.e. $\lambda \in J$, the set $K + C_\lambda$ has positive Lebesgue measure.*

   In fact, after this text was finished in principal, we learned that Conjecture 4.1 was proven by Damanik and Gorodetski, see a recent preprint [13]

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Appendix A

Proof of Proposition 3.1

This material is taken directly from [14], and reproduced here for the convenience of the reader. Before we can provide the proof of Proposition 3.1, will provide some preliminaries and the proof of another proposition which is similar, but weaker to Proposition 3.1, which we will need.

Let $\mathcal{A}^{\mathbb{Z}_+}$, with $|\mathcal{A}| = \ell \geq 2$ and $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, be the standard symbolic space, equipped with the product topology. Let A be a primitive 0–1 matrix of size $\ell \times \ell$, and $\Sigma_A^\ell \subset \mathcal{A}^{\mathbb{Z}_+}$ the one-sided topological Markov chain associated with A. That is, $\omega = \omega_0\omega_1 \ldots \in \mathcal{A}^{\mathbb{Z}_+}$ belongs to Σ_A^ℓ if and only if $A_{\omega_m\omega_{m+1}} = 1$ for all $m \geq 0$. The shift transformation $\sigma_A : \Sigma_A^\ell \to \Sigma_A^\ell$ acts as $(\sigma_A\omega)_n = \omega_{n+1}$. Let μ be a probability measure on Σ_A^ℓ.

Let $J = [\lambda_0, \lambda_1] \subset \mathbb{R}$ be a parameter interval. We assume that we are given a family of continuous maps

$$\Pi_\lambda : \Sigma_A^\ell \to \mathbb{R},\ \lambda \in J,$$

such that $\Sigma_\lambda = \Pi_\lambda(\Sigma_A^\ell)$ are the Cantor sets and

$$\nu_\lambda = \Pi_\lambda(\mu) := \mu \circ \Pi_\lambda^{-1}$$

are the measures of interest.

Let η be another (fixed) compactly supported Borel probability measure on the real line. We will assume that η is exact-dimensional, having the local dimension equal to d_η at η-a.e. x, that is,

$$\lim_{r \downarrow 0} \frac{\log \eta(B_r(x))}{\log r} = d_\eta \quad \text{for } \eta\text{-a.e. } x. \tag{A.1}$$

Here $B_r(x) = [x - r, x + r]$. In the first proposition below we assume a stronger, uniform Hölder condition for the measure η; it is subsequently relaxed to (A.1) using a truncation argument.

For a word $u \in \mathcal{A}^n$, $n \geq 0$, we denote by $|u| = n$ its length and by $[u]$ the cylinder set of elements of Σ_A^ℓ that have u as a prefix. More precisely, $[u] = \{\omega \in \Sigma_A^\ell : \omega_0 \ldots \omega_{n-1} = u\}$. For $\omega, \tau \in \Sigma_A^\ell$, we write $\omega \wedge \tau$ for the maximal common prefix of ω and τ (which is empty if $\omega_0 \neq \tau_0$; we set the length of the empty word to be zero). Furthermore, for $\omega, \tau \in \Sigma_A^\ell$, let

$$\phi_{\omega,\tau}(\lambda) := \Pi_\lambda(\omega) - \Pi_\lambda(\tau).$$

We write *Leb* for the one-dimensional Lebesgue measure.

Proposition A.1. *Suppose that there exist constants $C_1, C_2, C_3, C_4, \alpha, \beta, \gamma > 0$ and $k_0 \in \mathbb{Z}_+$ such that*

(a) $\max_{\lambda \in J} |\phi_{\omega,\tau}(\lambda)| \leq C_1 \ell^{-\alpha|\omega \wedge \tau|}$ *for all* $\omega, \tau \in \Sigma_A^\ell,\ \omega \neq \tau$*;*

(b) $\sup_{v \in \mathbb{R}} Leb\big(\{\lambda \in J : |v + \phi_{\omega,\tau}(\lambda)| \leq r\}\big) \leq C_2 \ell^{|\omega \wedge \tau|\beta} r$ *for all* $\omega, \tau \in \Sigma_A^\ell,\ \omega \neq \tau$*, such that* $|\omega \wedge \tau| \geq k_0$*, and*

(c) $\max_{u \in \mathcal{A}^n} \mu([u]) \leq C_3 \ell^{-\gamma n}$ *for all* $n \geq 1$*.*

(d) Further, let η be a compactly supported Borel probability measure on $\mathbb{R}$ satisfying $\eta(B_r(x)) \leq C_4 r^{d_\eta}$ *for all* $x \in \mathbb{R},\ r > 0$*.*

Then

$$\left(d_\eta + \frac{\gamma}{\beta} > 1 \textbf{ and } d_\eta > \frac{\beta-\gamma}{\alpha}\right) \implies \eta * \nu_\lambda \ll Leb \ \ \textit{with density in } L^2 \textit{ for a.e.-}\lambda \in J.$$

Remark A.1. *Condition (b) is the hardest to check in the applications we envision (in particular the ones given in this paper). It follows from*

$$\inf_{\lambda_1,\lambda_2\in J,\, \lambda_1\neq\lambda_2} \frac{|\phi_{\omega,\tau}(\lambda_1)-\phi_{\omega,\tau}(\lambda_2)|}{|\lambda_1-\lambda_2|} \geq C_2' \ell^{-|\omega\wedge\tau|\beta} \quad \textit{for all} \ \ \omega,\tau\in\Sigma_A^\ell,\ \omega\neq\tau.$$

An estimate of this kind in turn follows from a suitable lower bound on $|\frac{d}{d\lambda}\phi_{\omega,\tau}(\lambda)|$.

Proof of Proposition A.1. We follow closely the argument of [53, Theorem 2.1]. For a word $u \in \mathcal{A}^{k_0}$, let $\nu_\lambda^u = \mu|_{[u]} \circ \Pi_\lambda^{-1}$. Then $\nu_\lambda = \sum_{|u|=k_0} \nu_\lambda^u$, so it is enough to prove that $\eta * \nu_\lambda^u$ is absolutely continuous for all $u \in \mathcal{A}^{k_0}$ and a.e. λ. We fix such a u of length k_0 for the remainder of the proof.

Consider the lower density of $\eta * \nu_\lambda^u$,

$$\underline{D}(\eta * \nu_\lambda^u, x) = \liminf_{r\downarrow 0}\, (2r)^{-1}(\eta * \nu_\lambda^u)[B_r(x)].$$

As in [33, 9.7], if

$$\mathcal{J}_\lambda := \int_{\mathbb{R}} \underline{D}(\eta * \nu_\lambda^u, x)\, d(\eta * \nu_\lambda^u)(x) < \infty,$$

then $\underline{D}(\eta * \nu_\lambda^u, x)$ is finite for $(\eta * \nu_\lambda^u)$-a.e. x, and $\eta * \nu_\lambda^u$ is absolutely continuous, with a Radon-Nikodym derivative in L^2. Thus, it is enough to show that

$$\mathcal{S} := \int_J \mathcal{J}_\lambda\, d\lambda < \infty\,.$$

By Fatou's Lemma,

$$\mathcal{S} \leq \mathcal{S}_1 := \liminf_{r\downarrow 0}\, (2r)^{-1} \int_J \int_{\mathbb{R}} (\eta * \nu_\lambda^u)[B_r(x)]\, d(\eta * \nu_\lambda^u)(x)\, d\lambda.$$

Using the definition of convolution and making a change of variable, we obtain

$$\mathcal{S}_1 = \liminf_{r\downarrow 0}\,(2r)^{-1}\int_J\int_{\mathbb{R}}\int_{[u]}(\eta * \nu_\lambda^u)[B_r(y+\Pi_\lambda(\omega))]\,d\mu(\omega)\,d\eta(y)\,d\lambda. \tag{A.2}$$

Next we have, denoting by $\mathbf{1}_S$ the indicator function of a set S,

$$\begin{aligned}(\eta * \nu_\lambda^u)[B_r(y+\Pi_\lambda(\omega))] &= \int_{\mathbb{R}}\mathbf{1}_{B_r(y+\Pi_\lambda(\omega))}(w)\,d(\eta * \nu_\lambda^u)(w)\\ &= \int_{\mathbb{R}}\int_{[u]}\mathbf{1}_{\{(z,\tau):\ z+\Pi_\lambda(\tau)\in B_r(y+\Pi_\lambda(\omega))\}}(z,\tau)\,d\mu(\tau)\,d\eta(z).\end{aligned}$$

Substituting this into (A.2) and reversing the order of integration yields

$$\mathcal{S}_1 = \liminf_{r\downarrow 0}\,(2r)^{-1}\int_{\mathbb{R}}\int_{[u]}\int_{\mathbb{R}}\int_{[u]} Leb(\Lambda_r(y,z,\omega,\tau))\,d\mu(\tau)\,d\eta(z)\,d\mu(\omega)\,d\eta(y), \tag{A.3}$$

where

$$\begin{aligned}\Lambda_r(y,z,\omega,\tau) &:= \{\lambda\in J:\ |(y+\Pi_\lambda(\omega))-(z+\Pi_\lambda(\tau))|\le r\}\\ &= \{\lambda\in J:\ |y-z+\phi_{\omega,\tau}(\lambda)|\le r\}.\end{aligned}$$

Note that for $\omega,\tau\in[u]$, we have $|\omega\wedge\tau|\ge k_0$. Hence

$$Leb(\Lambda_r(y,z,\omega,\tau)) \le \widetilde{C}_2\min\{1,\ell^{|\omega\wedge\tau|\beta}r\} \tag{A.4}$$

by (b), where $\widetilde{C}_2 = \max\{C_2,|J|\}$. Now we consider the integral in (A.3), use Fubini, and split it according to the distance between y and z:

$$\begin{aligned}&\int_{\mathbb{R}}\int_{[u]}\int_{\mathbb{R}}\int_{[u]} Leb(\Lambda_r(y,z,\omega,\tau))\,d\mu(\tau)\,d\eta(z)\,d\mu(\omega)\,d\eta(y) \qquad (A.5)\\ &= \iint_{\{|y-z|<2r\}}\iint_{[u]\times[u]} + \iint_{\{|y-z|\ge 2r\}}\iint_{[u]\times[u]}\\ &=: \mathcal{I}_1+\mathcal{I}_2.\end{aligned}$$

To complete the proof, it is sufficient to show that $\mathcal{I}_1 \lesssim r$ and $\mathcal{I}_2 \lesssim r$ (here and below the symbol $\lesssim$ means inequality up to a multiplicative constant independent of r). In view of

(A.4),

$$\mathcal{I}_1 \lesssim \iint_{[u]\times[u]} \iint_{\{|y-z|<2r\}} \min\{1, \ell^{|\omega\wedge\tau|\beta} r\}\, d\eta(y)\, d\eta(z)\, d\mu(\omega)\, d\mu(\tau). \tag{A.6}$$

Note that the integrand does not depend on y, z, and we can estimate

$$(\eta \times \eta)\{(y,z):\ |y-z| < 2r\} \leq \int \eta(B(y,2r))\, d\eta(y) \lesssim r^{d_\eta},$$

by (d). Recalling that $|u| = k_0$ we obtain

$$\mathcal{I}_1 \lesssim r^{d_\eta} \sum_{k=k_0}^{\infty} \min\{1, \ell^{k\beta} r\} \cdot (\mu \times \mu)\{(\omega,\tau):\ |\omega \wedge \tau| = k\}. \tag{A.7}$$

Observe that (c) implies

$$\begin{aligned}(\mu \times \mu)\{(\omega,\tau):\ |\omega\wedge\tau| = k\} &\leq \sum_{|v|=k} \mu([v])^2 \\ &\leq C_3 \ell^{-\gamma k} \sum_{|v|=k} \mu([v]) \\ &= C_3 \ell^{-\gamma k}.\end{aligned} \tag{A.8}$$

Hence

$$\mathcal{I}_1 \lesssim r^{d_\eta} \sum_{k=k_0}^{\infty} \min\{1, \ell^{k\beta} r\} \cdot \ell^{-\gamma k}.$$

Setting $k_r = \frac{\log(1/r)}{\beta \log \ell}$, we get

$$\mathcal{I}_1 \lesssim r^{d_\eta} \left[\sum_{k \leq k_r} r\ell^{k\beta} \cdot \ell^{-\gamma k} + \sum_{k > k_r} \ell^{-\gamma k} \right].$$

Summing the geometric series and using $\ell^{-\beta k_r} = r$, we obtain

$$\mathcal{I}_1 \lesssim r^{d_\eta + \frac{\gamma}{\beta}}, \tag{A.9}$$

which is $\lesssim r$ by assumption.

It remains to estimate $\mathcal{I}_2$, the second integral in (A.5). If $|\omega\wedge\tau| = k$ and $C_1 \ell^{-\alpha k} < |y-z|/2$, then by (a), $|\phi_{\omega,\tau}(\lambda)| < |y-z|/2$ for all $\lambda \in J$, and $|y - z + \phi_{\omega,\tau}(\lambda)| > |y-z|/2$. When

$|y-z| \geq 2r$, this implies that the set $\Lambda_r(y,z,\omega,\tau)$ is empty. Denote

$$\kappa(y,z) = -\frac{\log \frac{|y-z|}{2C_1}}{\alpha \log \ell}\,.$$

We obtain, using (A.4) and (A.8):

$$\begin{aligned}
\mathcal{I}_2 &\lesssim \iint_{\mathbb{R}^2} \sum_{k \leq \kappa(y,z)} r\ell^{k\beta} \cdot \ell^{-\gamma k}\, d\eta(y)\, d\eta(z) \\
&\lesssim r \iint_{\mathbb{R}^2} \ell^{(\beta-\gamma)\kappa(y,z)}\, d\eta(y)\, d\eta(z) \\
&\lesssim r \iint_{\mathbb{R}^2} |y-z|^{-\frac{\beta-\gamma}{\alpha}}\, d\eta(y)\, d\eta(z).
\end{aligned}$$

But $\frac{\beta-\gamma}{\alpha} < d_\eta$ by assumption, so the last integral converges by (d) and the fact that η is compactly supported. Thus, $\mathcal{I}_2 \lesssim r$, and the proof of the proposition is complete. $\square$

And now for the proof of Proposition 3.1, which we restate below.

Proposition 3.1. *Let η be a compactly supported Borel probability measure on $\mathbb{R}$ of exact local dimension d_η. Suppose that for any $\varepsilon > 0$, there exists a subset $\Omega_\varepsilon \subset \Sigma_A^\ell$ such that $\mu(\Omega_\varepsilon) > 1-\varepsilon$ and the following holds; there exist constants $C_1, C_2, C_3, \alpha, \beta, \gamma > 0$ and $k_0 \in \mathbb{Z}_+$ such that*

(a) $d_\eta + \frac{\gamma}{\beta} > 1$ *and* $d_\eta > \frac{\beta-\gamma}{\alpha}$;

(b) $\max_{\lambda \in J} |\phi_{\omega,\tau}(\lambda)| \leq C_1 \ell^{-\alpha|\omega\wedge\tau|}$ *for all* $\omega, \tau \in \Omega_\varepsilon$, $\omega \neq \tau$;

(c) $\sup_{v \in \mathbb{R}} Leb\big(\{\lambda \in J : |v + \phi_{\omega,\tau}(\lambda)| \leq r\}\big) \leq C_2 \ell^{|\omega\wedge\tau|\beta} r$ *for all* $\omega, \tau \in \Omega_\varepsilon$, $\omega \neq \tau$ *such that* $|\omega \wedge \tau| \geq k_0$, *and*

(d) $\max_{u \in \mathcal{A}^n, [u]\cap\Omega_\varepsilon \neq \emptyset} \mu([u]) \leq C_3 \ell^{-\gamma n}$ *for all* $n \geq 1$.

Then, $\eta * \nu_\lambda \ll Leb$ *for a.e.-*$\lambda \in J$.

Proof. By Egorov's Theorem, we can find for any $\varepsilon > 0$, a Borel set $S_\varepsilon \subset \mathbb{R}$ such that $\eta(S_\varepsilon) > 1 - \varepsilon$ and $\eta_\varepsilon := \frac{1}{\eta(S_\varepsilon)} \cdot \eta|_{S_\varepsilon}$ satisfies

$$\eta_\varepsilon(B_r(x)) \leq C(\varepsilon) r^{d_\eta - \varepsilon} \text{ for all } x \in \mathbb{R},\ r > 0,$$

so that η_ε satisfies Proposition A.1 (d), with d_η replaced by $d_\eta - \varepsilon$. The verbatim repetition of the proof of Proposition A.1 shows that if μ_ε is the restriction of the measure μ to the set Ω_ε and $\nu_{\lambda,\varepsilon} = \Pi_\lambda(\mu_\varepsilon)$, then $\eta_\varepsilon * \nu_{\lambda,\varepsilon} \ll Leb$ for a.e.-$\lambda \in J$, for $\varepsilon > 0$ sufficiently small. Now Proposition 3.1 follows from the following simple lemma, which we state without proof. □

Lemma A.1. *Let η, ν be compactly supported probability measures on $\mathbb{R}$. Suppose that for any $\varepsilon > 0$, there are subsets $\Omega_{\eta,\varepsilon}, \Omega_{\nu,\varepsilon} \subset \mathbb{R}$ such that $\eta(\Omega_{\eta,\varepsilon}) > 1 - \varepsilon$, $\nu(\Omega_{\nu,\varepsilon}) > 1 - \varepsilon$, and $\eta|_{\Omega_{\eta,\varepsilon}} * \nu|_{\Omega_{\nu,\varepsilon}}$ is absolutely continuous with respect to Lebesgue measure. Then $\eta * \nu$ is absolutely continuous with respect to Lebesgue measure.*