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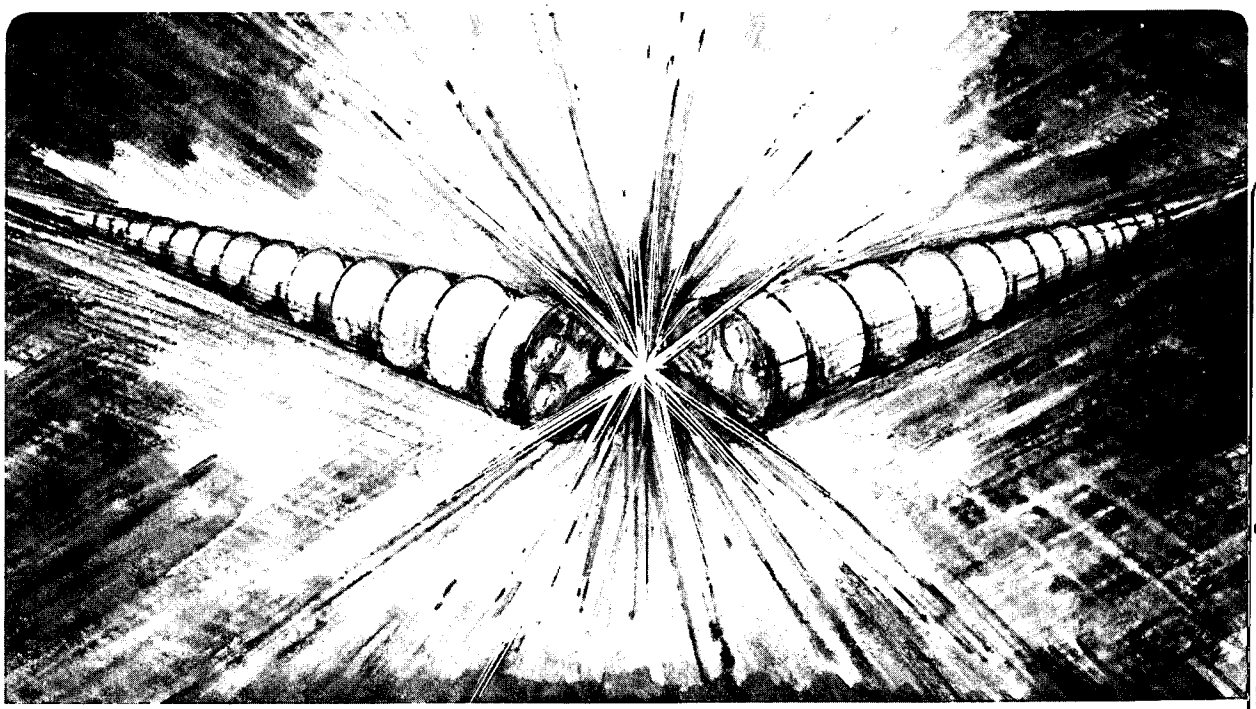

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Symplectic Methods In Circular Accelerators*

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Symplectic Methods In Circular Accelerators

Étienne Forest

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Abstract

By now symplectic integration has been applied to many problems in classical mechanics. It is my conviction that the field of particle simulation in circular rings is ideally suited for the application of symplectic integration. In this paper, I present a short description symplectic tools in circular storage rings.

In the field of circular ring dynamics, we distinguish two different but related topics:

1. Symplectic Integration or Symplectic Modelling
2. "Symplectification" of Maps

There is an important implication: Perturbation theory should be based on the maps produced by the integrator rather than the original Hamiltonian. But this is the topic of an other talk.

1 Symplectic Integration

In symplectic integration we write down a local Hamiltonian for each magnet or piece of magnet which is connected with the rest of the ring through various canonical transformations. The map for the entire ring has the following form:

$$\begin{aligned} \mathcal{M} &= \prod_{n=1}^{N} \mathcal{M}_{i\to i+1} \\ \mathcal{M}_{i\to i+1} &= \mathcal{E}_i \mathcal{F}_i \mathcal{S}_i \mathcal{F}_{i+1} \mathcal{E}_{i+1}. \end{aligned} \tag{1}$$

The maps $\mathcal{E}_i$ and $\mathcal{F}_i$ are elements of the Euclidian groups and fringe effects respectively. Each magnet is represented by a Hamiltonian which is separable:

$$H_i = \sum_{n=1}^{Q_i} H_i^n. \tag{2}$$

$$\mathcal{S}_i = \exp\ (-L_i : H_i :) \to \prod_{k=1}^{N_d} \exp\ \left(-L_i c^{n(k)} : H_i^{n(k)} :\right). \tag{3}$$

For example second order decomposition:

$$\mathcal{S}_i = \exp\ (-L_i : H_i :) \to e^{-\frac{L_i}{2}:H_i^Q:} e^{-\frac{L_i}{2}:H_i^{Q-1}:} \ldots e^{-L_i:H_i^1:} \ldots e^{-\frac{L_i}{2}:H_i^{Q-1}:} e^{-\frac{L_i}{2}:H_i^Q:}. \tag{4}$$

In the case of the ideal shifted quadrupole magnet, the ultrarelativistic Hamiltonian is for the body is given by

$$\begin{aligned} H &= -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} + \frac{qB}{p_0} x + k\frac{x^2 - y^2}{2} \\ \text{where} \qquad & [x, p_x] = 1 \quad [y, p_y] = 1 \quad [\delta, \ell] = 1, \end{aligned} \tag{5}$$

and it can be decomposed into two pieces for the purpose of symplectic integration:

$$\begin{aligned} H_1 &= -\sqrt{(1+\delta)^2 - p_x^2 - p_y^2} \\ H_2 &= \frac{qB}{p_0} x + k\frac{x^2 - y^2}{2}. \end{aligned} \tag{6}$$

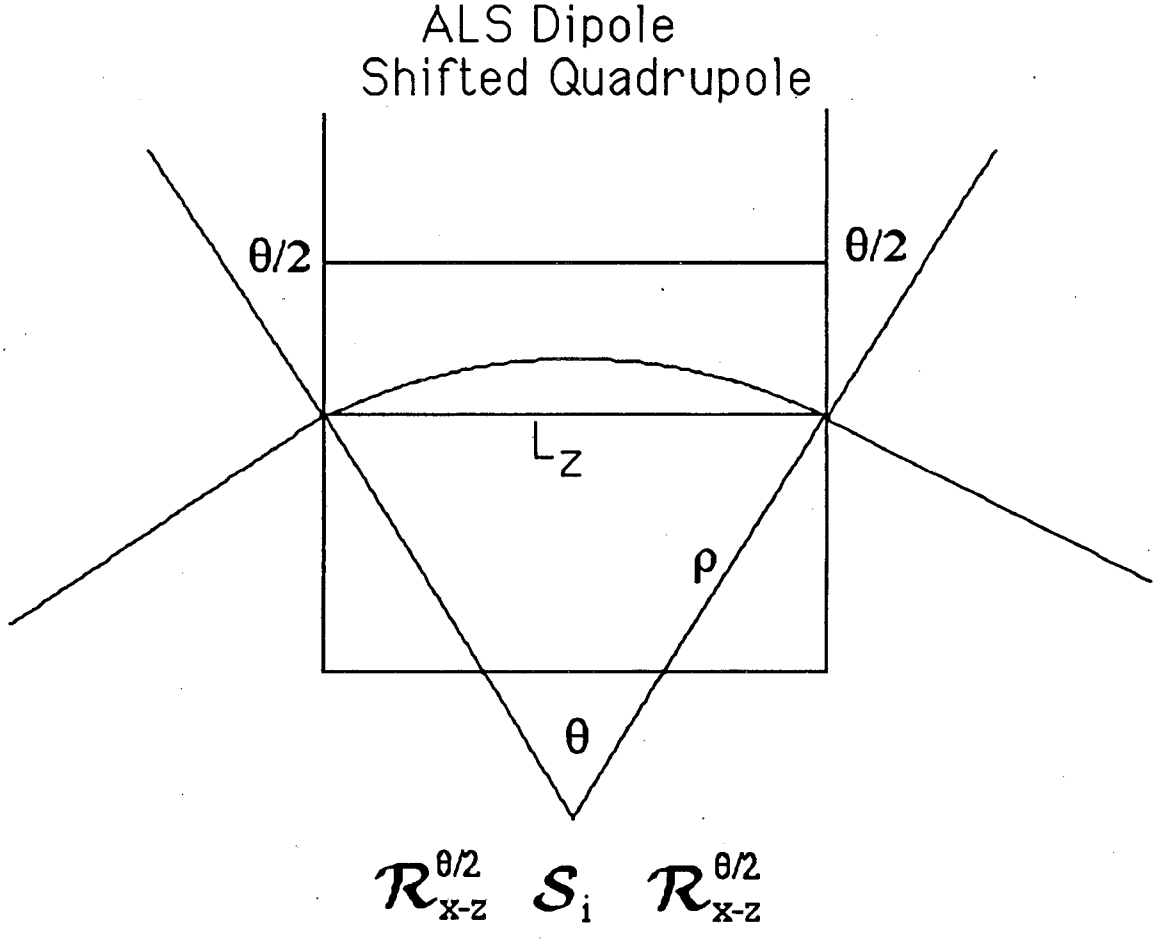
ALS Dipole
Shifted Quadrupole
θ/2
θ/2
L_z
ρ
θ
$\mathcal{R}^{\theta/2}_{x\text{-}z}$ $\mathcal{S}_i$ $\mathcal{R}^{\theta/2}_{x\text{-}z}$

Figure 1:

In general, one can still write a Hamiltonian for the full system using a pseudo-time like variable.

$$H_{total}(\vec{z};\sigma) = \sum_{\substack{Magnets \\ Coordinate\ patches}} h_i(\sigma) \tag{7}$$

This Hamiltonian is highly localized in the variable σ. Therefore, it probably does not have any non trivial constant of the motion.

Given the fact that:

1. The Hamiltonian is highly localized (not smooth)
2. We don't know the real Hamiltonian anyway because of construction errors
3. We are interested in stability issues and the general structure of phase space
4. We are not interested in the precise time depedent position of a given particle

then we conclude that the approximate solution provided by the symplectic integrator contains all the physical information we will ever hope to know.

In most applications, we replace the "real system" by the symplectic integrator. Therefore, it is appropriate to say that we are doing "Symplectic Modelling."

Finally, in electron machines, we can include radiative effects in the Hamiltonian. These effects are added to the symplectic integrator. Since they are tiny numerically, but qualitatively important, it is imperative that the integrator be Hamiltonian in the absence of radiation.

Important Comment: Simulation codes are equipped with an automatic differentiation algorithm. Taylor series maps around any orbit can be computed **exactly** and manipulated into any Lie representation routinely.

2 Symplectification of Maps

In accelerators, we are often tempted to produce an approximation of the one turn map provided by the integrator. The reasons are two-fold:

1. Hamiltonian perturbation theory is easier on the one turn map because of the complexity of the "s" or time dependence.
2. Tracking with the integrator is too slow.

There are at this point two categories of approximations used in our field.

↪I. Taylor Series around the central fixed point.

↪II. Fitted maps in action-angle variables (Warnock's work).

↪III. A combination of the above.

2.1 Taylor Series

First, it is important to state that Taylor series maps can be extracted from a symplectic integrator routinely using automatic differentiation. Secondly, all the manipulations, mentioned or hinted below, are all implemented or implementable as sub-libraries of an automatic differentiation package.

2.1.1 Factored Maps and Generating Functions

The nonlinear part of the map can be integrated using various methods. Here is one method using generating functions.

$$\begin{aligned} \mathcal{M}_{\vec{\zeta}_s} &= \exp\,(:h_2:)\exp\,(:h_{k\geq 3}:) \\ &\downarrow \\ \mathcal{M}_{\vec{\zeta}_s} &= \exp\,(:h_2:)\left\{\exp\left(:\frac{h_{k\geq 3}}{n}:\right)\right\}^n \end{aligned} \tag{8}$$

The total map is made of a linear map and n identical nonlinear maps:

$$\vec{\zeta}_s = \underbrace{\vec{\zeta}_s^{1/n} \circ \cdots \circ \vec{\zeta}_s^{1/n}}_{\text{n times}} \circ \vec{\zeta}_s^{1}$$

$$\begin{aligned} \text{where } \vec{\underline{\zeta}}_s^1 &= \exp\ (:h_2:)\,\vec{I} \\ \text{and } \vec{\underline{\zeta}}_s^{1/n} &= \exp\left(:\frac{h_{k\geq 3}}{n}:\right)\vec{I}. \end{aligned} \tag{9}$$

The nonlinear map $\vec{\underline{\zeta}}_s^{1/n}$ can be approximated by a generating function.

2.1.2 Jolt Tracking

The first type of "Lie algebraic" tracking ever implemented is due originally to Irwin [1, 4]. In its original form it was formulated in terms of "kicks" and rotations.

$$\begin{aligned} \mathcal{M}_{\text{Nonlinear}} &= \exp\left(:\mathcal{R}_1\chi^1:\right)\exp\left(:\mathcal{R}_2\chi^2:\right) \quad \exp\left(:\mathcal{R}_{\widehat{N}}\chi^{\widehat{N}}:\right) \\ \text{where } \mathcal{R}_i &= \exp\,(:-\vec{\alpha}^i\cdot\vec{J}:) \text{ and } J_i = \frac{q_i^2+p_i^2}{2} \\ \chi^i &= \sum_{\vec{m}}\beta^i_{\vec{m}}\ \underbrace{q_1^{m_1}\ldots q_N^{m_N}}_{\substack{\text{Position only} \\ \vec{q}=(x_1,x_3,..)}} \end{aligned} \tag{10}$$

The minimal value of the integer $\widehat{N}$ is a function of the degree of the power series and of the phase space dimension. It is also a function of the choice of linear maps $\mathcal{R}_i$, which are rotations in the original treatment of Irwin.

For each individual functional map of Equation (10), one can find the image of the identity function:

$$\vec{\Xi}^i = \exp\left(:\mathcal{R}_i\chi^i:\right)\vec{I} = \vec{I} + \left[\mathcal{R}_i\chi^i, \vec{I}\right] \tag{11}$$

The series terminates exacly. Therefore, the one-turn map $\vec{\underline{\zeta}}_s$ is approximated symplectically as

$$\vec{\underline{\zeta}}_s \approx \underbrace{\vec{\Xi}^{\widehat{N}} \circ \cdots \circ \vec{\Xi}^1}_{\text{Nonlinear part}} \circ \underbrace{\vec{\underline{\zeta}}_s^1}_{\text{Linear part}} \tag{12}$$

The good points of the Irwin factorization are

- Exactly symplectic and defined all over phase space.

- Obvious exact inverse.
- Can be put on an integer grid without ambiguity.

Unfortunately there remain a lot of unanswered questions with this representation.

- What should be the group chosen for the maps $\mathcal{R}_i$? Irwin picked the group of linear phase advances. One can show that the drifts would suffice.
- Secondly, given a group, how to choose the elements for a given number $\widehat{N}$. How do they foliate the polynomials into different equivalence classes?
- What is the normal form in each equivalence class leaf?
- Do all the above while minizing the difference between the maps; i.e. $\left\| \vec{\zeta}_s(\vec{x}) - \left(\vec{\Xi}^{\widehat{N}} \circ \ldots \circ \vec{\Xi}^1 \circ \vec{\zeta}_s^{-1} \right)(\vec{x}) \right\|$ must remain small within the aperture.

Strangely and unexpectedly, these questions are mathematically very hard to answer. Dragt and Abell have looked into these questions [2] and have found that the goodness of the approximation is very sensitive to the set chosen. In other words, the set of angles $\left\{ \vec{\alpha}^i \middle| \, i = 1, \widehat{N} \right\}$ is critical in determining the goodness of the approximation.

In conclusion, the topic of factorization in terms of jolts or any other exactly solvable functions is certainly wide open. But, as in symplectic integration, it is bound to a view of the ring based on maps and on the use of Lie methods. However, unlike symplectic integration, it remains to be seen how useful it is.

2.1.3 Monomial Tracking

Besides the jolt factorization, there is another way to do symplectic "Lie" tracking with Taylor series maps. It is based on the realization that monomial maps are exactly solvable [3]. Consider the map

$$\begin{aligned} \vec{\kappa}_{\vec{m}} &= \exp(: P_{\vec{m}} :)\vec{I} \\ \text{where } P_{\vec{m}}(\vec{q},\vec{p}) &= q_1^{m_1} \cdots q_N^{m_N}. \end{aligned} \tag{13}$$

It can be shown that the monomial map $\vec{\kappa}_{\vec{m}}$ involves finite Taylor series map (kick), exponential functions as well as roots and ratios of polynomials. So, the map $\mathcal{M}_{\underline{\zeta}_s}$ can be expressed as follows:

$$\begin{array}{c} \mathcal{M}_{\underline{\zeta}_s} = \prod_{n=1}^{\widehat{N}} \exp(: P_{\vec{m}_n} :) \\ \text{or equivalently} \\ \vec{\zeta}_s = \vec{\kappa}_{\vec{m}_{\widehat{N}}} \circ \cdots \circ \vec{\kappa}_{\vec{m}_1} \end{array} \tag{14}$$

It is easy to check that the various monomial maps have poles and therefore this technique suffers probably from the combined diseases of the generating functions and the jolts. Nevertheless, it can be very useful for short term tracking and to represent complex elements symplectically. Again, with the help of automatic differentiation, it has been implemented [4].

2.1.4 Combining Symplectic Integration and Symplectification

It has been verified (numerical experiments) that the Dragt-Finn factorization for a map leads to a good approximation if the underlying Taylor series is good.

$$\forall \vec{x} \in A \quad \begin{cases} \left\| \vec{\zeta}_{Dragt-Finn}(\vec{x}) - \vec{\zeta}_{exact}(\vec{x}) \right\| < \delta \ll 1 \\ \left\| \vec{\zeta}_{Taylor\ Series}(\vec{x}) - \vec{\zeta}_{exact}(\vec{x}) \right\| < \delta \ll 1 \end{cases} \tag{15}$$

Here A is the "aperture" or region of phase space under study. Therefore it seems that it would be desireable to approximate the Dragt-Finn map by a symplectic map without introducing higher order spurious terms. These spurious terms have been disastrous when using other methods (in particular the jolt factorization).

Recently, Yan and Shi at SSC, have combined kicks and monomials in trying to decompose a homogenous polynomial of degree n into a minimum number of exactly solvable terms.

$$\begin{array}{rcl} \text{Dragt} - \text{Finn map} & = & \exp\ (: f_3 :) \cdots \exp\ (: f_{No} :) \\ & \Downarrow & \\ f_i & = & \sum_{n=1}^{Q_i} f_i^n. \end{array} \tag{16}$$

We then integrate f_i using symplectic integration, i.e. fractal decomposition.

$$\begin{aligned} \exp\ (:\ f_i\ :) &= \exp\ \left(:\sum_{n=1}^{Q_i} f_i^n\ :\right) \\ &\cong \prod_{k=1}^{N_d} \exp\ \left(c^{n(k)}\ :\ f_i^{n(k)}\ :\right). \end{aligned} \tag{17}$$

The integrator of Equation (3) is used to approximate the Dragt-Finn map.

2.2 The Concept of a Fitted Map

Let us assume for a moment that we are interested in a special region of phase space defined in terms of some preconditioned action-angle variables[1] $\left(\vec{\Phi}, \vec{J}\right)$:

$$U = \left\{\vec{\Phi}, \vec{J} \mid 0 < J_{a;i} < J_i < J_{b;i}\ ,\ \Phi_i \in [0, 2\pi]\ ,\ i = 1, ..., N\right\} \tag{18}$$

The set U is a product of annuli in phase space. Let us assume that we are interested in the motion within the set U and that we will consider a particle to be lost if it leaves the said set. Then our goal is to represent the map within that set as accurately as possible (see [5]). To do this we simply take tracking data normalized by a canonical transformation $\vec{\omega}_s$. It is important to note that in many applications the transformation $\vec{\omega}_s$ need not be accurate. The action-angle variables are defined in a natural way:

$$\begin{aligned} \omega_{s;2i-1}^{-1}(\vec{x}) &= \sqrt{2J_i}\cos\ (\Phi_i) \\ \omega_{s;2i}^{-1}(\vec{x}) &= -\sqrt{2J_i}\sin\ (\Phi_i) \\ \\ \vec{x} &\rightarrow \text{tracking data} \end{aligned} \tag{19}$$

The map itself can be expressed as follows:

$$\begin{aligned} \vec{\Phi}^1 &= \vec{\Phi} + \vec{\Theta}\left(\vec{\Phi}, \vec{J}\right) \qquad \text{(20a)} \\ \vec{J}^1 &= \vec{J} + \vec{R}\left(\vec{\Phi}, \vec{J}\right) \qquad \text{(20b)} \end{aligned} \tag{20}$$

[1] Defined through some power series transformation $\vec{\omega}_s$ as shown in Equation (19).

The fitted map is obtained by expressing the functions $\vec{\Theta}$ and $\vec{R}$ as a Fourier series in the angles and using a spline representation for the actions. However, in long term tracking applications, we are interested in a symplectic map to computer accuracy. In the case of fitted maps, this can be achieved with generating functions. We now describe this process.

2.2.1 Fitting a Generating Function Map

The map is defined to be a transformation from the "old" variables $(\vec{\Phi}, \vec{J})$ to the "new" variables $(\vec{\Phi}^1, \vec{J}^1)$ as shown in Equation (20). The generating function in this case will be in terms of old action and new angle variables:

$$G\left(\vec{\Phi}^1, \vec{J}\right) = \sum_{\vec{m}} g_{\vec{m}}(\vec{J})\, e^{i\vec{m}\cdot\vec{\Phi}^1} \tag{21}$$

The resulting transformation is then just:

$$\begin{aligned} &\left(\vec{\Phi}, \vec{J}\right) \mapsto \left(\vec{\Phi}^1, \vec{J}^1\right) \\ &\vec{\Phi} = \vec{\Phi}^1 + \partial_{\vec{J}} G\left(\vec{\Phi}^1, \vec{J}\right) \qquad \text{(22a)} \\ &\vec{J}^1 = \vec{J} + \partial_{\vec{\Phi}^1} G\left(\vec{\Phi}^1, \vec{J}\right) \qquad \text{(22b)} \end{aligned} \tag{22}$$

We start with a "source map," which gives the final variables as an explicit function of the initial variables as given symbolically by Equation (20). This map will usually be defined as the result of tracking over one turn and preconditioning the data as explained in Equation (19). The Fourier coefficients are obtained from (22b) and (20b) as:

$$\begin{aligned} g_{\vec{m}}(\vec{J}) &= \tfrac{1}{(2\pi)^d\, i m_k} \int_0^{2\pi} d\vec{\Phi}^1\ \partial_{\Phi_k^1} G(\vec{\Phi}^1, \vec{J})\ e^{-i\vec{m}\cdot\vec{\Phi}^1} \\ &= \tfrac{1}{(2\pi)^d\, i m_k} \int_0^{2\pi} d\vec{\Phi}^1 R_k\left(\vec{\Phi}(\vec{\Phi}^1, \vec{J}), \vec{J}\right)\ e^{-i\vec{m}\cdot\vec{\Phi}^1} \end{aligned} \tag{23}$$

Since one does not know $\vec{R}$ as a function of $\vec{\Phi}^1$, one performs a change of variables in the integral to get an integral over $\vec{\Phi}$:

$$g_{\vec{m}}(\vec{J}) = \int_0^{2\pi} \frac{d\vec{\Phi}}{(2\pi)^d i m_k} R_k\left(\vec{\Phi}, \vec{J}\right)\ e^{-i\vec{m}\cdot\vec{\Phi}} e^{-i\vec{m}\cdot\vec{\Theta}(\vec{\Phi},\vec{J})}\ \det(1 + \partial_{\vec{\Phi}}\vec{\Theta}(\vec{\Phi}, \vec{J})\ ) \tag{24}$$

The integral is then discretized to obtain

$$g_{\vec{m}}(\vec{J}) = \frac{1}{im_k \prod_n N_n} \sum_{\vec{j}} R_k\left(\vec{\Phi}_{\vec{j}}, \vec{J}\right) e^{-i\vec{m}\cdot\vec{\Phi}_{\vec{j}}} e^{-i\vec{m}\cdot\vec{\Theta}(\vec{\Phi}_{\vec{j}},\vec{J})} \det\left(1 + \partial_{\vec{\Phi}}\vec{\Theta}(\vec{\Phi}_{\vec{j}}, \vec{J})\right) \tag{25}$$

where N_n is the number of Φ_n mesh points in the n dimension, and the summation is over the integer vectors $\vec{j}$ such that $j_n \in \{0, ..., N_n - 1\}$. The $\vec{m} = \mathbf{0}$ mode must be handled differently. We instead must use $\vec{\Theta}$ values. The resulting summation is

$$g_0(\vec{J}) = -\frac{1}{\prod_n N_n} \sum_{\vec{j}} \vec{\Theta}\left(\vec{\Phi}_{\vec{j}}, \vec{J}\right) \det\left(1 + \partial_{\vec{\Phi}}\vec{\Theta}(\vec{\Phi}_{\vec{j}}, \vec{J})\right) \tag{26}$$

To increase the speed of evaluation of the map, Fourier modes that are smaller than the expected or desired accuracy of the map can be removed from the generating function.

3 Conclusion

Symplectic integration and the symplectification of maps are essential tools in a modern treatment of circular rings. Our field is perversely lucky to have a localized Hamiltonian in the time-like variable. Therefore numerically induced chaos as well as other types of problems produced by symplectic integration are probably of no concern to accelerator physicists.

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[1] J. Irwin, SSC-228, (1989).

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[3] I. Gjaja, Monomial Factorization of Symplectic maps, Dynamical Systems and Accelerator Theory Group,

[4] E. Forest and J. Bengtsson have worked out a library based on automatic differentiation which allows power series normalization, various factorizations (Dragt-Finn, Irwin) and the computation of generating functions. It also normalizes maps in the nonsymplectic case which is useful in electron machines.

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