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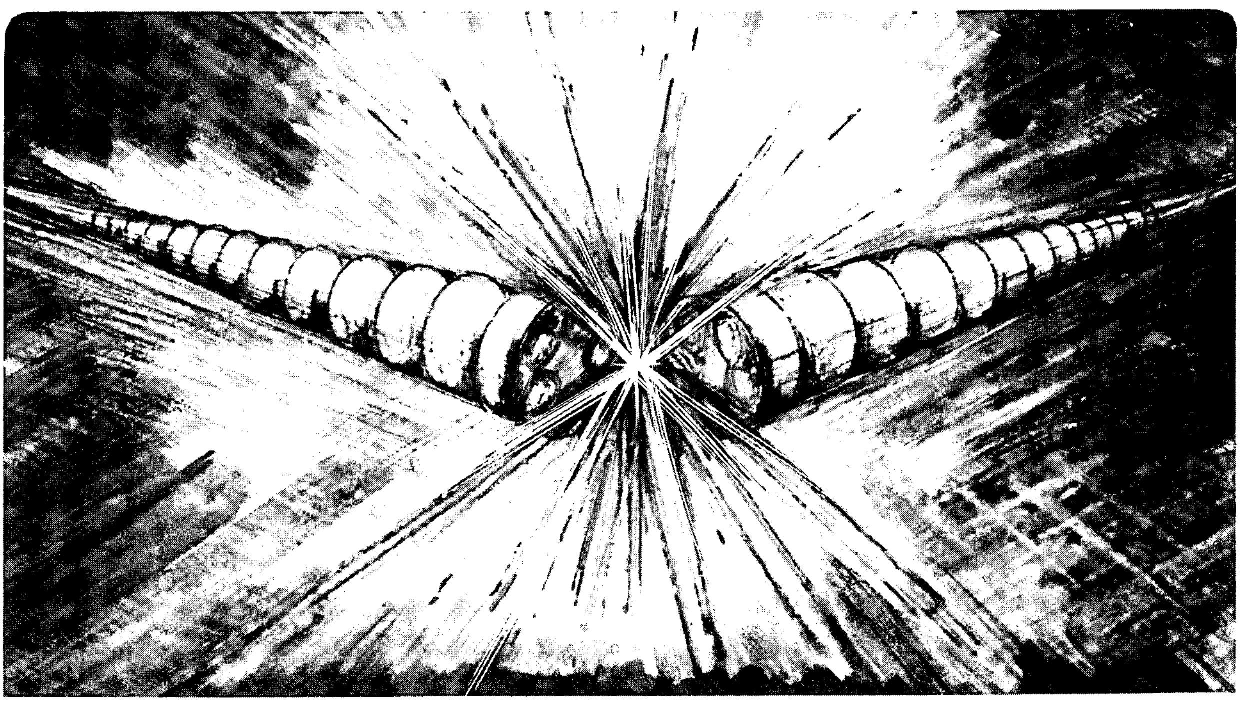

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PROOF THAT THE NEUMANN GREEN FUNCTION IN ELECTROSTATICS CAN BE SYMMETRIZED*

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Abstract

We prove by construction that the Green function satisfying the Neumann boundary conditions in electrostatic problems can be symmetrized. An illustrative example is given.

I. *Introduction*

It is well-known that the Green function satisfying the Dirichlet boundary conditions in electrostatic problems is symmetric in its arguments. The symmetry property is often very useful in constructing an explicit representation of the Green function. It is stated in Jackson [1] that, for a Green function satisfying the Neumann boundary conditions, the symmetry is not automatic but can be imposed as a separate requirement. However, an explicit proof that Neumann Green function can indeed be symmetrized does not appear to be readily available in published references. Here, we offer such a proof, and present an illustrative example.

II. *Proof*

Following the discussion in Jackson [1], we consider the electrostatic boundary value problem in a volume V bounded by a surface S. The Green function satisfies the following equation for $\boldsymbol{x}$ and $\boldsymbol{x}'$ in V:

$$\nabla'^2 G_{D,N}(\boldsymbol{x},\boldsymbol{x}') = -4\pi\delta(\boldsymbol{x}-\boldsymbol{x}') \tag{1}$$

We distinguish two different Green functions G_D and G_N. For a Dirichlet problem,

$$G_D(\boldsymbol{x},\boldsymbol{x}') = 0 \ \ for \ \ \boldsymbol{x}' \ on \ S. \tag{2}$$

For a Neumann problem, we must satisfy the Gauss theorem constraint, $\oint_S \frac{\partial G}{\partial n'} da' = -4\pi$. The simplest way to satisfy the requirement is to impose

$$\frac{\partial G_N}{\partial n'}(\boldsymbol{x},\boldsymbol{x}') = -\frac{4\pi}{S} \ \ for \ \boldsymbol{x}' \ on \ S \ and \ \boldsymbol{x} \ within \ V. \tag{3}$$

Here $\partial/\partial n'$ is the normal derivative at the surface S directed outwards from inside the volume V, and S in Eq. (3) is also the total area of the boundary surface. The solution to the Neumann boundary value problem is then

$$\Phi(x) = \langle\Phi\rangle_s + \int_V \rho(x')\, G_N(x,x')d^3x' + \frac{1}{4\pi}\int_S \frac{\partial\Phi}{\partial n'}\, G_N \tag{4}$$

The symmetry property of the Dirichlet Green function G_D can be proved by means of the second Green's identity

$$\int_v \left(\varphi(y)\, \nabla^2\, \psi(y) - \psi(y)\nabla^2\, \varphi(y)\right) d^3y = \oint \left(\varphi\,\frac{\partial\psi}{\partial n} - \psi\,\frac{\partial\varphi}{\partial n}\right) da\,. \tag{5}$$

Thus by setting $\varphi = G_D(x,y)$ and $\psi = G_D(x',y)$ in the above, one obtains readily $G_D(x,x') = G_D(x',x)$.

For a Neumann problem the Green function $G_N(x,x)$ is in general not symmetric in x and x'. However, we can show that a symmetrized Green function G_N^S can always be constructed. For this purpose, set $\varphi = G_N(x,y)$ and $\psi = G_N(x',y)$ in Eq. (5) to obtain

$$G_N(x,x') - G_N(x',x) = F(x) - F(x'), \tag{6}$$

where

$$F(x) = \frac{1}{S}\oint_S G_N(x,y)\, da_y\ , \tag{7}$$

apart from the possibility of an added constant. Equation (6) implies that the combination

$$G_N^S(x,x') = G_N(x,x') - F(x) \tag{8}$$

is symmetric in x and x'. Furthermore, since $G_N^S(x,x')$ differs from $G_N(x,x')$ by a function $F(x)$ that depends only on x, it satisfies both Eq. (1) and Eq. (3). Therefore $G_N^S(x,x')$ is a Neumann Green function which is symmetric in x and x'. One might be concerned that the additional $F(x)$ changes the solution, Eq. (4). However, Gauss's law saves the day because the added contribution to the potential from $F(x)$ is

$$\Delta\Phi(x) = F(x)\left[\int_V \rho(x')d^3x' + \frac{1}{4\pi}\oint_S \frac{\partial\Phi}{\partial n'}\,da'\right]$$

The first integral is the total charge within V, while the second is the negative of the total electric flux leaving V (divided by 4π); the sum vanishes. We note that the function F(x) defined by Eq. (7) is what is needed to make the Green function symmetric, but that any function of x can be added to $G_N(x,x')$ without affecting the result for the potential.

III. *An Example*

As an example, we consider the Neumann Green function for the volume V between two concentric spheres of radii a and b $(a < b)$. We have the following expansion in spherical harmonics:

$$G_N(x,x') = \sum_{\ell=0}^{\infty} g_\ell(r,r')\,P_\ell(\cos\gamma)\,, \qquad (9)$$

where r and r' are the radial components of x and x', respectively, and γ is the angle between x and x'. The function g_ℓ can be written in the following form:

$$g_\ell(r,r') = \frac{r_<^\ell}{r_>^{\ell+1}} + \alpha_\ell(r)\,r'^{\,\ell} + \beta_\ell(r)\,\frac{1}{r'^{\,\ell+1}}\,. \qquad (10)$$

Here $r_<$ $(r_>)$ is the smaller (greater) of r and r'. The first term in the right hand side of Eq. (10) gives rise to the delta function when inserted to Eq. (1). The unknown functions $\alpha_\ell(r)$ and

$\beta_\ell(r)$ are to be determined from the boundary condition. The requirement Eq. (3) involves only $g_o(r,r')$, the spherically symmetric term. The boundary conditions at $r' = b$ and $r' = a$ are, respectively

$$\left.\frac{\partial g_o}{\partial n'}\right|_{r'=b} = \frac{\partial}{\partial r'}\left(\frac{1+\beta_o(r)}{r'}\right)\Bigg|_{r'=b} = -\frac{1}{a^2+b^2}, \tag{11}$$

$$\left.\frac{\partial g_o}{\partial n'}\right|_{r'=a} = -\frac{\partial}{\partial r'}\left(\frac{1}{r}+\frac{\beta_o(r)}{r'}\right)\Bigg|_{r'=a} = -\frac{1}{a^2+b^2}. \tag{12}$$

It is easy to see that both Eq. (11) and Eq. (12) lead to

$$\beta_o(r) = -\frac{a^2}{a^2+b^2}. \tag{13}$$

Thus we have

$$g_o(r,r') = \frac{1}{r_>} + \alpha_o(r) - \frac{a^2}{a^2+b^2}\frac{1}{r'}. \tag{14}$$

The function $\alpha_o(r)$ is left undetermined, and the function g_o is in general not symmetric in r and r'. As noted above, the form of $a_o(r)$ is of no consequence for the solution of the potential problem.

For $\ell > 0$, one obtains by analogous calculation $\left(\frac{\partial g_e}{\partial r'} = 0 \text{ at } r' = a,b\right)$

$$\alpha_\ell(r) = \frac{1}{b^{2\ell+1}-a^{2\ell+1}}\left[\left(\frac{\ell+1}{\ell}\right)r^\ell + \frac{a^{2\ell+1}}{r^{\ell+1}}\right], \tag{15}$$

$$\beta_\ell(r) = \frac{1}{b^{2\ell+1}-a^{2\ell+1}}\left[a^{2\ell+1}r^\ell + \frac{\ell}{\ell+1}\frac{(ab)^{2\ell+1}}{r^{\ell+1}}\right], \tag{16}$$

$$g_\ell(r,r') = \frac{r_<^\ell}{r_>^{\ell+1}} + \frac{1}{b^{2\ell+1}-a^{2\ell+1}}\left[\frac{\ell+1}{\ell}(rr')^\ell + \frac{\ell}{\ell+1}\frac{(ab)^{2\ell+1}}{(rr')^{\ell+1}} + a^{2\ell+1}\left(\frac{r^\ell}{r'^{\ell+1}} + \frac{r'^\ell}{r^{\ell+1}}\right)\right]. \tag{17}$$

Note that $g_\ell(r,r')$ for $\ell > 0$ is symmetric in r and r'.

The integral in Eq. (7) defining F($\boldsymbol{x}$) receives contributions only from the term $\ell = 0$. We find

$$F(\boldsymbol{x}) = \frac{1}{a^2+b^2}\left\{a^2\left(\frac{1}{r} + \alpha_o(r) - \frac{a}{a^2+b^2}\right) + b^2\left(\frac{1}{b} + \alpha_o(r) - \frac{a^2}{a^2+b^2}\,\frac{1}{b}\right)\right\}. \qquad (18)$$

The symmetrized Green function Eq. (7) becomes in this case

$$G_N^S(\boldsymbol{x},\boldsymbol{x}') = \sum_{\ell=0}^{\infty} g_\ell^S(r,r')\,P_\ell(\cos\gamma)\,, \qquad (19)$$

where

$$g_o^S(r,r') = \frac{1}{r_>} - \frac{a^2}{a^2+b^2}\left(\frac{1}{r} + \frac{1}{r'}\right) + \frac{a^3-b^3}{(a^2+b^2)^2}\,, \qquad (20)$$

$$g_\ell^S(r,r') = g_\ell(r,r') \;\; for \;\; \ell > 0\,. \qquad (21)$$

Note that the left hand side of Eq. (20) is explicitly symmetric. The last term $(a^3-b^3)/(a^2+b^2)^2$ is a constant and thus can be omitted.

Acknowledgments

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