UNIVERSITY OF CALIFORNIA, SAN DIEGO

Topics in Supersymmetric Gauge Theories

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by

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ABSTRACT OF THE DISSERTATION

Topics in Supersymmetric Gauge Theories

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This dissertation consists of two parts, each of which improves our understanding of supersymmetric field theories. In the first part I use the technique of "a-maximization" to study the RG flows of these theories. In doing so I find evidence for the strongest form of Cardy's a-theorem. In doing so we move closer to a proof of Cardy's a-theorem by the removal of a loophole in the argument for the theorem. I also examine the remaining loopholes in the argument. I then apply a-maximization in the case where the superconformal field theory has a product gauge group. In these situations I find that the dynamics of one of the gauge

groups can dramatically alter the behavior of the other. I give a detailed analysis of the possible RG flows and fixed points.

The focus of the second part of the dissertation is on extending and furthering our understanding of a-maximization. I construct an alternative to a-maximization, called τ_{RR}-minimization, which also determines the anomalous dimensions of chiral operators. This technique is not as powerful as a-maximization because τ_{RR} receives quantum corrections. This method is extendable beyond four dimensions. I find the the geometrical analog of a-maximization, Z-minimization, gives the same results as τ_{RR}-minimization, and show how the two techniques are related. This allows for the computation of exact anomalous dimensions, in theories with known AdS/CFT duals, outside of four dimensions.

I

Introduction

In this chapter I will attempt to introduce the ideas discussed in this dissertation to the non-expert. This chapter should be accessible to anyone who has taken an introductory physics sequence. Those with a strong background in the ideas discussed are encouraged to skip this chapter. This introduction covers many topics and is necessarily incomplete. Those who are interested are encouraged to investigate further.

Quantum mechanics is a consistent and successful theory. It explained many phenomena, not describable by classical physics. There are, however, reasons why we know quantum mechanics is incomplete. One problem is that quantum mechanics is a one particle theory. By this I mean that the probability of finding a particle somewhere is 100% at all times. Because of this, quantum mechanics cannot describe, for example, the radioactive decay or certain nuclei. In the decay, the nucleus in question will shoot out a particle or some sort, leaving a different nucleus behind. Clearly, after the decay the probability of finding the original nucleus is no longer 100%. Similarly, quantum mechanics provides no mechanism to describe the creation of the emitted particle. Quantum mechanics is also not compatible with Einstein's theory of relativity, which a full description of our world should be.

The solution to these problems is Quantum Field Theory. Fields are

nothing new in physics. Anybody who has taken an introductory course on physics has encountered the electric field. The electric field was a vector function of position and time introduced to describe the interaction of two particles. We describe the motion of one particle by considering its interaction with the electric field created by the other. The basic idea of quantum field theory is that we associate a field with every particle.

We describe the field as an infinite number of interacting harmonic oscillators, or springs for those not familiar with the physics jargon. The excitations of these oscillators are interpreted as particles of definite properties such as mass, spin and charge. In addition to the excitations corresponding to the particles that we can see, there will be random excitations. These random excitations will be interpreted as what are called "virtual" particles. These are particles which we do not observe directly, but we can infer there existence indirectly. These are responsible for the nebulous "quantum effects" of which one often hears.

In this framework one is able to describe the creation and annihilation of particles and hence problems like radioactive decay and the spontaneous emission of a photon from an atom. It is compatible with special relativity. The list of successes of quantum field theory is long. By accepting relativistic quantum field theory we have had some rules forced on us. First, for any particle, there must exist an antiparticle with opposite charge, but with all other properties the same. Note that for an uncharged particle, it can be its own antiparticle. Also, particles come in two types fermions, which have half integer spin, and bosons, which have integer spin.

If we look at the various discovered particles we find that nature is naturally divided into these two types. Fermions make up matter. Electrons are fermions, as are quarks which make up protons and neutrons. Bosons are associated with forces between particles; two particles will exchange a boson, which will manifest itself as the corresponding force. The photon, the particle associated with the electromagnetic force, is an example of the boson. Imagine two electrons

moving along. One emits a photon, which the other absorbs. We see this as two electrons moving along, and repelling each other. The photon in this case was one of the previously mentioned "virtual" particles. We do not directly see it, but we see its effects in the repulsion of the two electrons. There is one boson not associated with a force, the Higgs boson. The Higgs boson is responsible for giving all other particles there mass.

Fermions and bosons clearly have very different properties, but are both necessary to the operation of our universe. Supersymmetry is a proposed symmetry between these two very different types of particles. If we require our quantum field theory to be supersymmetric then the particles will come in groups, generally called multiplets. The simplest multiplets contain one boson and one fermion, which have equal mass and charge. More complicated multiplets will have several of each, but the idea is the same. This may seem an odd symmetry to propose. As mentioned above, bosons and fermions play very different roles. We also know that the universe does not appear to be supersymmetric. That is we do not see any particles with the same mass and charge, but different spin, which is what would be required for them to form a multiplet. Despite these facts, there are many reasons to study supersymmetric theories.

There are two approaches one can take when studying supersymmetric theories. The first is that supersymmetry is a real symmetry of nature. Despite the fact that we do not see supersymmetry, this is not completely crazy. Since it is not observed, it must be a "broken" symmetry. That is, at the energies which we look at the universe, there is no supersymmetry, but the universe may be supersymmetric at higher energies. Once we understand that supersymmetry is possible, the question now becomes: what benefits does supersymmetry provide? There are many nice features of supersymmetric theories. One exciting feature is unification. If one modifies the current standard model of particle physics by adding a partner for each particle we see, making it supersymmetric, then at some energy scale the strong, weak and electromagnetic force will be equal in strength.

There is also a problem in the standard model. Above it was mentioned that the Higgs particle gives mass to other particles, but the higgs can have any mass. The problem is that when you take quantum effects into account the higgs mass becomes infinite. Supersymmetry fixes this because the quantum effects due the supersymmetric partners to the standard model particles are equal in magnitude, but with the opposite sign. The mass would no longer be driven to infinity. This type of cancellation is a generic feature of supersymmetric theories.

The second approach one can take to supersymmetric theories is that maybe supersymmetry is not a symmetry of nature, but it introduces many simplifications which enables us to extract a lot more information than we can from a non-supersymmetric theory. We can then think of supersymmetric theories as a "warm-up" to attacking the real world. By introducing this extra symmetry into the problem we dramatically restrict the answers to many questions that we might ask. This allows us to extract much more information than we would be able to had we not introduced supersymmetry. One particularly intriguing simplification provided by supersymmetric models is that there are many quantities that we can calculate exactly, where as in a non-supersymmetric theory we would only be able to calculate the same quantity to a certain precision and in some limit. When taking this approach, we expect some of the details to be different in the real world theory and the supersymmetric theory, but we also expect that a lot of the "big picture" will be similar. In this way we learn something about the real world, by studying theory which does not describe our world.

In chapters II and III of this dissertation we use sypersymmetry to study renormalization group (RG) flows. Renormalization is a procedure which allows us to determine what interactions and processes are important at different length scales and what the appropriate fields that describe the system are. The idea behind renormalization is fairly simple to understand. Consider the air in the room you are in. If we are interested in describing the motion of the electrons in the molecules of air, then we are looking at a length scale which is the size of the atoms,

since the electrons are confined within the atom. This length scale is much larger than the size of the nucleus. The equations describing the motion of the electrons will contain parameters depending on the details of what happens at scales the size of the nucleus. However, most of the information about what happens at the nuclear scale is unimportant. The exact details of how the protons and neutrons, which make up the nucleus, move and interact within the nucleus is not terribly important. What is relevant is a few parameters, such as the total charge of the nucleus and a few other "over-all" features. We could also ask about the speed of sound in the air. Here we are still studying the same system, but it seems very different. Now the important length scale is going to be larger than the distances we expect individual molecules to travel. At this scale the motion of the electrons is no longer important. The appropriate fields to describe the system are now the familiar thermodynamic fields of pressure and temperature. The equation for the speed of sound depends on a couple of parameters which encompass the details, which are important at the larger scale, of what happens on the smaller scales. As we varied the length scale at which we looked, the parameters, equations, and even the fields we use to describe it change dramatically. The idea of the renormalization group is to continuously change the length scale of the problem and see how the equations and parameters morph from those describing the protons and neutrons all the way to those describing the speed of sound in air.

As we change the length scale at which we are interested in studying a quantum field theory, the field theory can change drastically. The strengths and types of interactions will change, as will the fields which are present. We call this the RG flow. One can imagine that this process becomes very complicated, and in general we cannot determine the exact behavior of a quantum field theory as we change the length scale. The complexity of the problem will prevent us from changing the length scale too much. In general we refer to smaller length scales as UV and the larger scales as IR.

It can happen that as we vary the length scale along the the RG flow that

we find that the theory ceases changing as we continue to vary the length scale. This is what is called a RG fixed point. Once we reach a fixed point the flow stops. We often refer to a fixed point as UV (or IR) attractive. By this we mean that as we go to shorter (longer) length scales we approach the fixed point.

Note that in performing the renormalization procedure we have not solved any problems. We have just replaced an old problem with a new one. This can be useful, however. The quantum field theories studied in this dissertation are what are called non-Abelian gauge theories. These theories have the property that as you decrease the length scale, the interactions in the theory become weaker. We have tools available to study very weakly interacting field theories. We do not have as many tools to study strongly interacting theories. Renormalization allows us to start with a weakly coupled theory in the UV and flow to the IR, in the process extracting information we would otherwise be unable to access.

In these theories, it is possible that we encounter an IR fixed point as we flow. The quantum field theories at these fixed points can be either strongly and weakly coupled. In chapters II and III of this dissertation we examine the former. Specifically we obtain information about a strongly interacting field theory at the IR fixed point, using information about the weakly interacting theory in the UV. In particular we obtain what is called the scaling dimension of various fields in the theory, which tells us how the fields and interactions behave as we alter the length scale.

In order to extract this information we use a powerful tool known as "a-maximization."[14] We can use this tool in four dimensional supersymmetric field theories. The quantity "a" is called a central charge and is defined for any conformal quantum field theory. Loosely speaking, conformal means that the theory is scale invariant, it lives at a RG fixed point. For supersymmetric conformal field theories, the central charge "a" is a cubic function of the scaling dimensions of the fields in the theory[13]. In the days before a-maximization one could not always exactly determine the scaling dimensions of the various fields, but one could iden-

tify a set of possibilities. There was no way to determine which possibility was the correct one. It was shown that the correct scaling dimensions are those which give the maximum value of the central charge "a." a-maximization completely determines all of the scaling dimensions in the problem. The reason a-maximization is so powerful is that we can relate the quantity "a" to what are called 't Hooft anomalies. These have the feature that they do not change along RG flows. This means we can calculate at whatever energy scale is easiest, and the answer will still be true everywhere along the RG flow.

a-maximization also plays a role in what is called the "a-theorem." The a-theorem states that the central charge "a" should always decrease when flowing from the UV to the IR. The reason that this is expected to be true is that the quantity "a" should characterize the degrees of freedom of the system. The number of degrees of freedom is roughly the number of fields needed to characterize it. We expect that as we flow to larger length scales, we lose some of the details of the short length scale theory. In the example of air in the room, when we describe the large scale properties of the gas, we lost a lot of the details about the behavior of the protons, neutrons and electrons. We reduce all of that information to a few numbers and constants like pressure and viscosity. There is a similar statement, that has been proven, in two dimensions that the central charge, called "c" always decreases when flowing from the UV to the IR. a-maximization comes very close to proving the a-theorem. The a-theorem can be stated in several ways with differing levels of strength. In chapter II of this thesis, we provide evidence of the strongest form of the a-theorem.

In chapter V of this dissertation we make use of a certain aspect of string theory called AdS/CFT. When discussing field theories above, the starting point was that particles were point like. We then described the point particle as an excitation of a field. In string theory we instead postulate that particles are one dimensional objects called strings. This simple change has lead to many exciting results. String theory naturally contains gravity and, as it turns out, contains all

of the structure of quantum field theory. By considering one dimensional objects instead of point particles string theory forces us to accept other objects as well. String theory contains membrane like objects of many dimensions called Dp-branes. These objects have D space directions; for example a D2-brane would be a sheet and a D3-brane would have 3 spatial directions like the universe that we see. We often say that a quantum field theory is living on a brane. By this we mean that the physics an observer confined to the brane observes is described by that quantum field theory.

The AdS/CFT [66] correspondence states that a supersymmetric quantum field theory living on the brane is exactly equivalent to string theory in ten dimensions. The geometry, loosely the shape and the size, of the ten dimensions determine what field theory lives on the brane. Any quantities that we can calculate in the supersymmetric field theory can also, in principle, be calculated in the string theory and both will give the same result. The most well studied example relates a superconformal field theory in four dimensions to string theory with five of the ten dimensions taking the form of what is called Anti-deSitter space (AdS). The other 5 dimensions are very small and called compact. The details of the shape and size of these small dimensions determine many of the properties of the field theory living on the brane.

According to the AdS/CFT correspondence any computation done in the superconformal field theory should have a corresponding calculation in the string theory. One could ask what the calculation corresponding to a-maximization is. It has been shown that the central charge "a" is proportional to one divided by the volume of the compact dimensions. So one would expect the volume of that space to be minimized, since "a" is maximized. The quantity "a" is a function of the scaling dimensions of the fields. In order to carry out the string theory calculation we need to know what quantities, corresponding to the scaling dimensions, the volume of the compact space depends on. For a certain group of models the answer to this was found, and the dual calculation to a-maximization goes by the

name of "Z-minimization."[58] The "Z" which is minimized is proportional to the volume of the compact space, and is a function of what is called the Reeb vector. Using Z-minimization one can determine "a" and all of the scaling dimensions of the fields in corresponding quantum field theory and the answers match those calculated with a-maximization.

The discovery of Z-minimization brought another puzzle. One can use Z-minimization for different geometries than the type listed above, and for these geometries the corresponding field theory is no longer four dimensional. The corresponding field theories can be two or three dimensional, but a-maximization only works for four dimensional theories. In chapters IV and V we show why Z-minimization gives the correct values for the scaling dimensions in these theories. To do this we introduce another method of obtaining the scaling dimensions, which goes by the name of τ_{RR}-minimization. The quantity τ_{RR} does change along RG flows making it very difficult to compute, so this is not as powerful a a-maximization. In cases where we know the corresponding geometry problem, however, the string theory calculation is easier. τ_{RR}-minimization does work in superconformal field theories of dimension other than four and allows us to show why Z-minimization gives the proper scaling dimensions of the fields.

II

Evidence for the Strongest Version of the 4d a-Theorem via a-Maximization Along RG Flows

II.A Introduction

There is an intuition that RG flows are a one-way process, with information about the UV modes lost as one coarse-grains. More precisely (since even an RG fixed point conformal field theory (CFT) has UV modes going above the cutoff), the intuition is that non-trivial RG flows should always decrease the number of *massless* degrees of freedom: relevant deformations will lift some massless degrees of freedom, and RG flow to the IR coarse-grains away these lifted modes, with no new modes becoming massless.

Let us distinguish several possibilities:

1. One can define a quantity, c, that properly counts the massless degrees of freedom of a CFT (e.g. $c > 0$ for all unitarity CFTs, and $c = c_1 + c_2$ for two decoupled CFTs) such that the endpoints of all (unitarity) RG flows satisfy $c_{IR} < c_{UV}$.

2. A stronger claim is that c can be extended to a monotonically decreasing

"c-function" $c(g(t))$ along the entire RG flow to the IR:

$$\dot{c}(g) = -\beta^I(g)\frac{\partial c}{\partial g^I} \leq 0, \tag{II.1}$$

with $\dot{c} = 0$ iff the theory is conformal. Here $\dot{} = \frac{d}{dt}$, with $t = -\log\mu$ the RG "time", increasing towards the IR, and $\dot{g}^I(t) = -\beta^I(g)$, with $g^I(t)$ the running couplings.

3. The strongest possibility is that RG flow is gradient flow of the c-function,

$$\beta^I(g) = G^{IJ}(g)\frac{\partial c(g)}{\partial g^J}, \qquad \text{and} \qquad \frac{\partial c(g)}{\partial g^I} = G_{IJ}(g)\beta^J(g), \tag{II.2}$$

(here $G^{IJ} \equiv (G_{IJ})^{-1}$) with $G^{IJ}(g) > 0$ a positive definite metric (all eigenvalues positive) on the space of coupling constants. Eqn. (II.2) then implies $\dot{c} \leq 0$,

$$\dot{c}(g(t)) = -\beta^I\frac{\partial c}{\partial g^I} = -G_{IJ}\beta^I\beta^J \leq 0, \tag{II.3}$$

with $\dot{c} = 0$ iff the theory is conformal.

The possibility that RG flow is gradient flow with positive definite metric was proposed (and verified to 3-loop order in 4d multi-component $\lambda\phi^4$ theory) by Wallace and Zia [1]. In 2d, Zamolodchikov [2] defined a function $c(g)$, equal to the central charge of the Virasoro algebra for CFTs, which he proved satisfies (II.3) with $G_{IJ}(g) > 0$ (for unitary theories). G_{IJ} is determined from the two-point functions $\langle\mathcal{O}_I(x)\mathcal{O}_J(y)\rangle$ of the operators that g^I and g^J source. This proves version (2) above in 2d, and suggests the strongest version (3) (if the dot product with β^I could be eliminated from both sides of (II.3)). It was also demonstrated [2] that the strongest version (II.2) is indeed true, at least in conformal perturbation theory, in the vicinity of any 2d RG fixed point.

The apparent generality of these intuitions suggest that analogous statements should apply for RG flows in any spacetime dimension. Cardy [3] conjectured that an[1] appropriate quantity for counting the number of massless degrees

[1]This candidate doesn't have an analog for odd spacetime dimensions, unfortunately.

of freedom of 4d CFTs is the conformal anomaly a on a curved spacetime[2]:

$$a \sim \int_{S^4} \langle T^\mu_\mu \rangle. \tag{II.4}$$

The weakest version of the 4d a-theorem conjecture is then that the conformal anomaly a satisfies $a > 0$ for every (unitary) 4d RG fixed point, and $a_{UV} > a_{IR}$ for the endpoints of all (unitary) 4d RG flows. Every known computable example (both non-supersymmetric and using SUSY exact results) is strikingly, and often highly non-trivially, compatible with this conjecture. It would be very interesting and powerful if this a-theorem conjecture is indeed a completely general property of all (unitary) 4d RG flows. At present, however, there is not yet a general, and generally accepted, proof of the conjectured 4d a-theorem. See e.g. [4], [6] , [7], [8], [9] for further discussion of the a-theorem conjecture.

Given the striking successes of the weaker version of the 4d a-theorem, it is natural to consider the 4d analogs of the stronger possibilities (2) and (3) above: perhaps a can be extended to a monotonically decreasing "a-function" $a(g^I)$ along the entire RG flow, and perhaps the beta functions are gradients of this a-function, with positive definite metric, as in (II.2). Osborn and collaborators [10], [11] investigated this in perturbation theory for 4d QFTs (by considering renormalization with spatially dependent couplings) and indeed found a candidate a-function $a(g)$ which satisfies a relation similar to (II.2):

$$\frac{\partial a(g)}{\partial g^I} = (G_{IJ} + \partial_I W_J - \partial_J W_I)\beta^J, \quad \text{where} \quad a(g) = a_{conf.}(g) + W_I(g)\beta^I(g). \tag{II.5}$$

The candidate a-function $a(g_I)$ coincides with the conformal anomaly[3] $a_{conf}(g)$ at the endpoints of the RG flow. The possible term $\partial_{[I} W_{J]}$ in (II.5), a possible difference from gradient flow (II.2), was found to vanish in every example, to all

[2]A general curved 4d spacetime background has two independent anomaly coefficients, $\langle T^\mu_\mu \rangle = a(Euler) + c(Weyl)^2$, but $(Weyl)^2 = 0$ vanishes on a conformally flat background such as S^4. This is just as well, since its coefficient c (so named because it also appears in $\langle T_{\mu\nu}(x) T_{\rho\sigma}(0) \rangle$ in flat space) is known to not have definite monotonicity under RG flow [4], [5]. So we won't discuss c further, and will replace "c" with "a" in the conjectured 4d analogs of the above statements.

[3]To avoid repeatedly writing 3/32, we rescale a relative to other references, $a_{here} = (32/3) a_{usual}$, and write our a-function as $a_{here}(g) = (32/3)\widetilde{a}_{Osborn}(g)$. To avoid a factor of 4/3 which would then show up in (II.5), we also rescale our G_{IJ} relative to [10], [11]: $G^{here}_{IJ} = \frac{4}{3} G^{there}_{IJ}$.

orders checked. Also, it's not manifest in this approach that $G_{IJ}(g) > 0$ ($G_{IJ}(g)$ is defined via beta functions $\beta_{\mu\nu} \sim G_{IJ}(g)\partial_\mu g^I \partial_\nu g^J$ upon taking the couplings to be spatially dependent), but $G_{IJ} > 0$ was verified to be true in every example, to all orders checked [10], [11].

Here we'll explore these ideas in supersymmetric theories, where it's possible to obtain exact results. Supersymmetry relates the stress tensor to a particular R-symmetry, which we'll refer to as the superconformal R-symmetry (even when the theory isn't conformal). The matter chiral superfields Q_i have superconformal $U(1)_R$ charge

$$R(Q_i) = \frac{2}{3}\Delta(Q_i) = \frac{2}{3}(1 + \frac{1}{2}\gamma_i), \tag{II.6}$$

related to Q_i's anomalous dimension. The exact beta functions are related to the violations of the superconformal R-symmetry. For example, the NSVZ exact beta function [12] for the gauge coupling of gauge group G, with matter fields Q_i in representations r_i, is

$$\beta_{NSVZ}(g) = \left(\frac{3g^3/16\pi^2}{1 - \frac{g^2 T(G)}{8\pi^2}}\right)\widehat{\beta}_G(R), \quad \widehat{\beta}_G(R) \equiv -\left[T(G) + \sum_i T(r_i)(R_i - 1)\right], \tag{II.7}$$

with $T(G)$ the quadratic Casimir of the adjoint and $T(r_i)$ that of representation r_i. Likewise, the exact beta function for the coupling h of a superpotential term $W = h\prod_i Q_i^{n(W)_i}$ can be written as (using $\Delta(h) = 3 - \Delta(W)$ to write $h \sim \mu^{(3/2)(R(W)-2)}$):

$$\beta_W(h) \equiv -\dot{h} = \frac{3}{2}h\widehat{\beta}_W(R), \qquad \widehat{\beta}_W(R) \equiv R(W) - 2 = \sum_i n(W)_i R(Q_i) - 2. \tag{II.8}$$

$\widehat{\beta}_G(R)$ and $\widehat{\beta}_W(R)$ are simply linear combinations of the R-charges, independent of the coupling constants. They are defined to have the same sign as the full beta functions, and represent the violation of the R-symmetry by the interactions: $\widehat{\beta}_G(R)$ is the coefficient $\mathrm{Tr}RG^2$ of the $U(1)_R$ current's ABJ anomaly, and $\widehat{\beta}_W(R)$ gives the violation of the R-symmetry by the superpotential.

At the superconformal endpoints of RG flow, the superconformal R-current evolves to a conserved $U(1)_{R_*} \subset SU(2,2|1)$, as the interactions flow to

a zero of their beta functions. The superconformal R-charges of the fields determine the exact operator dimensions of gauge invariant chiral primary operators via $\Delta(\mathcal{O}) = \frac{3}{2}R_*(\mathcal{O})$ (computable in terms of $R_*(Q_i)$ since R-charges are simply additive). Moreover, as shown in [5], [13], the 't Hooft anomalies of $U(1)_{R_*}$ determine the exact central charge of the SCFT:

$$a_{SCFT} = 3\mathrm{Tr}R_*^3 - \mathrm{Tr}R_*. \tag{II.9}$$

It was shown in [14] how to uniquely pick out the special $U(1)_{R_*} \subset SU(2,2|1)$, from among all possible conserved R-symmetries (satisfying $\widehat{\beta}(R) = 0$): it is that which *maximizes* the combination of 't Hooft anomalies

$$a_{trial}(R) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R. \tag{II.10}$$

At the unique local maximum, the function (II.10) coincides with the conformal anomaly a_{SCFT} (II.9), hence the name "a-maximization." E.g. for a free chiral superfield $a_{trial}(R) = 3(R-1)^3 - (R-1)$, as plotted in fig. II.1, with local maximum at point (A). The same qualitative picture of fig. II.1 applies for interacting theories. The function $a_{trial}(R)$, and its local maximum R_* and value a_*, can be exactly computed, even for strongly interacting RG fixed points, via the power of 't Hooft anomaly matching. See e.g. [15] , [16], [17], [18], [19], [20], [21] for some extensions and applications of a-maximization.

a-maximization has several immediate general corollaries. E.g. it implies [14] in complete generality, for *any* 4d $\mathcal{N} = 1$ SCFT[4], that the superconformal R_* charges, and hence the exact scaling dimension of chiral primary operators and the central charges a_* and c_*, are necessarily very special numbers: *quadratic irrationals*, of the general form

$$R_*,\ a_* \in \{\frac{n+\sqrt{m}}{p} \ \mid \ n \in \mathbb{Z},\ m \in \mathbb{Z}_{\geq 0},\ p \in \mathbb{Z}_{\neq 0}\}. \tag{II.11}$$

[4]Theories with accidental symmetries could be exceptions to these general statements, though all known such examples, for example those associated with singular points of $\mathcal{N} = 2$ Seiberg-Witten curves [22], [23], still satisfy the above general statements.

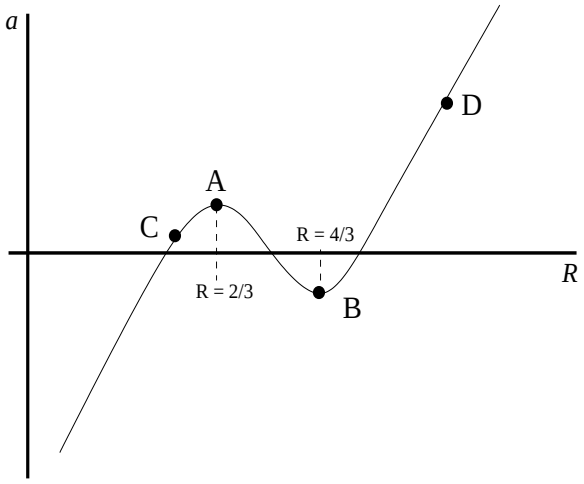

Figure II.1: The trial central charge $a_{trial}(R)$ (with R_* values indicated for free field case).

Quadratic irrational numbers are a measure zero subset of the reals[5], with special properties (e.g. precisely they have continued fraction form that's periodic). The result (II.11) implies that the superconformal $U(1)_R$ charges and central charge a_* cannot vary continuously; therefore, for any SCFT, they cannot depend on any continuous moduli.

As also discussed in [14], a-maximization gives a two line "almost proof" of the a-theorem for supersymmetric RG flows: relevant deformations will break some of the flavor symmetries, placing additional constraints on the IR R-symmetry as compared with the UV one, $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$, and maximizing a function over a subspace leads to smaller maximal value, hence $a_{IR} < a_{UV}$–**QED**! However, as also pointed out in [14], each of these two lines has possible exceptions. First of all, the IR SCFT can have additional accidental symmetries not present in the UV theory, in which case $\mathcal{F}_{IR} \not\subset \mathcal{F}_{UV}$; the result of [14] implies that a_{trial} should be maximized over *all* flavor symmetries, including all accidental ones, so

[5]Rational numbers are a subset of the quadratic irrationals. SCFTs with string dual descriptions are typically limited to this subset, though recently string geometry examples were obtained for which the R-charges are not rational [24], though they're indeed quadratic irrational, compatible with (II.11) (and the general prediction from (II.11) is that any (generally singular) H_5, such that $AdS_5 \times H^5$ is dual to a $N = 1$ SCFT, must have quadratic irrational volumes). There are many SUSY gauge theory examples with R-charges that are quadratic irrational but not rational.

it's crucial that accidental symmetries be properly included. The two-line proof needs to be supplemented with additional physical information to apply to cases with accidental symmetries. The caveat for the second line of the proof is the fact that the maximum is only a local one. E.g. in fig. II.1, suppose that the UV theory is at local maximum (A): perturbing away from there will reduce a, but we need to rule out the possibility that the deformation might eventually drive the value of a up to a point such as (D) in the IR, with $a_{(D)} > a_{(A)}$, violating $a_{IR} < a_{UV}$.

In [20] Kutasov made a very interesting proposal, which helps close the second loophole by extending a-maximization away from the RG fixed points. Assuming that $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$ (in sect. II.D, we'll discuss an extension for certain accidental symmetries), the idea is to implement the additional constraints associated with $\mathcal{F}_{IR} \subset \mathcal{F}_{UV}$ via Lagrange multipliers. We'll write this generally as

$$a(R, \lambda_I) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R + \sum_I \lambda_I \widehat{\beta}^I(R), \tag{II.12}$$

with $\widehat{\beta}^I(R)$ the linear constraints on the R-charges mentioned above, and $\widehat{\beta}_I = 0$ at the IR SCFT. Extremizing (II.12) w.r.t. R, holding the Lagrange multipliers λ_I fixed, yields $R(\lambda_I)$, and plugging back into (II.12) gives

$$a(\lambda_I) \equiv a(R(\lambda_I), \lambda_I) \quad \text{such that} \quad \frac{\partial a(\lambda)}{\partial \lambda_I} = \widehat{\beta}^I(R(\lambda)), \tag{II.13}$$

using the fact that $R(\lambda_I)$ solves $\partial a/\partial R = 0$. The observation now is that the function $a(\lambda_I)$ interpolates between a_{UV} and a_{IR}, and (II.13) suggests that $a(\lambda_I)$ is monotonic, using the physical intuition that beta functions are expected to have a definite sign along the entire RG flow: once a coupling hits a zero of the beta function, it just stops running (e.g. it doesn't overshoot a zero).

It was conjectured in [20] that the Lagrange multipliers λ_I are to be identified with the running coupling constants g_I^2 in some scheme. The extremizing solution $R(\lambda)$ of (II.12) is interpreted as the RG flow of the superconformal R-charges, and $a(\lambda)$ (II.13) is interpreted as a monotonically decreasing a-function along the RG flow to the IR. For relevant interactions, $\dot{\lambda}_I > 0$, so (II.13) with

$\widehat{\beta}^I < 0$ implies that $\dot{a} < 0$. Likewise, for irrelevant interactions, $\dot{\lambda}_I < 0$ and (II.13), with $\widehat{\beta}^I > 0$, again leads to $\dot{a} < 0$.

We will expand upon and further check the interpretation of (II.13) as defining a monotonically decreasing a-function along the RG flow. Our main point is that this proposal suggests the strongest version (3) of the a-theorem conjecture: that the exact RG flows are indeed gradient flows of the a-function (II.13), as in (II.2), with metric on the space of coupling constants given by

$$G_{IJ}(g) = f_J^K(g)\frac{\partial \lambda_K(g)}{\partial g^I}, \quad \text{where} \quad \widehat{\beta}^K(R) = f_J^K(g)\beta^J(g). \tag{II.14}$$

A sufficient condition for this metric to be positive definite is that the $f_J^K(g)$ are positive, e.g. g doesn't flow beyond the apparent pole in the denominator of $\beta_{NSVZ}(g)$ in (II.7), and the relation (scheme change) between the λ_K and the g^J is monotonic.

In section II.B.1 and II.B.2, we review the RG flow of the R-symmetry in the stress tensor supermultiplet, and the a-maximization method [14] for determining the superconformal R-charge at RG fixed points, as well as the extension of [16] for cases with accidental symmetries. In section II.B.3 we review Kutasov's proposal for a-maximization with Lagrange multipliers [20], first for the case of gauge interactions only. In sect. II.B.4, we use (II.6) and the R-charges $R(\lambda)$ obtained by extremizing (II.12) to compute the anomalous dimensions

$$\gamma_i(\lambda) = 3R(\lambda_I)_{Q_i} - 2 = 1 - \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}, \tag{II.15}$$

comparing with perturbative computations of $\gamma_i(g)$. This provides both a non-trivial check of a-maximization and its extensions, and also a means to determine the relation, $\lambda_I(g)$, of λ_I to the coupling constant g in a given scheme, e.g. that of the NSVZ beta function. In sect. II.B.4, we will check (II.15) to three loops, comparing with the computations of [25] (the one-loop check was already verified in [14], and the two-loop check was discussed and verified in [20]). In sect. II.B.5 we will discuss a-maximization along the RG flow for superpotential interactions,

obtaining the one-loop (scheme independent part) relation between the Lagrange multiplier and the superpotential Yukawa coupling. In sect. II.B.6, after reviewing a-maximization with Lagrange multipliers for $SU(N_c)$ SQCD (which was discussed in [20]), we apply this method to its magnetic $SU(N_f - N_c)$ Seiberg [26] dual. Analyzing the magnetic theory, we point out that the $R(\lambda_I)$ which extremizes (II.12) is a solution of a quadratic equation and that, in the RG flow of $R(\lambda_I)$ to the IR, λ can flow from increasing on one branch to decreasing λ on the other branch.

In section II.C, we point out that (II.13), with the Lagrange multipliers interpreted as the running coupling constants, demonstrates that RG flow is indeed gradient flow, with metric (II.14). We compute this metric for gauge (this case already appears in [20]) and Yukawa interactions. In the perturbative limit, we compare these metrics with those computed by Freedman and Osborn [11], and find perfect agreement for the leading, scheme independent coefficients. In other words, the a-function (II.13), computed by a-maximization with Lagrange multipliers, agrees with that proposed and computed perturbatively in [10], [11] (at least to leading perturbative order).

In section II.D, we propose an extension of the Lagrange multiplier method of [20] to apply for RG flows with accidental symmetries associated with gauge invariant operators hitting their unitarity bound and becoming free. This extension leads to a monotonically decreasing a-function for such RG flows, showing in particular that a-maximization indeed ensures that $a_{IR} < a_{UV}$ for these RG flows too. We also comment in sect. II.D on the challenge of finding a natural, monotonically decreasing a-function for RG flows associated with the Higgs mechanism: there are contributions (the eaten matter fields) whose effect is to reduce a in the IR, as well as contributions (the uneaten matter fields) whose effect is to increase a in the IR, and the challenge is to find an interpolating function which makes it manifest that the former always outweighs the latter.

Finally, in section II.E, we illustrate some of these ideas with a new

class of 4d SCFTs, which are simply a deformation of SQCD, where some general fraction of the flavors are coupled to added singlets. These theories generalize and interpolate between SQCD and its magnetic Seiberg dual [26], which are the special cases of none or all flavors coupled to singlets. As we discuss, these new SCFTs have a dual description, obtained as a deformation of Seiberg duality [26]. Though these new SCFTs are simply related to SQCD, they could not have been analyzed before the introduction of the a-maximization method [14]. In ordinary SQCD, mesons hitting their unitarity bound coincides with the entire magnetic dual being IR free [26]. In our "SSQCD" (for singlets + SQCD) generalizations, on the other hand, mesons can decouple with the rest of the SCFT remaining interacting. In the magnetic dual description, this happens when only the superpotential term involving that meson becomes irrelevant, with the rest of the dual theory remaining interacting.

Note added: The results of our section II.B.4 (including, in particular, the scheme change with $\partial \ln F_i/\partial g \sim C(r_i)^2 g^3 + O(g^5)$) were subsequently independently obtained in [27].

II.B The superconformal R-symmetry, a-maximization, and Lagrange multipliers

II.B.1 The flowing R-charges

$\mathcal{N} = 1$ supersymmetry puts the stress-energy tensor $T_{\mu\nu}$ into a current supermultiplet, $T_{\alpha\dot{\alpha}}(x, \theta, \overline{\theta})$, whose first component is a $U(1)_R$ current (and other components include the supercharge currents). For superconformal theories, this R-current is conserved, and is the $U(1)_R \subset SU(2,2|1)$ in the superconformal algebra. For non-conformal theories, supersymmetry relates the dilatation current divergence T^μ_μ to that of this R-current, via

$$\overline{\nabla}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \nabla_\alpha L_T, \tag{II.16}$$

with L_T the chiral superfield trace anomaly, e.g.

$$L_T = -\frac{\widehat{\beta}(R)}{64\pi^2}(W^\alpha W_\alpha)_{gauge} - \frac{\tau_{IJ}}{96\pi^2}(W^I_\alpha W^{\alpha J})_{flavor} + \frac{c}{24\pi^2}\mathcal{W}^2 - \frac{a}{24\pi^2}\Xi, \quad \text{(II.17)}$$

with the first term the gauge beta function, the second the contribution associated with background fields coupled to flavor currents, and the last two terms the contributions associated with a background metric and gauge field coupled to the superconformal R-current. See [13] for a discussion of the latter terms. We'll refer to the $U(1)_R$ current in $T_{\alpha\dot{\alpha}}$ as the superconformal R-current, whether or not the theory is conformal, keeping in mind that in the non-conformal case this R-symmetry is violated.

Whether or not the theory is conformal, supersymmetry relates the superconformal R-charges to the scaling dimensions of the fields:

$$R(Q_i) = \frac{2}{3}\Delta(Q_i) = \frac{2}{3}(1 + \frac{1}{2}\gamma_i), \quad \text{(II.18)}$$

with γ_i the anomalous dimension of field Q_i. Consider a RG flow, e.g. with asymptotically free gauge fields and matter in the UV, to an interacting RG fixed point in the IR. Along this RG flow we can write the superconformal R-current as

$$R^\mu = R^\mu_{cons} + X^\mu_{flow}, \quad \text{(II.19)}$$

with R^μ_{cons} a conserved current, and X^μ_{flow} not conserved. The current X^μ_{flow} gets an anomalous dimension, and becomes irrelevant, flowing to zero in the IR, so the R-symmetry in the stress tensor supermultiplet flows as $R \to R_{cons}$ in the IR.

As an example, consider SQCD: $SU(N_c)$ gauge theory with N_f fundamental flavors Q_f and $\widetilde{Q}_{\tilde{f}}$ (taking N_f in the superconformal window [26] $\frac{3}{2}N_c < N_f < 3N_c$). There is a unique conserved R-symmetry that commutes with all the flavor symmetries and charge conjugation, $R_{cons}(Q_f) = R_{cons}(\widetilde{Q}_{\tilde{f}}) = 1 - \frac{N_c}{N_f}$. This R-symmetry is conserved along the entire RG flow, but it is only the R-symmetry in the stress tensor supermultiplet at the IR SCFT fixed point. Along the RG flow, the R-symmetry in the stress tensor supermultiplet is the sum of terms

(II.19), with $X^{\mu}_{flow} \to 0$ in the IR (see e.g. [28]). The superconformal R-charges evolve along the RG flow, from $R_{UV}(Q_f) = R_{UV}(\widetilde{Q}_{\tilde{f}}) = R_{free} = 2/3$ (asymptotic freedom), to those of the IR SCFT, $R_{IR}(Q_f) = R_{IR}(\widetilde{Q}_{\tilde{f}}) = R_{cons} = 1 - \frac{N_c}{N_f}$.

Using the result of [5], [13], the conformal anomaly at the UV and IR endpoints of the RG flow are given by $a_{UV} = 3\mathrm{Tr}R^3_{UV} - \mathrm{Tr}R_{UV}$ and $a_{IR} = 3\mathrm{Tr}R^3_{IR} - \mathrm{Tr}R_{IR}$. 't Hooft anomaly matching does not equate a_{UV} and a_{IR}, because the R-charges themselves are different in the UV and the IR, with the R-current in $T_{\alpha\dot{\alpha}}$ not even conserved along the RG flow. E.g. for SQCD (with N_f in the superconformal window)

$$a_{UV} = 2(N_c^2 - 1) + 2N_cN_f\left(3\left(-\frac{1}{3}\right)^3 + \frac{1}{3}\right) = 2(N_c^2 - 1) + \frac{2}{9}(2N_cN_f), \quad \text{(II.20)}$$

the free-field contribution expected by asymptotic freedom ($a_V^{free} = 2$ and $a_Q^{free} = 2/9$ in our normalizations). At the IR endpoint of the RG flow, the conformal anomaly is

$$\begin{aligned} a_{IR} &= 2(N_c^2 - 1) + 2N_cN_f\left(3\left(-\tfrac{N_c}{N_f}\right)^3 + \tfrac{N_c}{N_f}\right) = 4N_c^2 - 2 - \tfrac{6N_c^4}{N_f^2} \qquad \text{(II.21)} \\ &\equiv a_{SQCD}(N_c, N_f), \end{aligned}$$

where we used $R_{IR} = R_{cons}$. 't Hooft anomaly matching is used to evaluate these R_{IR} 't Hooft anomalies using the weakly coupled degrees of freedom of the UV endpoint of the flow (since R_{IR}, unlike the R-symmetry in $T_{\alpha\dot{\alpha}}$, is here conserved along the entire RG flow). As predicted by the a-theorem conjecture, $a_{UV} > a_{IR}$. In the UV, the matter fields are at point (A) in fig. II.1, and in the IR they're at a lower point such as (C) in fig. II.1.

It's non-trivial that $a_{SCFT} > 0$, even at strongly coupled RG fixed points, as desired for a count of massless d.o.f.. E.g. expression (II.21) satisfies $a_{SQCD}(N_c, N_f) > a_{SQCD}(N_c, N_f - 1)$, as expected by the a-theorem conjecture, since we can RG flow from the theory with N_f flavors in the UV to one with $N_f - 1$ flavors in the IR by giving a mass to a flavor. If continued to sufficiently small N_f, (II.21) would give negative a. But N_f never gets sufficiently small to

violate $a > 0$, because for $N_f \leq \frac{3}{2}N_c$ something different happens, as can be seen from the fact that the mesons $M = Q\widetilde{Q}$ hit the unitarity bound $R(M) \geq 2/3$; in fact, the entire magnetic dual then becomes free [26].

II.B.2 a-maximization at RG fixed points

Let us briefly recall the argument of [14], that the exact superconformal R-symmetry maximizes $a_{trial} = 3\mathrm{Tr}R_t^3 - \mathrm{Tr}R_t$. We write the general trial $U(1)_R$ symmetry as $R_t = R_0 + \sum_I s_I F_I$, where R_0 is an arbitrary R-symmetry, the F_I are non-R flavor symmetries, and s_I are real coefficients. The superconformal R-symmetry $U(1)_{R_*} \subset SU(2,2|1)$ corresponds to some particular values of the s_{*I}, that we'd like to determine. The result of [14] is that they're uniquely determined by the 't Hooft anomaly relations

$$9\mathrm{Tr}R_*^2 F_I = \mathrm{Tr}F_I \qquad \text{for all } F_I, \tag{II.22}$$

$$\mathrm{Tr}R_* F_I F_J = -\frac{1}{3}\tau_{IJ} < 0. \tag{II.23}$$

Relation (II.22) is equivalent to the statement that the exact superconformal R-symmetry extremizes $a_{trial} = 3\mathrm{Tr}R_t^3 - \mathrm{Tr}R_t$; because a_{trial} is a cubic function, (II.22) is a quadratic equation for R in each variable s_I. The inequality (II.23) then implies that the correct extremum is the unique one which locally maximizes a_{trial}.

Relation (II.22) was obtained in [14] by using supersymmetry to relate the two corresponding anomaly triangle diagrams, $\langle F_I RR\rangle$ and $\langle F_I TT\rangle$. A non-R flavor supercurrent J_I is at one vertex and the super-stress tensor $T_{\alpha\dot{\alpha}}$, containing both the superconformal $U(1)_R$ current and the stress tensor, is at the other two vertices. Using a result of [29], the $\langle J_I(z_1)T_{\alpha\dot{\alpha}}(z_2)T_{\beta\dot{\beta}}(z_3)\rangle$ three-point function, and hence its anomaly, is uniquely determined by the superconformal Ward identities up to an overall normalization coefficient; this implies that the anomalies on the two sides of (II.22) have fixed ratio, and the factor of 9 can then be fixed by considering

the free-field case, where the fermions have $R = -1/3$. Another way to obtain (II.22) is to consider the anomalous violation of the flavor supercurrent J_I upon turning on a background coupled to $T_{\alpha\dot{\alpha}}$, i.e. a background metric and background gauge fields coupled to the superconformal R-current: (II.22) is obtained upon arguing that $\overline{D}^2 J_I = k_I \mathcal{W}^2$, with no contribution proportional to the chirally projected super Euler density Ξ [14].

The equality in (II.23), obtained in [5], relates the 't Hooft anomaly for $\langle R F_I F_J \rangle$ to the coefficients τ_{IJ} of the flavor current two-point functions $\langle J_I^\mu(x) J_J^\nu(y) \rangle$. The inequality in (II.23) then follows upon using unitarity to argue that the current-current two-point function coefficients are a positive definite matrix, $\tau_{IJ} > 0$. The extremum condition (II.22) is a quadratic equation, and inequality (II.23) determines that the correct solution is uniquely determined to be that which locally maximizes a_{trial}.

For a general $\mathcal{N} = 1$ SUSY gauge theory, with gauge group G and matter chiral superfields Q_i in representations r_i of G, (II.22) constrains the superconformal R-charges $R(Q_i) \equiv R_i$ to satisfy

$$\sum_i |r_i| (F_I)_i \left(9(R_i - 1)^2 - 1 \right) = 0. \tag{II.24}$$

$(F_I)_i \equiv F_I(Q_i)$ are any flavor charges of the matter fields, which must be G-anomaly free:

$$\mathrm{Tr} F_I G^2 = \sum_i (F_I)_i T(r_i) = 0, \tag{II.25}$$

with $T(r_i)$ the quadratic Casimir of representation r_i. Superpotential interactions further constrain the charges $(F_I)_i$; for now, consider the case of gauge interactions only. The general solution for R_i, satisfying (II.22) for any flavor charges $(F_I)_i$ satisfying (II.25), is

$$R_i = 1 - \frac{1}{3} \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}}. \tag{II.26}$$

λ_* is a parameter that is fixed by the constraint that $U(1)_R$ be anomaly free:

$$\mathrm{Tr}RG^2 = T(G) + \sum_i T(r_i)(R_i - 1) = T(G) - \frac{1}{3}\sum_i \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}} = 0. \quad \text{(II.27)}$$

The branch of the square-roots are determined by (II.23), which for gauge interactions has sign corresponding to negative anomalous dimensions, since (II.26) and (II.6) yield for the RG fixed point anomalous dimensions:

$$\gamma_i(g_*) = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda_* T(r_i)}{|r_i|}} = 1 - \sqrt{1 + \frac{\lambda_* C(r_i)}{|G|}}. \quad \text{(II.28)}$$

As standard, we define group theory factors as

$$\mathrm{Tr}_{r_i}(T^A T^B) = T(r_i)\delta^{AB}, \ \sum_{A=1}^{|G|} T^A_{r_i} T^A_{r_i} = C(r_i)\mathbf{1}_{|r_i|\times|r_i|}, \text{ so } C(r_i) = \frac{|G|T(r_i)}{|r_i|}, \quad \text{(II.29)}$$

normalizing quadratic Casimirs so that $T(G) = N_c$ and $T(Fund) = \frac{1}{2}$ for $SU(N_c)$.

As discussed in [14], a non-trivial check of a-maximization is that (II.28) indeed reproduces the correct anomalous dimensions for perturbatively accessible RG fixed points:

$$\gamma_i(g) = -\frac{g^2}{4\pi^2} C(r_i) + O(g^4). \quad \text{(II.30)}$$

Expanding the exact result (II.28) for small λ and comparing with (II.30) yields

$$\lambda_* = \frac{g_*^2 |G|}{2\pi^2} + O(g_*^4), \quad \text{(II.31)}$$

with both λ_* and g_* determined at the RG fixed point in terms of the group theory factors [14] by the condition that $U(1)_{R_*}$ be anomaly free (equivalently, $\beta_{NSVZ} = 0$).

The above results are valid as long as there are no accidental symmetries in the IR. They require modification when IR accidental symmetries are present [16], because we must a-maximize over all flavor symmetries, including all accidental symmetries. Restricting the landscape of allowed R-charges, by not accounting for the possibility of mixing with all accidental symmetries, would lead to incorrect results. A crucial issue then becomes how one can determine what accidental symmetries might be present.

One particular type of accidental symmetry, which is under control, is that associated with gauge invariant composite operators hitting a unitarity bound, and becoming free. To be concrete, suppose that $dim(M)$ operators $M = Q\widetilde{Q}$ become free, with an accidental $U(1)_M$ symmetry, under which only the composite operators M are charged; the $U(1)_M$ charge is F_M, with $F_M(M) = 1$ and all other fields neutral. a-maximization must include mixing with $U(1)_M$: $R_{trial} = R^{(0)}_{trial} + s_M F_M$. $a_{trial} = 3\mathrm{Tr}R^3_{trial} - \mathrm{Tr}R_{trial}$ can be computed using 't Hooft anomaly matching. Maximizing over s_M yields $R_*(M) = 2/3$, as appropriate for a free field, with $R_*(M) \neq R_*(Q) + R_*(\widetilde{Q})$ because of the mixing with $U(1)_M$. There is an important residual effect on the quantity to be maximized for determining $y \equiv R(Q)$ and $\widetilde{y} \equiv R(\widetilde{Q})$ [16] (see [17] for a derivation along the lines sketched here):

$$a^{(1)}(y, \widetilde{y}, \ldots) = a^{(0)}(y, \widetilde{y}, \ldots) + dim(M) \left(\frac{2}{9} - 3(y + \widetilde{y} - 1)^3 + y + \widetilde{y} - 1 \right). \quad \text{(II.32)}$$

The additional term in (II.32) vanishes when $R_0(M) \equiv y + \widetilde{y} = 2/3$, as does its first derivative. This ensures that a-maximization yields R_* charges and central charge a_{CFT} that are continuous and smooth (first derivative continuous, though higher derivatives are generally discontinuous) across a transition where the operators M become free (say as a function of parameters that can be varied, such as N_c/N_f).

II.B.3 a-maximization with Lagrange multipliers

We first review Kutasov's proposal [20] for the case of gauge interactions only. The idea is to implement the constraint that the superconformal $U(1)_R$ be anomaly free at the IR fixed point via a Lagrange multiplier λ, maximizing (II.12)

$$\begin{aligned} a(R_i, \lambda) \;&= 2|G| + \textstyle\sum_i |r_i|[3(R_i - 1)^3 - (R_i - 1)] \\ &\quad -\lambda \left(T(G) + \textstyle\sum_i T(r_i)(R_i - 1) \right). \end{aligned} \quad \text{(II.33)}$$

Extremizing (II.33) w.r.t. R_i yields

$$R_i(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda T(r_i)}{|r_i|}} = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda C(r_i)}{|G|}}. \tag{II.34}$$

Plugging back into (II.33) yields

$$a(\lambda) \equiv a(R_i(\lambda), \lambda) = 2|G| - \lambda T(G) + \frac{2}{9}\sum_i |r_i| \left(1 + \frac{\lambda T(r_i)}{|r_i|}\right)^{3/2}. \tag{II.35}$$

Because $R_i(\lambda)$ solves $\partial a/\partial R_i = 0$, we have

$$\frac{d}{d\lambda}a(\lambda) = \frac{\partial}{\partial\lambda}a(R_i, \lambda) = -T(G) - \sum_i T(r_i)(R_i - 1) \equiv \widehat{\beta}_G(R_i). \tag{II.36}$$

Extremizing now in λ has solution λ_*, where (II.36) vanishes, and $R_i(\lambda_*)$ are the same as in (II.26). Also, evaluating (II.33) with both R_i and λ extremized yields $a(R(\lambda_*), \lambda_*) = a_{SCFT}$, since the additional term proportional to λ in (II.33) vanishes at $\lambda = \lambda_*$.

The proposal of [20] is to interpret (II.34) and (II.35) as the running R-charges and a-function, along the *entire RG flow*, from the UV to the IR, with the Lagrange multiplier λ interpreted as the running gauge coupling g^2 in some scheme. The RG flow from UV to IR corresponds to $\lambda : 0 \to \lambda_*$. Since λ is increasing along the RG flow to the IR, $\dot{\lambda} > 0$, and the beta function along the RG flow is negative, (II.36) implies that this a-function is monotonically decreasing along the RG flow, $\dot{a} \leq 0$, with $\dot{a} = 0$ at precisely the IR SCFT, where the beta function vanishes.

The RG flow can be pictured using fig. II.1. In the UV, $\lambda = 0$ and the matter chiral superfields all have $R_i = 2/3$, at point (A). Extremizing (II.33) w.r.t. R_i implies that R_i should sit at a point where the slope of the function in fig. II.1 equals $\lambda T(r_i)$, giving (II.34). Increasing λ thus takes R_i to where the slope is positive, i.e. down the hill to the left of point (A), reducing a. Eventually the flow hits a zero of the beta function and stops, with $R(Q_i)$ at some point (C) in fig. II.1.

II.B.4 Comparing with the explicit perturbative computations of Jack, Jones, and North [25].

The proposal is that (II.34) gives the exact R-charges along the entire RG flow. Hence the exact anomalous dimensions, along the entire RG flow, are given by

$$\gamma_i(\lambda) = 2(\Delta(Q_i) - 1) = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda C(r_i)}{|G|}}. \tag{II.37}$$

In this subsection, we will compare this with explicit perturbative computations, extending the higher-loop check made in [20]. Note that the expression (II.37) is obviously compatible with the a-maximization result (II.28) for the exact anomalous dimension at RG fixed points. The check here is thus also a higher-loop extension of the check in [14] between the exact a-maximization results and explicit perturbative computations, for those RG fixed point theories which are perturbatively accessible.

Expanding (II.37) in $\widehat{\lambda} \equiv \lambda/2|G|$ yields (for uniform notation, we take $(-1)!! \equiv 1$)

$$\gamma_i(\lambda) = \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!}(-\widehat{\lambda})^p C(r_i)^p = -\widehat{\lambda}C(r_i) + \frac{\widehat{\lambda}^2}{2}C(r_i)^2$$
$$-\frac{\widehat{\lambda}^3}{2}C(r_i)^3 + \frac{5\widehat{\lambda}^4}{8}C(r_i)^4 + \ldots. \tag{II.38}$$

Comparing with the 1-loop anomalous dimensions (II.30) then yields

$$\widehat{\lambda} \equiv \frac{\lambda}{2|G|} = \frac{g^2}{4\pi^2} + \sum_{q=2}^{\infty} A_q g^{2q}, \tag{II.39}$$

the analog of (II.31), now interpreted as applying along the entire RG flow; (II.39) is indeed compatible with the interpretation of λ as corresponding to the running coupling. The undetermined coefficients $A_{q\geq 2}$ in (II.39) reflect the standard renormalization scheme freedom to reparametrize the coupling constant. In general, if one scheme has coupling g and wavefunction renormalization factors $Z_i(g)$, another could have coupling $g'(g)$ and wavefunction renormalization $Z'_i(g') = Z_i(g)F_i(g)$.

The anomalous dimensions and beta function of the two schemes are then related by

$$\gamma_i'(g') = \gamma_i(g) + \frac{1}{2}\beta(g)\frac{\partial \ln F_i(g)}{\partial g}, \quad \text{and} \quad \beta'(g') = \frac{\partial g'(g)}{\partial g}\beta(g). \tag{II.40}$$

We will compare the prediction (II.38) with the explicit higher loop computations of [25], assuming initially that the only scheme difference is a change of coupling constant $\lambda = \lambda(g)$, as in (II.39), assuming initially that $F_i(g) = constant$ in (II.40).

Keeping arbitrary A_q in (II.39), (II.38) yields

$$\gamma_i(g) = \sum_{p=1}^{\infty} \frac{(2p-3)!!}{p!}\left(-\frac{g^2 C(r_i)}{4\pi^2} - \sum_{q=2}^{\infty} A_q C(r_i) g^{2q}\right)^p. \tag{II.41}$$

Expanding yields predicted expressions for the p-loop anomalous dimensions:

$$\gamma_i^{(1)} = -\frac{C(r_i)}{4\pi^2}g^2, \quad \gamma_i^{(2)} = \left(\frac{C(r_i)^2}{32\pi^4} - A_2 C(r_i)\right) g^4,$$

$$\gamma_i^{(3)} = \left(-\frac{C(r_i)^3}{128\pi^6} + A_2\frac{C(r_i)^2}{4\pi^2} - A_3 C(r_i)\right) g^6,$$

$$\gamma^{(4)} = \left(\frac{5}{8}\frac{C(r_i)^4}{(4\pi^2)^4} - \frac{3}{2}A_2\frac{C(r_i)^3}{(4\pi^2)^2} + \frac{1}{2}(2\frac{A_3}{4\pi^2} + A_2^2)C(r_i)^2 - A_4 C(r_i)\right) g^8, \quad \text{etc.} \tag{II.42}$$

The prediction, for general p-loops, is that the highest power of $C(r_i)$ is $C(r_i)^p$. The coefficient of this highest power term is hence scheme independent, and predicted to be:

$$\gamma_i^{(p)}(g) = \left(\frac{(2p-3)!!}{p!}\left(-\frac{C(r_i)}{4\pi^2}\right)^p + \sum_{\ell=1}^{p-1}(\text{scheme dependent coeffs.})C(r_i)^\ell\right) g^{2p}. \tag{II.43}$$

Moreover, for each p, the scheme dependent coefficients of $C(r_i)^\ell$ in (II.43) are fixed in terms of those of lower orders of perturbation theory for $2 \leq \ell < p$ (only the coefficient of the $\ell = 1$ term isn't already determined by the results from lower orders in perturbation theory). The structure of the scheme dependent coefficients is predicted to be such that there exists a particular scheme, corresponding to setting all $A_{q>2} = 0$, in (II.39) in which the p-loop anomalous dimension has only the $C(r_i)^p$ term in (II.43).

As discussed in [20], the predicted $\gamma^{(2)}$ in (II.42) indeed agrees with that obtained from explicit computation of the Feynman diagrams: the scheme independent $C(r_i)^2$ term indeed has the same coefficient[6], and matching the coefficient of the $C(r_i)$ term fixes the coefficient A_2 in the expression (II.39) for λ in the particular scheme adopted in [25]:

$$A_2 = \frac{b_1}{64\pi^4}, \quad \text{with } b_1 \equiv 3T(G) - \textstyle\sum_i T(r_i), \quad \text{in the particular scheme of [25].} \tag{II.44}$$

We can now go to *three loops*, comparing the prediction (II.42) with the perturbative results of [25]. We indeed find precise agreement for the scheme independent coefficient of the $g^6 C(r_i)^3$ term! However, using (II.44) in (II.42), our prediction for the (scheme dependent) coefficient of the $g^6 C(r_i)^2$ term in $\gamma_i^{(3)}$ is twice that obtained in [25]. Fortunately, this difference (as in (II.44)) is proportional to (the leading term of) $\beta(g)$. Thus (II.42) can be salvaged by including a further scheme difference (II.40), between that of the Lagrange multiplier method and that of [25], coming from a non-trivial difference in the wavefunction renormalization starting at two loops: $\partial \ln F_i / \partial g \sim C(r_i)^2 g^3$.

II.B.5 Including superpotential interactions

Let's now consider the case of both gauge interactions and those associated with a superpotential term $W = h \prod_i Q_i^{n(W)_i}$. If this W is relevant, the IR SCFT has the added constraint that the superpotential[7] has total R-charge 2, which can again be implemented with a Lagrange multiplier. The prescription is then to modify (II.33) by adding a term $\lambda_W(R(W)-2)$, with $R(W) = \sum_i R_i n(W)_i$. Extremizing $a(R_i, \lambda_G, \lambda_W)$ w.r.t. the R_i, holding λ_G and λ_W fixed, then modifies

[6] In comparing with [25], note that we define anomalous dimensions as $\Delta(Q_i) = 1 + \frac{1}{2}\gamma_i$, whereas the definition in [25] wouldn't have the $\frac{1}{2}$, so $\gamma_{here} = 2\gamma_{there}$.

[7] We use the fact that the form of the superpotential is not renormalized along the RG flow: the only renormalization is that of the overall coupling h (coming from the renormalization of the kinetic terms). Non-perturbative corrections to the superpotential are avoided if there is sufficient matter, so that $\sum_i T(r_i) \geq T(G)$.

(II.34) to

$$R_i(\lambda_G, \lambda_W) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{n(W)_i \lambda_W}{|r_i|}}. \tag{II.45}$$

Plugging $R_i(\lambda_G, \lambda_W)$ back into $a(R_i, \lambda_G, \lambda_W)$ yields the a-function

$$\begin{aligned} a(\lambda_G, \lambda_W) &= 2|G| - \lambda_G T(G) + \lambda_W (n(W) - 2) \\ &+ \frac{2}{9}\sum_i |r_i| \left(1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{n(W)_i \lambda_W}{|r_i|}\right)^{3/2}, \end{aligned} \tag{II.46}$$

with $n_W = \sum_i n(W)_i$ the degree of the superpotential. This a-function satisfies

$$\frac{\partial a}{\partial \lambda_G} = \widehat{\beta}_G, \qquad \text{and} \qquad \frac{\partial a}{\partial \lambda_W} = \widehat{\beta}_W, \tag{II.47}$$

proportional to the exact gauge and Yukawa beta functions, as defined in (II.7) and (II.8).

The conjecture is again that λ_W can be interpreted as the running superpotential Yukawa coupling h^2, in some appropriate scheme. Using (II.34) for the exact R-charges yields exact anomalous dimensions

$$\gamma_i = 3R_i - 2 = 1 - \sqrt{1 + \frac{\lambda_G T(r_i)}{|r_i|} - \frac{\lambda_W n(W)_i}{|r_i|}}. \tag{II.48}$$

We can again write this exact expression for the anomalous dimensions as

$$\gamma_i = 1 - \sqrt{1 - 2\gamma_i^{(1)}}, \tag{II.49}$$

with

$$\gamma_i^{(1)} = -\frac{\lambda_G T(r_i)}{2|r_i|} + \frac{n(W)_i \lambda_W}{2|r_i|}, \tag{II.50}$$

to be identified with the one-loop anomalous dimension. Comparing with explicit perturbative computations allows us to check this result, e.g. verifying the $1/|r_i|$ dependence in (II.48) and (II.50), and to find the leading relation between λ_W and h^2.

To fix the normalization, let's first compare (II.50) with perturbation theory for a single chiral superfield Q, with cubic superpotential $W = \frac{1}{6}hQ^3$ (so $n(W) = 3$ in (II.50)):

$$\gamma_Q^{(1)} = \frac{|h|^2}{16\pi^2} = \frac{3\lambda_W}{2} \qquad \text{hence} \qquad \lambda_W = \frac{|h|^2}{24\pi^2} + \mathcal{O}(h^4). \tag{II.51}$$

With many chiral superfields Q_i and superpotential $W = \frac{1}{6}h^{ijk}Q_iQ_jQ_k$, the one-loop anomalous dimension matrix is

$$\gamma^{(1)}{}^i_j = \frac{h^{ikl}h^*_{jkl}}{16\pi^2}. \tag{II.52}$$

Suppose that the matter fields form distinct irreps of a group, with $h^{ijk} = hT^{r_ir_jr_k}$, with $T^{r_ir_jr_k}$ an invariant tensor to contract the group indices of those irreps. Schur's lemma then ensures that the anomalous dimension matrix (II.52) is diagonal and proportional to the identity matrix for each irrep, and taking the trace fixes the coefficient to be

$$\gamma^{(1)}{}^i_j = \delta^i_j \frac{(h^{klm}h^*_{klm})}{16\pi^2|r_i|} \qquad \left(\text{with} \quad h^{klm}h^*_{klm} = |h|^2T^{r_ir_jr_k}T^*_{r_ir_jr_k} \equiv |h|^2|T|^2\right), \tag{II.53}$$

giving $\gamma^{(1)} \sim 1/|r_i|$, as predicted from (II.48). Comparing (II.48) and (II.53) yields,

$$\lambda_W = \frac{|h|^2|T|^2}{24\pi^2} + \text{higher loop (scheme dependent) corrections.} \tag{II.54}$$

As in the previous subsection, one can do higher-loop comparisons with the results of [25], where the anomalous dimensions were computed to three loops, including the contributions from Yukawa couplings. But there is significant scheme freedom in redefining the Yukawa couplings, including their tensor structure, so we will not here explicitly discuss the higher order dictionary (II.54) between λ_W and the Yukawa couplings in the scheme of [25].

II.B.6 An example: electric and magnetic SQCD

For $SU(N_c)$ SQCD, with N_f fundamental flavors Q_f, $\widetilde{Q}_{\tilde{f}}$, (II.34) gives [20]

$$R_Q(\lambda) = R_{\widetilde{Q}}(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c}}, \tag{II.55}$$

and thus the a-function along the flow is [20]

$$a(\lambda) = 2(N_c^2 - 1) - \lambda_G N_c + \frac{4}{9}N_cN_f\left(1 + \frac{\lambda_G}{2N_c}\right)^{3/2}. \tag{II.56}$$

The asymptotically free UV theory corresponds to $\lambda = 0$, and the RG flow to the IR corresponds to $\lambda : 0 \to \lambda^*$, where

$$\frac{\lambda_{G*}}{2N_c} = \left(\frac{3N_c}{N_f}\right)^2 - 1 \tag{II.57}$$

is where the R-charges (II.55) are anomaly free, and hence (II.56) is critical and $\beta_{NSVZ} = 0$.

The magnetic dual [26] is $\widetilde{G} = SU(\widetilde{N}_c) \equiv SU(N_f - N_c)$ SCQD, with N_f dual quarks q^f, $\widetilde{q}^f$, and N_f^2 added singlets $M_{f\tilde{g}}$, with superpotential

$$W = hM_{f\tilde{g}} q_f \widetilde{q}^{\tilde{g}}. \tag{II.58}$$

The quantity to maximize for the RG flow of the dual theory is

$$\begin{aligned} a \;\; &= 2(\widetilde{N}_c^2 - 1) + 2\widetilde{N}_c N_f \left(3(R_q - 1)^3 - (R_q - 1)\right) + N_f^2 \left(3(R_M - 1)^3 - (R_M - 1)\right) \\ &\quad -\lambda_{\widetilde{G}} \left(\widetilde{N}_c + N_f(R_q - 1)\right) + \lambda_h \left(2R_q + R_M - 2\right). \end{aligned} \tag{II.59}$$

Extremizing in R_q and R_M, holding $\lambda_{\widetilde{G}}$ and λ_h fixed yields

$$R(q) = R(\widetilde{q}) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_{\widetilde{G}}}{2\widetilde{N}_c} - \frac{\lambda_h}{\widetilde{N}_c N_f}}, \qquad R(M) = 1 - \epsilon\frac{1}{3}\sqrt{1 - \frac{\lambda_h}{N_f^2}}. \tag{II.60}$$

Increasing $\lambda_{\widetilde{G}}$, and hence the magnetic gauge group coupling $\widetilde{g}^2$, lowers $R(q)$, whereas increasing λ_h increases $R(q)$ and $R(M)$. Plugging back into (II.59) yields a-function

$$a(\lambda_{\widetilde{G}}, \lambda_M) = 2(\widetilde{N}_c^2 - 1) - \lambda_{\widetilde{G}}\widetilde{N}_c + \frac{4}{9}\widetilde{N}_c N_f \left(1 + \frac{\lambda_{\widetilde{G}}}{2\widetilde{N}_c}\right)^{3/2} + \epsilon\frac{2}{9}N_f^2 \left(1 - \frac{\lambda_h}{N_f^2}\right)^{3/2}, \tag{II.61}$$

whose λ gradients give $\widehat{\beta}_{\widetilde{G}}$ and $\widehat{\beta}_W$.

The $\epsilon = \pm$ in (II.60) corresponds to the choice of branch sign in the square root, and is a main point of this subsection. Taking $N_f > \frac{3}{2}N_c$, the magnetic theory is asymptotically free, and the UV limit has the free-field R-charges $R(q) \to 2/3$ and $R(M) \to 2/3$, and hence $\lambda_{\widetilde{G}} \to 0$ and $\lambda_h \to 0$, with $\epsilon = +1$ in (II.60). As the magnetic theory RG flows to the IR, λ_h increases, and hence $R(M)$ moves to

$R(M) > 2/3$ (unitarity requires $R(M) \geq 2/3$, with equality iff it's a free field). In fig. II.1, $R(M)$ flows from point (A) towards point (B). If the IR fixed point is sufficiently strong coupling, $R(M)$ can increase past $R(M) = 1$, in which case λ_h must first increase to N_f^2 on the $\epsilon = +1$ branch of (II.60), and then we must switch to the $\epsilon = -1$ branch, after which λ_h must decrease as we flow farther in the IR.

As an extreme example, for $N_f \approx 3N_c$ (just below) the electric theory is barely asymptotically free and hence weakly coupled in the IR, whereas the magnetic dual is very strongly coupled in the IR. At the RG fixed point, we know from the electric side that $R_{IR}(Q) \approx 2/3$, and thus $R_{IR}(M) \approx 4/3$, i.e. $R(M)$ in the magnetic theory flows from $R_{UV}(M) = 2/3$ to $R_{IR}(M) \approx 4/3$. Using (II.60), the flow starts in the UV with $\epsilon = +1$ and λ_h increasing from zero to its maximal value $\lambda_h = N_f^2$, after which the continued flow to the IR is on the $\epsilon = -1$ branch, with λ_h decreasing, with $\lambda_h \to 0$ at the IR fixed point. Though $\lambda_h \approx 0$ at the IR fixed point, the magnetic dual is certainly strongly coupled, and we expect that h_*^2 isn't small. As we'll discuss in the next section, in order to have positive definite metric G_{IJ} and monotonically decreasing a-function, we expect that the jacobian $\frac{\partial \lambda_K}{\partial g^I}$ should be positive (positive eigenvalues); assuming the off-diagonal terms to be negligible, this requires $d\lambda_h/dh^2 > 0$, suggesting the "shark fin" shape of fig. II.2.

The slope of the beta function at a RG fixed point, $\beta'(\alpha_*)$, is a scheme independent quantity, which gives the anomalous dimension of the leading irrelevant operator along which we flow into a RG fixed point (i.e. $F_{\mu\nu}F^{\mu\nu}$ for gauge interactions). For SUSY gauge theories, $\beta'(\alpha_*)$ was argued to be related to the anomalous dimension of the Konishi current at the RG fixed point [30]. Using a claimed map of this current to that of the magnetic dual it was argued that $\beta'(g_*^2)_{elec} = \beta'_{min}(g_*^2, h_*^2)_{mag}$ [30]. For $N_f/N_c = \frac{3}{2} + \delta$, with $\delta \ll 1$, the magnetic RG fixed point is weakly coupled and $\beta'_{min}(g_*^2, h_*^2)_{mag}$ can be perturbatively computed; doing so, the claim of [30] leads to a prediction for $\beta'(\alpha_*)$ in the corresponding, strongly coupled electric theory [30], $\beta'(\alpha_*) = (28/3)\delta^2$. We do

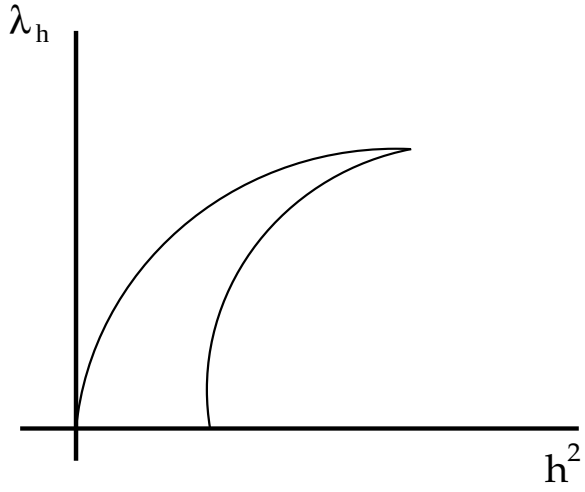

Figure II.2: Hypothetical plot of $\lambda_h(h^2)$, with $\epsilon = +1$ on the top part and $\epsilon = -1$ on the bottom.

not, however, find this qualitative behavior, of having $\beta'(\alpha_*) \to 0$ as $\delta \to 0$, in $(d\widehat{\beta}/d\lambda)_{\lambda_*} = (N_f/6N_c)^2$, as computed using (II.56) and (II.57). The factor from $\beta_{NSVZ}/\widehat{\beta}$ in (II.7) doesn't help (if anything, it's large in this limit); the only apparent way to get $\beta' \to 0$ would be if $(d\lambda/d\alpha)|_{\alpha_*} \to 0$ as $\delta \to 0$. We do not know whether or not this is the case.

II.C RG flow = gradient flow: evidence for the strongest version of the a-theorem

Writing the general a-function again as $a(\lambda) = a(R(\lambda), \lambda)$ with

$$a(R, \lambda_I) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R + \sum_I \lambda_I \widehat{\beta}^I(R), \tag{II.62}$$

and $R(\lambda)$ obtained by extremizing in R, the λ_K gradients of this function give

$$\frac{\partial a(\lambda)}{\partial \lambda_K} = \widehat{\beta}^K(R(\lambda)). \tag{II.63}$$

The $\widehat{\beta}^K(R)$ are are proportional to the exact beta functions, which we'll write as

$$\widehat{\beta}^K(R) = f^K_J(g)\beta^J(g). \tag{II.64}$$

Thus (II.63) demonstrates that the exact RG flow is indeed gradient flow! Writing the λ_I as functions of the couplings g^J in a general scheme, we have

$$\frac{\partial a}{\partial g^I} = \frac{\partial a}{\partial \lambda_K}\frac{\partial \lambda_K}{\partial g^I} = f_J^K(g)\frac{\partial \lambda_K}{\partial g^I}\beta^J(g) \equiv G_{IJ}(g)\beta^J(g). \tag{II.65}$$

This gives the beta-functions as gradients of the a-function, as in (II.2), with metric for the space of g^I coupling constants

$$G_{IJ}(g) = f_J^K(g)\frac{\partial \lambda_K}{\partial g^I}. \tag{II.66}$$

A sufficient condition for $G_{IJ}(g) > 0$ and the strongest version of the a-theorem is $f_J^K(g) > 0$ (e.g. we don't continue past the apparent pole associated with the denominator of β_{NSVZ}) and the coupling constant reparametrization $\lambda_K(g)$ is monotonic, $\frac{\partial \lambda_K}{\partial g^I} > 0$.

Using (II.66) and (II.7), the exact metric for gauge couplings is (this case appears already in [20])

$$G_{gg} = \frac{\widehat{\beta}}{\beta}\frac{d\lambda_G}{dg} = \frac{16\pi^2}{3g^3}\left(1 - \frac{g^2 T(G)}{8\pi^2}\right)\frac{d\lambda_G}{dg}, \tag{II.67}$$

with $\lambda_G(g)$ that for the NSVZ g scheme. As long as $g^2T(G) < 8\pi^2$ and $\lambda_G(g)$ is monotonic, (II.67) satisfies $G_{gg} > 0$. Using (II.39) and (II.44), for weak coupling we approximate:

$$\begin{aligned} G_{gg} &= \frac{16\pi^2}{3g^3}\left(1 - \frac{g^2T(G)}{8\pi^2}\right)\left(\frac{g|G|}{\pi^2} + \frac{|G|g^3 b_1}{8\pi^4} + \dots\right) \\ &\approx \frac{16|G|}{3g^2}\left(1 + \frac{g^2}{8\pi^2}(b_1 - T(G))\right). \end{aligned} \tag{II.68}$$

Likewise, for Yukawa couplings, using (II.66) and (II.8), the exact metric is

$$G_{hh} = \frac{\widehat{\beta}}{\beta}\frac{d\lambda_h}{dh} = \frac{4}{3}\frac{d\lambda_h}{d(h^2)}, \tag{II.69}$$

which satisfies $G_{hh} > 0$ as long as $\lambda_h(h)$ is monotonic. Using (II.54), we can approximate for weak coupling

$$G_{hh} = \frac{4}{3}\frac{d\lambda_h}{d(h^2)} \approx \frac{4}{3}\left(\frac{1}{24\pi^2} + \mathcal{O}(h^2)\right). \tag{II.70}$$

Consider e.g. the magnetic dual of SQCD, with gauge group $SU(\widetilde{N}_c)$, with gauge coupling $\widetilde{g}$, and superpotential (II.58), with Yukawa coupling h. The a-function (II.59) gives the beta functions as gradient flow:

$$\begin{pmatrix} \frac{\partial a}{\partial \widetilde{g}} \\ \frac{\partial a}{\partial h} \end{pmatrix} = \frac{4}{3} \begin{pmatrix} \frac{\partial \lambda_{\tilde{G}}}{\partial \widetilde{g}} & \frac{\partial \lambda_h}{\partial \widetilde{g}} \\ \frac{\partial \lambda_{\tilde{G}}}{\partial h} & \frac{\partial \lambda_h}{\partial h} \end{pmatrix} \begin{pmatrix} 4\pi^2 g^{-3}(1 - \frac{\widetilde{g}^2 T(\widetilde{G})}{8\pi^2}) & 0 \\ 0 & (2h)^{-1} \end{pmatrix} \begin{pmatrix} \beta_{NSVZ}(\widetilde{g}) \\ \beta_W(h) \end{pmatrix}. \quad \text{(II.71)}$$

A sufficient condition for positive metric in (II.71) is positivity of the jacobian $\frac{d\lambda_K}{dg^I}$ and $\widetilde{g}^2 T(G) < 8\pi^2$. Assuming that the off-diagonal components of the metric aren't appreciable (they're zero in perturbation theory), positivity of the jacobian requires $d\lambda_h/dh^2 > 0$, which motivated the shark-fin shape of fig. II.2, for the case of $N_f \approx 3N_c$.

As we discussed in the introduction, we can compare metrics G_{IJ}, as computed above, with those computed by Osborn and collaborators in the context of 4d field theories on curved spacetime, with spatially dependent couplings. The supersymmetric case was considered by Freedman and Osborn in [11]. To compare expressions, we need to account for our rescalings mentioned in footnote 3, $a_{here}(g) = (32/3)\widetilde{a}_{there}(g)$, and $G^{here}_{IJ}(g) = \frac{4}{3} G^{there}_{IJ}(g)$. We then find that the leading, scheme independent, term in both the metric G_{gg} (II.68), and also the Yukawa coupling metric (II.70), agree precisely with those found by Freedman and Osborn [11]! (The coefficient of the subleading, scheme dependent term in (II.68), however, does not agree with that obtained in [11]: rather than $b_1 - T(G)$ of (II.68), the coefficient obtained in [11] was $\frac{5}{2} b_1 - T(G)$. The apparent difference, $\sim b_1$, could be completed at higher orders into a difference $\sim \beta(g)$, which would at least vanish at the endpoints of the RG flow. More work is needed to verify if this is a real difference in the metric and a-function, or perhaps associated with a scheme discrepancy.)

The method of Osborn was to consider renormalization for spatially dependent coupling constants, e.g. with G_{IJ} coming from beta functions $\beta_{\mu\nu} \sim G_{IJ}(g) \partial_\mu g^I \partial_\nu g^J$. This is very reminiscent of the AdS/CFT correspondence, where coupling constants correspond to fields in the bulk, with G_{IJ} naturally associated

with the sigma model metric G_{IJ}^{bulk} of these bulk fields. Indeed, in [31] it was argued that the AdS holographic RG flow leads to $\dot{c} = -G_{IJ}\beta^I\beta^J$, with metric $G_{IJ} = 2cG_{IJ}^{bulk}$. This again suggests that RG flow is gradient flow, with positive definite metric, though it's important to emphasize that the AdS/CFT correspondence seems limited to a very restricted subset of all possible CFTs. In any case, $G_{IJ} = 2cG_{IJ}^{bulk}$ gives a nice insight into the result for the leading perturbative metric, $G_{gg} \sim |G|/g^2$ (II.68): it matches with the ($SL(2,Z)$ invariant) dilaton kinetic terms in the bulk: $\mathcal{L}_{bulk} = -\frac{1}{2}(\tau_2)^{-2}\partial_\mu\tau\partial^\mu\overline{\tau}$ (here $\tau = \frac{\theta}{2\pi} + 4\pi i g^{-2}$, so $\frac{1}{4}(d(\log\tau_2))^2 = (d(\log g))^2$).

II.D a-maximization along RG flows with accidental symmetries, and comments about Higgsing

The Lagrange multiplier method needs to be extended in order to apply to RG flows with accidental symmetries, or those associated with Higgsing [20]. In this section, we'll discuss an extension of the proposal of [20] for the case of accidental symmetries associated with gauge invariant operators hitting the unitarity bound and becoming free. This extension defines a monotonically decreasing a-function along such RG flows. This shows, in particular, that a-maximization indeed ensures that $a_{UV} > a_{IR}$ is automatically satisfied for such RG flows. We'll next discuss Higgsing RG flows, where we do not yet have a good candidate a-function, or general argument for $a_{UV} > a_{IR}$.

II.D.1 Accidental symmetries

Accidental symmetries, present in the IR but not in the UV, challenge the a-theorem conjecture. Additional symmetries broaden the landscape over which we're maximizing a_{trial}, increasing the value of a_{IR}. To avoid violating $a_{IR} < a_{UV}$ thus requires that the IR theory must not have too much accidental symmetry; at present, however, we do not know of a general way to prove that the possible

accidental symmetries are always sufficiently bounded so as to be compatible with $a_{IR} < a_{UV}$. Here we will limit our discussion to a particular type of accidental symmetry, that of a gauge invariant operator hitting its unitarity bound and becoming free (without additional free fields, such as free magnetic quarks and gluons, whose existence would have been hard to predict from the spectrum of gauge invariant operators of the UV theory).

Near the UV start of the RG flow, we'll use for the a-function, following [20],

$$a^{(0)}(R, \lambda_I) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R + \sum_I \lambda_I \widehat{\beta}_I(R). \tag{II.72}$$

Extremizing this in the R_i has solution $R_i^{(0)}(\lambda_I)$, and plugging back in gives a-function $a^{(0)}(\lambda_I) = a^{(0)}(R_i(\lambda_I), \lambda_I)$. We propose that these $R^{(0)}(\lambda_I)$ and $a^{(0)}(\lambda_I)$ give the R-charges and the a-function initially along the RG flow, up until the point where the accidental symmetry arises: until the flow hits a value of the Lagrange multiplier/coupling constants $\lambda_I^{(0)}$ where a gauge invariant composite operator M hits $R(M) = 2/3$. At that point on the RG flow, including the effect of the accidental $U(1)_M$ means patching onto another a-function, with the correction term of [16] added to (II.72):

$$a^{(1)}(R_i, \lambda_I) = a^{(0)}(R, \lambda_I) + dim(M)\left(\frac{2}{9} - 3(R_M - 1)^3 + R_M - 1\right), \tag{II.73}$$

with $R_M = \sum_i R_i m_I$ for $M = \prod_i Q_i^{m_i}$. Now (II.73) is extremized to find $R_i^{(1)}(\lambda_I)$, and plugging these back into (II.73) gives a-function $a^{(1)}(\lambda_I) = a^{(1)}(R_i^{(1)}(\lambda_I), \lambda_I)$. If other operators M' hit $R(M') = 2/3$ further down the RG flow, we'd similarly patch onto the a-function $a^{(2)}$ obtained by adding the analogous correction term to (II.73).

So the running R-charges $R_i(\lambda_I)$ and a-function $a(\lambda_I)$ along the entire RG flow are proposed to be given by this patching procedure, with the patches occurring at every place along the RG flow where some gauge invariant operator hits the unitarity bound. The important point is that, despite the patching together, the $R_i(\lambda_I)$ and $a(\lambda_I)$ thus obtained are continuous along the entire RG flow, as

presumably are $\dot{R}_i(\lambda_I)$ and $\dot{a}(\lambda_I)$, because the added term in (II.73) vanishes at the patching location, where $R_M = 2/3$, as does its first derivative w.r.t. R_M. Moreover, the patched-together a-function still satisfies

$$\frac{\partial a(\lambda_I)}{\partial \lambda_I} = \widehat{\beta}_I(R),$$

with $\widehat{\beta}_I(R)$ the same linear combinations of the (patched together) R-charge R_i, proportional to the exact beta functions, as in (II.7) and (II.8). Thus the patched-together a-function continues to satisfy $\dot{a}(\lambda_I) < 0$. In particular, for the endpoints of the RG flow, this demonstrates that a-maximization automatically ensures that the accidental symmetries of the above type never violate $a_{IR} < a_{UV}$.

Here is a suggestive way to obtain this same patching-together prescription. Consider coupling the N_f^2 composite, gauge invariant meson operators $Q_f \widetilde{Q}_{\tilde{g}}$ to the same number of added sources, $L^{f\tilde{g}}$, and also introduce into the theory the same number of added gauge invariant fields $M_{f\tilde{g}}$, with added superpotental

$$W = L^{f\tilde{g}} Q_f \widetilde{Q}_{\tilde{g}} + h L^{f\tilde{g}} M_{f\tilde{g}}. \tag{II.74}$$

We think of the second term, with coupling h, as a perturbation. Starting at $h = 0$, we have $R(M) = 2/3$ and $R(L) = 2 - R(Q\widetilde{Q})$, so the h perturbation is relevant if $R(Q\widetilde{Q}) > 2/3$. In this case, the effect of the two terms in (II.74) is that L and M are both massive, and hence should be integrated out. The L e.o.m. sets $M_{f\tilde{g}} = Q_f \widetilde{Q}_{\tilde{g}}$, the M e.o.m. sets $L^{f\tilde{g}} = 0$, and the upshot is that we're back to were we would have been had we not included the $2N_f^2$ additional fields $L^{f\tilde{g}}$ and $M_{f\tilde{g}}$. In particular, these massive fields make cancelling contributions to 't Hooft anomalies and hence to the a-function $a = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$.

On the other hand, if $R(Q\widetilde{Q}) < 2/3$, the second term in (II.74) is irrelevant, and the N_f^2 fields $M_{f\tilde{g}}$ are then decoupled free fields, with $R(M) = 2/3$. This gives the 2/9 term in (II.73), and the remaining additional terms in (II.73) are the contribution of the fields $L^{f\tilde{g}}$ (whose R-charge is fixed by the first term in (II.74) to be $R(L) = 2 - R(Q\widetilde{Q})$). The a-function computed with these added fields and superpotential interactions involves additional Lagrange multipliers, associated with

the added superpotential terms, but should be equivalent to the patched-together prescription described above.

II.D.2 Higgsing

Giving a chiral superfield an expectation value breaks the gauge group $G \to H$. There is then a Higgsing RG flow, from the unbroken G theory in the UV (as the vev's then negligible), to the H theory in the IR, with the massive G/H fields decoupled. We do not have a candidate a-function, or a general argument that $a_{IR} < a_{UV}$, for Higgsing RG flows. We'll simply illustrate the challenge here, taking $W_{tree} = 0$ for simplicity.

When $G \to H$, the G matter fields Q_i decompose into H representations as $Q_i \to \sum_\mu Q_{i\mu}$, some of which are eaten. As with other RG flows, we can compute $\Delta a \equiv a_{IR} - a_{UV}$ from the IR vs UV R-charges of the chiral superfields, with the gauge field contribution unchanged and canceling in Δa. The fact that the low energy group does change, from G to H, is accounted for by the contribution to Δa of the $|G| - |H|$ matter fields eaten by the Higgs mechanism. At the IR fixed point, these eaten matter fields will have $R_{IR}(Q_{eaten}) = 0$, as seen by the fact that their fermionic components pair up to get a mass with the G/H gauginos; their contribution to Δa then correctly accounts for $G \to H$. We'll write the total Δa as $\Delta a = \Delta a_{eaten} + \Delta a_{uneaten}$. The a-theorem conjecture predicts $\Delta a < 0$. The eaten contribution satisfies $\Delta a_{eaten} < 0$ if $R_{UV}(Q_{eaten}) > 0$, e.g. at point (C) in fig. II.3, which is the case for RG fixed points with $W_{tree} = 0$ and sufficient matter to avoid generating W_{dyn}. (Theories with W_{tree} can have matter with negative R-charge, as seen e.g. in [16] for the theory with $W_{tree} = \mathrm{Tr}X^{k+1}$.)

Very generally, however, $\Delta a_{uneaten} > 0$, because Higgsing leads to an IR theory that is less asymptotically free than the UV theory. The uneaten matter fields move up the hill of fig. II.3 (which is a blown-up portion of fig. II.1), from point (C) in the UV, to a larger value in the IR. Those that are H-charged move partially up the hill, and those that are H-singlets are IR free, and hence move all

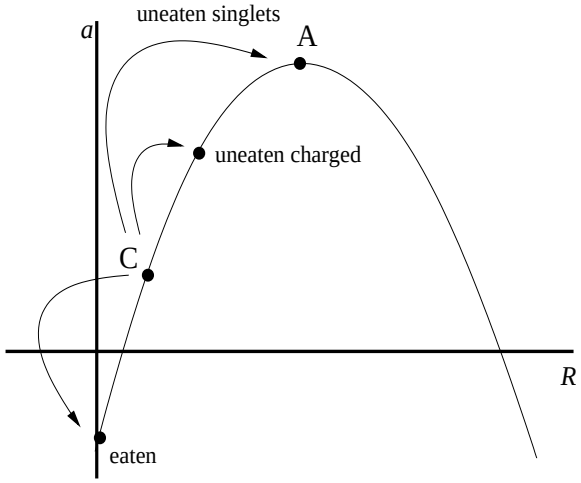

Figure II.3: Eaten and uneaten matter fields contribute oppositely to Δa.

the way up to point (A) in the IR. The a-theorem prediction that $\Delta a < 0$ thus requires that Δa_{eaten} be sufficiently negative, to compensate for $\Delta a_{uneaten} > 0$.

To illustrate all this, consider $SU(N_c)$ SQCD with N_f flavors in the superconformal window range $\frac{3}{2}N_c < N_f < 3N_c$. As reviewed in sect. II.B, this theory has

$$a_{SCFT} = a_{SQCD}(N_c, N_f) \equiv 2(N_c^2 - 1) + 2N_c N_f \left(\frac{N_c}{N_f} - 3\frac{N_c^3}{N_f^3} \right). \qquad \text{(II.75)}$$

Giving an expectation value to one of the flavors yields a $SU(N_c) \to SU(N_c - 1)$ Higgsing RG flow, with $N_f \to N_f - 1$, and a-theorem prediction

$$a_{SQCD}(N_c, N_f) > a_{SQCD}(N_c - 1, N_f - 1) + \frac{2}{9}(2N_f - 1), \qquad \text{(II.76)}$$

with the last term from the $2N_f - 1$ uneaten singlets (decomposing $(\mathbf{N_c}) \to (\mathbf{N_c} - \mathbf{1}) + (\mathbf{1})$). This inequality can be thought of as a statement about the contributions of the $2N_cN_f$ matter fields to $\Delta a \equiv a_{IR} - a_{UV}$. In the UV limit of the Higgsing flow, all of these fields start at point (C) in fig. II.3, with $R_{UV} = 1 - (N_c/N_f)$. In the IR limit, the $2(N_c - 1)(N_f - 1)$ uneaten charged matter fields move slightly up the hill of fig. II.3 (to $R_{IR} = 1 - (N_c - 1/N_f - 1)$), contributing to an increase in a. The $2N_f - 1$ uneaten singlets also contribute positively to Δa, moving up the hill in fig. II.3 from point (C) to point (A), with $R = 2/3$. Only the $|G| - |H| = 2N_c - 1$

eaten matter fields contribute to a decreased value of a_{IR}, moving down the hill of fig. II.3 from point (C) to $R_{IR}(Q_{eaten}) = 0$.

Since $\Delta a_{uneaten} > 0$, it's non-trivial to prove that the eaten matter field contribution is sufficient to ensure that $\Delta a < 0$. Indeed, (II.76) would be violated for N_f sufficiently small, if we didn't account for the effect of accidental symmetries for $N_f \leq \frac{3}{2}N_c$. Upon taking into account these accidental symmetries, $\Delta a < 0$ is satisfied. Proving that Higgsing RG flows always satisfy $\Delta a < 0$ thus generally requires accounting for accidental symmetries. Perhaps it's possible to prove that $a_{IR} < a_{UV}$ is satisfied whenever the unitarity bound condition is satisfied by all gauge invariant operators, with accidental symmetries giving $R = 2/3$ for any gauge invariant operators appearing to violate the unitarity bound, but we have not found an effective way to implement this.

An attempt to generalize the proposal of [20] for defining a flowing a-function for Higgsing RG flows would be to introduce several Lagrange multipliers, to interpolate along each of the three flows depicted in fig. II.3, λ_e for the eaten matter fields, $\lambda_{u.c.}$ for the uneaten charged matter, and $\lambda_{u.s.}$ for uneaten singlet matter fields. The Higgsing RG flow would then correspond to some path $\lambda_e(t)$, $\lambda_{u.c.}(t)$, $\lambda_{u.s.}(t)$, along which we'd like to find a monotonically decreasing a-function. Some clever choice of path would be required, since only the flow associated with λ_e has the needed sign of decreasing a.

II.E New SCFTs from SQCD with singlets: SSQCD

In this section, we illustrate some of the points discussed in the previous sections with a new set of examples. Consider $SU(N_c)$ SQCD with N_f fundamental flavors Q_i and $\widetilde{Q}_{\tilde{i}}$ (with $i = 1 \ldots N_f$), and N'_f additional flavors $Q'_{i'}$ and $\widetilde{Q}'_{\tilde{i}'}$ (with $i' = 1 \ldots N'_f$), with the N'_f flavors coupled to ${N'_f}^2$ singlets $S^{i'\tilde{j}'}$ by a superpotential term

$$W = hS^{i'\tilde{j}'}Q'_{i'}\tilde{Q}'_{\tilde{j}'}. \tag{II.77}$$

For $h = 0$, the theory is just SQCD, with $N_f + N_f'$ flavors, which flows to an interacting SCFT in the superconformal window $\frac{3}{2}N_c < N_f + N_f' < 3N_c$. The superpotential (II.77) is a relevant deformation of these SCFTs, $h : 0 \to h_* \neq 0$, driving a RG flow to a new family of SCFTs in the IR, labeled by (N_c, N_f, N_f'). The usual SQCD RG fixed points are the special case $N_f' = 0$ (electric description) or $N_f = 0$ (dual, magnetic description).

The $SU(N_f + N_f' - N_c)$ Seiberg dual [26] of the theory with $h = 0$ can be deformed by the superpotential (II.77), whose effect in the dual is simply a mass term that pairs up the $N_f'^2$ added singlets S with the $N_f'^2$ mesons M' (which $Q'\tilde{Q}'$ map to). The dual description of the new RG fixed points associated with (II.77) is thus simply a deformation of Seiberg duality, where we integrate out the massive gauge singlets S' and M'. What's left is an $SU(\widetilde{N}_c)$ gauge theory, with $\widetilde{N}_c \equiv N_f + N_f' - N_c$, with N_f flavors of dual quarks, q', and $\tilde{q}'$ (if $Q \in \mathbf{N_f}$ of $SU(N_f)_L$, then $q' \in \overline{\mathbf{N}}_\mathbf{f}$), and N_f' flavors q, and $\tilde{q}$ (if $Q' \in \mathbf{N_f'}$ of $SU(N_f')$, then $q \in \overline{\mathbf{N}}_\mathbf{f}'$), and N_f^2 gauge singlets $M_{i\tilde{j}}$, and $2N_fN_f'$ singlets $P_{i\tilde{j}'}$, and $P'_{\tilde{i}j'}$, with superpotential (suppressing flavor and color indices)

$$W = Mq'\tilde{q}' + Pq'\tilde{q} + P'\tilde{q}'q. \tag{II.78}$$

The first term in (II.78) is similar to the superpotential (II.77) of the electric theory, with an exchange $N_f \leftrightarrow N_f'$ in the number of flavors coupled to singlets. But the additional P and P' terms in (II.78) distinguish the magnetic duals from the original electric theory (II.77, so the duality does *not* simply equate the SCFT, obtained from the electric theory with (N_c, N_f, N_f'), to that obtained from the electric theory with $(N_f + N_f' - N_c, N_f', N_f)$. Duality equates these two SCFTs only for the special case of SQCD, $N_fN_f' = 0$; for $N_fN_f' \neq 0$, the electric (N_c, N_f, N_f') and $(N_f + N_f' - N_c, N_f', N_f)$ theories are distinct (each with their own, distinct, magnetic dual). The duality map for mesons, singlets, and baryonic operators is

$$Q\widetilde{Q} \to M, \quad S \to -q\tilde{q}, \quad Q\widetilde{Q}' \to P, \quad Q'\widetilde{Q} \to P', \qquad Q^r Q'^{N_c - r} \leftrightarrow q'^{N_f - r} q^{N_f' - N_c + r}, \tag{II.79}$$

(with r an arbitrary integer).

Both the electric and magnetic theories have an $SU(N_f)_L \times SU(N_f)_R \times SU(N'_f)_L \times SU(N'_f)_R \times U(1)_B \times U(1)_{B'} \times U(1)_F \times U(1)_{R_0}$ flavor symmetry. E.g. taking $h \neq 0$ in (II.77) breaks the axial $SU(N_f+N'_f)$ to $SU(N_f)\times SU(N'_f)\times U(1)_F$, so the $U(1)_F$ charges are $F(Q) = F(\widetilde{Q}) = N'_f/(N_f + N'_f)$ and $F(Q') = F(\widetilde{Q}') = -N_f/(N_f + N'_f)$. It is straightforward to list all of the flavor charges in the electric and magnetic duals, and to verify that they are compatible with the mappings (II.79), and also to verify that all of their 't Hooft anomalies match. All of these checks are guaranteed to work, because they worked for the original Seiberg duality [26], and the above new SCFTs and duality are obtained from those via a relevant deformation and its map to the dual description.

Despite the fact that these new SCFTs are such a simple deformation of those associated with SQCD, they could not have been quantitatively analyzed prior to the introduction [14] of the a-maximization method for determining the superconformal R-charges. The reason is that there are three independent R-charges, $R(Q) = R(\widetilde{Q}) \equiv y$, $R(Q') = R(\widetilde{Q}') \equiv y'$, and $R(S) \equiv z$, but only two constraints among them, anomaly freedom and the constraint that the superpotential (II.77) respect the R-symmetry:

$$N_c + N_f(R(Q)-1) + N'_f(R(Q')-1) = 0, \qquad \text{and} \qquad R(S) + 2R(Q') = 2. \quad \text{(II.80)}$$

This is because the R-symmetry can mix with the $U(1)_F$ flavor symmetry, whose effect is to allow $R(Q)$ and $R(Q')$ to differ. We'll first discuss a-maximization at the RG fixed points, imposing (II.80) at the outset, and then next a-maximization along the RG flow, with (II.80) imposed along the lines of [20], with Lagrange multipliers.

II.E.1 a-maximization at the RG fixed point

Before getting started, it's worth noting that the superconformal R-charges, obtained via a-maximization in the above electric and magnetic dual

theories, will be compatible with the duality maps (II.79, which require

$$2R_*(Q) = R_*(M), \qquad R_*(S) = 2R_*(q), \qquad R_*(Q) + R_*(Q') = R_*(P). \quad \text{(II.81)}$$

The two duals have the same flavor symmetries and 't Hooft anomalies, so we're maximizing the same function $a_{trial}(s)$ in both descriptions. The result is that the superconformal R-charges of the electric and magnetic theories are related by

$$R_*(q') = 1 - R_*(Q), \qquad R_*(q) = 1 - R_*(Q'), \quad \text{(II.82)}$$

which imply (II.81).

In the electric theory we have $R(Q) = R(\widetilde{Q}) \equiv y$, $R(Q') = R(\widetilde{Q}') \equiv y'$, and $R(S) \equiv z$, which are subject to the constraints (II.80) at the RG fixed point. We use these to eliminate y' and z in favor of y, and we then obtain y at the RG fixed point by maximizing $a_{trial} = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$, which we write as (taking N_c, N_f, and N_f' all large, to simplify the expressions, holding fixed $x \equiv N_c/N_f$ and $n \equiv N_f'/N_f$):

$$\frac{a}{2N_fN_f'}(x,n,y) = \frac{x}{n}\left[3(y-1)^3 - y + 1\right] + x\left[3\left(\frac{1-y-x}{n}\right)^3 - \frac{1-y-x}{n}\right]$$
$$+\frac{n}{2}\left[3\left(2\left(\frac{x+y-1}{n}\right)-1\right)^3 - \left(2\left(\frac{x+y-1}{n}\right)-1\right)\right] + \frac{x^2}{n}. \quad \text{(II.83)}$$

Maximizing this with respect to y determines the superconformal R-charge to be

$$y = \frac{1}{3x - 3n(4+nx)}\Big(-3(2n(2+n) + (n(n-4)-1)x + x^2)$$
$$+\sqrt{n^2(9x^2(x-2n)^2 + 8n(1-n^2)x + 4n^2)}\Big). \quad \text{(II.84)}$$

The result (II.84) is only valid over a range of x and n for which no gauge invariant operator violates the unitarity bound. The first operator to hit the unitarity bound is the meson $M = Q\widetilde{Q}$, which hits the unitarity bound when $R(Q) = 1/3$; solving (II.84) for the value $x_M(n)$ such that $y(x_M(n)) = 1/3$, the unitarity bound is hit at $x_M(n) = \frac{1}{3}(1 + 5n - \sqrt{1 - 14n + 13n^2})$. So (II.84) is

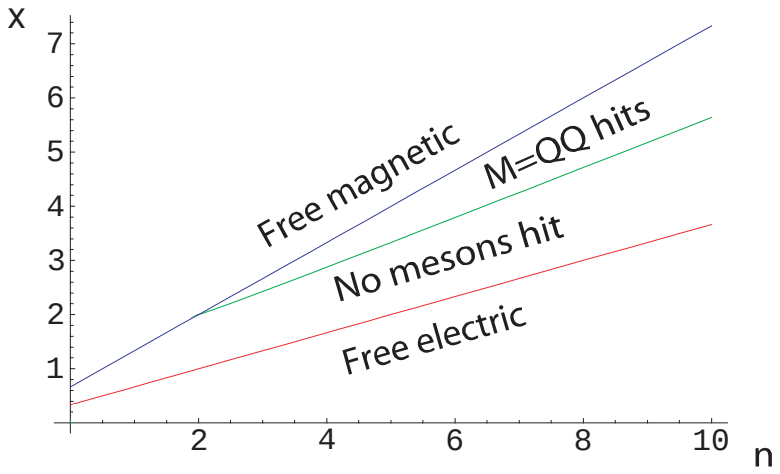

Figure II.4: Phases of SSQCD.

valid for $x < x_M(n)$, and needs correction to account for the accidental symmetry associated with the free-fields M when $x \geq x_M(n)$.

We also know that, when $N_f + N_f' \leq \frac{3}{2}N_c$, i.e. when $x \geq x_{FM}(n) \equiv \frac{2}{3}(1+n)$, the theory is in a free magnetic phase, with IR free quarks, $SU(N_f + N_f' - N_c)$ gluons, and singlets M, P, P'. The phases are as in fig. II.4: for $n = N_f'/N_f < 2$, (e.g. for the usual SQCD, where $n = 0$) the theory goes directly from having no accidental symmetries to free magnetic phase, where the entire magnetic theory is IR free. On the other hand, for $n \geq 2$, there is a wedge in the (x, n) parameter space where the field $Q\tilde{Q}$ hits its unitarity bound, while the dual is still asymptotically free. In this wedge, the IR theory remains an interacting SCFT, with only the field M becoming free and decoupled.

In the wedge $x_M < x < x_{FM}$, where $M = Q\tilde{Q}$ hits the unitarity bound, but the theory is not in the free magnetic phase, the effect of the accidental $U(1)_M$ symmetry is, as in [16], simply to replace the M field contributions with those of free fields: we instead maximize the quantity

$$a^{(1)} = a^{(0)} + \left(\frac{2}{9} - 3(2y-1)^3 + (2y-1)\right) N_f^2. \tag{II.85}$$

The maximizing solution for the superconformal R-charges, and the maximal value a for the central charge, are pasted-together with the solution (II.84) at $x = x_M(n)$.

Because the added quantity in (II.85) has a second order zero at $y = 2/3$, these pasted together quantities are continuous and smooth (first derivatives match) at $x = x_M(n)$.

The magnetic description of the decoupling of M in the wedge $x_M(n) < x < x_{FM}(n)$ is very simple, the term $Mq'\widetilde{q}'$ in the dual superpotential (II.78) is then irrelevant: when its coefficient is small, $R(Mq'\widetilde{q}') > 2$, because $R(M) \approx 2/3$ and $R(q') > 2/3$ for $x > x_M(n)$. In the IR, this irrelevant term goes away, and the dual superpotential becomes

$$W_{mag} = Pq'\widetilde{q} + P'\widetilde{q}'q. \tag{II.86}$$

When we now compute $\widetilde{a}_{trial}$ in the magnetic theory, with superpotential (II.86), we obtain the same result as on the electric side, reproducing the correction term in (II.85).

II.E.2 a-function, via a-maximization with Lagrange multipliers

For the electric theory, a-maximization along the RG flow, imposing (II.80) with Lagrange multipliers, yields

$$R(Q) = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c}}, \quad R(Q') = 1 - \frac{1}{3}\sqrt{1 + \frac{\lambda_G}{2N_c} - \frac{\lambda_S}{N_c N'_f}},$$

$$R(S) = 1 - \frac{1}{3}\epsilon\sqrt{1 - \frac{\lambda_S}{N_f'^2}}, \tag{II.87}$$

with both branches $\epsilon = \pm 1$ generally needed, as we discussed in sect. II.B.6. Plugging these back into $a(R_i, \lambda_I)$ yields $a(\lambda_G, \lambda_S)$,

$$a = \frac{4}{9}N_c N_f \left(1 + \frac{\lambda_G}{2N_c}\right)^{3/2} + \frac{4}{9}N_c N'_f \left(1 + \frac{\lambda_G}{2N_c} - \frac{\lambda_S}{N_c N'_f}\right)^{3/2} + \frac{2}{9}N_f'^2\epsilon\left(1 - \frac{\lambda_S}{N_f'^2}\right)^{3/2}$$
$$+2N_c^2 - \lambda_G N_c + \lambda_S. \tag{II.88}$$

It would be interesting to determine the RG flow path of the gauge coupling and superpotential coupling Lagrange multipliers, $\lambda_G(t)$ and $\lambda_S(t)$ to their eventual IR

values, where (II.88) is critical. It's gradient flow, as discussed in sect. II.C, but to actually determine the full trajectory requires knowing the full $\lambda_I(g)$.

Similarly, a-maximization along the RG flow, with Lagrange multipliers, in the magnetic dual yields

$$R(q) = 1 - \frac{1}{3}\sqrt{1 + \frac{\widetilde{\lambda}_G}{\widetilde{N}_c} - \frac{\widetilde{\lambda}_P}{2\widetilde{N}_c N'_f}}, \qquad R(M) = 1 - \frac{1}{3}\epsilon_M\sqrt{1 - \frac{\widetilde{\lambda}_M}{N_f^2}}$$
$$R(q') = 1 - \frac{1}{3}\sqrt{1 + \frac{\widetilde{\lambda}_G}{\widetilde{N}_c} - \frac{\widetilde{\lambda}_M}{\widetilde{N}_c N_f} - \frac{\widetilde{\lambda}_P}{2\widetilde{N}_c N_f}} \qquad R(P) = 1 - \frac{1}{3}\epsilon_P\sqrt{1 - \frac{\widetilde{\lambda}_P}{2N_f N'_f}}. \tag{II.89}$$

In the wedge $x_M(n) < x < x_{FM}(n)$, where M decouples but the theory is otherwise interacting, the RG fixed point has $\widetilde{\lambda}^*_M = 0$. This happens when $R(q') > 2/3$, hence $\widetilde{\lambda}_P/2N_f > \widetilde{\lambda}_G$ in (II.89).

II.E.3 Predictions and Checks of the a-theorem

Having obtained the superconformal R-charge R_* via a-maximization, as discussed above, we can compute $a(N_c, N_f, N'_f) = 3\mathrm{Tr}R_*^3 - \mathrm{Tr}R_*$ for our new SCFTs. There are many RG flows associated with these theories, and in this subsection we'll discuss and check some of the $a_{UV} > a_{IR}$ predictions.

First, there is the RG flow associated with superpotential (II.77). In the UV limit of this flow, $h \to 0$, and the theory is the SCFT associated with ordinary SQCD with $N_f + N'_f$ flavors plus the N'^2_f decoupled singlets, so $a_{UV} = a_{SQCD}(N_c, N_f + N'_f) + \frac{2}{9}N'^2_f$. The IR limit is our new SSQCD superconformal field theory, with $a_{IR} = a(N_c, N_f, N'_f)$, so $a_{UV} > a_{IR}$ means

$$2N_c^2 + 2N_c(N_f + N'_f)\left(3(-\frac{N_c}{N_f + N'_f})^3 - (-\frac{N_c}{N_f + N'_f})\right) + \frac{2}{9}N'^2_f > a(N_c, N_f, N'_f). \tag{II.90}$$

For simplicity, we again consider the limit of large N_c, N_f, and N'_f, holding fixed $x \equiv N_c/N_f$ and $n \equiv N'_f/N_f$. Defining $\widehat{a}(x,n) \equiv a(N_c, N_f, N'_f)/2N_f N'_f$, (II.90) becomes

$$\frac{x^2}{n} + x(1 + \frac{1}{n})\left(-3(\frac{x}{1+n})^3 + \frac{x}{1+n}\right) + \frac{n}{9} > \widehat{a}(x,n). \tag{II.91}$$

We have verified numerically that this prediction is indeed satisfied.

Another RG flow is to start at our SSQCD fixed point and deform by giving a Q flavor a mass. The IR theory is again SSQCD, but with $N_f \to N_f - 1$, and $a_{UV} > a_{IR}$ becomes

$$a(N_c, N_f, N_f') > a(N_c, N_f - 1, N_f'). \tag{II.92}$$

In the limit discussed above, this becomes

$$\widehat{a}(x, n) > (1 - \epsilon)\widehat{a}(x(1 + \epsilon), n(1 + \epsilon)), \tag{II.93}$$

with $\epsilon \equiv 1/N_f > 0$. The order ϵ term then gives

$$0 > \left(x\frac{\partial}{\partial x} + n\frac{\partial}{\partial n} - 1 \right) \widehat{a}(x, n), \tag{II.94}$$

which must hold for all x and n in the conformal window, $3x > 1 + n > \frac{3}{2}x$. In fig. II.5, we have plotted the function on the right hand side of (II.94). The plane at the top of the graph indicates both the conformal window as well as where the right hand side of (II.94) would equal 0, so $a_{IR} < a_{UV}$ is indeed always satisfied in the conformal window.

Now consider giving a mass to one of the q' flavors, which is equivalent to giving, say $S_{N_f' N_f'}$ a non-zero expectation value. This drives the theory in the IR to a similar RG fixed point, with $N_c \to N_c$, $N_f \to N_f$, and $N_f' \to N_f' - 1$. In addition, the IR fixed point has $2N_f' - 1$ decoupled free singlets, coming from the $S_{iN_f'}$. The a-theorem thus requires

$$a(N_c, N_f, N_f') > a(N_c, N_f, N_f' - 1) + \frac{2}{9}(2N_f' - 1). \tag{II.95}$$

As above, we divide both sides by $2N_f N_f'$ and take the term proportional to $\epsilon \equiv 1/N_f > 0$ to write this inequality as

$$\widehat{a} + \frac{\partial \widehat{a}}{\partial n} > \frac{2}{9}n. \tag{II.96}$$

Once again, we find numerically that (II.96) is satisfied.

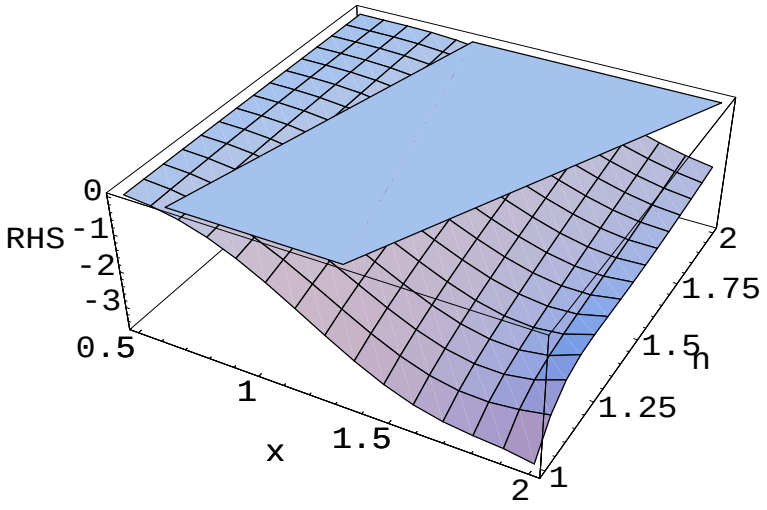

Figure II.5: Q mass RG flow, checking $a_{IR} < a_{UV}$, i.e. $0 > (x\frac{\partial}{\partial x} + n\frac{\partial}{\partial n} - 1)\widehat{a}$ in the conformal window.

Now consider giving $Q_{N_f}\widetilde{Q}_{N_f}$ a non-zero expectation value. This leads to

$$a(N_c, N_f, N_f') > a(N_c - 1, N_f - 1, N_f') + \frac{2}{9}(2N_f + 2N_f' - 1), \tag{II.97}$$

with the last term from the uneaten $SU(N_c - 1)$ singlets, which are IR free. We can write (II.97) as

$$\widehat{a}(x, n) > (1 - \epsilon)\widehat{a}((x - \epsilon)(1 + \epsilon), n(1 + \epsilon)) + \frac{2}{9}(1 + \frac{1}{n})\epsilon, \tag{II.98}$$

so, taking the ϵ term,

$$0 > -(1 + (1 - x)\frac{\partial}{\partial x} - n\frac{\partial}{\partial n})\widehat{a} + \frac{2}{9}(1 + \frac{1}{n}). \tag{II.99}$$

This inequality is shown in fig. II.6, where there appears to be a region where it's violated. But within the conformal window, the inequality is indeed satisfied. (Outside of the conformal window, additional contributions of free fields come to the rescue.)

There is a similar Higgsing RG flow upon giving $Q_f\widetilde{Q}'_{N_f}$ an expectation value (i.e. P in the dual), and $a_{UV} > a_{IR}$ is

$$a(N_c, N_f, N_f') > a(N_c - 1, N_f, N_f' - 1) + \frac{2}{9}(2N_f), \tag{II.100}$$

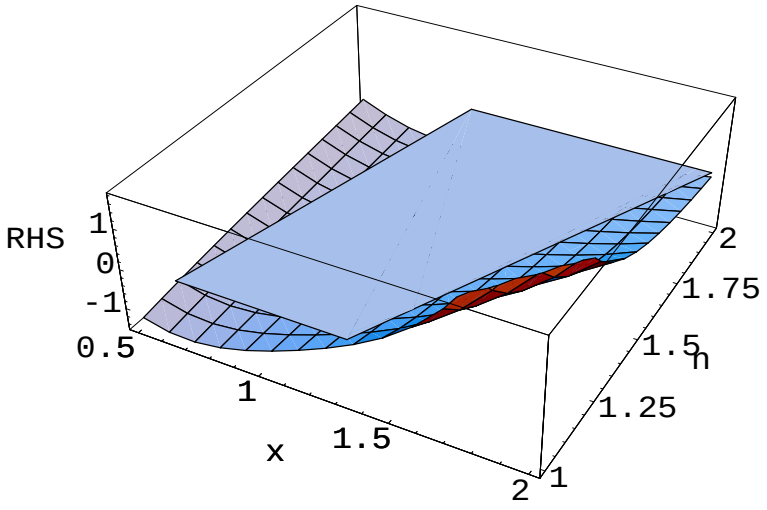

Figure II.6: Q vev Higgsing satisfies $a_{IR} < a_{UV}$ in the conformal window.

where there are fewer singlets than in (II.97) because some pair up with the $S_{i'N'_f}$ to get a mass. We write (II.100) as

$$\widehat{a}(x,n) > \left(1 - \frac{1}{n}\epsilon\right) \widehat{a}(x-\epsilon, n-\epsilon) + \frac{2}{9n}\epsilon, \tag{II.101}$$

and hence

$$(\frac{1}{n} + \frac{\partial}{\partial x} + \frac{\partial}{\partial n})\widehat{a} > \frac{2}{9n}. \tag{II.102}$$

Once again, we numerically verified that this inequality is true.

This chapter is a reprint of the material as it appears in E. Barnes, K. Intriligator, B. Wecht and J. Wright, "Evidence for the strongest version of the 4d a-theorem, via a-maximization along RG flows," Nucl. Phys. B **702**, 131 (2004), and on the preprint archive, http://www.arxiv.org/hep-th/0408156.

III

$\mathcal{N}$=1 RG flows, Product Groups, and a-Maximization

III.A Introduction

Asymptotically free gauge theories have various possible IR phases, one being the "non-Abelian Coulomb phase," which is an interacting conformal field theory RG fixed point, where all beta functions vanish. A classic example is $\mathcal{N} = 1$ $SU(N_c)$ SQCD with N_f massless flavors, which flows to a SCFT in the IR for N_f within the Seiberg superconformal window [26] $\frac{3}{2}N_c < N_f < 3N_c$. For $N_f \leq \frac{3}{2}N_c$, the theory is instead in a free magnetic $SU(N_c - N_f)$ phase in the IR. (See e.g. [32] for a review and references.) Our prejudice is that the interacting SCFT phase is rather generic for asymptotically free SUSY gauge theories with enough massless matter to avoid dynamical superpotentials, i.e. with massless matter representation R such that $T(G) < T(R) < 3T(G)$, with $T(R)$ the quadratic Casimir of R and $T(G)$ that of the adjoint. The theory at the origin is then either a non-trivial free field solution of 't Hooft anomaly matching (as in the free magnetic phase) or an interacting SCFT. Unfortunately, unless one has a conjectured dual description[1], there is no simple test for directly determining if the IR phase is a

[1]Even with a non-trivial free field solution of 't Hooft anomaly matching, e.g. as in the example of [42], there's the possibility that the matching is a fluke, and that the theory actually flows to an

SCFT or (fully or partially) IR free magnetic.

There is an essentially endless landscape of possible RG fixed point SCFTs to explore, coming from various gauge groups, including product groups, and matter representations. Here we'll consider examples with product gauge groups, e.g. the theory

$$\begin{array}{lccl} & \text{gauge group:} & SU(N_c)\times SU(N_c') & \\ \text{matter:} & X\bigoplus\widetilde{X} & (\square,\overline{\square})\bigoplus(\overline{\square},\square), & \\ & Q_f\bigoplus\widetilde{Q}_{\tilde{f}} & (\square,\mathbf{1})\bigoplus(\overline{\square},\mathbf{1}) & (f,\tilde{f}=1\ldots N_f), \\ & Q'_{f'}\bigoplus\widetilde{Q}'_{\tilde{f}'} & (\mathbf{1},\square)\bigoplus(\mathbf{1},\overline{\square}) & (f',\tilde{f}'=1\ldots N'_f). \end{array} \tag{III.1}$$

We'll be interested in when this theory flows to an (either fully or partially) interacting SCFT and when various dualities apply, e.g. dualizing one gauge group with the other treated as a spectator. We'll also be interested in the superconformal window for a duality proposed in [36], for the above theory deformed by superpotential $W_{A_{2k+1}}=\mathrm{Tr}(X\widetilde{X})^{k+1}$.

With multiple couplings, e.g. the two gauge couplings of (III.1), the RG running of one coupling can radically affect that of the other, possibly driving it into another basin of attraction. For example, as depicted in fig. III.1, there can be saddle point IR fixed points (A) and (B) when one or the other coupling is tuned to precisely zero, but which are unstable to any perturbation in the other coupling: the generic RG flows then end up at point (C) in the IR, with both g_* and g'_* nonzero. Another possibility, shown in fig. III.2, is that g' is interacting in the IR only if $g=0$, but that any arbitrarily small, nonzero, g would eventually overwhelm g', and drive g' to be IR irrelevant, $g'\to 0$ in the IR; generic RG flows then end up at point (A), with $g'_*=0$. Fig. III.3 depicts an opposite situation, where an otherwise IR free coupling g' is driven to be interacting in the IR by the coupling g. Fig. III.4 depicts two separately IR free couplings, which can cure each other and lead to an interacting RG fixed point (this happens for e.g. the gauge and Yukawa couplings of the $\mathcal{N}=4$ theory, when we break to $\mathcal{N}=1$ by taking them to be unequal).

interacting SCFT after all, as was argued to be the case for another example in [43].

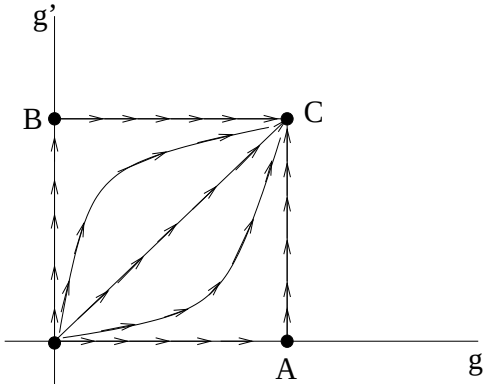

Figure III.1: A and B are saddle-points.

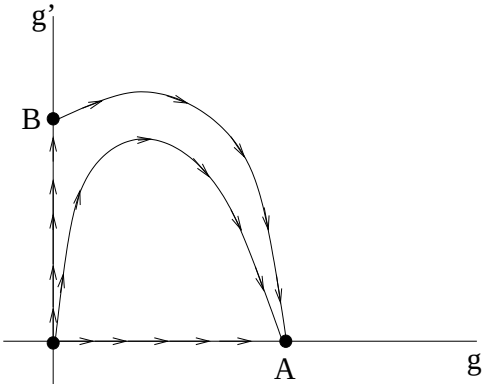

Figure III.2: The plop. B is a saddle-point.

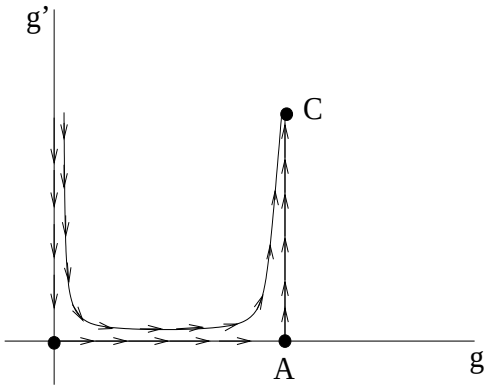

Figure III.3: The opposite of fig. III.2. g' is IR free for $g = 0$, but $g \neq 0$ drives g' IR interacting.

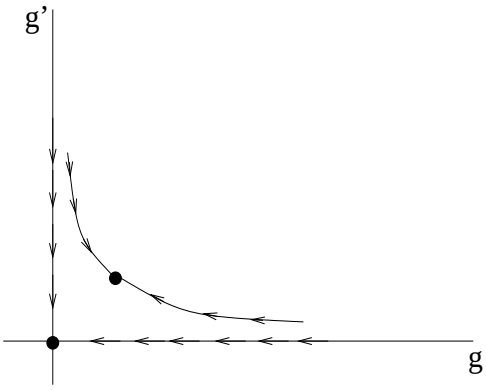

Figure III.4: Two separately irrelevant couplings combine to be interacting. $\mathcal{N} = 4$ SYM is such an example.

The theory (III.1) realizes the RG flows depicted in fig. III.1 or fig. III.2, depending on the values of the parameters (N_c, N_c', N_f, N_f'). With an additional superpotential, as is present if we dualize one of the factors in (III.1), the phenomenon of fig. III.3 is also realized.

We'll focus here on supersymmetric theories, such as (III.1) with $\mathcal{N} = 1$ supersymmetry, where some exact results can be obtained. However, we expect the phenomena of figs. III.1 and III.2 to occur even in non-supersymmetric $G \times G'$ gauge theories, with matter in mixed $G \times G'$ representations, at least when the matter content is chosen such that each group is just barely asymptotically free. There can then be RG fixed points in the perturbative regime, as can be seen by considering the beta functions to two loops:

$$\beta_\alpha = \frac{\alpha^2}{2\pi}(-b_1 + b_2\alpha + c_2\alpha') + O(\alpha^4), \quad \text{and} \quad \beta_{\alpha'} = \frac{\alpha'^2}{2\pi}(-b_1' + b_2'\alpha' + c_2'\alpha) + O(\alpha^4), \tag{III.2}$$

(writing $\alpha = g^2/4\pi$ and $\alpha' = g'^2/4\pi$, and $O(\alpha^4)$ refers to powers of either α or α'), where the c_2 and c_2' terms come from the matter in mixed representations; see e.g. [44]. Choosing the matter content to be such that the groups are barely asymptotically free, i.e. such that b_1 and b_1' are small positive numbers, it is then found that the two-loop coefficients (b_2, c_2, b_2', c_2') in (III.2) are positive and not especially small (e.g. in large N_c); this allows for RG fixed points to exist at relatively small values of the fixed point coupling, so that this argument for the RG fixed point's existence could be qualitatively reliable.

In particular, to two loops, we find zeros of the beta functions (III.2) at three points: point (A) at $(\alpha_*, \alpha_*')_A \approx (b_1/b_2, 0)$, point (B) at $(\alpha_*, \alpha_*')_B \approx (0, b_1'/b_2')$, and point (C), at

$$\begin{pmatrix} \alpha_* \\ \alpha'_* \end{pmatrix}_C \approx \frac{1}{b_2 b_2' - c_2 c_2'} \begin{pmatrix} b_2' & -c_2 \\ -c_2' & b_2 \end{pmatrix} \begin{pmatrix} b_1 \\ b_1' \end{pmatrix}. \tag{III.3}$$

For point (C) to actually exist, the values of α_* and α_*' in (III.3) must be positive. It is found that the determinant denominator in (III.3) is generally positive, so the

positivity condition for RG fixed point (C) to exist is thus

$$b_1 b_2' > b_1' c_2 \quad \text{and} \quad b_1' b_2 > b_1 c_2' \qquad \text{to have } g_* \neq 0 \text{ and } g_*' \neq 0. \tag{III.4}$$

These inequalities may or may not hold, depending on the choice of matter content. The intuition for these inequalities is that each gauge coupling makes the other less interacting in the IR (via the c_2 or c_2' terms), so there can only be a RG fixed point (C), with both interacting, if the couplings flow in balance: if either flows too much faster than the other, it can drive the other to be IR free. For example, if the matter content is such that $b_1 c_2' > b_1' b_2$, then $g' \to 0$ in the IR, as in fig. III.2, with the G dynamics overwhelming the G' dynamics in the IR. Likewise, if $b_1' c_2 > b_1 b_2'$, then G' wins, and drives $g \to 0$ in the IR. The inequality $b_2 b_2' > c_2 c_2'$ implies that both inequalities in (III.4) could not be reversed.

The criteria (III.4) for RG fixed point (C) to exist are equivalent to the condition that RG fixed points (A) and (B) be IR unstable to perturbations in the other coupling, as depicted in fig. III.5. For example, near (A), where $\alpha' = 0$ and $\alpha_* \approx b_1/b_2$, (III.2) gives $\beta_{\alpha'} = \alpha'^2(-b_1' + c_2' b_1/b_2)/2\pi + O(\alpha'^3)$. The second inequality in (III.4) is thus equivalent to having (A) be IR repulsive to α' perturbations, as in fig. III.1. If (A) and (B) are both IR repulsive to perturbations, generic couplings flow to having both interacting, and can end up at a fixed point (C), as in fig. III.1. If either inequality of (III.4) is violated, then either (A) or (B) is IR attractive, and then RG fixed point (C) does not exist (at least it does not exist within the basin of attraction of zero couplings). In that case, as depicted in fig. III.2, generic RG flows attract to the IR stable point (A) or (B). Because $b_2 b_2' > c_2 c_2'$, both inequalities in (III.4) could not be reversed, i.e. we can not have (A) and (B) both be IR attractive. As depicted in fig. III.6, such a hypothetical flow would have required an unstable separatrix ridge, depicted as a dashed line, dividing the RG flows into two different domains of attraction. In neither the perturbative analysis, nor the supersymmetric examples to follow, do we find examples of such flows.

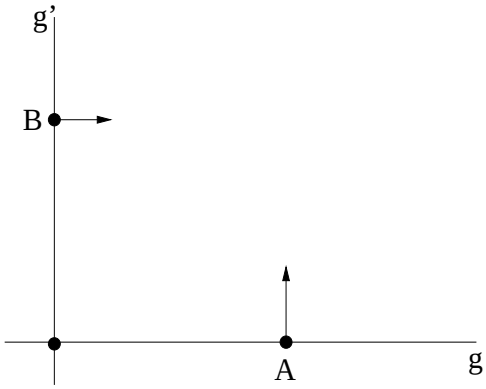

Figure III.5: If A and B are both IR unstable to perturbations, the theory flows to C, with both couplings interacting.

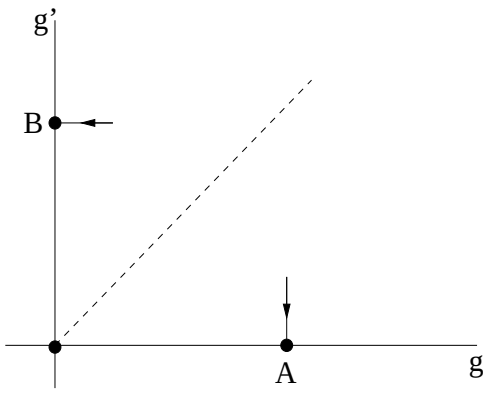

Figure III.6: We don't find examples of A and B both IR stable to perturbations. Would've required a separatrix between domains of attraction.

We can go beyond the above perturbative analysis in $\mathcal{N} = 1$ supersymmetric theories, where exact results can be obtained via the NSVZ [12] beta functions. For a general $\mathcal{N} = 1$ $G \times G'$ gauge theory, with matter chiral superfield in representations $\oplus_i(\mathbf{r_i}, \mathbf{r'_i})$, these are

$$\beta_g(g,g') = -\frac{g^3 f}{16\pi^2}\left(3T(G) - \sum_i T(r_i)|r'_i|(1-\gamma_i(g,g'))\right) = -\frac{3g^3 f}{16\pi^2}\text{Tr } GGR$$

$$\beta_{g'}(g,g') = -\frac{g'^3 f'}{16\pi^2}\left(3T(G') - \sum_i T(r'_i)|r_i|(1-\gamma_i(g,g'))\right) = -\frac{3g'^3 f'}{16\pi^2}\text{Tr } G'G'R. \tag{III.5}$$

In the NSVZ scheme, $f = (1 - \frac{g^2 T(G)}{8\pi^2})^{-1}$ and $f' = (1 - \frac{g'^2 T(G')}{8\pi^2})^{-1}$, while in other schemes these factors are replaced with other functions of the coupling [27], such that $f = 1 + O(g^2)$; these scheme-dependent prefactors are unimportant for our discussion, except for the fact that they should be strictly positive in our range of coupling constants.

The last equality in each line of (III.5) involves Tr GGR, which is the coefficient of the $U(1)_R$ ABJ triangle anomaly, with two external G gluons. This uses the fact that supersymmetry relates the dilatation current to a $U(1)_R$ current,

with the exact scaling dimensions of chiral fields related to their $U(1)_R$ charges:

$$\Delta = \frac{3}{2}R \quad \text{e.g.} \quad \Delta(Q_i) \equiv 1 + \frac{1}{2}\gamma_i = \frac{3}{2}R(Q_i). \tag{III.6}$$

When the theory is conformally invariant, this $U(1)_R$ is conserved, and part of the superconformal group $SU(2,2|1)$. When the theory is not conformally invariant, e.g. along the RG flow from the UV to the IR, supersymmetry still relates the stress tensor to an R-current, whose charges run with the anomalous dimensions according to (III.6), and whose anomaly is related to the beta function according to (III.5). Among all possible R-symmetries, the superconformal R-symmetry is that which locally maximizes $a_{trial}(R) \equiv 3\mathrm{Tr}R^3 - \mathrm{Tr}R$ [14]. This method for determining the superconformal R-charges is referred to as "a-maximization," because the value of a_{trial} at its unique local maximum equals the conformal anomaly coefficient a of the SCFT [13], [5] (we rescale a by a conventional factor of $3/32$). An extension of a-maximization [20], further explored in [27], [45], has been proposed for determining the running R-charges, along the RG flow from the UV to the IR. See e.g. [15],[16],[18],[17],[19],[21],[46],[47] for further applications and extensions of a-maximization.

For our particular example (III.1), the exact beta functions (III.5) are

$$\begin{aligned}
\beta_g(g,g') &= -\frac{3g^3 f}{16\pi^2}\mathrm{Tr}\ SU(N_c)^2 R = -\frac{g^3 f}{16\pi^2}\left(b_1 + N_f\gamma_Q + N_c'\gamma_X\right), \\
\beta_{g'}(g,g') &= -\frac{3g'^3 f'}{16\pi^2}\mathrm{Tr}\ SU(N_c')^2 R = -\frac{g'^3 f'}{16\pi^2}\left(b_1' + N_f'\gamma_{Q'} + N_c\gamma_X\right),
\end{aligned} \tag{III.7}$$

where $b_1 \equiv 3N_c - N_f - N_c'$ and $b_1' \equiv 3N_c' - N_f' - N_c$ are the one-loop beta functions. We'll take both groups to be asymptotically free, i.e. take $g = g' = 0$ to be UV attractive:

$$3N_c - N_f - N_c' > 0, \quad \text{and} \quad 3N_c' - N_f' - N_c > 0, \tag{III.8}$$

so that $g = g' = 0$ is IR repulsive, as in figs. III.1 and III.2. To have the theory flow to a SCFT in the IR, rather than dynamically generating a vev, from a dynamically generated superpotential or quantum moduli space constraint, we also require

$$N_f + N_c' > N_c \quad \text{and} \quad N_f' + N_c > N_c' \qquad \text{(stability)}. \tag{III.9}$$

Assuming that (III.8) and (III.9) hold, much as in the above perturbative analysis, we identify three possible RG fixed points:

$$(A)\ g_* \neq 0,\ g'_* = 0: \quad \text{where} \qquad \gamma_Q = \gamma_X = -\frac{b_1}{N_f + N'_c}, \quad \text{and} \quad \gamma_{Q'} = 0. \tag{III.10}$$

$$(B)\ g_* = 0,\ g'_* \neq 0: \quad \text{where} \qquad \gamma_{Q'} = \gamma_X = -\frac{b'_1}{N'_f + N_c}, \quad \text{and} \quad \gamma_Q = 0. \tag{III.11}$$

$$(C)\ g_* \neq 0,\ g'_* \neq 0: \quad \text{where} \quad b_1 + N_f\gamma_Q + N'_c\gamma_X = 0 = b'_1 + N'_f\gamma_{Q'} + N_c\gamma_X. \tag{III.12}$$

For point (A), we used the fact that there is an enhanced flavor symmetry which ensures that $\gamma_X = \gamma_Q$ when $g' = 0$, and that Q' is a free field for $g' = 0$, so $\gamma_{Q'} = 0$. Analogous considerations apply for RG fixed point (B). Seiberg duality [26] shows that (A) and (B) are actually interacting SCFTs only if

$$N_f + N'_c > \frac{3}{2}N_c, \quad \text{and} \quad N'_f + N_c > \frac{3}{2}N'_c, \tag{III.13}$$

respectively; otherwise (A) or (B) should be replaced with its free magnetic Seiberg dual.

Our interest here is in the possible RG fixed point (C). We'll discuss when it exists as an interacting SCFT. We'll find, for example, that (III.13) is modified, once the RG flow of both couplings is taken into account: the otherwise free magnetic dual can be driven interacting by the other gauge coupling, as depicted in fig. III.3.

Let us first discuss some simple necessary, though not sufficient, conditions for (C) to exist – at least within the domain of attraction of flows to the IR from the asymptotically free UV fixed point at zero couplings – by determining when the RG flow is as in fig. III.1, or as in fig. III.2, with one of the couplings driven IR free. (Our discussion here is essentially identical to one that already appeared in [48] for a chiral example similar to (III.1), having the field X but not $\widetilde{X}$.) As in fig. III.5, (C) exists within the domain of attraction of the UV fixed point only if (A) and (B) are both IR unstable to perturbations in the other coupling. Using (III.7), (A) is IR stable to g' perturbations if $\mathrm{Tr}SU(N'_f)^2R|_A < 0$, i.e. we

get $\beta_{g'} \sim -g'^3(b'_1 - N_c b_1/(N_f + N'_c)) + O(g'^5)$, with the second contribution from γ_X at (A), so g' is an IR irrelevant perturbation of (A) if $b'_1 < N_c b_1/(N_f + N'_c)$, i.e.

$$\text{(A) is IR attractive, with } g' \to 0, \text{ if} \qquad (3N'_c - N'_f)(N'_c + N_f) - 3N_c^2 < 0. \tag{III.14}$$

Similarly, g will be an irrelevant perturbation of (B) if Tr $SU(N_c)^2 R|_B < 0$, which gives

$$\text{(B) is IR attractive, with } g \to 0, \text{ if} \qquad (3N_c - N_f)(N_c + N'_f) - 3N_c'^2 < 0. \tag{III.15}$$

The two inequalities in (III.14) and (III.15) are mutually incompatible, so we do not find the situation of fig. III.6. The condition for RG fixed point (C) to exist (within the domain of attraction of the UV fixed point) is that neither (III.14) nor (III.15) holds, i.e. we have a flow as in fig. III.1 only if

$$(3N_c - N_f)(N_c + N'_f) - 3N_c'^2 > 0 \qquad \text{and} \qquad (3N'_c - N'_f)(N'_c + N_f) - 3N_c^2 > 0. \tag{III.16}$$

The inequalities (III.16) generally differ from the asymptotic freedom conditions (III.8) needed to have $g = g' = 0$ not be IR attractive. For the special case $N_c = N'_c$ and $N_f = N'_f$, (III.16) do reduce to the asymptotic freedom conditions (III.8).

When RG fixed point (C) does exist, the three independent anomalous dimensions, γ_Q, $\gamma_{Q'}$, and γ_X are under-constrained by the two constraints of (III.12), so a-maximization [14] is required to determine the exact anomalous dimensions of chiral operators at (C). When the RG fixed point is not at sufficiently strong coupling for there to be accidental symmetries, the a-maximization result can be written as

$$\gamma_Q = 1 - \sqrt{1 + \frac{\lambda_G}{2N_c}}, \quad \gamma_{Q'} = 1 - \sqrt{1 + \frac{\lambda_{G'}}{2N'_c}}, \quad \gamma_X = 1 - \sqrt{1 + \frac{\lambda_G}{2N_c} + \frac{\lambda_{G'}}{2N'_c}}, \tag{III.17}$$

with λ_G and $\lambda_{G'}$ determined by the two conditions in (III.12), for the two beta functions (III.7) to vanish. This way of writing the a-maximization result is motivated by the extension of a-maximization due to Kutasov [20], [27], [45], where the interaction constraints on the superconformal R-charges, e.g. that the ABJ anomalies

should vanish, are imposed with Lagrange multipliers. The conjecture is that the Lagrange multipliers can be interpreted as the running coupling constants along the flow to the RG fixed point. In particular, the claim is that (III.17) gives the running anomalous dimensions along the entire RG flow, from $g = g' = 0$ in the UV to the RG fixed point (C) in the IR, with $\lambda_G = g^2|G|/2\pi^2$ and $\lambda_{G'} = g'^2|G'|/2\pi^2$ the running couplings in some scheme.

This analysis needs to be supplemented when there are accidental symmetries [16], and we'll find that many accidental symmetries do arise in these theories for general (N_c, N_c', N_f, N_f'). a-maximization with many accidental symmetries is best left to a computer (we used Mathematica), and then it's simpler to do the a-maximization at the RG fixed point, imposing the constraints at the outset rather than with Lagrange multipliers. We simplify the analysis by considering the limit of large numbers of flavors and colors, for arbitrary fixed values of the ratios, which for the example (III.1) are

$$x \equiv \frac{N_c}{N_f}, \quad x' \equiv \frac{N_c'}{N_f}, \quad n \equiv \frac{N_f'}{N_f}. \tag{III.18}$$

In this limit, the operator scaling dimensions then only depend on these ratios. Depending on (x, x', n), a variety of accidental symmetries associated with gauge invariant operators hitting the unitarity bound are found to occur, and their effect on the a-maximization analysis [16] is accounted for in our numerical algorithm.

As a function of the parameters (x, x', n), the theory either flows in the IR to a fully interacting RG fixed point, or can be partially or fully free. Our motivation for considering the examples (III.1) is that they have various possible dualities, and the a-maximization results can give insight into when they're applicable. For example, we could Seiberg dualize [26] one of the groups in (III.1), treating the other as a weakly gauged spectator. As we'll discuss, there is a range of (x, x', n) for which this dual theory realizes the RG flow possibility of fig. III.3: an arbitrarily small non-zero coupling of the "spectator" group can drive an otherwise free magnetic group to be interacting in the IR. This also occurs in an example discussed in [21], which appeared during the course of the present work.

Knowing the exact dimensions of chiral operators, we can classify the relevant superpotential deformations of (C). In particular, we can now determine the "superconformal window" range of validity of a duality proposed in [36] for the theory (III.1) with added superpotential interaction $W_{A_{2k+1}} = \mathrm{Tr}(X\widetilde{X})^{k+1}$. The dual [36] has gauge group $SU((k+1)(N_f+N'_f) - N_f - N'_c) \times SU((k+1)(N_f+N'_f) - N'_f - N_c)$ with similar matter content and additional gauge singlets (corresponding to the mesons), and a dual analog of the $W_{A_{2k+1}}$ superpotential. The superconformal window, where both dual descriptions are useful, is the range of (N_c, N'_c, N_f, N'_f) within which both the electric $W_{A_{2k+1}}$, as well as its analog in the magnetic dual, are both relevant. The a-maximization results allow us to determine this subspace of (x, x', n) parameter space, as a function of k. For large k, we find that this subspace is necessarily close to the $x \approx x'$ slice, i.e. $N_c \approx N'_c$.

The outline of this paper is as follows. In sect. III.B we briefly review a-maximization, and apply it to determine the superconformal R-charges for the $SU(N_c) \times SU(N'_c)$ example (III.1). We find that there are accidental symmetries arising from gauge invariant operators hitting the unitarity bound $\Delta \geq 1$, and use the procedure of [16] to take these into account during a-maximization. We especially consider the parameter slice $x = x'$ (i.e. $N_c = N'_c$) for large x (i.e. $N_c \gg N_f$), and general n. In this slice and limit, $R(X) \to 0$. We account for the many accidental symmetries, associated with generalized mesons hitting their unitarity bound, in this limit (and numerically check that no baryon operators hit the unitarity bound.) As we discuss, if we set $n \equiv N'_f/N_f = 1$, our results should – and indeed do – coincide with those of [16].

In sect. III.C we consider the theory (III.1) deformed by the superpotential $W_{A_{2k+1}} = \mathrm{Tr}(X\widetilde{X})^{k+1}$, and the dual description of [36] of that theory. We use a-maximization to determine the exact chiral operator dimensions in the dual of [36]. Combining these results with those of sect. III.B, we can determine the superconformal window region of (x, x', n) parameter space, for any given value of k, within which the $W_{A_{2k+1}}$ superpotential of both the electric theory (III.1) and

its dual are both relevant. For large k, the superconformal window is necessarily close to the parameter slice $x \approx x'$. We check numerically that, for all k, there is always a non-empty superconformal window region of parameter space in which the duality of [36] is applicable.

In sect. III.D we consider Seiberg dualizing [26] one of the groups in (III.1), treating the other gauge group as a spectator. We'll discuss analogs $\widetilde{A}$, $\widetilde{B}$, and $\widetilde{C}$ of the possible RG fixed points in fig. III.1, when they exist, and when they're IR stable to perturbations. We find that there is a range of the parameters (III.18) (x, x', n) where an otherwise IR free magnetic gauge group is driven to be interacting for any non-zero gauge coupling of the "spectator" group. This is the phenomenon depicted in fig. III.3. As seen from the exact beta functions (III.5), positive anomalous dimensions are needed to turn a 1-loop IR irrelevant coupling into an IR relevant one. The superpotential of the Seiberg dual theory plays a crucial role here, together with the spectator gauge coupling, to get the positive anomalous dimensions needed for the effect of fig. III.3. The condition that the RG fixed point (C) of the original electric theory (III.1) be interacting, rather than flowing to a free magnetic dual, is that the fully interacting RG fixed point ($\widetilde{C}$) exists in the dual theory; this issue is analyzed by the dual analog of fig. III.5. When ($\widetilde{C}$) does exist, we expect that it's equivalent to the RG fixed point (C) of the electric description. We verify that their superconformal R-charges and central charges indeed agree.

In Sect. III.E we briefly conclude, and present a topic for further research.

In the Appendix, we note that all of the many duality examples of [36] have a non-zero superconformal window. The theories in [36] with a single gauge group (either $SU(N_c), SO(N_c)$, or $Sp(N_c)$) and matter in a two-index representation (e.g. adjoint, symmetric, or antisymmetric) are shown in the large N_c limit to all have the same superconformal R-charges, and superconformal window, as that of $SU(N_c)$ with an adjoint; we can directly borrow the results obtained in [16], with the central charge differing from that of [16] by just a fixed overall multiplica-

tive factor. We also note that all of the other examples in [36], involving product groups, all also have superconformal R-charges and superconformal window that reduce to those obtained in [16] for a 1d slice of their parameter space, when we take all of the group ranks equal and all numbers of flavors equal (e.g. taking $x = x'$ and $n = 1$ in (III.18)). This suffices to show that all of the duality examples of [36] indeed have a non-empty superconformal window.

III.B a-maximization analysis for the $SU(N_c) \times SU(N_c')$ theory (III.1)

The superconformal $U(1)_R$ symmetry is uniquely determined by the condition that it maximizes $a_{trial}(R) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$ among all possible R-symmetries [14]. The constraints on the superconformal R-symmetry, e.g. that it's ABJ anomaly free, can either be implemented at the outset, before maximizing $a(R)$ w.r.t. R, or via Lagrange multipliers λ [20]. a-maximization with the Lagrange multipliers yields simple general expressions for the R-charges of the fields $R_i(\lambda)$, with the conjectured interpretation of giving the running R-charges along the RG flow to the RG fixed point [20], [27], [45].

For example, for a general $\mathcal{N} = 1$ supersymmetric $G \times G'$ gauge theory, with zero superpotential, we determine the running R-charges by maximizing with respect to the R_i

$$a(\lambda, R) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R - \lambda_G \mathrm{Tr}\, G^2 R - \lambda_{G'} \mathrm{Tr}\, G'^2 R = 2|G| + 2|G'| - \lambda_G T(G) - \lambda_{G'} T(G')$$
$$+ \sum_i |\mathbf{r_i}||\mathbf{r'_i}| \left[3(R_i - 1)^2 - 1 - \lambda_G \frac{T(\mathbf{r_i})}{|\mathbf{r_i}|} - \lambda_{G'} \frac{T(\mathbf{r'_i})}{|\mathbf{r'_i}|} \right] (R_i - 1), \qquad \text{(III.19)}$$

holding fixed the Lagrange multipliers λ_G and $\lambda_{G'}$, which enforce the constraints that $U(1)_R$ not have ABJ anomalies, $\mathrm{Tr}GGR = \mathrm{Tr}G'G'R = 0$. This yields:

$$R_i(\lambda) = 1 - \frac{1}{3}\sqrt{1 + \lambda_G \frac{T(\mathbf{r}_i)}{|\mathbf{r}_i|} + \lambda_{G'} \frac{T(\mathbf{r'}_i)}{|\mathbf{r'}_i|}} \quad \text{i.e.} \quad \gamma_i = 1 - \sqrt{1 + \lambda_G \frac{T(\mathbf{r}_i)}{|\mathbf{r}_i|} + \lambda_{G'} \frac{T(\mathbf{r'}_i)}{|\mathbf{r'}_i|}}, \qquad \text{(III.20)}$$

where we used $\gamma_i = 3R_i - 2$ for the anomalous dimensions. The conjecture is that these expressions can be interpreted as giving the anomalous dimensions along the entire RG flow, with $\lambda_G = g^2|G|/2\pi^2$ and $\lambda_{G'} = g'^2|G'|/2\pi^2$ in some scheme. For the example (III.1) this gives the result (III.17). As in [20], using (III.20) in (III.19) yields a monotonically decreasing a-function $a(\lambda) = a(\lambda, R(\lambda))$ along the RG flow. The values of λ^*_G and $\lambda^*_{G'}$ at the IR fixed point are the extremal values of $a(\lambda)$; since $a(\lambda)$'s gradients are proportional to the exact beta functions [20], [27], [45], this is equivalent to the conditions that the anomalous dimensions (III.17) yield zeros of the beta functions (III.5).

Whenever a gauge invariant operator M hits or appears to violate the unitarity bound $R(M) \geq 2/3$, M becomes a decoupled free field. This affects the a-maximization analysis by introducing an additive correction to the quantity $a(R)$ to be maximized [16] (this can be derived from the presence of an accidental $U(1)_M$ symmetry, acting only on M [17]):

$$a_{trial}(R) \to a_{trial}(R) + \frac{1}{9} dim(M) \left(2 - 3R(M)\right)^2 \left(5 - 3R(M)\right). \qquad \text{(III.21)}$$

This correction can also be included in the a-maximization analysis with Lagrange multipliers [45], but it becomes unwieldy to do so when there are many such contributions from operators that hit the unitarity bound, as is the case in our examples for general values of the numbers of flavors and colors. For this reason, we will here do the a-maximization analysis at the RG fixed point, numerically, with the constraints implemented at the outset rather than via Lagrange multipliers.

We consider the example (III.1) in the range of the parameters (III.18) where it's possible to have the RG fixed point like (C) in fig. III.1, with both groups interacting. For asymptotic freedom of $g = g' = 0$ in the UV, and to avoid having it be attractive in the IR, we take

$$3x - x' - 1 > 0, \quad \text{and} \quad 3x' - x - n > 0. \qquad \text{(III.22)}$$

We also impose the condition (III.9), which is

$$-n < x - x' < 1 \qquad \text{(stability)}, \qquad \text{(III.23)}$$

to have the origin of the moduli space of vacua not be dynamically disallowed. Finally, to have the points (A) and (B) not be IR attractive, as in fig. III.2, we impose (III.16),

$$(3x-1)(x+n)-3x'^2>0 \quad \text{and} \quad (3x'-n)(x'+1)-3x^2>0. \tag{III.24}$$

If either inequality of (III.24) is not satisfied, one or the other group is driven IR free, to RG fixed point (A) or (B), with anomalous dimensions and R-charges given by (III.10) or (III.11). When both (III.24) are satisfied, RG flows generally end up with both couplings interacting, which can end up being a RG fixed point (C), where (III.12) is satisfied. As mentioned in the introduction, we do *not* impose the naive conditions (III.13) to avoid IR free magnetic dual groups: as we'll see in sect. III.D, the conditions (III.13) are generally dramatically modified by the dynamics of the other gauge group.

As always, the conditions in (III.12) for the exact beta functions to vanish are equivalent to requiring that the superconformal $U(1)_R$ have vanishing ABJ anomalies with respect to all of the interacting gauge groups. So at (C) we have the two anomaly free conditions

$$\begin{aligned} N_c+N_c'(R(X)-1)+N_f(R(Q)-1) &= 0 \\ N_c'+N_c(R(X)-1)+N_f'(R(Q')-1) &= 0 \end{aligned} \tag{III.25}$$

to have $\mathrm{Tr}SU(N_c)^2U(1)_R=\mathrm{Tr}SU(N_c')^2U(1)_R=0$. Enforcing (III.25) at the RG fixed point, we can solve for $R(X)$ and $R(Q')$ in terms of $y\equiv R(Q)$

$$R(X)=\frac{1-y}{x'}+1-\frac{x}{x'}, \qquad R(Q')=\frac{x}{nx'}(y-1)+\frac{x^2}{nx'}-\frac{x'}{n}+1, \tag{III.26}$$

where the parameters (x,x',n) of the theory are the ratios (III.18). We determine the superconformal R-charge $y(x,x',n)$ by a-maximization (in the single variable y).

Imposing (III.26), we compute $a_{trial}(R)=3\mathrm{Tr}R^3-\mathrm{Tr}R$ from the spectrum (III.1) to be

$$a^{(0)}/N_f^2=2x^2+2x'^2+6x(y-1)^3-2x(y-1)+6nx'\left[\frac{x}{nx'}(y-1)+\frac{x^2}{nx'}-\frac{x'}{n}\right]^3$$

$$-2nx'\left[\frac{x}{nx'}(y-1)+\frac{x^2}{nx'}-\frac{x'}{n}\right]+6xx'\left[\frac{1-y}{x'}-\frac{x}{x'}\right]^3-2xx'\left[\frac{1-y}{x'}-\frac{x}{x'}\right]. \quad \text{(III.27)}$$

We then compute the superconformal R-charges by locally maximizing (III.27) w.r.t. y, for general fixed values of the parameters (x, x', n); we'll denote the solution as $y^{(0)}(x, x', n)$. The superscript $^{(0)}$ is a reminder that these results are valid only in the range of (x, x', n) in which no gauge invariant operators have hit the unitarity bound; otherwise (III.27) will require corrections as in (III.21). Within this range of (x, x', n), we can also use the Lagrange multiplier approach. Imposing (III.25) with Lagrange multipliers yields the simple expressions (III.17), which can be interpreted as the running R-charges along the RG flow, coinciding with the R-charges obtained above from $y^{(0)}(x, x', n)$ at the RG fixed point.

The first gauge invariant, chiral, composite operators $\mathcal{O}$ to hit the unitarity bound $R(\mathcal{O}) \geq 2/3$ are the mesons $M \equiv Q\widetilde{Q}$ or $M' \equiv Q'\widetilde{Q}'$. M hits the unitarity bound when $y(x, x', n) \leq 1/3$; using (III.6), this happens when Q has the large, negative anomalous dimension, $\gamma_Q(x, x', n) < -1$, which can only happen if (x, x', n) are such that the RG fixed point values of the gauge couplings are large. The above result $y^{(0)}(x, x', n)$ is valid within the range of (x, x', n) where neither M or M' have hit their unitarity bound, i.e. the range of (x, x', n) where $y^{(0)}(x, x', n) > 1/3$ and where $R(Q') \geq 1/3$, with $R(Q')$ computed from $y^{(0)}(x, x', n)$ via (III.26). Outside of this range, the above a-maximization analysis has to be supplemented, as in (III.21), to account for the accidental symmetries associated with operators hitting the unitarity bound and becoming free fields.

For general (x, x', n) the gauge operators that will hit the unitarity bound are:

$$M_j = Q(\widetilde{X}X)^{j-1}\widetilde{Q}, \quad M'_j = Q'(\widetilde{X}X)^{j-1}\widetilde{Q}',$$
$$P_j = Q(\widetilde{X}X)^{j-1}\widetilde{X}\widetilde{Q}', \quad \widetilde{P}_j = \widetilde{Q}X(\widetilde{X}X)^{j-1}Q'. \quad \text{(III.28)}$$

For every integer $j \geq 1$, there are $N_f N'_f$ mesons P_j and $\widetilde{P}_j$, N_f^2 mesons M_j, and $N_f'^2$ mesons M'_j. We verified that it's self-consistent to assume that the baryon operators do not hit the unitarity bound; also, gauge invariants without fundamen-

tals, such as $\mathrm{Tr}(X\widetilde{X})^j$, contribute negligibly in the large N limit. The quantity to maximize in general is then

$$a^{(p)}/N_f^2 = \tilde{a}^{(0)}/N_f^2 + \frac{2n}{9}\sum_{j=1}^{p_P}\left[2-3R(P_j)\right]^2\left[5-3R(P_j)\right]$$

$$+\frac{1}{9}\sum_{j=1}^{p_M}\left[2-3R(M_j)\right]^2\left[5-3R(M_j)\right] + \frac{n^2}{9}\sum_{j=1}^{p_{M'}}\left[2-3R(M'_j)\right]^2\left[5-3R(M'_j)\right], \tag{III.29}$$

where p denotes $\{p_P, p_M, p_{M'}\}$, with $p_P = p_{\widetilde{P}}$ the number of P (and also $\widetilde{P}$ type) mesons which have hit the unitarity bound. The quantities such as $R(M_j)$ in (III.29) are given by e.g. $R(M_j) = R[Q(X\widetilde{X})^{j-1}\widetilde{Q}] = 2y + 2(j-1)R(X)$, with $R(X)$ given by (III.26); so the corrections in (III.29) are complicated functions of the variable y that we're maximizing with respect to, along with the parameters (x, x', n). Maximizing (III.29) yields $y^{(p)}(x, x', n)$, and $y(x, x', n)$ is obtained by pasting these together, with the appropriate values of p depending on (x, x', n), increasing e.g. p_M every time another value of j is obtained such that $R(M_j)$ hits $2/3$. We numerically implemented this process to obtain $y(x, x', n)$, but it's difficult to produce an illuminating plot of a function of three variables.

Let us discuss some qualitative aspects of our results. From $y(x, x', n)$ we can compute the anomalous dimensions γ_Q, γ_X, and $\gamma_{Q'}$, using (III.6), and we find that all are negative within the range (III.24) where the RG fixed point (C) can exist. This is to be expected, since our theory (III.1) has only gauge interactions, and no superpotential (gauge interactions yield negative anomalous dimension, and superpotentials yield positive contributions to the anomalous dimensions). When either inequality (III.24) is violated, the theory flows not to (C), but rather to RG fixed points (A) or (B), as in fig. III.2 and the above a-maximization analysis, which assumed in (III.25) that both groups are interacting, is inapplicable. At the boundaries of (III.24), where either inequality is saturated, our a-maximization results properly reduce to (III.10) or (III.11).

It is interesting to note that there is a 1d slice of the (x, x', n) parameter space, given by $x = x'$ and $n = 1$, for which the a-maximization analysis of this

theory coincides with that of [16] for $SU(N_c)$ SQCD with N_f fundamentals and an added adjoint. In this slice, for every contribution to the quantity a_{trial} to maximize in [16], we have here two analogous matter fields in the spectrum of our theory, with the same R-charges: twice as many gauge fields, twice as many fundamentals (Q and Q' and conjugate), the X and $\widetilde{X}$ fields contribute as two adjoints (using (III.26) for $x = x'$ and $n = 1$), and all the mesons (III.28) hitting the unitarity bound map to two copies of the mesons hitting the unitarity bound in [16]. Thus, for $x = x'$ and $n = 1$, (III.29) is exactly twice the expression obtained in [16] for the theory considered there. Since a_{trial} is the same function of y, up to a factor of 2, it is maximized by the same solution $y_{KPS}(x)$ obtained by the analysis of [16]. So the superconformal R-charges of our theory (III.1) satisfy $y(x, x', n)|_{x=x',n=1} = y_{KPS}(x)$.

For $x \approx x'$ taken to be very large, i.e. $N_c \approx N_c' \gg N_f$, the superconformal R-charge of the field X goes to zero, $R(X) \to 0$, for arbitrary fixed values of $n \equiv N_f'/N_f$, as seen from (III.26), and the fact that y remains finite in this limit. The asymptotic value $y_{as}(n)$ in this $x = x' \to \infty$ limit is determined by our numerical a-maximization analysis, but it can also be approximated analytically. Because many mesons contribute to the sums (III.29), the sums can be approximated as integrals (following [16]):

$$\frac{1}{9}\sum_{j=1}^{p}[2-3R_j]^2[5-3R_j] \approx \frac{1}{27\beta}\int_0^{2-3\alpha} u^2(3+u)du = \frac{1}{18\beta}(2-3\alpha)^3(1-\frac{1}{2}\alpha), \quad \text{(III.30)}$$

where α and β are defined by $R_j \equiv \alpha + (j-1)\beta$ (and $u \equiv 2 - 3R_j$). The upper limit p in the sum is solved for by setting $R_p = \alpha+(p-1)\beta$ equal to $2/3$. Applying (III.30) to the sums in (III.29), $\beta = R(X\widetilde{X}) = 2R(X)$ for all three, and for the first sum in (III.29) $\alpha = R(Q\widetilde{X}\widetilde{Q}') = y + R(Q') + R(X)$, while for the second and third $\alpha = 2y$ and $\alpha = 2R(Q')$ respectively; here, $R(X)$ and $R(Q')$ are to be written in terms of the variable y and the parameters (x, x', n) using (III.26).

Setting $x = x'$ and taking both large, (III.29) then becomes

$$a/N_f^2 \simeq 6x\left[1 + \frac{1}{n^2}\right](y-1)^3 - 20x(y-1)$$

$$+\frac{x}{36}\left[2-6y\right]^3+\frac{xn}{36}\left[\frac{6}{n}(1-y)-4\right]^3+\frac{xn}{36}\left(1+\frac{1}{n}\right)\left[3(1-y)\left(1+\frac{1}{n}\right)-4\right]^3. \tag{III.31}$$

The first line of (III.31) is the large $x = x'$ limit of (III.27), and the second line contains the meson sums of (III.29), evaluated using (III.30). Note that every term in (III.31) is linear[2] in x in this limit, so maximizing (III.31) w.r.t. y yields an asymptotic value, $y_{as}(n)$, that's independent of x in this limit of large $x = x'$. This asymptotic value depends on $n \equiv N'_f/N_f$, and the conditions (III.24) needed for neither gauge coupling to drive the other to be IR free here require n to lie in the range

$$3 > n > \frac{1}{3} \qquad \text{for } x = x' \to \infty. \tag{III.32}$$

As expected from the discussion above, for $n = 1$ (III.31) reduces to twice the expression obtained in the large x analysis of [16], and for $n = 1$ our expression for $y_{as}(n)$ coincides with the asymptotic large x value of y obtained there: $y_{as}(n)|_{n=1} = (\sqrt{3}-1)/3$.

The asymptotic value $y_{as}(n)$ will be used in the next section to find the minimal value of $x \approx x'$ needed for the superpotential $\Delta W_{A_{2k+1}} \equiv \mathrm{Tr}(X\widetilde{X})^{k+1}$ to be a relevant deformation of RG fixed point (C) in the limit of large k. This gives one side of the superconformal window for the duality of [36] (see fig. III.7). We have also checked, including away from the strict $x = x'$ limit, that $R(X)$ is nowhere negative, i.e. using (III.26) that the a-maximizing solution $y(x, x', n)$ satisfies $1 - y(x, x', n) + x' - x > 0$. This is important for the self-consistency of our analysis since, if $R(X)$ were negative, baryonic operators, formed by dressing the quarks with many powers of $X\widetilde{X}$, would hit the unitarity bound and lead to additional contributions analogous to (III.21).

[2]The fact that the expression in (III.31) grows for large x only linearly is a check of the conjectured a-theorem. Any greater exponent would've led to a-theorem violations, e.g. along a Higgs flat direction where $X\widetilde{X}$ gets an expectation value, Higgsing each $SU(N_c)$ gauge group factor to products of similar factors. This flat direction is analogous to that considered in a non-trivial check of the a-theorem in [16] (where it's also pointed out that the sub-leading constant term must be – and indeed is – negative for the a-theorem to hold for this Higgs RG flow).

III.C The theory with $W_{A_{2k+1}} = \mathbf{Tr}(X\widetilde{X})^{k+1}$ and its dual

In [36] it was proposed that our theory (III.1), together with a superpotential $W_{A_{2k+1}} = \mathrm{Tr}(X\widetilde{X})^{k+1}$ has a dual given by a similar theory:

$$\begin{array}{lll} \text{gauge group:} & SU(\widetilde{N}_c) \times SU(\widetilde{N}'_c) & \\ \text{matter:} \quad Y \oplus \widetilde{Y} & (\square, \overline{\square}) \oplus (\overline{\square}, \square), & \\ q_f \oplus \widetilde{q}_{\tilde{f}} & (\square, \mathbf{1}) \oplus (\overline{\square}, \mathbf{1}) & (f, \tilde{f} = 1 \dots N'_f), \\ q'_{f'} \oplus \widetilde{q}'_{\tilde{f}'} & (\mathbf{1}, \square) \oplus (\mathbf{1}, \overline{\square}) & (f', \tilde{f}' = 1 \dots N_f), \end{array} \tag{III.33}$$

where $\widetilde{N}_c = (k+1)(N_f + N'_f) - N_f - N'_c$ and $\widetilde{N}'_c = (k+1)(N_f + N'_f) - N'_f - N_c$. There are also singlets P_j, for $j = 1 \dots k$, and M_j and M'_j for $j = 1 \dots k+1$, with superpotential

$$W = \mathrm{Tr}(Y\widetilde{Y})^{k+1} + \sum_{j=1}^{k} \left[P_j q \widetilde{Y} (Y\widetilde{Y})^{k-j} \widetilde{q}' + \widetilde{P}_j \widetilde{q}' (Y\widetilde{Y})^{k-j} Y q \right]$$

$$+ \sum_{j=1}^{k+1} \left[M_j q' (\widetilde{Y}Y)^{k-j+1} \widetilde{q}' + M'_j \widetilde{q} (Y\widetilde{Y})^{k-j+1} q \right]. \tag{III.34}$$

The first term is the dual analog of the $W_{A_{2k+1}}$ superpotential, and the remaining terms are analogs of the $Mq\widetilde{q}$ superpotential in Seiberg duality [26].

The duality is useful within a superconformal window, which is the range of the parameters (x, x', n), where the superpotential $W_{A_{2k+1}}$ and its dual analog in (III.34) are both relevant in controlling the IR dynamics, i.e. when the following conditions are satisfied:

$$(i) \qquad R(X) = \frac{1 - y(x, x', n)}{x'} + 1 - \frac{x}{x'} < \frac{1}{k+1}, \tag{III.35}$$

$$(ii) \qquad R(Y) = \frac{1 - \widetilde{y}(\widetilde{x}, \widetilde{x}', \widetilde{n})}{\widetilde{x}'} + 1 - \frac{\widetilde{x}}{\widetilde{x}'} < \frac{1}{k+1}, \tag{III.36}$$

$$(iii) \qquad (k+1)(N_f + N'_f) - N_f - N'_c > 0, \quad \text{and} \quad (k+1)(N_f + N'_f) - N'_f - N_c > 0. \tag{III.37}$$

If (i) is not satisfied, $W_{elec} = \mathrm{Tr}(X\widetilde{X})^{k+1}$ is an irrelevant deformation of RG fixed point (C), and thus $W_{elec} \to 0$ in the IR; this fact is obscured in the magnetic dual description. Likewise, if (ii) is not satisfied, one should use the magnetic

description, with the coefficient of the $\mathrm{Tr}(Y\widetilde{Y})^{k+1}$ superpotential term flowing to zero in the IR; the electric description then doesn't readily describe the true RG fixed point. Finally, condition (*iii*) is the "stability bound," needed for the RG fixed point to exist (and for the dual groups (III.33) to have positive ranks): if (III.37) are not satisfied, the electric theory (III.1) with $W_{A_{2k+1}}$ superpotential dynamically generates a superpotential, spoiling conformal invariance.

Using the results of the previous subsection, we can now determine the range of (x, x', n) in which condition (III.35) is satisfied, for $\mathrm{Tr}(X\widetilde{X})^{k+1}$ to be relevant. The $k = 0$ case is a mass term and (III.35) is then always satisfied (starting with $W = 0$ at (C), all fields have $R \leq 2/3$). For all $k > 1$, (III.35) is only satisfied in subspaces of the (x, x', n) parameter space for which the RG fixed point is at sufficiently strong enough coupling to give X a sufficiently negative anomalous dimension. The larger k is, the more strongly coupled the RG fixed point must be in order to have (III.35) be satisfied. For arbitrarily large k, there's a non-empty range of (x, x', n) in which (III.35) is satisfied: as we saw in the previous subsection, $R(X) \to 0$ in parts of the parameter space. Let's consider, for example, the parameter slice $x = x'$ and ask when (III.35) is satisfied for large values of k. Since satisfying (III.35) for large k requires large x, we can replace $y(x, x', n)$ in (III.35) with the asymptotic value $y_{as}(n)$ obtained by maximizing (III.31). Then the condition (III.35) for the superpotential to be relevant becomes

$$x > x_{min}(n) \approx k\,(1 - y_{as}(n)) \qquad \text{for } k \gg 1. \tag{III.38}$$

The above analysis of the electric theory gives one edge of the superconformal window of the parameters (x, x', n) for the duality of [36]. The other edge of the window is obtained by determining the range of these parameters in which (III.36) is satisfied, for the $W_{A_{2k+1}}$ superpotential to be relevant in the magnetic theory. Again, we simplify the analysis by taking the numbers of flavors and colors in the electric theory to be large, so the same is true in the magnetic theory. The ratios on the magnetic side are defined to be $\tilde{x} \equiv \widetilde{N}_c/N'_f$, $\tilde{x}' \equiv \widetilde{N}'_c/N'_f$, and

$\tilde{n} \equiv N_f/N_f'$, which are related to the electric ones (III.18) as

$$\tilde{x} = (k+1)(1+n^{-1}) - n^{-1} - x'n^{-1}, \quad \tilde{x}' = (k+1)(1+n^{-1}) - 1 - xn^{-1}, \quad \tilde{n} = n^{-1}. \tag{III.39}$$

In the magnetic theory (III.33), we assume that both magnetic gauge groups remain interacting. The superconformal R-charge is constrained by the magnetic analog of (III.25), that it be anomaly free w.r.t. both gauge groups. As in (III.26), we can use this to solve for $R(Y) = R(\widetilde{Y})$ and $R(q') = R(\widetilde{q}') \equiv \widetilde{y}'$ in terms of $R(q) = R(\widetilde{q}) \equiv \widetilde{y}$:

$$R(Y) = \frac{1-\widetilde{y}}{\widetilde{x}'} + 1 - \frac{\widetilde{x}}{\widetilde{x}'}, \qquad \widetilde{y}' = \frac{\widetilde{x}}{\widetilde{n}\widetilde{x}'}(\widetilde{y}-1) + \frac{\widetilde{x}^2}{\widetilde{n}\widetilde{x}'} - \frac{\widetilde{x}'}{\widetilde{n}} + 1. \tag{III.40}$$

The contribution to the magnetic $\widetilde{a}_{trial}$ from the fields in (III.33) is

$$\tilde{a}^{(0)}/N_f'^2 = 2\tilde{x}^2 + 2\tilde{x}'^2 + 6\tilde{x}(\tilde{y}-1)^3 - 2\tilde{x}(\tilde{y}-1) + 6\tilde{n}\tilde{x}'\left[\frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y}-1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}}\right]^3$$

$$-2\tilde{n}\tilde{x}'\left[\frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y}-1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}}\right] + 6\tilde{x}\tilde{x}'\left[\frac{1-\tilde{y}}{\tilde{x}'} - \frac{\tilde{x}}{\tilde{x}'}\right]^3 - 2\tilde{x}\tilde{x}'\left[\frac{1-\tilde{y}}{\tilde{x}'} - \frac{\tilde{x}}{\tilde{x}'}\right]. \tag{III.41}$$

To this we must add the contributions from the singlets P_i, $\widetilde{P}_j$, M_i, M_j'. Each of these fields couples only via a superpotential term in (III.34) and, initially taking that singlet's R-charge to be 2/3, that superpotential term may be relevant or irrelevant in the IR. If it's relevant, then the singlet's R-charge is determined by the requirement that the superpotential term have $R = 2$ total in the IR. If it's irrelevant, the singlet is a free field, with $R = 2/3$. If we assume that the last p_P P_j's (and $\widetilde{P}_j$'s), the last p_M M_j's, and the last $p_{M'}$ M_j''s are interacting, then the quantity to maximize is

$$\tilde{a}^{(p)}/N_f'^2 = \tilde{a}^{(0)}/N_f'^2 + \frac{2\tilde{n}}{9}\sum_{j=1}^{p_P}(2-3\alpha_j^P)^2(5-3\alpha_j^P) + \frac{\tilde{n}^2}{9}\sum_{j=1}^{p_M}(2-3\alpha_j^M)^2(5-3\alpha_j^M)$$

$$+\frac{1}{9}\sum_{j=1}^{p_{M'}}(2-3\alpha_j^{M'})^2(5-3\alpha_j^{M'}) + \frac{4\tilde{n}}{9}(k-2p_P) + \frac{2\tilde{n}^2}{9}(k+1-2p_M) \tag{III.42}$$

$$+\frac{2}{9}(k+1-2p_{M'}),$$

where we define

$$\alpha_j^P \equiv \tilde{y} + \frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y}-1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} + 1 + (2j-1)\left[\frac{1-\tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'}\right],$$

$$\alpha_j^M \equiv 2\left[\frac{\tilde{x}}{\tilde{n}\tilde{x}'}(\tilde{y}-1) + \frac{\tilde{x}^2}{\tilde{n}\tilde{x}'} - \frac{\tilde{x}'}{\tilde{n}} + 1\right] + 2(j-1)\left[\frac{1-\tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'}\right], \qquad \text{(III.43)}$$

$$\alpha_j^{M'} \equiv 2\tilde{y} + 2(j-1)\left[\frac{1-\tilde{y}}{\tilde{x}'} + 1 - \frac{\tilde{x}}{\tilde{x}'}\right].$$

The additional terms in (III.42) are the contributions from the singlets (see sect. 6 of [17] for a detailed discussion of an analogous example). The full solution $\tilde{y}(\tilde{x}, \tilde{x}', \tilde{n})$ is obtained by patching together the maximizing solutions of (III.42) with the appropriate values of p_M, $p_{M'}$, and p_P, depending on $(\tilde{x}, \tilde{x}', \tilde{n})$, given by the largest integer j's such that the α_j in (III.43) satisfy $\alpha_j \leq 4/3$ (where the corresponding superpotential term becomes irrelevant).

For any given value of k, we can use the numerical a-maximization analysis to determine the range of $(\widetilde{x}, \widetilde{x}', \widetilde{n})$, and thus the range of electric parameters (x, x', n), in which the condition (III.36) for $\Delta W = \mathrm{Tr}(Y\widetilde{Y})^{k+1}$ to be relevant is satisfied. Using (III.39), we'll express this in terms of the electric parameters (x, x', n). To be concrete, let us consider the limit of large k. The condition (III.35) on the electric side gave the inequality (III.38), which shows that $x \approx x'$ must get large, linearly in k, in the large k limit, while n is restricted to the range (III.32). Then (III.39) gives $\tilde{x} \approx \tilde{x}' \approx k(1+n^{-1}) - xn^{-1}$, and the condition (III.36) will require $\tilde{x}$ to also be large. In this limit of large $\tilde{x} \approx \tilde{x}'$, (III.43) becomes

$$\begin{aligned}\tilde{a}/N_f'^2 \simeq\ & 6\tilde{x}\left[1+\frac{1}{\tilde{n}^2}\right](\tilde{y}-1)^3 - 20\tilde{x}(\tilde{y}-1) + \frac{\tilde{x}\tilde{n}}{36}\left(1+\frac{1}{\tilde{n}}\right)\left[3(1-\tilde{y})\left(1+\frac{1}{\tilde{n}}\right) - 4\right]^3 \\ & + \frac{4\tilde{x}}{9}\left[\frac{\tilde{n}\tilde{y}}{1-\tilde{y}} - 1\right] + \frac{\tilde{x}\tilde{n}}{36}\left[\frac{6}{\tilde{n}}(1-\tilde{y}) - 4\right]^3 + \frac{2\tilde{n}\tilde{x}}{9}\left[\frac{\tilde{n}}{1-\tilde{y}} - 2\right] \qquad \text{(III.44)} \\ & + \frac{\tilde{x}}{36}\left[2-6\tilde{y}\right]^3 + \frac{2\tilde{x}(2\tilde{y}-1)}{9(1-\tilde{y})} + \frac{2k}{9}(\tilde{n}+1)^2.\end{aligned}$$

The first two terms are the large $\tilde{x} \approx \tilde{x}'$ limit of (III.41), and the rest are the remaining terms in (III.42), with sums evaluated using (III.30) (modifying the lower limit of the integral (III.30) to be $2-3R = -2$, rather than 0, since $\alpha_j = 4/3$

is the limit where the superpotential term becomes irrelevant). Maximizing (III.44) with respect to $\widetilde{y}$ gives $\widetilde{y}_{as}(\tilde{n})$.

The condition (III.36) for $\mathrm{Tr}(Y\widetilde{Y})^{k+1}$ to be relevant can then be written for $k \gg 1$ as

$$\frac{(1-\widetilde{y}_{as})n}{k(n+1)-x} < \frac{1}{k}. \tag{III.45}$$

Rearranging and combining with (III.38), the electric and magnetic conditions (III.35) and (III.36) can be written together for $k \gg 1$ as

$$1 - y_{as}(n) < \frac{x}{k} < 1 + n\widetilde{y}_{as}(n) \qquad \text{for } x \approx x' \text{ and } k \gg 1. \tag{III.46}$$

For the duality (III.33) of [36] to be useful, and the superconformal window be non-empty, the inequalities at the two ends of (III.46) had better be compatible with each other. This is verified to indeed be the case, as seen in the plots in fig. III.7, for the entire allowed range (III.32) of n. The vertical axis of fig. III.7 gives the allowed values of x/k, for a given value of n, and the superconformal window is the region between the lower two curves on fig. III.7. There is also the stability bound conditions (III.37), which in our $k \gg 1$ limit, with $x \approx x'$ scaling linearly in k, can both be written as $x/k < 1+n$. In the plot of fig. III.7, the upper line is the stability bound, and the values of x/k in the superconformal window, between the lower two curves, is indeed always safely below the stability bound for the entire allowed range of n. All of these successes can be viewed as reassuring checks of the duality of [36].

For the case $x = x'$, $n = 1$, the conformal window plotted in fig. III.7 coincides with that obtained in [16] for SQCD with an adjoint, for the reason discussed above.

III.D Dualizing one gauge group

With product gauge groups, such as (III.1), we can consider dualizing one of the gauge groups, treating the other gauge group as a spectator. The validity

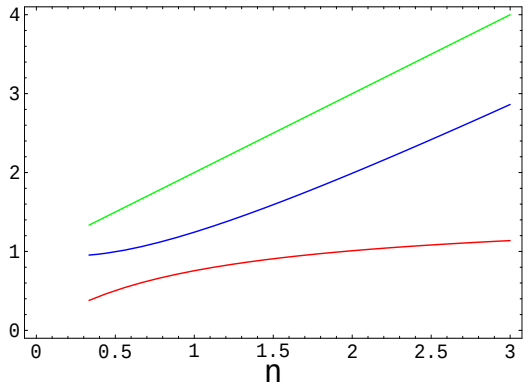

Figure III.7: The x/k conformal window: the upper line is the stability bound $1+n$, the middle line is $1+n\widetilde{y}_{as}(n)$ and the lower line is $1-y_{as}(n)$.

of doing this deserves scrutiny, because duality is only exact at the IR fixed point. Dualizing away from the extreme IR can be potentially justified if the dualized group's dynamical scale is far above that of the other "spectator" group, $\Lambda_d \gg \Lambda_s$ (and then holomorphic quantities can be analytically continued in Λ_d/Λ_s) or if one group gets strong while the other gets weak in the IR (as in string theory examples, see e.g. [49], [50]).

Let's consider the $SU(N_c)\times SU(N_c')$ theory (III.1), and consider dualizing $SU(N_c)$, supposing that it's valid to treat $SU(N_c')$ as a weakly gauged flavor symmetry spectator. We'll suppose that the original electric theory satisfies (III.24), so that both electric couplings RG flow to non-zero values. (If the second inequality (III.24) is violated, $SU(N_c')$ is IR free, and thus reasonably treated as a spectator. But if the first inequality in (III.24) is violated then $SU(N_c)$ is actually IR free, and the validity of dualizing it with $SU(N_c')$ treated as a spectator is questionable.) The $SU(N_c)$ group has N_f+N_c' flavors and its Seiberg [26] dual has $SU(\widetilde{N}_c)$ gauge group, with $\widetilde{N}_c \equiv N_f+N_c'-N_c$, with N_f+N_c' flavors of dual quarks and $(N_f+N_c')^2$ singlet mesons. The stability condition (III.23) ensures that $\widetilde{N}_c > 0$. Gauging $SU(N_c')_{mag}$, with the subscript as a reminder that its spectrum now differs

from that of (III.1), the dual is

$$\begin{array}{lccl} & \text{gauge group:} & SU(\widetilde{N}_c) \times SU(N'_c)_{mag} & \\ \text{matter:} & Y \oplus \widetilde{Y} & (\square, \overline{\square}) \oplus (\overline{\square}, \square), & \\ & q_f \oplus \widetilde{q}_{\tilde{f}} & (\square, \mathbf{1}) \oplus (\overline{\square}, \mathbf{1}) & (f, \tilde{f} = 1 \dots N_f), \\ & Q'_{f'} \oplus \widetilde{Q}'_{\tilde{f}'} & (\mathbf{1}, \square) \oplus (\mathbf{1}, \overline{\square}) & (f', \tilde{f}' = 1 \dots N'_f), \\ & F'_{n'} \sim X\widetilde{Q} \oplus c.c. & (\mathbf{1}, \square) \oplus (\mathbf{1}, \overline{\square}) & (n', \tilde{n}' = 1 \dots N_f), \\ & M_{f,\tilde{g}} \sim Q\widetilde{Q} & (\mathbf{1}, \mathbf{1}) & (f, \tilde{g} = 1 \dots N_f), \\ & \Phi \sim X\widetilde{X} & (\mathbf{1}, \mathbf{Adj}) \oplus (\mathbf{1}, \mathbf{1}), & \end{array} \tag{III.47}$$

with the superpotential of [26] yielding

$$W = Mq\tilde{q} + YF'\tilde{q} + \widetilde{Y}q\widetilde{F}' + \Phi Y\widetilde{Y}. \tag{III.48}$$

The one loop beta function coefficients of the electric theory (III.1) were

$$b_1 = 3N_c - N'_c - N_f, \quad \text{and} \quad b'_1 = 3N'_c - N_c - N'_f \tag{III.49}$$

(writing $b_1 > 0$ if asymptotically free), and those of the dual (III.47) are

$$b_1{}^{mag} = 2N_f + 2N'_c - 3N_c, \quad \text{and} \quad b'_1{}^{mag} = N'_c + N_c - 2N_f - N'_f. \tag{III.50}$$

Note that $b'_1{}^{mag}$ differs from b'_1, because the $SU(N'_c)_{mag}$ fields in (III.47) differ from those of the original $SU(N_c) \times SU(N'_c)$ theory (III.1); in fact, $b'_1 - b'_1{}^{mag} = 2(N_f + N'_c - N_c) = 2\widetilde{N}_c > 0$, so $SU(N'_c)_{mag}$ is always less asymptotically free than the electric $SU(N'_c)$ in the UV. Ignoring the $SU(N'_c)_{mag}$ dynamics, we'd conclude that the magnetic $SU(\widetilde{N}_c)$ is IR free if $N_f + N'_c < \frac{3}{2}N_c$; we'll discuss here how the $SU(N'_c)_{mag}$ dynamics can dramatically affect when the magnetic group is actually IR free.

The important quantities for the IR dynamics are the exact beta functions for the theory (III.47), which are

$$\beta_{g_{mag}} = -\frac{3g^3_{mag}f}{16\pi^2}\text{Tr } SU(\widetilde{N}_c)^2 R, \quad \beta_{g'_{mag}} = -\frac{3g'^3_{mag}f'}{16\pi^2}\text{Tr } SU(N'_c)^2_{mag} R, \tag{III.51}$$

where again f and f' are unimportant, positive, scheme dependent factors. The beta functions (III.51) can be written in the usual NSVZ form using (III.6), which gives

$$\begin{aligned} 3\mathrm{Tr}\ SU(\widetilde{N}_c)^2 R &= b_1^{mag} + N_c'\gamma_Y + N_f\gamma_q, \\ 3\mathrm{Tr}\ SU(N_c')_{mag}^2 R &= b_1'^{mag} + \widetilde{N}_c\gamma_Y + N_f'\gamma_{Q'} + N_f\gamma_{F'} + N_c'\gamma_\Phi. \end{aligned} \tag{III.52}$$

As with the electric theory, the dual (III.47) has three possible RG fixed points,

$$(\widetilde{A})\ g_{mag*} \neq 0,\ g'_{mag*} = 0, \quad \text{i.e. } SU(N_c')_{mag} \text{ free and } \mathrm{Tr}\, SU(\widetilde{N}_c)^2 R|_{\widetilde{A}} = 0, \tag{III.53}$$

$$(\widetilde{B})\ g_{mag*} = 0,\ g'_{mag*} \neq 0, \quad \text{i.e. } SU(\widetilde{N}_c) \text{ free and } \mathrm{Tr}\, SU(N_c')_{mag}^2 R|_{\widetilde{B}} = 0, \tag{III.54}$$

$$(\widetilde{C})\ g_{mag*} \neq 0, g'_{mag*} \neq 0 \quad \text{so } \mathrm{Tr}\, SU(\widetilde{N}_c)^2 R|_{\widetilde{C}} = \mathrm{Tr}\, SU(N_c')_{mag}^2 R|_{\widetilde{C}} = 0, \tag{III.55}$$

which are depicted in fig. III.8. RG fixed point $(\widetilde{A})$ is simply the Seiberg dual description of RG fixed point (A) of the original electric theory (with $SU(N_c')$ part of the global flavor symmetry). We expect that RG fixed point $(\widetilde{C})$, when it exists, is an equivalent, dual description of the SCFT at RG fixed point (C) of the original electric theory (III.1). The qualifier "when it exists" is because, as in the electric description, the RG flow may look like that of fig. III.2 rather than that of fig. III.1. In the electric description, the condition for the RG fixed point (C) to exist is (III.16). We will determine its analog in the magnetic theory (III.47), for $(\widetilde{C})$ to exist, by analyzing the IR stability of the RG fixed points $(\widetilde{A})$ and $(\widetilde{B})$ to small non-zero perturbations in the couplings that are set to zero in (III.53) and (III.54), in analogy with fig. III.5. We will find that, for a particular range of flavors and colors, the theory (III.47) with superpotential (III.48) realizes the RG flow depicted in fig. III.3: even if the one-loop beta function might suggest that $SU(\widetilde{N}_c)$ is IR free, it can be driven to be interacting by the interactions of the other gauge group and the superpotential.

Finally, we note that RG fixed point $(\widetilde{B})$ is *not* the dual of RG fixed point (B): as duality exchanges strong and weak coupling, the RG fixed point $(\widetilde{B})$, where

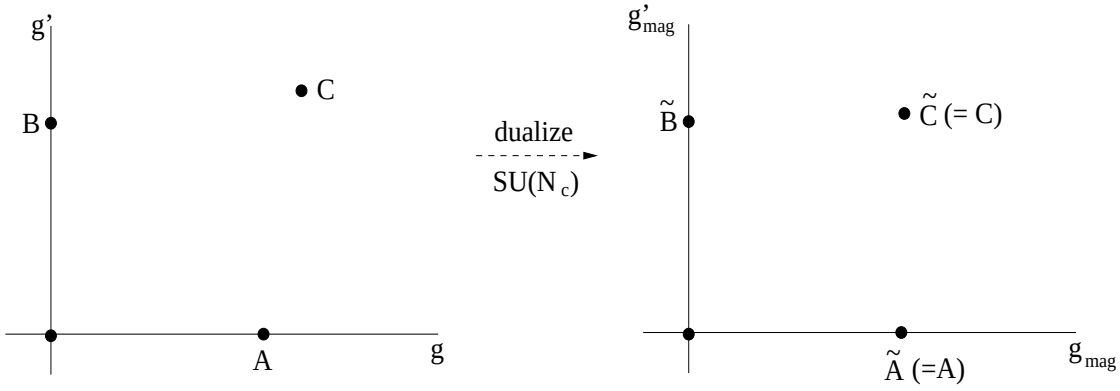

Figure III.8: The process of dualizing one group.

the magnetic $SU(\widetilde{N}_c)$ is free, corresponds to strongly coupled electric $SU(N_c)$. In cases where RG fixed point $(\widetilde{B})$ is IR stable, our interpretation is that the electric side appears to flow to interacting RG fixed point (C), but the magnetic dual reveals that the theory actually flows instead to having a free magnetic $SU(\widetilde{N}_c)$, at the RG fixed point $(\widetilde{B})$.

III.D.1 The RG fixed point $(\widetilde{A})$ and its IR stability to g'_{mag} perturbations.

RG fixed point $(\widetilde{A})$ is the Seiberg dual description of RG fixed point (A) of the original electric theory. All of the superconformal R-charges at $(\widetilde{A})$ are immediately computable from the dual matter content (III.47) and superpotential, or from the Seiberg duality map [26] and the superconformal R-charges at RG fixed point (A) in the electric description. Either way, the result is: $R(Y) = R(q) = N_c/(N_f + N'_c)$, $R(M) = R(F') = R(\Phi) = 2 - 2N_c/(N_f + N'_c)$, and $R(Q') = 2/3$. Using (III.51), we see that g'_{mag} is an IR relevant perturbation of $(\widetilde{A})$ if $\mathrm{Tr}\, SU(N'_c)^2_{mag} R|_{\widetilde{A}}$ is positive, or an IR irrelevant perturbation if it's negative. This 't Hooft anomaly is easily directly computed, or we can use the fact that 't Hooft anomalies match in Seiberg duality [26] (since $SU(N'_c)$ is a subgroup of the flavor group), so $\mathrm{Tr}\, SU(N'_c)^2_{mag} R|_{\widetilde{A}} = \mathrm{Tr}\, SU(N'_c)^2 R|_A$. The RG fixed point $(\widetilde{A})$ of the dual theory is thus IR stable if precisely the same inequality (III.14) found in the

electric description holds. So our first necessary condition for RG fixed point ($\widetilde{C}$) to exist, at least within the domain of attraction of the UV fixed point at zero couplings, is that the opposite inequality of (III.14) should hold,

$$(3N_c' - N_f')(N_c' + N_f) - 3N_c^2 > 0, \tag{III.56}$$

to have ($\widetilde{A}$) be IR repulsive. It is satisfying to see that the magnetic ($\widetilde{A}$) RG fixed point is IR repulsive precisely when the electric RG fixed point (A) is. It is hard to imagine how it could have been otherwise, given that the RG fixed points (A) and ($\widetilde{A}$) are identified.

III.D.2 The RG fixed point ($\widetilde{B}$) and its IR stability to g_{mag} perturbations.

This case is considerably more difficult than that of the previous subsection, as a-maximization is needed to determine the superconformal R-charges at RG fixed point ($\widetilde{B}$). Notice that, with the $SU(\widetilde{N}_c)$ gauge coupling set to zero at ($\widetilde{B}$), the $SU(N_c')_{mag}$ spectrum in (III.47) is the same as that analyzed in [16]: SQCD with an additional adjoint. But here the a-maximization analysis is further complicated by the presence of the superpotential in (III.47), which couples some of the $SU(N_c')_{mag}$ fundamentals Y to the adjoint Φ, and also to fundamentals F' and $SU(N_c')_{mag}$ singlets q. Rather than maximizing a_{trial} as a function of one variable, y, depending on one parameter, x, as in [16], we'll have here to maximize a_{trial} as function of two variables, $R(Q') \equiv u$ and $R(\Phi) \equiv v$, depending on the three parameters (x, x', n) of (III.18).

Let's consider the constraints on the superconformal $U(1)_R$ at ($\widetilde{B}$). Having $\beta_{g'_{mag}} = 0$ requires $\mathrm{Tr}\, SU(N_c')^2_{mag} R|_{\widetilde{B}} = 0$ (III.54):

$$N_c' + \widetilde{N}_c(R(Y)-1) + N_f'(R(Q')-1) + N_f(R(F')-1) + N_c'(R(\Phi)-1) = 0. \tag{III.57}$$

The superpotential terms (III.48) further impose

$$R(Y) + R(F') + R(q) = 2, \quad \text{and} \quad R(\Phi) + 2R(Y) = 2. \tag{III.58}$$

Note that the first term in the superpotential (III.48) is irrelevant for $g_{mag} = 0$, since none of its fields are charged under $SU(N_c')$, so M is a free field, with $R(M) = 2/3$. The constraints (III.57) and (III.58) are three constraints on five R-charges; they can be solved for

$$R(Y) = 1-\frac{1}{2}v, \quad R(q) = \frac{1}{2}(x+x')v+n(u-1), \quad R(F') = 1+n(1-u)+\frac{1}{2}(1-x-x')v, \tag{III.59}$$

with $R(Q') \equiv u$, $R(\Phi) \equiv v$, and (x, x', n) defined as in (III.18). a-maximization w.r.t. u and v is needed to determine the values of $u(x, x', n)$ and $v(x, x', n)$.

Once we've determined the superconformal R-charges at $(\widetilde{B})$, we can determine whether or not $(\widetilde{B})$ is stable to non-zero g_{mag} perturbations. We see from $\beta_{g_{mag}}$ in (III.51) that g_{mag} is a relevant perturbation of $(\widetilde{B})$ if $\mathrm{Tr}\, SU(\widetilde{N}_c)^2 R|_{\widetilde{B}} > 0$, i.e. if

$$3\widetilde{N}_c-N_f-N_c'+N_f\gamma_q+N_c'\gamma_Y > 0, \quad \text{i.e. if} \quad -N_c+N_fR(q)+N_c'R(Y) > 0. \tag{III.60}$$

This condition, together with (III.56), are the necessary conditions for RG fixed point $(\widetilde{C})$ to exist (at least within the domain of attraction of zero couplings). If the inequality in (III.60) is not satisfied, RG fixed point $(\widetilde{B})$ is IR attractive, and then we expect RG flows from generic values of the couplings to end up there in the IR. So if (III.60) is not satisfied, the original electric theory (III.1) flows to having a free magnetic $SU(\widetilde{N}_c)$ in the IR.

The condition (III.60), for $SU(\widetilde{N}_c)$ to not be free magnetic in the IR, is generally very different from the naive criterion, $N_f + N_c' > \frac{3}{2}N_c$, based on when $SU(\widetilde{N}_c)$ is asymptotically free in the UV. The difference is that (III.60) accounts for the $SU(N_c')_{mag}$ dynamics. If the numbers of flavors and colors are chosen such that the $SU(N_c')_{mag}$ matter spectrum is just barely asymptotically free (i.e. $b_1'^{mag}$ in (III.50) is small and positive), then the RG fixed point coupling g'_{mag} at $(\widetilde{B})$ is small. In this case, the the $SU(N_c')_{mag}$ dynamics doesn't much affect the running of the $SU(\widetilde{N}_c)$ coupling g_{mag}. In particular, when $(\widetilde{B})$ is at weak coupling, the a-maximization results properly give $R(q) \approx 2/3$ and $R(Y) \approx 2/3$, since these

fields are approximately free. We then find that (III.60) gives approximately the standard condition from Seiberg duality [26] for the magnetic dual $SU(\widetilde{N}_c)$ to be interacting rather than IR free, $N_f + N'_c > \frac{3}{2}N_c$, which is the condition that g_{mag} be an IR relevant perturbation of free theory at $g_{mag} = g'_{mag} = 0$.

On the other hand, when the number of flavors and colors are such that $SU(N'_c)_{mag}$ is very much asymptotically free, i.e. ${b'_1}^{mag}$ in (III.50) is positive and large, the RG fixed point $(\widetilde{B})$ is at strong $SU(N'_c)_{mag}$ coupling. In this case, the $SU(N'_c)_{mag}$ dynamics can radically affect whether or not the $SU(\widetilde{N}_c)$ coupling g_{mag} is relevant. Indeed, depending on the values of (x, x', n), this theory can realize the flow of fig. III.3: even if $N_f + N'_c \leq \frac{3}{2}N_c$, so $SU(\widetilde{N}_c)$ is IR free around $g_{mag} = g'_{mag} = 0$, the condition (III.60) for $SU(\widetilde{N}_c)$ to be an IR interacting deformation of $(\widetilde{B})$ can nevertheless be satisfied. In short, the $SU(N'_c)_{mag}$ has driven an otherwise IR free $SU(\widetilde{N}_c)$ theory to instead be IR interacting. We'll focus on this phenomenon of fig. III.3 in the rest of this subsection.

To have (III.60) be satisfied when $b_1^{mag} < 0$ requires that q and/or Y have *positive* anomalous dimension, i.e. R-charge greater than 2/3, at the RG fixed point (B). Positive anomalous dimensions are possible provided that there is a W_{tree} superpotential, as in the case here (III.48): gauge interactions make negative contributions to the anomalous dimensions, and superpotential interactions make positive contributions. As we'll now discuss, there is indeed a range of parameter space of flavors and colors for the magnetic theory (III.47) where the anomalous dimensions are sufficiently positive so as to have (III.60) satisfied, despite having $b_1^{mag} < 0$, realizing the effect of fig. III.3. (See also the example [21].)

Though the a-maximization analysis at RG fixed point $(\widetilde{B})$ is standard, it's computationally intensive. Using the spectrum (III.47) and (III.59), we compute the combination of 't Hooft anomalies $a^{(0)} = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$, as a function of the two variables, (u, v), and the parameters (x, x', n). Depending on (x, x', n), we have to also add the additional contributions (III.21) for any gauge invariant operators with $R \leq 2/3$. (The operators hitting the unitarity bound are found to be

$Q'\Phi^{j-1}\widetilde{Q}'$, $Q'\Phi^{j-1}\widetilde{F}'$, and $F'\Phi^{j-1}\widetilde{F}'$ for values of $j = 1\ldots$ increasing with x and x', as the RG fixed point ($\widetilde{B}$) becomes more and more strongly coupled.) Again, we implemented this a-maximization analysis numerically, using Mathematica. While the numerics are similar in spirit to our previous cases, the fact that here we're maximizing a function of two, rather than one, variables, as a function of the three parameters, greatly prolongs the required computational timescale.

Let's focus on an interesting range of parameter space, where we take the parameters x and x' to be large. This is an interesting region of parameter space because then RG fixed point ($\widetilde{B}$) is at very strong $SU(N'_c)_{mag}$ gauge coupling (as seen from the fact that the one-loop beta function is very large). For large x and x', there are terms quartic in x in $a^{(0)} = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$, coming from the $O(x)$ terms in $R(q)$ and $R(F')$ in (III.59). Note, however, that the $O(x)$ terms in (III.59) all appear multiplied by v, so the leading terms for large x transform homogeneously, with degree one, under $x \to \lambda x$ and $v \to \lambda^{-1}v$, e.g. the quartic term in $a^{(0)}$ is $\sim x^4v^3$. When we include the contributions from the meson hitting the unitarity bound, we find that they also have a leading large x term which is degree one under this scaling. Because of this homogeneity, when we maximize w.r.t. v for large x, we find $v \sim 1/x$, and the value of a at its maximum is linear in x, since it's degree one in the above rescaling. (The fact that the central charge of the SCFT grows linearly in x is a check of the a-theorem conjecture, since there's a Higgsing RG flow analogous to that of [16] which would violate the a-theorem with any higher degree.) To study the limit of large x, we can thus scale $\lambda \to \infty$, keeping only the terms of degree one in λ.

In this scaling limit, $R(\Phi) \equiv v \to \lambda^{-1}v \to 0$ for $\lambda \to \infty$. Then (III.59) gives $R(Y) \to 1$, while $R(Q') \equiv u$, $R(q)$, and $R(F')$ asymptote to some finite values that are determined by a-maximization. The a-maximization answer for these quantities, in our limit of large x and x', can also be obtained by borrowing the a-maximization results of [16]. The idea is that the $SU(N'_c)_{mag}$ gauge coupling at ($\widetilde{B}$) is very strong for large x and x', and the $SU(N'_c)_{mag}$ matter content that it

couples to coincides with that of [16], and adjoint and fundamentals. The theory at $(\widetilde{B})$ differs from that of [16] only because of the superpotential interactions (III.48). If not for the superpotential interactions, the a-maximization result of [16] would tell us that $R(Y)$, $R(Q')$ and $R(F')$ all asymptote to $(\sqrt{3}-1)/3 \approx 0.244$ for $x \to \infty$. The superpotential non-negligibly affects $R(Y)$: the $\Phi Y \widetilde{Y}$ term requires that $R(Y) \to 1$, since $R(\Phi) \to 0$ for large x. But the superpotential negligibly affects $R(Q')$ and $R(F')$ in the strong $SU(N'_c)_{mag}$ coupling, large x limit. For example, though there is a term $YF'\widetilde{q}$ in the superpotential, its effect is to determine the R-charge of the otherwise free field q, leaving $R(Y) \to 1$ and $R(F') \to (\sqrt{3}-1)/3 \approx 0.244$ unaffected. So in this large x limit we obtain (and the detailed a-maximization analysis bears this out):

$$R(Y) \to 1, \quad \text{and} \quad R(q) \to 1 - \left(\frac{\sqrt{3}-1}{3}\right) \approx 0.756, \quad \text{for large } x \text{ and } x'. \tag{III.61}$$

The condition (III.60) for g_{mag} to be relevant at $\widetilde{B}$ is then (recalling (III.18))

$$N'_c - N_c + N_f(0.756) > 0, \quad \text{i.e.} \quad x' - x + 0.756 > 0, \quad \text{for large } x \text{ and } x'. \tag{III.62}$$

This is very different from the condition that g_{mag} be asymptotically free for $g'_{mag} = 0$, $b_1^{mag} > 0$, i.e. $1 + x' - \frac{3}{2}x > 0$, and III.62) can be satisfied even when $b_1^{mag} < 0$, i.e. we can have

$$1 + x' - \frac{3}{2}x < 0 \quad \text{but nevertheless} \quad x' - x + 0.756 > 0; \tag{III.63}$$

for example, we can take $x \approx x' \to \infty$. For values of (x, x', n) such that the inequalities in (III.63) both hold, the RG flow is as in fig. III.3: if $SU(N'_c)_{mag}$'s coupling were set to exactly zero, then $SU(\widetilde{N}_c)$ would be IR free, but any non-zero $SU(N'_c)_{mag}$ coupling would eventually drive $SU(\widetilde{N}_c)$ to be instead interacting in the IR.

III.D.3 The RG fixed point $(\widetilde{C})$

RG fixed point $(\widetilde{C})$ exists if $(\widetilde{A})$ and $(\widetilde{B})$ are both IR unstable to perturbations in the other coupling; we found this to be the case if (III.56) holds, and if, say for large x and x', (III.62) holds, respectively. When $(\widetilde{C})$ exists, we expect that it's an equivalent, dual description of the RG fixed point (C) of the original electric theory. We'll here check that the superconformal R-charges are compatible with this identification.

At $(\widetilde{C})$, the six independent superconformal R-charges, on the six lines of (III.47), are subject to five constraints: for vanishing $\beta_{g_{mag}}$ and $\beta_{g'_{mag}}$, we require (III.55), $\mathrm{Tr}\, SU(\widetilde{N}_c)^2 R|_{\widetilde{C}} = \mathrm{Tr}\, SU(N'_c)^2_{mag} R|_{\widetilde{C}} = 0$, along with three more constraints from requiring that the superpotential terms (III.48) all have total $R(W) = 2$. (All terms in (III.48) are relevant deformations of the $W = 0$ theory when g_{mag*} and g'_{mag*} are both non-zero.) There is thus a one-variable family of R-charges, as for the electric RG fixed point (C). These constraints are compatible with the duality map identification of the fields F', M, and Φ in (III.47): the R-charges of the dual theory (III.47) can be related to those of the original electric theory (III.1), with $R(Q) \equiv y$ as before, by

$$R(M) = 2y,\ R(F') = R(X) + y,\ R(\Phi) = 2R(X),\ R(q) = 1 - y,\ R(Y) = 1 - R(X), \tag{III.64}$$

with $R(X)$ and $R(Q')$ given by (III.26) in terms of the variable y and parameters (x, x', n).

We compute the same function $a^{(0)}_{trial} = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$ to maximize w.r.t. y as in the electric theory (III.27), as expected from the 't Hooft anomaly matching for the global flavor symmetries in Seiberg duality [26]. Compatible with our claim that the electric RG fixed point (C) is equivalent to the dual one $(\widetilde{C})$, there is a one-to-one mapping of the operators that have hit the unitarity bound. Corresponding

to the operators (III.28) we have

$$M_{j=1} \leftrightarrow M,\ M_{j>1} \leftrightarrow \widetilde{F}\Phi^{j-2}F,\ M'_j \leftrightarrow Q'\Phi^{j-1}\widetilde{Q}', P_j \leftrightarrow F'\Phi^{j-1}\widetilde{Q}',\ \widetilde{P}_j \leftrightarrow \widetilde{F}'\Phi^{j-1}Q'. \tag{III.65}$$

So, even including the contributions of the operators hitting their unitarity bound, we find the same a_{trial} function of y and (x, x', n) to maximize w.r.t. y, and hence the same superconformal R-charges are given by (III.64) with $y(x, x', n)$ the same superconformal R-charge as obtained by analyzing the electric theory (III.1).

III.E Conclusions and Comments

A general potential pitfall in applying a-maximization is that one must really have the full symmetry group under control, including all accidental symmetries, to obtain correct results. Overlooking some symmetries will lead to a value of the central charge a_{SCFT} that is too low. Seiberg duality [26] shows that there can be highly non-obvious accidental symmetries, such as those acting on the free magnetic $SU(N_f - N_c)$ quarks and gluons when $N_f < \frac{3}{2}N_c$. More generally, without knowing the dual, we do not presently have a way to look for such accidental symmetries, which do not act on any of the "obvious" gauge invariant operators of the theory.

Ignoring the interplay of the two gauge couplings, the superconformal window of [26] for each gauge group in (III.1) separately is

$$\frac{3}{2}N_c < N_f + N'_c < 3N_c, \qquad \frac{3}{2}N'_c < N'_f + N_c < 3N'_c. \tag{III.66}$$

These are the conditions for points (A) and (B) to be interacting SCFTs, respectively. The upper limits are needed for the electric coupling to not be driven to zero in the IR, and the lower limits are for the couplings of the dual [26] to not be driven to zero in the IR.

Accounting for the g and g' interplay, the conditions for point (C) to exist as a fully interacting SCFT differ from (III.66). The upper limits of (III.66) should

be replaced with the conditions (III.16), for neither electric gauge coupling to be driven to zero in the IR. Similarly, the duality of sect. III.D (assuming its validity) shows how the lower limits of (III.66) are modified, in order for neither magnetic gauge coupling to be driven to zero in the IR.

For example, taking $x \equiv N_c/N_f$ and $x' \equiv N'_c/N_f$ large, we found in sect. III.D that $(\widetilde{B})$, with $g_{mag} \to 0$, is IR attractive if

$$N'_c - N_c + (0.756)N_f < 0, \quad \text{i.e. if} \quad x' - x + 0.756 < 0. \tag{III.67}$$

In this case, rather than flowing to the fully interacting RG fixed point (C), the theory flows to the free magnetic point $(\widetilde{B})$ in the IR, where the original electric $SU(N_c)$ is very strongly coupled, but its $SU(N_f + N'_c - N_c)$ magnetic dual is IR free. There is then a large, non-obvious, accidental symmetry of the original electric theory when (III.67) holds. Likewise, dualizing the $SU(N'_c)$ factor of (III.1), we find for large x and x' that the apparent RG fixed point (C) of the electric theory instead flows to having a free magnetic $SU(N'_f + N_c - N'_c)$ group when

$$N_c - N'_c + (0.756)N'_f < 0, \quad \text{i.e. if} \quad x - x' + (0.756)n < 0. \tag{III.68}$$

So, for RG fixed point (C) to be fully interacting, rather than partially free magnetic, the lower limits in (III.66) are replaced, for large x and x', with the conditions

$$-(0.756)n < x - x' < 0.756. \tag{III.69}$$

The range (III.69) is a subset of the stability range (III.23). Outside of the range (III.69), there are non-obvious accidental symmetries. Within the range (III.69), we have no evidence for non-obvious accidental symmetries. If there had been any such non-obvious accidental symmetries, our a-maximization analysis of sects. III.B and III.C would have to be appropriately modified. In particular, in our parameter slice of special interest in sects. III.B and III.C, $x = x'$, i.e. $N_c = N'_c$, the magnetic duals remain fully interacting.

As we noted, for $x = x'$ and $n = 1$, the a-maximization analysis of our product group example (III.1) coincides with that in [16] for $SU(N_c)$ with an

adjoint and N_f fundamentals. In the analysis of [16] of that latter theory, it was assumed that the only accidental symmetries are the obvious ones, associated with gauge invariant operators hitting the unitarity bound. But, as in the example of [21], there's a possibility of a non-obvious accidental symmetry, associated with a free-magnetic gauge group in a deconfining dual. The idea of the deconfining dual [51] is that the dual (III.33) of our theory (III.1) would look quite a lot like $SU(N_c')$ SQCD with an adjoint if we chose the flavors and colors such that $\widetilde{N}_c = 1$ in (III.47). And a slight modification of the theory (III.1), with added fields and superpotential terms (designed to eliminate the analog of (III.48) in the dual), will lead to precisely SQCD with an adjoint and fundamentals, with no superpotental; see table 8 of [52] for the needed field content. It would be interesting to carry out the a-maximization analysis of that theory, and its duals, to determine whether or not any of the gauge groups of the deconfining duals can become IR free.

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IV

The Exact Superconformal R-symmetry Minimizes τ_{RR}

IV.A Introduction

Our interest here will be in the coefficients τ_{IJ} of two-point functions of globally conserved currents J_I^μ (I labels the various currents) in d-dimensional CFTs:

$$\langle J_I^\mu(x) J_J^\nu(y)\rangle = \frac{\tau_{IJ}}{(2\pi)^d}(\partial^2\delta^{\mu\nu} - \partial^\mu\partial^\nu)\frac{1}{(x-y)^{2(d-2)}}. \tag{IV.1}$$

The general form (IV.1) of the correlator is completely fixed by conformal invariance, with the specific dynamics of the theory entering only in the coefficients τ_{IJ}. Unitarity restricts τ_{IJ} to be a positive matrix (positive eigenvalues). For 4d CFTs, τ_{IJ} give [54], [55] the violation of scale invariance, $\langle T^\mu_\mu\rangle = \frac{1}{4}\tau_{IJ}(F^I)_{\mu\nu}(F^J)^{\mu\nu}$, when the global currents are coupled to background gauge fields.

We'll here consider field theories with four supercharges: $\mathcal{N}=1$ in 4d, and $\mathcal{N}=2$ in 3d (one could also consider $\mathcal{N}=(2,2)$ in 2d), and their renormalization group fixed point SCFTs (where there are an additional four superconformal supercharges). The stress tensor of these theories lives in a supermultiplet $T_{\alpha\dot\beta}(x,\theta,\overline\theta)$ (in 4d Lorentz spinor notation; for $d<4$ the dot on $\dot\beta$ is unnecessary), which also contains a $U(1)_R$ current – this is "the superconformal $U(1)_R$ sym-

metry". Supersymmetry relates this current and its divergence to the dilitation current and its divergence. The scaling dimension of chiral operators are related to their superconformal $U(1)_R$ charge by

$$\Delta = \frac{d-1}{2} R. \tag{IV.2}$$

For a chiral superfield, writing $\Delta = \frac{1}{2}d - 1 + \frac{1}{2}\gamma$, with γ the anomalous dimension, (IV.2) yields

$$R = \frac{d-2}{d-1} + \frac{1}{d-1}\gamma. \tag{IV.3}$$

There are often additional non-R flavor currents, whose charges we'll write as F_i, with i labeling the flavor symmetries. In superspace, these currents reside in a different kind of supermultiplet, which we'll write as $J_i(x, \theta, \overline{\theta})$. When there are such additional flavor symmetries, the superconformal $U(1)_R$ of RG fixed point SCFTs can not be determined by the symmetries alone, as the R-symmetry can mix with the flavor symmetries. Some additional dynamical information is then needed to determine precisely which, among all possible R-symmetries, is the superconformal one, in the $T_{\alpha\dot{\beta}}$ supermultiplet.

We will here present a new condition that, in principle, completely determines which is the superconformal $U(1)_R$. We write the most general possible trial R-symmetry as

$$R_t = R_0 + \sum_i s_i F_i, \tag{IV.4}$$

where R_0 is any initial R-symmetry, and F_i are the non-R flavor symmetries. The subscript "t" is for "trial", with the s_i arbitrary real parameters. The superconformal R-symmetry, which we'll write as R without the subscript, corresponds to some special values s_i^* of the coefficients in (IV.4), that we'd like to determine, $R = R_t|_{s_j = s_j^*}$.

As we'll discuss, the fact that the superconformal R-symmetry and the non-R flavor symmetries reside in different kinds of supermultiplets, implies that their current-current two-point function necessarily vanishes, $\langle J_R^\mu(x) J_{F_i}^\nu(y)\rangle = 0$,

i.e.

$$\tau_{Ri} = 0 \qquad \text{for all non-R symmetries } F_i. \tag{IV.5}$$

This condition uniquely characterizes the superconformal R-symmetry among all possibilities (IV.4). To see this, use (IV.4) to write (IV.5) as

$$0 = \tau_{Ri} = \tau_{R_t i}|_{s_j = s_j^*} = \tau_{R_0 i} + \sum_j s_j^* \tau_{ij} \quad \text{for all i.} \tag{IV.6}$$

Here $\tau_{R_0 i}$ is the coefficient of the $\langle J^\mu_{R_0}(x) J^\nu_{F_i}(y)\rangle$ current-current two-point function of the currents for R_0 and F_i, and τ_{ij} is the coefficient of the $\langle J^\mu_{F_i}(x) J^\nu_{F_j}(y)\rangle$ of the current-current two-point function for the non-R flavor symmetries F_i and F_j. The conditions (IV.6) is a set of linear equations which uniquely determines the s_j^*, if the coefficients $\tau_{R_0 i}$ and τ_{ij} are known. Unitarity implies that the matrix τ_{ij} is necessarily positive, with non-zero eigenvalues, so it can be inverted, and the solution of (IV.6) is

$$s_j^* = -\sum_i (\tau^{-1})_{ij} \tau_{R_0 i}. \tag{IV.7}$$

The conditions (IV.6) can be phrased as a minimization principle: *the exact superconformal R-symmetry is that which minimizes the coefficient $\tau_{R_t R_t}$ of its two-point function among all trial possibilities (IV.4).* Using (IV.4), the coefficient of the trial R-current R_t two-point function is a quadratic function of the parameters s_j:

$$\tau_{R_t R_t}(s) = \tau_{R_0 R_0} + 2\sum_i s_i \tau_{R_0 i} + \sum_{ij} s_i s_j \tau_{ij}. \tag{IV.8}$$

Our result (IV.5) implies that the exact superconformal R-symmetry extremizes this function,

$$\frac{\partial}{\partial s_i} \tau_{R_t R_t}(s)|_{s_j = s_j^*} = 2\tau_{Ri} = 0. \tag{IV.9}$$

The unique solution of (IV.9) is a global minimum of the function (IV.8) since

$$\frac{\partial^2}{\partial s_i \partial s_j} \tau(s) = 2\tau_{ij} > 0, \tag{IV.10}$$

with the last inequality following from unitarity.

The value of $\tau_{R_t R_t}$ at its unique minimum is the coefficient τ_{RR} of the superconformal R-current two-point function. As is well known, supersymmetry relates this to the coefficient, "c", of the stress tensor two-point function, $\tau_{RR} \propto c$; as we'll discuss, the proportionality factor is

$$\tau_{RR} = \frac{(2\pi)^d}{d(d^2-1)(d-2)} C_T \qquad \text{or, for } d=4, \quad \tau_{RR} = \frac{16}{3} c. \tag{IV.11}$$

τ_{RR} minimization immediately implies some expected results. For non-Abelian flavor symmetry, (IV.5) is automatically satisfied for all flavor currents with traceless generators, if the superconformal R-symmetry is taken to commute with these generators. This shows, as expected, that the superconformal R-symmetry does not mix with such non-Abelian flavor symmetries. Similarly, (IV.5) is automatically satisfied by any baryonic flavor currents which are odd under a charge conjugation symmetry, taking the superconformal $U(1)_R$ to be even under charge conjugation. So, as expected, the superconformal $U(1)_R$ does not mix with baryonic symmetries which are odd under a charge conjugation symmetry.

As a simple example of τ_{RR} minimization, consider a single, free, chiral superfield Φ in d spacetime dimensions. The R-symmetry can mix with a non-R $U(1)_F$ flavor current, under which Φ has charge 1 (the "Konishi current"). Write the general trial R-charges for the scalar and fermion components as $R(\phi) = R_t$, $R(\psi) = R_t - 1$. As we'll review, the free field two-point function of this R-current is

$$\tau_{R_t R_t} = \frac{\Gamma(\frac{d}{2})^2 2^{d-2}}{(d-1)(d-2)} \left(\frac{1}{d-2} R_t^2 + (R_t - 1)^2 \right) \tag{IV.12}$$

with the two terms the scalar and fermion contributions. Taking the derivative w.r.t. R_t,

$$\tau_{R_t F} = \frac{1}{2} \frac{d}{dR_t} \tau_{R_t R_t} = \frac{\Gamma(\frac{d}{2})^2 2^{d-2}}{(d-1)(d-2)} \left(\frac{R_t}{d-2} + R_t - 1 \right). \tag{IV.13}$$

Requiring $\tau_{RF} = 0$ then gives the correct result (IV.3), with anomalous dimension $\gamma = 0$, for a free chiral superfield in d spacetime dimensions.

The above considerations all apply independent of space-time dimension; they are equally applicable for 4d $\mathcal{N} = 1$ SCFTs as with 3d $\mathcal{N} = 2$ SCFTs.

For 4d $\mathcal{N} = 1$ SCFTs, there is already a known method for determining the superconformal R-symmetry: a-maximization [14]. It was shown in [14] that the s_i^* can be determined by a-maximization, maximizing w.r.t. the s_i in (IV.4) the combination of 't Hooft anomalies

$$a_{trial}(R_t) = \frac{3}{32}(3\mathrm{Tr}R_t^3 - \mathrm{Tr}R_t), \tag{IV.14}$$

(where we decided here to include the conventional normalization prefactor). For example, for a free 4d chiral superfield we locally maximize the function

$$a_{trial}(R_t) = \frac{3}{32}(3(R_t - 1)^3 - (R_t - 1)). \tag{IV.15}$$

The local maximum of (IV.15) is at $R = 2/3$, which indeed coincides with the global minimum of (IV.12), but it's illustrative to see how the functions themselves differ.

a-maximization in 4d is much more powerful than $\tau_{R_tR_t}$ minimization, because one can use the power of 't Hooft anomaly matching to exactly compute $a_{trial}(R_t)$ (IV.14), whereas the current two-point functions τ_{R_0i} and τ_{ij} needed for $\tau_{R_tR_t}$ minimization receive quantum corrections. Actually, once the exact superconformal R-symmetry is known, there is a nice way to evaluate τ_{ij} in terms of 't Hooft anomalies [13]:

$$\tau_{ij} = -3\mathrm{Tr}RF_iF_j, \tag{IV.16}$$

as we'll review in what follows. (The result (IV.16) generally can not be turned around, and used as a way to determine the superconformal $U(1)_R$, because plugging (IV.4) in (IV.16) can not always be inverted to solve for the s^*.)

In the context of the AdS/CFT correspondence, the criterion (IV.6) for determining the superconformal R-symmetry becomes more useful and tractable, because the AdS duality gives a weakly coupled dual description of τ_{R_0i} and τ_{ij}: these quantities become the coefficients of gauge field kinetic terms in the AdS bulk [56]. As we'll discuss in a separate paper [57], these coefficients are computable by reducing SUGRA on the corresponding Sasaki-Einstein space. We'll show in [57] that the conditions (IV.6) are in fact equivalent to the "geometric dual of a-maximization" of Martelli, Sparks, and Yau [58].

There is no known analog of a-maximization for 3d $\mathcal{N} = 1$ SCFTs, and in 3d there is no useful analog of 't Hooft anomalies and matching (aside from a Z_2 parity anomaly matching [59]). $\tau_{R_t R_t}$ minimization gives an alternative to a-maximization in 4d, which applies equally well to 3d $\mathcal{N} = 2$ SCFTs.

a-maximization in 4d ties the problem of finding the superconformal $U(1)_R$ together with Cardy's conjecture [3], that the conformal anomaly a counts the degrees of freedom of a quantum field theory, with $a_{UV} > a_{IR}$ and $a_{CFT} > 0$. The result that a is maximized over its possibilities implies that relevant deformations decrease a [14], in agreement with Cardy's conjecture. Unfortunately, we have not gained any new insights here into general RG inequalities from our τ_{RR} minimization result. Indeed, τ_{RR} is related to the conformal anomaly c in 4d, which is known to not have any general behavior, neither generally increasing nor generally decreasing, in RG flows to the IR. And there is no analogous argument to that of [14], to conclude that τ_{RR} generally increases in RG flows in the IR, from the fact that τ_{RR} is minimized among all possibilities: the quantum corrections to τ_{RR}, coming from the relevant interactions, can generally have either sign. (The difference is that the argument of [14] was based on 't Hooft anomalies, which do not get any quantum corrections for conserved currents).

Our τ_{RR} minimization result applies for SCFTs at their RG fixed point. It would be interesting to extend τ_{RR} minimization to study RG flows away from the RG fixed point. Perhaps this can be done by using Lagrange multipliers, as in [20], to impose the constraint that one minimize only over currents that are conserved by the relevant interactions.

IV.B Current two point functions; free fields and normalization conventions

Two point functions of currents and stress tensors for free bosons and fermions in d-spacetime dimensions were worked out, e.g. in [60]. To compare

with [60], rewrite (IV.1) as

$$\langle J_I^\mu(x) J_J^\nu(y)\rangle = \tau_{IJ} \frac{2(d-1)(d-2)}{(2\pi)^d} \frac{I_{\mu\nu}(x-y)}{(x-y)^{2(d-1)}}, \tag{IV.17}$$

with $I_{\mu\nu}(x) \equiv \delta_{\mu\nu} - 2x_\mu x_\nu (x^2)^{-1}$. The normalization conventions of [60] is

$$\langle J_\mu(x) J_\nu(0)\rangle = \frac{C_V}{x^{2(d-1)}} I_{\mu\nu}(x), \qquad \langle T_{\mu\nu}(x) T_{\rho\sigma}(0)\rangle = \frac{C_T}{x^{2d}} I_{\mu\nu,\sigma\rho}(x), \tag{IV.18}$$

with $I_{\mu\nu,\sigma\rho}(x) = \frac{1}{2}(I_{\mu\sigma}(x) I_{\nu\rho}(x) + I_{\mu\rho}(x) I_{\nu\sigma}(x)) - d^{-1}\delta_{\mu\nu}\delta_{\sigma\rho}$. Thus $C_V = 2\tau(d-1)(d-2)/(2\pi)^d$. With these normalizations, the coefficients (IV.18) for a single complex scalar are

$$C_V = \frac{2}{d-2}\frac{1}{S_d^2}, \qquad C_T = \frac{2d}{d-1}\frac{1}{S_d^2}, \tag{IV.19}$$

where $S_d \equiv 2\pi^{\frac{1}{2}d}/\Gamma(\frac{1}{2}d)$ and the current was normalized to give ϕ and ϕ^* charges ± 1. The coefficients for a free fermion having the same number of components as a 4d complex chiral fermion (half the components of a Dirac fermion) the coefficients are

$$C_V = 2\frac{1}{S_d^2}, \qquad C_T = d\frac{1}{S_d^2} \tag{IV.20}$$

(we don't have the factors of $2^{d/2}$ of [60], because we're here considering a fermion with the same number of components as the dimensional reduction of a 4d chiral fermion for all d).

More generally, let current $J_I(x)$ give charges $q_{I,b}$ to the complex bosons and charges $q_{I,f}$ to the chiral fermions. Using (IV.19) and (IV.20), we have

$$\tau_{IJ}^{\text{free field}} = \frac{\Gamma(\frac{d}{2})^2 2^{d-2}}{(d-1)(d-2)} \left(\frac{1}{d-2} \sum_{\text{bosons } b} q_{I,b} q_{J,b} + \sum_{\text{fermions } f} q_{I,f} q_{J,f} \right). \tag{IV.21}$$

In particular, for a $U(1)_R$ symmetry, this gives (IV.12). For $d = 4$, $\Gamma(d/2)^2 2^{d-2}/(d-1)(d-2) = 2/3$, so e.g. a 4d $U(1)_F$ non-R symmetry which assigns charge q to a single chiral superfield has $\tau_{FF}^{free field} = q^2$.

IV.C Supersymmetric field theories

Supersymmetry relates the superconformal R-symmetry to the stress tensor: both reside in the supercurrent supermultiplet

$$T_{\alpha\dot\alpha}(x,\theta,\overline\theta) \sim J_{R,\alpha\dot\alpha}(x) + S_{\alpha\dot\alpha\beta}(x)\theta^\beta + \overline S_{\alpha\dot\alpha\dot\beta}(x)\overline\theta^{\dot\beta} + T_{\alpha\dot\alpha\beta\dot\beta}(x)\theta^\beta\overline\theta\dot\beta + \ldots, \quad \text{(IV.22)}$$

whose first component is the superconforal $U(1)_R$ current and whose $\theta\overline\theta$ component is the stress energy tensor (we're omitting numerical coefficients here). Our notation is for the 4d case; similar results hold for 3d $\mathcal{N}=2$ theories, with $\overline\theta^{\dot\alpha}$ replaced with a second flavor of θ^α. For superconformal theories, the stress tensor is traceless, and the superconformal R-current is conserved. As discussed in [29], the supercurrent two-point function is then of a completely determined form, with the only dependence on the theory contained in a single overall coefficient C:

$$\langle T_{\alpha\dot\alpha}(z_1) T_{\beta\dot\beta}(z_2)\rangle = C\frac{(x_{1\overline 2})_{\alpha\dot\beta}(x_{2\overline 1})_{\beta\dot\alpha}}{(x^2_{\overline 2 1}x^2_{\overline 1 2})^{d/2}}; \quad \text{(IV.23)}$$

see [29] for an explanation of the superspace notation in (IV.23).

Expanding out (IV.23) in superspace, the LHS includes both the R-current two-point function and the stress-tensor two-point function. So (IV.23) shows that the coefficient $C \propto \tau_{RR}$, and also $C \propto C_T$, and so it follows that $\tau_{RR} \propto C_T$. We could determine the precise coefficients in these relations by being careful with the coefficients in (IV.22) and in expanding both sides of (IV.23); instead we will fix these universal proportionality factors by considering the particular example of a free chiral superfield. Using (IV.19) and (IV.20) to get C_T, and comparing with the free-field value of τ_{RR} computed from (IV.21), gives the general proportionality factor that we quoted in (IV.11; e.g. for $d=3$ it's $\tau_{RR} = \pi^3 C_T/3$. In 4d, $C_T \propto c$, one of the conformal anomaly coefficients, and the proportionality can again be fixed by considering the case of a free 4d $\mathcal{N}=1$ chiral superfield, for which $c=1/24$ and (IV.21) gives $\tau_{RR}=2/9$ (or a free 4d $\mathcal{N}=1$ vector superfield, for which $c=1/8$ and (IV.21) gives $\tau_{RR}=2/3$); this gives the relation quoted in (IV.11).

The non-R global flavor currents $J_i^\mu(x)$ are the $\theta^\alpha\overline{\theta}^{\dot{\alpha}}$ components of superfields $J_i(x,\theta,\overline{\theta})$, whose first component is a scalar. We can write their two-point functions in superspace [29], with the coefficients given by that of the flavor current correlators, τ_{ij}:

$$\langle J_i(z_1)J_j(z_2)\rangle = \frac{\tau_{ij}}{(2\pi)^d}\frac{1}{(x_{\overline{2}1}^2 x_{\overline{1}2}^2)^{(d-2)/2}}. \tag{IV.24}$$

In general d dimensional CFTs, two-point functions of primary operators vanish unless the operators have conjugate Lorentz spin and the same operator dimension. Noting that the first component of the supermultiplet (IV.22) has dimension $\Delta(T_{\alpha\dot{\beta}}) = d-1$, and the first component of the current J_i has dimension $\Delta(J_i) = d-2$ (since the $\theta^\alpha\overline{\theta}^{\dot{\alpha}}$ component is the current, with dimension $d-1$), the two-point function of the first components of these two different supermultiplets must vanish. Because there is no non-trivial nilpotent invariant for two-point functions [29] , this implies that two-point function of the entire supermultiplets must vanish:

$$\langle T_{\alpha\dot{\alpha}}(z_1)J_i(z_2)\rangle = 0. \tag{IV.25}$$

I.e. the two-point function of any operator in the $T_{\alpha\dot{\alpha}}$ supermultiplet and any operator in the J_i supermultiplet vanishes; in particular, this implies that the two-point function of the superconformal $U(1)_R$ current and all non-R flavor currents necessarily vanish, $\tau_{RF_i} = 0$. We thus have the general result (IV.5), and this same argument applies equally for $d = 4$ $\mathcal{N} = 1$ as well as lower dimensional SCFTs with the same number of supersymmetries.

IV.C.1 4d $\mathcal{N} = 1$ SCFTs: relating current correlators to 't Hooft anomalies

The superspace version of an anomaly in the dilatation current is

$$\overline{\nabla}^{\dot{\alpha}} T_{\alpha\dot{\alpha}} = \nabla_\alpha L_T, \tag{IV.26}$$

with L_T the trace anomaly, which is the variation of the effective action with respect to the chiral compensator chiral superfield [61].

On a curved spacetime, there is the conformal anomaly

$$\langle T^\mu_\mu \rangle = \frac{1}{120}\frac{1}{(4\pi)^2}\left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler})\right), \tag{IV.27}$$

(there can also be an $a'\partial^2 R$ term, whose coefficient a' is ambiguous, which was discussed in detail in [62]). The coefficient "c" is that of the stress tensor two-point function in flat space, whereas the coefficient "a" can be related to a stress tensor 3-point function in flat space. The superspace version of this anomaly, including also background gauge fields coupled to the superconformal R-current, is as in (IV.26), with $L_T = (c\mathcal{W}^2 - a\Xi_c)/24\pi^2$ [13]. Taking components of this superspace anomaly equation relates the conformal anomaly coefficients a and c to the 't Hooft anomalies of the superconformal $U(1)_R$ symmetry [13]:

$$a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R) \qquad c = \frac{1}{32}(9\text{Tr}R^3 - 5\text{Tr}R). \tag{IV.28}$$

An alternate derivation [29] of these relations follows from the fact that, in flat space, the 3-point function $\langle T_{\alpha\dot\alpha}(z_1)T_{\beta\dot\beta}(z_2)T_{\gamma\dot\gamma}(z_3)\rangle$ is of a form that's completely determined by the symmetries and Ward identities, up to two overall normalization coefficients, with one linear combination of these coefficients proportional to the coefficient (IV.23) of the $T_{\alpha\beta}$ two-point function. In components, this relates the stress tensor three-point functions, and hence a and c, and to the R-current 3-point functions, and hence the $\text{Tr}U(1)_R$ and $\text{Tr}U(1)_R^3$ 't Hooft anomalies, to these two coefficients. It follows that a and c can be expressed as linear combinations of $\text{Tr}U(1)_R$ and $\text{Tr}U(1)_R^3$, and the coefficients in (IV.28) can easily be determined by considering the special cases of free chiral and vector superfields.

Combining (IV.11) and (IV.28), we have

$$\tau_{RR} = \frac{3}{2}\text{Tr}R^3 - \frac{5}{6}\text{Tr}R. \tag{IV.29}$$

It was also argued in [13] that the two-point functions τ_{ij} of non-R flavor currents are related to 't Hooft anomalies, as

$$\tau_{ij} = -3\text{Tr}RF_iF_j. \tag{IV.30}$$

Again, this can be argued for either by turning on background fields, or by considering correlation functions in flat space. In the former method, one uses the fact that coupling background field strengths to the non-R currents leads to $\Delta L_T = C_{ij} W_{\alpha i} W_j^\alpha$, in (IV.26), for some coefficients C_{ij}. In components, (IV.26) then gives $\delta\langle T_\mu^\mu\rangle \sim C_{ij} F_{\mu\nu,i} F_j^{\mu\nu}$ and $\delta\langle \partial_\mu J_R^\mu\rangle \sim C_{ij} F_{\mu\nu i} \widetilde{F}_j^{\mu\nu}$. The former gives $C_{ij} \sim \tau_{ij}$ and the latter gives $C_{ij} \sim \mathrm{Tr} R F_i F_j$, so $\tau_{ij} \propto \mathrm{Tr} R F_i F_j$. The coefficient in (IV.30) is again easily determined by considering the special case of free field theory.

The alternate derivation would be to consider the flat space 3-point function of the stress tensor and two flavor currents, $\langle T_{\alpha\dot\alpha}(z_1) J_i(z_2) J_j(z_3)\rangle$. It was shown in [54] that such 3-point functions are completely determined by the symmetries and Ward identities, up to two overall coefficients, and that one linear combination of these coefficients is proportional to the current-current two point functions, and hence τ_{ij}. In our supersymmetric context, that same linear combination should be related by supersymmetry to $\langle \partial_\mu J_R^\mu(x_1) J_{F_i}^\rho(x_2) J_{F_j}^\sigma(x_3)\rangle$, and hence to the $\mathrm{Tr} R F_i F_j$ 't Hooft anomaly.

The a-maximization [14] constraint on the superconformal R-symmetry follows from the fact that supersymmetry relates the $\mathrm{Tr} R^2 F_i$ and $\mathrm{Tr} F_i$ 't Hooft anomalies:

$$9\mathrm{Tr} R^2 F_i - \mathrm{Tr} F_i = 0, \tag{IV.31}$$

which again can be argued for either by considering again an anomaly with background fields, or by considering current correlation functions in flat space [14]. In the former method, one considers the anomaly of the flavor current coming from a curved background metric and background gauge field coupled to the superconformal R-current, $\overline{\nabla}^2 J \propto \mathcal{W}^2$. With the latter method, one uses the result of [29] that the flat space 3-point function $\langle T_{\alpha\dot\alpha}(z_1) T_{\beta\dot\beta}(z_2) J_i(z_3)\rangle$ is completely determined by the symmetries and superconformal Ward identities, up to a single overall normalization constant.

We note that supersymmetry does not relate τ_{Ri} to the 't Hooft anomaly

$\mathrm{Tr}R^2F_i$. Naively, one might have expected some such relation, in analogy with the above arguments, for example by trying to use (IV.26) to relate a term $\delta\langle T^\mu_\mu\rangle \sim \tau_{Ri}F_{R,\mu\nu}F_i^{\mu\nu}$ to a term $\delta\langle\partial_\mu J^\mu_R\rangle \sim (\mathrm{Tr}R^2F_i)F_{R,\mu\nu}\widetilde{F}_i^{\mu\nu}$, when background fields are coupled to both $U(1)_R$ and $U(1)_{F_i}$ currents. But there is actually no way to write such combined contributions of the $U(1)_R$ and $U(1)_{F_i}$ background fields to (IV.26), because the former resides in the spin 3/2 chiral super field strength $\mathcal{W}_{\alpha\beta\gamma}$, and the latter resides in the spin 1/2 chiral super field strength $W_{\alpha i}$, and there is no way to combine the two of them into the spin zero chiral object L_T. Likewise, in flat space, a relation between τ_{Ri} and $\mathrm{Tr}R^2F_i$ would occur if the 3-point function $\langle T_{\alpha\dot\alpha}(z_1)T_{\beta\dot\beta}(z_2)J_i(z_3)\rangle$, which includes a term proportional to $\mathrm{Tr}R^2F_i$, were related to the two-point function $\langle T_{\beta\dot\beta}(z_2)J_i(z_3)\rangle$, which is proportional to τ_{Ri} (and, as we have argued above, vanishes). It sometimes happens that 3-point functions with a stress tensor are simply proportional to the 2-point function without the stress tensor, e.g. this is the case when the other two operators are chiral and anti-chiral primary [29]. But the the $\langle T_{\alpha\dot\alpha}(z_1)T_{\beta\dot\beta}(z_2)J_i(z_3)\rangle$ 3-point function in [29] is not related to the $\langle T_{\beta\dot\beta}(z_2)J_i(z_3)\rangle$ two-point function. Indeed, the free field example discussed in the introduction illustrates that $\mathrm{Tr}R^2F_i$ and τ_{Ri} are not related by supersymmetry, as $\mathrm{Tr}R^2F_i \neq 0$ for this example but, as always, $\tau_{Ri} = 0$.

IV.C.2 Using $\tau_{Ri} = 0$ to determine the superconformal R-symmetry

As discussed in the introduction, using (IV.4), we have for a general trial R-symmetry

$$\tau_{R_ti} = \tau_{R_0i} + \sum_j s_j\tau_{ij}. \tag{IV.32}$$

Imposing $\tau_{R_ii} = 0$ gives a set of linear equations, which determines the particular values s_j^* of the parameters for which the trial R-symmetry is the superconformal R-symmetry. As discussed in the introduction, this can equivalently be expressed as "the exact superconformal R-symmetry minimizes its two-point function coefficient $\tau_{R_tR_t}(s)$, which is given by (IV.8), and which we can re-write using $\tau_{Ri} = 0$ for the

superconformal R-symmetry as

$$\tau_{R_tR_t}(s) = \tau_{RR} + \sum_{ij}(s_i - s_i^*)(s_j - s_j^*)\tau_{ij}, \tag{IV.33}$$

making it manifest that $\tau_{R_tR_t}$ has a unique global minimum, when the s_j are set to the particular value s_j^*. At $s_j = s_j^*$, the general R-symmetry R_t in (IV.4) becomes the superconformal R-symmetry, in the supermultiplet stress tensor $T_{\alpha\dot{\alpha}}$.

The function $\tau_{R_tR_t}(s)$ to minimize and the function $a_{trial}(s)$ to locally maximize in 4d are different. Let us compare the values of them and their derivatives at the extremal point $s_i = s_i^*$. For (IV.32), we have:

$$\begin{aligned} \tau_{R_tR_t}|_{s^*} = \tau_{RR} = \frac{16}{3}c = \frac{3}{2}\mathrm{Tr}R^3 - \frac{5}{6}\mathrm{Tr}R, \\ \frac{\partial}{\partial s_i}\tau_{R_tR_t}|_{s^*} = 0, \\ \frac{\partial^2}{\partial s_i \partial s_j}\tau_{R_tR_t} = 2\tau_{ij}, \end{aligned} \tag{IV.34}$$

whereas for $\frac{16}{3}a_{trial}(R_t) \equiv \frac{1}{2}(3\mathrm{Tr}R_t^3 - \mathrm{Tr}R_t)$ we have:

$$\begin{aligned} \frac{16}{3}a_{trial}(R_t)|_{s^*} = \frac{16}{3}a = \frac{3}{2}\mathrm{Tr}R^3 - \frac{1}{2}\mathrm{Tr}R, \\ \frac{\partial}{\partial s_i}\frac{16}{3}a_{trial}(R_t)|_{s^*} = \frac{9}{2}\mathrm{Tr}R^2F_i - \frac{1}{2}\mathrm{Tr}F_i = 0, \\ \frac{\partial^2}{\partial s_i \partial s_j}\frac{16}{3}a_{trial}(R_t)|_{s^*} = 9\mathrm{Tr}RF_iF_j = -3\tau_{ij}. \end{aligned} \tag{IV.35}$$

The derivatives of both functions of s vanish at the same values s^*. The values of the two functions in (IV.34) and (IV.35) differ, except for SCFTs with $a = c$, i.e. $\mathrm{Tr}R = 0$, as is the case for SCFTs with AdS duals [1] The second derivatives of the functions in (IV.34) and (IV.35) are proportional, though with opposite sign, reflecting the fact that the exact superconformal R-symmetry minimizes $\tau_{R_tR_t}$ and maximizes $a_{trial}(R_t)$.

[1] Quite generally, quiver gauge theories with only bi-fundamental matter have $\mathrm{Tr}R = 0$, and hence $a = c$ [15] , [63].

For the sake of comparison, let's also consider the function $\frac{16}{3}c_{trial}(R_t) \equiv \frac{3}{2}R_t^3 - \frac{5}{6}R_t$; the value of this function and its first two derivatives at $R_t = R$, i.e. $s_i = s_i^*$, are

$$\begin{aligned} \frac{16}{3}c_{trial}(R_t)|_{s^*} &= \frac{16}{3}c = \frac{3}{2}\mathrm{Tr}R^3 - \frac{5}{6}\mathrm{Tr}R, \\ \frac{\partial}{\partial s_i}\frac{16}{3}c_{trial}(R_t)|_{s^*} &= \frac{9}{2}\mathrm{Tr}R^2F_i - \frac{5}{6}\mathrm{Tr}F_i = -\frac{1}{3}\mathrm{Tr}F_i, \\ \frac{\partial^2}{\partial s_i \partial s_j}\frac{16}{3}c_{trial}(R_t)|_{s^*} &= 9\mathrm{Tr}RF_iF_j = -3\tau_{ij}. \end{aligned} \tag{IV.36}$$

The value of $\tau_{R_tR_t}$ and $c_{trial}(R_t)$ coincide at $R_t = R$. The value of their first derivatives differ for any flavor symmetries with $\mathrm{Tr}F_i \neq 0$. General SCFTs can have flavor symmetries with $\mathrm{Tr}F_i = 0$, but SCFTs with AdS duals always have $\mathrm{Tr}F_i = 0$, and $\mathrm{Tr}F_i = 0$ for general superconformal quivers with only bifundamental matter [15], [63]. The second derivatives in (IV.36) differ from those of (IV.34) by a factor of $-3/2$, coinciding with those of (IV.35).

As a further comparison of a-maximization in 4d with τ_{RR} minimization, let's consider the equations for the case where the superconformal $U(1)_R$ can mix with a single non-R flavor symmetry, $R_t = R_0 + sF$. a-maximization gives the value s^* for the superconformal $U(1)_R$ as a solution of the quadratic equation

$$s^2\mathrm{Tr}F^3 + 2s\mathrm{Tr}R_0F^2 + \mathrm{Tr}R_0^2F - \frac{1}{9}\mathrm{Tr}F = 0. \tag{IV.37}$$

τ_{RR} minimization gives s^* as (IV.7)

$$s^* = -\tau_{R_0F}/\tau_{FF}. \tag{IV.38}$$

If $\mathrm{Tr}F^3$ is non-zero, s^* can also be obtained from (IV.16), which here gives

$$s^* = -\left[\mathrm{Tr}R_0F^2 + \frac{1}{3}\tau_{FF}\right]/\mathrm{Tr}F^3. \tag{IV.39}$$

For any given choice of R_0 and F, the value of s^* obtained in these three different ways must agree. It would be nice to have a direct proof of the relations that this implies. E.g. comparing (IV.39) with (IV.38) gives the identity $\tau_{R_0F}\mathrm{Tr}F^3 = \tau_{FF}\left(\frac{1}{3}\tau_{FF} + \mathrm{Tr}R_0F^2\right)$ which, evidently, must hold for any choice of the R-symmetry R_0 (taking R_0 to equal the superconformal $U(1)_R$, both sides vanish).

IV.D SQCD Example

4d $\mathcal{N} = 1$ SCQD, with gauge group $SU(N_c)$ and N_f fundamental and anti-fundmantal flavors, Q and $\widetilde{Q}$, has been argued to flow to a SCFT in the IR for the flavor range $\frac{3}{2}N_c < N_c < 3N_c$ [26]. Taking the superconformal $U(1)_R$ to be the anomaly free R-symmetry, the superconformal R-charges are $R(Q) = R(\widetilde{Q}) = 1 - (N_c/N_f)$. Let's also consider the baryonic $U(1)_B$ symmetry, with $B(Q) = -B(\widetilde{Q}) = 1/N_c$. Using the 't Hooft anomaly relations,

$$\tau_{RR} = \frac{3}{2}\mathrm{Tr}R^3 - \frac{5}{6}\mathrm{Tr}R = \frac{3}{2}\left[N_c^2 - 1 - 2\frac{N_c^4}{N_f^2}\right] + \frac{5}{6}\left[N_c^2 + 1\right], \tag{IV.40}$$

$$\tau_{BB} = -3\mathrm{Tr}RBB = 6. \tag{IV.41}$$

For $N_f \approx 3N_c$, where the RG fixed point is at weak coupling as in [33], [38], these expressions reduce to the free field values.

There is a unique, anomaly free $U(1)_R$ symmetry that commutes with charge conjugation and the $SU(N_f)$ global symmetries. Our $\tau_{R_tR_t}$ minimization condition immediately leads to the same conclusion. $\tau_{R_tR_t}$ is minimized by having $\tau_{RB} = 0$ and $\tau_{RF_i} = 0$ for the $U(1)_B$ and $SU(N_f)$ global symmetries. Taking the $U(1)_R$ to be even under charge conjugation ensures that $\tau_{RB} = 0$, because the $U(1)_B$ current is odd, so charge conjugation symmetry gives $\tau_{RB} = -\tau_{RB}$. Likewise $\tau_{RF_i} = 0$ for the $SU(N_f)$ flavor currents, simply by the tracelessness of the generators, if $U(1)_R$ is taken to commute with $SU(N_f)$.

IV.E Perturbative analysis

Consider a general 4d $\mathcal{N} = 1$ SCFT with gauge group G and matter chiral superfields Q_f in representations r_f (of dimension $|r_f|$) of G, with no superpotential, $W = 0$. If the theory is just barely asymptotically free, there can be a RG fixed point at weak gauge coupling, where perturbative results can be valid. We will verify that the leading order pertubative expression for the anomalous

dimension for fields,

$$\gamma_f(g) = -\frac{g^2}{4\pi^2}C(r_f) + \mathcal{O}(g^4), \qquad \text{i.e} \qquad R_f = \frac{2}{3} - \frac{g^2}{12\pi^2}C(r_f) + O(g^4). \quad \text{(IV.42)}$$

agrees with τ_{RR} minimization. As standard, we define group theory factors as

$$\mathrm{Tr}_{r_f}(T^A T^B) = T(r_f)\delta^{AB}, \quad \sum_{A=1}^{|G|} T^A_{r_f} T^A_{r_f} = C(r_f)\mathbf{1}_{|r_f|\times|r_f|}, \quad \text{so} \quad C(r_f) = \frac{|G|T(r_f)}{|r_f|}. \quad \text{(IV.43)}$$

The RG fixed point value g_* of the coupling is determined by the constraint that the R-symmetry be anomaly free, $T(G) + \sum_f T(r_f)(R_f - 1) = 0$.

For the free UV theory, we minimize τ_{RR} over all possible R charges R_f of the matter chiral superfields, which are unconstrained for $g = 0$. As we discussed in the introduction, this gives the free-field term $R_f^{(0)} = 2/3$. For $g \neq 0$, we write $R_f = R_f^{(0)} + R_f^{(1)} + \ldots$, with $R_f^{(1)}$ the $O(g^2)$ term that we'd like to find via τ_{RR} minimization. For $g \neq 0$, τ_{RR} should be minimized subject to the constraint that the symmetries be anomaly free, i.e. we impose $\tau_{Ri} = 0$ over all anomaly free $U(1)_R$ and $U(1)_{F_i}$ symmetries, with R charges R_f, and flavor F_i charges $q_i(r_f)$ constrained to satisfy

$$T(G) + \sum_f T(r_f)(R_f - 1) = 0, \qquad \text{and} \qquad \sum_f T(r_f) q_i(r_f) = 0. \quad \text{(IV.44)}$$

The $U(1)_R$ current assigns charges R_f to the squark and $R_f - 1$ to the quarks components of Q_f. The $U(1)_{F_i}$ non-R current assigns charges $q_i(r_f)$ to both the quark and squark components of Q_f. To compute τ_{RF_i}, we consider the diagrams for the two point function $\langle J_R^\mu(x_1) J_{F_i}^\nu(x_2)\rangle$. Because we take the currents to be conserved, they have vanishing anomalous dimension, so we anticipate that the various diagrams sum such that all apparent divergences cancel, and we're left with only finite contributions to τ_{RF_i}. The $O(g^2)$ contributions can be written as

$$\tau_{Ri}^{(1)} = \sum_f q_i(r_f)\left[(\frac{1}{3}R_f^{(1)} + \frac{2}{3}R_f^{(1)})|r_f| + R_f^{(0)}(A_f^{(1)} + C_f^{(1)}) + (R_f^{(0)} - 1)(B_f^{(1)} + C_f^{(1)})\right]. \quad \text{(IV.45)}$$

The first two terms come from the leading diagrams, without interactions, exactly as in the free-field result (IV.13), but weighted by the $O(g^2)$ R-charges $R_f^{(1)}$. The

first term is from connecting the currents at x_1 and x_2, with squark ϕ_f propagators, and the second from connecting them with quark ψ_f propagators. The remaining contributions in (IV.45) are $O(g^2)$ because they involve $O(g^2)$ interaction diagrams, and the R-charge weighting is thus taken as $R^{(0)} = 2/3$. Here $A_f^{(1)}$ is the contribution of all $O(g^2)$ 1PI diagrams connecting squark ϕ_f, at x_1, to squark ϕ_f at x_2. $B_f^{(1)}$ is similarly the contribution from all $O(g^2)$ diagrams connecting quark ψ_f at x_1 to quark ψ_f at x_2. $C_f^{(1)}$ is the contributions of diagrams connecting squark ϕ_f at x_1 to quark ψ_f at x_2 (or vice-versa). We note that the group theory factors in all of these diagrams with $O(g^2)$ interactions is the same: $\mathrm{Tr}_{r_f} \sum_{A=1}^{|G|} T_{r_f}^A T_{r_f}^A = |r_f| C(r_f) = |G| T(r_f)$, i.e. $A_f^{(1)} = |G| T(r_f) A^{(1)}$, $B_f^{(1)} = |G| T(r_f) B^{(1)}$, and $C_f^{(1)} = |G| T(r_f) C^{(1)}$, where $A^{(1)}$, $B^{(1)}$, and $C^{(1)}$ are independent of the gauge group and representation, e.g. they could be computed in $U(1)$ SQED.

Using the second constraint in (IV.44), $\sum_f T(r_f) q_i(r_f) = 0$, it immediately follows, without even having to compute $A^{(1)}$, $B^{(1)}$, and $C^{(1)}$, that their contributions to $\tau_{Ri}^{(1)}$ in (IV.45) all vanish, for all anomaly free flavor symmetries F_i. The only contributions remaining in (IV.45) are the $R_f^{(1)}$ ones, $\tau_{Ri}^{(1)} = \sum_f q_i(r_f) R_f^{(1)} |r_f|$. Our τ_{RR} minimization result implies that this must vanish, for any $q_i(r_f)$ satisfying the anomaly free constraint in (IV.44). This implies that $R_f^{(1)} = \alpha C(r_f)$ for some constant α that's independent of the rep. r_f.

We have thus used $\tau_{R_t R_t}$ minimization to re-derive the group theory dependence of the $O(g^2)$ term in the anomalous dimension (IV.42). The coefficient is also fixed to agree with (IV.42), at the fixed point g_*, by using the condition in (IV.44) that the R-symmetry be anomaly free to solve for α (which is appropriately small when the matter content is such that the theory is barely asymptotically free). This reproduces the $O(g^2)$ contribution to the R-charges in (IV.42) at the RG fixed point.

In principle, one could extend this analysis, and use τ_{RR} minimization to compute the anomalous dimensions to all orders. Using a-maximization [14]

(assuming that the RG fixed point has no accidental symmetries), the general result can be written as [20]

$$R_f = \frac{2}{3}(1+\frac{1}{2}\gamma_f(g_*)) = 1-\frac{1}{3}\sqrt{1+\frac{\lambda_* T(r_f)}{|r_f|}} = 1-\frac{1}{3}\sqrt{1+\frac{\lambda_* C(r_f)}{|G|}}, \qquad \text{(IV.46)}$$

where λ_* is a Lagrange multiplier [20], which is determined by the constraint that the R-symmetry be anomaly free, $T(G)+\sum_f T(r_f)(R_f-1)=0$. The result (IV.46) was successfully compared [27], [45] with the results for the anomalous dimensions to 3-loops of [25]. But, because current two-point functions get quantum corrections, τ_{RR} minimization does not seem to be a very efficient way to compute anomalous dimensions. Indeed, the higher order quantum corrections to τ_{Ri} include diagrams where the currents at x_1 and x_2 are connected by renormalized propagators, with all quantum corrections from the interactions, and computing such τ_{Ri} contributions is already tantamount to directly computing the anomalous dimensions $\gamma_f(g)$.

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V

Current Correlators and AdS/CFT Geometry

V.A Introduction

This work is devoted to the geometry / gauge theory interrelations of the AdS/CFT correspondence [66], [67], [68], which has been much developed and checked over the past year (a sample of recent references is [24], [69], [46], [47], [70], [58], [71], [72]).

In the AdS/CFT correspondence [66], [67], [68], global currents J_I^μ (I labels the various currents) of the d-dimensional CFT couple to gauge fields in the AdS_{d+1} bulk. The current two-point functions of the CFT are of fixed form,

$$\langle J_I^\mu(x) J_J^\nu(y)\rangle = \frac{\tau_{IJ}}{(2\pi)^d}(\partial^2\delta_{\mu\nu} - \partial_\mu\partial_\nu)\frac{1}{(x-y)^{2(d-2)}}, \tag{V.1}$$

with only the coefficients τ_{IJ} depending on the theory and its dynamics. Unitarity restricts τ_{IJ} to be a positive matrix (positive eigenvalues). The coefficients τ_{IJ} map to the coupling constants of the corresponding gauge fields in AdS_{d+1}: writing their kinetic terms as

$$S_{AdS_{d+1}} = \int d^d z dz_0 \sqrt{g}\left[-\frac{1}{4} g_{IJ}^{-2} F^I_{\mu\nu} F^{\mu\nu J} + \ldots\right], \tag{V.2}$$

the relation is [56]:

$$\tau_{IJ} = \frac{2^{d-2}\pi^{\frac{d}{2}}\Gamma[d]}{(d-1)\Gamma[\frac{d}{2}]}L^{d-3}g_{IJ}^{-2}, \tag{V.3}$$

where L is the AdS_{d+1} length scale. Our main interest here will be in the quantities τ_{IJ}, and comparing field theory results with the AdS relation (V.3).

We will here consider 4d $\mathcal{N}=1$ superconformal field theories, 3d $\mathcal{N}=2$ SCFTs, and their AdS duals, coming, respectively, from IIB string theory on $AdS_5 \times Y_5$, 11d SUGRA or M-theory on $AdS_4 \times Y_7$. Supersymmetry requires Y_5 and Y_7 to be Sasaki-Einstein. In general, a Sasaki-Einstein space Y_{2n-1} is the horizon of a non-compact local Calabi-Yau n-fold $X_{2n} = C(Y_{2n-1})$, with conical metric

$$ds^2(C(Y_{2n-1})) = dr^2 + r^2 ds^2(Y_{2n-1}). \tag{V.4}$$

The gauge theories come from N $D3$ or $M2$ branes at the tip of the cone. In the large N dual, the radial r becomes that of AdS_{d+1}. The dual to 4d $\mathcal{N}=1$ SCFTs is IIB on

$$AdS_5 \times Y_5: \qquad ds_{10}^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + L^2 ds^2(Y_5), \tag{V.5}$$

and the dual to 3d $\mathcal{N}=2$ SCFTs is 11d SUGRA or M-theory with metric background

$$AdS_4 \times Y_7: \qquad ds_{11}^2 = \frac{r^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{r^2}dr^2 + (2L)^2 ds^2(Y_7). \tag{V.6}$$

The SCFTs have a conserved, superconformal $U(1)_R$ current, in the same supermultiplet as the stress tensor. The scaling dimensions of chiral operators are related to their superconformal $U(1)_R$ charges by

$$\Delta = \frac{d-1}{2}R. \tag{V.7}$$

There are also typically various non-R flavor currents, whose charges we'll write as F_i, with i labeling the flavor symmetries. The superconformal $U(1)_R$ of RG fixed point SCFTs is then not determined by the symmetries alone, as the R-symmetry can mix with the flavor symmetries. Some additional dynamical information is

then needed to determine precisely which, among all possible R-symmetries, is the superconformal one, in the stress tensor supermultiplet.

On the field theory side, we presented a new condition in [73], which, in principle, uniquely determines the superconformal $U(1)_R$: among all possible trial R-symmetries,

$$R_t = R_0 + \sum_i s_i F_i, \tag{V.8}$$

the superconformal one is that which *minimizes* the coefficient $\tau_{R_t R_t}$ of its two point function (V.1). An equivalent way to state this is that the two-point function of the superconformal R-current with all non-R flavor symmetries necessarily vanishes:

$$\tau_{Ri} = 0 \qquad \text{for all non-R symmetries } F_i. \tag{V.9}$$

(Our notation will always be that capital I runs over all symmetries, including the superconformal $U(1)_R$, and lower case i runs over the non-R flavor symmetries.) We refer to the field theory condition of [73] as "τ_{RR} minimization". The minimal value of $\tau_{R_t R_t}$ is then the coefficient, τ_{RR}, of the superconformal $U(1)_R$ current two-point function, which is related by supersymmetry to the coefficient of the stress-tensor two-point function,

$$\tau_{RR} \propto C_T. \tag{V.10}$$

For the case of 4d $\mathcal{N} = 1$ SCFTs, a-maximization [14] gives another way, besides τ_{RR} minimization, to determine the superconformal $U(1)_R$: the exact superconformal R-symmetry is that which (locally) maximizes the combination of 't Hooft anomalies

$$a_{trial}(R_t) = \frac{3}{32}(3\mathrm{Tr}R^3 - \mathrm{Tr}R). \tag{V.11}$$

Equivalently, the superconformal $U(1)_R$ satisfies the 't Hooft anomaly identity [14]

$$9\mathrm{Tr}R^2 F_i = \mathrm{Tr}F_i \qquad \text{for all flavor symmetries } F_i. \tag{V.12}$$

a-maximization does not apply for 3d SCFTs, as there are there no 't Hooft anomalies.

The global symmetries of the $SCFT_d$ map to the following gauge symmetries in the AdS_{d+1} bulk:

1. The graviphoton, which maps to the superconformal $U(1)_R$, is a Kaluza-Klein gauge field, associated with the "Reeb" Killing vector isometry of Sasaki-Einstein Y_{2n-1}. The R-charge is normalized so that superpotential terms, which are related to the holomorphic n form of X_{2n}, have charge $R = 2$.

2. Any other Kaluza-Klein gauge fields, from any additional isometries of Y_{2n-1}. These can be taken to be non-R symmetries, by taking the holomorphic n-form to be neutral. We refer to these as "mesonic, non-R, flavor symmetries," because mesonic operators (gauge invariants not requiring an epsilon tensor) of the dual gauge theory can be charged under them. When Y_{2n-1} is toric, there is always (at least) a $U(1)^{n-1}$ group of mesonic, non-R flavor symmetries.

3. Baryonic $U(1)^{b_*}$ gauge fields, from reducing Ramond-Ramond gauge fields on non-trivial cycles of Y_{2n-1}. In particular, for IIB on $AdS_5 \times Y_5$, there are $U(1)^{b_3}$ baryonic gauge fields come from reducing C_4 on the $b_3 = \dim(H_3(Y_5))$ non-trivial 3-cycles of Y_5. These are also non-R symmetries. Baryonic $U(1)$ symmetries have the distinguishing property in the gauge theory that only baryonic operators, formed with an epsilon tensor, are charged under them. It was pointed out in [15] that 4d baryonic symmetries have another distinguishing property: their cubic 't Hooft anomalies all vanish, $\mathrm{Tr}U(1)_B^3 = 0$, as seen from the fact that it's not possible to get the needed Chern-Simons term [68] $A_B \wedge dA_B \wedge dA_B$ from reducing 10d string theory on Y_5.

In field theory, the superconformal $U(1)_R$ can, and generally does mix with the mesonic and baryonic [1] flavor symmetries. The correct superconformal

[1] A point of possible confusion: as pointed out in [14], the superconformal $U(1)_R$ does not mix with those baryonic symmetries which transform under charge conjugation symmetry. But the superconformal gauge theories associated with general Y_{2n-1} are chiral, with no charge conjugation symmetries. So the superconformal $U(1)_R$ can mix with these baryonic $U(1)$'s.

$U(1)_R$ can, in principle, be determined by τ_{RR} minimization [73]. τ_{RR} minimization is not especially practical to implement in field theory, because the coefficients (V.9) get quantum corrections. But, on the AdS dual side, τ_{RR} minimization becomes more useful and tractable, because the AdS duality gives a weakly coupled dual description of $\tau_{R_0 i}$ and τ_{ij}, via (V.3).

The problem of determining the superconformal $U(1)_R$ in the field theory maps to a corresponding problem in the geometry: determining which $U(1)$, out of the $U(1)^n$ geometric isometries of toric Sasaki-Einstein spaces, is that of the Reeb vector. A solution of this mathematical problem was recently found by Martelli, Sparks, and Yau [58]: the correct Reeb vector is that which minimizes the Einstein-Hilbert action on Y_{2n-1} – this is referred to as "Z-minimization," [58]. The mathematical result of [58] was shown, on a case-by-case basis, to always lead to the same superconformal R-charges as found from a-maximization [14] in the corresponding field theory, but there was no general proof as to why Z-minimization in geometry implements a-maximization in field theory. In addition, Z-minimization applies to general Y_{2n-1}, whereas a-maximization is limited to 4d SCFTs, and hence the case of $AdS_5 \times Y_5$.

Our main result will be to show that the Z-minimization of Martelli, Sparks, and Yau [58] is precisely equivalent to ensuring that the τ_{RR} minimization conditions (V.9) of [73] are satisifed, i.e. *Z-minimization = τ_{RR} minimization.* This demonstrates that Z-minimization in the geometry indeed determines the correct superconformal R-symmetry of the dual SCFT, not only for 4d SCFTs, but also for 3d SCFTs with dual (V.6). We will also explain why it's OK that the $U(1)^{b_*}$ baryonic $U(1)$ symmetries did not enter into the geometric Z-minimization of [58]: the condition (V.9) is automatically satisfied in the string theory constructions for all baryonic symmetries.

The outline of this paper is as follows. In sect. V.B, we review relations in 4d $\mathcal{N} = 1$ field theory for the current two-point functions, and the 't Hooft anomalies of the superconformal $U(1)_R$. We then show that these relations are

satisfied by the effective AdS_5 bulk SUGRA theory, thanks to the structure of real special geometry. In particular, the kinetic terms in the AdS_5 bulk are related to the Chern-Simons terms, which yield the 't Hooft anomalies of the dual SCFT. In the following sections, we discuss how these kinetic terms are obtained from the geometry of Y; it would be interesting to also directly obtain the Chern-Simons terms from the geometry of Y, but that will not be done here. In sect. V.C, we discuss the contributions to the kinetic terms in the AdS bulk. As usual, Kaluza-Klein gauge fields get a contribution, with coefficient $(g_{ij}^{-2})^{KK}$, from reducing the Einstein term in the action on Y. Because of the background flux in Y, there is also a contribution $(g_{IJ}^{-2})^{CC}$ from reducing the Ramond-Ramond C field kinetic terms on Y. We point out (closely following [74]) that these two contributions always have the fixed ratio: $(g_{IJ}^{-2})^{CC} = \frac{1}{2}(D_c - 1)(g_{IJ}^{-2})^{KK}$, for any Einstein manifold Y of dimension D_c. This relation will be used, and checked, in following sections. For the baryonic gauge fields, there is only the contribution $(g_{IJ}^{-2})^{CC}$, from reducing the Ramond-Ramond kinetic term on Y.

In sect. V.D, we discuss generally how the gauge fields A_I alter Ramond-Ramond flux background, and thereby alter the Ramond-Ramond field at linearized level, as $\delta C = \sum_I \omega_I \wedge A_I$, for some particular $2n-3$ forms ω_I on Y. We discuss how the A_I charges of branes wrapped on supersymmetric cycles can be obtained by integrating ω_I over the cycle, and how the Ramond-Ramond contribution to the gauge kinetic terms is written as $\sim \int_Y \omega_I \wedge *\omega_J$. In sect. V.E, we review some aspects of Sasaki-Einstein geometry, and the analysis of [75] for how to determine the form ω_R for the $U(1)_R$ gauge field. In sect. V.F, we generalize this to determine the forms ω_I for the non-R isometry and baryonic gauge fields. In sect. V.G, we give expressions for the gauge kinetic terms g_{IJ}^{-2}, and thereby the current-current two-point function coefficients τ_{IJ} that we are interested in, in terms of integrals $\sim \int_Y \omega_I \wedge *\omega_J$ of these forms. We note that this immediately implies that there is never any mixing in the kinetic terms between Kaluza-Klein

isometry gauge fields and the baryonic gauge fields, i.e. that

$$\tau_{IJ} = 0 \quad \text{automatically, for } I = \text{Kaluza-Klein and } J = \text{baryonic.} \tag{V.13}$$

This shows that our condition (V.9) for the $U(1)_R$ is automatically satisfied, for all baryonic symmetries, by taking $U(1)_R$ to be purely a Kaluza-Klein isometry gauge field, without any mixing with the baryonic symmetries. For the mesonic, non-R isometry gauge fields, the condition (V.9) becomes

$$\int_Y g_{ab} K^a K_i^b vol(Y) = 0, \tag{V.14}$$

which give conditions to determine the $U(1)_R$ isometry Killing vector K^a. The condition (V.14) must hold for every non-R isometry Killing vector of Y, i.e. for every Killing vector K_i^a under which the the holomorphic n form of $C(Y_{2n-1})$ is neutral.

In sect. V.H, we summarize the results of Martelli, Sparks, and Yau [58] for toric $C(Y)$. Then Y_{2n-1} always has at least $U(1)^n$ isometry, associated with shifts of toric coordinates ϕ_i, and the $U(1)_R$ Killing Reeb vector K^a is given by some components b_i, $i = 1 \ldots n$, in this basis. The volume of Y and its supersymmetric cycles are completely determined by the b_i, without needing to know the metric on Y. And the b_i are themselves determined by Z-minimization [58], which is minimization of the Einstein-Hilbert action on Y. In sect. V.I, we point out that Z-minimization is precisely equivalent to τ_{RR} minimization. We also discuss the flavor charges of wrapped branes. In sect. V.J, we illustrate our results for the $Y^{p,q}$ examples of [24], [69]. We find the forms ω_I, and thereby use the flavor charges of wrapped branes. We also compute from the geometry of Y the gauge kinetic term coefficients, and thus the current-current two-point function coefficients τ_{IJ}. These quantities, computed from the geometry of Y, match with those computed in the dual field theory of [47]; this gives new checks of the AdS/CFT correspondence for these theories.

In the final stages of writing up this paper, the very interesting work [76] appeared, in which it was mathematically shown that the Z-function [58] of 5d toric

Sasaki-Einstein Y_5 and the a_{trial} function [14] of the dual quiver 4d gauge theory are related by $Z(x,y) = 1/a(x,y)$ (even before extremizing). The approach and results of our paper are orthogonal and complementary to those of [76]. Also in the final stages of writing up this paper, the work [77] appeared, which significantly overlaps with the approach of section V.B of our paper, and indeed goes further along those lines than we did here.

V.B 4d $\mathcal{N} = 1$ SCFTs and real special geometry

This section is somewhat orthogonal to the rest of the paper. The rest of this paper is devoted to deriving the AdS bulk gauge field kinetic terms g_{IJ}^{-2} in (V.2) and (V.3) directly from the geometry of Y. In the present section, without explicitly considering Y, we will discuss how the various identities of 4d $\mathcal{N} = 1$ SCFTs are guaranteed to also show up in the effective AdS_5 SUGRA theory, thanks to the structure of real, special geometry.

Because the superconformal R-current is in the same supermultiplet as the stress tensor, their two-point function coefficients are proportional, $\tau_{RR} \propto C_T$. Also, in 4d $C_T \propto c$, with c the conformal anomaly coefficient in

$$\langle T^{\mu}_{\mu} \rangle = \frac{1}{120} \frac{1}{(4\pi)^2} \left(c(\text{Weyl})^2 - \frac{a}{4}(\text{Euler}) \right). \tag{V.15}$$

So $\tau_{RR} \propto c$; more precisely,

$$\tau_{RR} = \frac{16}{3} c, \tag{V.16}$$

with c normalized such that $c = 1/24$ for a free $\mathcal{N} = 1$ chiral superfield. Supersymmetry also relates a and c in (V.15) to the 't Hooft anomalies of the superconformal $U(1)_R$ [5]:

$$a = \frac{3}{32}(3\text{Tr}R^3 - \text{Tr}R) \qquad c = \frac{1}{32}(9\text{Tr}R^3 - 5\text{Tr}R). \tag{V.17}$$

Combining (V.16) and (V.17), we have

$$\tau_{RR} = \frac{3}{2}\text{Tr}R^3 - \frac{5}{6}\text{Tr}R, \tag{V.18}$$

The flavor current two-point functions are also given by 't Hooft anomalies [5]:

$$\tau_{ij} = -3\mathrm{Tr}RF_iF_j. \tag{V.19}$$

There are precise analogs to the above relations in the effective[2] 5d $\mathcal{N}=2$ bulk gauged U(1) supergravity; this is not surprising given that, on both sides of the duality, these relations come from the same $SU(2,2|1)$ superconformal symmetry group.

The bosonic part of the effective 5d Lagrangian is [78] (also see e.g. [79])

$$\mathcal{L}^{\text{bosonic}} = \sqrt{|g|}\left[\frac{1}{2}R - \frac{1}{2}G_{ij}\partial_\mu\phi^i\partial^\mu\phi^j - \frac{1}{4}g_{IJ}^{-2}F^I_{\mu\nu}F^{\mu\nu J} - V(X)\right] + \frac{1}{48}C_{IJK}A^I\wedge F^J\wedge F^K \tag{V.20}$$

where, to simplify expressions, we'll set the 5d gravitational constant $\kappa_5 = 1$ in this section. There are n_V+1 gauge fields, $I = 1\ldots n_V+1$, one of them being the graviphoton, which corresponds to the superconformal $U(1)_R$ in the 4d SCFT. The n_V gauge fields correspond to the non-R (i.e. the gravitino is neutral under them) flavor symmetries, which reside in current supermultiplets J_i, $i = 1\ldots n_V$; the first component of this supermultiplet is a scalar, which couples to the scalars ϕ^i in (V.20). The scalars of the n_V vector multiplets are constrained by real special geometry to the space

$$\mathcal{N} \equiv \frac{1}{6}C_{IJK}X^IX^JX^K = 1. \tag{V.21}$$

The kinetic terms are all determined by the Chern-Simons coefficients C_{IJK}. In particular, the gauge field kinetic term coefficients g_{IJ}^{-2} are given by

$$g_{IJ}^{-2} = -\frac{1}{2}\partial_I\partial_J\ln\mathcal{N}|_{\mathcal{N}=1} = -\frac{1}{2}(C_{IJK}X^K - X_IX_J), \tag{V.22}$$

where $X_I \equiv \frac{1}{2}C_{IJK}X^JX^K$. In a given vacuum, where X^I has expectation values satisfying (V.21), the n_V scalars in (V.20) are given by the tangents X_i^I to the surface (V.21), which satisfy

$$C_{IJK}X_i^IX^JX^K = 0. \tag{V.23}$$

[2]The 5d SUGRA theory suffices for studying current two-point functions, and relations to 't Hooft anomalies, even if there is no full, consistent truncation from 10d to an effective 5d theory.

This can be written as $X_I X_i^I = 0$. The vacuum expectation value X^I picks out the direction of the graviphoton A_R, and the tangents X_i^I pick out the direction of the non-R flavor gauge fields:

$$A^I = \alpha X^I A_R + X_i^I A_i, \tag{V.24}$$

with α a normalization factor, to ensure that the R-symmetry is properly normalized, to give the gravitinos charges ± 1. The correct value is $\alpha = 2L/3$, where L is the AdS_5 length scale, related to the value of the potential at its minimum by $\Lambda = -6/L^2$.

Using (V.24) and (V.22), we can compute the kinetic term coefficients for the graviphoton and non-R gauge fields. Using (V.3) to convert these into the current-current 2-point function coefficients, we have for the R-symmetry/graviphoton kinetic term

$$\tau_{RR} = 8\pi^2 L g_{RR}^{-2} = 8\pi^2 L \alpha^2 g_{IJ}^{-2} X^I X^J = 12\pi^2 L \alpha^2. \tag{V.25}$$

For the n_V non-R gauge fields, we have

$$\tau_{ij} = 8\pi^2 L g_{ij}^{-2} = 8\pi^2 L g_{IJ}^{-2} X_i^I X_j^J = -4\pi^2 L C_{IJK} X_i^I X_j^J X^K. \tag{V.26}$$

It also follows from (V.22) and (V.23), $X_I X_i^I = 0$, that there is no kinetic term mixing between the graviphoton and the non-R gauge fields:

$$\tau_{Ri} = 8\pi^2 L g_{Ri}^{-2} = 8\pi^2 L \alpha g_{IJ}^{-2} X_i^I X^J = 0 \qquad \text{for all } i = 1 \ldots n_V. \tag{V.27}$$

This matches with the general SCFT field theory result (V.9) of [73].

The Chern-Simons terms for the graviphoton and flavor gauge fields are similarly found from (V.24). We'll normalize them as $C_{IJK}/48 = k_{IJK}/96\pi^2$, where k_{IJK} is the properly normalized 5d Chern-Simons coefficients, which map [68] to the 't Hooft anomalies of the gauge theory:

$$\mathrm{Tr} R^3 = k_{RRR} = 2\pi^2 \alpha^3 C_{IJK} X^I X^J X^K = 12\pi^2 \alpha^3, \tag{V.28}$$

$$\mathrm{Tr} R^2 F_i = k_{RRj} = 2\pi^2 \alpha^2 C_{IJK} X^I X^J X_i^K = 0, \tag{V.29}$$

where we used (V.23), and also

$$\mathrm{Tr}RF_iF_j = k_{Rij} = 2\pi^2\alpha C_{IJK}X^I X_i^J X_j^K. \tag{V.30}$$

The field theories with (weakly coupled) AdS duals generally have $\mathrm{Tr}R = 0$ and also $\mathrm{Tr}F_i = 0$. The result (V.29) then reproduces the 't Hooft anomaly identity (V.12) of [14]. For $\mathrm{Tr}R = 0$, (V.18) becomes $\tau_{RR} = \frac{3}{2}\mathrm{Tr}R^3$, which is reproduced by (V.25) and (V.28) for $\alpha = 2L/3$ in (V.24). Also the relation (V.17) of [13], which for $\mathrm{Tr}R = 0$ is $a = c = \frac{9}{32}\mathrm{Tr}R^3$, is also reproduced by (V.28) for $\alpha = 2L/3$, since the result of [80] is $a = c = L^3\pi^2$ in $\kappa_5 = 1$ units. The relation (V.19) is also reproduced, for $\alpha = 2L/3$, by (V.26) and (V.30).

In later sections, we will be interested in computing the AdS_5 gauge field kinetic terms τ_{IJ} directly from IIB string theory on $AdS_5 \times Y_5$. To connect with the above expressions, we restore the factors of κ_5 via dimensional analysis, and convert using

$$\frac{L^3}{\kappa_5^2} = \frac{L^3}{8\pi G_5} = \frac{L^8 Vol(Y_5)}{8\pi G_{10}} = \frac{N^2}{4}\frac{\pi}{Vol(Y_5)}, \tag{V.31}$$

where $Vol(Y_5)$ is the dimensionless volume of Y_5, with factors of its length scale, which coincides with the AdS_5 length scale L, factored out. The last equality of (V.31) uses the flux quantization / brane tensions relation (see [81] and references therein)

$$2\sqrt{\pi}\kappa_{10}^{-1}L^4 Vol(Y_5) = \frac{L^4 Vol(Y_5)}{\sqrt{2G_{10}}} = N\pi. \tag{V.32}$$

E.g. using (V.31) the result of [80] becomes [82]

$$a = c = \frac{L^3\pi^2}{\kappa_5^2} = \frac{N^2}{4}\frac{\pi^3}{Vol(Y_5)}, \tag{V.33}$$

and (V.25) for $\alpha = 2L/3$ becomes

$$\tau_{RR} = \frac{16\pi^2}{3}\frac{L^3}{\kappa_5^2} = \frac{4N^2}{3}\frac{\pi^3}{Vol(Y_5)}. \tag{V.34}$$

In the following sections, we will directly compute the τ_{IJ} kinetic terms from reducing SUGRA on Y. One could also directly determine the Chern-Simons coefficients C_{IJK} from reduction on Y, but doing so would require going beyond

our linearized analysis, and we will not do that here. It would be nice to extend our analysis to compute the C_{IJK} from Y, and explicitly verify that the special geometry relations reviewed in the present section are indeed satisfied.

V.C Kaluza-Klein gauge couplings: a general relation for Einstein spaces

Our starting point is the Einstein action in $D_t = D + D_c$ spacetime dimensions, along with the Ramond-Ramond gauge field kinetic terms:

$$\frac{1}{16\pi G_{D_t}} \int \left(R_{D_t} * 1 - \frac{1}{4} F \wedge *F \right). \tag{V.35}$$

We'll be interested in fluctuations of this action around a background solution of the form $M_D \times Y$, with M_D non-compact and Y compact, of dimension $D_c \equiv p+2$, with flux

$$F_{p+2}^{bkgd} = (p+1) m^{-(p+1)} vol(Y), \tag{V.36}$$

and metric

$$ds^2 = ds_M^2 + m^{-2} ds_Y^2. \tag{V.37}$$

Here m^{-1} is the length scale of Y, which we'll always factor out explicitly; $vol(Y)$ is the volume form of Y, with the length scale m^{-1} again factored out. (We always use lower case $vol(Y)$ for a volume form, and upper case $Vol(Y)$ for its integrated volume.) Our units are such that the integrated flux is

$$\mu_p \int_Y F_{p+2}^{bkgd} \sim \mu_p m^{-(p+1)} Vol(Y) \sim N, \tag{V.38}$$

with μ_p the p-brane tension. Our particular cases of interest will be IIB on $AdS_5 \times Y_5$ and 11d SUGRA on $AdS_4 \times Y_7$, but we'll be more general in this section.

Metric fluctuations along directions of Killing vectors K_I^a of Y lead to Kaluza-Klein gauge fields A_I^μ in M. Fluctuations of the Ramond-Ramond gauge field background, reduced on non-trivial cycles of Y lead to additional, "baryonic" gauge fields that we'll also discuss. In general, Kaluza-Klein reduction involves

a detailed, and highly non-trivial, ansatz for how the Kaluza-Klein gauge fields affect the metric and background field strengths. But here we're simply interested in the coefficients g_{IJ}^{-2} of the gauge field kinetic term, and for these it's unnecessary to employ the full Kaluza-Klein ansatz: a linearized analysis suffices.

The linearized analysis will be presented in the following section. In this section, we'll note some general aspects, and discuss a useful relation that can be obtained by a generalization of an argument in [74], that was based on the non-trivial Kaluza-Klein ansatz for how the Kaluza-Klein gauge fields modify the backgrounds.

For Kaluza-Klein isometry gauge fields, both the Einstein term and the C field kinetic terms in (V.35) contribute to their gauge kinetic terms:

$$g_{IJ}^{-2} = (g_{IJ}^{-2})^{KK} + (g_{IJ}^{-2})^{CC}, \tag{V.39}$$

where $(g_{IJ}^{-2})^{KK}$ is the Kaluza-Klein contribution coming from the Einstein term in (V.35) and $(g_{IJ}^{-2})^{CC}$ is that coming from the Ramond-Ramond C field kinetic terms in (V.35). On the other hand, if either I or J is a baryonic gauge field, coming from C reduced on a non-trivial cycle of Y, then only the dC kinetic terms in (V.35) contribute

$$g_{IJ}^{-2} = (g_{IJ}^{-2})^{CC}, \qquad \text{if } I \text{ or } J \text{ is baryonic.} \tag{V.40}$$

Let's review how the Kaluza-Klein contribution in (V.39) is obtained, see e.g. [83]. Let y^a be coordinates on Y, and $K_I^a(y)$ isometric Killing vectors (I labels the isometry). The one-form $d\phi_I$ dual to K_I is shifted by the 1-form gauge field $A_I(x) = A_I^\mu dx^\mu$, with x^μ coordinates on M. This variation of the metric leads to variation of the Ricci scalar

$$R \to R - \frac{m^{-2}}{4} g_{ab}(y) K_I^a(y) K_J^b(y) (F_I)_{\mu\nu} (F_J)^{\mu\nu}, \tag{V.41}$$

where $ds_Y^2 = g_{ab} dy^a dy^b$ is the metric on Y, with the length scale m^{-1} factored out. Since (V.41) is already quadratic in A_I, we don't need to vary $\sqrt{|g|}$. The

contribution to the Kaluza-Klein gauge field kinetic terms coming from the Einstein action is thus

$$(g_{IJ}^{-2})^{KK} = \frac{m^{-(D_c+2)}}{16\pi G_{D_t}} \int_Y g_{ab} K_I^a K_J^b vol(Y). \tag{V.42}$$

In [83], the Killing vectors are normalized so that the gauge fields have canonical kinetic terms, and then what we're referring to as the "coupling" becomes the "charge" unit; here we'll normalize K_I^a and gauge fields so that the charge unit is unity, and then physical charges governing interactions are given by what we're calling the couplings g_{IJ}^{-2}.

As an example, it was shown [83] that reducing the Einstein action on a D_c dimensional sphere, $Y = S^{D_c}$ of radius m^{-1} leads to $SO(D_c+1)$ Kaluza-Klein gauge fields in the uncompactified directions, with coupling [83]

$$(g^{-2})^{KK} = \frac{1}{8\pi G_D (D_c+1) m^2} \qquad \text{for} \quad Y = S^{D_c}, \tag{V.43}$$

with $G_D = G_{D_t} m^{D_c}/Vol(Y)$ the effective Newton's constant in the uncompactified M_D.

In [74], it was pointed out that (V.43), applied to 11d SUGRA on S^7, with Freund-Rubin flux for the Ramond-Ramond gauge field, would be incompatible with the 4d $\mathcal{N} = 8$ $SO(8)$ SUGRA of [84], but that properly including the additional contribution from the Ramond-Ramond fields fixes this problem. In our notation above, it was shown in [74] that the full coupling of the $SO(8)$ gauge fields in the AdS_4 bulk is

$$g^{-2} = (g^{-2})_{KK} + (g^{-2})_{CC} = 4 g_{KK}^{-2} = \frac{1}{16\pi G_4 m^2}, \tag{V.44}$$

which is now perfectly compatible with the 4d $\mathcal{N} = 8$ theory of [84].

We here point out that, for general Freund-Rubin compactifications on any Einstein space Y of dimension D_c, there is always a fixed proportionality between the Einstein and Ramond-Ramond contributions to the Kaluza-Klein gauge kinetic terms:

$$(g_{IJ}^{-2})^{CC} = \frac{D_c - 1}{2} (g_{IJ}^{-2})^{KK}, \tag{V.45}$$

of which (V.44) is a special case. Our relation (V.45) follows from a generalization of the argument in [74]. In a KK ansatz like that of (V.44), the contribution to g_{IJ}^{-2} from the Ramond-Ramond kinetic term in (V.35) is

$$(g_{IJ}^{-2})^{CC} = \frac{m^{-(D_c+2)}}{16\pi G_{D_t}} \int_Y \frac{1}{2} g_{ab}\, \nabla_c K_I^a\, \nabla^c K_J^b vol(Y) = \frac{D_c - 1}{2} (g_{IJ}^{-2})^{KK}. \qquad \text{(V.46)}$$

In the last step, there was an integration by parts, use of $-\nabla_c \nabla^c K_I^a = R_c^a K_I^c$, use of $R_{ab} = (D_c - 1)m^2 g_{ab}$ since Y is taken to be Einstein, and comparison with (V.42). We will check and verify the relation (V.45) more explicitly in the following sections.

As a quick application, we find from (V.43) and (V.45) that reducing 10d IIB SUGRA on S^5 leads to a theory in the AdS_5 bulk with $SO(6)$ gauge fields with coupling

$$g_{SO(6)}^{-2} = (g_{SO(6)}^{-2})^{KK} + (g_{SO(6)}^{-2})^{CC} = 3(g_{SO(6)}^{-2})^{KK} = \frac{L^2}{16\pi G_5}, \qquad \text{(V.47)}$$

where $m^{-1} = L$ is the radius of the S^5, and also the length scale of the AdS_5 vacuum. The result (V.47) agrees with that found in [85] for 5d $\mathcal{N} = 8$ SUGRA: the $SO(5)$ invariant vacuum in eqn. (5.43) of [85] has, in $4\pi G_5 = 1$ units, $R_{\mu\nu} = g^2 g_{\mu\nu}$; thus $g^{-2} = L^2/4 = L^2/16\pi G_5$, in agreement with (V.47). Using (V.31), with $Vol(S^5) = \pi^3$, gives $\tau_{SO(6)} = 8\pi^2 L g^{-2} = \pi L^3/2G_5 = N^2$. On the other hand, (V.34) here gives $\tau_{RR} = 4N^2/3$. We can also verify $\tau_{RR} = 4N^2/3$ by direct computation in the $\mathcal{N} = 4$ theory (where the free field value is not renormalized). The apparent difference with the above $\tau_{SO(6)}$ is because of the different normalization of the $U(1)_R$ vs. $SO(6)$ generators.

The relation (V.45) will prove useful in what follows, because the Ramond-Ramond contribution $(g_{IJ}^{-2})^{CC}$ is sometimes, superficially, easier to compute than the Kaluza-Klein contribution (V.42). Thanks to the general relation (V.45), the full coefficient of the kinetic terms for Kaluza-Klein gauge fields can be computed from $(g_{IJ}^{-2})^{CC}$ as

$$g_{IJ}^{-2} = (g_{IJ}^{-2})^{KK} + (g_{IJ}^{-2})^{CC} = \frac{D_c + 1}{D_c - 1} (g_{IJ}^{-2})^{CC}. \qquad \text{(V.48)}$$

V.D Gauge fields and associated p-forms on Y

The linearized fluctuations of the gauge fields modify the background as

$$F_{p+2}^{bkgd} \to (p+1)m^{-(p+1)}vol(Y) + d\left(\sum_I \omega_I \wedge A_I\right), \tag{V.49}$$

and hence, writing $F = dC$,

$$C_{p+1} \to C_{p+1}^{bkgd} + \sum_I \omega_I \wedge A_I \tag{V.50}$$

Here A_I are all of the gauge fields, both Kaluza-Klein and the baryonic ones coming from reducing C_{p+1} on non-trivial p cycles of Y.

So every gauge field A_I enters into C_{p+1} at the linearized level, and we'll here be interested in determining the associated form ω_I in (V.50). The ω_I associated with Kaluza-Klein gauge fields A_I are found from the variation of $vol(Y)$ in (V.36) by the linearized shift of the 1-form, dual to the Killing vector isometry K_I, by A_I:

$$vol(Y) \to vol(Y) + d\left(\sum_I \widehat{\omega}_I \wedge A_I\right), \qquad \text{with} \qquad d\widehat{\omega}_I = i_{K_I} vol(Y). \tag{V.51}$$

Using this in (V.49) gives (V.50), with associated p-form $\omega_I \equiv (p+1)m^{-(p+1)}\widehat{\omega}_I$ on Y.

Note that this definition of the ω_I is ambiguous under shifts of the ω_I by any closed p form. Shifts of ω_I by any exact form will have no effect, so this ambiguity in defining the ω_I associated with Kaluza-Klein gauge fields is associated with the cohomology $H_p(Y)$ of closed, mod exact, p forms on Y.

The baryonic gauge fields A_I enter into (V.50) with ω_I running over a basis of the cohomology $H_p(Y)$ of closed, mod exact, p-forms on Y. The ambiguity mentioned above in the Kaluza-Klein gauge fields corresponds to the freedom in one's choice of basis of the global symmetries, as any linear combination of a "mesonic" flavor symmetry and any "baryonic" flavor symmetry is also a valid "mesonic" flavor symmetry.

Branes that are electrically charged under C_{p+1} have worldvolume coupling $\mu_p \int C_{p+1}$, with μ_p the brane tension. Wrapping these branes on the non-trivial cycles Σ of $H^p(Y)$ yield particles in the uncompactified dimensions, and (V.50) implies that these wrapped branes carry electric charge

$$q_I(\Sigma) = \mu_p \int_\Sigma \omega_I \tag{V.52}$$

under the gauge field A_I.

Plugging (V.50) into F_{p+2} kinetic terms in (V.35) gives what we called the $(g_{IJ}^{-2})^{CC}$ contribution to the gauge field kinetic terms to be

$$(g_{IJ}^{-2})^{CC} = \frac{1}{16\pi G_{D_t}} \int_Y \omega_I \wedge *\omega_J \equiv \frac{(p+1)^2 m^{-(p+4)}}{16\pi G_{D_t}} \int_Y \widehat{\omega}_I \wedge *\widehat{\omega}_J, \tag{V.53}$$

where $\omega_I \equiv (p+1)m^{-(p+1)}\widehat{\omega}_I$ and $*\omega_I \equiv (p+1)m^{-3} * \widehat{\omega}_I$

We will use (V.53), together with (V.48) for Kaluza-Klein gauge fields, or (V.40) for baryonic gauge fields, to compute the coefficients g_{IJ}^{-2} of the gauge field kinetic terms in AdS_{d+1}. These are then related to the coefficients, τ_{IJ}, of the current-current two-point functions in the gauge theory according to (V.3).

V.E Sasaki-Einstein Y, and the form ω_R for the R-symmetry.

The modification (V.50) for the $U(1)_R$ gauge field, coming from the $U(1)_R$ isometry of Sasaki-Einstein spaces, was found in [75], which we'll review in this section.

The metric of Sasaki-Einstein Y_{2n-1} can locally be written as

$$ds^2(Y) = (\frac{1}{n} d\psi' + \sigma)^2 + ds^2_{2(n-1)}, \tag{V.54}$$

with $ds^2_{2(n-1)}$ a local, Kahler-Einstein metric, and

$$d\sigma = 2J \qquad d\Omega = ni\sigma \wedge \Omega, \tag{V.55}$$

with J the local Kahler form and Ω the local holomorphic $(n-1,0)$ form for $ds^2_{2(n-1)}$. In [75] the coordinate $\psi = \psi'/q$ was used, in order to have the range $0 \leq \psi < 2\pi$; q is given by $nd\sigma = 2\pi q c_1$, with c_1 the first Chern class of the $U(1)$ bundle over the $n-1$ complex dimensional Kahler-Einstein space with metric $ds^2_{2(n-1)}$. The $U(1)_R$ isometry is associated with the Reeb Killing vector

$$K = n\frac{\partial}{\partial\psi'}. \tag{V.56}$$

It is convenient to define the unit 1-form, dual to the Reeb vector, of the $U(1)_R$ fiber

$$e^\psi \equiv \frac{1}{n}d\psi' + \sigma. \tag{V.57}$$

Note that $de^\psi = d\sigma = 2J$. The volume form of Y_{2n-1} is

$$vol(Y_{2n-1}) = \frac{1}{(n-1)!}e^\psi \wedge J^{n-1}. \tag{V.58}$$

Following [75], the linearized effect of the $U(1)_R$ isometry (V.56) Kaluza-Klein gauge field is found by shifting

$$e^\psi \to e^\psi + \frac{2}{n}A_R, \tag{V.59}$$

where the coefficient of A_R is chosen so that the $U(1)_R$ symmetry is properly normalized: the holomorphic n-form on $C(Y)$, which leads to superpotential terms, has R-charge 2. The shift (V.59) affects the volume form (V.58) as

$$vol(Y_{2n-1}) \to vol(Y_{2n-1}) + \frac{2}{n!}A_R \wedge J^{n-1} - \frac{1}{n!}dA_R \wedge e^\psi \wedge J^{n-2}, \tag{V.60}$$

where the last term in (V.60) was added to keep the form closed:

$$vol(Y_{2n-1}) \to vol(Y_{2n-1}) + d\left(\frac{1}{n!}e^\psi \wedge J^{n-2} \wedge A_R\right). \tag{V.61}$$

The shift (V.61) alters the Ramond-Ramond flux background F^{bkgd}_{2n-1} (V.49), and thus alters C_{2n-2} as in (V.50), $\delta C_{2n-2} = \omega_R \wedge A_R$, with the $2n-3$ form ω_R given by

$$\widehat{\omega}_R \equiv \frac{\omega_R}{(2n-2)m^{-(2n-2)}} = \frac{1}{n!}e^\psi \wedge J^{n-2}. \tag{V.62}$$

In particular, for type IIB on $AdS_5 \times Y_5$, the background flux is

$$F_5^{bkgd} = 4L^4 \left(vol(Y_5) + *vol(Y_5) \right), \tag{V.63}$$

and (V.61) alters the C_4 on Y_5 as in (V.50), with 3-form ω_R given by [75]:

$$\widehat{\omega}_R \equiv \frac{1}{4L^4}\omega_R = \frac{1}{6} e^{\psi} \wedge J, \qquad \text{for } Y_5. \tag{V.64}$$

For 11d SUGRA on $AdS_4 \times Y_7$, the effect of (V.61) on the Ramond-Ramond flux

$$F_7 = 6(2L)^6 vol(Y_7) \tag{V.65}$$

leads to a shift as in (V.50) of C_6, by $\omega_R \wedge A_R$, with 5-form ω_R given by [75]

$$\widehat{\omega}_R \equiv \frac{1}{6(2L)^6}\omega_R = \frac{1}{24} e^{\psi} \wedge J \wedge J. \tag{V.66}$$

Wrapping a brane on a supersymmetric $2n-3$ cycle Σ of Y yields a baryonic particle B_Σ in the AdS_{d+1} bulk, dual to a baryonic chiral operator in the gauge theory. It was verified in [75] that the R-charges assigned to such objects by the forms (V.64) and (V.66) are compatible with the relation (V.7) in the dual field theory. Using (V.62), the R-charge assigned to such an object is related to the operator dimension Δ as

$$\begin{aligned} R[B_\Sigma] = \mu_{2n-3} \int_{\Sigma_{2n-3}} \omega_R &= \frac{2}{n} \mu_{2n-3} m^{-(2n-2)} \int_\Sigma \frac{1}{(n-2)!} e^{\psi} \wedge J^{n-2} \\ &= \frac{2}{n} \mu_{2n-3} m^{-(2n-2)} Vol(\Sigma_{2n-3}) = \frac{2m^{-1}}{nL} \Delta[B_\Sigma]. \end{aligned} \tag{V.67}$$

In going from the first to the second line of (V.67), we used the fact that the supersymmetric $2n-3$ cycles in Y are calibrated, with $vol(\Sigma) = e^{\psi} \wedge J^{n-2}/(n-2)!$. For both IIB on $AdS_5 \times Y_5$ and M theory on $AdS_4 \times Y_7$, (V.67) matches with the relation (V.7) in the 4d and 3d dual, respectively [75]: in the former case, $m^{-1} = L$ and $n = 3$ in (V.67), and in the latter case $m^{-1} = 2L$ and $n = 4$.

The $\mu_{2n-3} m^{-(2n-2)}$ factor in (V.67) is proportional to $N/Vol(Y)$ by the flux quantization condition. For $AdS_5 \times Y_5$, using (V.32) then gives [75]

$$R(\Sigma_i) = \frac{2}{3} \mu_3 L^4 Vol(\Sigma_i) = \frac{\pi N}{3} \frac{Vol(\Sigma_i)}{Vol(Y_5)}. \tag{V.68}$$

For M theory on $AdS_4 \times Y_7$, the flux quantization condition (see e.g. the recent work [86])

$$6(2L)^6 Vol(Y_7) = (2\pi\ell_{11})^6 N, \tag{V.69}$$

where $16\pi G_{11} = (2\pi)^8 \ell_{11}^9$. Using the M5 tension $\mu_5 = 1/(2\pi)^4 \ell_{11}^6$, (V.67) then gives

$$R(\Sigma_i) = \frac{\pi^2 N}{3} \frac{Vol(\Sigma_i)}{Vol(Y_7)}. \tag{V.70}$$

V.F The forms ω_I for other symmetries

In this section, we find the forms entering in (V.50), for the non-R flavor symmetries. Those associated with non-R isometries are found in direct analogy with the discussion of [75], reviewed in the previous section, for ω_R. We re-write (V.58) as

$$vol(Y_{2n-1}) = \frac{1}{2^{n-1}(n-1)!} e^\psi \wedge (de^\psi)^{n-1}. \tag{V.71}$$

Under a non-R isometry, the form e^ψ (V.57) shifts by

$$e^\psi \to e^\psi + h_i(Y) A_{F_i}, \tag{V.72}$$

with the functions $h_i(Y)$ obtained by contracting the 1-form σ in (V.57) with the Killing vector K_i for the flavor symmetry,

$$h_i(Y) = i_{K_i}\sigma = g_{ab} K^a K_i^b. \tag{V.73}$$

The last equality follows from (V.54): $i_{K_i}\sigma$ can be obtained by contracting the Reeb vector K^a and the general Killing vector K_i^b, using the metric (V.54).

In the last section, for $U(1)_R$, only the first e^ψ factor in (V.71) was shifted, as that e^ψ factor is associated with the $U(1)_R$ fiber, where $U(1)_R$ acts. Conversely, since non-R isometries do not act on the $U(1)_R$ fiber, but rather in the Kahler Einstein base, we should not shift the first e^ψ factor in (V.71), but instead shift the $n-1$ factors of de^ψ in (V.71). Effecting this shift gives

$$\delta vol(Y_{2n-1}) = \frac{1}{2^{n-1}(n-2)!} \Big(e^\psi \wedge d(h_i(Y) A_{F_i}) \wedge (de^\psi)^{n-2}$$

$$-de^{\psi}\wedge h_i(Y)A_{F_i}\wedge(de^{\psi})^{n-2}\Big), \tag{V.74}$$

where the last term was added to keep the form closed:

$$\delta vol(Y_{2n-1}) = -d\left(\frac{1}{2(n-2)!}h_i(Y)e^{\psi}\wedge J^{n-2}\wedge A_{F_i}\right). \tag{V.75}$$

Effecting this shift in F^{bkgd} leads to $\delta C_{2n-2} = \omega_{F_i}\wedge A_{F_i}$, with $2n-3$ form ω_{F_i}:

$$\widehat{\omega}_{F_i} \equiv \frac{\omega_{F_i}}{(2n-2)m^{-(2n-2)}} = -\frac{1}{2(n-2)!}h_i(Y)e^{\psi}\wedge J^{n-2} = -\frac{n(n-1)}{2}h_i(Y)\widehat{\omega}_R. \tag{V.76}$$

Aside from the factor of $-\frac{1}{2}n(n-1)h_i(Y)$, ω_{F_i} is the same as for ω_R, as given in (V.62).

In particular, for IIB on $AdS_5\times Y_5$ we have

$$\widehat{\omega}_{F_i} \equiv \frac{\omega_{F_i}}{4L^4} = -\frac{1}{2}h_i(Y_5)e^{\psi}\wedge J = -3h_i(Y_5)\widehat{\omega}_R, \tag{V.77}$$

and for M theory on $AdS_4\times Y_7$ we have

$$\widehat{\omega}_{F_i} \equiv \frac{1}{6(2L)^6}\omega_{F_i} = -\frac{1}{4}h_i(Y_7)e^{\psi}\wedge J\wedge J = -6h_i(Y_7)\widehat{\omega}_R. \tag{V.78}$$

As reviewed in (V.67), the R-charge of branes wrapped on supersymmetric cycles Σ is

$$R[B_\Sigma] = \frac{2}{n}\mu_{2n-3}m^{-(2n-2)}\int_\Sigma vol(\Sigma). \tag{V.79}$$

Using (V.76), the flavor charges of these wrapped branes can similarly be written as

$$\begin{aligned} F_i[B_\Sigma] &= \mu_{2n-3}\int_\Sigma \omega_{F_i} = -(n-1)\mu_{2n-3}m^{-(2n-2)}\int_\Sigma h_i vol(\Sigma) \\ &= -\frac{n(n-1)}{2}\cdot R[B_\Sigma]\cdot\frac{\int_\Sigma h_i vol(\Sigma)}{\int_\Sigma vol(\Sigma)}. \end{aligned} \tag{V.80}$$

In particular, for IIB on $AdS_5\times Y_5$, we have

$$F_i[B_\Sigma] = -\frac{\pi N}{Vol(Y)}\int_{\Sigma_3} h_i vol(\Sigma) = -3R[B_\Sigma]\frac{\int_\Sigma h_i vol(\Sigma)}{\int_\Sigma vol(\Sigma)}. \tag{V.81}$$

The baryonic symmetries, coming from reducing C_{2n-2} on the non-trivial $(2n-3)$-cycles of Y_{2n-1}, also alter C_{2n-2} at linear order as in (V.50), $\delta C_{2n-2} =$

$\omega_{B_i} \wedge A_{B_i}$, where the $2n-3$ forms ω_{B_i} are representatives of the cohomology $H_{2n-3}(Y,\mathbb{Z})$. These can be locally written on Y_{2n-1} as

$$\omega_{B_i} = k_i e^{\psi} \wedge \eta_i, \tag{V.82}$$

where η_i are $2(n-2)$ forms on the Kahler-Einstein base, satisfying $d\eta_i = 0$, and $\eta_i \wedge J = 0$. The normalization constants k_i in (V.82) are chosen so that $\mu_{2n-3} \int_\Sigma \omega_{B_i}$ is an integer for all $(2n-3)$-cycles Σ of Y_{2n-1}.

As mentioned in sect. V.D, this construction of the forms ω_{F_i} involves integrating an expression for $d\omega_{F_i}$, so there's an ambiguity of adding an arbitrary closed form to ω_{F_i}. Since addition of an exact form would not affect the charges of branes wrapped on closed cycles, the interesting ambiguity corresponds precisely to the same cohomology class of forms as the ω_{B_j}. This is as it should be: there is an ambiguity in our basis for the mesonic flavor symmetries, as one can always re-define them by arbitrary additions of the baryonic flavor symmetries. The form (V.76) for ω_{F_i} corresponds to some particular choice of the basis for the mesonic flavor symmetries. In the field theory dual, it may look more natural to call this a linear combination of mesonic and baryonic flavor symmetries.

V.G Computing τ_{IJ} from the geometry of Y

The expressions (V.53) for the Ramond-Ramond kinetic term contribution $(g_{IJ}^{-2})^{CC}$ is

$$(g_{IJ}^{-2})^{CC} = \frac{1}{16\pi G_{D_t}} \int_Y \omega_I \wedge *\omega_J \equiv \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_{D_t}} \int_Y \widehat{\omega}_I \wedge *\widehat{\omega}_J \tag{V.83}$$

and the Einstein action contribution (V.42) is

$$(g_{IJ}^{-2})^{KK} = \frac{m^{-(2n+1)}}{16\pi G_{D_t}} \int_{Y_{2n-1}} g_{ab} K_I^a K_J^b vol(Y_{2n-1}); \tag{V.84}$$

again, the length scale m^{-1} is factored out of the metric and volume form. As discussed in sect. 3, for gauge fields associated with isometries of Y, and in particular the graviphoton, we add the two contributions, $g_{IJ}^{-2} = (g_{IJ}^{-2})^{CC} + (g_{IJ}^{-2})^{KK}$,

whereas for baryonic symmetries there is no contribution from the Einstein action, so $g_{IJ}^{-2} = (g_{IJ}^{-2})^{CC}$.

Our claimed general proportionality (V.45) here gives

$$(g_{IJ}^{-2})^{CC} = (n-1)(g_{IJ}^{-2})^{KK}, \tag{V.85}$$

which implies that

$$4(n-1)\int_{Y_{2n-1}} \widehat{\omega}_I \wedge *\widehat{\omega}_J = \int_{Y_{2n-1}} g_{ab} K_I^a K_J^b vol(Y_{2n-1}). \tag{V.86}$$

As we'll see, this relation can look non-trivial in the geometry.

To compute $(g_{IJ}^{-2})^{CC}$ from (V.83), we first note that (V.62) gives

$$*\widehat{\omega}_R \equiv \frac{*\omega_R}{(2n-2)m^{-3}} = \frac{1}{n!} * e^{\psi} \wedge J^{n-2} = \frac{n-2}{n!} J, \tag{V.87}$$

and then, using (V.58), gives

$$\widehat{\omega}_R \wedge *\widehat{\omega}_R = \frac{(n-2)}{n!n} vol(Y_{2n-1}). \tag{V.88}$$

In particular, for the $U(1)_R$ graviphoton, we obtain

$$(g_{RR}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_{D_t}} \frac{(n-2)}{n!n} Vol(Y_{2n-1}). \tag{V.89}$$

For the mixed kinetic term between $U(1)_R$ and non-R isometries $U(1)_{F_i}$,

$$(g_{RF_i}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_{D_t}} \frac{(n-2)}{n!n} \left(-\frac{n(n-1)}{2}\right) \int_Y h_i(Y) vol(Y). \tag{V.90}$$

For the $U(1)_{F_i}$ and $U(1)_{F_j}$ kinetic terms, we similarly obtain

$$(g_{F_iF_j}^{-2})^{CC} = \frac{(2n-2)^2 m^{-(2n+1)}}{16\pi G_{D_t}} \frac{(n-2)}{n!n} \left(\frac{n(n-1)}{2}\right)^2 \int_Y h_i(Y) h_i(Y) vol(Y). \tag{V.91}$$

For $U(1)_{B_i}$ symmetries, we have

$$g_{RB_i}^{-2} = \frac{1}{16\pi G_{D_t}} \int_Y \omega_{B_i} \wedge *\omega_R = \frac{(2n-2)m^{-(2n-2)}}{16\pi G_{D_t}} \frac{n-2}{n!} \int_Y k_i e^{\psi} \wedge \eta_i \wedge J = 0, \tag{V.92}$$

where we used (V.82) for ω_{B_i}, (V.87), and we get zero immediately from $\eta_i \wedge J = 0$. Likewise,

$$g_{F_jB_i}^{-2} = 0, \tag{V.93}$$

for any isometry symmetry F_i, since (V.76) gives $\omega_{F_j} \propto \omega_R$, so $*\omega_{F_i} \propto J$, and we immediately get zero in (V.93) again from $\eta_i \wedge J = 0$. As mentioned in the introduction, there is thus never any kinetic term mixing between any of the isometry Kaluza-Klein gauge fields and any of the gauge fields coming from reducing the C fields on non-trivial homology cycles of Y. Finally, for the baryonic kinetic terms, we have

$$g^{-2}_{B_iB_j} = \frac{1}{16\pi G_{D_t}} \int_Y k_i k_j e^{\psi} \wedge \eta_i \wedge *_B \eta_j, \tag{V.94}$$

where $*_B$ acts on the $2n-2$ dimensional Kahler-Einstein base.

For the isometry (non-baryonic) gauge fields, we have to add the Kaluza-Klein contributions, $(g^{-2}_{IJ})^{KK}$, from the Einstein action, to the kinetic terms. These can either be explicitly computed, using (V.84), or one can just use our relation (V.86) to the above Ramond-Ramond contributions. It's interesting to check that our relation (V.86) is indeed satisfied. For example, the Kaluza-Klein contribution $(g^{-2}_{RR})^{KK}$ is

$$\frac{m^{-(2n+1)}}{16\pi G_{D_t}} \int_{Y_{2n-1}} g_{ab} K^a K^b vol(Y_{2n-1}) = \frac{m^{-(2n+1)}}{16\pi G_{D_t}} \frac{4}{n^2} Vol(Y_{2n-1}), \tag{V.95}$$

where we used the local form of the metric (V.54), and $U(1)_R$ isometry Killing vector (V.56), rescaled by the factor in (V.59) to have $U(1)_R$ properly normalized. Comparing with (V.89), our relation (V.86) is indeed satisfied for both of our cases of interest, $n = 3$ and $n = 4$, appropriate for IIB on $AdS_5 \times Y_5$ and M theory on $AdS_4 \times Y_7$, respectively.

Our main point will be that the $\tau_{R_tR_t}$ minimization condition (V.9) of [73] requires (V.90) to vanish, $\tau_{RF_i=0}$, so we must have

$$\int_Y h_i(Y) vol(Y) = \int_Y i_{K_i} \sigma vol(Y) = \int_Y g_{ab} K^a K^b_i = 0, \tag{V.96}$$

for every non-R isometry Killing vector K^a_i. We know from the field theory argument of (V.9) that the conditions (V.96) must uniquely determine which, among all possible R-symmetries, is the superconformal R-symmetry. Correspondingly, (V.96) determines the isometry K, from among all possible mixing with the K^a_i.

As we'll discuss in the following sections, the Z-minimization of [58] precisely implements (V.96) (in the context of toric $C(Y)$). Also, (V.92) implies that the condition τ_{Ri} of [73] is automatically satisfied for baryonic $U(1)_{B_i}$. This is the reason why the Z-minimization method of [58] did not need to include any mixing of $U(1)_R$ with the baryonic $U(1)_B$ symmetries.

For future reference, we'll now explicitly write out the above formulae for our cases of interest. For IIB on $AdS_5 \times Y_5$, we have $n = 3$ and $m^{-1} = L$, so (V.83) is

$$\tau_{IJ}^{CC} \equiv 8\pi^2 L (g_{IJ}^{-2})^{CC} = \frac{8\pi L^8}{G_{10}} \int_{Y_5} \widehat{\omega}_I \wedge *\widehat{\omega}_J = \frac{16N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} \widehat{\omega}_I \wedge *\widehat{\omega}_J, \qquad \text{(V.97)}$$

where we used (V.32) to write the result in terms of N. For I or J baryonic, this is the entire contribution:

$$\tau_{IJ} = \frac{16N^2\pi^3}{Vol(Y_5)^2} \times \int_{Y_5} \widehat{\omega}_I \wedge *\widehat{\omega}_J, \qquad \text{for } I \text{ or } J \text{ baryonic.} \qquad \text{(V.98)}$$

For isometry gauge fields, we add this to

$$\tau_{IJ}^{KK} = \frac{8\pi^2 L^8}{16\pi G_{10}} \int_{Y_5} vol(Y_5) g_{ab} K_I^a K_J^b = \frac{N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} vol(Y_5) g_{ab} K_I^a K_J^b, \qquad \text{(V.99)}$$

or, using relation (V.45), we simply have

$$\tau_{IJ} = \frac{3}{2}\tau_{IJ}^{CC} = \frac{24N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} \widehat{\omega}_I \wedge *\widehat{\omega}_J, \qquad \text{for } I \text{ and } J \text{ Kaluza-Klein.} \qquad \text{(V.100)}$$

In particular, for the $U(1)_R$ kinetic term we compute

$$\tau_{RR}^{CC} = \frac{16N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} \omega_R \wedge *\omega_R = \frac{16N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} \frac{1}{36} e^{\psi} \wedge J \wedge J = \frac{8N^2\pi^3}{9Vol(Y_5)}, \qquad \text{(V.101)}$$

and

$$\tau_{RR}^{KK} = \frac{N^2\pi^3}{Vol(Y_5)^2} \int_{Y_5} \frac{4}{9} vol(Y_5) = \frac{4N^2\pi^3}{9Vol(Y_5)}, \qquad \text{(V.102)}$$

verifying (V.45). The total for the graviphoton kinetic term coefficient then gives

$$\tau_{RR} = \tau_{RR}^{CC} + \tau_{RR}^{KK} = \frac{4}{3}\frac{N^2\pi^3}{Vol(Y_5)}. \qquad \text{(V.103)}$$

This agrees perfectly with the relation (V.16) and (V.18), given (V.33).

For the kinetic terms for two mesonic non-R symmetries, (V.100) gives

$$\tau_{F_iF_j} = \frac{12N^2\pi^3}{Vol(Y_5)^2} \times \int_{Y_5} h_i h_j vol(Y_5). \tag{V.104}$$

The relation (V.45), $\tau_{IJ}^{KK} = \frac{1}{2}\tau_{IJ}^{CC}$, which was already used in (V.104) can be written as

$$\int_{Y_5} g_{ab} K_{F_i}^a K_{F_j}^b vol(Y_5) = 4\int_{Y_5} h_i h_j vol(Y_5) = 4\int_{Y_5} g_{ac} g_{bd} K^c K^d K_{F_i}^a K_{F_j}^b vol(Y_5). \tag{V.105}$$

Likewise, using (V.98), the kinetic terms for two baryonic flavor symmetries are

$$\tau_{B_iB_j} = \frac{16N^2\pi^3}{Vol(Y_5)^2} k_i k_j \int_{Y_5} e^{\psi} \wedge \eta_i \wedge *(e^{\psi} \wedge \eta_j). \tag{V.106}$$

For M theory on $AdS_4 \times Y_7$, we set $n = 4$ for Y_7, and $m^{-1} = 2L$ for its length scale, in the above expressions. Then we obtain from (V.83), using also (V.3) with $d = 3$,

$$\tau_{IJ}^{CC} \equiv 4\pi (g_{IJ}^{-2})^{CC} = 4\pi (6)^2 (2L)^9 \frac{1}{16\pi G_{11}} \int \widehat{\omega}_I \wedge *\widehat{\omega}_J. \tag{V.107}$$

Using the flux quantization relation (V.69), (V.107) becomes

$$\tau_{IJ}^{CC} = \frac{48\pi^2 N^{3/2}}{\sqrt{6}(Vol(Y_7))^{3/2}} \int_{Y_7} \widehat{\omega}_I \wedge \widehat{\omega}_J. \tag{V.108}$$

Using (V.42) we can also write the Kaluza-Klein contribution, as

$$\tau_{IJ}^{KK} \equiv 4\pi (g_{IJ}^{-2})^{KK} = \frac{4\pi^2 N^{3/2}}{3\sqrt{6}(Vol(Y_7))^{3/2}} \int_{Y_7} g_{ab} K_I^a K_J^b vol(Y_7). \tag{V.109}$$

For τ_{RR}, (V.89) gives

$$\tau_{RR}^{CC} = \frac{\pi^2 N^{3/2}}{\sqrt{6Vol(Y_7)}}. \tag{V.110}$$

The Kaluza-Klein contribution is given by (V.84), with $g_{ab} K_R^a K_R^b = (1/2)^2$ from (V.59), so

$$\tau_{RR}^{KK} = \frac{\pi^2 N^{3/2}}{3\sqrt{6Vol(Y_7)}}. \tag{V.111}$$

Comparing (V.110) and (V.111), we verify that $\tau_{RR}^{CC} = 3\tau_{RR}^{KK}$, in agreement with our general expression (V.46) (specializing $Y_7 = S^7$ gives the case analyzed in [74]). The total is

$$\tau_{RR} = \frac{4\pi^2 N^{3/2}}{3\sqrt{6Vol(Y_7)}}. \tag{V.112}$$

We can compare (V.112) with the 3d $\mathcal{N} = 2$ gauge theory proportionality relation

$$\tau_{RR} = \frac{\pi^3}{3} C_T \qquad \text{in } d = 3, \tag{V.113}$$

where C_T is the coefficient of the stress tensor two-point function. Along the lines of [80], [82], the central charge C_T is determined in the dual, from the Einstein term of M theory on $AdS_4 \times Y_7$, to be

$$C_T = \frac{(2N)^{3/2}}{\pi\sqrt{3Vol(Y_7)}}, \tag{V.114}$$

so (V.112) indeed satisfies (V.113). As a special case, for $Y_7 = S^7$, $Vol(S^7) = \pi^4/3$ and (V.112) gives $\tau_{RR} = (2N)^{3/2}/3$.

For two non-R isometries , we have from (V.107) and (V.85), for $AdS_4 \times Y_7$:

$$\tau_{F_i F_j} = \frac{4}{3}\tau_{F_i F_j}^{CC} = \frac{\pi^2 (2N)^{3/2}}{3\sqrt{3}(Vol(Y_7))^{3/2}} \int_{Y_7} (6)^2 h_i h_j vol(Y). \tag{V.115}$$

V.H Toric Sasaki-Einstein Geometry and Z-minimization

In this section, we'll briefly summarize some of the results of [58]. Consider a Sasaki-Einstein manifold Y_{2n-1}, of real dimension $2n-1$, whose metric cone $X = C(Y)$ (V.4) is a local Calabi-Yau n-fold. The condition that (V.4) be Kahler is equivalent to $Y = X|_{r=1}$ being Sasaki, which is needed for the associated field theory to be supersymmetric. The complex structure of X pairs the Euler vector $r\partial/\partial r$ with the Reeb vector K, $K = \mathcal{I}(r\partial/\partial r)$. This is the AdS dual version of the pairing, by supersymmetry, between the dilitation generator and the superconformal R-symmetry, respectively. The physical problem of determining

the superconformal R-symmetry among all possibilities (V.8) maps to the mathematical problem of determining the Reeb vector among all $U(1)$ isometries of Y.

When $X = C(Y)$ is toric, it can be given local coordinates (y^i, ϕ_i), $i = 1 \ldots n$, and both $C(Y)$ and Y have a $U(1)^n$ isometry group, associated with the torus coordinates $\phi_i \sim \phi_i + 2\pi$. It is useful to introduce both symplectic coordinates (y^i, ϕ_i) and complex coordinates (x_i, ϕ_i). In the symplectic coordinates, the symplectic Kahler form is simply $\omega = dy^i \wedge d\phi_i$, and the metric with toric $U(1)^n$ isometry takes the form

$$ds^2 = G_{ij} dy^i dy^j + G^{ij} d\phi_i d\phi_i, \tag{V.116}$$

with G^{ij} the inverse to $G_{ij}(y)$, and $G_{ij} = \partial^2 G / \partial y^i \partial y^j$ for some convex symplectic potential function $G(y)$. In the complex coordinates, $z_i = x_i + i\phi_i$, the metric is

$$ds^2 = F^{ij} dx_i dx_j + F^{ij} d\phi_i d\phi_i, \tag{V.117}$$

and $F^{ij} = \partial^2 F(x) / \partial x_i \partial x_j$, with $F(x)$ the Kahler potential. The two coordinates are related by a Legendre transform, $y^i = \partial F(x)/\partial x_i$ and $F^{ij}(x) = G^{ij}(y = \partial F/\partial x)$, with $F(x) = (y_i \partial G/\partial y_i - G)(y)$. The holomorphic n-form of the cone $X = C(Y)$ is

$$\Omega_n = e^{x_1 + i\phi_1} (dx_1 + i d\phi_1) \wedge \ldots \wedge (dx_n + i d\phi_n). \tag{V.118}$$

The Reeb vector can be expanded as

$$K = b_i \frac{\partial}{\partial \phi_i}, \tag{V.119}$$

and its symplectic pairing with $r\frac{\partial}{\partial r}$ implies that

$$b_i = 2 G_{ij} y^j, \qquad \text{note:} \quad b_i = \text{constant}. \tag{V.120}$$

The problem of determining the superconformal R-symmetry maps to that of determining the coefficients b_i, $i = 1 \ldots n$. The component b_1 is fixed to $b_1 = n$ by the condition that $\mathcal{L}_K \Omega_n = in\Omega_n$, which is the condition that $U(1)_R$ in the field

theory is properly normalized to give the superpotential charge $R(W) = 2$. The remaining $n-1$ components b_i are unconstrained by symmetry conditions, corresponding to the field theory statement that $U(1)_R$ can mix with an $U(1)^{n-1}$ group of non-R flavor symmetries.

The space $X = C(Y)$ is mapped by the moment map, μ, where one forgets the angular coordinates ϕ_i, to $\mathcal{C} = \{y|(y, v_a) \geq 0\}$, where $v_a \in \mathbb{Z}^n$, for $a = 1 \ldots d$, are the "toric data". The supersymmetric divisors D_a of X are mapped by μ to the subspaces $(y, v_a) = 0$; here $a = 1 \ldots d$ label the divisors (d here, of course, is unrelated to the spacetime dimension d of our other sections). The Sasaki-Einstein Y is given by $X|_{r=1}$, and $r = 1$ gives $1 = b_i b_j G^{ij} = 2(b, y)$. It is also useful to define $X_1 \equiv X|_{r\leq 1}$, with $\mu(X_1) = \Delta_b \equiv \{y|(y, v_a) \geq 0, \text{ and } (y, b) \leq \frac{1}{2}\}$. The supersymmetric $2n-3$ dimensional cycles Σ_a of Y, for $a = 1 \ldots d$, have cone $D_a = C(\Sigma_a)$ which are the divisors of X, and $\mu(\Sigma_a)$ is the subspace $\mathcal{F}_a$ of Δ_b with $(y, v^a) = 0$.

The volume of Y and its supersymmetric cycles Σ_a are found from considering their cones in X_1, which are calibrated by the Kahler form $\omega = dy^i \wedge d\phi_i$. This gives

$$Vol_b(Y) = 2n(2\pi)^n Vol(\Delta_b), \quad Vol_b(\Sigma_a) = (2n-2)(2\pi)^{n-1}\frac{1}{|v_a|}Vol_b(\mathcal{F}_a). \quad \text{(V.121)}$$

As shown in [58], $\sum_a \frac{1}{|v_a|}Vol_b(\mathcal{F}_a)(v_a)_i = 2nVol(\Delta_b)b_i$, from which it follows that these volumes satisfy $\pi \sum_a Vol(\Sigma_a) = n(n-1)Vol(Y)$. (This ensures that superpotential terms, associated in the geometry with the holomorphic n-form, have $R(W) = 2$.)

The key point [58] is that the full information of the Sasaki-Einstein metric on Y is not needed to determine the volumes (V.121); the weaker information of the Reeb vector b^i and the toric data v_a suffice.

Moreover, the Reeb vector b_i can be determined from the toric data [58]. This fits with the fact that the toric data determines the dual quiver gauge theory (see e.g. [71] and references cited therein), from which the superconformal R-charges can be determined. The Z-minimization method of [58] for determining

the Reeb vector is to start with the $2n-1$ dimensional Einstein-Hilbert action for the metric g on Y_{2n-1}:

$$S[g] = \int_Y (R_g + 2(n-1)(3-2n))vol(Y), \tag{V.122}$$

including the needed cosmological constant term associated with the added flux. Though (V.122) appears to be a functional of the metric, it was shown in [58] that it's actually only a function of only the Reeb vector:

$$S[g] = S[b] = 4\pi \sum_a Vol_b(\Sigma_a) - 4(n-1)^2 Vol_b(Y). \tag{V.123}$$

The full information of the metric is not needed, the weeker information of the Reeb vector suffices to evaluate the action.

As shown in [58], the condition that b be the correct Reeb vector, associated with a Sasaki-Einstein metric, is precisely the condition that the action (V.123) be extremal:

$$\frac{\partial}{\partial b_i} S[b] = 0. \tag{V.124}$$

Defining

$$Z[b] \equiv \frac{1}{4(n-1)(2\pi)^n} S[b] = (b_1 - (n-1))2nVol(\Delta_b), \tag{V.125}$$

the equation (V.124) for $i = 1$ gives $b_1 = n$, which is just the condition that the holomorphic n-form transforms as appropriate for a $U(1)_R$ symmetry. Following [58], define

$$\widetilde{Z}[b_2, \ldots b_n] = Z|_{b_1=n} = 2nVol_b(\Delta)|_{b_1=n}. \tag{V.126}$$

The equations (V.124) for $i \neq 1$ give, upon setting $b_1 = n$,

$$0 = \frac{\partial}{\partial b_i} \widetilde{Z}[b] = -2(n+1) \int_{\Delta_b} y_i dy_1 \ldots dy_n \qquad \text{for } i \neq 1. \tag{V.127}$$

These are the equations that determine the components b_i, for $i = 2 \ldots n$, of the Reeb vector, i.e. that pick out the superconformal $U(1)_R$ from the $U(1)^n$ isometry group [58]. The correct Reeb vector *minimizes* $\widetilde{Z}$, since the matrix of second derivatives is positive [58]

$$\frac{\partial^2 \widetilde{Z}}{\partial b_i \partial b_j} \propto \int_H y_i y_j d\sigma > 0. \tag{V.128}$$

V.I Z-minimization $= \tau_{RR}$ minimization.

Let's write (V.126) and (V.121) as

$$\tilde{Z}[b_2, \dots b_n] = 2nVol_b(\Delta) = \frac{1}{(2\pi)^n} Vol_b(Y)|_{b_1=n}, \tag{V.129}$$

so Z minimization corresponds to minimizing the volume of Y, over the choices of $b_2, \dots, b_n$, subject to $b_1 = n$. This can be directly related to τ_{RR} minimization [73], i.e. minimization of the $U(1)_R$ graviphoton's coupling, since

$$\tau_{RR} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} Vol(Y). \tag{V.130}$$

The constant C_n is obtained from adding the contributions (V.89) and (V.95) and using the relation (V.3). Let us now consider the quantity (V.130), but with $Vol(Y)$ promoted to the function $Vol_b(Y)$, depending on components $b_2, \dots b_n$ of the Reeb vector:

$$\widetilde{\tau}_{R_t R_t}[b_2, \dots, b_n] \equiv C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} Vol_b(Y) = C_n (2\pi)^n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} \cdot \tilde{Z}[b_2, \dots, b_n]. \tag{V.131}$$

For the superconformal $U(1)_R$ values of $b_2, \dots b_n$, $\widetilde{\tau}_{R_t R_t} = \tau_{RR}$.

If we hold $L^{d-3} m^{-(2n+1)}/G_{D_t}$ fixed, (V.131) suggests a direct relation between Z and τ_{RR} minimization. Physically, we should hold the number of flux units N fixed, i.e. use the flux quantization relation to eliminate $L^{d-3} m^{-(2n+1)}/G_{D_t}$ in favor of $N/Vol(Y)$. In particular, for IIB on $AdS_5 \times Y_5$ and M theory on $AdS_4 \times Y_4$,

$$AdS_5 \times Y_5 : \; C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} = \frac{4\pi^3}{3} \left(\frac{N}{Vol(Y)} \right)^2,$$

$$AdS_4 \times Y_7 : \; C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} = \frac{4\pi^2}{3\sqrt{6}} \left(\frac{N}{Vol(Y)} \right)^{3/2}. \tag{V.132}$$

Using these in (V.130) shows that, for fixed N, τ_{RR} is actually *inversely* related to $Vol(Y)$. From that perspective, it would seem that Z minimization instead *maximizes* τ_{RR}, which is opposite to the result of [73] that the exact superconformal $U(1)_R$ minimizes τ_{RR}. To avoid this, we do not promote the constant $Vol(Y)$ in

the flux relations (V.132) to the function $Vol_b(Y)$ of the Reeb vector, but instead there hold it fixed to its true, physical value. Then the function $\widetilde{\tau}_{R_t R_t}[b]$ (V.131) is simply a constant times the function $\widetilde{Z}[b]$ of [58].

To use the formulae of our earlier sections, consider the Killing vectors

$$\chi = \chi_i \frac{\partial}{\partial \phi_i} \tag{V.133}$$

for the $U(1)^n$ isometries of toric Y_{2n-1}. R-symmetries, and in particular the Reeb vector, have $\chi_1 = n$, and non-R isometries have $\chi_1 = 0$. As we discussed in sections V.E and V.F, the isometry $d\phi_\chi \to d\phi_\chi + A_\chi$ has an associated $2n-3$ form, which is found from the associated shift $e^\psi \to e^\psi + h_\chi(Y) A_\chi$. For the R-symmetry, this comes from the shift of $d\psi'$, and for non-R flavor symmetries the shift is via $h_\chi = i_\chi \sigma$. Using the second equality in (V.73), we have

$$h_\chi(Y) = F^{ij} b_i \chi_j = G^{ij} b_i \chi_j = 2 y^i \chi_i = 2\langle r^2 \theta, \chi \rangle, \tag{V.134}$$

with the inner product with $r^2\theta$ as in [58]. For the Reeb vector, (V.134) gives $h_K = 1$, since the cone $r = 1$ has $1 = b_i b_j G^{ij} = 2(b, y)$ [58].

For the non-R isometries, we can take as our basis of Killing vectors e.g. $\chi^{(i)} = \frac{\partial}{\partial \phi_i}$, so $\chi^{(i)}_j = \delta_{ij}$, for $i = 2 \ldots n$. Then (V.134) gives simply

$$h_{\chi^{(i)}} = 2 y^i. \tag{V.135}$$

In this basis, where $U(1)_{F_i}$ is associated with Killing vector $\frac{\partial}{\partial \phi_i}$, the F_i charge of a brane wrapped on cycle Σ is

$$\begin{aligned} F_i[B_\Sigma] &= -(n-1)\mu_{2n-3} m^{-(2n-2)} \int_\Sigma 2 y^i vol(\Sigma) \\ &= -n(n-1) \cdot R[B_\Sigma] \cdot \frac{\int_\Sigma y_i vol(\Sigma)}{\int_\Sigma vol(\Sigma)}. \end{aligned} \tag{V.136}$$

In particular, for IIB background $AdS_5 \times Y_5$, we have

$$F_i[B_\Sigma] = -\frac{2\pi N}{Vol(Y_5)} \int_{\Sigma_3} y_i vol(\Sigma), \tag{V.137}$$

and for M theory background $AdS_4 \times Y_7$ we have

$$F_i[B_\Sigma] = -\frac{4\pi^2 N}{Vol(Y_7)} \int_{\Sigma_5} y_i vol(\Sigma). \tag{V.138}$$

Using our formulae from sect. V.G, we can determine the kinetic terms g_{IJ}^{-2}, and hence τ_{IJ} in terms of the geometry of Y. In particular, using (V.90) and (V.135), we have

$$\tau_{RF_i} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} \left(-n(n-1)\right) \int_Y y^i vol(Y), \tag{V.139}$$

with C_n the same constant appearing in (V.130). Note that

$$\int_Y y^i vol(Y) = 2(n+1) \int_{X_1} y^i vol(X_1) = 2(n+1)(2\pi)^n \int_{\Delta_b} y^i dy^1 \dots dy^n, \tag{V.140}$$

($2(n+1)$ accounts for the extra r integral in X_1). Moreover, eqn. (3.21) of [58] gives

$$\int_{\Delta_b} y^i dy^1 \dots dy^n = -\frac{1}{2(n+1)} \frac{\partial}{\partial b_i} Vol_b(\Delta). \tag{V.141}$$

So (V.139) gives

$$\tau_{RF_i} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} (2\pi)^n \frac{(n-1)}{2} \frac{\partial}{\partial b_i} \widetilde{Z}[b_2, \dots b_n]. \tag{V.142}$$

As discussed, we take the factors in (V.132) to be b_i independent constants, so (V.142) can be written as

$$\tau_{RF_i} = \frac{(n-1)}{2} \frac{\partial}{\partial b_i} \widetilde{\tau}_{R_t R_t}[b_2 \dots b_n]. \tag{V.143}$$

The relation (V.142) shows that the $\tau_{R_t R_t}$ minimization equations, $\tau_{RF_i} = 0$, are indeed equivalent to the Z minimization equations (V.127) of [58].

We can similarly use our formula (V.90) and (V.134) to obtain the coefficient $\tau_{F_i F_j}$ for two flavor currents:

$$\tau_{F_i F_j} = C_n \frac{L^{d-3} m^{-(2n+1)}}{16\pi G_{D_t}} \left(n(n-1)\right)^2 \int_Y y^i y^j vol(Y), \tag{V.144}$$

with C_n the same constant appearing in (V.130). Note now that

$$\int_Y y^i y^j vol(Y) = 2(n+2) \int_{X_1} y^i y^j vol(X_1) = 2(n+2)(2\pi)^n \int_{\Delta_b} y^i y^j dy^1 \dots dy^n. \tag{V.145}$$

Moreover, in analogy with the derivation of (V.141), in eqn. (3.21) of [58], we find:

$$\int_{\Delta_b} y^i y^j dy^1 \dots dy^n = \frac{1}{4(n+1)(n+2)} \frac{\partial^2}{\partial b_i \partial b_j} Vol_b(\Delta). \tag{V.146}$$

We can then write (V.144) as

$$\tau_{F_i F_j} = \frac{n(n-1)^2}{4(n+1)} \frac{\partial^2}{\partial b_i \partial b_j} \widetilde{\tau}_{R_t R_t}[b_2 \dots b_n], \tag{V.147}$$

where again we take (V.132) as b independent.

Since $\widetilde{\tau}_{R_t R_t}$ is proportional to $\widetilde{Z}$, (V.147) provides a way to evaluate the current two-point function coefficients $\tau_{F_i F_j}$ entirely in terms of the Reeb vector and the toric data, without needing to know the metric.

In [73], we discussed the trial function $\tau_{R_t R_t}(s_i)$, which is quadratic in the parameters s_i, and satisfies

$$\tau_{R_t R_t}|_{s^*} = \tau_{RR}, \quad \frac{\partial}{\partial s_i} \tau_{R_t R_t}|_{s^*} = 2\tau_{Ri} = 0, \quad \frac{\partial^2}{\partial s_i \partial s_j} \tau_{R_t R_t}(s) = 2\tau_{ij}. \tag{V.148}$$

This can be compared with the function $\widetilde{\tau}_{R_t R_t}(b_i)$ defined above, which coincides with τ_{RR} for the minimizing values b_i^*, which are determined by setting the derivatives to zero, (V.143), and the second derivatives (V.147) are proportional to τ_{ij}, as in (V.148). The relation between s_i and b_i can be chosen to convert the coefficients in (V.147) to equal those of (V.148).

Let us now consider further the expression (V.136), or more explicitly (V.137) and (V.138), for the flavor charges of branes wrapped on cycles. We would like to evaluate these for the supersymmetric cycles $\Sigma_a \subset Y$, i.e. to evaluate

$$\int_{\Sigma_a} y^i vol(\Sigma) \tag{V.149}$$

in terms of the toric data and Reeb vector. Note that

$$\int_{\Sigma_a} y^i vol(\Sigma) = 2n \int_{C(\Sigma_a)} y^i vol(C(\Sigma_a)) = 2n(2\pi)^{n-1} \int_{\mathcal{F}_a} y^i d\sigma_a, \tag{V.150}$$

where the $2n$ factor is from the extra r integral in going from Σ_a to $C(\Sigma_a)$, and $d\sigma_a$ is the measure on $\mathcal{F}_a$, from $\int \delta((y, v_a)) dy^1 \dots dy^n$. In analogy with the derivation

of eqn. (3.21) in [58], it seems likely that the y^i in (V.149) and (V.150) can be obtained from the volume $Vol_b(\Sigma_a)$ in (V.121) by differentiating w.r.t. b_i. But completing this argument, accounting for all the potential new boundary terms, seems potentially subtle (to us).

Let us, instead, note a different way to compute the charges from the toric data. Consider the expression for $Vol_b(Y)$, as a function of both b *and* the toric data $(v_a)_i$. In the integral leading to $Vol_b(Y) = 2n(2\pi)^n Vol(\Delta_b)$ in (V.121), the vectors $(v_a)^i$ appear via the boundary of Δ_b, which has $(y, v_a) \geq 0$. Thinking of them as variables, taking the derivative w.r.t. v_a then gives a contribution only on the boundary $(y, v_a) = 0$:

$$\frac{\partial}{\partial (v_a)_i} Vol(\Delta_b) = -\int_{\mathcal{F}_a} y_i d\sigma_a. \tag{V.151}$$

Using (V.150) and (V.121) then gives

$$\int_{\Sigma_a} y^i vol(\Sigma) = -\frac{1}{2\pi} \frac{\partial}{\partial (v_a)_i} Vol_b(Y). \tag{V.152}$$

In the above expressions for $\widetilde{\tau}_{RR}$ and τ_{RF_i} and $\tau_{F_iF_j}$, the Ramond-Ramond and Kaluza-Klein contributions to g_{IJ}^{-2} were summed together, in the coefficient C_n. Using the relation (V.46), which here gives $(g_{IJ}^{-2})^{CC} = (n-1)(g_{IJ}^{-2})^{KK}$, those two contributions have a fixed ratio. Let us now examine that relation in the present context. For general Killing vectors $\chi^{(I)}$ and $\chi^{(J)}$, the contribution (V.53) to their mixed kinetic term is

$$(g_{IJ}^{-2})^{CC} \propto \int_Y 4 y^i y^j \chi_i^{(I)} \chi_j^{(J)} vol(Y). \tag{V.153}$$

The contribution (V.42) of the Einstein term is similarly

$$(g_{IJ}^{-2})^{KK} \propto \int_Y G^{ij} \chi_I^{(I)} \chi_j^{(J)} vol(Y). \tag{V.154}$$

Taking both I and J to be the R-symmetry, with χ_I and χ_J the Reeb vector, the relation from $(g_{IJ}^{-2})^{CC} = (n-1)(g_{IJ}^{-2})^{KK}$ is

$$\int_Y G^{ij} b_i b_j dy_1 \dots dy_n = 4 \int_Y (y^i b_i)^2 dy_1 \dots dy_n; \tag{V.155}$$

which is clearly satisfied, since $2b_i y^i = G^{ij} b_i b_j = 1$. For non-R flavor symmetries, the identity is less trivial. For general Y_{2n-1} it states that

$$\int_{Y_{2n-1}} G^{ij} vol(Y) = 4(n-1)^2 \int_{Y_{2n-1}} y^i y^j vol(Y) \qquad i,j \neq 1. \tag{V.156}$$

The extra factor of $(n-1)^2$, as compared with (V.155), is as in (V.90), coming from writing the volume form as $\sim e^\psi \wedge (de^\psi)^{n-1}$ and the fact that ω_R is found from the shift of the first e^ψ factor, whereas the non-R isometries are obtained by shifting the $n-1$ factors of $d(e^\psi)$. The relation (V.156) can indeed be verified to hold in the various examples. It can also be written in terms of integrals over Δ_b, by extending to X_1 and doing the extra r integrals, as

$$(n+1)\int_{\Delta_b} G^{ij} dy^1 \dots dy^n = 4(n-1)^2(n+2)\int_{\Delta_b} y^i y^j dy^1 \dots dy^n. \tag{V.157}$$

V.J Examples and checks of AdS/CFT: $Y^{p,q}$

The metric of [24], [69] is simply written in the basis of unit one-forms

$$\begin{aligned} e^\psi &= \frac{1}{3}(d\psi' - \cos\theta d\phi + y(d\beta + \cos\theta d\phi)) \\ e^\theta &= \sqrt{\frac{1-y}{6}} d\theta, \qquad e^\phi = \sqrt{\frac{1-y}{6}} \sin\theta d\phi, \\ e^y &= \frac{1}{\sqrt{wv}} dy, \qquad e^\beta = \frac{\sqrt{wv}}{6}(d\beta + \cos\theta d\phi), \end{aligned} \tag{V.158}$$

as $ds_Y^2 = (e^\theta)^2 + (e^\phi)^2 + (e^y)^2 + (e^\beta)^2 + (e^\psi)^2$. The coordinate y lives in the range $y_1 \leq y \leq y_2$, where y_1 and y_2 are the two smaller roots of $v(y) = 0$ [69] :

$$y_1 = \frac{1}{4p}\left(2p - 3q - \sqrt{4p^2 - 3q^2}\right), \qquad y_2 = \frac{1}{4p}\left(2p + 3q - \sqrt{4p^2 - 3q^2}\right). \tag{V.159}$$

The local Kahler form of the 4d base is

$$J = e^\theta \wedge e^\phi + e^y \wedge e^\beta. \tag{V.160}$$

The gauge symmetries in AdS_5 of IIB on $Y_{p,q}$, and the global symmetries of the dual SCFTs [47], are $U(1)_R \times SU(2) \times U(1)_F \times U(1)_B$. The first three

factors are associated with isometries of the metric, and $U(1)_B$ comes from the single representative of $H_3(Y_{p,q}, Z)$ (topologically, all are $S^2 \times S^3$). As usual, the superconformal $U(1)_R$ symmetry is associated with the shift in e^{ψ}: $\frac{1}{3}d\psi' \to \frac{1}{3}d\psi' + \frac{2}{3}A_R$, and the associated 3-form is that of [75]:

$$\widehat{\omega}_R \equiv \frac{1}{4L^4}\omega_R = \frac{1}{6}e^{\psi} \wedge J. \tag{V.161}$$

The $SU(2)$ is symmetry is an non-R isometry, associated with rotations of the spherical coordinates θ and ϕ. Finally, the $U(1)_F$ isometry is associated with shifts $d\beta + \cos\theta d\phi \to d\beta + \cos\theta d\phi + A_F$. $U(1)_\phi \subset SU(2)$ and $U(1)_F$ form a basis for the $U(1)^2$ non-R isometries, expected from the fact that $Y_{p,q}$ is toric [69]. The 3-forms associated with these flavor $U(1)^2$ are found from (V.73) and (V.77) to be

$$\widehat{\omega}_\phi \equiv \frac{1}{4L^4}\omega_\phi = -\cos\theta\widehat{\omega}_R \qquad \text{and} \qquad \widehat{\omega}_F \equiv \frac{1}{4L^4}\omega_F = -y\widehat{\omega}_R. \tag{V.162}$$

The 3-form associated with the $U(1)_B$ baryonic symmetry was already constructed in [70], restricting their form $\Omega_{2,1}$ on $C(Y_{p,q})$ to $Y_{p,q}$ by setting $r = 1$:

$$\mu_3\omega_B = \frac{9}{8\pi^2}(p^2 - q^2)e^{\psi} \wedge \eta \qquad \eta \equiv \frac{1}{(1-y)^2}(e^{\theta} \wedge e^{\phi} - e^{y} \wedge e^{\beta}), \tag{V.163}$$

where the normalization constant is to keep the periods of $\mu_3 \int C_4$ properly integral.

D3 branes wrapped on the various supersymmetric 3-cycles Σ_a of Y map to the di-baryons of the dual gauge theory [47] as:

$$\Sigma_1 \leftrightarrow \det Y, \quad \Sigma_2 \leftrightarrow \det Z, \quad \Sigma_3 \leftrightarrow \det U_\alpha, \quad \Sigma_4 \leftrightarrow \det V_\alpha. \tag{V.164}$$

The cycles Σ_1 and Σ_2 are given by the coordinates at $y = y_1$ and $y = y_2$ respectively [69]. The cycle Σ_3 is given by fixing θ and ϕ to constant values, which yields the $SU(2)$ collective coordinate of the di-baryon [70]. The cycle $\Sigma_4 \cong \Sigma_2 + \Sigma_3$.

As in [75], the R-charges of the wrapped D-3 branes, computed from $\mu_3 \int_{\Sigma_i} \omega_R$, are

$$R(\Sigma_i) = \frac{\pi N}{3Vol(Y_5)} \int_{\Sigma_i} vol(\Sigma) = \frac{\pi N}{3}\frac{Vol(\Sigma_i)}{Vol(Y_5)}. \tag{V.165}$$

It was verified in [69], [46], [47], [70] that the R-charges computed from the cycle volumes as in (V.165) agree perfectly with the map (V.164) and the superconformal

R-charges, computed in the field theory dual by using the a-maximization [14] method.

We can similarly verify that integrating the $U(1)_\phi$, $U(1)_F$ and $U(1)_B$ 3-forms (V.162) and (V.163) over the 3-cycles Σ_a agree with the map (V.164) and the corresponding charges of the dual field theory [47]. For $U(1)_B$ we have

$$B(\Sigma_i) = \mu_3 \int_{\Sigma_i} \omega_B = \frac{9}{8\pi^2}(p^2 - q^2) \int_{\Sigma_i} e^\psi \wedge \frac{1}{(1-y)^2}(e^\theta \wedge e^\phi - e^y \wedge e^\beta), \quad \text{(V.166)}$$

and, as already computed in [70], this gives (reversing Σ_1's orientation)

$$B(\Sigma_1) = (p-q), \qquad B(\Sigma_2) = (p+q), \qquad B(\Sigma_3) = p, \quad \text{(V.167)}$$

in agreement with the $U(1)_B$ charges of [47] for Y, Z, and U_α, respectively. One minor difference is that we normalize the $U(1)_B$ charges for the bi-fundamentals with a factor of $1/N$, so that the charges of the baryons are $O(1)$ rather than $O(N)$; this is natural when $U(1)_B$ is thought of as an overall $U(1)$ factor of a $U(N)$ gauge group, and also natural in terms of having the charges be properly quantized, so that $\int \mu_3 C_4$ and $\int B(Q_i) A_B$ are gauge invariant mod 2π under large gauge transformations.

We can compute the $U(1)_F$ charges of the wrapped D3 branes by using (V.81), here with $h = y/3$:

$$F(B_\Sigma) = -R(B_\Sigma) \frac{\int_\Sigma y vol(\Sigma)}{\int_\Sigma vol(\Sigma)}. \quad \text{(V.168)}$$

This gives

$$F(\Sigma_1) = y_1 R(\Sigma_1), \qquad F(\Sigma_2) = -y_2 R(\Sigma_2), \qquad F(\Sigma_3) = -\frac{1}{2}(y_1 + y_2) R(\Sigma_3). \quad \text{(V.169)}$$

The Σ_1 and Σ_2 cases follow immediately from (V.168), since $y = y_1$ and $y = y_2$ is constant (the Σ_1 integral gets an extra minus sign from the orientation), and $F(\Sigma_3)$ in (V.169) simply comes from $\int_{y_1}^{y_2} y dy / \int_{y_1}^{y_2} dy$. The charges (V.169) agree with the $U(1)_F$ charges of [47], up to the ambiguity that we have mentioned for redefining $U(1)_F$ by an arbitrary addition of $U(1)_B$, i.e. $U(1)_F^{here} = U(1)_F^{there} + \alpha U(1)_B$.

Using the metric [24], [69], we can explicitly compute the contributions τ_{IJ}^{CC} in (V.97) and the contributions τ_{IJ}^{KK} in (V.99), and verify that $\tau_{IJ}^{CC} = 2\tau_{IJ}^{KK}$, as expected from (V.45), for the $U(1)_R$ and $U(1)_\phi$ and $U(1)_F$ isometry gauge fields. For $U(1)_B$, there is only the τ_{IJ}^{CC} contribution to τ_{IJ}. For the superconformal $U(1)_R$, we find, as expected $\tau_{RR}^{KK} = 4N^2\pi^3/9Vol(Y_{p,q})$ and $\tau_{RR}^{CC} = 8N^2\pi^3/9Vol(Y_{p,q})$, with [69]

$$Vol(Y_{p,q}) = \frac{q^2[2p + (4p^2 - 3q^2)^{1/2}]}{3p^2[3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}]}\pi^3. \tag{V.170}$$

For τ_{FF}^{KK}, the metric [24], [69] gives $g_{ab}K_F^a K_F^b = \frac{1}{36}wq + \frac{1}{9}y^2 = \frac{1}{36}w(y)$, so (V.99) yields

$$\tau_{FF}^{KK} = \frac{N^2\pi^3}{36Vol(X_5)}\frac{\int dy w(y)(1-y)}{\int dy(1-y)} = \frac{N^2\pi^3}{18Vol(X_5)}\frac{\sqrt{4p^2-3q^2}}{p^2}\left(2p - \sqrt{4p^2-3q^2}\right). \tag{V.171}$$

Using $\widehat{\omega}_F$ of (V.162) in (V.97) we can also compute

$$\tau_{FF}^{CC} = \tau_{RR}^{CC}\frac{\int dy y^2(1-y)}{\int dy(1-y)} = \tau_{RR}^{CC}\frac{1}{16}\frac{\int dy w(y)(1-y)}{\int dy(1-y)} = 2\tau_{FF}^{KK}, \tag{V.172}$$

satisfying the relation (V.45). Combining (V.171) and (V.172) gives

$$\tau_{FF} = \frac{N^2\pi^3}{6Vol(Y_{p,q})}\frac{\sqrt{4p^2-3q^2}}{p^2}\left(2p - \sqrt{4p^2-3q^2}\right). \tag{V.173}$$

This result for τ_{FF} can be compared with the field theory prediction. The $U(1)_F$ charges of the bifundamentals are found from the $U(1)_F$ charges (V.169) of the dibaryons, and the map (V.164) (so the factor of N from (V.165) is eliminated), e.g. $F(Z) = -y_2 R(Z) = -y_2\pi Vol(\Sigma_2)/3Vol(Y_5)$, which looks rather ugly when written out in terms of p and q. From these charges and the $U(1)_R$ charges, we can compute the 't Hooft anomalies, and thereby compute τ_{FF} on the field theory side by using the relation $\tau_{FF} = -3\mathrm{Tr}RFF$. The result is found to agree perfectly with (V.173).

Let us now consider τ_{RF}. The Kaluza-Klein contribution is given as in (V.99), with $g_{ab}K_R^a K_F^b = y/9$, and the integral over y vanishes, so $\tau_{RF}^{KK} = 0$. Likewise, $\tau_{RF}^{CC} = 0$, because $\int y(1-y)$ vanishes. So, as expected, $\tau_{RF} = 0$.

As we discussed in the previous section, the $F_i[\Sigma_a]$ charges and τ_{IJ} can also be computed entirely from the toric data and Z-function of [58]. In the toric basis of [58],

$$v_1 = (1,0,0), \quad v_2 = (1, p-q-1, p-q), \quad v_3 = (1,p,p), \quad v_4 = (1,1,0). \tag{V.174}$$

The Z-function is, with $(b_1, b_2, b_3) \equiv (x, y, t)$, [58]

$$Z[x,y,t] = \frac{(x-2)p(p(p-q)x + q(p-q)y + q(2-p+q)t)}{2t(px - py + (p-1)t)((p-q)y + (1-p+q)t)(px + qy - (q+1)t)}, \tag{V.175}$$

which, imposing $x = 1$, is minimized for [58]:

$$b_{min} = \left(3, \frac{1}{2}(3p - 3q + \ell^{-1}), \frac{1}{2}(3p - 3q + \ell^{-1})\right),$$

$$\ell^{-1} = \frac{1}{q}\left(3q^2 - 2p^2 + p\sqrt{4p^2 - 3q^2}\right). \tag{V.176}$$

Our formula (V.147), for example, gives $\tau_{F_i F_j}$, for the F_i associated with the $\sim \frac{\partial}{\partial \phi_i}$ Killing vectors, in terms of the Hessian of second derivatives of the function (V.175), evaluated at (V.176). To connect the results in the toric basis for the flavor symmetries to those discussed above, we note that the Killing vector for shifting β can be related to those for shifting ϕ_1 and ϕ_2 as $\frac{\partial}{\partial \beta} = \frac{\ell^{-1}}{6}\left(\frac{\partial}{\partial \phi_2} + \frac{\partial}{\partial \phi_3}\right)$, so $U(1)_F = \frac{\ell^{-1}}{6}(U(1)_2 + U(1)_3)$.

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Appendix.

On the superconformal window of the other duals of [36]

Appendix..A Reviewing $SU(N_c)$ SQCD, with N_f fundamental flavors, and an adjoint X

Let us briefly review the a-maximization analysis of Kutasov, Parnachev, and Sahakyan (KPS) [16] for this theory, with $W_{tree} = 0$. The anomaly free superconformal R-charges of the fields are $R(Q) = R(\widetilde{Q}) \equiv y$ for the fundamentals and $R(X) = (1-y)/x$ for the adjoint, where $x \equiv N_c/N_f$. a-maximization determines $y(x)$ via maximizing

$$a^{(p)}_{KPS}(y,x)/N_f^2 = 2x^2 + x^2\left(3\left(\frac{1-y}{x}-1\right)^3 - \left(\frac{1-y}{x}-1\right)\right)$$

$$+2x\left(3(y-1)^3-(y-1)\right) + \sum_{j=0}^{p}\left(2y + j\frac{1-y}{x} - \frac{2}{3}\right)^2\left(5 - 3[2y + j\frac{1-y}{x}]\right), \tag{Appendix..1}$$

w.r.t. y; this has solution $y^{(p)}(x)$. The sums account for the generalized mesons $QX^j\widetilde{Q}$ hitting their unitarity bound, with p the greatest integer such that $R(QX^j\widetilde{Q})$ would naively violated the unitarity bound. The solution $y_{KPS}(x)$ is obtained by patching together the functions $y^{(p)}(x)$, with the appropriate value of p depending on x. The function $y_{KPS}(x)$ is monotonically decreasing, with asymp-

totic value $y(x \to \infty) \to y_{as} = (\sqrt{3}-1)/3$. $R(X) = (1-y)/x$ is also monotonically decreasing in x and, for $x \to \infty$, $R(X) \to (1 - y_{as})/x$.

The superpotential $W_{A_k} = \mathrm{Tr}X^{k+1}$ is a relevant deformation of the $W = 0$ SCFT if $R(X^{k+1}) < 2$ (since $\Delta(W) = \frac{3}{2}R(W)$), i.e. if $R(X) = (1-y)/x < 2/(k+1)$. Since $R(X)$ monotonically decreases with x, W_{A_k} can always be made relevant, by taking x sufficiently large, $x > x^{min}_{A_k}$, where $x^{min}_{A_k}$ is determined by the condition that $(1 - y(x^{min}_{A_k}))/x^{min}_{A_k} = 2/(k+1)$. Using the numerical solution for $y(x)$, the numerical values of $x^{min}_{A_k}$ can be obtained for arbitrary k. For large k, $x^{min}_{A_k}$ is large, and then $R(X) \approx (1 - y_{as})/x^{min}_{A_k} = 2/(k+1)$ gives $x^{min}_{A_k} \to \left(\frac{4-\sqrt{3}}{6}\right) k \approx 0.3780k$ [16].

The A_k theory has dual description [39], [40], [34] in terms of a magnetic $SU(\widetilde{N_c})$ gauge theory, with $\widetilde{N_c} = kN_f - N_c$. It has an adjoint field Y, N_f fundamental flavors q, $\tilde{q}$, and N_f^2 gauge singlet fields M_j, for $j = 1 \ldots k$. The superpotential is

$$W_{\widetilde{A_k}} = \mathrm{Tr}Y^{k+1} + \sum_{j=1}^{k} M_{k-j} q Y^{j-1} \tilde{q} \ . \qquad \text{(Appendix..2)}$$

The analysis of the dual theory is similar to that of the electric theory, with $x \to \widetilde{x} = \widetilde{N_c}/N_f = k - x$, though the specifics are not identical, because of the effect of the additional gauge singlets M_j and superpotential terms in (Appendix..2). We refer the reader to [16], for the details of the a-maximizing $\widetilde{y}(\widetilde{x})$ in the magnetic theory. The qualitative result is that $\widetilde{y}(\widetilde{x})$ drops to zero a little faster on the magnetic side than the electric $y(x)$, so the $\mathrm{Tr}Y^{k+1}$ term in (Appendix..2) is relevant for $\widetilde{x} > \widetilde{x}^{min}_{A_k}$, with $\widetilde{x}^{min}_{A_k} < x^{min}_{A_k}$. In particular, for $k \gg 1$, $\widetilde{x}^{min}_{A_k} \approx 0.3578k$ [16] .

The superconformal window, where both dual descriptions of the $A_k(N_c, N_f)$ SCFTs are useful, is $x^{min}_{A_k} < x < k - \widetilde{x}^{min}_{A_k}$; for $k \gg 1$, it's $0.3780k < x < 0.6422k$. Within this range the electric and magnetic theories have the same central charge a, as guaranteed by 't Hooft anomaly matching. Outside of this range, there are accidental symmetries that are manifest in one of

the dual descriptions but not in the other so, without accounting for these accidental symmetries, the central charge as computed by a-maximization for the electric and magnetic theories can appear to differ [16]. The larger of a_{elec} or a_{mag} is the correct one – it's larger because of maximizing a_{trial} over R-symmetries that can mix with the additional, accidental flavor symmetries.

Appendix..B Some immediate generalizations, with other groups and matter content

Many analogs of the duality of [39], [40], [34] were soon given in [36], [53], [41], all for theories with W_{A_k} type LG superpotential. Without the LG superpotential, those theories are expected to flow to other SCFTs, which can now be analyzed via a-maximization. Doing so determines when the W_{A_k} superpotential is relevant. Doing a similar a-maximization analysis of the duals of [36], [53], [41] determines when the dual A_k LG superpotentials are relevant. Combining the two bounds, as in the analysis of [16], reviewed in the last section, determines the superconformal window for where the duals of [36], [53], [41] are useful. In particular, we can verify that the superconformal window is non-empty for all k.

As in the analysis of [16], it is most convenient to consider the limit $N_c \gg 1$, $N_f \gg 1$, holding $x \equiv N_c/N_f$ fixed. However, as we'll now explain, the a-maximization analysis of all of the examples of [36], [53], [41] involving a single gauge group becomes simply identical to that of [16] in this limit, where we drop terms $O(1/N_c)$ or $O(1/N_f)$.

The examples of [36], [53], [41] involving a single gauge group are:

group	X	Q	# mesons QX^jQ	a/a_{KPS}
$SO(N_c)$	$\square\square$	$N_f \cdot \square$	$\frac{1}{2}N_f(N_f+1)$	$\frac{1}{2}$
$Sp(N_c)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$2N_f \cdot \square$	$N_f(2N_f-1)$	2
$SO(N_c)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}$	$N_f \cdot \square$	$\frac{1}{2}N_f(N_f+(-1)^j)$	$\frac{1}{2}$
$Sp(N_c)$	$\square\square$	$2N_f \cdot \square$	$N_f(2N_f-(-1)^j)$	2
$SU(N_c)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}\oplus\begin{smallmatrix}\overline{\square}\\\overline{\square}\end{smallmatrix}$	$N_f \cdot (\square \oplus \overline{\square})$	N_f^2 or $N_f(N_f-1)$	1
$SU(N_c)$	$\square\square\oplus\overline{\square\square}$	$N_f \cdot (\square \oplus \overline{\square})$	N_f^2 or $N_f(N_f+1)$	1
$SU(N_c)$	$\begin{smallmatrix}\square\\\square\end{smallmatrix}\oplus\overline{\square\square}$	$8\cdot\square \oplus N_f \cdot (\square \oplus \overline{\square})$	$\sim N_f^2$	1

(Appendix..3)

Our notation for $Sp(N_c)$ is that $SU(2) \cong Sp(1)$.

Let us compare the theory on the first line of (Appendix..3) with the $SU(N_c)$ with adjoint theory analyzed in [16]. The anomaly free R-symmetry is constrained to satisfy $2(N_c-2)+2N_f(R(Q)-1)+2(N_c+2)(R(X)-1)=0$. But in the $N_c \gg 1$ and $N_f \gg 1$ limit, holding fixed $x \equiv N_c/N_f$, this gives an identical relation, $R(X)=(1-y)/x$, with $x \equiv N_c/N_f$, as in the case reviewed in the previous subsection. Computing the analog of (Appendix..1) for the theory on the first line of (Appendix..3), we find that every term is now simply half of that in (Appendix..1), coming from the fact that the only difference (to leading order $O(1/N_c)$ and $O(1/N_f)$) is that there are half as many of each of the different fields. For example, the $2x^2$ term in (Appendix..1) is the contribution of the $|SU(N_c)| \approx N_c^2$ gauge fields, which here becomes a similar contribution from the $|SO(N_c)| = \frac{1}{2}N_c(N_c-1) \approx \frac{1}{2}N_c^2$ gauge fields. Similarly, there are here half as many X fields ($\frac{1}{2}N_c(N_c+1) \approx \frac{1}{2}N_c^2$ here, vs. $N_c^2-1 \approx N_c^2$ there), half as many Q fields (N_cN_f here, vs $2N_cN_f$ there) and half as many of the meson fields ($\frac{1}{2}N_f(N_f+1) \approx \frac{1}{2}N_f^2$ here, vs N_f^2 there). So the a-function to maximize for the theory in the first line of (Appendix..3) is simply half a_{KPS} (Appendix..1). Maximizing this obviously leads to the same solution for the superconformal R-charges as obtained in [16], $y(x)=y_{KPS}(x)$.

Likewise, all of the other theories in (Appendix..3) similarly lead to the

same results in the $N_c \gg 1$, $N_f \gg 1$ limit, for arbitrary $x \equiv N_c/N_f$. In this limit, the anomaly free condition on the superconformal R-symmetry gives $R(X) = (1-y)/x$, with $R(Q) \equiv y$, for all of them. For all of these theories, the analog of every term in (Appendix..1) becomes approximately simply the same as in (Appendix..1), up to an overall factor, which is given in the last column of (Appendix..3). For example, for the theory in the last line of (Appendix..3), the generalized mesons QX^jQ which can hit the unitarity bound are given for even j by $\widetilde{Q}(X\widetilde{X})^rQ$ (which are $N_f(N_f+8) \approx N_f^2$ in number) and for odd j by $Q(\widetilde{X}X)^2\widetilde{X}Q$ (which are $\frac{1}{2}(N_f+8)(N_f+9) \approx \frac{1}{2}N_f^2$ in number) and $\widetilde{Q}X(\widetilde{X}X)^r\widetilde{Q}$ (which are $\frac{1}{2}N_f(N_f-1) \approx \frac{1}{2}N_f^2$ in number) so, whether j is even or odd, there are approximately the same number N_f^2 of mesons, leading to the same contributions as in the last line of (Appendix..1).

There are several interrelations among the theories (Appendix..3) associated with giving X an expectation value (which we're free to do, since we're now discussing the theories with $W_{tree} = 0$), and it can be verified that all of these Higgsing RG flows satisfy $a_{IR} < a_{UV}$. These checks make use of the a/a_{KPS} factors in the last column of (Appendix..3). For example, consider the $SO(N_c)$ theory with adjoint X, on the third line of (Appendix..3). Giving X an expectation value, there is a RG flow connecting this $SO(N_c)$ theory in the UV to an IR theory with gauge group $U(\frac{1}{2}N_c)$, adjoint matter X_{low}, and N_f fundamental flavors ($\square \oplus \overline{\square}$). Using the last column of (Appendix..3), the UV theory has $a_{UV} \approx \frac{1}{2}N_f^2\widehat{a}_{KPS}(x)$. The IR theory has $a_{IR} \approx N_f^2 a_{KPS}(\frac{1}{2}x)$, because the IR theory has $N_c/2$ colors. The a-theorem conjecture thus requires $\frac{1}{2}\widehat{a}_{KPS}(x) > \widehat{a}_{KPS}(\frac{1}{2}x)$, which can be verified to be satisfied.

Since all of the theories in (Appendix..3) have the same R-charge $R(X)$, given by $R(X) = (1-y_{KPS}(x))/x$, the minimal values $x_{A_k}^{min}$ for the $W_{A_k} = \mathrm{Tr}X^{k+1}$ superpotential to be relevant is the same, for all of these theories[1], as was obtained in [16] for the $SU(N_c)$ with adjoint theory. E.g. for $k \gg 1$, all have $x_{A_k}^{min} \to \left(\frac{4-\sqrt{3}}{6}\right)k$.

[1]The theories in the third through sixth line of (Appendix..3) must have $k = odd$, and that in the last line of (Appendix..3) must have $k+1 = 0$ (mod) 4.

We can similarly analyze the magnetic duals of the above theories [36], [53], [41]. For example, the theory in the last line of (Appendix..3), upon deforming by superpotential $W_{A_k} = \mathrm{Tr}(X\widetilde{X})^{\frac{1}{2}(k+1)}$ (with $k+1 = 0$ mod 4 here), was argued to be dual to a similar theory, with gauge group $SU(\widetilde{N}_c)$, with $\widetilde{N}_c \equiv k(N_f+4)-N_c$, along with some additional gauge singlets and superpotential terms. We can use a-maximization to analyze this dual $SU(\widetilde{N}_c)$ theory for $W_{tree} = 0$, and thereby determine when the various terms in the superpotential appearing in the duality of [36] are relevant. In particular, we can determine $\widetilde{x}^{min}_{A_k}$, the lower bound on $\widetilde{x} \equiv \widetilde{N}_c/N_f$ in order for the superpotential $\widetilde{W}_{A_k}$ to be relevant. As on the electric side, in the limit of large N_c and N_f, the a-maximization analysis becomes identical to that of [16] for the magnetic dual of the adjoint theory with superpotential W_{A_k}: the above $\widetilde{N}_c$ becomes $\widetilde{N}_c \approx kN_f - N_c$, as in the adjoint theory, and every term in the a-maximization analysis here maps to a corresponding term there. In this limit, the values here of the $\widetilde{x}^{min}_{A_k}$ are the same as those obtained in [16] for the adjoint theory.

Thus, at least in the $N_c \gg 1$ and $N_f \gg 1$ limit, all of the above theories have exactly the same superconformal window as obtained in [16] for the $SU(N_c)$ theory with adjoint.

We also note that all of the other theories in [36], with product gauge groups, also have the same superconformal window range of x, as long as we take all the groups to have the same (large) rank, and take all to have the same (large) number of fundamental flavors. This generalizes our observation of Sect. III.B, that the $SU(N_c) \times SU(N_c')$ theory gives the superconformal window obtained in [16] for the slice of parameter space $N_c = N_c'$ and $N_f = N_f'$.

For example, consider the $SU(M) \times SO(N) \times SO(N')$ duality discussed in sect. 11 of [36]. In the parameter slice, $M = N = N' \equiv N_c$, and $m = n = n' \equiv N_f$, taking $N_c \gg 1$ and $N_f \gg 1$ large, holding fixed the ratio $x \equiv N_c/N_f$, we find that every term in the quantity $a_{trial}(y,x) = 3\mathrm{Tr}R^3 - \mathrm{Tr}R$ equals twice a corresponding term in the corresponding function of [16], $a_{trial}(y,x) = 2a_{KPS}(y,x)$

(even including the contributions of the gauge invariant operators that hit the unitarity bound). This is because there is a correspondence in this limit between every field, with the same R-charges and twice as many copies for the $SU \times SO \times SO$ theory of [36] as compared with that of [16]. Since $a_{trial} = 2a_{KPS}$, it is maximized by the same function, $y = y_{KPS}(x)$. The anomalous dimensions are thus the same of those in [16] for this parameter slice. There is an analogous equality, up to the same factor of 2, between the function a_{trial} for the magnetic duals. It thus follows that the duality of [36] for this product group has a non-empty superconformal window, which reduces to the x interval of [16] in this 1d subspace of the full parameter space of flavors and colors. Likewise, all the dualities of [36] have a non-empty superconformal window, for any k, which reduces to the x interval of [16] in a 1d subspace of the full parameter space of flavors and colors.

Bibliography

[1] D. J. Wallace and R. K. P. Zia, "Gradient Flow And The Renormalization Group," Phys. Lett. A **48**, 325 (1974). D. J. Wallace and R. K. P. Zia, "Gradient Properties Of The Renormalization Group Equations In Multicomponent Systems," Annals Phys. **92**, 142 (1975).

[2] A. B. Zamolodchikov, "'Irreversibility' Of The Flux Of The Renormalization Group In A 2-D Field Theory," JETP Lett. **43**, 730 (1986) [Pisma Zh. Eksp. Teor. Fiz. **43**, 565 (1986)].

[3] J. L. Cardy, "Is There A C Theorem In 4d?," Phys. Lett. B **215**, 749 (1988).

[4] A. Cappelli, D. Friedan and J. I. Latorre, "C Theorem And Spectral Representation," Nucl. Phys. B **352**, 616 (1991).

[5] D. Anselmi, J. Erlich, D. Z. Freedman and A. A. Johansen, "Positivity constraints on anomalies in supersymmetric gauge theories," Phys. Rev. D **57**, 7570 (1998) [arXiv:hep-th/9711035].

[6] D. Friedan, lecture at the Nordisk Forskarkurs (1990) Iceland, unpublished.

[7] S. Forte and J. I. Latorre, "A proof of the irreversibility of renormalization group flows in four dimensions," Nucl. Phys. B **535**, 709 (1998) [arXiv:hep-th/9805015].

[8] H. Osborn and G. M. Shore, "Correlation functions of the energy momentum tensor on spaces of constant curvature," Nucl. Phys. B **571**, 287 (2000) [arXiv:hep-th/9909043].

[9] D. Anselmi, "Inequalities for trace anomalies, length of the RG flow, distance between the fixed points and irreversibility," Class. Quant. Grav. **21**, 29 (2004) [arXiv:hep-th/0210124].

[10] H. Osborn, "Derivation Of A Four-Dimensional C Theorem," Phys. Lett. B **222**, 97 (1989); H. Osborn, "Weyl Consistency Conditions And A Local Renormalization Group Equation For General Renormalizable Field Theories," Nucl. Phys. B **363**, 486 (1991); I. Jack and H. Osborn, "Analogs

For The C Theorem For Four-Dimensional Renormalizable Field Theories," Nucl. Phys. B **343**, 647 (1990).

[11] D. Z. Freedman and H. Osborn, "Constructing a c-function for SUSY gauge theories," Phys. Lett. B **432**, 353 (1998) [arXiv:hep-th/9804101].

[12] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "Exact Gell-Mann-Low Function Of Supersymmetric Yang-Mills Theories From Instanton Calculus," Nucl. Phys. B **229**, 381 (1983).

[13] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, "Nonperturbative formulas for central functions of supersymmetric gauge theories," Nucl. Phys. B **526**, 543 (1998) [arXiv:hep-th/9708042].

[14] K. Intriligator and B. Wecht, "The exact superconformal R-symmetry maximizes a," Nucl. Phys. B **667**, 183 (2003) [arXiv:hep-th/0304128].

[15] K. Intriligator and B. Wecht, "Baryon charges in 4D superconformal field theories and their AdS duals," Commun. Math. Phys. **245**, 407 (2004) [arXiv:hep-th/0305046].

[16] D. Kutasov, A. Parnachev and D. A. Sahakyan, "Central charges and U(1)R symmetries in N = 1 super Yang-Mills," JHEP **0311**, 013 (2003) [arXiv:hep-th/0308071].

[17] K. Intriligator and B. Wecht, "RG fixed points and flows in SQCD with adjoints," Nucl. Phys. B **677**, 223 (2004) [arXiv:hep-th/0309201].

[18] C. P. Herzog and J. Walcher, "Dibaryons from exceptional collections," JHEP **0309**, 060 (2003) [arXiv:hep-th/0306298].

[19] S. Franco, Y. H. He, C. Herzog and J. Walcher, "Chaotic duality in string theory," arXiv:hep-th/0402120.

[20] D. Kutasov, "New results on the 'a-theorem' in four dimensional supersymmetric field theory," arXiv:hep-th/0312098.

[21] C. Csaki, P. Meade and J. Terning, "A mixed phase of SUSY gauge theories from a-maximization," JHEP **0404**, 040 (2004) [arXiv:hep-th/0403062].

[22] P. C. Argyres and M. R. Douglas, "New phenomena in SU(3) supersymmetric gauge theory," Nucl. Phys. B **448**, 93 (1995) [arXiv:hep-th/9505062].

[23] P. C. Argyres, M. Ronen Plesser, N. Seiberg and E. Witten, "New N=2 Superconformal Field Theories in 4d," Nucl. Phys. B **461**, 71 (1996) [arXiv:hep-th/9511154].

[24] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, "Sasaki-Einstein metrics on S(2) x S(3)," arXiv:hep-th/0403002.

[25] I. Jack, D. R. T. Jones and C. G. North, "$N = 1$ supersymmetry and the three loop anomalous dimension for the chiral superfield," Nucl. Phys. B **473**, 308 (1996) [arXiv:hep-ph/9603386]; I. Jack, D. R. T. Jones and C. G. North, "Scheme dependence and the NSVZ beta-function," Nucl. Phys. B **486**, 479 (1997) [arXiv:hep-ph/9609325].

[26] N. Seiberg, "Electric - magnetic duality in supersymmetric nonAbelian gauge theories,"Nucl. Phys. B **435**, 129 (1995)[arXiv:hep-th/9411149].

[27] D. Kutasov and A. Schwimmer, "Lagrange Multipliers and Couplings in Supersymmetric Field Theory," arXiv:hep-th/0409029.

[28] I. I. Kogan, M. A. Shifman and A. I. Vainshtein, "Matching conditions and duality in N=1 SUSY gauge theories in the conformal window," Phys. Rev. D **53**, 4526 (1996) [arXiv:hep-th/9507170].

[29] H. Osborn, "N = 1 superconformal symmetry in four-dimensional quantum field theory," Annals Phys. **272**, 243 (1999) [arXiv:hep-th/9808041].

[30] D. Anselmi, M. T. Grisaru and A. Johansen, "A Critical Behaviour of Anomalous Currents, Electric-Magnetic Universality and CFT_4," Nucl. Phys. B **491**, 221 (1997) [arXiv:hep-th/9601023].

[31] D. Anselmi, L. Girardello, M. Porrati and A. Zaffaroni, "A note on the holographic beta and c functions," Phys. Lett. B **481**, 346 (2000) [arXiv:hep-th/0002066].

[32] K. A. Intriligator and N. Seiberg, "Lectures on supersymmetric gauge theories and electric-magnetic duality," Nucl. Phys. Proc. Suppl. **45BC**, 1 (1996) [arXiv:hep-th/9509066].

[33] D. J. Gross and F. Wilczek, "Asymptotically Free Gauge Theories. 2," Phys. Rev. D **9**, 980 (1974).

[34] D. Kutasov, A. Schwimmer and N. Seiberg, "Chiral Rings, Singularity Theory and Electric-Magnetic Duality," Nucl. Phys. B **459**, 455 (1996) [arXiv:hep-th/9510222].

[35] K. A. Intriligator, "New RG fixed points and duality in supersymmetric SP(N(c)) and SO(N(c)) gauge theories," Nucl. Phys. B **448**, 187 (1995) [arXiv:hep-th/9505051].

[36] K. A. Intriligator, R. G. Leigh and M. J. Strassler, "New examples of duality in chiral and nonchiral supersymmetric gauge theories," Nucl. Phys. B **456**, 567 (1995) [arXiv:hep-th/9506148].

[37] K. A. Intriligator and N. Seiberg, "Phases of N=1 supersymmetric gauge theories in four-dimensions," Nucl. Phys. B **431**, 551 (1994) [arXiv:hep-th/9408155].

[38] T. Banks and A. Zaks, "On The Phase Structure Of Vector - Like Gauge Theories With Massless Fermions," Nucl. Phys. B **196**, 189 (1982).

[39] D. Kutasov, "A Comment on duality in N=1 supersymmetric nonAbelian gauge theories," Phys. Lett. B **351**, 230 (1995) [arXiv:hep-th/9503086].

[40] D. Kutasov and A. Schwimmer, "On duality in supersymmetric Yang-Mills theory," Phys. Lett. B **354**, 315 (1995) [arXiv:hep-th/9505004].

[41] R. G. Leigh and M. J. Strassler, "Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory," Nucl. Phys. B **447**, 95 (1995) [arXiv:hep-th/9503121].

[42] K. A. Intriligator, N. Seiberg and S. H. Shenker, "Proposal for a simple model of dynamical SUSY breaking," Phys. Lett. B **342**, 152 (1995) [arXiv:hep-ph/9410203].

[43] J. H. Brodie, P. L. Cho and K. A. Intriligator, "Misleading anomaly matchings?," Phys. Lett. B **429**, 319 (1998) [arXiv:hep-th/9802092].

[44] M. E. Machacek and M. T. Vaughn, "Two Loop Renormalization Group Equations In A General Quantum Field Theory. 1. Wave Function Renormalization," Nucl. Phys. B **222**, 83 (1983).

[45] E. Barnes, K. Intriligator, B. Wecht and J. Wright, "Evidence for the strongest version of the 4d a-theorem, via a-maximization along RG flows," arXiv:hep-th/0408156.

[46] M. Bertolini, F. Bigazzi and A. L. Cotrone, "New checks and subtleties for AdS/CFT and a-maximization," JHEP **0412**, 024 (2004) [arXiv:hep-th/0411249].

[47] S. Benvenuti, S. Franco, A. Hanany, D. Martelli and J. Sparks, "An infinite family of superconformal quiver gauge theories with Sasaki-Einstein duals," arXiv:hep-th/0411264.

[48] E. Poppitz, Y. Shadmi and S. P. Trivedi, "Duality and Exact Results in Product Group Theories," Nucl. Phys. B **480**, 125 (1996) [arXiv:hep-th/9605113].

[49] I. R. Klebanov and M. J. Strassler, "Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities," JHEP **0008**, 052 (2000) [arXiv:hep-th/0007191].

[50] F. Cachazo, B. Fiol, K. A. Intriligator, S. Katz and C. Vafa, 'A geometric unification of dualities," Nucl. Phys. B **628**, 3 (2002) [arXiv:hep-th/0110028].

[51] M. Berkooz, "The Dual of supersymmetric SU(2k) with an antisymmetric tensor and composite dualities," Nucl. Phys. B **452**, 513 (1995) [arXiv:hep-th/9505067].

[52] M. A. Luty, M. Schmaltz and J. Terning, "A Sequence of Duals for Sp(2N) Supersymmetric Gauge Theories with Adjoint Matter," Phys. Rev. D **54**, 7815 (1996) [arXiv:hep-th/9603034].

[53] K. A. Intriligator, "New RG fixed points and duality in supersymmetric SP(N(c)) and SO(N(c)) gauge theories," Nucl. Phys. B **448**, 187 (1995) [arXiv:hep-th/9505051].

[54] J. Erdmenger and H. Osborn, "Conserved currents and the energy-momentum tensor in conformally invariant theories for general dimensions," Nucl. Phys. B **483**, 431 (1997) [arXiv:hep-th/9605009].

[55] D. Anselmi, "Central functions and their physical implications," JHEP **9805**, 005 (1998) [arXiv:hep-th/9702056].

[56] D. Z. Freedman, S. D. Mathur, A. Matusis and L. Rastelli, "Correlation functions in the CFT(d)/AdS($d+1$) correspondence," Nucl. Phys. B **546**, 96 (1999) [arXiv:hep-th/9804058].

[57] E. Barnes, E. Gorbatov, K. Intriligator and J. Wright, "Current correlators and AdS/CFT geometry," Nucl. Phys. B **732**, 89 (2006) [arXiv:hep-th/0507146].

[58] D. Martelli, J. Sparks and S. T. Yau, "The geometric dual of a-maximisation for toric Sasaki-Einstein manifolds," arXiv:hep-th/0503183.

[59] O. Aharony, A. Hanany, K. A. Intriligator, N. Seiberg and M. J. Strassler, "Aspects of N = 2 supersymmetric gauge theories in three dimensions," Nucl. Phys. B **499**, 67 (1997) [arXiv:hep-th/9703110].

[60] H. Osborn and A. C. Petkou, "Implications of conformal invariance in field theories for general dimensions," Annals Phys. **231**, 311 (1994) [arXiv:hep-th/9307010].

[61] S. J. Gates, M. T. Grisaru, M. Rocek and W. Siegel, "Superspace, Or One Thousand And One Lessons In Supersymmetry," Front. Phys. **58**, 1 (1983) [arXiv:hep-th/0108200].

[62] D. Anselmi, "Anomalies, unitarity, and quantum irreversibility," Annals Phys. **276**, 361 (1999) [arXiv:hep-th/9903059].

[63] S. Benvenuti and A. Hanany, "New results on superconformal quivers," arXiv:hep-th/0411262.

[64] J. Erlich and D. Z. Freedman, "Conformal symmetry and the chiral anomaly," Phys. Rev. D **55**, 6522 (1997) [arXiv:hep-th/9611133].

[65] M. Baker and K. Johnson, "Applications of conformal symmetry in quantum electrodynamics," Physics 96A (1979) 120.

[66] J. M. Maldacena, "The large N limit of superconformal field theories and supergravity," Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[67] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "Gauge theory correlators from non-critical string theory," Phys. Lett. B **428**, 105 (1998) [arXiv:hep-th/9802109].

[68] E. Witten, "Anti-de Sitter space and holography," Adv. Theor. Math. Phys. **2**, 253 (1998) [arXiv:hep-th/9802150].

[69] D. Martelli and J. Sparks, "Toric geometry, Sasaki-Einstein manifolds and a new infinite class of AdS/CFT duals," arXiv:hep-th/0411238.

[70] C. P. Herzog, Q. J. Ejaz and I. R. Klebanov, "Cascading RG flows from new Sasaki-Einstein manifolds," JHEP **0502**, 009 (2005) [arXiv:hep-th/0412193].

[71] S. Franco, A. Hanany, D. Martelli, J. Sparks, D. Vegh and B. Wecht, "Gauge theories from toric geometry and brane tilings," arXiv:hep-th/0505211.

[72] A. Butti, D. Forcella and A. Zaffaroni, "The dual superconformal theory for L(p,q,r) manifolds," arXiv:hep-th/0505220.

[73] E. Barnes, E. Gorbatov, K. Intriligator, M. Sudano and J. Wright, "The exact superconformal R-symmetry minimizes tau(RR)," Nucl. Phys. B **730**, 210 (2005) [arXiv:hep-th/0507137].

[74] M. J. Duff, C. N. Pope and N. P. Warner, "Cosmological And Coupling Constants In Kaluza-Klein Supergravity," Phys. Lett. B **130**, 254 (1983).

[75] D. Berenstein, C. P. Herzog and I. R. Klebanov, "Baryon spectra and AdS/CFT correspondence," JHEP **0206**, 047 (2002) [arXiv:hep-th/0202150].

[76] A. Butti and A. Zaffaroni, "R-charges from toric diagrams and the equivalence of a-maximization and Z-minimization," arXiv:hep-th/0506232.

[77] Y. Tachikawa, "Five-dimensional supergravity dual of a-maximization," arXiv:hep-th/0507057.

[78] M. Gunaydin, G. Sierra and P. K. Townsend, "The Geometry Of N=2 Maxwell-Einstein Supergravity And Jordan Algebras," Nucl. Phys. B **242**, 244 (1984).

[79] K. Behrndt and M. Cvetic, "Anti-de Sitter vacua of gauged supergravities with 8 supercharges," Phys. Rev. D **61**, 101901 (2000) [arXiv:hep-th/0001159].

[80] M. Henningson and K. Skenderis, "The holographic Weyl anomaly," JHEP **9807**, 023 (1998) [arXiv:hep-th/9806087].

[81] C. P. Herzog, I. R. Klebanov and P. Ouyang, "D-branes on the conifold and N = 1 gauge / gravity dualities," arXiv:hep-th/0205100.

[82] S. S. Gubser, "Einstein manifolds and conformal field theories," Phys. Rev. D **59**, 025006 (1999) [arXiv:hep-th/9807164].

[83] S. Weinberg, "Charges From Extra Dimensions," Phys. Lett. B **125**, 265 (1983).

[84] B. de Wit and H. Nicolai, "N=8 Supergravity With Local SO(8) X SU(8) Invariance," Phys. Lett. B **108**, 285 (1982).

[85] M. Gunaydin, L. J. Romans and N. P. Warner, "Compact And Noncompact Gauged Supergravity Theories In Five-Dimensions," Nucl. Phys. B **272**, 598 (1986).

[86] J. P. Gauntlett, S. Lee, T. Mateos and D. Waldram, "Marginal deformations of field theories with AdS(4) duals," arXiv:hep-th/0505207.