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F-Theory, T-Duality on K3 Surfaces and N = 2 Supersymmetric Gauge Theories in Four Dimensions

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Abstract

We construct T-duality on $K3$ surfaces. The T-duality exchanges a 4-brane R-R charge and a 0-brane R-R charge. We study the action of the T-duality on the moduli space of 0-branes located at points of $K3$ and 4-branes wrapping it. We apply the construction to F-theory compactified on a Calabi-Yau 4-fold and study the duality of $N = 2$ $SU(N_c)$ gauge theories in four dimensions. We discuss the generalization to the $N = 1$ duality scenario.

1 Introduction

The string interpretation of the duality between four dimensional $N = 1$ supersymmetric gauge theories has been studied recently [1–4]. It has been suggested in [1] that the duality between four dimensional $N = 1$ supersymmetric gauge theories [5, 6] may be understood as a consequence of T-duality in string theory. The crucial point for understanding the $N = 1$ duality in this framework is the meaning of T-duality of a Kähler surface which is not a torus and is embedded in a Calabi-Yau space. Our aim in this paper is to try to gain an understanding of the required generalization of the notion of T-duality and its implications.

The framework for studing the duality phenomena will be the same as suggested in [1]. Consider a compactification of F-theory on a Calabi-Yau 4-fold elliptically fibered over a 3-fold base B. This leads to an $N = 1$ theory in four dimensions. Let S be a complex surface in B along which the elliptic fibration acquires singularity of the A_{N_c-1} type. We consider a 7-brane with worldvolume $\mathbf{R}^4 \times S$ on which we have an $SU(N_c)$ gauge symmetry. In addition there are $h^{1,0}(S) + h^{2,0}(S)$ chiral multiplets in the adjoint representation. We will also add N_f 3-branes with world volume $\mathbf{R}^4$ which are located at points of the surface S. The open strings stretching between the 3-branes and the 7-brane give N_f hypermultiplets in the fundamental representation of the gauge group.

The Higgs branch of the supersymmetric gauge theory on $\mathbf{R}^4$ is constructed as the moduli space of 0-branes and 4-branes on S. T-duality maps this moduli space to another D-brane moduli space which describes the Higgs branch of the dual theory. In section 2 we will begin by defining the D-brane moduli space as a space of vector bundles on S. In particular we will see that we are forced to generalize the notion of a vector bundle to that of a sheaf, as suggested in [7]. We will discuss the modification for the study of the D-brane moduli space when S is embedded in a curved space. In section 3 we will construct a generalization of T-duality for $K3$ surfaces, which maps a 0-brane charge to a 4-brane charge and vice versa. We will study its properties, check its consistency with the duality between the heterotic string on T^4 and type IIA string theory on $K3$, and compare it to the mirror transform of $K3$. We will then study the implications to the $N = 2$ duality. Finally, we will discuss the case when S is a rational surface, which is the relevant surface for the study of $N = 1$ duality.

2 D-Brane Moduli Space

Let us consider type II string theory compactified on a manifold X of real dimension $2d$. We are interested in the moduli space of D-branes wrapping supersymmetric cycles in X. BPS states are associated with the cohomology classes of the D-brane moduli space. Consider a configuration of $2d$-branes wrapped on X. It carries charges for various RR fields which, as shown in [8, 7], takes the following form

$$Q = v(E) = \mathrm{ch}(E)\sqrt{\hat{\mathrm{A}}(X)} \ . \tag{2.1}$$

$\mathrm{ch}(E)$ is the Chern character of the vector bundle E

$$\mathrm{ch}(E) = \mathrm{Tr}\exp\left[\frac{1}{2\pi}(F - B)\right] \ , \tag{2.2}$$

where F is the field strength of the gauge field on the brane and B is the bulk NS-NS 2-form. It has an expansion in terms of the Chern classes

$$\mathrm{ch}(E) = \mathrm{rank}(E) + c_1(E) + \frac{1}{2}(c_1^2(E) - 2c_2(E)) + \ldots \ . \tag{2.3}$$

$\hat{\mathrm{A}}(X)$ is the A-roof genus and it has an expansion in terms of the Pontrjagin classes

$$\hat{\mathrm{A}}(X) = 1 - \frac{p_1(X)}{24} + \ldots \ . \tag{2.4}$$

$v(E)$ is what is known as the Mukai vector of the vector bundle E on X [1]. In the following we will be interested in the case where X is a complex surface. In this case $v(E) \in H^0(X, \mathbf{Z}) \oplus H^2(X, \mathbf{Z}) \oplus H^4(X, \mathbf{Z})$, and expanding (2.2) using (2.3) and (2.4) we have

$$v(E) = \Big(\mathrm{rank}(E), c_1(E), \frac{1}{2}c_1^2(E) - c_2(E) - \frac{p_1(X)}{48}\mathrm{rank}(E)\Big) \ . \tag{2.5}$$

Consider now one 4-brane wrapped on X. It corresponds to a flat $U(1)$ bundle on X. However, if $p_1(X) \neq 0$ the 4-brane induces a 0-brane charge via the term $\frac{1}{48}\int_X p_1(X)A_1$ in its effective action, where A_1 is the RR 1-form. Indeed, the Mukai vector corresponding to a 4-brane is $v(E) = (1, 0, -\frac{p_1}{48})$. In this paper, we take the convention that the charge vector of the 0-brane is $(0, 0, -1)$. For instance, after integrating $p_1(X)$ over the surface X the Mukai vector for a 4-brane wrapping T^4 is $v(E) = (1, 0, 0)$, while the Mukai vector for a 4-brane wrapping $K3$ is $v(E) = (1, 0, 1)$ and induces the 0-brane charge -1.

[1] In [9], the Mukai vector is defined as $v(E) = \mathrm{ch}(E)\sqrt{\mathrm{Td}(X)}$. This coincides with (2.1) when X is a Calabi-Yau space.

The D-brane moduli space can be viewed as the moduli space of vector bundles E on X. To be more precise, we need to consider not only vector bundles but also coherent sheaves[2]. A *coherent sheaf* on X is represented as a cokernel of a map of vector bundles on X. A notable difference between coherent sheaves and vector bundles is that while the dimension of the fiber of a vector bundle is constant as we move along the base X, the dimension of the fiber of a coherent sheaf is allowed to jump. For illustration, consider a configuration with one 4-brane wrapped on a $K3$ surface X and n 0-branes at points in X. It has the charge vector $(1, 0, 1-n)$. There is no vector bundle whose Mukai vector is $v(E) = (1, 0, 1-n)$, namely no line bundle can have non-zero second Chern number n. But there is indeed such a sheaf. It is a sheaf of holomorphic functions on X vanishing at n points. (This is an element of the so called *Hilbert scheme of n-points in X*.) This simple example indicates that the use of this generalized notion of a vector bundle enables us to describe the D-brane moduli spaces of various charges on the same footing, including those whose charge vector is not realized as the Mukai vector of a vector bundle. As to terminology, we will still use the notion of vector bundles, although it should be clear from the above that in some of the cases the objects are really coherent sheaves.

A 0-brane looks like a zero size instanton on a 4-brane wrapping S [11–13]. While coherent sheaves are objects of algebraic geometry, instantons are objects of differential geometry. However, the intuitive relation between small instantons and coherent sheaves is correct [3].

Let us consider D branes (partially) wrapped on cycles in a manifold X which is embedded in a curved manifold. In particular, $X = S$ in the base B of an elliptic Calabi-Yau 4-fold defining an F-theory vacuum. Then the formula (2.1) for the RR charge vector will be in general modified. In such a case the scalar and fermionic fields on the worldvolume of the brane are in general twisted [15]. If X was embedded in a manifold for type II compactification, the scalars would be sections of the normal bundle while the fermions would be sections of the spin bundle tensored by the square root of the normal bundle. Since the normal bundle to the worldvolume of the brane is in general non-trivial the scalars and the fermions are twisted.

In the framework that we want to study, in which $X = S$ embedded in the base B of F-theory compactification, we do not know in detail how to twist the fields. Nevertheless the twist can be uniquely determined [16]. On a flat 7-brane, we would have the $N = 1$

[2]It has been advocated in [7] that the appropriate objects are coherent simple semistable sheaves.

[3]Small instantons are needed for the (Uhlenbeck) compactification of the instanton moduli space, while the coherent sheaves are needed for the (Gieseker) compactification of the moduli space of stable vector bundles, and on algebraic complex surfaces the two compactifications are related [14].

supersymmetry in eight dimensions. Our requirement is to have $N = 1$ supersymmetry on the uncompactified direction $\mathbf{R}^4$ of the 7-brane wrapped on $S \times \mathbf{R}^4$. On a Kähler manifold with spin structure, spinors are $(0,p)$ forms with values in the square root of the canonical line bundle $K^{\frac{1}{2}}$. This implies that we twist the fermions by $K^{-\frac{1}{2}}$ and therefore they transform as $(0,p)$ forms. For X being T^4 or $K3$ the canonical class is trivial and therefore (2.1) is not modified. This is not the case for the rational surfaces which are of interest to us for the case of $N = 1$ duality. For example, for the Hirzebruch surface S with $p_1(S) = 0$, the formula (2.1) without modification would show that the 4-brane does not induce 0-brane charge and that T-duality proposed in [1] does not lead to $N = 1$ duality.

3 $N = 2$ Duality

3.1 Fourier-Mukai Transform for $K3$

Our aim is to generalize the concept of T-duality to surfaces other than T^4. In this section we will construct a generalization of T-duality for $K3$ surfaces. The generalization will be a natural extension of the Nahm transform [17, 18] which is a way of viewing T-duality on T^4 in the differential geometric language, and is known as the Fourier-Mukai transform [10] in the algebraic geometry framework.

Let us first discuss T-duality on T^4 and the action of T-duality on the moduli space of D-branes on T^4. In particular we are interested in the action of T-duality on 0-branes located at points on the T^4 and 4-branes wrapping it. In the language of the previous section the torus is the moduli space of a 0-brane on T^4 with charge vector $(0,0,-1)$. The dual torus $\widehat{T}^4$ is the moduli space of flat $U(1)$ bundles on T^4 or line bundles on T^4 with Mukai vector $v = (1,0,0)$. In other words, the dual torus $\widehat{T}^4$ is the moduli space of a 4-brane wrapping T^4. Given a vector bundle E on T^4 which describes a configuration of D-branes on T^4, the dual bundle $\widehat{E}$ on $\widehat{T}^4$ is constructed as the (negative) index bundle $-\mathrm{Ind}D$ of a family of Dirac operators $D_{\hat{t}}$ associated with the twisted vector bundles $E_{\hat{t}} = E \otimes L_{\hat{t}}$. $L_{\hat{t}}$ are line bundles on T^4 with Mukai vector $(1,0,0)$ parametrized by the dual torus, $\hat{t} \in \widehat{T}^4$. One can compute the Mukai vector of $\widehat{E}$ by using the family index theorem

$$\mathrm{ch}(\mathrm{Ind}D) = \int_{T^4} \mathrm{ch}(E \otimes \mathcal{Q})\widehat{\mathrm{A}}(T^4) \ , \tag{3.1}$$

where $\mathcal{Q}$ is the so called Poincaré bundle over $T^4 \times \widehat{T}^4$ such that its restriction on $T^4 \times \{\hat{t}\}$

is $L_{\hat{t}}$. As computed explicitly in [17, 19], for $c_1(E) = 0$ we have

$$\mathrm{rank}(\widehat{E}) = c_2(E), \qquad c_2(\widehat{E}) = \mathrm{rank}(E) \ . \tag{3.2}$$

This is what we expect from T-duality under which 0-branes and 4-branes are exchanged.

In order to generalize the above construction of T-duality to $K3$ we first have to define the dual $K3$. There are many ways to define the dual $K3$ [9] but only one corresponds to the required T-duality on all four coordinates.[1] Later we will construct for comparison the dual $K3$ that is obtained by a mirror transform.

We can view $K3$ as the moduli space of a 0-brane on $K3$ with RR charge vector $(0, 0, -1)$. Naively we may think that the dual $K3$ is the moduli space of a 4-brane wrapping $K3$. This cannot be correct on dimensional ground. The complex dimension of the moduli space of vector bundles on $K3$ with Mukai vector $v = (r, l, s)$ is $l^2 - 2rs + 2$ [9]. As we saw in the previous section, the Mukai vector of a 4-brane wrapping $K3$ is $v = (1, 0, 1)$ and the dimension of the moduli space of a 4-brane wrapping $K3$ is zero, thus it cannot be a dual to $K3$.

Indeed, in analogy with the torus case, the correct dual should be the moduli space of sheaves with Mukai vector $v = (1, 0, 0)$. Such a Mukai vector corresponds to one 0-brane and one 4-brane. This means that T-duality on $K3$ does not map a 0-brane to a 4-brane, but rather a 0-brane to a 4-brane plus a 0-brane. In other words T-duality on $K3$ does not map a physical 0-brane to a physical 4-brane but rather a 0-brane charge to a 4-brane charge, and vice versa. A sheaf with Mukai vector $(1, 0, 0)$ has rank one, $c_1 = 0$ and $c_2 = 1$. It cannot be a vector (line) bundle. Rather, as remarked previously, it is a sheaf of holomorphic functions vanishing at a point. By assigning such a point to each sheaf, we obtain a bijection of the moduli space of sheaves with Mukai vector $(1, 0, 0)$ to the original $K3$. This is the Hilbert scheme of one point on $K3$.

Given a vector bundle E on a $K3$ surface X which describes a configuration of D-branes on X, we wish to construct the dual bundle $\widehat{E}$ as the (negative) index bundle of a Dirac operator associated with $E_{\hat{x}} = E \otimes L_{\hat{x}}$ where $L_{\hat{x}}$ are sheaves on X with Mukai vector $(1, 0, 0)$ parametrized by $\hat{x} \in \widehat{X}$. However, as $L_{\hat{x}}$ is not locally free (i.e. not a vector bundle), it is not obvious how to define the Dirac operator. Now we recall that on a $K3$ surface, the positive and negative spin bundles are $S_+ = \Omega^{0,0} \oplus \Omega^{0,2}$ and $S_- = \Omega^{0,1}$ respectively, where $\Omega^{0,p}$ is the bundle of anti-holomorphic p-forms, and the Dirac operator is essentially the $\overline{\partial}$ operator. Thus, the index bundle of the Dirac operator

[1] A Fourier-Mukai transform for reflexive $K3$ surfaces has been derived in a rigorous way in [20]. However, the case studied in that paper does not correspond to the required T-duality.

can be considered as the bundle of Dolbeault cohomology groups $H^{0,0} - H^{0,1} + H^{0,2}$. With a twisted coefficient $\mathcal{E}$, this is the same as the bundle of cohomology groups $H^0(X, \mathcal{E}) - H^1(X, \mathcal{E}) + H^2(X, \mathcal{E})$, which can be extended to the case where $\mathcal{E}$ is not locally free. Applying to the case $\mathcal{E} = E_{\hat{x}}$, we can define the dual bundle $\widehat{E}$ as such an index bundle with its sign inverted.

Applying the Grothendieck-Riemann-Roch theorem, which is an analog of the family index theorem, we can compute the Chern character of $\widehat{E}$:

$$\mathrm{ch}(-\widehat{E}) = \int_X \mathrm{ch}(E \otimes \mathcal{Q})\mathrm{Td}(X). \tag{3.3}$$

The Poincaré bundle $\mathcal{Q}$ is a bundle on $X \times \widehat{X}$ such that the restriction to $X \times \{\hat{x}\}$ is $L_{\hat{x}}$. It is a sheaf of holomorphic functions on $X \times \widehat{X}$ vanishing on the diagonal $\Delta \cong X$ (recall that $\widehat{X}$ is canonically isomorphic to X). Since the restriction of $\mathcal{Q}$ to $X \times \{\hat{x}\}$ is $L_{\hat{x}}$ whose Chern character is $1 - w_X$ where w_X is the 4-form of X with volume one, $\mathrm{ch}(\mathcal{Q})$ must have the term $1 - w_X$ (pulled back to $X \times \widehat{X}$). Similarly, it must have $1 - w_{\widehat{X}}$ and thus, it must contain the term $1 - w_X - w_{\widehat{X}}$. For the purpose of our calculation, we want to know the coefficient of the term $w_X w_{\widehat{X}}$ in $\mathrm{ch}(\mathcal{Q})$. Note that we have an exact sequence of sheaves

$$0 \longrightarrow \mathcal{Q} \longrightarrow \mathcal{O}_{X \times \widehat{X}} \longrightarrow \mathcal{O}_\Delta \longrightarrow 0\ , \tag{3.4}$$

where $\mathcal{O}_{X \times \widehat{X}}$ is the sheaf of holomorphic functions on $X \times \widehat{X}$, and $\mathcal{O}_\Delta$ is the sheaf of holomorphic functions supported on Δ. From this we obtain $\chi(X \times \widehat{X}, \mathcal{O}) = \chi(X, \mathcal{O}) + \chi(X \times \widehat{X}, \mathcal{Q})$. Since $h^{0,0} = h^{0,2} = 1$ and $h^{0,1} = 0$ for $X \cong \widehat{X}$, we have $\chi(X, \mathcal{O}) = 2$, and thus, we see that $\chi(X \times \widehat{X}, \mathcal{Q}) = 2$. Applying the Riemann-Roch formula $\chi(X \times \widehat{X}, \mathcal{Q}) = \int_{X \times \widehat{X}} \mathrm{ch}(\mathcal{Q})\mathrm{Td}(X \times \widehat{X})$, and using $\mathrm{Td}(X) = 1 + 2w_X$ and $\mathrm{Td}(\widehat{X}) = 1 + 2w_{\widehat{X}}$ together with the property $\mathrm{Td}(X \times \widehat{X}) = \mathrm{Td}(X)\mathrm{Td}(\widehat{X})$, we see that

$$\mathrm{ch}(\mathcal{Q}) = 1 - w_X - w_{\widehat{X}} + 2w_X w_{\widehat{X}} \tag{3.5}$$

up to a possible term in $H^2(X) \wedge H^2(\widehat{X})$ which does not contribute to the index for $c_1(E) = 0$. Then, the formula (3.3) yields $-\mathrm{ch}(\widehat{E}) = \mathrm{ch}_2(E) + \mathrm{rank}(E) - w_{\widehat{X}}\mathrm{ch}_2(E)$ in the case $c_1(E) = 0$. Namely, we have seen that

$$\mathrm{rank}(\widehat{E}) = c_2(E) - \mathrm{rank}(E), \qquad c_2(\widehat{E}) = c_2(E)\ . \tag{3.6}$$

Equation (3.6) describes the action of T-duality on the moduli space of 4-branes wrapping the $K3$ surface and 0-branes located at points on it.

It is instructive, for a comparison with T-duality, to define mirror symmetry of $K3$ surfaces in the above language. Following [21–24], we define the mirror of $K3$ as the

moduli space of 2-branes wrapping supersymmetric 2-cycles with a topology of T^2 in $K3$ (holomorphically embedded elliptic curves). The Mukai vector for such a brane is $v = (0, u, 0)$ where $u^2 = 0$ is the self-intersection number of T^2 in $K3$. Since $K3$ can be viewed as the moduli space of a 0-brane on it, with Mukai vector $v = (0, 0, -1)$ we see that mirror symmetry transforms $v = (0, 0, -1) \rightarrow v = (0, u, 0), u^2 = 0$. Given a bundle E on $K3$ describing a configuration of D-branes, the dual bundle $\widehat{E}$ can be constructed as before as the index bundle of $E_{\hat{x}}$ and has a $\text{rank}(\widehat{E}) = -c_1(E)u$. For instance the 2-brane with Mukai vector $(0, u, 0)$, $u^2 = -2$, wrapping a rational curve (supersymmetric 2-cycle S^2) which intersect with the T^2 transversally is mapped to a 4-brane with Mukai vector $(1, 0, 0)$.

The duality between the heterotic string on T^4 and the type IIA string on $K3$ is a useful way to gain some further understanding of the meaning of T-duality on $K3$. We will now show that a particular T-duality on T^4 at the heterotic side corresponds to the above T-duality on $K3$. The integer homology lattice of $K3$ can be decomposed as $\Gamma_{3,19} \oplus \Gamma_{1,1}$ where $\Gamma_{3,19}$ corresponds to $H_2(K3, \mathbf{Z})$ and $\Gamma_{1,1}$ to $H_0(K3, \mathbf{Z}) \oplus H_4(K3, \mathbf{Z})$. We can decompose that Narain lattice $\Gamma_{4,20}$ in the heterotic side in a similar way as $\Gamma_{3,19} \oplus \Gamma_{1,1}$, and let (p_R, p_L) denote the momenta in the $\Gamma_{1,1}$ part.

T-duality on the torus maps $p_R \pm p_L \rightarrow p_R \mp p_L$. We argued that T-duality on $K3$ exchanges 0-brane charge and 4-brane charge. It is natural to ask whether T-duality on T^4 and T-duality on $K3$ are consistent with the heterotic-type IIA duality. This will be the case if $\frac{1}{\sqrt{2}}(p_R + p_L)$ corresponds to 0-brane charge and $\frac{1}{\sqrt{2}}(p_R - p_L)$ corresponds to 4-brane charge, or vice versa. It is easy to see that this is correct. The product $\frac{1}{2}(p_R + p_L)(p_R - p_L)$ is the length of a vector (p_R, p_L) in $\Gamma_{1,1}$. This is mapped by the heterotic-type IIA duality to the intersection number of 0-branes and 4-branes on $K3$, or more accurately taking into account the induced 0-brane charge from a 4-brane on $K3$, to the product of 0-brane charge and 4-brane charge [25]. Thus we see that the dual $K3$ that we constructed is natural from the viewpoint of string duality.

Note that since the construction of the dual $K3$ is not affecting the $\Gamma_{3,19}$ lattice of $K3$, it is natural to expect that the T-duality on $K3$ preserves its complex structure. We have already observed this since the Hilbert scheme of one point on X is the same as X itself $\widehat{X} \cong X$. Note that when constructing the mirror to $K3$ we also affect the $\Gamma_{3,19}$ part of the lattice and therefore change the complex structure, in accord with the mirror transform.

Let us now discuss what happens to the volume of $K3$ after T-duality. We expect that the volume of the dual $K3$ will be proportional to the inverse of the original $K3$. In

order to show that consider the decomposition of a vector $B' \in \mathbf{R}^{4,20}$ as [26]

$$B' = \alpha w + w^* + B \ , \tag{3.7}$$

where $B \in \mathbf{R}^{3,19}$ is the NS-NS two form, and $w, w^* \in \Gamma_{1,1}$ satisfy $w \cdot w = w^* \cdot w^* = 0, w^* \cdot w = 1$. It is argued in [26] that α is the volume of the $K3$ surface. T-duality for $K3$ as constructed above exchanges $w^* \leftrightarrow w$ and indeed, as seen from (3.7), it inverts the volume of the $K3$ surface $\alpha \to 1/\alpha$, as expected.

In closing this section let us comment how we can see from the orbifold viewpoint that the T-duality maps 4-brane charge to 0-brane charge and not physical 4-branes to physical 0-branes. On the surface X the coupling to the R-R 1-form A_1 and 5-form A_5 has the structure

$$(-c_2(E) - \frac{p_1(X)}{48})A_1 + \mathrm{rank}E \int_X A_5 \ , \tag{3.8}$$

where the term multiplying A_1 is the 0-brane charge while the term multiplying $\int_X A_5$ is the 4-brane charge. When X is an orbifold we can still use flat coordinates. In particular, the R-R forms are constructed using the zero modes $\frac{1}{2}(\psi_0^\mu \pm \tilde{\psi_0}^\mu)$. T-duality maps $\frac{1}{2}(\psi_0^\mu \pm \tilde{\psi_0}^\mu) \to \frac{1}{2}(\psi_0^\mu \mp \tilde{\psi_0}^\mu)$. This exchanges the R-R fields A_1 with A_5, and since the (3.8) has to be preserved (if T-duality is a symmetry) the 4-brane and 0-brane charges must be exchanged.

3.2 $N = 2$ Duality

When $S = K3$, since $h_{2,0}(K3) = 1$ we get an $N = 2$ supersymmetry in the uncompactified direction $\mathbf{R}^4$ of the worldvolume of the 7-brane wrapping $S \times \mathbf{R}^4$. We can approximate the F-theory configuration near the 7-brane by a perturbative type IIB string theory compactified on $K3$ with parallel N_c 7-branes wrapped on $K3 \times \mathbf{R}^4$. Indeed, such a configuration yields $N = 2$ supersymmetry on the uncompactified direction $\mathbf{R}^4$ of the worldvolume. The gauge group is $SU(N_c)$ and the matter content is N_f hypermultiplets in the fundamental representation.

In the model that we consider the D-brane moduli space describes vector bundles E with $\mathrm{rank}(E) = N_c, c_1(E) = 0, c_2(E) = N_f$. In principle, there is another gauge group $U(N_f)$ corresponding to the N_f 3-branes. However, we are looking at worldvolume dynamics of the 7-brane. Thus, the $U(N_f)$ group appears in this framework as a global symmetry.

In the following discussion, neglecting the uncompactified direction $\mathbf{R}^4$ for a while, we will use the words 4-branes and 0-branes instead of 7-branes and 3-branes respectively.

The Mukai vector describing N_c 4-branes wrapping $K3$ and N_f 0-branes located at points on $K3$ is

$$v(E) = (N_c, 0, N_c - N_f) \ . \tag{3.9}$$

The moduli space of N_c 4-branes wrapping S and N_f 0-branes located at points on S is the moduli space of vector bundles on $K3$ with Mukai vector (3.9). The complex dimension of this space is

$$\dim M_{v=(N_c,0,N_c-N_f)}(K3) = 2N_cN_f - 2(N_c^2 - 1) \ . \tag{3.10}$$

The description of 0-branes on the 4-branes as instantons suggests that the moduli space of N_f 0-brane on N_c 4-branes wrapping $K3$ $M_{v=(N_c,0,N_c-N_f)}(K3)$ is closely related to the moduli space of $SU(N_c)$ N_f-instantons on $K3$.

The link between the D-branes and the supersymmetric gauge theory in $\mathbf{R}^4$ is the identification of the D-brane moduli space and the Higgs branch of the gauge theory. This presumably requires some limit such as large volume of the surface. The Higgs branch of $N = 2$ $SU(N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation contains two kinds of branches: The Baryonic branch and the non-Baryonic branch [27]. Only in the Baryonic branch the gauge group is completely Higgsed and one has a pure Higgs branch. The non Baryonic branch extends to a mixed branch. The Baryonic and non-Baryonic branches intersect classically, and are separated due to instanton correction in the quantum theory. On dimensional ground, we expect that the D-brane moduli space describes the Baryonic branch.

Using the results of the previous section (3.6), T-duality on $K3$ maps the Mukai vector (3.9) to

$$v(\widehat{E}) = (N_f - N_c, 0, -N_c) \ . \tag{3.11}$$

The moduli space of D-branes on $K3$ and the moduli space of D-branes on the dual $K3$ are isomorphic. Thus, the T-duality suggests that the Baryonic branch of $N = 2$ $SU(N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation is identical to the Baryonic branch of $N = 2$ $SU(N_f - N_c)$ gauge theory with N_f hypermultiplets in the fundamental representation.

The Higgs branch of $N = 2$ supersymmetric QCD was studied in [28] where it was claimed that the part of the moduli space corresponding to complete Higgsing (open dense subset of the Baryonic branch) of $N = 2$ $SU(N_c)$ SQCD with N_f flavors is given by the cotangent bundle of the total space of the determinant line bundle of the Grassmannian $Gr(N_c, N_f)$ with its zero section deleted. This claim is correct up to a subtle point, which we will clarify in the following.

Let us denote by $Q, \tilde{Q}$ the pair of $N = 1$ chiral superfields that constitute a hypermultiplet of $N = 2$ supersymmetry. Here we consider $\tilde{Q}$ as a map from $\mathbf{C}^{N_c}$ to $\mathbf{C}^{N_f}$ and Q as a map from $\mathbf{C}^{N_f}$ to $\mathbf{C}^{N_c}$. The Higgs branch is constructed as the set of $SL(N_c, \mathbf{C})$ orbits of solutions of the F-flatness equation

$$Q\tilde{Q} \propto \mathbf{1}_{N_c} \ . \tag{3.12}$$

When rank$\tilde{Q} = N_c$, $\tilde{Q}$ defines a non-zero point in the determinant line bundle of $Gr(N_c, N_f)$. Then, Q defines a linear form $\delta\tilde{Q} \mapsto \mathrm{Tr}(Q\delta\tilde{Q})$ vanishing on the $sl(N_c, \mathbf{C})$ variation of $\tilde{Q}$, as seen from the F-flatness (3.12). Thus, the part of the Higgs branch where the rank of $\tilde{Q}$ is N_c can be identified with the cotangent bundle of the (non-zero) determinant bundle of $Gr(N_c, N_f)$. This is an open dense subset of the Baryonic branch. Note, however, that there are vacua such that rank$\tilde{Q} < N_c$ and rank$Q = N_c$ [27], and hence the above subset is a proper subset of the moduli space corresponding to complete Higgsing.

There is an isomorphism (as complex manifolds) between the determinant of $Gr(N_c, N_f)$ and that of $Gr(N_f - N_c, N_f)$. The isomorphism can be constructed as follows. Let $Gr(N_c, N_f)$ be realized as the space of N_c planes in a vector space V of dimension N_f, and let $Gr(N_f - N_c, N_f)$ be realized as the space of $N_f - N_c$ planes in its dual V^*. We fix an element $v^1 \wedge \cdots \wedge v^{N_f}$ of the top exterior power $\wedge^{N_f} V^*$. To an element $w_1 \wedge \cdots \wedge w_{N_c}$ in the determinant line over the N_c plane $W \subset V$ spanned by $w_1, \ldots, w_{N_c}$, we associate an element $i_{w_1} \cdots i_{w_{N_c}}(v^1 \wedge \cdots \wedge v^{N_c})$ in the determinant line over the $N_f - N_c$ plane $W^\perp \subset V^*$ orthogonal to W. Here, i_v is the interior product mapping q-th exterior power of V^* to $q-1$-th. Thus, open dense subsets of the Baryonic branches of the $SU(N_c)$ and the $SU(N_f - N_c)$ QCDs with N_f flavors are holomorphically identical.

The above discussion suggests that $N = 2$ duality is only a duality of the Baryonic branches. It is also clear that since the D-brane moduli space that we consider describes only part of the Higgs branch of the $SU(N_c)$ gauge theory, we are unable in this model to make any predictions about the the behavior of the Coulomb branch of the $N = 2$ theory under T-duality.

The complex structure of the D-brane moduli space depends on the complex structure of the $K3$ surface. On the the other hand the complex structure of the Baryonic branch of the $N = 2$ theory on $\mathbf{R}^4$ is fixed by the D-term and F-term equations that determine the branch as a hyperkähler quotient. This seems puzzling, since we wish to identify the Baryonic branch with the D-brane moduli space. To this puzzle, two resolutions are possible. One possibility is that the supersymmetric Lagrangian field theory as we formulate it corresponds to picking one complex structure of the D-brane moduli space but there are other field theories that correspond to picking other complex structures.

The other possibility is that if we appropriately take the field theory limit the dependence on the complex structure of $K3$ disappears, and all will yield the same result.

3.3 Comments on $N = 1$ Duality

If the surface S is rational, the gauge theory on $\mathbf{R}^4$ is $N = 1$ supersymmetric [1]. By rational surface we mean a complex surface birationally equivalent to $\mathbf{P}^2$. A rational surface S satisfies $h_{1,0}(S) = h_{2,0}(S) = 0$. Consider for example the Hirzebruch surfaces F_n.

As in the $K3$ case, we consider F_n as the moduli space of a 0-brane on F_n with charge vector $(0, 0, -1)$. The dual to F_n is the moduli space of vector bundles with Mukai vector $v = (1, 0, 0)$. As we discussed in section 2, since the canonical class of S is non trivial, the definition of Mukai vector (2.1) has to be modified in order to take into account the fact that the fermions and scalars on the surface S are twisted. This implies that bundles with Mukai vector $v = (1, 0, 0)$ have rank one, $c_1 = 0$, $c_2 = 1$ [1]. The moduli space of bundles on F_n with such a Mukai vector is the Hilbert scheme of one point and is isomorphic (as a complex manifold) to F_n. We can now follow the same steps as in the $K3$ case in order to construct T-duality. This, however, does not lead to the required exchange of 0-brane and 4-brane charges. For the required exchange of charges, it seems that we have to define the dual F_n as the moduli space of flat line bundles on F_n. This cannot be the case since the latter moduli space is trivial. Similar analysis can be carried for other rational surfaces such as blow-up of P^2 at points. As in the F_n case, the results indicate that some modification of the scenario is needed in order to make the $N = 1$ duality to work.

The duality between heterotic string theory on T^4 and type IIB string theory on $K3$ was useful in order to gain an understanding of T-duality on $K3$ using our knowledge of T-duality on T^4. Similarly, it is likely that the duality between heterotic string theory on $K3$ and type IIB string theory on F_n (in the appropriate F-theory context) [29] can be used to gain an understanding of the generalization of T-duality on $K3$ surfaces, as constructed in this paper, to the required T-duality on F_n.

[1] We use for the twisted case the Mukai vector $v(E) = \mathrm{ch}(E \otimes K^{-\frac{1}{2}})\sqrt{\widehat{\mathrm{A}}(X)}$.

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