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PRODUCTS OF TOPOLOGICAL AMPLITUDES: AN ALGEBRAIC SOLUTION*

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ABSTRACT

Given topological amplitudes with specific genus and boundary structure we define a product of these amplitudes, also specified by a genus and a boundary structure. We show that this product is independent of the particular graphical representation of the members of the product; we describe an explicit construction to obtain the boundary structure of this product without using graphs and finally we derive the general formula to compute the genus of the product.

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I. Introduction

The basic assumption of the topological expansion[1] of mesonic amplitudes is that physical scattering amplitudes can be written as an infinite sum:

$$M = \sum_{h} \sum_{\substack{\text{boundary} \\ \text{structure}}} M_h(\text{boundary structure}) .$$

Each term of the sum represents a "topological amplitude" characterized by a particular singularity structure that is simpler than the singularity structure of the full amplitude.

The adjective "topological" is due to the usual representation of the "topological amplitude" by means of a graph whose genus (number of handles) is h and whose faces reproduce the boundary structure. These topological properties completely determine the singularity structure of the associated complex function.

The boundary structure, which determines the poles of the amplitude, is characterized by a partition of the set of external particles into b subsets called boundaries. The particles belonging to a particular boundary are cyclicly ordered but no order is specified between particles belonging to different boundaries.

Let $\Lambda_b^i = (\lambda_1^i,\ldots,\lambda_b^i)$ represent the i^{th} partition of the set of N external particles into b boundaries λ_j^i $(j = 1,\ldots,b)$.

Then the physical amplitude can be written as:

$$M = \sum_{h} \sum_{i=1}^{N!} M_h(\Lambda_b^i) \tag{1}$$

where $\sum_{i=1}^{N!}$ stands for the sum over all possible partitions of the N particles.*

The purpose of such a "topological expansion" of the physical amplitude is to obtain discontinuity formulae that are easier to handle. The normal threshold discontinuity formula for the physical amplitude can symbolically be written as:

$$\text{disc } M = M^{+} \times M$$

where the symbol "disc" refers to a specific discontinuity (i. e. set of intermediate particles) in a specific channel. Therefore:

$$\sum_{h,i} \text{disc } M_h(\Lambda_b^i) = \left(\sum M_{h_1}(\Lambda_{b_1}^{i_1})\right)\left(\sum M_{h_2}(\Lambda_{b_2}^{i_2})\right)$$

which is then decomposed into the infinite set of equations:

$$\text{disc } M_h(\Lambda_b^i) = \sum (M_{h_1}(\Lambda_{b_1}^{i_1})\, M_{h_2}(\Lambda_{b_2}^{i_2})). \tag{2}$$

* The fact that there exist N! different partitions of N was pointed out by Y. Eylon.

The sum on the right hand side includes all terms resulting in a complex function whose singularities are characterized by h and the boundary structure (poles) Λ_b^i.

Several conjectures were made along the line:

1. Existence: The product on the right hand side of (2) can be characterized by a genus and a boundary structure (Veneziano).

2. Uniqueness: The product on the right hand side (2) is independent of the particular (graphical) representation used to express the amplitudes (Chew).

3. Usefulness: The resulting discontinuity formulae will be easier to use (i. e., the number of terms on the right hand side is finite).

The objective of this paper is to prove these conjectures and to describe a general method to obtain the boundary structure and the genus (h) of a product of two "topological amplitudes" without using a graphical representation.

In the next section we discuss various equivalent representations. In Section 3 we describe the method to obtain the boundary structure of a product, first on a simple example, then for the general case. In Section 4 we derive the formula for the genus of the product of two general amplitudes. Finally, in the Appendix we discuss some aspects of graphical amplitude representation.

II. Representation of Amplitudes

II.1 Graphical Representation

Chronologically, the first representation of topological amplitude was given in terms of quark diagrams[1] as shown on Fig. 1a (for $h = 1$, $b = 1$). Later, when the concept of order became clearer[2] the same amplitude was represented in terms of particle diagrams as shown in Fig. 1c. Here the lines represent mesons and not quarks. A hybrid representation was also used from time to time as shown in Fig. 1b. There, certain edges represent mesons (edges connecting the "bubbles") whereas others represent quarks (edges forming the rings). Although this hybrid notation may appear confusing it helps an intuitive understanding of the equivalence of quark diagrams with particle diagrams. This equivalence is formally shown in the Appendix. Each vertex of Fig. 1c is ordered in the sense that edges sticking out of it have a definite cyclic order on the imbedding two-dimensional surface.

II.2 Edmond's Permutation Technique

In order to find the boundary structure and the genus of an amplitude or a product of amplitudes in a discontinuity formula represented by a particle diagram (or a hybrid diagram) one can apply Edmond's permutation technique.[2,3] The method consists of finding the orbits of the graph (corresponding to the windows and boundaries of quark diagrams) as follows: We start from one vertex

(or ring in the hybrid notation) and turn clockwise around it until we reach a connecting edge. We then follow the edge to the next vertex (or ring) and continue clockwise around the latter until we reach a connecting edge (i. e., particle edge) and so on until we return to the starting point. This path is called an "orbit". An example is given in Fig. 2 for a cylinder ($b = 2$, $h = 0$).

The genus of the graph is then readily obtained from Euler's formula:

$$h = \tfrac{1}{2}(2 - v + e - f) \qquad (3)$$

where v is the number of vertices (or rings), e the number of connecting edges (i. e., particle edges only in the hybrid diagrams) and f the number of orbits. Note that this method is equivalent to following the quark lines in a quark diagram. An orbit is either a boundary or a window. That is, an orbit need not contain twigs (or vertices) representing external particles (window). If it does contain external particles (boundary) then the cyclic order of the external particles must be retained.

II.3 Algebraic notation

Since the assumption is that the number of handles and the boundary structure are sufficient to define the amplitude, Chew[2] has proposed an algebraic notation whereby the particles of a given boundary are listed sequentially modulo a cyclic permutation. Sets of particles belonging to different boundaries are separated by a + sign. For instance, a 4-point cylinder amplitude is represented by M_o (AB + CD) while a 4-point torus amplitude (one handle) is denoted by M_1 (ABCD). The subscript always refers to the genus.

The advantage of such a notation is its independence of any particular graphical representation.

III. The Boundary Structure

III.1 An Example

We shall now describe a method to obtain the boundary structure of the product of two amplitudes using Chew's notation. Since the general case is quite difficult to manipulate we start with a specific simple example defined in Fig. 3.

The external particles are labelled a, b, c, d, e and the intermediate particles are labelled i_1, i_2, i_3. By applying Edmond's permutations technique on $P = A \times B$ one can see that the resulting discontinuity is $M_2(a + b + ced)$.

III.2 Constructing the Boundary Structure of the Product

We now apply the algebraic method to this particular case:

(1) We start from an external particle, say a, and record it in the final product.

(2) We continue to record all external particles following a on the same boundary until we reach an intermediate particle (if there is no intermediate particle on the boundary carrying a, we write a + sign thus

terminating the list of this boundary and start on another boundary of A or B). In our example we reach i_1. We slash it and jump to i_1 in B; we also slash i_1 here and continue on the same boundary. If there were any external particles between i_1 and i_2 on B we would have listed them after a. Here we reach immediately another intermediate particle i_2, which we slash and jump back to i_2 on A. We slash i_2 on A and continue on the same boundary on A. We find a which means we are back to the starting point. This completes the first boundary of the product (+ sign). In our example, the situation after this step is as follows:

$$M_1\ (a\ \cancel{i_1}\ b\ \cancel{i_2} + i_3\ c)\ \otimes\ M_0\ (\cancel{i_1}\cancel{i_2} + d\ i_3\ e)\ =\ M_0(a + .$$

We then write the next external particle, b, and repeat the process. The various steps in our example are detailed below. The analysis is completed when each intermediate particle has been slashed twice:

$$M_1\ (a\ \cancel{i_1}\ b\ \cancel{i_2} + c\ i_3)\ \otimes\ M_0(\cancel{i_1}\cancel{i_2} + d\ i_3\ e)\ =\ M_0\ (a + b +$$

and finally:

$$M_1(a\,\cancel{i_1}\;b\,\cancel{i_2} + c\,\cancel{i_3}) \otimes M_0(\cancel{i_1}\,\cancel{i_2} + d\,\cancel{i_3}\;e) = M_0(a + b + ced).$$

The value of h is obtained using Eq.(9) derived in Section 4:

$$2h = 2(1 + 0) + (2 + 2 - 3) + (3 - 2) = 4,$$

hence

$$h = 2.$$

III.3 The General Product

We can now write the general product as follows:

$$P = M_{h_1}\left(\sum_{\alpha=1}^{b_1} a_1^\alpha\; i_1^\alpha \ldots a_{m_\alpha}^\alpha\; i_{m_\alpha}^\alpha\right) \otimes \tag{4}$$

$$\otimes \quad M_{h_2}\left(\sum_{\beta=1}^{b_2} b_1^\beta\; i_1^\beta \ldots b_{m_\beta}^\beta\; i_{m_\beta}^\beta\right)$$

$$= M_h\left(\sum_{\gamma=1}^{b} c_1^\gamma\, c_2^\gamma \;\ldots c_{m_\gamma}^\gamma\right),$$

where i_ℓ^α stands for an intermediate particle in boundary α. The "length" of the boundary (i. e., the number of intermediate particles in α) is m_α.

Similarly a_ℓ^α stands for the group of adjacent external particles lying between $i_{\ell-1}^\alpha$ and i_ℓ^α (the subscripts are understood modulo m_α); a_ℓ^α may represent one, several or no external particle whereas i_ℓ^α always represents one particle. Finally, the c_k^γ represent some a_ℓ^α or b_n^β or a combination thereof.

The procedure to obtain the boundary structure of the product is as explained in the particular case discussed above. We start from some a_ℓ^α which we list in the product, we then slash the following i_ℓ^α and go the corresponding particle in the other amplitude. We slash it and continue. The principle is to always remain within a boundary in one member of the product until we encounter an intermediate particle. We slash it before and after the "jump" to the other amplitude and continue on whatever boundary we have "landed" until we return to the starting point. This constitutes an orbit.

In the event a given boundary α or β has only external particles say a_1^α (or b_1^β) then this boundary is recorded by itself in the final product.

In the event no external particles are found in a particular orbit of the product, a zero must be entered (This corresponds to a window.). This point needs to be clarified: Although amplitudes never contain "windows"as part of the boundary structure, products of amplitudes - which enter in discontinuity formulae for a specific set of intermediate particles may contain "windows" characteristic of the particular discontinuity involved. In other words, one must be careful not to associate immediately a product of amplitudes with an amplitude but rather with an amplitude discontinuity.

IV. The Genus of the Product

We now derive the formula to compute h. This number will be seen to depend on the number of boundaries b_1 and b_2 and on the number of orbits in the product but not on the boundary structure. Let us consider the particle diagram (Fig. 1c) for a moment and let us assume that the first amplitude A is represented by v_1 vertices. We thus have, using Euler formula (Eq. 3):

$$v_1 - e_1 + b_1 = 2 - 2h_1 \tag{5}$$

where e_1 is the number of edges connecting the v_1 vertices and forming a graph with b_1 orbits and h_1 handles. Similarly let B be represented by a graph with v_2 vertices, then:

$$v_2 - e_2 + b_2 = 2 - 2h_2 \tag{6}$$

where e_2 is defined similarly.

Now, the product $P = A \times B$ will have v vertices, e connecting edges, f orbits and h handles. These quantities are again related by the Euler formula:

$$v - e + f = 2 - 2h, \tag{7}$$

and furthermore we have

$$v = v_1 + v_2 \tag{8}$$

$$e = e_1 + e_2 + I,$$

where I is the number of edges between A taken as a whole and B taken as a whole. In other words, I is the number of intermediate particles in the product under consideration.

Inserting (5), (6) and (8) into (7) we get:

$$2h = 2(h_1 + h_2) + (b_1 + b_2 - f) + (I - 2), \tag{9}$$

which is the general formula giving h in terms of h_1, h_2, b_1, b_2 and f.*

It is crucial to realize that f is the number of orbits involved in the product (not the number of boundaries). This product may contain some "windows" as a result of the particular discontinuity

* In some schemes little or no distinction is made between graphs representing amplitudes and graphs representing discontinuities (i. e., products of amplitudes). As a result, certain amplitudes may be represented by graphs containing windows. Equation (9) obviously accomodates these schemes if b_1 and b_2 are replaced by f_1 and f_2 respectively (f = boundaries + windows) and proper account is taken of these new orbits in computing f.

(i. e., intermediate state). Although these "windows" should be dropped in characterizing the amplitude whose discontinuity contains the product, they must be taken into account when computing f.

Conclusions

We have completed the task set at the beginning of this paper. We have shown that given topological amplitudes with specified genus and boundary structure we can define a product also specified by a genus and a boundary structure. We have shown that this product is independent of the graphical representation of the members of the product and we have described an explicit construction to obtain both the boundary structure and the genus h.

The "usefulness" of the topological expansion also stems from Eq. (9). Suppose we are interested in a particular discontinuity of a topological amplitude: disc $M_h(\Lambda_b^i)$, then owing to the inequality

$$h \geqslant h_1 + h_2$$

we see that only a finite number of terms can participate in the discontinuity formula. Furthermore one can readily describe a systematic way to generate all terms that should participate. We first fix one member of the product (which yields h_1 and b_1) and then, for each boundary structure of the other member,

Equation (9) will yield the appropriate genus (h_2). By summing first over all possible boundary structures of the second member and then over all possible graphs (first member) with $h_1 \leqslant h$ we get all possible terms.

After completion of this work, it was pointed out to the author that Ciafaloni, Marchesini and Veneziano[4] (1975) have discussed some of the problems addressed in this paper.*

Acknowledgments

I am grateful to Professor G. F. Chew for continuous encouragement, countless suggestions, and enlightening discussions during the course of this work.

* These authors proposed an inequality, $2h \geqslant 2(h_1 + h_2) + (b_1 + b_2 - 2)$, but counterexamples can easily be found.

Appendix

We first establish the equivalence of the three graphical notations used in the text (cf. Fig. 1).

(1) The equivalence of the quark diagrams and hybrid diagrams is trivial. The only difference between the two systems of graphs is that the "gap" between quark lines in the quark diagram is transformed into a vertex in the hybrid diagram and internal (intermediate) particles are represented by a single line instead of two quark-lines.

Edmond's rule insures that boundaries, windows and handles are in complete correspondence.

(2) The equivalence of hybrid diagrams and particle diagrams is also trivial and is obtained by shrinking the rings of the hybrid representation into simple vertices.

Formally, the equivalence can be deduced from Euler's formula. For each ring of the hybrid representation, the number of vertices v equals the number of quark edges: e_q. Thus, for a hybrid diagram we have

$$v - (e_q + e_p) + f = 2 - 2h,$$

where e_p = number of intermediate particle edges. Hence,

$$-e_p + f = 2 - 2h.$$

Now, let $f = f_{int} + f_{ext}$ where f_{int} refers to the faces inside the rings; so we have:

$$f_{int} - e_p + f_{ext} = 2 - 2h.$$

On the other hand, for particle diagrams, Euler's formula yields:

$$v - e_p + g = 2 - 2h,$$

when v = number of vertices; g = number of orbits. The equivalence is obtained by realizing that:

$$v = f_{int}$$

and

$$f_{ext} = g.$$

Graphical Representation of Amplitudes

Ordered amplitudes are usually represented by a ring whereas higher order amplitudes are generally represented by two connected rings. In fact, it is easy to see that all amplitudes can be represented by one ring (or vertex). Using Euler's formula again we get:

$$2h = e - b + (2 - v).$$

If $v = 2$, the minimum number of edges for a given amplitude is

$$e = 2h + b.$$

If $v = 1$, this number is

$$e = 2h + b - 1,$$

which is always feasible since $2h + b \geqslant 1$. We have represented a cylinder and a torus (one handle) 4-point amplitude in Fig. 4a and 4b respectively.

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FIGURE CAPTIONS

Fig. 1: Amplitude representation: (a) quark diagrams, (b) hybrid diagrams, (c) particle diagrams.

Fig. 2: Orbits according to Edmond's method. Cylinder in particle diagram notation (a), and in hybrid notation (b).

Fig. 3: Product of two amplitudes.

Fig. 4: One-vertex representation of amplitudes: (a) a cylinder; (b) a torus.

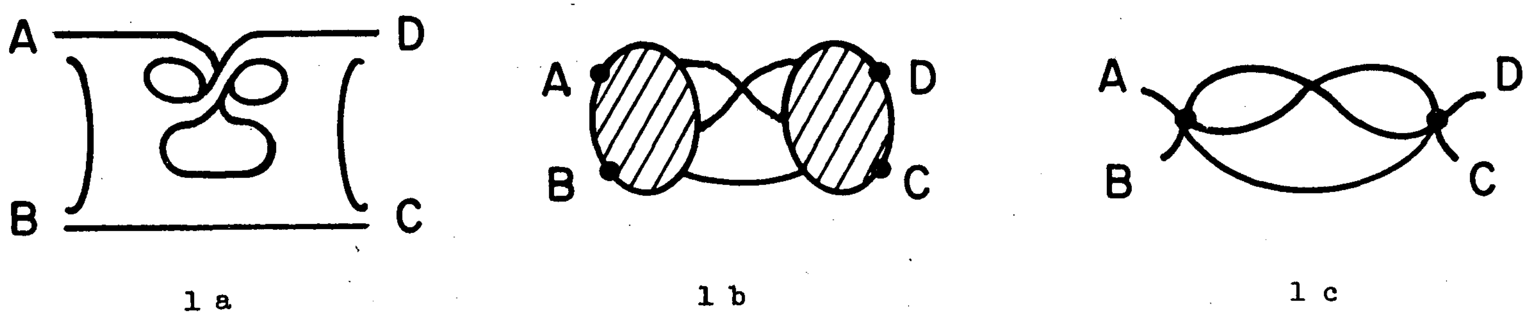

Fig. 1

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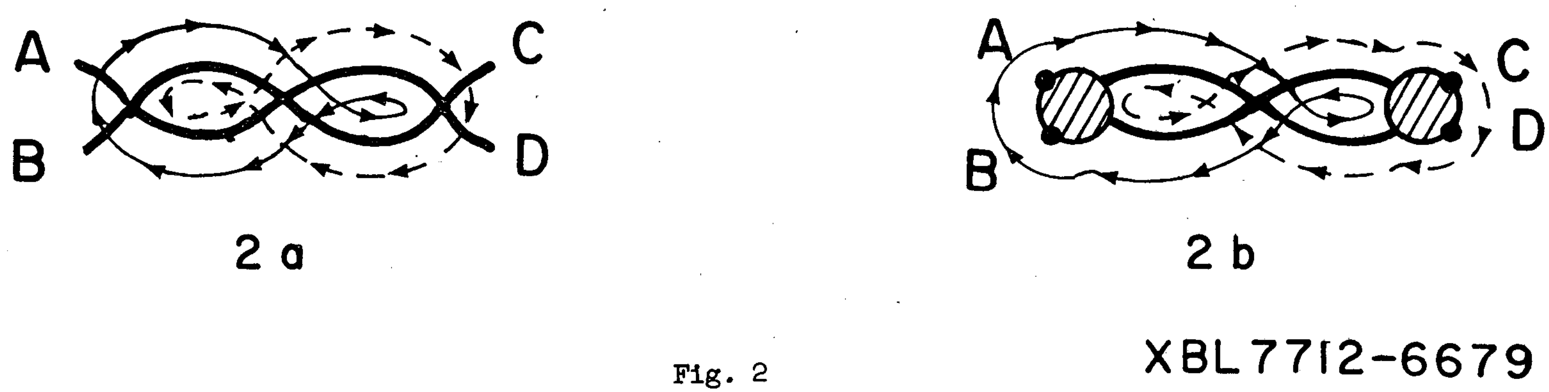
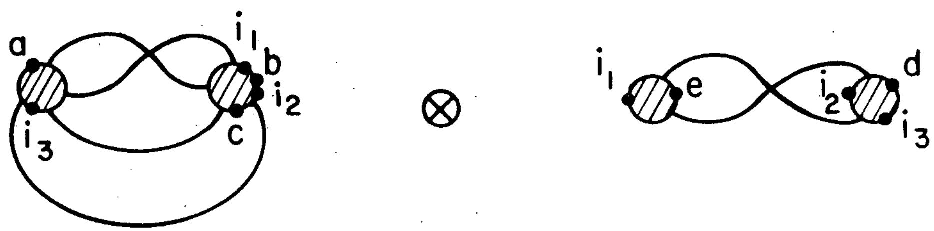
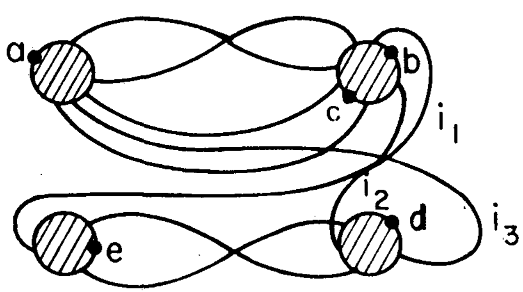


Fig. 2



$$\left[A \equiv M_1(a i_1 b i_2 + i_3 c)\right] \otimes \left[B \equiv M_0(\, i_1\, i_2 + d\, i_3 e\,\right]$$



$P = A \otimes B$

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Fig. 3

Fig. 4

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