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Sums of $SL(3, Z)$ Kloosterman Sums

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Jack Buttcane

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Abstract of the Dissertation

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Professor William Duke, Chair

We show that sums of the $SL(3, \mathbb{Z})$ long element Kloosterman sum against a smooth weight function have cancellation due to the variation in argument of the Kloosterman sums, when each modulus is at least the square root of the other. Our main tool is Li's generalization of the Kuznetsov formula on $SL(3, \mathbb{R})$, which has to date been prohibitively difficult to apply. We first obtain analytic expressions for the weight functions on the Kloosterman sum side by converting them to Mellin-Barnes integral form. This allows us to relax the conditions on the test function and to produce a partial inversion formula suitable for studying sums of the long-element $SL(3, \mathbb{Z})$ Kloosterman sums.

The dissertation of Jack Buttcane is approved.

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William Duke, Committee Chair

University of California, Los Angeles

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To my family.

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CHAPTER 1

Introduction

The classical Kloosterman sums originated in 1926 in the context of applying the circle method to counting representations of integers by the four-term quadratic form $ax^2 + by^2 + cz^2 + dt^2$ [18]; they are defined by

$$S(a,b,c) = \sum_{\substack{x \pmod c \\ (x,c)=1 \\ x\bar{x}\equiv 1 \pmod c}} e\left(\frac{ax+b\bar{x}}{c}\right), \qquad e\left(x\right) = e^{2\pi i x},$$

and they enjoy a multiplicativity relation: If $(c,c')=1$, then

$$S(a,b,cc') = S(\overline{c'}a, \overline{c'}b, c)S(\bar{c}a, \bar{c}b, c'),$$

where $c'\overline{c'} \equiv 1 \pmod c$, $c\bar{c} \equiv 1 \pmod{c'}$. In 1927, Kloosterman [17] used these sums to estimate Fourier coefficients of modular forms, as did Rademacher in 1937 [30]. Optimal estimates for individual Kloosterman sums were obtained in 1948 by André Weil [41]: $|S(a,b,c)| \le d(c)\sqrt{(a,b,c)}\sqrt{c}$, where $d(c)$ is the number of positive divisors of c and (a,b,c) is the greatest common divisor. In 1963, Linnik published a paper outlining methods for problems in additive number theory [22] in which he noted the importance of sums of Kloosterman sums and made the conjecture that such sums should have good cancellation between terms:

Conjecture 1 (Linnik)**.** *Let N be large and $C > N^{\frac{1}{2}-\epsilon}$, then*

$$\sum_{c\le C} S(1,N,c) \ll C^{1+\epsilon}.$$

One should compare this to Weil's estimate which gives $C^{\frac{3}{2}+\epsilon}$.

On a parallel track, between 1932 and 1940, Petersson [29], Rankin [31] and Selberg [35] connected Fourier coefficients of modular forms to sums of Kloosterman sums by studying Poincaré series. This led to Kuznetsov's trace formulas [19] which relate sums of Kloosterman sums to sums of Fourier coefficients of $SL(2,\mathbb{Z})$ automorphic forms, and using these formulas in 1980, Kuznetsov was able to make progress towards Linnik's conjecture:

Theorem 2 (Kuznetsov)**.**

$$\sum_{c\le T}\frac{1}{c}S(n,m,c)\ll_{n,m} T^{\frac{1}{6}}\left(\ln T\right)^{\frac{1}{3}}.$$

As Weil's estimate here gives $T^{\frac{1}{2}+\epsilon}$, we must be seeing cancellation between terms as Linnik predicted.

This second track has been quite fruitful for the followers of Iwaniec – sums of arithmetic functions, usually related to quadratic forms in some sense, can sometimes be decomposed into sums of Kloosterman sums, e.g. [5], and similarly, exponential sums related to quadratic forms can often be decomposed into Poincaré series, e.g. [8]. The Kuznetsov trace formulas then play the role of Poisson summation, allowing one to substitute a sum of Fourier coefficients of automorphic forms for a sum of Kloosterman sums and visa versa. Iwaniec in particular has made good use of a sort of double application of Kuznetsov's formulas; using positivity to study averages of Fourier coefficients of automorphic forms via the Kuznetsov formula and then applying these estimates to sums of Kloosterman sums via the second form of the Kuznetsov formula, e.g. [5].

Finally, we note that the Fourier coefficients of automorphic forms which are also eigenfunctions of the Hecke operators give rise to L-functions. By applying the Kuznetsov formulas in this situation we may obtain results on averages of L-functions and all of the problems to which such things apply, e.g. [7].

Now having noted the strong connection between analysis on $SL(2,\mathbb{R})$ and quadratic forms, it is hoped that analysis on $SL(3,\mathbb{R})$ will play a similar role in the study of cubic forms, and the analysis of Hecke operators on $SL(3,\mathbb{R})$ automorphic forms is also known to give rise to L-functions. A paper of Jacquet, Piatetski-Shapiro and Shilika [14] and a book of Bump [2] (which is essentially his dissertation), form the foundations of the L-function approach; and a paper of Bump, Friedberg and Goldfeld [3] initiates the study of Poincaré series and Kloosterman sums on $SL(3,\mathbb{Z})$.

The BFG paper notes that the Fourier coefficients of Poincaré series are given by sums of two new types of exponential sums in addition to the classical sums of Kloosterman himself; we will primarily be concerned with the long-element sum which we denote $S_{w_l}(\psi_m, \psi_n, c)$, for reasons which will be made clear later, and is given by the sum

$$S_{w_l}(\psi_{m_1,m_2}, \psi_{n_1,n_2}, (A_1, A_2)) = \sum^*_{\substack{B_1,C_1 \pmod{A_1} \\ B_2,C_2 \pmod{A_2}}} e\left(m_2 \frac{Z_2B_1 - Y_2A_1}{A_2} + m_1 \frac{Y_1A_2 - Z_1B_2}{A_1} + n_2 \frac{B_1}{A_1} + n_1 \frac{-B_2}{A_2} \right),$$

here the sum $\sum^*$ is restricted to those quadruples of B_1, C_1, B_2, C_2 satisfying

$$(A_1, B_1, C_1) = (A_2, B_2, C_2) = 1, \quad A_1C_2 + B_1B_2 + C_1A_2 \equiv 0 \pmod{A_1A_2},$$

and the numbers Y_1, Z_1, Y_2, Z_2 are defined by

$$Y_1B_1 + Z_1C_1 \equiv 1 \pmod{A_1}, \qquad Y_2B_2 + Z_2C_2 \equiv 1 \pmod{A_2}.$$

In BFG, the authors list a number of basic properties of this new Kloosterman sum, which generally relate to its well-definedness and interchanging indices of characters or moduli, but the most important is a type of multiplicativity:

Lemma 3 (BFG). *If* $(c_1c_2, c_1'c_2') = 1$ *and*

$$\overline{c_1}c_1 \equiv \overline{c_2}c_2 \equiv 1 \pmod{c_1'c_2'}, \qquad \overline{c_1'}c_1' \equiv \overline{c_2'}c_2' \equiv 1 \pmod{c_1c_2},$$

then

$$S_{w_l}(\psi_m, \psi_n, (c_1c_1', c_2c_2')) = S_{w_l}(\psi_{m'}, \psi_n, (c_1, c_2))S_{w_l}(\psi_{m''}, \psi_n, (c_1', c_2')),$$

where $m' = \left(\overline{c_1'}^2 c_2' m_1, c_1'\overline{c_2'}^2 m_2\right)$, *and* $m'' = (\overline{c_1}^2 c_2 m_1, c_1\overline{c_2}^2 m_2)$.

Similarly, we have Weil-quality estimates for these sums, courtesy of Stevens [38]:

Theorem 4 (Stevens)**.**

$$|S_{w_l}(\psi_m, \psi_n, (A_1, A_2))|^2 \le d(A_1)^2 d(A_2)^2 \left(|m_1 n_2|, D\right)\left(|m_2 n_1|, D\right)(A_1, A_2)A_1A_2,$$

where $D = \frac{A_1A_2}{(A_1,A_2)}$.

Dabrowski and Fisher [4] have improved these estimates in most cases, but we expect that the exponents $(A_1A_2)^{\frac{1}{2}}$ are sharp in the general case, though the author is unaware of any such proof.

The hope that these generalized Kloosterman sums will play a similar role to their classical counterparts leads us to make Linnik-type conjectures for cancellation between terms in a sum of $SL(3,\mathbb{Z})$ Kloosterman sums, and the main result of this paper confirms this for a smooth weight function when the moduli are roughly the same size:

Theorem 5. *Let* $f \in C_c^8((\mathbb{R}^+)^2)$, *and take* X *and* Y *to be large parameters, with* ψ_m *and* ψ_n *non-degenerate characters, then*

$$\sum_{\varepsilon\in\{\pm1\}^2}\sum_{c_1,c_2\ge1}\frac{S_{w_l}(\psi_m,\psi_{\varepsilon n},c)}{c_1c_2} f\left(X\frac{4\pi^2 c_2|m_1n_2|}{c_1^2}, Y\frac{4\pi^2 c_1|m_2n_1|}{c_2^2}\right)$$
$$\ll_{f,m,n,\epsilon} (XY)^\epsilon\left((XY)^{\frac{5}{14}} + X^{\frac{1}{2}} + Y^{\frac{1}{2}}\right).$$

If we instead apply Stevens' estimate for the individual Kloosterman sums, we are led to the bound $(XY)^{\frac{1}{2}+\epsilon}$, so we are seeing cancellation between terms in the sum. The $(XY)^{\frac{5}{14}}$ comes from the Kim-Sarnak bound on the (real part of the) Langlands parameters of $SL(3,\mathbb{R})$ cusp forms, and the $X^{\frac{1}{2}}$ and $Y^{\frac{1}{2}}$ terms come from some second-term asymptotics which present a difficulty in our partial inversion of a two-dimensional integral transform. If the generalized Ramanujan-Selberg conjecture (a.k.a. generalized Selberg eigenvalue conjecture) holds, then our bound becomes $(XY)^\epsilon \left(X^{\frac{1}{2}} + Y^{\frac{1}{2}}\right)$, but we expect that the optimal bound would be $(XY)^\epsilon$ if one had a full inversion formula.

We expect that the most interesting examples should have $c_1 \asymp c_2$, i.e. when $X = Y$, and in this case the dominant term becomes $X^{\frac{5}{7}+\epsilon}$, which is entirely controlled by the Kim-Sarnak bound. Again, under the generalized Ramanujan-Selberg conjecture, this becomes $X^{\frac{1}{2}+\epsilon}$ and the optimal bound should be X^ϵ.

We have not chosen to track the dependence on the indices m and n here, but it is simple to do so. The resulting bound is not close to optimal; essentially, we are multiplying the bound by powers of m and n. For comparison, Sarnak and Tsimerman [32] have made Theorem 2 explicit in m and n with the bound

$$\left(x^{\frac{1}{6}} + (mn)^{\frac{1}{6}} + (m+n)^{\frac{1}{8}}(mn)^{\frac{7}{128}}\right)(mnx)^\epsilon,$$

and the third term may be removed if we assume the Ramanujan-Selberg Conjecture. Similar bounds for the long-element Kloosterman sums on $SL(3)$ would require a great deal more work, and optimal bounds are not possible with the current method, again because of the error terms.

Finally, there is another new type of Kloosterman sum on $SL(3)$ which arises in the same manner, but is much smaller in summation. There is some contention over whether this second type also has good cancellation in sums: If it behaves as the examples we have studied so far, the answer should be yes, but Bump, Friedberg and Goldfeld have put forth a competing theory in [3, Conjecture 1.2] to the effect that the Kloosterman zeta function of this sum should have poles on the boundary of its region of absolute convergence; in particular, this region would coincide with the region of conditional convergence, and there would be no significant cancellation between terms.

The methods here come from harmonic analysis on symmetric spaces. Specifically, these results are obtained by studying a generalization of the Kuznetsov formula to $SL(3,\mathbb{R})$: Starting from a proof of Kuznetsov's trace formula on $SL(2,\mathbb{R})$ by Zagier, and using the

Fourier coefficient decomposition of automorphic forms on $SL(n,\mathbb{R})$ by Friedberg (generalizes that of BFG on $SL(3,\mathbb{R})$), Li has given a generalization of the first of Kuznetsov's trace formulas to $SL(n,\mathbb{R})$ and this appears in Goldfeld's book on automorphic forms on $SL(n,\mathbb{R})$ [11]. So far, only the most basic of estimates have come out of the $SL(n)$ Kuznetsov formula and only for $SL(3)$, these may be found in a paper of Li herself [21], but in general, the integral transforms appearing in her formula are too complex to use effectively. Blomer has been able to push somewhat farther by developing his own generalization of Kuznetsov's first formula [1].

Using the Kuznetsov formula, we are able to express the integral transforms as an integral of the original test function against a function in Mellin-Barnes integral form. With this representation, we can produce a sort of first-term inversion for the integral transform attached to the sum of long-element Kloosterman sums, which gives us a sort of incomplete generalization of Kuznetsov's second trace formula, and the proof of Theorem 5 then proceeds much as in Kuznetsov's original paper.

The central idea is that the spectral parameters of the $SL(3,\mathbb{R})$ automorphic forms occur in a strip which is positive distance from the region of absolute convergence of the long-element Kloosterman zeta function. The aforementioned difficulties with the second-term asymptotics prevent us from obtaining the analytic continuation of the Kloosterman zeta function, but a similar path of shifting contours outside the region of absolute convergence yields the above results.

The BFG paper contains an alternate approach; they state, but do not prove, the meromorphic continuation of the *unweighted* Kloosterman zeta function (the main object of study in the paper is weighted by a type of generalized Bessel function, much as the sum appearing in the spectral Kuznetsov formula), which would in principle give the above results without the error terms $X^{1/2}$ and $Y^{1/2}$, if one could control the growth of the Kloosterman zeta function on vertical lines in the complex plane. On $SL(2)$, this method was started by Selberg [34] (see [33] as well as the Göttingen lecture in the second volume) and completed by Goldfeld and Sarnak [10].

Similarly, Yangbo Ye [42] has given a third approach starting directly with sums of the long-element Kloosterman sums. He provides a spectral interpretation which could be used to provide bounds in much the same manner as the current paper. The difficulty with his Kuznetsov formula, as with Li's, lies in the complexity of the generalized Bessel functions, hence an analysis of the functions occurring in his formula, as we are about to provide for Li's, should produce similar results. It would be an interesting problem for future research to compare the two.

CHAPTER 2

Background

To do harmonic analysis, we require a space with two properties:

1. A measure on the space.

2. A commutative group of differential operators which act on smooth functions on the space.

The first allows us to discuss the L^2 space, and the second allows us to decompose the L^2 space into eigenfunctions, a.k.a. harmonics.

More specifically, we are interested in harmonic analysis on symmetric spaces: Originally, a symmetric space referred to a Riemannian manifold with a geodesic-reversing isometry, but in Lie group theory, we define a symmetric space as having a continuous, transitive group action, where the stabilizer of any point is an open subgroup of the fixed point set of an involution of the group. We will not be concerned with the exact definition of a symmetric space, except to say that the groups we study are Lie groups, and the spaces are symmetric spaces under both definitions, so we may borrow theorems from these areas as necessary. In summary, we require:

3. A smooth, transitive group action on the space; the measure and differential operators should be invariant under this action.

Lastly, as we are studying number theory on these groups, we need:

4. A discrete subgroup.

Goldfeld's ν Parameters			
ν_1	$= \frac{\mu_1-\mu_2+1}{3}$	μ_1	$= 2\nu_1+\nu_2-1$
ν_2	$= \frac{1+\mu_1+2\mu_2}{3}$	μ_2	$= -\nu_1+\nu_2$
		μ_3	$= 1-\nu_1-2\nu_2$

Terras' s Parameters			
s_1	$= \frac{1+\mu_1-\mu_2}{2}$	μ_1	$= -1+\frac{2s_1-2s_2}{3}$
s_2	$= -\frac{2\mu_1+\mu_2}{2}-1$	μ_2	$= -\frac{4s_1+2s_2}{3}$
s_3	$= \frac{1+\mu_1+\mu_2}{2}$	μ_3	$= 1+\frac{2s_1+4s_2}{3}$

Terras' ia Parameters = Jorgensen and Lang's $i\lambda$ Parameters			
λ_1	$= a_1 = -2i\mu_3$	μ_1	$= \frac{ia_3}{2}$
λ_2	$= a_2 = -2i\mu_2$	μ_2	$= \frac{ia_2}{2}$
λ_3	$= a_3 = -2i\mu_1$	μ_3	$= \frac{ia_1}{2}$

Table 2.1: Conversion of Spectral Parameters.

We will then study the L^2 space of functions on the quotient space of our symmetric space modulo this discrete subgroup.

The setting that we are most concerned with is $SL(3,\mathbb{R})$ and its symmetric space at full level, i.e. the discrete subgroup is $SL(3,\mathbb{Z})$. We will first review $SL(2,\mathbb{R})$ and discuss the above four properties in a setting which is hopefully most natural to the reader, before generalizing to $SL(n,\mathbb{R})$ and then specializing back to $SL(3,\mathbb{R})$. A good reference here is Goldfeld's book [11] for the automorphic forms side. For the harmonic analysis on symmetric spaces, the author learned from Terras' book [39], but would like to recommend Jorgensen and Lang [15].

Before we begin, we have two notes on notation: First, we are considering $SL(n,\mathbb{R})$ embedded as the matrices of positive determinant in $GL(n,\mathbb{R})/\mathbb{R}^+$, so when we discuss matrices of (positive) determinant other than one, we simply mean to divide by the n-th root of the determinant. Second, we are expressing everything in terms of the Langlands parameters $\mu = (\mu_1,\ldots,\mu_n)$ with $\mu_1+\ldots+\mu_n = 0$, which are the analytic parameters of Whittaker functions and Eisenstein series . This differs from all of the referenced texts, but it is difficult to give a coherent presentation using the ν parameters of Goldfeld's book, which are the analytic parameters of L functions, and the extra factor of $\frac{i}{2}$ in the harmonic analysis books becomes annoying to deal with (clearly, the ia and $i\lambda$ parameters are best suited for harmonic analysis). Regardless, the reader must convert between multiple parameter definitions when referencing the texts, so we give the conversions for $SL(3,\mathbb{R})$ in Table 2.1.

2.1 A Review of $SL(2,\mathbb{R})$

For classical automorphic forms on $G = SL(2,\mathbb{R})$, the symmetric space is the upper half-plane $\mathbb{H} = \{z = x + iy \in \mathbb{C} : y > 0\}$, the measure is $\frac{dx\,dy}{y^2}$, the group action is by fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+c}{bz+d},$$

the group is differential operators is $\mathbb{C}[\Delta]$, where $\Delta = -y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)$ is the hyperbolic Laplacian, and we are operating at full level, so the discrete subgroup is $\Gamma = SL(2,\mathbb{Z})$. A calculation shows that the measure and Δ are invariant under the action of $SL(2,\mathbb{R})$.

To connect with the general case, we take a moment to rephrase this: Given a matrix $g \in SL(2,\mathbb{R})$, we may apply the Gram-Schmidt procedure on the rows, starting at the bottom, to obtain $g = uk$ where u is upper-triangular, and $k \in K = SO(2,\mathbb{R})$ is orthogonal. We can further decompose $u = rxy$ where $r \in \mathbb{R}^+$, x is upper unipotent and y is positive diagonal with a one in the bottom right:

$$g \equiv \begin{pmatrix} 1 & x \\ & 1 \end{pmatrix} \begin{pmatrix} y & \\ & 1 \end{pmatrix} \pmod{K},$$

where $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$, this is called the Iwasawa decomposition. (Recall that we ignore the determinant, so the y matrix need not have determinant 1.) This space G/K may then be identified with $\mathbb{H}$ by $g \mapsto x + iy$, and if we allow G to act on G/K by left translation, we see that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} g \equiv \begin{pmatrix} 1 & \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{y}{(cx+d)^2+c^2y^2} & 0 \\ 0 & 1 \end{pmatrix} \pmod{K}$$

(by Gram-Schmidt), and

$$\frac{az+b}{cz+d} = \frac{(ax+b)(cx+d)+acy^2}{(cx+d)^2+c^2y^2} + i\frac{y}{(cx+d)^2+c^2y^2},$$

so the G actions agree.

2.1.1 Selberg Spectral Decomposition

We mentioned above that our first goal is a decomposition of the space $L^2(\Gamma\backslash\mathbb{H})$ by eigenfunctions of the Laplacian; a theorem of Selberg gives

$$L^2(\Gamma\backslash\mathbb{H}) = \mathbb{C} \oplus L_{\text{cusp}} \oplus L_{\text{Eisenstein}},$$

where L_{cusp} is spanned by the Maass cusp forms, and $L_{\text{Eisenstein}}$ is spanned by integrals of the Eisenstein series. Before we go into more detail, we need to give the definitions of these functions.

We start with the simplest function on $\mathbb{H}$: the power function. Let $p_{\frac{1}{2}+\mu}(x+iy) = y^{\frac{1}{2}+\mu}$, then $p_{\frac{1}{2}+\mu}$ is a function on $\mathbb{H}$, but it is not Γ invariant. It is, however, an eigenfunction of Δ, with $\Delta p_{\frac{1}{2}+\mu} = \left(\frac{1}{4} - \mu^2\right) p_{\frac{1}{2}+\mu}$.

Maass forms are defined by three conditions: A non-zero function $\phi : \mathbb{H} \to \mathbb{C}$, which is square-integrable on $\Gamma\backslash\mathbb{H}$, is called a Maass form of type μ if it is

(a) Automorphic, $\phi(\gamma z) = \phi(z)$ for all $\gamma \in \Gamma$,

(b) Harmonic, $\Delta\phi = \left(\frac{1}{4} - \mu^2\right)\phi$, and

(c) Cuspidal, $\int_0^1 \phi(x+iy)dx = 0$.

The $SL(2,\mathbb{Z})$ Eisenstein series is initially defined by

$$E(z,\mu) = \frac{1}{2}\sum_{\gamma\in\Gamma_\infty\backslash\Gamma} p_{\frac{1}{2}+\mu}(\gamma z), \qquad \Gamma_\infty = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\}, \qquad \text{Re}(\mu) > \frac{1}{2}.$$

Its completion is $E^*(z,\mu) = \pi^{-\left(\frac{1}{2}+\mu\right)}\Gamma\left(\frac{1}{2}+\mu\right)\zeta\left(1+2\mu\right)E(z,\mu)$, which has analytic continuation to $\mathbb{C}\setminus\left\{\pm\frac{1}{2}\right\}$, simple poles at $\mu = \pm\frac{1}{2}$, and functional equation $E^*\left(z,\mu\right) = E^*\left(z,-\mu\right)$.

Theorem 6 (Selberg)**.** *Let $\{\phi_j\}$ be an orthonormal basis of $SL(2,\mathbb{R})$ cusp forms with ϕ_0*

constant. Then for any $\Phi \in L^2(\Gamma\backslash G/K)$*, we have*

$$\Phi(z) = \sum_{j=0}^{\infty} \langle \Phi, \phi_j \rangle \phi_j(z) + \frac{1}{4\pi i} \int_{\mathrm{Re}(\mu)=0} \langle \Phi, E(\cdot;\mu) \rangle E(z;\mu) d\mu.$$

2.1.2 Location of the cusp forms & The Weyl Law

The eigenvalues of the Laplacian Δ are of the form $\frac{1}{4} - \mu^2$ where μ is the Langlands parameter of an eigenfunction. Now cusp forms are square-integrable and integration by parts shows that Δ is a positive operator:

$$\langle \Delta f, f \rangle = \int_{\Gamma\backslash\mathbb{H}} \left(\left| \frac{\partial f}{\partial x} \right|^2 + \left| \frac{\partial f}{\partial y} \right|^2 \right) dx\, dy = \langle f, \Delta f \rangle,$$

so the eigenvalues of cusp forms must be non-negative. An actual calculation can show that these eigenvalues are larger than $\frac{1}{4}$ (due to Roelcke and Selberg, but see [11, Thm 3.7.2] for an easy proof), so we have $\mu = iy$ for some $y \in \mathbb{R}$.

Now that we know the (general) location of the Langlands parameters of cusp forms, we also want to be able to count the cusp forms; this was also done by Selberg:

Theorem 7 (Selberg)**.** *Let* $\{\phi\}$ *be a basis of* $SL(2,\mathbb{R})$ *cusp forms with eigenvalues* $\lambda_\phi = \frac{1}{4} - \mu_\phi^2$*, and let* $N(T)$ *be the counting function*

$$N(T) = \#\{\phi : \lambda_\phi \le T\},$$

then

$$N(T) \sim \frac{\mathrm{vol}(\Gamma\backslash\mathbb{H})}{4\pi} T.$$

This sort of theorem is called a Weyl Law.

2.1.3 Bruhat decomposition, Plücker coordinates & Kloosterman sums

Notice that a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ with $c \neq 0$, can be written uniquely as

$$\gamma = \begin{pmatrix} 1 & \frac{a}{c} \\ & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{c} & \\ & c \end{pmatrix} \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \begin{pmatrix} 1 & \frac{d}{c} \\ & 1 \end{pmatrix},$$

so we define the Kloosterman sums for $c \in \mathbb{N}$ by

$$S(m,n,c) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma / \Gamma_w \\ \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)}} e\left(m\frac{a}{c} + n\frac{d}{c}\right),$$

by $ad - bc = \det \gamma = 1$, we have $(a,c) = 1$ and $ad \equiv 1 \pmod{c}$, and the Bruhat decomposition tells us that this is really a sum over $a, d \pmod{c}$, so

$$S(m,n,c) = \sum_{\substack{a \pmod{c} \\ (a,c)=1}} e\left(\frac{ma + n\bar{a}}{c}\right),$$

where $a\bar{a} \equiv 1 \pmod{c}$. This is the classical Kloosterman sum.

Notice that the bottom row of a matrix is invariant under left translation by a matrix in Γ_∞.

2.1.4 Fourier coefficients & Whittaker functions

For any Maass cusp form ϕ, we have

$$\phi(z+1) = \phi\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z\right) = \phi(z),$$

so ϕ has a Fourier expansion in x, but the zeroth term is zero by the cuspidality condition, so

$$\phi(x+iy) = \sum_{0\neq n\in\mathbb{Z}} A_n(y) e\left(nx\right), \qquad e\left(x\right) = e^{2\pi i x}.$$

If we define the Whittaker function by

$$W(z,\mu,n) = \int_{-\infty}^{\infty} p_{\frac{1}{2}+\mu}(w(z+u)) e\left(-nu\right) du, \qquad w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \mathrm{Re}(\mu) > 0,$$

we can show that

$$W(z,\mu,n) = \frac{\sqrt{4\pi y}}{(\pi\left|n\right|)^{-\mu}\Gamma\left(\frac{1}{2}+\mu\right)} e\left(nx\right) K_\mu\left(2\pi\left|n\right|y\right),$$

and that actually the $A_n(y)$ above is some constant multiple of the Whittaker function $W(y,\mu,n)$. Thus if ϕ is of type μ, we have the Fourier-Whittaker expansion

$$\phi(z) = \sum_{0\neq n\in\mathbb{Z}} a(n) W(z,\mu,n).$$

Here K_μ denotes the K-Bessel function.

There is some question of normalization of Fourier coefficents, and we now choose a particular normalization: For any Maass cusp form ϕ, we define the normalized Fourier coeffcients ρ_ϕ by

$$\frac{\rho_\phi(m)}{|m|} W^*(\left|m\right| y,\mu,1) = \int_0^1 \phi(x+iy) e\left(-mx\right) dx,$$

where $W^*(y,\mu,1) = 2y^{1/2} K_\mu\left(2\pi y\right)$ is the normalized Whittaker function.

One may show by direct computation that the Eisenstein series has a Fourier-Whittaker expansion

$$E(z,\mu) = p_{\frac{1}{2}+\mu}(y) + \sqrt{\pi}\frac{\Gamma(\mu)\zeta(2\mu)}{\Gamma\left(\frac{1}{2}+\mu\right)\zeta(1+2\mu)}p_{-\mu}(y) + \sum_{0\neq n\in\mathbb{Z}}\frac{\sigma_{-2\mu}(n)}{\zeta(1+2\mu)}W(z,\mu,n),$$

$$\sigma_s(n) = \sum_{\substack{d|n\\ d>0}} d^s.$$

We let $\eta(m,\mu)$ be the Fourier coefficients of the Eisenstein series with the same normalization as above.

2.1.5 The Kuznetsov formula

The formula most central to this thesis will be the Kuznetsov formula on $SL(3,\mathbb{R})$, so in preparation to discuss that generalization, we give the original Kuznetsov formula for $SL(2,\mathbb{R})$, a good reference here is Iwaniec and Kowalski ch. 16 [13]. Let $\{\phi\}$ be an o.n.b. of cusp forms with Langlands parameters μ_ϕ and normalized Fourier coefficients ρ_ϕ; similarly, let η be the normalized Fourier coefficents of the Eisenstein series, then the form of Kuznetsov's formula having an arbitrary test function on the spectral side is:

Theorem 8 (Kuznetsov Trace Formula, Spectral Form)**.** *Let h be holomorphic on $-\frac{1}{2}-\delta \leq \mathrm{Re}(\mu) \leq \frac{1}{2}+\delta$, $h(\mu) = h(-\mu)$, and $h(\mu) \ll (1+|\mu|)^{-2-\delta}$, for some $\delta > 0$, then for $m,n > 0$,*

$$\sum_\phi \frac{h(\mu_\phi)}{\cos\pi\mu_\phi}\rho_\phi(n)\overline{\rho_\phi}(m) + \frac{1}{4\pi i}\int_{\mathrm{Re}(\mu)=0}\frac{h(\mu)}{\cos\pi\mu}\eta(n,\mu)\overline{\eta}(m,\mu)d\mu$$
$$= \delta_{mn}H_I(h,m) + \sum_{c=1}^{\infty} S(m,n,c)H_w(h,m,n,c),$$

where

$$H_I(h,m) = \frac{|m|\,i}{4\pi^3}\int_{\mathrm{Re}(\mu)=0} h(\mu)\pi\mu\tan\pi\mu\,d\mu,$$

and

$$H_w(h,m,n,c) = \frac{|mn|^{1/2}}{2\pi^2 ic}\int_{\mathrm{Re}(\mu)=0}\frac{h(\mu)}{\cos\pi\mu}J_{2\mu}\left(\frac{4\pi\sqrt{mn}}{c}\right)\pi\mu\,d\mu.$$

Here J_μ denotes the J-Bessel function. A quick construction of this formula may be found in [12], but beware the different normalizations:

$$\nu_{\mathfrak{a}j}(n) = \rho_\phi(n)\sqrt{\frac{2\pi}{|n|\cos\pi\mu}}.$$

The spectral form will be our starting point here, but our goal will be the arithmetic (a.k.a. geometric) form. Let $\psi_{j,k}$ be the Fourier coefficients of the jth element of an o.n.b. of S_{2k}, the space of holomorphic cusp forms of weight $2k$, then the form of Kuznetsov's formula having an arbitrary test function on the Kloosterman sum side is:

Theorem 9 (Kuznetsov Trace Formula, Arithmetic Form)**.** *Let f be twice continuously differentiable on $[0,\infty)$ with $f(0) = 0$ and $f^{(a)}(x) \ll (1+x)^{-\alpha}$ for $a = 0, 1, 2$, and some $\alpha > 2$, then for $m, n > 0$,*

$$\begin{aligned} &4\pi\sqrt{mn}\sum_{c=1}^{\infty}\frac{S(m,n,c)}{c}f\left(\frac{4\pi\sqrt{mn}}{c}\right) \\ &= \sum_{\phi}\frac{F(\mu_\phi)}{\cos\pi\mu_\pi}\rho_\phi(n)\overline{\rho_\phi}(m) + \frac{1}{4\pi i}\int_{\mathrm{Re}(\mu)=0}\frac{F(\mu)}{\cos\pi\mu}\eta(n,\mu)\overline{\eta}(m,\mu)d\mu \\ &\quad + \frac{16\pi(2k-1)!}{(4\pi i)^{2k}(mn)^{k-1}}\sum_{k=1}^{\infty}G(k)\sum_{j=1}^{\dim S_k}\psi_{j,k}(n)\overline{\psi_{j,k}}(m), \end{aligned}$$

where

$$F(\mu) = \int_0^\infty \frac{J_{-2\mu}(x) - J_{2\mu}(x)}{2\sin\pi\mu}f(x)\frac{dx}{x},$$

and

$$G(k) = \int_0^\infty J_{2k-1}(x)f(x)\frac{dx}{x}.$$

This formula is produced from the first by inverting the integral transform H_w and applying a formula of Peterson. Similar formulas apply for $mn < 0$ and $m, n < 0$.

2.1.6 Spherical Inversion

The Selberg transform is something that arises in spectral analysis on $SL(n,\mathbb{R})$ and we will need it and its inversion formula for Li's construction of the generalized Kuznetsov formula.

For a function $k : \mathbb{H} \to \mathbb{C}$ of sufficient decay, we define the Selberg transform

$$\hat{k}(\mu) = \int_{\mathbb{H}} k(z)\overline{p_{\frac{1}{2}+\mu}(z)}dz,$$

and it has the following inversion due to Selberg :

Theorem 10 (Selberg Inversion Formula for $SO(2,\mathbb{R})$ Invariant Functions)**.** *If $k(gz) = k(z)$ for all $g \in SO(2,\mathbb{R})$, the Selberg transform $k \mapsto \hat{k}$ has the inversion*

$$k(z) = \frac{1}{4\pi^2 i}\int_{\mathrm{Re}(\mu)=0} \hat{k}(\mu)h_\mu(z)\pi\mu\tan\pi\mu\, d\mu,$$

where $h_\mu(z)$ is the spherical function

$$h_\mu(x+yi) = P_{-\frac{1}{2}-\mu}\left(\frac{1+x^2+y^2}{2y}\right).$$

Here $P_s(y)$ is the Legendre P function. This is also called Helgason-Fourier or spherical inversion. The inversion extends to any nice $\hat{k}$ on $\mathrm{Re}(\mu) = 0$ satisfying $\hat{k}(\mu) = \hat{k}(-\mu)$.

2.2 The General Case: $SL(n,\mathbb{R})$

For the higher-rank automorphic forms on $G = SL(n,\mathbb{R})$, we have alluded to the fact that the symmetric space is G/K with $K = SO(n,\mathbb{R})$. Clearly, G acts by left translation on this space, and the discrete subgroup of interest is $\Gamma = SL(n,\mathbb{Z})$, i.e. we are interested in "full level". Applying the Gram-Schmidt procedure, we may compute the Iwasawa decomposition: If $z \in G$, then $z = rxyk$ with $r \in \mathbb{R}^+$, $k \in K$, and

$$x = \begin{pmatrix} 1 & x_{1,2} & \dots & \dots & x_{1,n} \\ & 1 & \ddots & & \vdots \\ & & \ddots & \ddots & \vdots \\ & & & 1 & x_{n-1,n} \\ & & & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1\cdots y_{n-1} & & & & \\ & y_1\cdots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix},$$

$y_i \in \mathbb{R}^+$, $x_{i,j} \in \mathbb{R}$. Again ignoring the determinant, we have $z \equiv xy \pmod{K}$. We denote the space of such x as $U(\mathbb{R}) = \mathbb{R}^{\frac{n(n-1)}{2}}$ and such y as $Y(\mathbb{R}) = Y(\mathbb{R}^+) = (\mathbb{R}^+)^{n-1}$.

The G-invariant measure, a.k.a Haar measure on a quotient space, has the form $dz = dx\, dy$ where

$$dx = \prod_{1 \le i < j \le n} dx_{i,j}, \qquad dy = \prod_{k=1}^{n-1} \frac{dy_k}{y_k^{k(n-k)+1}}.$$

For the differential operators, we start with the space of G-invariant differential operators acting on smooth functions $G/K \to \mathbb{C}$ and take its center, call it $\mathfrak{D}$. It can be shown that $\mathfrak{D} = \mathbb{C}[\Delta_1, \ldots, \Delta_{n-1}]$ where Δ_i is given by an explicit formula, called a Casimir operator, and Δ_1 in particular generalizes the Laplacian, which we will discuss when we talk about the geometric location of the Maass cusp forms on $SL(3, \mathbb{R})$.

2.2.1 Langlands Spectral Decomposition

After Selberg, Langlands was able to show a very general spectral decomposition:

Theorem 11 (Langlands)**.**

$$L^2(\Gamma \backslash G/K) = \mathbb{C} \oplus L_{cusp} \oplus L_{residual} \oplus L_{Eisenstein},$$

where L_{cusp} is spanned by Maass cusp forms, $L_{Eisenstein}$ is spanned by Eisenstein series, and $L_{residual}$ is spanned by residues of the Eisenstein series at points in the complex plane, all of which are eigenfunctions of all of $\mathfrak{D}$.

The constant function, Maass cusp forms, and residues of Eisenstein series form the discrete spectrum of $\mathfrak{D}$, while the Eisenstein series cover the continuous spectrum. Actually, Langlands spectral decomposition applies in much greater generality than we are using here, see [27].

It is easiest to define the power function in the μ parameters by looking at diagonal matrices

$$a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix}.$$

For $\mu = (\mu_1, \dots, \mu_n)$ with $\sum \mu_i = 0$, we set

$$p_{\rho+\mu}(a) = \prod_{i=1}^{n} a_i^{\rho_i+\mu_i},$$

and choose ρ so that

$$p_{\rho+0}(a)^2 = \prod_{1\le i<j\le n} \frac{a_i}{a_j} \Rightarrow \rho_i = \frac{(n-i)-(i-1)}{2} = \frac{n+1}{2} - i.$$

The conditions on ρ and μ give $p_{\rho+\mu}(rI) = 1$ for $r \in \mathbb{R}^+$, so it is unaffected by the determinant, and we can explicitly write it as a function of y and extend to G/K by the Iwasawa decomposition, as before:

$$p_{\rho+\mu}(xy) = p_{\rho+\mu}(y) = \prod_{i=1}^{n-1} y_i^{s_i},$$

$$s_i = \sum_{j=1}^{n-i}(\rho_j + \mu_j) = \frac{i(n-i)}{2} + \mu_1 + \ldots + \mu_{n-i}.$$

We say a non-zero function $\varphi : G/K \to \mathbb{C}$ is a Maass cusp form if it is

(a) Automorphic: $\varphi(\gamma z) = \varphi(z)$ for all $\gamma \in \Gamma$,

(b) Harmonic: $D\varphi = \lambda_D \varphi$ for some $\lambda_D \in \mathbb{C}$ for each $D \in \mathfrak{D}$,

(c) Cuspidal:

$$\int_{U^*(\mathbb{Z})\backslash U^*(\mathbb{R})} f(uz)du = 0,$$

for all upper-triangular groups

$$U^* = \left\{ \begin{pmatrix} I_{n_1} & & * \\ & \ddots & \\ & & I_{n_r} \end{pmatrix} \right\}, \qquad \sum_{i=1}^{r} n_i = n,$$

and square-integrable, i.e. $\varphi \in L^2(\Gamma \backslash G/K)$. We parameterize cusp forms by the Langlands parameters: A Maass cusp form is of type μ if it shares the eigenvalues under $\mathfrak{D}$ of the power function at μ, i.e. if $Dp_{\rho+\mu} = \lambda_D p_{\rho+\mu}$, then also $D\varphi = \lambda_D \varphi$. It can be shown that the eigenvalues of $p_{\rho+\mu}$ form a basis for the symmetric polynomials in μ, hence they are sufficient to describe the eigenvalues of the cusp forms, which are also symmetric in μ.

The power function is harmonic and, for certain values of μ, it is square-integrable, but not automorphic (and not cuspidal). It is, however, a character of the group of upper-triangular matrices: If $z = xy$ and $z' = x'y'$ are upper triangular, then

$$p_{\rho+\mu}(zz') = p_{\rho+\mu}(x(yx'y^{-1})yy') = p_{\rho+\mu}(yy') = p_{\rho+\mu}(y)p_{\rho+\mu}(y') = p_{\rho+\mu}(z)p_{\rho+\mu}(z').$$

Langlands Eisenstein series are much more complicated. In general, they split into two types: Parabolic Eisenstein series, and Eisenstein series twisted by Maass cusp forms of lower rank. The Eisenstein series are constructed by summing the power function and possibly a Maass cusp form over quotients of Γ, so they are harmonic, and the quotients are chosen so the resulting function is automorphic. Since we are only interested in these series to the extent that they appear in the spectral decomposition, we will defer their construction to the section on $SL(3, \mathbb{R})$.

For ease of notation, we will denote the combined spectral basis $\mathcal{B}$, and write the spectral expansion of $\Phi \in L^2(\Gamma \backslash G/K)$ as

$$\Phi(z) = \int_{\mathcal{B}} \xi(z) \langle \Phi, \xi \rangle \, d\xi.$$

The $\int_{\mathcal{B}} d\xi$ is a place-holder for the sum over cusp forms and residual spectrum, integrals of parabolic Eisenstein series, and sums of integrals of Eisenstein series twisted by Maass forms of lower rank.

2.2.2 Location of the cusp forms & The Weyl law

The operators Δ_i can be taken to be symmetric or antisymmetric, and Δ_1, the generalized Laplacian, is distinguished as a negative operator – in particular, for $SL(2,\mathbb{R})$, $\Delta_1 = -\Delta$. As with $SL(2,\mathbb{R})$, we believe that the Langlands parameters of cusp forms should be purely imaginary – called the Strong Ramanujan-Selberg Conjecture, but this is not known for $n > 2$. Following the same track, the eigenvalues of $-\Delta_1$, which turn out to be $\frac{n^3-n}{24} - \frac{\mu_1^2+\ldots+\mu_n^2}{2}$ are at least $\frac{n^3-n}{24}$ by a theorem of Miller [26]. This shows that the Langlands parameters are not all real, but is no longer sufficient to imply that they are purely imaginary (on $SL(2,\mathbb{R})$ this works because there is essentially only one parameter). We do have results on the size of the real part of the Langlands parameters, the current best is due to Luo, Rudnik, and Sarnak [23, 24]: $|\mathrm{Re}(\mu_i)| \leq \frac{1}{2} - \frac{1}{n^2+1}$ for the Langlands parameters of a Maass cusp form. This can be improved in special cases, in particular, a result of Kim and Sarnak is $|\mathrm{Re}(\mu_i)| \leq \frac{1}{2} - \frac{1}{\frac{n(n+1)}{2}+1}$ for $n = 3, 4$ [16]. Taking these results and the symmetry of the operators, we can solve for constraints on the possible location of the Langlands parameters of cusp forms. This is best done on a case-by-case basis, which we defer to the section on $SL(3,\mathbb{R})$.

Sarnak conjectured a form for the Weyl Law on $SL(n,\mathbb{R})$ which was recently proven by Müller [28]:

Theorem 12 (Müller)**.** *Let $\{\varphi\}$ be a basis of $SL(n,\mathbb{R})$ cusp forms with eigenvalues $-\Delta_1\varphi = \lambda_\varphi\varphi$, and let $N(T)$ be the counting function*

$$N(T) = \#\left\{\varphi : \lambda_\varphi \leq T\right\},$$

then

$$N(T) \sim \frac{\mathrm{vol}(\Gamma\backslash G/K)}{(4\pi)^{d/2}\Gamma\left(d+\frac{d}{2}\right)} T^{d/2},$$

where $d = \dim_{\mathbb{R}} G/K$.

This is actually not the most general theorem that we would like: The Maass cusp forms on $SL(n,\mathbb{R})$ are parameterized by $n-1$ variables – the Langlands Parameters μ, but the above theorem counts points in a one-dimensional space – $\lambda \in \mathbb{R}^+$. Lapid and Müller [20] have a much more general Weyl law which essentially counts Langlands parameters in a region as the size of the region becomes large, but their proof is currently only for level at least 3.

2.2.3 Bruhat decomposition, Plücker coordinates & Kloosterman sums

Let W be the group of permutation matrices in G, we call this the Weyl group. The Bruhat decomposition of G tells us that

$$G = \bigcup_{w\in W} G_w, \qquad G_w = BwB,$$

where $B \subset G$ is the group of (all) upper triangular matrices, a.k.a. the Borel subgroup. We are more interested in the following consequence, which we also call the Bruhat decomposition: Let $U(\mathbb{R})$ be the set of real upper unipotent matrices, and for each $w \in W$, let $U_w = (w^{-1}Uw) \cap U$, $\overline{U}_w = (w^{-1t}Uw) \cap U$ and $\Gamma_w = \overline{U}_w(\mathbb{Z})$ (forces zeros in some of the coordinates), then

Theorem 13 (Bruhat Decomposition). *A matrix in $\Gamma \cap G_w$ can be represented by a product of the form $b_1 cvwb_2$ where $b_1, b_2 \in U(\mathbb{Q})$, v in the group of diagonal orthogonal matrices V, and*

$$c = \begin{pmatrix} \frac{1}{c_1} & & & & \\ & \frac{c_1}{c_2} & & & \\ & & \ddots & & \\ & & & \frac{c_{n-2}}{c_{n-1}} & \\ & & & & c_{n-1} \end{pmatrix}, c_i \in \mathbb{N}.$$

The decomposition is unique if we require $b_2 \in \overline{U}_w(\mathbb{Q})$.

The characters of $U(\mathbb{R})$ are exactly the functions

$$\psi_m \begin{pmatrix} 1 & x_{n-1} & & * \\ & \ddots & \ddots & \\ & & \ddots & x_1 \\ & & & 1 \end{pmatrix} = e\left(m_1 x_1 + \ldots + m_{n-1} x_{n-1}\right)$$

(note the order of indices!), $m_i \in \mathbb{R}$. If all $m_i \in \mathbb{Z}$, then ψ_m is a function on $U(\mathbb{Z})\backslash U(\mathbb{R})$ and $U(\mathbb{R})/U(\mathbb{Z})$, and by abuse of terminology, we call ψ_m a character on these quotient spaces. It is vital to note that for $n > 2$, these are not groups! If all $m_i \neq 0$, then we call ψ_m non-degenerate.

The v matrix contains the sign information which would otherwise be part of the c matrix; we want to give a definition of the Kloosterman sum c matrices with $0 \neq c_i \in \mathbb{Z}$ which handles this in a convenient, but algebraically nice manner: then we define the Kloosterman sums for G by

$$S_w(\psi_m, \psi_n, cv) = \sum_{\substack{\gamma \in \Gamma_\infty \backslash \Gamma \cap G_w / V\Gamma_w \\ \gamma = b_1 cvwb_2}} \psi_m(b_1)\psi_n(b_2)$$

when the sum does not depend on the choice of Bruhat decompositions – where we allow $b_1, b_2 \in U(\mathbb{R})$, and 0 otherwise. If we let $v' = w^{-1}vw$, and conjugate $v'b_2v' \mapsto b_2$, we see that $S_w(\psi_m, \psi_n, cv) = S_w(\psi_m, \psi_n^{v'}, c)$, where $\psi_n^{v'}(b) = \psi_n(v'b_2v')$ (note that v' is its own inverse). Comparing this to our previous definition of the Kloosterman sums, we find the two definitions match on $SL(2, \mathbb{R})$ when $w = \left(\begin{smallmatrix} & -1 \\ 1 & \end{smallmatrix}\right)$.

The condition that $S_w(\psi_m, \psi_n, cv)$ not depend on the choice of Bruhat decompositions is called the compatibility condition, and it can be given explicitly from two facts: First, for any $u \in U_w(\mathbb{R})$ and any Bruhat decomposition $\gamma = b_1 cvwb_2$, we also have the Bruhat decomposition $\gamma = b_1' cvwb_2'$ with $b_1' = b_1(cvw)u(cvw)^{-1}$ and $b_2' = u^{-1}b_2$ gives us the necessary condition $\psi_m((cvw)u(cvw)^{-1})\psi_n(u^{-1}) = 1$ for all $u \in U_w(\mathbb{R})$. Second, the fact that the Bruhat decomposition is unique if $b_2 \in \overline{U}_w(\mathbb{Q})$ tells us that any b_2' differs from b_2 by an element $u \in U_w(\mathbb{R})$ so setting $b_1 cvwb_2 = b_1' cvwub_2$ gives $b_1 = b_1'(cvw)u(cvw)^{-1}$ and the condition is sufficient.

These Kloosterman sums first arose in the Fourier coefficients of $SL(n,\mathbb{R})$ Poincaré series, and they will appear in the Kuznetsov formula as one side is essentially the Fourier coefficient of a Poincaré series. We will list the specific types which occur on $SL(3,\mathbb{R})$ in that section.

In expressing the Kloosterman sums, we will need something called Plücker coordinates; these are invariants of $U(\mathbb{R})\backslash G$ (and so also of $U(\mathbb{Z})\backslash\Gamma$), and they are defined as the set of all $j\times j$ minors taken from the bottom j rows for $j=1\ldots n-1$. For $SL(2,\mathbb{Z})$, these are just the coordinates of the bottom row of the matrix and there are no linear relations among the variables. In general, the $j\times j$ minors of a matrix in Γ will not have any common factor, and there will be linear relations between the minors for different values of j.

2.2.4 Fourier coefficients & Whittaker functions

Shalika [36] has shown that an automorphic, harmonic function satisfying certain growth properties has Fourier coefficients of the form

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(uy)\psi_m(u)du = \frac{\rho_\varphi(m)}{|m_1\cdots m_n|}W^*(|m|\,y,\mu,\psi_{1,\ldots,1})$$

for non-degenerate characters ψ_m, where $\rho_\varphi(m)$ is some constant depending on φ and m,

$$W(z,\mu,\psi_m) = \int_{U(\mathbb{R})} p_{\rho+\mu}(w_l uz)\psi_m(u)du, \qquad w_l = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix},$$

is the Jacquet-Whittaker function, and W^* is its completion – this particular normalization of ρ_φ is chosen to simplify later formulae. Maass forms meet these three conditions. Notice also that if any $m_i=0$, then $\rho_\varphi(m)=0$ for Maass cusp forms by the cuspidality condition because we are integrating over the upper-triangular group

$$\left\{ \begin{pmatrix} I_{n-i} & * \\ 0 & I_i \end{pmatrix} \right\} \subset U.$$

There is an accompanying Fourier-Whittaker expansion for automorphic functions, but it is not relevant to our purposes.

The Langlands Eisenstein series are automorphic and harmonic and meet the growth properties, so their non-degenerate Fourier coefficients have the form given above. In the case of the parabolic Eisenstein series, one may compute the Fourier-Whittaker coefficients directly, and both types of Eisenstein series may be handled by the use of Hecke operators, but again, we defer this discussion to the section on $SL(3,\mathbb{R})$.

2.2.5 The Kuznetsov Formula

We collect the disparate pieces in Li's construction of the Kuznetsov formula on $SL(n,\mathbb{R})$ here, because its solitary appearance is in Goldfeld [11] mostly as an outline. Let $k \in C_c^\infty(K\backslash G/K)$, and let ψ_m, ψ_n be non-degenerate characters, then set

$$K(z,z') = \sum_{\gamma\in\Gamma} k(z^{-1}\gamma z'),$$

and we evaluate

$$P(k,y,y',\psi_m,\psi_n) = \int_{U(\mathbb{R})/U(\mathbb{Z})} \int_{U(\mathbb{R})/U(\mathbb{Z})} K(xy,x'y')\,\psi_m(x)\overline{\psi_n(x')}dx'\,dx$$

in two ways: first analytically, then algebraically.

Lemma 14 (Pre-Kuznetsov Formula Spectral Decomposition)**.** *P has a spectral decomposition of the form*

$$P = \int_{\mathcal{B}} \hat{k}(\mu_\xi) \int_{U(\mathbb{R})/U(\mathbb{Z})} \xi(x'y')\overline{\psi_n(x')}dx' \int_{U(\mathbb{R})/U(\mathbb{Z})} \overline{\xi(xy)}\psi_m(x)dx\,d\xi,$$

where $\hat{k}$ is the generalized Selberg transform

$$\hat{k}(\mu) = \int_G k(z)\overline{p_{\rho+\mu}(z)}dz.$$

Proof. Use the Langlands spectral decomposition for K in the z' variable, giving

$$K(z,z') = \int_{\mathcal{B}} \xi(z') \int_{\Gamma\backslash G/K} K(z,u)\overline{\xi(u)}du\, d\xi = \int_{\mathcal{B}} \xi(z') \int_{G/K} k(z^{-1}u)\overline{\xi(u)}du\, d\xi.$$

Examining the inner integral, we may substitute $u \mapsto zu$, integrate over K on the right of u, and separate the K integral on the left of G so

$$\int_{G/K} k(z^{-1}u)\overline{\xi(u)}du = \int_{K\backslash G} k(u) \int_K \overline{\xi(zku)}dk\, du.$$

Taking $\overline{f(u)}$ to be the inner integral, we have that f is an eigenfunction of all of $\mathcal{D}$ with eigenvalues matching $p_{\rho+\mu_\xi}$ by left-translation invariance, and $f(k_1 g k_2) = f(g)$ for all $k_1, k_2 \in K$, $g \in G$, but these two properties uniquely define the spherical function

$$h_\mu(z) = \int_K p_{\rho+\mu}(kz)dk$$

up to a constant multiple [39, Ch 4, eq. 2.27 and Theorem 3 (4)], and taking $u = I$ with the fact $h_\mu(I) = 1$ gives $f(u) = \xi(z)h_{\rho+\mu_\xi}(u)$, so

$$\int_{G/K} k(z^{-1}u)\overline{\xi(u)}du = \overline{\xi(z)} \int_{K\backslash G} k(u)\overline{h_{\mu_\xi}(u)}du = \overline{\xi(z)} \int_G k(u)\overline{p_{\rho+\mu_\xi}(u)}du,$$

by applying the above integral representation of h_μ. □

Lemma 15 (Pre-Kuznetsov Formula Arithmetic Decomposition)**.** *For each $w \in W$, let $R_w \subset U(\mathbb{Q}) \times CV \times \overline{U}_w(\mathbb{Q})$ be a complete set of representatives for $U(\mathbb{Z})\backslash\Gamma \cap G_w/\overline{U}_w(\mathbb{Z})$,* *that is*

$$U(\mathbb{Z})\backslash\Gamma \cap G_w/\overline{U}_w(\mathbb{Z}) = \left\{U(\mathbb{Z})b_1cvwb_2\overline{U}_w(\mathbb{Z}) : (b_1, cv, b_2) \in R_w\right\},$$
$$(b_1, cv, b_2) \neq (b_1', c'v', b_2') \in R_w \Rightarrow U(\mathbb{Z})b_1cvwb_2\overline{U}_w(\mathbb{Z}) \neq U(\mathbb{Z})b_1'c'v'w'b_2'\overline{U}_w(\mathbb{Z}),$$

then

$$P = \sum_{w\in W}\sum_{v\in V}\sum_{c_1,c_2\in\mathbb{N}} \left(\int_{U_w(\mathbb{R})/U_w(\mathbb{Z})} \psi_m\left((cw)u(cw)^{-1}\right)\psi_n^v(u)du\right) \left(\sum_{(b_1,cwvw^{-1},b_2)\in R_w} \psi_m(b_1)\psi_n(b_2)\right) \int_{U(\mathbb{R})}\int_{\overline{U}_w(\mathbb{R})} k\left(y^{-1}x^{-1}cwx'y'\right)\psi_m(x)\overline{\psi_n^v(x')}dx'\,dx.$$

Note that P is an integral of a Poincaré series, and though we are not presenting it in that manner, we are essentially following a construction first given on $SL(3,\mathbb{R})$ by Bump, Friedberg and Goldfeld [3] and later extended to $SL(n,\mathbb{R})$ by Friedberg [9].

Proof. Apply the Bruhat decomposition and unfold:

$$P = \sum_{w,c,v}\sum_{(b_1,cv,b_2)\in R_w}\int_{U(\mathbb{R})}\int_{U(\mathbb{R})/U(\mathbb{Z})} k\left(y^{-1}x^{-1}b_1cvwb_2x'y'\right)\psi_m(x)\overline{\psi_n(x')}dx'\,dx.$$

First, we substitute $x \mapsto b_1x$ and $x' \mapsto b_2^{-1}x'$; then conjugating by any element $w \in W$ leaves $w^{-1}Vw = V$ intact, so we may move w one step to the left and conjugate v completely to the right, sending $vx'v \mapsto x'$:

$$P = \sum_{w,v,c}\left(\sum_{(b_1,cwvw^{-1},b_2)\in R_w}\psi_m(b_1)\psi_n(b_2)\right) \int_{U(\mathbb{R})}\int_{U(\mathbb{R})/U(\mathbb{Z})} k\left(y^{-1}x^{-1}cwx'y'\right)\psi_m(x)\overline{\psi_n^v(x')}dx'\,dx,$$

where $\psi_n^v(x') = \psi_n(vx'v)$. Next substitute $x' \mapsto ux'$ where now $x' \in \overline{U}_w(\mathbb{R})$ and u is in some fixed fundamental domain of $U_w(\mathbb{R})/U_w(\mathbb{Z})$, so that $(cw)u(cw)^{-1} \in U(\mathbb{R})$, and we substitute on x to remove this from the main integral. □

Lemma 16 (Friedberg)**.** *Independent of the choice of Bruhat decompositions R_w, the product*

$$\left(\int_{U_w(\mathbb{R})/U_w(\mathbb{Z})} \psi_m\left((cw)u(cw)^{-1}\right)\psi_n^v(u)du\right)\left(\sum_{(b_1,cwvw^{-1},b_2)\in R_w} \psi_m(b_1)\psi_n(b_2)\right)$$

is $S_w(\psi_m,\psi_n^v,c)$.

Proof. We denote the sum and integral above by S_{cvw} and I_{cvw}, respectively. From the definition and the remarks on the compatibility condition, $S_{cvw} = S_w(\psi_m,\psi_n^v,c)$ exactly when $\psi^*(u) = \psi_m\left((cw)u(cw)^{-1}\right)\psi_n^v(u)$ is the trivial character on $U_w(\mathbb{R})$, and in that case, the integral is one, so the main point here is that if $\psi^*(u)$ is non-trivial on $U_w(\mathbb{R})$, then the product is zero. Friedberg [9] gives two arguments that the sum is zero unless $\psi^\dagger(u) = \psi_m\left((cw)u(cw)^{-1}\right)$ is trivial on $U_w(\mathbb{Z})$; if that is the case, and $\psi^*(u)$ is non-trivial, then the integral must be 0: If $\psi^*(a)\neq 1$, then we translate by $u\mapsto au$ in the integral (which requires ψ^* to be well-defined on the quotient space and not just a fundamental domain), giving $I_{cvw} = \psi^*(a)I_{cvw}$ so that $I_{cvw}=0$.

Friedberg's first proof that $\psi^\dagger(u)$ is trivial on $U_w(\mathbb{Z})$ is the easiest, but least informative: For each $w\in W$, $c\in C$, and $v\in V$, the integral P_{cvw} given by

$$\begin{aligned}&\sum_{(b_1,cwvw^{-1},b_2)\in R_w}\int_{U(\mathbb{R})/U(\mathbb{Z})}\int_{U(\mathbb{R})/U(\mathbb{Z})} k\left((xy)^{-1}b_1cwvb_2(x'y')\right)\psi_m(x)\overline{\psi_n(x')}dx'\,dx\\ &= I_{cvw}S_{cvw}\int_{U(\mathbb{R})}\int_{\overline{U}_w(\mathbb{R})} k\left(y^{-1}x^{-1}cwx'y'\right)\psi_m(x)\overline{\psi_n^v(x')}dx'\,dx\end{aligned}$$

is independent of the choice of fundamental domains in I_{cvw}. As before, we translate by an element $a\in U_w(\mathbb{Z})$ giving $P_{cvw} = \psi^\dagger(a)P_{cvw}$.

For his second proof, he finds a sum of $\psi^\dagger$ over a subgroup inside the sum S_{cvw}, which is much more constructive, but also more difficult. We will not give that argument here. □

To move from the Pre-Kuznetsov formula to the Kuznetsov formula, we need to integrate away some extra variables on the spectral side, which leads us to the following theorem of

Stade [37]:

Theorem 17 (Stade)**.** *For* $\mathrm{Re}(s) \geq 1$,

$$2^{n-1}\Gamma\left(\frac{ns}{2}\right)\int_{Y(\mathbb{R})} W^*(y,\mu,\psi_{1,\ldots,1})W^*(y,\mu',\psi_{1,\ldots,1})d\nu_s(y)$$
$$=\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{s+\mu_j+\mu'_k}{2}\right),$$

where

$$d\nu_s(y)=\prod_{j=1}^{n-1}(\pi y_j)^{(n-j)s}\,\frac{dy_j}{y_j^{1+j(n-j)}}=\left(\prod_{j=1}^{n-1}(\pi y_j)^{(n-j)s}\right)dy.$$

The spectral side of the Pre-Kuznetsov formula has products of two Fourier coefficients of automorphic forms, which we know are multiples of Whittaker functions, so we choose values $y=|m|^{-1}t$ and $y'=|n|^{-1}t$, and integrate the formula against the measure $d\nu_1(t)$, then Stade's formula will replace the Whittaker functions with a product of gamma functions:

$$\prod_{j=1}^{n}\prod_{k=1}^{n}\Gamma\left(\frac{1+\mu_j+\overline{\mu_k}}{2}\right)=\Gamma\left(\frac{1}{2}\right)^n\prod_{j<k}\Gamma\left(\frac{1+\mu_j-\mu_k}{2}\right)\Gamma\left(\frac{1+\mu_k-\mu_j}{2}\right)$$
$$=\frac{\pi^{n^2/2}}{\prod_{j<k}\cos\frac{\pi}{2}(\mu_k-\mu_j)},$$

when $-\overline{\mu}$ is some permutation of μ.

Corollary 18. *Suppose* $-\overline{\mu}$ *is some permutation of* μ*, then*

$$\int_{Y(\mathbb{R})} W^*(y,\mu,\psi_{1,\ldots,1})W^*(y,\overline{\mu},\psi_{1,\ldots,1})\left(\prod_{j=1}^{n-1}(\pi y_j)^{n-j}\right)dy$$
$$=\frac{\pi^{n^2/2}}{2^{n-1}\Gamma\left(\frac{n}{2}\right)\prod_{j<k}\cos\frac{\pi}{2}(\mu_k-\mu_j)}.$$

Putting everything together gives

Theorem 19 (Kuznetsov Formula)**.**

$$\int_{\mathcal{B}}\frac{\hat{k}(\mu_\xi)}{C(\mu_\xi)}\rho_\xi(n)\overline{\rho_\xi}(m)\,d\xi=\frac{2^{n-1}\Gamma\left(\frac{n}{2}\right)}{\pi^{n/2}}\sum_{w\in W}\sum_{v\in V}\sum_{c\in C}S_w(\psi_m,\psi_n^v,c)H_w(k,\psi_m,\psi_n^v,c),$$

where $C(\mu) = \prod_{j<k} \cos\frac{\pi}{2}(\mu_k - \mu_j)$ *if* $-\overline{\mu}$ *is a permutation of* μ *and* H_w *is given by the integral*

$$\int_{Y(\mathbb{R})}\int_{U(\mathbb{R})}\int_{\overline{U}_w(\mathbb{R})} k\left(|m|\, t^{-1}x^{-1}cwx't\, |n|^{-1}\right)\psi_m(x)\overline{\psi_n^v(x')}dx'\, dx \left(\prod_{j=1}^{n-1} |m_j n_j|\, t_j^{n-j}\right) dt.$$

We will write out the formula completely for $SL(3,\mathbb{R})$ below. The condition that $-\overline{\mu}$ be a permutation of μ is met for the spectral decomposition on $GL(2)$ ($-\overline{iy} = iy$) and $GL(3)$ [40] and seems to be a general symmetry statement for all $GL(n)$, but the author is unware of the proof of this fact.

Li also notes that one may apply spherical inversion to allow $\hat{k}$ in some larger space of analytic functions, provided both sides converge absolutely.

2.2.6 Spherical Inversion

The generalization of the Selberg transform which occurs in the generalized Kuznetsov formula has the following inversion due to Helgason, Harish-Chandra, and Bhanu-Murthy (see Sect 4.3 Theorem 1 of [39] and equation 3.23 in particular for the K-invariant form on $SL(n,\mathbb{R})$):

Theorem 20 (Spherical Inversion)**.** *For* k *as above, the Selberg transform* $k \mapsto \hat{k}$ *has the inversion*

$$k(z) = 2\pi i \omega_n \int_{\mathrm{Re}(\mu)=(0,\ldots,0)} \frac{\hat{k}(\mu)}{|c_n(\mu)|^2} h_\mu(z) d\mu,$$

where

$$\omega_n = \prod_{j=1}^{n} \frac{\Gamma\left(\frac{j}{2}\right)}{j(2\pi i)\pi^{j/2}},$$

$$|c_n(\mu)|^{-2} = \left|\prod_{i<j} \frac{B\left(\frac{\mu_j-\mu_i}{2}, \frac{1}{2}\right)}{B\left(\frac{1}{2}, \frac{1}{2}\right)}\right|^{-2} = \prod_{i<j} \frac{\pi}{2}(\mu_j - \mu_i)\tan\frac{\pi}{2}(\mu_j - \mu_i)$$

for $\mathrm{Re}(\mu) = 0$*.* $\hat{k}$ *may be taken to be any Schwartz-class function on* $\mathrm{Re}(\mu) = 0$ *which is invariant under permutations of the* μ *variables.*

To discuss the action of the Weyl group, we need to remove ρ from our definition of the power function: $p_{\rho+\mu} = p_\rho p_\mu$, where p_s, a character of the group of positive diagonal matrices, has the form

$$p_s \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} = \prod_{i=1}^{n} a_i^{s_i}.$$

It is trivial to see that conjugating by an element of the Weyl group permutes the entries of a, so the action of the Weyl group is given by

$$p_\mu^w(a) := p_\mu(w^{-1}aw) =: p_{\mu^w}(a),$$

where now μ^w is some permutation of the coordinates of μ. That is, the Weyl group acts by permutation on the coordinates of μ. The reason why this is necessary is that one may show the Selberg transform is invariant under the action of the Weyl group, $\hat{k}(\mu^w) = \hat{k}(\mu)$.

The first theorem of Harish-Chandra [15, Ch VIII, Sect 6, Thm 6.1] is that the Selberg transform (a.k.a. spherical transform) is an isomorphism

$$C_c^\infty(K\backslash G/K) \xrightarrow{\sim} \mathrm{PW}^W(\mu),$$

where the second space is the Paley-Wiener space of entire functions of finite order in the real part and rapid decay in the imaginary part which are invariant under the action of the Weyl group:

$$f(\mu) \ll_N \frac{e^{c|\mathrm{Re}(\mu)|}}{(1+|\mathrm{Im}(\mu)|)^N},$$

for all $N \in \mathbb{N}$, where c is some fixed constant.

The second theorem of Harish-Chandra [15, Ch X, Sect 5, Thm 5.6] is that the Selberg transform extends to an isomorphism

$$\mathrm{HCS}(K\backslash G/K) \xrightarrow{\sim} \mathrm{SCH}^W(\mu).$$

The first space is the Harish-Chandra Schwartz space defined by the property that f belongs to the space whenever $f : G \to \mathbb{C}$ is bi-K-invariant, smooth in the coordinates of G, and for each bi-K-invariant differential operator D on G and any $N \in \mathbb{N}$, we have

$$|f(a)| \ll_{D,N} \frac{h_0(a)}{(1 + |\log \chi_1(a)|)^N},$$

for all positive diagonal a. The second space is the Schwartz space of real-analytic functions on $\mathrm{Re}(\mu) = 0$ which are invariant under the action of the Weyl group.

2.3 $SL(3, \mathbb{R})$ in Particular

As $G = SL(3, \mathbb{R})$ is the particular case which this thesis is about, we wish to carefully spell out the above concepts for it here: The symmetric space is G/K with $K = SO(3, \mathbb{R})$, and G acts by left translation on this space – the action is sufficiently complex that we have no desire to write it out. We are interested strictly in $\Gamma = SL(3, \mathbb{Z})$, i.e. "full level". The Iwasawa decomposition for $z \in G$ is $z \equiv xy \pmod{K}$, where

$$x = \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix},$$

$y_i \in \mathbb{R}^+$ and $x_i \in \mathbb{R}$ – we will not often write out the form of the x and y matrices, so please note the ordering of the indices here. The G-invariant measure has the form

$$dz = dx\, dy, \qquad dx = dx_1\, dx_2\, dx_3, \qquad dy = \frac{dy_1\, dy_2}{(y_1 y_2)^3}.$$

The center of the space of G-invariant differential operators is $\mathfrak{D} = \mathbb{C}[\Delta_1, \Delta_2]$ where

$$\Delta_1 = y_1^2 \frac{\partial^2}{\partial {y_1}^2} + y_2^2 \frac{\partial^2}{\partial {y_2}^2} - y_1 y_2 \frac{\partial^2}{\partial y_1 \partial y_2} + y_1^2 (x_2^2 + y_2^2) \frac{\partial^2}{\partial {x_3}^2} + y_1^2 \frac{\partial^2}{\partial {x_1}^2} + y_2^2 \frac{\partial^2}{\partial {x_2}^2} + 2 y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3}$$

is the generalized Laplacian, and

$$\begin{aligned}\Delta_2 = &- y_1^2 y_2 \frac{\partial^3}{\partial y_1^2 \partial y_2} + y_1 y_2^2 \frac{\partial^3}{\partial y_1 \partial y_2^2} - y_1^3 y_2 \frac{\partial^3}{\partial x_3^2 \partial y_1} + y_1 y_2^2 \frac{\partial^3}{\partial x_2^2 \partial y_1} - 2y_1^2 y_2 x_2 \frac{\partial^3}{\partial x_1 \partial x_3 \partial y_2} \\ &+ (-x_2^2 + y_2^2) y_1^2 y_2 \frac{\partial^3}{\partial x_3^2 \partial y_2} - y_1^2 y_2 \frac{\partial^3}{\partial x_1^2 \partial y_2} + 2y_1^2 y_2^2 \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} + 2y_1^2 y_2^2 x_2 \frac{\partial^3}{\partial x_2 \partial x_3^2} \\ &+ y_1^2 \frac{\partial^2}{\partial {y_1}^2} - y_2^2 \frac{\partial^2}{\partial {y_2}^2} + 2y_1^2 x_2 \frac{\partial^2}{\partial x_1 \partial x_3} + (x_2^2 + y_2^2) y_1^2 \frac{\partial^2}{\partial {x_3}^2} + y_1^2 \frac{\partial^2}{\partial {x_1}^2} - y_2^2 \frac{\partial^2}{\partial {x_2}^2}.\end{aligned}$$

2.3.1 Langlands Spectral Decomposition

We have $\rho = (1, 0)$, so the power function has the form $p_{\rho+\mu}(xy) = y_1^{1+\mu_1+\mu_2} y_2^{1+\mu_1}$. The power function can also be realized as

$$p_{\rho+\mu}(z) = |Y_1|^{\frac{1+\mu_1-\mu_2}{2}} |Y_2|^{-\frac{2\mu_1+\mu_2}{2}-1} |Y_3|^{\frac{1+\mu_1+\mu_2}{2}},$$

where $|Y_j|$ is the determinant of the upper-left $j \times j$ minor of $Y = {}^t z^{-1} z^{-1}$; clearly this is right-invariant by $SO(3, \mathbb{R})$ and independent of $\det(z)$, block multiplication shows that it does not depend on x, and a simple calculation shows that has the appropriate dependence on y_1 and y_2. We mention this here as it is the definition used by Terras [39].

The definition of Maass cusp forms on $SL(3, \mathbb{R})$ becomes: $\varphi : G/K \to \mathbb{C}$ is a Maass cusp form if $\varphi(\gamma z) = \varphi(z)$ for all $\gamma \in \Gamma$, $\varphi \in L^2(\Gamma \backslash G/K)$,

$$\int_{U_i(\mathbb{Z}) \backslash U_i(\mathbb{R})} \varphi(uz) du = 0,$$

for the upper-triangular groups

$$U_1 = \left\{ \begin{pmatrix} 1 & 0 & * \\ & 1 & * \\ & & 1 \end{pmatrix} \right\}, \qquad U_2 = \left\{ \begin{pmatrix} 1 & * & * \\ & 1 & 0 \\ & & 1 \end{pmatrix} \right\},$$

and φ is an eigenvalue of all the differential operators in $\mathfrak{D}$. A Maass form is of type $\mu = (\mu_1, \mu_2, \mu_3)$ (where $\mu_1 + \mu_2 + \mu_3 = 0$) if it shares the eigenvalues under $\mathfrak{D}$ of the power function at μ:

$$-\Delta_1 p_{\rho+\mu} = \left(1 - \frac{\mu_1^2+\mu_2^2+\mu_3^2}{2}\right) p_{\rho+\mu}, \qquad \Delta_2 p_{\rho+\mu} = -\mu_1\mu_2\mu_3 p_{\rho+\mu}.$$

For the Langlands Eisenstein series, we have two types:

$$E(z,\mu) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} p_{\rho+\mu}(\gamma z), \qquad E_\phi(z,\mu_1) = \sum_{\gamma\in P_{2,1}\backslash\Gamma} p_{\mu_1,\phi}(\gamma z),$$

where

$$\Gamma_\infty = \left\{\begin{pmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1\end{pmatrix} \in \Gamma\right\}, \qquad P_{2,1} = \left\{\begin{pmatrix} * & * & * \\ * & * & * \\ 0 & 0 & 1\end{pmatrix} \in \Gamma\right\},$$

and $p_{\mu_1,\phi}(g) = (y_1^2 y_2)^{\frac{1}{2}+\mu_1}\phi(x_2+iy_2)$ with ϕ any $SL(2,\mathbb{R})$ cusp form. The Eisenstein series are smooth functions on $\Gamma\backslash G/K$ which are eigenvalues of all of $\mathfrak{D}$, but not square-integrable. They have meromorphic continuation to all of $\mathbb{C}$ in each μ_i.

The residual spectrum is literally the residues of Eisenstein series at points in the μ parameters, and a basis for the residual spectrum is E_{ϕ_0}.

Theorem 21 (Langlands)**.** *Let $\{\varphi_j\}$ be an orthonormal basis of the $SL(3,\mathbb{R})$ cusp forms with φ_0 constant, and $\{\phi_j\}$ an orthonormal basis of $SL(2,\mathbb{R})$ cusp forms with ϕ_0 constant. Then for any $\Phi \in L^2(\Gamma\backslash G/K)$, we have*

$$\begin{aligned}\Phi(z) &= \sum_{j=0}^{\infty} \langle \Phi, \varphi_j\rangle \varphi_j(z) \\ &+ \frac{1}{4\pi i}\sum_{j=0}^{\infty}\int_{\mathrm{Re}(\mu_1)=0} \left\langle \Phi, E_{\phi_j}(\cdot,\mu_1)\right\rangle E_{\phi_j}(z,\mu_1)d\mu_1 \\ &+ \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\mu)=(0,0)} \langle \Phi, E(\cdot;\mu)\rangle E(z;\mu)d\mu.\end{aligned}$$

2.3.2 Location of the cusp forms & The Weyl Law

The operator $-\Delta_1$ (the generalized Laplacian) is homogeneous of degree 2 and symmetric with respect to Haar measure, and integration by parts shows its eigenvalues on L^2 are non-negative. Again, one can show that these eigenvalues are greater than 1. The second operator has no homogeneity properties and is anti-symmetric, so its eigenvalues on L^2 are purely imaginary. These three constraints imply that the Langlands parameters of a $GL(3)$ cusp form are (some permutation of) either $(iy_1, iy_2, -iy_1 - iy_2)$ ("tempered at infinity" = good) or $(x+iy, -x+iy, -2iy)$ (not "tempered at infinity" = bad) for some $y_1, y_2, x, y \in \mathbb{R}$, see [40]. The Kim-Sarnak result shows that the real part of the Langland's parameters is at most $\frac{5}{14}$, so we have $|x| \le \frac{5}{14}$ as well (expect $x = 0$, i.e. the second case never happens).

Even though we don't have the results we want for the location of the Langlands parameters of the cusp forms, we do have a counting function for the eigenvalues of the generalized Laplacian. This was first proved by Miller in his thesis (using upper bounds of Donnelly) (see [25] for an article version).

Theorem 22 (Weyl Law for $SL(3,\mathbb{R})$)**.** *Let $\{\varphi_j\}$ be a basis of $SL(3,\mathbb{R})$ cusp forms with eigenvalues $-\Delta_1\varphi_j = \lambda_j\varphi_j$, and let $N(T)$ be the counting function*

$$N(T) = \#\left\{j : \lambda_j \le T\right\},$$

then

$$N(T) \sim \frac{\mathrm{vol}(\Gamma\backslash G/K)}{(4\pi)^{5/2}\Gamma\left(\frac{7}{2}\right)} T^{5/2}.$$

A simple consequence of the $SL(3,\mathbb{R})$ Kuznetsov formula are some mean value estimates for Fourier-Whittaker coefficients of $SL(3,\mathbb{Z})$ automorphic forms, which will help us evaluate convergence of the spectral side of the Kuznetsov formula:

Theorem 23 (Blomer)**.** *For $\mu^\dagger$ and $T \ge 1$ fixed, the quantities*

$$\sum_{\substack{j\ge 1\\ \left\|\mu_j-\mu^\dagger\right\|\le T}} \frac{\left|\rho_{\varphi_j}(1,1)\right|^2}{C(\mu_j)},$$

$$\sum_{j\geq 1} \int_{\substack{\mathrm{Re}(\mu_1)=0 \\ \left\|\left(\mu_1-\mu_j',-2\mu_1\right)-\mu^\dagger\right\|\leq T}} \frac{\left|\eta_j((1,1);\mu_1)\right|^2}{C\left(\mu_1-\mu_j',-2\mu_1\right)} d\mu_1,$$

and

$$\int_{\substack{\mathrm{Re}(\mu)=(0,0) \\ \left\|\mu_j-\mu^\dagger\right\|\leq T}} \frac{\left|\eta((1,1);\mu)\right|^2}{C(\mu)} d\mu$$

are all bounded by

$$T^2\left(T+\left\|\mu^\dagger\right\|\right)^3.$$

This is essentially a theorem of Blomer, but one can obtain these by applying Theorem 31 with a test function $\hat{k}$ which is non-negative on the spectrum and decays rapidly away from the desired regions then applying bounds for the $J_{w,\mu}$ functions. Such test functions are constructed in [6] – see [21], and we will prove bounds of similar nature in the final section of this paper. Blomer demonstrates results of this type by applying a Kuznetsov formula of his own, the purpose of which is the same; that is, to make a sufficiently simplified version of the Kuznetsov formula on $SL(3,\mathbb{R})$. One can extend this to the $m \neq (1,1)$ Fourier coefficents by applying the second half of the Kim-Sarnak result, $\frac{\rho_\phi(m)}{\rho_\phi(1,1)} \ll (m_1 m_2)^{\frac{5}{14}+\epsilon}$, or by applying the Kuznetsov formula directly, which results in a slightly different bound.

2.3.3 Bruhat decomposition, Plücker coordinates & Kloosterman sums

The c matrix takes the form

$$c = \begin{pmatrix} \frac{1}{c_2} & & \\ & \frac{c_2}{c_1} & \\ & & c_1 \end{pmatrix}, c_1, c_2 \in \mathbb{N},$$

the Weyl group has 6 elements

$$I = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad w_2 = \begin{pmatrix} & 1 & \\ -1 & & \\ & & 1 \end{pmatrix}, \quad w_3 = \begin{pmatrix} 1 & & \\ & & -1 \\ & 1 & \end{pmatrix},$$

$$w_4 = \begin{pmatrix} & 1 & \\ & & 1 \\ 1 & & \end{pmatrix}, \quad w_5 = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}, \quad w_l = \begin{pmatrix} & & 1 \\ & -1 & \\ 1 & & \end{pmatrix},$$

and V has four

$$I, \quad \begin{pmatrix} -1 & & \\ & -1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} -1 & & \\ & 1 & \\ & & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & -1 & \\ & & -1 \end{pmatrix}.$$

We define characters of $U(\mathbb{R})$ by

$$\psi_m \begin{pmatrix} 1 & x_2 & x_3 \\ 0 & 1 & x_1 \\ 0 & 0 & 1 \end{pmatrix} = e\left(m_1 x_1 + m_2 x_2\right),$$

and before we define the Kloosterman sums, we return to the discussion of Plücker coordinates. Coset representatives

$$\begin{pmatrix} * & * & * \\ d & e & f \\ a & b & c \end{pmatrix} \in U(\mathbb{Z}) \backslash \Gamma$$

are characterized by six invariants: The bottom row $A_1 = a, B_1 = b, C_1 = c$ having $(A_1, B_1, C_1) = 1$, and the first set of minors $A_2 = bd - ae$, $B_2 = af - cd$, $C_2 = ce - bf$ having $(A_2, B_2, C_2) = 1$ and subject to $A_1 C_2 + B_1 B_2 + C_1 A_2 = 0$. In [3], the Bruhat decomposition for each element of the Weyl group were computed using these invariants – note that membership in a particular Bruhat cell imposes certain requirements on the Plücker

	Bruhat	Compatibility	$S_w(\psi_m, \psi_n, (c_1, c_2))$
I	$c_1 = c_2 = 1$	$m = n$	1
w_2	$c_1 = 1$	$m_1 = n_1 = 0$	$S(-m_2, -n_2, c_2)$
w_3	$c_2 = 1$	$m_2 = n_2 = 0$	$S(m_1, n_1, c_1)$

Table 2.2: Degenerate $SL(3,\mathbb{Z})$ Kloosterman Sums

coordinates, giving explicit forms to the $SL(3, Z)$ Kloosterman sums. The three degenerate sums are listed in Table 2.2.

The w_4 Kloosterman sum is a new exponential sum. Its Bruhat condition is $c_2|c_1$ and the compatibility condition is $m_2 c_1 = n_1 c_2^2$. Explicitly,

$$S_{w_4}(\psi_m, \psi_n, (A_1, B_2)) = \sum_{\substack{C_2 \pmod{B_2} \\ C_1 \pmod{A_1} \\ (A_1/B_2, C_1)=(B_2,C_2)=1}} e\left(-m_2 \frac{\overline{C_2} C_1}{B_2} - m_1 \frac{\overline{C_1} B_2}{A_1} - n_2 \frac{C_2}{B_2}\right).$$

The w_5 Kloosterman sum is essentially the same as for w_4. Its Bruhat condition is $c_1|c_2$ and the compatibility condition is $m_1 c_2 = n_2 c_1^2$, and we have

$$S_{w_5}(\psi_m, \psi_n, (c_1, c_2)) = S_{w_4}(\psi_{-m_2, m_1}, \psi_{n_2, -n_1}, (c_2, c_1)).$$

The second new exponential sum is the long-element Kloosterman sum. Its Bruhat and compatibility conditions are vacuously true, and it is given by

$$S_{w_l}(\psi_m, \psi_n, (A_1, A_2)) = \sum^*_{\substack{B_1, C_1 \pmod{A_1} \\ B_2, C_2 \pmod{A_2}}} e\left(m_2 \frac{Z_2 B_1 - Y_2 A_1}{A_2} + m_1 \frac{Y_1 A_2 - Z_1 B_2}{A_1} + n_2 \frac{B_1}{A_1} + n_1 \frac{-B_2}{A_2}\right),$$

here the sum $\sum^*$ is restricted to those quadruples of B_1, C_1, B_2, C_2 satisfying

$$(A_1, B_1, C_1) = (A_2, B_2, C_2) = 1, \quad A_1 C_2 + B_1 B_2 + C_1 A_2 \equiv 0 \pmod{A_1 A_2},$$

and the numbers Y_1, Z_1, Y_2, Z_2 are defined by

$$Y_1 B_1 + Z_1 C_1 \equiv 1 \pmod{A_1}, \qquad Y_2 B_2 + Z_2 C_2 \equiv 1 \pmod{A_2}.$$

There are Weil-quality bounds for these Kloosterman sums due to Larsen (in the BFG paper) and Stevens (after some work):

Theorem 24 (Larsen)**.**

a) $|S_{w_4}(\psi_m, \psi_n, c)| \le \min\left\{d(c_2)^{\varkappa}\left(|m_1|, \frac{c_1}{c_2}\right)c_2^2, d(c_1)(|m_1|, |n_2|, c_2)c_1\right\}$,

b) $|S_{w_5}(\psi_m, \psi_n, c)| \le \min\left\{d(c_1)^{\varkappa}\left(|m_2|, \frac{c_2}{c_1}\right)c_1^2, d(c_2)(|m_2|, |n_1|, c_1)c_2\right\}$,

where $\varkappa = \frac{\log 3}{\log 2}$.

As Stevens is completely unconcerned with the dependence of his estimate on the indices m and n, we take a moment to go through his proof and keep closer track of this:

Proof of Theorem 4. In the proof of Theorem (5.9), on page 49, use instead the estimates

$$\begin{aligned}\left(\left|\nu_1 p^{s-a}\right|_p^{-1}, \left|\nu_2' p^{r-b}\right|_p^{-1}, p^r\right) &\le \left(\left|\nu_1\nu_2'\right|_p^{-1}, p^r\right)\left(p^{s-a}, p^{r-b}\right) \\ &\le \left(\left|\nu_1\nu_2'\right|_p^{-1}, p^r\right) p^{\frac{s-a+r-b}{2}},\end{aligned}$$

and similarly

$$\left(\left|\nu_2 p^{2r-s-b}\right|_p^{-1}, \left|\nu_1' p^{r-a}\right|_p^{-1}, p^r\right) \le \left(\left|\nu_2\nu_1'\right|_p^{-1}, p^r\right) p^{\frac{2r-s-b+r-a}{2}},$$

so that (5.13) becomes

$$S_{w_0}\left(\theta_{a,b}^{\lambda}; r\right) \le 4\left(\left|\nu_1\nu_2'\right|_p^{-1}, p^r\right)^{1/2}\left(\left|\nu_2\nu_1'\right|_p^{-1}, p^r\right)^{1/2} p^{2r-\frac{a+b}{2}}.$$

Since this section of the proof assumes $r \ge s$, we now replace his Theorem (5.9) with the statement

$$|S_{a,b}(n, \psi, \psi')| \le \left(\left|\nu_1\nu_2'\right|_p^{-1}, p^l\right)^{1/2}\left(\left|\nu_2\nu_1'\right|_p^{-1}, p^l\right)^{1/2} p^{\sigma+\frac{a+b}{2}},$$

where $l = \max\{r, s\}$.

Then the C in the statement and proof of his Theorem (5.1) becomes

$$C = \left(|\nu_1 \nu_2'|_p^{-1}, p^l \right)^{1/2} \left(|\nu_2 \nu_1'|_p^{-1}, p^l \right)^{1/2},$$

where again $l = \max\{r, s\}$. Multiplicativity then gives Theorem 4 as stated. $\square$

To apply the Kuznetsov formula, we will also need to know when the various sums and integrals converge. From the Weil-quality bounds on the Kloosterman sums, we may investigate the absolute convergence of the corresponding Kloosterman zeta functions:

Proposition 25.

$$\text{a)} \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_4}(\psi_m, \psi_n, c))|}{c_1 c_2} \left(\frac{c_2^3 |n_1|}{|m_1 m_2^2 n_2|} \right)^u$$

$$\ll \frac{(|n_1|, |m_2|)^{1-3u-2\epsilon} \left(|m_1|, |n_2|, \frac{|m_2|}{(|n_1|,|m_2|)} \right)}{|m_1 n_2|^{u-\epsilon} |m_2|^{1-u-\epsilon} |n_1|^{-u-\epsilon}} \textit{ for } u < 0,$$

$$\text{b)} \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_5}(\psi_m, \psi_n, c))|}{c_1 c_2} \left(\frac{c_1^3 |n_2|}{|m_1^2 m_2 n_1|} \right)^u$$

$$\ll \frac{(|n_2|, |m_1|)^{1-3u-2\epsilon} \left(|m_2|, |n_1|, \frac{|m_1|}{(|n_2|,|m_1|)} \right)}{|m_2 n_1|^{u-\epsilon} |m_1|^{1-u-\epsilon} |n_2|^{-u-\epsilon}} \textit{ for } u < 0,$$

$$\text{c)} \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_l}(\psi_m, \psi_n, c)|}{c_1 c_2} \left(\frac{c_1^2}{c_2 |m_1 n_2|} \right)^{u_1} \left(\frac{c_2^2}{c_1 |m_2 n_1|} \right)^{u_2}$$

$$\ll |m_1 n_2|^{-u_1+\epsilon} |m_2 n_1|^{-u_2+\epsilon} \textit{ for } 2u_1 - u_2, -u_1 + 2u_2 < -\frac{1}{2}.$$

(These normalizations are unusual, but will make more sense once we start evaluating the integral transforms.) The proof of this proposition is given in appendix A.

2.3.4 Whittaker functions

As before, we define the Jacquet-Whittaker function as

$$W(z;\mu,\psi_m) = \int_{U(\mathbb{R})} p_{\rho+\mu}(w_l uz)\psi_m(u)du.$$

(Note that the location of the -1 in the w_l matrix does not affect $p_{\rho+\mu}(w_l uz)$.) Its completion is given by $W^*(z;\mu,\psi_m) = \Lambda(\mu)W(z;\mu,\psi_m)$, where

$$\Lambda(\mu) = \pi^{-\frac{3}{2}+\mu_3-\mu_1}\Gamma\left(\frac{1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{1+\mu_1-\mu_3}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right).$$

We will need a number of basic facts about the Whittaker function, so we collect them here: The Whittaker function is harmonic, square-integrable, cuspidal exactly when ψ_m is non-degenerate, shares the eigenvalues of $p_{\rho+\mu}$ (but isn't automorphic), and

$$W(z;\mu,\psi_m) = \psi_m(x)W(y;\mu,\psi_m), \qquad x \in u(\mathbb{R}).$$

It is also easy to check that if $0 \neq t_1, t_2 \in \mathbb{R}$,

$$\begin{aligned} W(y;\mu,\psi_{t_1 t_2}) &= p_{-2\rho}(t)p_{\rho+\mu}(w_l t^{-1} v w_l^{-1})W(tvy;\mu,\psi_{11}) \\ &= p_{-\rho-\mu^{w_l}}(t)W(ty;\mu,\psi_{11}), \end{aligned}$$

by sending $u \mapsto t^{-1}vutv$, where

$$t = \begin{pmatrix} |t_1|\,|t_2| & & \\ & |t_1| & \\ & & 1 \end{pmatrix}, \qquad v = \begin{pmatrix} \mathrm{sign}(t_1) & & \\ & \mathrm{sign}(t_1 t_2) & \\ & & \mathrm{sign}(t_2) \end{pmatrix},$$

and $\mu^{w_l} = (\mu_3, \mu_2, \mu_1)$. This means that we need only analyze the behavior of the function $W(y,\mu,\psi_{11})$.

We have the double Mellin transform pair [11, p155 6.1.4 and 6.1.5]

$$W^*(y,\mu,\psi_{11}) = -\frac{1}{16\pi^4}\int_{\mathrm{Re}(t)=(2,2)} G(t,\mu)(\pi y_1)^{1-t_1}(\pi y_2)^{1-t_2}dt,$$
$$G(t,\mu) = \frac{4}{\pi^2}\int_{Y(\mathbb{R})} W^*(y,\mu,\psi_{11})(\pi y_1)^{1+t_1}(\pi y_2)^{1+t_2}dy,$$

where

$$G(t,\mu) = \frac{\Gamma\left(\frac{t_1-\mu_1}{2}\right)\Gamma\left(\frac{t_1-\mu_2}{2}\right)\Gamma\left(\frac{t_1-\mu_3}{2}\right)\Gamma\left(\frac{t_2+\mu_1}{2}\right)\Gamma\left(\frac{t_2+\mu_2}{2}\right)\Gamma\left(\frac{t_2+\mu_3}{2}\right)}{\Gamma\left(\frac{t_1+t_2}{2}\right)}.$$

The inverse Mellin form of the Whittaker function also gives the following asymptotics: By shifting the t_1 integral to the right (as we may by the exponential decay of $G(t,s)$), we see that the Whittaker function decays faster than any power of y_1 as $y_1\to\infty$; by shifting to the left past the first pole of the G function, we see that the Whittaker function is bounded by $y_1^{1-c_1}$ where $c_1 = \max\{\mathrm{Re}(-\mu_1),\mathrm{Re}(-\mu_2),\mathrm{Re}(-\mu_3)\}$ as $y_1\to 0$. The same reasoning applies for y_2 with $c_2 = \max\{\mathrm{Re}(\mu_1),\mathrm{Re}(\mu_2),\mathrm{Re}(\mu_3)\}$. In particular, the Mellin transform of the Whittaker function converges absolutely for $\mathrm{Re}(t_1) > c_1$,$\mathrm{Re}(t_2) > c_2$.

Aside: Given the number of different parameterizations of the Whittaker function, the author has found it useful to verify the Λ function as follows:

$$\begin{aligned}
&\lim_{y\to 0} p_{-\rho-\mu^{w_l}}(y)W(y;\mu,\psi_{11})\\
=&\lim_{y\to 0}\int_{U(\mathbb{R})} p_{\rho+\mu}(w_l u)\psi_y(u)du\\
=&\int_{U(\mathbb{R})}\left(1+x_1^2+(x_1x_2-x_3)^2\right)^{\frac{-1+\mu_2-\mu_1}{2}}\left(1+x_2^2+x_3^2\right)^{\frac{-1+\mu_3-\mu_2}{2}}du\\
=&\int_{U(\mathbb{R})}\left(1+x_1^2\right)^{\frac{-1+\mu_2-\mu_1}{2}}\left(1+x_2^2\right)^{\frac{-1+\mu_3-\mu_2}{2}}\left(1+x_3^2\right)^{\frac{-1+\mu_3-\mu_1}{2}}du\\
=&\pi^{3/2}\frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_2-\mu_3}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right)}{\Gamma\left(\frac{1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)\Gamma\left(1+\frac{\mu_1-\mu_3}{2}\right)}
\end{aligned}$$

for $\mathrm{Re}(\mu_3) < \mathrm{Re}(\mu_2) < \mathrm{Re}(\mu_1)$ using the substitutions $x_1' \mapsto \frac{x_1'}{\sqrt{1+x_2'^2}}$, $x_3' \mapsto x_3'\sqrt{1+x_2'^2}$ and then $x_1' \mapsto x_2'x_3' + x_1'\sqrt{1+x_3'^2}$. We now compute $\lim_{y\to 0} p_{-\rho-\mu^{w_l}}(y)W^*(y;\mu,\psi_{11})$ from the inverse Mellin transform representation using the G function: Shifting the t contours to the left, we pick up the first poles at $t_1 = \mu_1$ and $t_2 = -\mu_3$ and computing the residue there gives

$$\lim_{y\to 0}(\pi y_1)^{-1+\mu_1}(\pi y_2)^{-1-\mu_3} W^*(y;\mu,\psi_{11}) =$$
$$\frac{4(2\pi i)^2}{-16\pi^4}\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_2-\mu_3}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right),$$

so $\Lambda(\mu)$ is the ratio of W^* over W.

A related product which will occur frequently is

$$\frac{1}{\Lambda(\mu)\Lambda(-\mu)} = \cos\frac{\pi}{2}(\mu_1-\mu_2)\cos\frac{\pi}{2}(\mu_1-\mu_2)\cos\frac{\pi}{2}(\mu_2-\mu_3).$$

2.3.5 Fourier coefficients

Again, a Maass cusp form has Fourier coefficients of the form

$$\int_{U(\mathbb{Z})\backslash U(\mathbb{R})} \varphi(uy)\psi_m(u)du = \frac{\rho_\varphi(m)}{|m_1 m_2|} W^*(|m|\, y;\mu,\psi_{11}),$$

where $\rho_\varphi(m)$ is some constant depending on φ.

It is interesting to write out the Fourier-Whittaker expansion of the minimal parabolic Eisenstein series: Let $\eta(y,m,\mu) = \int_{U(\mathbb{Z})\backslash U(\mathbb{R})} E(xy;\mu), \psi_m(x)dx$, then [11, pp303-306],

$$\eta(y,m,\mu) = \sum_{w\in W}\sum_{v\in V} \delta_{m,w} W_w(y,\mu,\psi_m^v)\zeta_w(\psi_{00},\psi_m^v,\mu),$$

where $\delta_{m,w} = 1$ if ψ_m is trivial on $U_w(\mathbb{R})$ and zero otherwise,

$$W_w(z,\mu,\psi_m) = \int_{\overline{U}_w(\mathbb{R})} p_{\rho+\mu}(wuz)\overline{\psi_m(u)}du$$

is the Whittaker function associated to the w Bruhat cell (note $W_{w_l} = W$; the other cases are termed "degenerate" Whittaker functions), and

$$\zeta_w(\psi_n, \psi_m, \mu) = \sum_{c \in \mathbb{N}^2} p_{\rho+\mu}(c) S_w(\psi_n, \psi_m, c)$$

is the $SL(3, \mathbb{R})$ Kloosterman zeta function for the w Bruhat cell. This is computed by applying the Bruhat decomposition in a manner similar to that of the Kuznetsov formula that we give in the next section.

If ψ_m is non-degenerate, then it is non-trivial on $U_w(\mathbb{R})$ for all w except the long element so $\eta(y, m, \mu) = W(y, \mu, \psi_m)\zeta_{w_l}(\psi_{00}, \psi_m, \mu)$, and we normalize the Fourier-Whittaker coefficients as $\eta(m, \mu) = |m_1 m_2| \frac{\zeta_{w_l}(\psi_{00}, \psi_m, \mu)}{\Lambda(\mu)}$ so that $\eta(z, m, \mu) = \frac{\eta(m,\mu)}{|m_1 m_2|} W^*(z, \mu, \psi_m)$.

The Fourier-Whittaker coefficients of Eisenstein series twisted by cusp forms are more difficult, but they may be computed by considering the Hecke operators. We choose not to do so here.

2.3.6 The Kuznetsov formula

Bringing the pieces together, we have Li's Kuznetsov formula for $SL(3, \mathbb{R})$, which we will discuss in detail section 5:

Theorem 26 (Li). *Let $\{\varphi\}$ be an orthonormal basis of the $SL(3, \mathbb{R})$ cusp forms with Langlands parameters μ_φ, and $\{\phi\}$ an orthonormal basis of $SL(2, \mathbb{R})$ cusp forms with Langlands* *parameters* μ_ϕ. *Let* $k \in C_c^\infty(K\backslash G/K)$, *and* m, n *pairs of non-zero integers. Then*

$$\begin{aligned}
&\sum_{\varphi} \frac{\hat{k}(\mu_\varphi)}{C(\mu_\varphi)} \rho_\varphi(n)\overline{\rho_\varphi}(m) \qquad (2.1)\\
&\quad + \frac{1}{4\pi i}\sum_{\phi}\int_{\mathrm{Re}(\mu_1)=0} \frac{\hat{k}\left(\mu_1-\mu_\phi, -2\mu_1\right)}{C\left(\mu_1-\mu_\phi,-2\mu_1\right)}\eta_\phi(n;\mu_1)\overline{\eta_\phi}(m;\mu_1)d\mu_1\\
&\quad + \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\mu)=(0,0)} \frac{\hat{k}(\mu)}{C(\mu)}\eta(n;\mu)\overline{\eta}(m;\mu)d\mu\\
&= \delta_{mn} H_I(k,\psi_m,\psi_n,c)\\
&\quad + \sum_{w\in\{w_4,w_5,w_l\}}\sum_{v\in V}\sum_{c_1,c_2\in\mathbb{N}} S_w(\psi_m,\psi_n^v,c)H_w(k,\psi_m,\psi_n^v,c),
\end{aligned}$$

where

$$C(\mu) = \cos\frac{\pi}{2}(\mu_1-\mu_2)\cos\frac{\pi}{2}(\mu_1-\mu_3)\cos\frac{\pi}{2}(\mu_2-\mu_3),$$

ρ_φ, η_ϕ, *and* η *are the Fourier-Whittaker coefficients of* φ, E_ϕ, *and* E, *respectively, and* H_w *is given by the integral*

$$\frac{2\left|m_1m_2n_1n_2\right|}{\pi}\int_{Y(\mathbb{R})}\int_{U(\mathbb{R})}\int_{\overline{U}_w(\mathbb{R})} k\left(|m|\,(xt)^{-1}\,cw(x't)\,|n|^{-1}\right)\psi_m(x)\overline{\psi_n(x')}dx'\,dx\,t_1^2t_2\,dt. \quad (2.2)$$

Note that $C(\mu)$ is real, since $\overline{\mu}$ is some permutation of $-\mu$ and $C(\mu)$ is invariant under both operations.

2.3.7 Spherical Inversion

The spherical inversion formula for $SL(3,\mathbb{R})$ has the form:

Theorem 27 (Spherical Inversion)**.** *For* $k \in \mathrm{HCS}(K\backslash G/K)$, *the Selberg transform* $k \mapsto \hat{k}$ *has the inversion*

$$k(z) = -\frac{1}{48\pi^4}\int_{\mathrm{Re}(\mu)=(0,0)} \frac{\hat{k}(\mu)}{\left|c_3(\mu)\right|^2}h_\mu(z)d\mu,$$

where

$$\left|c_3(\mu)\right|^{-2} = \prod_{1\le i<j\le 3}\frac{\pi}{2}(\mu_i-\mu_j)\tan\frac{\pi}{2}(\mu_i-\mu_j).$$

$\hat{k}$ *may be taken to be any Schwartz-class function on* $\mathrm{Re}(\mu) = 0$ *which is invariant under permutations of the coordinates of* μ.

CHAPTER 3

The Method on $SL(2,\mathbb{R})$

Before diving into the process on $SL(3,\mathbb{R})$, we demonstrate the method in the more familiar setting of $SL(2,\mathbb{R})$. This appears to be a novel approach – though there are similarities with [10] – to proving the following theorem, originally due to Kuznetsov:

Theorem 28 (Kuznetsov)**.** *Suppose $g:\mathbb{R}^+\to\mathbb{C}$ is smooth and compactly supported, and $T>0$, then*

$$\sum_{c=1}^{\infty}\frac{S(m,n,c)}{c}g\left(T\frac{\pi^2\,|mn|}{c^2}\right)\ll_{m,n,g,\epsilon} T^{\epsilon}.$$

Starting with Zagier's form of the $SL(2,\mathbb{Z})$ Kuznetsov formula, before the simplification of the weight functions on the arithmetic side:

Theorem 29 (Kuznetsov Trace Formula, Spectral Form)**.** *Suppose $k:\mathbb{H}\to\mathbb{C}$ is sufficiently nice with $k(gz)=k(z)$ for all $g\in SO(2,\mathbb{R})$, then*

$$\sum_{\phi}\frac{\hat{k}(\mu_\phi)}{\cos\pi\mu_\phi}\rho_\phi(n)\overline{\rho_\phi}(m)+\frac{1}{4\pi i}\int_{\mathrm{Re}(\mu)=0}\frac{\hat{k}(\mu)}{\cos\pi\mu}\eta(n,\mu)\overline{\eta}(m,\mu)d\mu$$
$$=\delta_{mn}H_I(k)+\sum_{c=1}^{\infty}S(m,n,c)H_{w_l}(k,m,n,c),$$

where

$$\hat{k}(\mu)=\int_{\mathbb{H}}k(z)y^{\frac{1}{2}-\mu}\frac{dx\,dy}{y^2},$$

$$H_I=4\,|mn|\int_{\mathbb{R}^+}\int_{\mathbb{R}}k\left(-\frac{|m|}{t}x+i\right)e\,(mx)\;dx\,t\,\frac{dt}{t^2},$$

and

$$H_{w_l} = 4\,|mn| \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} k\left(-\frac{|m|}{t}x + \frac{|mn|}{c^2}\frac{-\frac{|n|}{t}x' + i}{t^2 + n^2x'^2}\right) e\,(mx)\, e\,(-nx')\, dx'\, dx\, t\, \frac{dt}{t^2}.$$

Note that the action of V is trivial.

We want to express the H_I and H_{w_l} functions as integrals of $\hat{k}$ against a function in Mellin-Barnes integral form. In both integrals, we send $x \mapsto -\frac{t}{|m|}x$ and in the second, $x' \mapsto \frac{t}{n}x'$:

$$H_I = 4\,|n| \int_{\mathbb{R}^+} \int_{\mathbb{R}} k\,(x+i)\, e\,(-v_1 tx)\; dx\, t^2\, \frac{dt}{t^2},$$

$$H_{w_l} = 4 \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} k\left(x + \frac{-v_2\alpha x' + \alpha i}{t^2(1+x'^2)}\right) e\,(-v_1 tx)\, e\,(-tx')\, dx'\, dx\, t^3\, \frac{dt}{t^2}.$$

where $v_1 = \mathrm{sign}(m)$, $v_2 = \mathrm{sign}(n)$ and $\alpha = \frac{|mn|}{c^2}$.

For H_{w_l}, interchange the x and x' integrals, and send $x \mapsto x + \frac{v_2\alpha x'}{t^2(1+x'^2)}$, so

$$\begin{aligned} H_{w_l} =& 4 \int_{\mathbb{R}^+} \int_{\mathbb{R}} \int_{\mathbb{R}} k\left(x + \frac{\alpha}{t^2(1+x'^2)}i\right) \\ & e\,(-v_1 tx)\, e\left(-v_1 v_2 \frac{\alpha}{t}\frac{x'}{1+x'^2}\right) e\,(-tx')\, dx\, dx'\, t^3\, \frac{dt}{t^2}. \end{aligned}$$

Now we apply Selberg inversion (Theorem 10) and interchange the x and μ integrals; we see both H_I and H_{w_l} involve an integral of the form

$$X = \int_{\mathbb{R}} h_\mu\,(x+yi)\, e\,(ax)\; dx,$$

so we take the integral formula for the spherical function

$$h_\mu\,(z) = \int_0^\pi p_{\frac{1}{2}+\mu}\left(\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} z\right) \frac{d\theta}{\pi},$$

and for reasons which will become clear momentarily, make the substitution $u = -\cot\theta$:

$$\cos\theta = \frac{u}{\sqrt{1+u^2}}, \quad \sin\theta = \frac{1}{\sqrt{1+u^2}}, \quad d\theta = \frac{du}{u^2+1},$$

$$h_\mu(x+yi) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{y}{y^2+(u-x)^2}\right)^{\frac{1}{2}+\mu}\left(1+u^2\right)^{-\frac{1}{2}+\mu} du.$$

Though the combined u and x integral does not converge absolutely, we can justify the interchange of integrals through integration by parts in x, so we send $x \mapsto x+u$ and the integrals separate leaving us (formally) with

$$X = \frac{1}{\pi}\int_{\mathbb{R}}\left(\frac{y}{y^2+x^2}\right)^{\frac{1}{2}+\mu} e(ax)\; dx \int_{-\infty}^{\infty}\left(1+u^2\right)^{-\frac{1}{2}+\mu} e(au)\, du.$$

These integrals are both the Whittaker function

$$\int_{\mathbb{R}}\left(\frac{y}{y^2+x^2}\right)^{\frac{1}{2}+\mu} e(ax)\; dx = 2\pi^{\frac{1}{2}+\mu}|a|^{\mu} y^{\frac{1}{2}}\frac{K_{-\mu}(2\pi|a|y)}{\Gamma\left(\frac{1}{2}+\mu\right)},$$

so we have

$$X = 4y^{1/2}\frac{K_{-\mu}(2\pi y|a|)K_{\mu}(2\pi|a|)}{\Gamma\left(\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}-\mu\right)}.$$

Applying this gives

$$H_I = \frac{4|n|}{\pi^2 i}\int_{\mathbb{R}^+}\int_{\mathrm{Re}(\mu)=0}\hat{k}(\mu)\frac{K_{-\mu}(2\pi t)K_{\mu}(2\pi t)}{\Gamma\left(\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}-\mu\right)}\pi\mu\tan\pi\mu\, d\mu\, dt,$$

$$\begin{aligned} H_{w_l} =& \frac{4\alpha^{1/2}}{\pi^2 i}\int_{\mathbb{R}^+}\int_{\mathbb{R}}\int_{\mathrm{Re}(\mu)=0}\hat{k}(\mu)K_{-\mu}\left(2\pi\frac{\alpha}{t(1+x'^2)}\right)K_{\mu}(2\pi t)\,\mu\sin\pi\mu\, d\mu \\ & e\left(-v_1v_2\frac{\alpha}{t}\frac{x'}{1+x'^2}\right)e(-tx')\,\frac{dx'}{(1+x'^2)^{1/2}}\, dt. \end{aligned}$$

Where we have applied $\Gamma\left(\frac{1}{2}+\mu\right)\Gamma\left(\frac{1}{2}-\mu\right) = \pi\sec\pi\mu$ to the long-element term.

We may now finish our computation of H_I using the integral formula

$$\int_{\mathbb{R}^+} K_{-\mu}(2\pi t) K_\mu(2\pi t)\, dt = \frac{\pi}{8\cos\pi\mu},$$

which is Stade's formula at $s = 1$, giving

$$H_I = \frac{|mn|^{1/2}}{2\pi^2 i} \int_{\mathrm{Re}(\mu)=0} \tilde{k}(\mu)\pi\mu \tan\pi\mu\, d\mu.$$

For the long-element term, we apply

$$\begin{aligned} K_\mu(2t) =& \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=1} \Gamma\left(\frac{u+\mu}{2}\right)\Gamma\left(\frac{u-\mu}{2}\right) t^{-u}\, du, \qquad \mathfrak{u} > |Re(\mu)|, \\ =& \Gamma(\mu) t^\mu + \Gamma(-\mu) t^{-\mu} \\ &+ \frac{1}{2\pi i} \int_{\mathrm{Re}(u)=-\frac{1}{2}-4\epsilon} \Gamma\left(\frac{u+\mu}{2}\right)\Gamma\left(\frac{u-\mu}{2}\right) t^{-u}\, du, \end{aligned}$$

for $|\mathrm{Re}(\mu)| < \frac{1}{2} + 4\epsilon$, $\epsilon > 0$, and using the symmetry $\mu \mapsto -\mu$ of the remaining terms in the integrand, we have

$$\begin{aligned} H_{w_l} =& \frac{1}{2\pi i} \int_{\mathrm{Re}(\mu)=-\frac{1}{4}-3\epsilon} \tilde{k}(\mu) J_{w_l,\mu}(\alpha, v) \tan\pi\mu\, d\mu, \\ J_{w_l,\mu}(\alpha, v) =& \Gamma(-\mu) X'(\alpha, v, \mu)\mu\cos\pi\mu \\ &+ \frac{\mu\cos\pi\mu}{2\pi i} \int_{\mathrm{Re}(u)=-\frac{1}{4}-4\epsilon} \Gamma\left(\frac{u+\mu}{2}\right)\Gamma\left(\frac{u-\mu}{2}\right) X'(\alpha, v, u)\, du, \\ X'(\alpha, v, \mu, u) =& \frac{16\alpha^{\frac{1}{2}-u}}{\pi^{u-1}} \int_{\mathbb{R}^+}\int_{\mathbb{R}} e\left(-v_1 v_2 \frac{\alpha}{t}\frac{x'}{1+x'^2}\right) e(-tx') \\ & (1+x'^2)^{u-\frac{1}{2}}\, dx'\, K_\mu(2\pi t)\, t^u\, dt, \end{aligned}$$

which converges absolutely. One could use a similar idea to [10] and replace

$$e\left(-v_1 v_2 \frac{\alpha}{t}\frac{x'}{1+x'^2}\right) = 1 + \left(e\left(-v_1 v_2 \frac{\alpha}{t}\frac{x'}{1+x'^2}\right) - 1\right),$$

with the main term coming from the 1, and the second term being an error term, but we will instead explicitly compute a Mellin-Barnes expansion of X' as this will generalize more effectively.

We will eventually want a first-term asymptotic for X' as $\alpha \to 0$:

$$\begin{aligned}\lim_{\alpha\to 0} \alpha^{-\frac{1}{2}+\mu} X'(\alpha, v, \mu, \mu) =& \frac{16}{\pi^{\mu-1}} \int_{\mathbb{R}^+} \int_{\mathbb{R}} e\left(-tx'\right) (1+x'^2)^{\mu-\frac{1}{2}}\, dx'\, K_\mu\left(2\pi t\right) t^\mu\, dt,\\ =& \frac{32}{\pi^{2\mu-\frac{3}{2}}\Gamma\left(\frac{1}{2}-\mu\right)} \int_{\mathbb{R}^+} K_\mu\left(2\pi t\right) K_\mu\left(2\pi t\right)\, dt\\ =& \frac{4}{\pi^{2\mu-\frac{1}{2}}\Gamma\left(\frac{1}{2}-u\right)\cos\pi\mu},\end{aligned}$$

by dominated convergence and Stade's formula. Note that our main term also derives from replacing that particular complex exponential with 1.

Now we inverse-Mellin expand the exponentials in X' by taking the limit as $\theta \to \frac{\pi}{2}$ (outside H_{w_l} by dominated convergence) of

$$e\left(x \exp\left(i\left(\frac{\pi}{2}-\theta\right)\operatorname{sign}(x)\right)\right) = \frac{1}{2\pi i}\int_{\operatorname{Re}(u)=\mathfrak{u}} \left|2\pi x\right|^{-u} e^{iu\theta \operatorname{sign}(x)} \Gamma\left(u\right)\, du,$$

for $x \neq 0$, $0 < \theta < \frac{\pi}{2}$ and $\mathfrak{u} > 0$. Now we have $X'(\alpha, v, u) = \lim_{\theta\to\frac{\pi}{2}} X'_\epsilon\left(\alpha, v, u, \theta\right)$, where

$$\begin{aligned}&X'_\delta(\alpha, v, \mu, u, \theta) =\\ &\frac{8\alpha^{\frac{1}{2}-u}}{\pi^{1+u}(2\pi i)^2}\int_{\operatorname{Re}(s)=(\delta,\epsilon)} (2\pi)^{s_1+s_2}\alpha^{-s_1}\Gamma\left(s_1\right)\Gamma\left(s_2\right)\\ &\int_{\mathbb{R}^+} K_\mu\left(2\pi t\right) t^{u+s_1-s_2}\, dt\\ &\int_{\mathbb{R}} \left|x'\right|^{-s_1-s_2}\left(1+x'^2\right)^{s_1+u-\frac{1}{2}} \exp\left(-i\theta \operatorname{sign}(x')(v_1v_2s_1+s_2)\right)\, dx'\, ds.\end{aligned}$$

We split the x' integral by sign and apply the Mellin transform

$$\int_0^\infty (1+x^2)^u x^t dx = \frac{1}{2}B\left(\frac{t+1}{2}, \frac{-2u-t-1}{2}\right),$$

for $-1 < \mathrm{Re}(t) < -1 - 2\mathrm{Re}(u)$, where $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ is the beta function, so

$$\begin{aligned} X'_\delta(\alpha, v, \mu, u, \theta) =& \frac{2\alpha^{\frac{1}{2}-u}}{\pi^{2+2u}(2\pi i)^2} \int_{\mathrm{Re}(s)=(\delta,\epsilon)} 2^{s_1+s_2}\pi^{2s_2}\alpha^{-s_1} \cos\left(\theta(v_1 v_2 s_1 + s_2)\right) \\ & \Gamma(s_1)\Gamma(s_2)\Gamma\left(\frac{1+u+s_1-s_2+\mu}{2}\right)\Gamma\left(\frac{1+u+s_1-s_2-\mu}{2}\right) \\ & B\left(\frac{-s_1-s_2+1}{2}, \frac{-2u-s_1+s_2}{2}\right) ds, \end{aligned}$$

which should converge absolutely at $\theta = \frac{\pi}{2}$ on $\mathrm{Re}(u) + 2\mathrm{Re}(s_1) < -\frac{1}{2}$ provided the contours miss the poles of the gamma functions, so dominated convergence gives

$$X'(\alpha, v, \mu, u) = X'_0(\alpha, v, \mu, u) + X'_{-\frac{1}{4}-\epsilon}(\alpha, v, \mu, u),$$

where $X'_\delta(\alpha, v, \mu, u) := X'_\delta\left(\alpha, v, \mu, u, \frac{\pi}{2}\right)$ and $X'_0(\alpha, v, \mu, u)$ indicates taking the residue at $s_1 = 0$. Absolute convergence of the remaining integral in X'_0, i.e. the s_2 integral, is obvious from the exponential decay factors.

Having already computed the primary asymptotic of $X'(\alpha, v, \mu, \mu)$ as $\alpha \to 0$, the secondary asymptotics, which will become error terms in our partial inversion formula, can be obtained by shifting the s_1 contour to the left as we have already done – the integral over u in $J_{w_l,\mu}$ is small compared to $\alpha^{\frac{1}{2}-\mu}$. The pole at $s_1 = 0$ gives the primary asymptotic, and the next pole is at $s_1 = -1 + s_2 - 2\mu$ which has real part $-\frac{1}{2} + 7\epsilon$:

$$\begin{aligned} J_{w_l,\mu}(\alpha, v) =& -\frac{4(\pi^2\alpha)^{\frac{1}{2}-\mu}}{\pi} \frac{\Gamma(1-\mu)}{\Gamma\left(\frac{1}{2}-\mu\right)} + E_{1,\mu}(\alpha, v) + E_{2,\mu}(\alpha, v), \\ E_{1,\mu}(\alpha, v) =& \Gamma(-\mu) X'_{-\frac{1}{4}-\epsilon}(\alpha, v, \mu, \mu) \sin \pi\mu, \\ E_{2,\mu}(\alpha, v) =& \frac{\sin \pi\mu}{2\pi i} \int_{\mathrm{Re}(u)=-\frac{1}{2}-4\epsilon} \Gamma\left(\frac{u+\mu}{2}\right)\Gamma\left(\frac{u-\mu}{2}\right) X'(\alpha, v, \mu, u)\, du. \end{aligned}$$

(The exponential growth of $\sin \pi\mu$ in $E_{2,\mu}$ may look daunting, but the gamma factors $\Gamma\left(\frac{u+\mu}{2}\right)\Gamma\left(\frac{u-\mu}{2}\right)$ and $\Gamma\left(\frac{1+u+s_1-s_2+\mu}{2}\right)\Gamma\left(\frac{1+u+s_1-s_2-\mu}{2}\right)$ have enough exponential decay to compensate.)

With the integrals evaluated and the primary asymptotics found, we take our test function

$\tilde{k}(\mu) = \tilde{k}_1(\mu)$ so that the integral in μ becomes, to a first-term approximation, a Mellin inversion:

$$\begin{aligned}\tilde{k}_\delta(\mu) =& -\frac{1}{2\pi i}\int_{\mathrm{Re}(q)=-\delta}\hat{f}(q)\frac{2q}{(q-\mu)(q+\mu)}\frac{\pi\Gamma\left(\frac{1}{2}-q\right)}{4\Gamma\left(1-q\right)\tan\pi q}dq\\ =&\hat{f}(\mu)\frac{\Gamma\left(\frac{1}{2}-\mu\right)}{4\pi^{\frac{1}{2}}\Gamma\left(-\mu\right)\mu\tan\pi\mu}+\tilde{k}_0(\mu),\end{aligned}$$

where

$$\hat{f}(q) = \int_0^\infty f(t)t^{q-1}dt$$

is the Mellin transform of some sufficiently nice function $f : \mathbb{R}^+ \to \mathbb{C}$. For such a test function, we have

$$\begin{aligned}H_{w_l} =& f(\pi^2\alpha)\sqrt{\pi^2\alpha}+\sum_{j=1}^{3}F_j(\alpha,v),\\ F_1(\alpha,v) =& \frac{1}{2\pi i}\int_{\mathrm{Re}(\mu)=-\frac{1}{2}-3\epsilon}\tilde{k}_0(\mu)J_{w_l,\mu}(\alpha,v)\tan\pi\mu\,d\mu,\\ F_2(\alpha,v) =& \frac{1}{2\pi i}\int_{\mathrm{Re}(\mu)=0}\hat{f}(\mu)\frac{\pi\Gamma\left(\frac{1}{2}-\mu\right)}{4\Gamma\left(1-\mu\right)}E_{1,\mu}(\alpha,v)\,d\mu,\\ F_3(\alpha,v) =& \frac{1}{2\pi i}\int_{\mathrm{Re}(\mu)=0}\hat{f}(\mu)\frac{\pi\Gamma\left(\frac{1}{2}-\mu\right)}{4\Gamma\left(1-\mu\right)}E_{2,\mu}(\alpha,v)\,d\mu,\end{aligned}$$

so the partial inversion formula becomes (after shifting $\hat{k}_1 \mapsto \hat{k}_\epsilon$ on the spectral side):

Theorem 30 (Partial Kuznetsov Inversion Formula on $SL(2,\mathbb{R})$)**.** *Suppose $f : \mathbb{R}^+ \to \mathbb{C}$ is sufficiently nice (which we can quantify, but choose not to), then*

$$\begin{aligned}&\pi\sqrt{|mn|}\sum_{c=1}^{\infty}\frac{S(m,n,c)}{c}f\left(\frac{\pi^2\,|mn|}{c^2}\right)\\ &= -\delta_{mn}H_I(\tilde{k})-\sum_{j=1}^{3}\sum_{c=1}^{\infty}S(m,n,c)F_j\left(\frac{\pi^2\,|mn|}{c^2},(\,\mathrm{sign}(m),\,\mathrm{sign}(n))\right),\\ &+\sum_{\phi}\frac{\tilde{k}_\epsilon(\mu_\phi)}{\cos\pi\mu_\phi}\rho_\phi(n)\overline{\rho_\phi}(m)+\frac{1}{4\pi i}\int_{\mathrm{Re}(\mu)=0}\frac{\tilde{k}_\epsilon(\mu)}{\cos\pi\mu}\eta(n,\mu)\overline{\eta}(m,\mu)d\mu,\end{aligned}$$

using the functions constructed above.

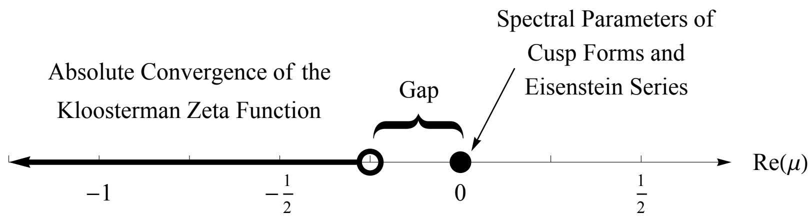

Figure 3.1: Location of the $SL(2)$ Spectral Parameters

One should compare this to the full arithmetic Kuznetsov formula on $GL(2)$, i.e. Theorem 9: First, our formula here applies regardless of the signs of m and n. Second, the full inversion formula requires the use of Petersson's trace formulas and Fourier coefficents of holomorphic modular forms; as these are zeros of the Laplacian, in some sense they lie on the line $\mathrm{Re}(\mu) = 0$ with the other automorphic forms and our result confirms that the continuous part of the inversion formula should be large compared to this discrete series. Third, our error terms limit the study of the Kloosterman zeta function to $\mathrm{Re}(\mu) < -\epsilon$; in other words, the full Kuznetsov formula can be used to give the meromorphic continuation of the Kloosterman zeta function with poles at $\mu = \mu_\phi$ on the line $\mathrm{Re}(\mu) = 0$, but our error terms prevent us from reaching this line.

Now suppose $f(t) = g(tT)$ for some smooth, compactly supported $g : \mathbb{R}^+ \to \mathbb{C}$, $T > 0$, then

$$\hat{f}(t) \ll_A \frac{T^{-\mathrm{Re}(q)}}{|q|^A}, \qquad A > 0,$$

and applying this bound to the absolutely convergent integrals and sums above gives

$$\sum_{c=1}^{\infty} \frac{S(m,n,c)}{c} g\left(T\frac{\pi^2 |mn|}{c^2}\right) \ll_{m,n,g,\epsilon} T^\epsilon.$$

We visualize the relevant parameters in figure 3.1.

CHAPTER 4

The Method on $SL(3,\mathbb{R})$

We start with Li's generalization of the Kuznetsov formula, whose complexity leads us to our first technical theorem; in chapter 5 we prove:

Theorem 31. *Let $\hat{k}(\mu)$ be symmetric in μ, holomorphic in each variable on* $\mathrm{Re}(\mu) = \eta \in \left[-\frac{1}{2}-\epsilon, \frac{1}{2}+\epsilon\right]^2$*, of sufficient decay that the integral*

$$\int_{\mathrm{Re}(\mu)=\eta} \left|\hat{k}(\mu)\right| \prod_{i<j} |\mu_i - \mu_j|^{\frac{13}{8}+100\epsilon} |d\mu|$$

converges, then we have the formula (2.1)*, where now*

$$H_I(k, \psi_m, \psi_n, (1,1)) = -\frac{1}{2^8 3} \int_{\mathrm{Re}(\mu)=(0,0)} \hat{k}(\mu) \prod_{j<k} \frac{(\mu_k - \mu_j)^2}{\sqrt{9-(\mu_j-\mu_k)^2}} d\mu,$$

$$H_{w_4}(k, \psi_m, \psi_n, c) = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\eta} \hat{k}(\mu) J_{w_4,\mu}\left(\frac{8\pi^3 m_1 m_2^2 n_2}{c_2^3 n_1}\right) d\mu,$$

$$H_{w_5}(k, \psi_m, \psi_n, c) = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\eta} \hat{k}(\mu) J_{w_5,\mu}\left(\frac{8\pi^3 m_1^2 m_2 n_1}{c_1^3 n_2}\right) d\mu,$$

$$H_{w_l}(k, \psi_m, \psi_n, c) = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\eta} \hat{k}(\mu) J_{w_l,\mu}\left(\frac{4\pi^2 c_2 m_1 n_2}{c_1^2}, \frac{4\pi^2 c_1 m_2 n_1}{c_2^2}\right) d\mu,$$

with $J_{w,\mu}$ given by the Mellin-Barnes integrals (5.7), (5.8), *and* (5.6), *and $C(\mu)$ replaced with*

$$C^*(\mu) = \prod_{j<k} \frac{\sqrt{9-(\mu_j-\mu_k)^2}\sin\frac{\pi}{2}(\mu_k-\mu_j)}{\frac{1}{2}(\mu_k-\mu_j)}.$$

This should be regarded as a theorem on the higher-rank hypergeometric functions, in the style of Stade; it assigns to the weight functions H_w good complex analytic expressions, in the form of Mellin-Barnes integral representations, though we strongly suspect that we have not achieved the optimal such representations. We are being quite wasteful in the exponent $\frac{13}{8}+100\epsilon$ for our assumptions on $\hat{k}$; it should properly be 1, but here we only prove the larger exponent is sufficient. Lastly, we needed to replace $C(\mu)$ because the pole at $\mu_1-\mu_2=-1$ also shows up on the arithmetic side and interfers with the absolute convergence of the sum of the long-element Kloosterman sums; we accomplish this change by applying Stade's formula at $s=2$ instead of $s=1$, and renormalizing to obtain the proper asymptotics ($C(\mu)\asymp C^*(\mu)$).

Having the Mellin-Barnes representation of Theorem 31, in section 6.1 we compute a type of first-term asymptotic for $J_{w_l,\mu}$:

Proposition 32. *$J_{w_l,\mu}(y)=|y_1|^{-\mu_1}|y_2|^{\mu_2}K_{w_l}(\mu)+\sum_{j=1}^{7}E_{w_l,j}(\mu,y)$, where*

$$K_{w_l}(\mu)=\frac{\pi^{\frac{1}{2}+3\mu_1}}{2^{5-2\mu_1+2\mu_2}}\frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right)\Gamma\left(\frac{\mu_3-\mu_2}{2}\right)}{\Gamma\left(\frac{1+\mu_2-\mu_1}{2}\right)\Gamma\left(\frac{1+\mu_3-\mu_1}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)}\prod_{j<k}\frac{(\mu_k-\mu_j)^2}{\sqrt{9-(\mu_j-\mu_k)^2}^{2}},$$

and the $E_{w_l,j}$ are given explicitly by equations (6.2)-(6.8) *and satisy*

$$E_{w_l,j}(\mu,y)=o\left(|y_1|^{-\mathrm{Re}(\mu_1)}|y_2|^{\mathrm{Re}(\mu_2)}\right) \tag{4.1}$$

as $y\to 0$ with $\mathrm{Re}(\mu_1)\le\mathrm{Re}(\mu_3)\le\mathrm{Re}(\mu_2)$.

The asymptotics (4.1) are actually power-saving bounds over $|y_1|^{-\mathrm{Re}(\mu_1)}|y_2|^{\mathrm{Re}(\mu_2)}$ which are vital to our purposes, but their dependence on $\mathrm{Re}(\mu_1)$ and $\mathrm{Re}(\mu_2)$ is unfortunately quite complicated. Note that the only zeros of $K_{w_l}(\mu)$ in $|\mathrm{Re}(\mu_i)|<\frac{1}{2}+\epsilon$ occur when one of $\mu_2-\mu_3$, $\mu_3-\mu_1$, $\mu_2-\mu_1$ are 0 or -1, and it has no poles on this region.

With this asymptotic of $J_{w_l,\mu}$ and the accompanying explicit error terms, in section 6.2 we produce a type of partial inversion to the H_{w_l} transform:

Theorem 33. *Let* $f : (\mathbb{R}^+)^2 \to \mathbb{C}$ *such that*

$$\hat{f}(q) := \int_{(\mathbb{R}^+)^2} f(y) y_1^{q_1} y_2^{q_2} \frac{dy_1\, dy_2}{y_1 y_2}$$

is holomorphic on $-\frac{1}{2} - \epsilon < \mathrm{Re}(q_i) \le 0$ *and satisfies* $\hat{f}(q) \ll |q_1 q_2|^{-8}$ *there. Then we again have the formula* (2.1)*, where now*

$$\begin{aligned} H_{w_l}(k, \psi_m, \psi_n, c) =& \frac{1}{c_1 c_2} f\left(\frac{4\pi^2 c_2 m_1 n_2}{c_1^2}, \frac{4\pi^2 c_1 m_2 n_1}{c_2^2}\right) \\ &+ \frac{1}{c_1 c_2} \sum_{j=1}^{10} F_j\left(\hat{f}; \left(\frac{4\pi^2 c_2 m_1 n_2}{c_1^2}, \frac{4\pi^2 c_1 m_2 n_1}{c_2^2}\right)\right), \end{aligned}$$

using

$$\begin{aligned} \hat{k}(\mu) = \hat{k}_{\mathfrak{q}}(\mu) :=& \frac{1}{2\pi i} \int_{\mathrm{Re}(q)=\mathfrak{q}} \frac{\hat{f}(q)}{K_{w_l}(q_1, -q_2)} k_{conv}(\mu, q) \\ & \frac{(q_1+q_2)^2 (2q_1 - q_2)(2q_2 - q_1)}{(q_1-\mu_1)(q_1-\mu_2)(q_1-\mu_3)(q_2+\mu_1)(q_2+\mu_2)(q_2+\mu_3)} dq, \end{aligned} \tag{4.2}$$

where $k_{conv}(\mu, q)$ *is chosen in* (B.3) *to be holomorphic on* $\mathrm{Re}(\mu_i), \mathrm{Re}(q_i) \in (-2, 2)$ *with* $k_{conv}(\mu, (\mu_1, -\mu_2)) = 1$*, and the* F_j *are given explicitly by equations* (6.10)-(6.13) *and satisy*

$$F_j\left(\hat{f}; y\right) = o\left(|y_1|^{-\mathfrak{q}_1} |y_2|^{-\mathfrak{q}_2}\right) \tag{4.3}$$

as $y \to 0$ *with* $\mathrm{Re}(\mu_1) \le \mathrm{Re}(\mu_3) \le \mathrm{Re}(\mu_2)$*.*

Again, (4.3) does not quite do justice to the actual bounds we obtain. The construction of $\hat{k}_{\mathfrak{q}}(\mu)$ is such that the double pole at $q_1 = \mu_1$ and $q_2 = -\mu_2$ gives the residue

$$\hat{k}_{\mathfrak{q}}(\mu) = \frac{\hat{f}(\mu_1, -\mu_2)}{K_{w_l}(\mu)} + \text{error terms},$$

and paired with our first-term asymptotic for $J_{w_l,\mu}$, H_{w_l} looks like the inverse Mellin transform of $\hat{f}$. The motivation for the existence of k_{conv} is to control the convergence of the μ integral and for ease of proof in the section on bounds; again, the exponent -8 is certainly not best possible – optimal is likely $-2-\epsilon$, but it is convenient.

One should note that this is an incomplete generalization to $SL(3,\mathbb{R})$ of the second form of Kuznetsov's formula on $SL(2,\mathbb{R})$; it allows us to study sums of Kloosterman sums by applying knowledge of the Fourier-Whittaker coefficients of automorphic forms. As the asymptotic in Proposition 32 is for $y_1, y_2 \to 0$, this partial inversion formula is effective when studying sums of Kloosterman sums with $\frac{1}{2} < \frac{\log c_1}{\log c_2} < 2$, i.e. when each of the moduli is at least the square-root of the other. One would expect that in practice, the remaining sums, i.e. those over $c_1 < \sqrt{c_2}$ or $c_2 < \sqrt{c_1}$, will be small. Similarly, we expect that the sums of Kloosterman sums for the intermediate Weyl elements w_4 and w_5 will tend to be small compared to the long-element sum and the trivial term H_I.

Lastly, by comparison with the method on $SL(2)$, one might wonder if the two error terms at $\left(-\frac{1}{2},0\right)$ and $\left(0,\frac{1}{2}\right)$ indicate the need for some form of discrete series in the full inversion formula. This is a somewhat tenuous connection, however.

The formulae used above and in Theorem 5 depend strictly on the location of the contours in the Mellin-Barnes integrals for each $J_{w,\mu}$, so we include as appendix B some bounds to demonstrate their absolute convergence at the relevant locations, which are unfortunately not entirely trivial. The bounds we obtain are most likely not optimal in their dependence on μ, but they are sufficient for our purposes here:

Proposition 34. *For the contours given in Table 6.1, we have absolute convergence of all of the weight functions with*

$$\begin{aligned}
\left|J_{w_l,\mu}(y)\right| \ll &\left|y_1\right|^{\frac{1}{2}+\epsilon}\left|y_2\right|^{\frac{1}{2}+\epsilon} \\
&\sum_{w\in W} \left|\mu^w_3-\mu^w_1\right|^{\frac{3}{2}+100\epsilon}\left|\mu^w_3-\mu^w_2\right|^{\frac{3}{2}+100\epsilon}\left|\mu^w_2-\mu^w_1\right|^{-\frac{1}{2}+100\epsilon}, \\
\left|J_{w_4,\mu}(y)\right| \ll &\left|y\right|^{\epsilon}\sum_{w\in W} \left|\mu^w_3-\mu^w_1\right|^{\frac{13}{8}+100\epsilon}\left|\mu^w_3-\mu^w_2\right|^{\frac{13}{8}+100\epsilon}\left|\mu^w_2-\mu^w_1\right|^{\frac{3}{4}+100\epsilon}, \\
\left|J_{w_5,\mu}(y)\right| \ll &\left|y\right|^{\epsilon}\sum_{w\in W} \left|\mu^w_3-\mu^w_1\right|^{\frac{13}{8}+100\epsilon}\left|\mu^w_3-\mu^w_2\right|^{\frac{13}{8}+100\epsilon}\left|\mu^w_2-\mu^w_1\right|^{\frac{3}{4}+100\epsilon},
\end{aligned}$$

$$|E_{w_l,1}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{\frac{11}{8}+100\epsilon} |\mu_3^w - \mu_2^w|^{\frac{11}{8}+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{3}{4}+100\epsilon},$$

$$|E_{w_l,2}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{\frac{11}{8}+100\epsilon} |\mu_3^w - \mu_2^w|^{\frac{11}{8}+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{3}{4}+100\epsilon},$$

$$|E_{w_l,3}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{\frac{3}{2}+100\epsilon} |\mu_3^w - \mu_2^w|^{\frac{3}{2}+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{1}{2}+100\epsilon},$$

$$|E_{w_l,4}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{1+100\epsilon} |\mu_3^w - \mu_2^w|^{1+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{1}{2}+100\epsilon},$$

$$|E_{w_l,5}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{1+100\epsilon} |\mu_3^w - \mu_2^w|^{1+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{1}{2}+100\epsilon},$$

$$|E_{w_l,6}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{1+100\epsilon} |\mu_3^w - \mu_2^w|^{1+100\epsilon} |\mu_2^w - \mu_1^w|^{100\epsilon},$$

$$|E_{w_l,7}(\mu,y)| \ll |y_1|^{\frac{1}{2}+\epsilon} |y_2|^{\frac{1}{2}+\epsilon} \sum_{w\in W} |\mu_3^w - \mu_1^w|^{1+100\epsilon} |\mu_3^w - \mu_2^w|^{1+100\epsilon} |\mu_2^w - \mu_1^w|^{-\frac{1}{2}+100\epsilon},$$

Finally, in section 6.3, we obtain the results of the introduction. These results follow essentially immediately from the locations of the spectral parameters of the objects in the partial inversion formula Theorem 33, and the absolute convergence of the weight functions in the desired locations using Proposition 34. We visualize the spectral parameters in figure 4.1.

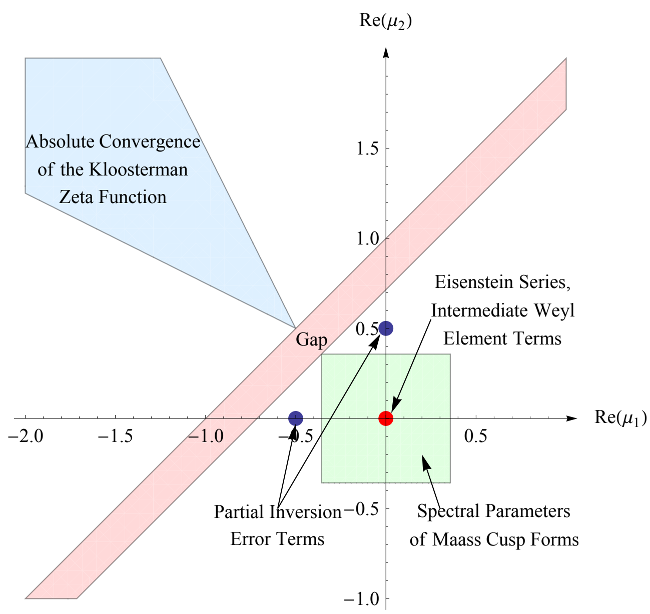

Figure 4.1: Location of the $SL(3)$ Spectral Parameters

CHAPTER 5

Evaluation of the Integral Transforms

We will need a number of elementary results to obtain the Kuznetsov formula above, so we collect them here.

5.1 Mellin Transforms

Define

$$e_\theta\left(x\right) = \begin{cases} e\left(x \exp\left(i\left(\frac{\pi}{2}-\theta\right)\operatorname{sign}(x)\right)\right) & \text{if } x \neq 0, \\ 1 & \text{otherwise.} \end{cases}$$

Note that e_θ is no longer a character of $\mathbb{R}$.

We have the Mellin transform

$$\int_0^\infty e\left(ay\right) y^{t-1} dy = \left(-2\pi i a\right)^{-t} \Gamma\left(t\right),$$

for $\mathrm{Re}(t) > 0$ and $\mathrm{Im}(a) > 0$ (since it holds for $-2\pi i a \in \mathbb{R}^+$ by substitution in the Euler integral representation of the gamma function and extends by analytic continuation), and Mellin inversion gives

$$e\left(ay\right) = \frac{1}{2\pi i}\int_{(c)} \left(-2\pi i a\right)^{-t} \Gamma\left(t\right) y^{-t} dt,$$

for $c > 0$ and $\mathrm{Im}(a) > 0$.

As we are dealing with the principal value of the power function, we have

$$\left(-2\pi i x \exp\left(i\left(\frac{\pi}{2}-\theta\right)\ \mathrm{sign}(x)\right)\right)^{-t} = \left(2\pi\left|x\right|\exp\left(-i\theta\,\mathrm{sign}(x)\right)\right)^{-t}$$
$$= e^{-t(\log|2\pi x|-i\theta\,\mathrm{sign}(x))}$$
$$= \left|2\pi x\right|^{-t} e^{it\theta\,\mathrm{sign}(x)}.$$

The construction of $e_\theta(x)$ is such that the argument of the exponential always has a negative real part, so applying the previous two formulae gives

$$e_\theta\left(x\right) = \frac{1}{2\pi i}\int_{\mathrm{Re}(t)=c} \left|2\pi x\right|^{-t} e^{it\theta\,\mathrm{sign}(x)}\Gamma\left(t\right)\,dt, \tag{5.1}$$

for $x \neq 0$ and $c > 0$.

We also have the Mellin transform

$$\int_0^\infty (1+x^2)^u x^t dx = \frac{1}{2}B\left(\frac{t+1}{2}, \frac{-2u-t-1}{2}\right), \tag{5.2}$$

for $-1 < \mathrm{Re}(t) < -1-2\mathrm{Re}(u)$. Here $B(u,v) = \frac{\Gamma(u)\Gamma(v)}{\Gamma(u+v)}$ is the beta function.

5.2 The G Function

We start with $G^*(u,\mu) := \frac{G(u,\mu)}{\Lambda(\mu)}$. The G function has poles at $u_1 = \mu_i$ and $-u_2 = \mu_i$, so up to permutations, we may assume a pole is at $u_1 = \mu_1$, then the residue of $G^*(u,\mu)$ there is given by

$$G_l^*(1,u_2,\mu) := 2\pi^{\frac{3}{2}+\mu_1-\mu_3}\frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right)}{\Gamma\left(\frac{1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{1+\mu_1-\mu_3}{2}\right)}\frac{\Gamma\left(\frac{u_2+\mu_2}{2}\right)\Gamma\left(\frac{u_2+\mu_3}{2}\right)}{\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)}.$$

For the other poles, we let $G_l^*(j,u_2,\mu)$ be the residue at $u_1=\mu_j$, and $G_r^*(j,u_1,\mu)$ the residue at $-u_2=\mu_j$. In particular,

$$G_r^*(2,u_1,\mu) := 2\pi^{\frac{3}{2}+\mu_1-\mu_3} \frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_3-\mu_2}{2}\right)}{\Gamma\left(\frac{1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)} \frac{\Gamma\left(\frac{u_1-\mu_1}{2}\right)\Gamma\left(\frac{u_1-\mu_3}{2}\right)}{\Gamma\left(\frac{1+\mu_1-\mu_3}{2}\right)}.$$

The residue at $u_1=\mu_1$ again has poles at $-u_2=\mu_2,\mu_3$, and we assume $-u_2=\mu_2$, giving the residue

$$G_b^*(1,2,\mu) := 4\pi^{\frac{3}{2}+\mu_1-\mu_3} \frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right)\Gamma\left(\frac{\mu_3-\mu_2}{2}\right)}{\Gamma\left(\frac{1+\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{1+\mu_1-\mu_3}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)}.$$

In general, let $G_b^*(j,k,\mu)$ be the residue at $(u_1,-u_2)=(\mu_j,\mu_k)$ for $j\neq k$.

5.3 The General Term

In Li's construction of the Kuznetsov formula, the final step involved integrating away some extra variables on the spectral side, using Stade's formula (Theorem 17) at $s=1$, but this will make it impossible to obtain a function which is nicely holomorphic in the region we require. Instead we will apply Stade's formula at $s=2$, which results in a weight function on the spectral side which is actually too large. So we replace $\hat{k}$ with

$$\hat{k}(\mu) = \frac{32\pi^4\tilde{k}(\mu)}{\prod_{j<k}\sqrt{9-(\mu_j-\mu_k)^2}},$$

taking the branch cuts of the square root to be outside the strip $|\mathrm{Re}(\mu_i)| < \frac{1}{2}+\epsilon$, and the spectral side is now of the correct magnitude as a function of $\tilde{k}$.

Before we truly start the simplification process, we must engage in a series of transformations: Since $\psi_m(x)=\psi_{11}(mxm^{-1})$, and $|m|\,m^{-1}\in V$ (up to a multiple of -1), conjugating by m^{-1} and by tn^{-1} we have

$$H_w = \frac{2\,|m_1m_2n_1n_2|}{\pi(m_1m_2)^2C_w(n)} \int_{(\mathbb{R}^+)^2}\int_{U(\mathbb{R})}\int_{\overline{U}_w(\mathbb{R})} k\left(t^{-1}x^{-1}\alpha w t x'\right)\psi_{11}(x)\overline{\psi_t(x')}dx'\,dx\,C_w(t)t_2^4t_1^2\,dt,$$

where

$$\alpha = mcwn^{-1}w^{-1} \equiv \begin{pmatrix} \alpha_1\alpha_2 & & \\ & \alpha_1 & \\ & & 1 \end{pmatrix} \quad (\text{mod } \mathbb{R}^+),$$

and $C_w(y)$ is the Jacobian of the change of variables $u \mapsto yuy^{-1}$ for $u \in \overline{U}_w(\mathbb{R})$. Note that α_1 and α_2 may be negative! Now interchange the x and x' integrals. Then if $wx' \equiv x^* y^*$ (mod K) and $t^w = wtw^{-1}$, we may translate and invert $x^{-1}(\alpha t^w)x^*(\alpha t^w)^{-1} \mapsto x$.

$$H_w = \frac{2\,|m_1m_2n_1n_2|}{\pi(m_1m_2)^2 C_w(n)} \int_{(\mathbb{R}^+)^2} \int_{\overline{U}_w(\mathbb{R})} \int_{U(\mathbb{R})} k\left(t^{-1}x\alpha t^w y^*\right) \overline{\psi_{11}(x)}\psi_{\alpha t^w}(x^*)\overline{\psi_t(x')}dx\,dx'$$
$$C_w(t)t_2^4 t_1^2\,dt.$$

Since k is a function on $K\backslash G/K$ (essentially the space of diagonal matrices by the singular value decomposition), it is invariant under transposition of its argument, so send $(\alpha t^w y^*)^{-1}x(\alpha t^w y^*) \mapsto x$ and transpose, giving

$$H_w = \frac{2\,|m_1m_2n_1n_2|}{\pi(m_1m_2)^2 C_w(n)} \int_{(\mathbb{R}^+)^2} \int_{\overline{U}_w(\mathbb{R})} \int_{U(\mathbb{R})} k\left({}^t x\alpha t^w y^* t^{-1}\right) \overline{\psi_{\alpha t^w y^*}}(x)dx$$
$$\psi_{\alpha t^w}(x^*)\overline{\psi_t}(x')p_{2\rho}(\alpha t^w y^*)dx'\,C_w(t)t_2^4 t_1^2\,dt.$$

We wish to apply spherical inversion, which will require some care with respect to convergence of the integrals, but the integral in x can be evaluated explicitly as in the following lemma.

Lemma 35 (Fourier Transform of the Spherical Function)**.** *For* $\text{Re}(\mu_1), \text{Re}(\mu_2) \in \left(-\frac{1}{8}, 0\right)$, *let*

$$X(y,\mu,\psi) = \int_{U(\mathbb{R})} h_\mu({}^t xy)\psi(x)dx,$$

where the integrals over x_1 *and* x_2 *are taken in the limit sense* $\int_{-\infty}^{\infty} = \lim_{R\to\infty}\int_{-R}^{R}$*, then*

$$X(y,\mu,\psi) = \kappa W(y^{-1}, -\mu, \psi)W(I,\mu,\psi),$$

where

$$\frac{1}{\kappa} = \prod_{1\le i<j\le 3} B\left(\frac{1}{2}, \frac{j-i}{2}\right) = 2\pi^2.$$

Note: We expect this formula to hold on $SL(n,\mathbb{R})$ for arbitrary n, but the interchange of integrals might be difficult to justify.

Applying spherical inversion, we may shift the integrals in μ_1 and μ_2 slightly to the left of the 0 line and apply the above lemma. Note that, despite appearances, we have an expression for $|c_3(\mu)|^2$ which is analytic in μ. After moving μ back to the 0 lines, we have

$$\begin{aligned}H_w = -\frac{|m_1 m_2 n_1 n_2|}{48\pi^7 (m_1 m_2)^2 C_w(n)} \int_{(\mathbb{R}^+)^2} \int_{\overline{U}_w(\mathbb{R})} \int_{\mathrm{Re}(\mu)=(0,0)} \frac{\hat{k}(\mu)}{|c_3(\mu)|^2} \\ W((\alpha t^w y^*)^{-1} t, -\mu, \overline{\psi_{\alpha t^w y^*}}) W(I, \mu, \overline{\psi_{\alpha t^w y^*}}) d\mu \\ \psi_{\alpha t^w}(x^*) \overline{\psi_t}(x') p_{2\rho}(\alpha t^w y^*) dx'\, C_w(t) t_2^4 t_1^2\, dt.\end{aligned}$$

We then return $\alpha t^w y^*$ to the argument of the Whittaker function:

$$\begin{aligned}H_w = -\frac{|m_1 m_2 n_1 n_2|}{48\pi^7 (m_1 m_2)^2 C_w(n)} \int_{(\mathbb{R}^+)^2} \int_{\overline{U}_w(\mathbb{R})} \int_{\mathrm{Re}(\mu)=(0,0)} \frac{\hat{k}(\mu)}{|c_3(\mu)|^2} \\ W(t, -\mu, \psi_{11}) W(\alpha t^w y^*, \mu, \psi_{11}) d\mu\, \psi_{\alpha t^w}(x^*) \overline{\psi_t}(x') dx'\, C_w(t) t_2^4 t_1^2\, dt.\end{aligned} \tag{5.3}$$

For all but the trivial term, we will use the Mellin expansion of the second Whittaker function. We will shift the lines of integration in this Mellin expansion $\mathrm{Re}(u) \mapsto -\frac{1}{2} - 10\epsilon$, picking up poles at $u_1 = \mu_i$ and $-u_2 = \mu_i$. As $\hat{k}$, c_3, and the product

$$W(z, -\mu, \psi_{11}) W(z', \mu, \psi_{11}) = \frac{W^*(z, -\mu, \psi_{11}) W^*(z', \mu, \psi_{11})}{\Lambda(-\mu)\Lambda(\mu)}$$

are invariant under permutations of μ, we may collect like terms, leaving us with four pieces: The pole at $u_1 = \mu_2$, $u_2 = -\mu_2$; the pole at $u_1 = \mu_1$ with an integral along $\mathrm{Re}(u_2) = -\frac{1}{2} - 10\epsilon$; the pole at $u_2 = -\mu_2$ with an integral along $\mathrm{Re}(u_1) = -\frac{1}{2} - 10\epsilon$; the double integral along $\mathrm{Re}(u) = -\frac{1}{2} - 10\epsilon$. Then we shift $\mathrm{Re}(\mu_1, \mu_2) \mapsto \left(-\frac{1}{2} - 9\epsilon, \frac{1}{2} + 9\epsilon\right)$, which is possible since

$\frac{|c_n(\mu)|^{-2}}{\Lambda(\mu)\Lambda(-\mu)}$ has no poles. Thus we are evaluating

$$H_w = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\left(-\frac{1}{2}-9\epsilon, \frac{1}{2}+9\epsilon\right)} \tilde{k}(\mu) J'_{w,\mu}(\alpha)\, d\mu,$$

$$J'_{w,\mu}(\alpha) = \frac{\pi^2 |m_1 m_2 n_1 n_2| c_1 c_2}{3(m_1 m_2)^2 C_w(n)} \left(\prod_{j<k} \tan^2 \frac{\pi}{2}(\mu_j - \mu_k)\right) \int_{(\mathbb{R}^+)^2} W(t, -\mu, \psi_{11}) \tag{5.4}$$

$$\left(\frac{-1}{16\pi^4} \int_{\mathrm{Re}(u)=-\frac{1}{2}-10\epsilon} G^*(u,\mu) X'_w(u,v,\beta,t) (\pi\beta_1 t_1^w)^{1-u_1} (\pi\beta_2 t_2^w)^{1-u_2} du\right.$$
$$+ \frac{-3i}{8\pi^3} (\pi\beta_1 t_1^w)^{1-\mu_1} \int_{\mathrm{Re}(u_2)=-\frac{1}{2}-10\epsilon} G_l^*(1,u_2,\mu) X'_w((\mu_1,u_2),v,\beta,t) (\pi\beta_2 t_2^w)^{1-u_2} du$$
$$+ \frac{-3i}{8\pi^3} (\pi\beta_2 t_2^w)^{1+\mu_2} \int_{\mathrm{Re}(u_1)=-\frac{1}{2}-10\epsilon} G_r^*(2,u_1,\mu) X'_w((u_1,-\mu_2),v,\beta,t) (\pi\beta_1 t_1^w)^{1-u_1} du$$
$$\left.+ \frac{6}{4\pi^2} (\pi\beta_1 t_1^w)^{1-\mu_1} (\pi\beta_2 t_2^w)^{1+\mu_2} G_b^*(1,2,\mu) X'_w((\mu_1,-\mu_2),v,\beta,t)\right) C_w(t) t_2^4 t_1^2\, dt,$$

$$X'_w(u,v,\beta,t) = \int_{\overline{U}_w(\mathbb{R})} \psi_{v\beta t^w}(x^*) \overline{\psi_t}(x') y_1^{*1-u_1} y_2^{*1-u_2} dx',$$

where $\beta = |\alpha|$, $v = \mathrm{sign}(\alpha)$, and we justify the interchange of integrals by explicitly computing y^*, which shows that, in general, X'_w converges absolutely for some region in $\mathrm{Re}(u_1), \mathrm{Re}(u_2) < 0$. This will further show that the t integral converges absolutely as well as the sum of Kloosterman sums. This step was a technical necessity, as we did not know it was safe to pull the t integral inside the sum of Kloosterman sums until right now, but having done so, we may forget about the sum of Kloosterman sums.

The function X'_w is a type of generalized hypergeometric function. For fixed v, it is a function of four variables $\beta_1, \beta_2, t_1, t_2$ with two parameters u_1 and u_2. The object of the remainder of the analysis will be to obtain a Mellin-Barnes integral representation for this function; this is accomplished by brute force: By judicious use of the e_θ function, and absolute convergence of the t, μ, u, and x' integrals, we will write H_w as a limit over θ:

$$J'_{w,\mu}(\alpha) = \frac{1}{12\pi^6}\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_j-\mu_k)\right)\lim_{\theta\to\frac{\pi}{2}-}\int_{(\mathbb{R}^+)^2} W(t,-\mu,\psi_{11})$$
$$\left(\frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(u)=-\frac{1}{2}-10\epsilon} G^*(u,\mu)X'_w(u,v,\beta,t,\theta)(\pi\beta_1 t_1^w)^{-u_1}(\pi\beta_2 t_2^w)^{-u_2}du\right.$$
$$+\frac{3}{2\pi i}(\pi\beta_1 t_1^w)^{-\mu_1}\int_{\mathrm{Re}(u_2)=-\frac{1}{2}-10\epsilon} G_l^*(1,u_2,\mu)X'_w((\mu_1,u_2),v,\beta,t,\theta)(\pi\beta_2 t_2^w)^{-u_2}du$$
$$+\frac{3}{2\pi i}(\pi\beta_2 t_2^w)^{\mu_2}\int_{\mathrm{Re}(u_1)=-\frac{1}{2}-10\epsilon} G_r^*(2,u_1,\mu)X'_w((u_1,-\mu_2),v,\beta,t,\theta)(\pi\beta_1 t_1^w)^{-u_1}du$$
$$\left.+6(\pi\beta_1 t_1^w)^{-\mu_1}(\pi\beta_2 t_2^w)^{\mu_2}G_b^*(1,2,\mu)X'_w((\mu_1,-\mu_2),v,\beta,t,\theta)\right)(\pi t_1)^5(\pi t_2)^3\,dt,$$

since in every case, we have

$$\frac{|m_1m_2n_1n_2|\,\beta_1\beta_2}{(m_1m_2)^2C_w(n)} = \frac{1}{c_1c_2}, \qquad C_w(t)(t_1^w t_2^w)t_2^4t_1^2 = t_1^5t_2^3.$$

Suppose that we have

$$X'_w(u,v,\beta,t,\theta) = \frac{1}{(2\pi i)^4}\int_{\mathrm{Re}(s)=\eta}\int_{\mathrm{Re}(r)=\eta'} T'_w(u,s,r,v,\theta)(\pi\beta_1)^{-s_1}(\pi\beta_2)^{-s_2}(\pi t_1)^{r_1}(\pi t_2)^{r_2}dr\,ds,$$

where $r=(r_1,\dots,r_k)$ and η, η' are chosen to maintain absolute convergence. Explicitly, the absolute convergence of the t integral requires $\mathrm{Re}(4+r_1)>\frac{1}{2}$ and $\mathrm{Re}(2+r_2)>\frac{1}{2}$. Then we may again apply the Mellin transform of the Whittaker function, so

$$H_w = \frac{1}{(2\pi i)^2c_1c_2}\int_{\mathrm{Re}(\mu)=\eta}\tilde{k}(\mu)J'_{w,\mu}(\alpha)\,d\mu, \tag{5.5}$$

where

$$J'_{w,\mu}(y) = \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(s)=\mathfrak{s}}|\pi y_1|^{-s_1}|\pi y_2|^{-s_2}N'_w(s,\mu,\mathrm{sign}(y))\,ds,$$

$$\begin{aligned} N'_w(s,\mu,v) =& \Bigg(\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(u)=\mathfrak{u}} G^*(u,\mu) R'_w(s,u,v)\, du \\ &+ \frac{3}{2\pi i} \int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} G^*_l(1,u_2,\mu) R'_w(s,(\mu_1,u_2),v)\, du_2 \\ &+ \frac{3}{2\pi i} \int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} G^*_r(2,u_1,\mu) R'_w(s,(u_1,-\mu_2),v)\, du_1 \\ &+ 6G^*_b(1,2,\mu) R'_w(s,(\mu_1,-\mu_2),v) \Bigg) \prod_{j<k} \frac{(\mu_k-\mu_j)\tan\frac{\pi}{2}(\mu_k-\mu_j)}{\sqrt{9-(\mu_j-\mu_k)^2}}, \end{aligned}$$

$$R'_w(s,u,v) = \frac{\pi^{u_1^w+u_2^w}}{48\pi^4} \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(r)=\mathfrak{r}} G^*((4,2)+r-u^w,-\mu) T'_w\left(u, s-u, r, v, \frac{\pi}{2}\right) dr.$$

with u^w defined by $t_1^{u_1^w} t_2^{u_2^w} = (t_1^w)^{u_1}(t_2^w)^{u_2}$, assuming we have absolute convergence at $\theta = \frac{\pi}{2}$, for which we have the contents of appendix B. Note that the zero of $\frac{1}{\Lambda(\mu)}$ at $\mu_1-\mu_2=-1$ cancels the pole of $\tan\frac{\pi}{2}(\mu_2-\mu_1)$ there, so the last term of N'_w contributes no poles on the strip $|\mathrm{Re}(\mu_i)| < \frac{1}{2}+\epsilon$.

This will conclude the construction of the formula.

5.3.1 Fourier Transform of the Spherical Function

The proof of Lemma 35 has three parts: First we give an integral formula for the spherical function; this is a slight extension of a formula in Terras [39, p. 3.30]. Then we justify an interchange of integrals in the absence of absolute convergence; essentially, we find an integral formula for the Jacquet-Whittaker function having a slightly larger region of absolute convergence. Lastly, some translation is needed to show the integrals we find are actually each the Jacquet-Whittaker function.

We define the K-part function on G as $K(xyk)=k$, then the power function identity [39, p. 3.17]

$$p_{\rho+\mu}(K({}^t x)z) = p_{\rho+\mu}({}^t xz) p_{-\rho-\mu}({}^t x)$$

comes from the decomposition ${}^t x = x_1 y_1 k_1$:

$$p_{\rho+\mu}({}^t xz) = p_{\rho+\mu}(x_1 y_1 k_1 z) = p_{\rho+\mu}(y_1) p_{\rho+\mu}(k_1 z).$$

We also need the change of variables formula [39, Lemma 4.3.2]

$$\int_{K/V} f(\bar{k}) d\bar{k} = \kappa \int_{U(\mathbb{R})} f(K({}^t x)V) p_{2\rho}({}^t x) dx,$$

where the measure on K/V is again normalized to $\int_{K/V} d\bar{k} = 1$ Then we expand

$$\begin{aligned} h_\mu(z) &= \int_{K/V} \sum_{v \in V} p_{\rho+\mu}(\bar{k} v z) \frac{d\bar{k}}{|V|} \\ &= \int_{K/V} p_{\rho+\mu}(\bar{k} z) d\bar{k} \\ &= \kappa \int_{U(\mathbb{R})} p_{\rho+\mu}(K({}^t u) z) p_{2\rho}({}^t u) du \\ &= \kappa \int_{U(\mathbb{R})} p_{\rho+\mu}({}^t u z) p_{\rho-\mu}({}^t u) du, \end{aligned}$$

by substituting $v\bar{k}v \mapsto \bar{k}$ in the first integral. Intuitively, if we could pull the x integral of X inside the u integral of h_μ and send $xu \mapsto x$, then we would have a product of two Whittaker functions; however, we have absolute convergence of the combined x and u integral for no values of μ.

Applying this integral representation of h_μ to X, we have

$$X(y, \mu, \psi) = \kappa \lim_{R \to \infty} \int_{[-R,R]^2 \times \mathbb{R}} \int_{U(\mathbb{R})} p_{\rho+\mu}({}^t u {}^t x y) p_{\rho-\mu}({}^t u) \, du \, \psi(x) dx,$$

and the integrals inside the limit converge absolutely. We may then interchange the integrals and send $xu \mapsto x$, and for convenience, we also send ${}^t x \mapsto y {}^t x y^{-1}$, giving

$$X = \kappa p_{-\rho+\mu}(y) \lim_{R \to \infty} \int_{U(\mathbb{R})} \int_{\mathcal{X}(u,y,R) \times \mathbb{R}} p_{\rho+\mu}({}^t x) \psi^*(x) \, dx \, p_{\rho-\mu}({}^t u) \overline{\psi(u)} \, du,$$

where $\mathcal{X}(u, y, R)$ is the result of applying these transforms to the box $[-R, R]^2$, and $\psi^*(x) =$

$\psi(y^{-1}xy)$.

Now we need only to rearrange the x integral into an absolutely convergent form as the u integral will then converge absolutely by our assumptions on μ. To that end, we separate the x_3 integral

$$\begin{aligned} X_3(x_1,x_2,\mu) &= \int_{\mathbb{R}} p_{\rho+\mu}({}^t x)dx_3 \\ &= \int_{\mathbb{R}} \left(1+x_1^2+x_3^2\right)^{-\frac{1+\mu_1+2\mu_2}{2}} \left(1+x_2^2+(x_3-x_1x_2)^2\right)^{-\frac{1+\mu_1-\mu_2}{2}} dx_3, \end{aligned}$$

and for convenience, we write $X_3(s_1,s_2) = X_3(x_1,x_2,\mu)$ where $s_1 = -\frac{1+\mu_1+2\mu_2}{2}$ and $s_2 = -\frac{1+\mu_1-\mu_2}{2}$.

A quick and useful bound for X_3 comes from applying Cauchy-Schwarz:

$$\begin{aligned} |X_3(s_1,s_2)| \leq& \sqrt{\int_{\mathbb{R}} (1+x_1^2+x_3^2)^{2\mathrm{Re}(s_1)}\, dx_3}\sqrt{\int_{\mathbb{R}} (1+x_2^2+(x_3-x_1x_2)^2)^{2\mathrm{Re}(s_2)}\, dx_3} \\ =& \sqrt{\pi\frac{\Gamma\left(-\frac{1}{2}-2\mathrm{Re}(s_1)\right)\Gamma\left(-\frac{1}{2}-2\mathrm{Re}(s_2)\right)}{\Gamma\left(-2\mathrm{Re}(s_1)\right)\Gamma\left(-2\mathrm{Re}(s_2)\right)}} \left(1+x_1^2\right)^{\mathrm{Re}(s_1)+\frac{1}{4}} \left(1+x_2^2\right)^{\mathrm{Re}(s_2)+\frac{1}{4}} \\ \ll_s& \left(1+x_1^2\right)^{\mathrm{Re}(s_1)+\frac{1}{4}} \left(1+x_2^2\right)^{\mathrm{Re}(s_2)+\frac{1}{4}}, \end{aligned}$$

assuming $\mathrm{Re}(s_1), \mathrm{Re}(s_2) < -\frac{1}{4}$.

Substituting $x_3 \mapsto x_1x_3$ gives

$$X_3 = x_1^{2(s_1+s_2)+1} \int_{\mathbb{R}} \left(x_1^{-2}+1+x_3^2\right)^{s_1} \left(\frac{1+x_2^2}{x_1^2}+(x_3-x_2^2)^2\right)^{s_2} dx_3,$$

so

$$\frac{\partial X_3}{\partial x_1} = \frac{2(s_1+s_2)+1}{x_1}X_3(s_1,s_2) - 2\frac{s_1}{x_1}X_3(s_1-1,s_2) - 2s_2\frac{1+x_2^2}{x_1}X_3(s_1,s_2-1).$$

Similarly,

$$\frac{\partial X_3}{\partial x_2} = \frac{2(s_1+s_2)+1}{x_2}X_3(s_1,s_2) - 2s_1\frac{1+x_1^2}{x_2}X_3(s_1-1,s_2) - 2\frac{s_2}{x_2}X_3(s_1,s_2-1),$$

and

$$\begin{aligned}
\frac{\partial^2 X_3}{\partial x_2 \partial x_1} = {} & \frac{(2(s_1+s_2)+1)^2}{x_1 x_2} X_3(s_1, s_2) \\
& - 2s_1 \frac{2(s_1+s_2)(2+x_1^2)+x_1^2}{x_1 x_2} X_3(s_1-1, s_2) \\
& - 2s_2 \frac{2(s_1+s_2)(2+x_2^2)+x_2^2}{x_1 x_2} X_3(s_1, s_2-1) \\
& + 4s_1 s_2 \frac{1+(1+x_1^2)(1+x_2)^2}{x_1 x_2} X_3(s_1-1, s_2-1) \\
& + 4s_1(s_1-1) \frac{1+x_1^2}{x_1 x_2} X_3(s_1-2, s_2) \\
& + 4s_2(s_2-1) \frac{1+x_2^2}{x_1 x_2} X_3(s_1, s_2-2).
\end{aligned}$$

By comparing the powers of x_1 and x_2 in each of the three partial derivatives of X_3 against the given bound for the corresponding $X_3(s_1 - a, s_2 - b)$, we see that integration by parts causes problems near zero, but will give us convergence on an integral which is bounded away from zero, so we now split the plane into four regions (nine total components) as x_1 and x_2 have magnitude smaller or larger than 1. On the region $|x_1| \le 1, |x_2| \le 1$, we do nothing, as this integral converges absolutely without our help. On the region $|x_1| \le 1, |x_2| > 1$, we integrate by parts in x_2. On the region $|x_1| > 1, |x_2| \le 1$, we integrate by parts in x_1. On the region $|x_1| > 1, |x_2| > 1$, we integrate by parts in both x_1 and x_2.

Note that the only dependence on u in the x integral is to position the center of the box $\mathcal{X}$. The integrals over the regions, after the approriate integration by parts, now converge absolutely (assuming $\epsilon < \frac{1}{4}$), hence the integral over the interior of the box is bounded and converges pointwise in u as $R \to \infty$. We have two types of boundary coming from the integration by parts: The first is the boundary of the box, whose integral is bounded and tends to 0 for each fixed u as $R \to \infty$, after the appropriate integration by parts. The second set are the boundaries of each of the above regions: For the lines $x_1 = \pm 1$, integrate by parts in x_1 to obtain an absolutely convergent integral, and for $x_2 = \pm 1$, integrate by parts in x_2; again, the integral over the portion of these lines which falls in the box now converges absolutely, hence is bounded and converges pointwise in u. Lastly, the value of X_3 at the intersection of the box and these lines is bounded and tends to zero pointwise in u. Thus, by dominated convergence, we may move the limit inside the s and u integrals to obtain an absolutely convergent integral.

After pulling the limit as $R \to \infty$ inside the u integral, the x and u integrals separate, with the u integral converging absolutely. Now we undo the substitution ${}^t x \mapsto y {}^t x y^{-1}$. To finish, we first note that a symmetry of the power function [39, Prop. 4.2.1 (4)]: With $\mu^{w_l} = (\mu_3, \mu_2, \mu_1)$, we have

$$p_\mu(xy) = p_\mu(y) = p_{-\mu^{w_l}}(w_l y^{-1}) = p_{-\mu^{w_l}}(w_l {}^t (xy)^{-1}),$$

so the Jacquet-Whittaker function may be written as

$$W(y^{-1}, -\mu^{w_l}, \psi) = \int_{U(\mathbb{R})} p_{\rho+\mu}({}^t x y) \psi(x) dx,$$

by sending $x \mapsto v x^{-1} v$ for $v = \begin{pmatrix} 1 & & \\ & -1 & \\ & & 1 \end{pmatrix}$ and noticing that $-\rho^{w_l} = \rho$. Thus the rearranged x integral (now consisting of integrals over nine regions, twelve lines, and four points in the x_1, x_2 plane) is, by construction, an analytic continuation of the Whittaker function to the double half-plane $\mathrm{Re}(\mu_1), \mathrm{Re}(\mu_2) > -\frac{1}{8}$.

Lastly, the product

$$W(z, -\mu, \psi_{11}) W(z', \mu, \psi_{11}) = \frac{W^*(z, -\mu, \psi_{11}) W^*(z', \mu, \psi_{11})}{\Lambda(-\mu)\Lambda(\mu)}$$

is permutation-invariant in μ, so we may replace $\mu^{w_l} \mapsto \mu$.

5.4 Trivial Element Term

Only occurs when $m = n$ and only for the $c = I$ term; the integral over $\overline{U}_w(\mathbb{R})$ is trivial as well. $C_w(y)$ is just 1 since we didn't actually do any substituting, $\alpha = I$, and $x^* = I$, $y^* = I$ since II is already of the form $x^* y^*$, so pulling the t integral inside in (5.3) (justified by the absolute convergence of the interchanged form) gives

$$H_w = -\frac{|m_1m_2n_1n_2|}{48\pi^7(m_1m_2)^2}\int_{\mathrm{Re}(\mu)=(0,0)}\frac{\hat{k}(\mu)}{|c_3(\mu)|^2\,\Lambda(\mu)\Lambda(-\mu)}$$
$$\int_{(\mathbb{R}^+)^2} W^*(t,-\mu,\psi_{11})W^*(t,\mu,\psi_{11})t_1^4t_2^2\,dt\,d\mu$$
$$= -\frac{1}{2^7 3\pi^4}\int_{\mathrm{Re}(\mu)=(0,0)}\frac{\hat{k}(\mu)}{|c_3(\mu)|^2\,\Lambda(\mu)\Lambda(-\mu)}\prod_{j<k}\frac{\frac{1}{2}(\mu_j-\mu_k)}{\sin\frac{\pi}{2}(\mu_j-\mu_k)}d\mu$$
$$= -\frac{1}{2^8 3}\int_{\mathrm{Re}(\mu)=(0,0)}\tilde{k}(\mu)\prod_{j<k}\frac{(\mu_k-\mu_j)^2}{\sqrt{9-(\mu_j-\mu_k)^2}}d\mu,$$

by Stade's formula.

5.5 Long Element Term

The computational data that is required is

$$\overline{U}_w(\mathbb{R}) = U(\mathbb{R}), \qquad C_w(y) = (y_1y_2)^2, \qquad t^w = \left(\frac{1}{t_2},\frac{1}{t_1}\right), \qquad u^w = (-u_2,-u_1)$$

$$\alpha_1 = \frac{c_2m_1n_2}{c_1^2}, \qquad \alpha_2 = \frac{c_1m_2n_1}{c_2^2},$$
$$x_1^* = -\frac{x_2'+x_1'x_3'}{1+x_2'^2+x_3'^2}, \qquad x_2^* = -\frac{x_1'+x_2'(x_1'x_2'-x_3')}{1+x_1'^2+(x_1'x_2'-x_3')^2},$$
$$y_1^* = \frac{\sqrt{1+x_1'^2+(x_1'x_2'-x_3')^2}}{1+x_2'^2+x_3'^2}, \qquad y_2^* = \frac{\sqrt{1+x_2'^2+x_3'^2}}{1+x_1'^2+(x_1'x_2'-x_3')^2},$$

so that we are evaluating

$$X'_{w_l}(u,v,\beta,t) = \int_{U(\mathbb{R})} e\left(-v_1\frac{\beta_1}{t_2}x_1^* - v_2\frac{\beta_2}{t_1}x_2^* + t_1x_1' + t_2x_2'\right)(y_1^*)^{1-u_1}(y_2^*)^{1-u_2}\,dx'.$$

We wish to separate the three x' variables, so we start by noticing that $(x_1'x_2'-x_3')^2 + x_1'^2+1 = (1+x_2'^2)x_1'^2 - 2x_1'x_2'x_3' + x_3'^2 + 1$; sending $x_1' \mapsto \frac{x_1'}{\sqrt{1+x_2'^2}}$ and $x_3' \mapsto x_3'\sqrt{1+x_2'^2}$ the expression becomes $(x_1' - x_2'x_3')^2 + x_3'^2 + 1$ and lastly we send $x_1' - x_2'x_3' \mapsto x_1'\sqrt{1+x_3'^2}$:

$$X'_{w_l} = \int_{U(\mathbb{R})} e\left(v_1 \frac{\beta_1}{t_2} \frac{x_2'}{1+x_2'^2} + v_1 \frac{\beta_1}{t_2} \frac{x_1'x_3'}{(1+x_2'^2)\sqrt{1+x_3'^2}} + v_2 \frac{\beta_2}{t_1} \frac{x_1'\sqrt{1+x_2'^2}}{(1+x_1'^2)\sqrt{1+x_3'^2}}\right)$$
$$e\left(t_1 \frac{x_2'x_3'}{\sqrt{1+x_2'^2}} + t_1 \frac{x_1'\sqrt{1+x_3'^2}}{\sqrt{1+x_2'^2}} + t_2 x_2'\right)$$
$$\left(1+x_1'^2\right)^{\frac{-1-u_1+2u_2}{2}} \left(1+x_2'^2\right)^{\frac{-1+2u_1-u_2}{2}} \left(1+x_3'^2\right)^{\frac{-1+u_1+u_2}{2}} dx'.$$

For each of the six terms in the exponential, we replace $e(\cdot) \mapsto e_\theta(\cdot)$ and apply its Mellin expansion (5.1) (interchange by absolute convergence):

$$X'_{w_l}(u,v,\beta,t,\theta)$$
$$= \frac{1}{(2\pi i)^6} \int_{\mathrm{Re}(r)=\nu} \int_{U(\mathbb{R})} (4\pi^2\beta_1)^{-r_1-r_2} (4\pi^2\beta_2)^{-r_3} (2\pi t_1)^{r_3-r_4-r_5} (2\pi t_2)^{r_1+r_2-r_6}$$
$$|x_1'|^{-r_2-r_3-r_5} |x_2'|^{-r_1-r_4-r_6} |x_3'|^{-r_2-r_4}$$
$$\exp -i\theta \left(r_1 v_1 \operatorname{sign}(x_2') + r_2 v_1 \operatorname{sign}(x_1'x_3') + r_3 v_2 \operatorname{sign}(x_1')\right)$$
$$\exp -i\theta \left(r_4 \operatorname{sign}(x_2'x_3') + r_5 \operatorname{sign}(x_1') + r_6 \operatorname{sign}(x_2')\right)$$
$$\left(1+x_1'^2\right)^{\frac{-1-u_1+2u_2+2r_3}{2}} \left(1+x_2'^2\right)^{\frac{-1+2u_1-u_2+2r_1+2r_2-r_3+r_4+r_5}{2}}$$
$$\left(1+x_3'^2\right)^{\frac{-1+u_1+u_2+r_2+r_3-r_5}{2}} dx' \left(\prod_{j=1}^{6} \Gamma(r_j)\right) dr.$$

Collecting by sign gives

$$X'_{w_l} = \frac{4}{(2\pi i)^6} \int_{\mathrm{Re}(r)=\nu} (4\pi^2\beta_1)^{-r_1-r_2} (4\pi^2\beta_2)^{-r_3} (2\pi t_1)^{r_3-r_4-r_5} (2\pi t_2)^{r_1+r_2-r_6}$$
$$\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)\Gamma(r_4)\Gamma(r_5)\Gamma(r_6) A'_{w_l}(r,v,\theta)$$
$$\int_{(\mathbb{R}^+)^3} x_1'^{-r_2-r_3-r_5} x_2'^{-r_1-r_4-r_6} x_3'^{-r_2-r_4} \left(1+x_1'^2\right)^{\frac{-1-u_1+2u_2+2r_3}{2}}$$
$$\left(1+x_2'^2\right)^{\frac{-1+2u_1-u_2+2r_1+2r_2-r_3+r_4+r_5}{2}} \left(1+x_3'^2\right)^{\frac{-1+u_1+u_2+r_2+r_3-r_5}{2}} dx'\, dr,$$

where

$$
\begin{aligned}
A'_{w_l} =& \frac{1}{4} \sum_{\varepsilon_1,\varepsilon_2,\varepsilon_3\in\{\pm1\}} \exp -i\theta \left(r_1 v_1 \varepsilon_2 + r_2 v_1 \varepsilon_1 \varepsilon_3 + r_3 v_2 \varepsilon_1 + r_4 \varepsilon_2 \varepsilon_3 + r_5 \varepsilon_1 + r_6 \varepsilon_2\right) \\
=& \cos\theta(r_2 v_1 + r_3 v_2 + r_5) \cos\theta(r_1 v_1 + r_4 + r_6) \\
&\qquad + \cos\theta(-r_2 v_1 + r_3 v_2 + r_5) \cos\theta(r_1 v_1 - r_4 + r_6)
\end{aligned}
$$

where $\nu_1, \ldots, \nu_6 = \epsilon$ are small compared to $\mathrm{Re}(u_1 - 2u_2) > \frac{1}{2}$, $\mathrm{Re}(-2u_1 + u_2) > \frac{1}{2}$, and $\mathrm{Re}(-u_1 - u_2) > \frac{1}{2}$. The inner integral may be evaluated by (5.2), so we have

$$
\begin{aligned}
X'_{w_l} = & \frac{1}{(2\pi i)^6} \int_{\mathrm{Re}(r)=\nu} (4\pi^2\beta_1)^{-r_1-r_2} (4\pi^2\beta_2)^{-r_3} (2\pi t_1)^{r_3-r_4-r_5} (2\pi t_2)^{r_1+r_2-r_6} \\
& A'_{w_l}(r, v, \theta) \\
& B\left(\frac{1-r_2-r_3-r_5}{2}, \frac{u_1-2u_2+r_2-r_3+r_5}{2}\right) \\
& B\left(\frac{1-r_1-r_4-r_6}{2}, \frac{-2u_1+u_2-r_1-2r_2+r_3-r_5+r_6}{2}\right) \\
& B\left(\frac{1-r_2-r_4}{2}, \frac{-u_1-u_2-r_3+r_4+r_5}{2}\right) dr,
\end{aligned}
$$

which converges absolutely for $\theta < \frac{\pi}{2}$ because of the exponential decay of the A'_{w_l} function.

Sending $(r_1 + r_2, r_3, r_3 - r_4 - r_5, r_1 + r_2 - r_6, r_2, r_4 - r_2) \mapsto (s_1, s_2, r_1, r_2, t_1, t_2)$, we may read off

$$
\begin{aligned}
T'_{w_l} =& \frac{4(4\pi)^{-s_1-s_2}}{(2\pi i)^2} \int_{\mathrm{Re}(t)=\eta_4} 2^{r_1+r_2} \Gamma\left(s_1 - t_1\right) \Gamma\left(t_1\right) \Gamma\left(s_2\right) \\
& \Gamma\left(t_1 + t_2\right) \Gamma\left(s_2 - t_1 - t_2 - r_1\right) \Gamma\left(s_1 - r_2\right) \\
& A'_{w_l}((s_1 - t_1, t_1, s_2, t_1 + t_2, s_2 - t_1 - t_2 - r_1, s_1 - r_2), v, \theta) \\
& B\left(\frac{1-2s_2+r_1+t_2}{2}, \frac{u_1-2u_2-r_1-t_2}{2}\right) \\
& B\left(\frac{1-2s_1+r_2-t_2}{2}, \frac{-2u_1+u_2+r_1-r_2+t_2}{2}\right) \\
& B\left(\frac{1-2t_1-t_2}{2}, \frac{-u_1-u_2-r_1}{2}\right) dt,
\end{aligned}
$$

and sending $1 + r - u^{w_l} \mapsto r$ and normalizing the powers of π in (5.5), we have

$$H_{w_l} = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\eta} \tilde{k}(\mu) J_{w_l,\mu}\left(\frac{4\pi^2 c_2 m_1 n_2}{c_1^2}, \frac{4\pi^2 c_1 m_2 n_1}{c_2^2}\right) d\mu,$$

where

$$J_{w_l,\mu}(y) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(s)=\mathfrak{s}} |y_1|^{-s_1} |y_2|^{-s_2} N_{w_l}(s, \mu, \operatorname{sign}(y))\, ds, \tag{5.6}$$

$$\begin{aligned}
&N_{w_l}(s,\mu,v) = \\
&\Bigg(\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(u)=\mathfrak{u}} G^*(u + (\mu_1, -\mu_2), \mu)\Gamma(s_2 - u_2 + \mu_2)\, T_{w_l,1}(s, u + (\mu_1, -\mu_2), v)\, du \\
&+ \frac{3}{2\pi i} \int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} G_l^*(1, (\mu_1, u_2 - \mu_2), \mu)\Gamma(s_2 - u_2 + \mu_2)\, T_{w_l,1}(s, (\mu_1, u_2 - \mu_2), v)\, du_2 \\
&+ \frac{3}{2\pi i} \int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} G_r^*(2, (u_1 + \mu_1, -\mu_2), \mu)\Gamma(s_2 + \mu_2)\, T_{w_l,1}(s, (u_1 + \mu_1, -\mu_2), v)\, du_1 \\
&+ 6G_b^*(1, 2, \mu)\Gamma(s_2 + \mu_2)\, T_{w_l,1}(s, (\mu_1, -\mu_2), v)\Bigg) \prod_{j<k} \frac{(\mu_k - \mu_j)\tan\frac{\pi}{2}(\mu_k - \mu_j)}{\sqrt{9 - (\mu_j - \mu_k)^2}},
\end{aligned}$$

$$\begin{aligned}
T_{w_l,1}(s,u,v) =& \frac{1}{2\pi i} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} \Gamma(t_1)\Gamma(s_1 - u_1 - t_1)\, R_{w_l}(s,u,v,t_1)\, dt_1, \\
R_{w_l}(s,u,v,t_1) =& \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(r)=\mathfrak{r}} G^*((3,1) + r, -\mu)\Gamma(1 + s_1 - r_2)\, T_{w_l,2}(r,s,u,v,t_1)\, dr,
\end{aligned}$$

$$\begin{aligned}
&T_{w_l,2}(r,s,u,v,t_1) = \\
&\frac{2^{u_1+u_2+r_1+r_2}}{48\pi^4(2\pi i)} \int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} A_{w_l}(r,s,t,u,v)\Gamma(t_1 + t_2)\Gamma(1 + s_2 - r_1 - t_1 - t_2) \\
&B\left(\frac{u_1 - 2s_2 + r_1 + t_2}{2}, \frac{1 + u_1 - u_2 - r_1 - t_2}{2}\right) \\
&B\left(\frac{u_1 - 2s_1 + r_2 - t_2}{2}, \frac{-u_1 + r_1 - r_2 + t_2}{2}\right) \\
&B\left(\frac{1 - 2t_1 - t_2}{2}, \frac{1 - u_1 - r_1}{2}\right) dt_2,
\end{aligned}$$

$$
\begin{aligned}
A_{w_l} = \cos\frac{\pi}{2}\left(t_1v_1+v_2(s_2-u_2)+s_2-u_2-t_1-t_2-r_1\right)& \\
\cos\frac{\pi}{2}\left(v_1(s_1-u_1-t_1)+t_1+t_2+s_1-u_1-r_2\right)& \\
+\cos\frac{\pi}{2}\left(-t_1v_1+v_2(s_2-u_2)+s_2-u_2-t_1-t_2-r_1\right)& \\
\cos\frac{\pi}{2}\left(v_1(s_1-u_1-t_1)-t_1-t_2+s_1-u_1-r_2\right),&
\end{aligned}
$$

and

$$
\eta = \left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right), \quad \mathfrak{u}=-\epsilon, \quad \mathfrak{s}=-\frac{1}{2}-\epsilon, \mathfrak{r}=\left(\frac{1}{2}-4\epsilon, 100\epsilon\right), \quad \mathfrak{t}=\epsilon
$$

are sufficient to maintain positivity of the arguments of all the gamma functions.

We should mention that this is the point where we run into difficulties using Stade's formula at $s=1$: Doing so would replace the $(3,1)$ in the argument of the second G function with $(1,0)$, and the conflicting inequalities are $\mathrm{Re}(r_2+\mu_1)>0$, $\mathrm{Re}(1+s_1-r_2)>0$, $\mathrm{Re}(t_1)>0$, $\mathrm{Re}(s_1-\mu_1-t_1)>0$, $\mathrm{Re}(s_1)<-\frac{1}{2}$; the last three imply $\mathrm{Re}(\mu_1)<-\frac{1}{2}$ and so the first two become $\mathrm{Re}(r_2)>\frac{1}{2}$ and $\mathrm{Re}(r_2)<\frac{1}{2}$. It may, however, be worthwhile using Stade's formula at say $1+O(\epsilon)$ to reduce the variation between the minimum and average exponents in the bounds of Proposition 34.

Additionally, the reader might be wondering how we are shifting contours around without justification: Since our contours are vertical lines in the complex plane, the requirement that the real part of the argument of every gamma function be greater than zero (or between -1 and 0 in the case of $G^*(u+(\mu_1,-\mu_2),\mu)$, etc.) forms a set of linear inequalities on the parameter space $\mathrm{Re}(\mu)$, $\mathrm{Re}(u)$, $\mathrm{Re}(r)$, $\mathrm{Re}(s)$, $\mathrm{Re}(t)$, hence defines an open, convex (therefore connected) subset of the same. Thus we have some finite process of shifting integrals to get between any two points in this region, and we need only know that both the start and end points are contained in the region and that the integrals each converge to a holomorphic function of the remaining variables on some compact set containing the two points and the path between them. For $\theta<\frac{\pi}{2}$, we have exponential decay in every variable, hence each integral grows at most polynomially, to be compensated by the exponential decay in the next (and we are assuming for the moment that $\hat{k}$ is Schwartz-class), thus every integral converges absolutely and uniformly on compact subsets of the above region, and we need only know that both the start and end points are contained therein.

5.6 The w_4 Term

The computational data that is required is

$$\overline{U}_w(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x_2 & x_3 \\ & 1 & 0 \\ & & 1 \end{pmatrix} : x_2, x_3 \in \mathbb{R} \right\},$$

$$C_w(y) = y_1 y_2^2, \quad t^w = \left(\frac{1}{t_1 t_2}, t_1 \right), \quad u^w = (u_2 - u_1, -u_1),$$

$$\alpha_1 = \frac{c_2 m_1 n_1 n_2}{c_1^2} = \frac{m_1 m_2^2 n_2}{c_2^3 n_1}, \qquad \alpha_2 = \frac{c_1 m_2}{c_2^2 n_1} = 1$$

$$x_1^* = \frac{x_3'}{1 + x_2'^2 + x_3'^2}, \qquad x_2^* = -\frac{x_2' x_3'}{1 + x_2'^2},$$

$$y_1^* = \frac{\sqrt{1 + x_2'^2}}{1 + x_2'^2 + x_3'^2}, \qquad y_2^* = \frac{\sqrt{1 + x_2'^2 + x_3'^2}}{1 + x_2'^2},$$

so that we are evaluating

$$X_{w_4}'(u, v, \beta, t) = \int_{\overline{U}_w(\mathbb{R})} e\left(-v_1 \frac{\beta_1}{t_1 t_2} x_1^* - t_1 x_2^* + t_2 x_2' \right) (y_1^*)^{1-u_1} (y_2^*)^{1-u_2} \, dx'.$$

Sending $x_3' \mapsto x_3' \sqrt{1 + x_2'^2}$ gives

$$X_{w_4}'(u, v, \beta, t) = \int_{\overline{U}_w(\mathbb{R})} e\left(-v_1 \frac{\beta_1}{t_1 t_2} \frac{x_3'}{\sqrt{1 + x_2'^2}(1 + x_3'^2)} + t_1 \frac{x_2' x_3'}{\sqrt{1 + x_2'^2}} + t_2 x_2' \right)$$
$$\left(1 + x_2'^2\right)^{\frac{-1+u_1+u_2}{2}} \left(1 + x_3'^2\right)^{\frac{-1+2u_1-u_2}{2}} \, dx'.$$

As above, we send $e(\cdot) \mapsto e_\theta(\cdot)$ and apply the Mellin expansion (5.1):

$$
\begin{aligned}
&X'_{w_4}(u,v,\beta,t,\theta)\\
&= \frac{1}{(2\pi i)^3}\int_{\mathrm{Re}(r)=\nu}\int_{\overline{U}_w(\mathbb{R})} (8\pi^3\beta_1)^{-r_1}(2\pi t_1)^{r_1-r_2}(2\pi t_2)^{r_1-r_3}\\
&\qquad \exp -i\theta\left(-r_1 v_1 \operatorname{sign}(x'_3) + r_2 \operatorname{sign}(x'_2 x'_3) + r_3 \operatorname{sign}(x'_2)\right)\\
&\qquad |x'_2|^{-r_2-r_3}\,|x'_3|^{-r_1-r_2}\left(1+x'^2_2\right)^{\frac{-1+u_1+u_2+r_1+r_2}{2}}\left(1+x'^2_3\right)^{\frac{-1+2u_1-u_2+2r_1}{2}}\,dx'\\
&\qquad \Gamma(r_1)\,\Gamma(r_2)\,\Gamma(r_3)\,dr.
\end{aligned}
$$

Splitting by sign and applying the Mellin transform (5.2) gives

$$
\begin{aligned}
X'_{w_4} = \frac{1}{(2\pi i)^3}\int_{\mathrm{Re}(r)=\nu} &(8\pi^3\beta_1)^{-r_1}(2\pi t_1)^{r_1-r_2}(2\pi t_2)^{r_1-r_3}\\
&\Gamma(r_1)\,\Gamma(r_2)\,\Gamma(r_3)\,A'_{w_4}(r,v,\theta)\\
&B\left(\frac{1-r_2-r_3}{2}, \frac{-u_1-u_2-r_1+r_3}{2}\right)\\
&B\left(\frac{1-r_1-r_2}{2}, \frac{-2u_1+u_2-r_1+r_2}{2}\right)dr,
\end{aligned}
$$

$$
\begin{aligned}
A'_{w_4}(r,v,\theta) =& \frac{1}{4}\sum_{\varepsilon_2,\varepsilon_3\in\{\pm 1\}} \exp -i\theta\left(-r_1 v_1 \varepsilon_3 + r_2\varepsilon_2\varepsilon_3 + r_3\varepsilon_2\right)\\
=& \cos\theta r_1 \cos\theta r_2 \cos\theta r_3 - i v_1 \sin\theta r_1 \sin\theta r_2 \sin\theta r_3.
\end{aligned}
$$

Sending $r_1 \mapsto s$, $r_2 \mapsto s - r_1$ and $r_3 \mapsto s - r_2$, we again read off

$$
\begin{aligned}
T'_{w_4} = (8\pi^2)^{r_1-s}&\Gamma(s)\,\Gamma(s-r_1)\,\Gamma(s-r_2)\,A'_{w_4}((s,s-r_1,s-r_2),v,\theta)\\
&B\left(\frac{1-2s+r_1+r_2}{2}, \frac{-u_1-u_2-r_2}{2}\right)\\
&B\left(\frac{1-2s+r_1}{2}, \frac{-2u_1+u_2-r_1}{2}\right)
\end{aligned}
$$

(here we have dropped the $\frac{1}{2\pi i}\int_{\mathrm{Re}(s_2)=\eta_2} ds_2$), and we have

$$H_{w_4} = \frac{1}{(2\pi i)^2 c_1 c_2}\int_{\mathrm{Re}(\mu)=\eta} \tilde{k}(\mu) J_{w_4,\mu}\left(\frac{8\pi^3 m_1 m_2^2 n_2}{c_2^3 n_1}\right) d\mu,$$

where

$$J_{w_4,\mu}(y) = \frac{1}{2\pi i}\int_{\mathrm{Re}(s)=\mathfrak{s}} |y|^{-s} N_{w_4}(s,\mu,\,\mathrm{sign}(y))\,ds, \tag{5.7}$$

$$\begin{aligned} N_{w_4}(s,\mu,v) =& \Bigg(\frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(u)=\mathfrak{u}} G^*(u+(\mu_1,-\mu_2),\mu) R_{w_4}(s,u+(\mu_1,-\mu_2),v)\,du \\ & \quad + \frac{3}{2\pi i}\int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} G_l^*(1,u_2,\mu) R_{w_4}(s,(\mu_1,u_2-\mu_2),v)\,du_2\Bigg) \\ & \prod_{j<k} \frac{(\mu_k-\mu_j)\tan\frac{\pi}{2}(\mu_k-\mu_j)}{\sqrt{9-(\mu_j-\mu_k)^2}}, \end{aligned}$$

$$\begin{aligned} R_{w_4}(s,u,v) =& \frac{2^{3r_1+3u_1}\pi^{2r_1+u_2}}{48\pi^4}\Gamma(s-u_1) \\ & \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(r)=\mathfrak{r}} 2^{r_1+r_2}\Gamma(s-u_1-r_1)\Gamma(s-u_1-r_2) A_{w_4}(r,s,u,v) \\ & \quad G^*((4+u_1-u_2+r_1, 2+u_1+r_2),-\mu) \\ & \quad B\left(\frac{1+2u_1-2s+r_1+r_2}{2}, \frac{-u_1-u_2-r_2}{2}\right) \\ & \quad B\left(\frac{1+2u_1-2s+r_1}{2}, \frac{-2u_1+u_2-r_1}{2}\right) dr, \end{aligned}$$

$$\begin{aligned} A_{w_4}(r,s,u,v) =& \cos\frac{\pi}{2}(s-u_1)\cos\frac{\pi}{2}(s-u_1-r_1)\cos\frac{\pi}{2}(s-u_1-r_2) \\ & - iv\sin\frac{\pi}{2}(s-u_1)\sin\frac{\pi}{2}(s-u_1-r_1)\sin\frac{\pi}{2}(s-u_1-r_2), \end{aligned}$$

and

$$\eta = (-2\epsilon, 0), \qquad \mathfrak{u} = (-\epsilon,\epsilon), \qquad \mathfrak{s} = -\epsilon, \qquad \mathfrak{r} = -\frac{1}{2}+3\epsilon$$

are sufficient to maintain positivity of the arguments of the gamma functions. We have chosen to reabsorb the poles in u_2 since they do not affect the asymptotics in y.

5.7 The w_5 Term

The computational data that is required is

$$\overline{U}_w(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & 0 & x_3 \\ & 1 & x_1 \\ & & 1 \end{pmatrix} : x_1, x_3 \in \mathbb{R} \right\},$$

$$C_w(y) = y_1^2 y_2, \quad t^w = \left(t_2, \frac{1}{t_1 t_2}\right), \quad u^w = (-u_2, u_1 - u_2),$$

$$\alpha_1 = \frac{c_2 m_1}{c_1^2 n_2} = 1, \qquad \alpha_2 = \frac{c_1 m_2 n_1 n_2}{c_2^2} = \frac{m_1^2 m_2 n_1}{c_1^3 n_2}$$

$$x_1^* = -\frac{x_1' x_3'}{1 + x_1'^2}, \qquad x_2^* = \frac{x_3'}{1 + x_1'^2 + x_3'^2},$$

$$y_1^* = \frac{\sqrt{1 + x_1'^2 + x_3'^2}}{1 + x_1'^2}, \qquad y_2^* = \frac{\sqrt{1 + x_1'^2}}{1 + x_1'^2 + x_3'^2},$$

so that we are evaluating

$$X_{w_5}'(u, v, \beta, t) = \int_{\overline{U}_w(\mathbb{R})} e\left(-t_2 x_1^* - v_2 \frac{\beta_2}{t_1 t_2} x_2^* + t_1 x_1'\right) (y_1^*)^{1-u_1} (y_2^*)^{1-u_2} \, dx'.$$

This matches X_{w_4}' with $x_2' \mapsto x_1'$, $\alpha_1 \mapsto \alpha_2$, and the coordinates of u and t permuted, so one may just propagate these changes, but we give the entire construction instead. Sending $x_3' \mapsto x_3' \sqrt{1 + x_1'^2}$ gives

$$X_{w_5}'(u, v, \beta, t) = \int_{\overline{U}_w(\mathbb{R})} e\left(t_2 \frac{x_1' x_3'}{\sqrt{1 + x_1'^2}} - v_2 \frac{\beta_2}{t_1 t_2} \frac{x_3'}{\sqrt{1 + x_1'^2}\,(1 + x_3'^2)} + t_1 x_1'\right)$$
$$\left(1 + x_1'^2\right)^{\frac{-1+u_1+u_2}{2}} \left(1 + x_3'^2\right)^{\frac{-1-u_1+2u_2}{2}} \, dx'.$$

As above, we send $e(\cdot) \mapsto e_\theta(\cdot)$ and apply the Mellin expansion (5.1):

$$\begin{aligned}
&X'_{w_5}(u,v,\beta,t,\theta) \\
&= \frac{1}{(2\pi i)^3} \int_{\mathrm{Re}(r)=\nu} \int_{\overline{U}_w(\mathbb{R})} (8\pi^3\beta_2)^{-r_2} (2\pi t_1)^{r_2-r_3} (2\pi t_2)^{-r_1+r_2} \\
&\quad \exp -i\theta \left(r_1 \operatorname{sign}(x'_1 x'_3) - r_2 v_2 \operatorname{sign}(x'_3) + r_3 \operatorname{sign}(x'_1)\right) \\
&\quad |x'_1|^{-r_1-r_3} |x'_3|^{-r_1-r_2} \left(1+x'^2_1\right)^{\frac{-1+u_1+u_2+r_1+r_2}{2}} \left(1+x'^2_3\right)^{\frac{-1-u_1+2u_2+2r_2}{2}} \, dx' \\
&\quad \Gamma(r_1)\Gamma(r_2)\Gamma(r_3)\, dr.
\end{aligned}$$

Splitting by sign and applying the Mellin transform (5.2) gives

$$\begin{aligned}
X'_{w_5} = \frac{1}{(2\pi i)^3} &\int_{\mathrm{Re}(r)=\nu} (8\pi^3\beta_2)^{-r_2} (2\pi t_1)^{r_2-r_3} (2\pi t_2)^{-r_1+r_2} \\
&\Gamma(r_1)\Gamma(r_2)\Gamma(r_3)\, A'_{w_5}(r,v,\theta) \\
&B\left(\frac{1-r_1-r_3}{2}, \frac{-u_1-u_2-r_2+r_3}{2}\right) \\
&B\left(\frac{1-r_1-r_2}{2}, \frac{u_1-2u_2+r_1-r_2}{2}\right) dr,
\end{aligned}$$

$$\begin{aligned}
A'_{w_5}(r,v,\theta) =& \frac{1}{4} \sum_{\varepsilon_1,\varepsilon_3 \in \{\pm 1\}} \exp -i\theta \left(r_1\varepsilon_1\varepsilon_3 - r_2 v_2 \varepsilon_3 + r_3\varepsilon_1\right) \\
=& \cos\theta r_1 \cos\theta r_2 \cos\theta r_3 - i v_2 \sin\theta r_1 \sin\theta r_2 \sin\theta r_3.
\end{aligned}$$

Sending $r_2 \mapsto s$, $r_1 \mapsto s - r_1$ and $r_3 \mapsto s - r_2$, we again read off

$$\begin{aligned}
T'_{w_5} = (8\pi^2)^{r_1-s}&\Gamma(s)\Gamma(s-r_1)\Gamma(s-r_2)\, A'_{w_5}((s, s-r_1, s-r_2), v, \theta) \\
&B\left(\frac{1-2s+r_1+r_2}{2}, \frac{-u_1-u_2-r_2}{2}\right) \\
&B\left(\frac{1-2s+r_1}{2}, \frac{u_1-2u_2-r_1}{2}\right),
\end{aligned}$$

and we have

$$H_{w_5} = \frac{1}{(2\pi i)^2 c_1 c_2} \int_{\mathrm{Re}(\mu)=\eta} \tilde{k}(\mu) J_{w_5,\mu}\left(\frac{8\pi^3 m_1^2 m_2 n_1}{c_1^3 n_2}\right) d\mu,$$

where

$$J_{w_5,\mu}(y) = \frac{1}{2\pi i} \int_{\mathrm{Re}(s)=\mathfrak{s}} |y|^{-s} N_{w_5}(s, \mu, \mathrm{sign}(y))\, ds, \tag{5.8}$$

$$\begin{aligned} N_{w_5}(s,\mu,v) = &\Bigg(\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(u)=\mathfrak{u}} G^*(u+(\mu_1,-\mu_2),\mu) R_{w_5}(s, u+(\mu_1,-\mu_2), v)\, du \\ &\quad + \frac{3}{2\pi i} \int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} G_r^*(2,u_1,\mu) R_{w_5}(s,(u_1+\mu_1,-\mu_2),v)\, du_1\Bigg) \\ &\prod_{j<k} \frac{(\mu_k-\mu_j)\tan\frac{\pi}{2}(\mu_k-\mu_j)}{\sqrt{9-(\mu_j-\mu_k)^2}}, \end{aligned}$$

$$\begin{aligned} R_{w_5}(s,u,v) = &-\frac{2^{3r_1+3u_2}\pi^{2r_1+u_1}}{48\pi^4}\Gamma(s-u_2) \\ &\frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(r)=\mathfrak{r}} 2^{r_1+r_2}\Gamma(s-u_2-r_1)\Gamma(s-u_2-r_2) A_{w_5}(r,s,u,v) \\ &\quad G^*((4+u_2+r_1, 2-u_1+u_2+r_2), -\mu) \\ &\quad B\left(\frac{1+2u_2-2s+r_1+r_2}{2}, \frac{-u_1-u_2-r_2}{2}\right) \\ &\quad B\left(\frac{1+2u_2-2s+r_1}{2}, \frac{u_1-2u_2-r_1}{2}\right) dr, \end{aligned}$$

$$\begin{aligned} A_{w_5}(r,s,u,v) = &\cos\frac{\pi}{2}(s-u_2-r_1)\cos\frac{\pi}{2}(s-u_2)\cos\frac{\pi}{2}(s-u_2-r_2) \\ &- iv\sin\frac{\pi}{2}(s-u_2-r_1)\sin\frac{\pi}{2}(s-u_2)\sin\frac{\pi}{2}(s-u_2-r_2), \end{aligned}$$

and

$$\eta = (0,-2\epsilon), \qquad \mathfrak{u} = (\epsilon,-\epsilon), \qquad \mathfrak{s} = -\epsilon, \qquad \mathfrak{r} = -\frac{1}{2}+3\epsilon$$

are sufficient to maintain positivity of the arguments of the gamma functions. Here we have reabsorbed the poles in u_1 as they do not affect the asymptotics in y.

5.8 Notes

(a) The above reasoning is sufficient to evaluate the trivial term of the Kuznetsov formula on $SL(n,\mathbb{R})$ for all n, provided one can justify the interchange of integrals in Lemma 35. Though this line of attack works well for the trivial term, it gets more difficult for the remaining terms. The author would like to point out a method of Zagier in his infamous "unpublished notes" for the Kuznetsov formula on $SL(2,\mathbb{R})$: He proceeded from the original formula by substituting $x' \mapsto x'x$ and then performing a substitution on xt to put $u = m^{-1}cwnx'$ in block diagonal form. For the positive discriminant case, one ends with an integral of essentially a Herz hypergeometric function. In the negative discriminant case, the author encountered an unexpected interaction with an off-diagonal term and was unable to complete the process. The purpose in mentioning this here is that up to that point the method appeared quite promising and readily generalizable; if one could overcome the technical difficulties, it should lead to formulas for the long-element term for the Kuznetsov formula on $SL(n,\mathbb{R})$ for general n. Also, Zagier's method does apply to the trivial term for all n. As the trivial and long-element terms tend to be the most important for applications, that would be quite useful.

(b) Again, assuming the interchange of integrals in Lemma 35 can be justified on $SL(n,\mathbb{R})$, the long-element weight function is the $SL(n,\mathbb{R})$ convolution

$$X'(\alpha,\mu) = \int_{G/K} W(z,-\mu,\psi_{11})W(\alpha w_l z,-\mu,\psi_{11})p_{11}(z)\,dz.$$

It would be nice to think that this satisfies some differential equation in α having a known solution. This would give $X'(\alpha,\mu) = g(\mu)f(\alpha,\mu)$ and later we will compute the limit

$$\lim_{\alpha\to 0} p_{-\rho-\mu^{w_l}}(\alpha)X'(\alpha,\mu) = \prod_{j<k} B\left(\frac{1}{2},\mu_j-\mu_k\right) \neq 0,$$

for $\operatorname{Re}(\mu_1), \operatorname{Re}(\mu_2) > 0$, which would fix the value of $g(\mu)$. The author did not have much luck finding such a differential equation, but still believes it should be related to the differential equations satisfied by the Whittaker function itself. (This would be obvious, except that we are *right*-translating α.)

(c) While we have made use of the gamma function to convert exponentials to powers (i.e. the Mellin expansion of the e_θ function), Stade was able to compute the Mellin transforms of the Jacquet-Whittaker functions by converting powers to exponentials (essentially the same trick in reverse), giving quadratics in the exponential terms, which can then be evaluated using the known Fourier transform of $\exp(-x^2)$. Attempting to do so here becomes complicated rather quickly.

Stade was also quite successful in applying the theory of Barnes integrals to reduce the number of extraneous integrals in the Mellin transform of the Whittaker functions. Again, we did not have any success with this method.

(d) One may reduce the number of extra integrals in the long-element weight function by 2 by sending $y \mapsto y\,|m|^{-1}$ and $y' \mapsto y'\,|n|^{-1}$ and integrating over both y and y' separately. (As opposed to sending $y \mapsto t\,|m|^{-1}$ and $y' \mapsto t\,|n|^{-1}$ and integrating over t.) This then requires finding an exponential decay factor in the weight function to compensate, which is somewhat difficult.

(e) It may be possible to attack the X' function as in [10] by writing

$$\begin{aligned} e\left(-\frac{\alpha_1}{t_2}x_1^* - \frac{\alpha_2}{t_1}x_2^*\right) =& 1 + \left(e\left(-\frac{\alpha_1}{t_2}x_1^*\right) - 1\right) + \left(e\left(-\frac{\alpha_2}{t_1}x_2^*\right) - 1\right) \\ &+ \left(e\left(-\frac{\alpha_1}{t_2}x_1^*\right) - 1\right)\left(e\left(-\frac{\alpha_2}{t_1}x_2^*\right) - 1\right), \end{aligned}$$

and simply bounding the resulting error terms directly. This could lead to a much cleaner derivation, if it is possible to use this method.

CHAPTER 6

Applications

6.1 Asymptotics of the $J_{w_l,\mu}$ Function

We want to achieve the highest power of the β variables possible – this gives the fastest convergence of the Kloosterman zeta function, so we want to move the s variables as negative as possible. As the s variables are indirectly bounded below by $(\mu_1, -\mu_2)$, any terms which allow us to cross below those lines will be considered small. Thus we only care about the $u = (0,0)$ residue in the N_{w_l} function. Then we shift the s integrals back, with poles at $s_1 = \mu_1 + t_1$ and $s_2 = -\mu_2$, and we shift the t_1 integral back, with a pole at $t_1 = 0$. So far, we have

$$J_{w_l,\mu}(y) \sim 6\, |y_1|^{-\mu_1} |y_2|^{\mu_2} G_b^*(1,2,\mu) \left(\prod_{j<k} \tan^2 \frac{\pi}{2}(\mu_k - \mu_j) \right) R_{w_l}((\mu_1,-\mu_2),(\mu_1,-\mu_2),v,0), \tag{6.1}$$

as $y \to 0$. This yields the error terms of Proposition 32:

$$\begin{aligned} E_{w_l,1}(\mu,y) =& \frac{3}{(2\pi i)^3} \left(\prod_{j<k} \tan^2 \frac{\pi}{2}(\mu_k - \mu_j) \right) \int_{\mathrm{Re}(s)=\mathfrak{s}} |y_1|^{-s_1} |y_2|^{-s_2} \\ & \int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} G_r^*(2,(u_1+\mu_1,-\mu_2),\mu) \Gamma\left(s_1 - u_1 - \mu_1\right) \\ & T_{w_l,1}(s,(u_1+\mu_1,-\mu_2),v)\, du_1\, ds, \end{aligned} \tag{6.2}$$

$$E_{w_l,2}(\mu,y) = \frac{3}{(2\pi i)^3}\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right)\int_{\mathrm{Re}(s)=\mathfrak{s}}|y_1|^{-s_1}|y_2|^{-s_2} \tag{6.3}$$
$$\int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} G_l^*(1,(\mu_1,u_2-\mu_2),\mu)\Gamma(s_1-\mu_1)$$
$$T_{w_l,1}(s,(\mu_1,u_2-\mu_2),v)\,du_2\,ds,$$

$$E_{w_l,3}(\mu,y) = \frac{1}{(2\pi i)^4}\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right)\int_{\mathrm{Re}(s)=\mathfrak{s}}|y_1|^{-s_1}|y_2|^{-s_2} \tag{6.4}$$
$$\int_{\mathrm{Re}(u)=\mathfrak{u}} G^*(u+(\mu_1,-\mu_2),\mu)\Gamma(s_1-u_1-\mu_1)$$
$$T_{w_l,1}(s,u+(\mu_1,-\mu_2),v)\,du\,ds,$$

$$E_{w_l,4}(\mu,y) = \frac{6}{2\pi i}G_b^*(1,2,\mu)\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right) \tag{6.5}$$
$$\int_{\mathrm{Re}(s_1)=\mathrm{Re}(\mu_1)-\epsilon}|y_1|^{-s_1}|y_2|^{\mu_2}$$
$$T_{w_l,1}((s_1,\mu_2),(\mu_1,-\mu_2),\,\mathrm{sign}(y))\,ds_2,$$

$$E_{w_l,5}(\mu,y) = \frac{6}{(2\pi i)^2}G_b^*(1,2,\mu)\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right) \tag{6.6}$$
$$\int_{\mathrm{Re}(s_2)=\mathrm{Re}(-\mu_2)-\epsilon}|y_2|^{-s_2}\Gamma(s_2+\mu_2)\int_{\mathrm{Re}(t_1)=\mathfrak{t}_1}$$
$$|y_1|^{-\mu_1-t_1}\Gamma(t_1)R_{w_l}((\mu_1+t_1,s_2),(\mu_1,-\mu_2),v,t_1)\,dt_1\,ds_1,$$

$$E_{w_l,6}(\mu,y) = \frac{6}{(2\pi i)^2}G_b^*(1,2,\mu)\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right) \tag{6.7}$$
$$\int_{\mathrm{Re}(s)=\mathrm{Re}(\mu_1,-\mu_2)-\epsilon}|y_1|^{-s_1}|y_2|^{-s_2}$$
$$\Gamma(s_1-\mu_1)T_{w_l,1}(s,(\mu_1,-\mu_2),\,\mathrm{sign}(y))\,ds,$$

$$E_{w_l,7}(\mu,y) = \frac{6}{2\pi i} G_b^*(1,2,\mu)\left(\prod_{j<k}\tan^2\frac{\pi}{2}(\mu_k-\mu_j)\right) \tag{6.8}$$

$$|y_1|^{-\mu_1}\int_{\mathrm{Re}(t_1)=-\epsilon}|y_2|^{\mu_2-t_1}\,\Gamma(t_1) \tag{6.9}$$

$$R_{w_l}((\mu_1+t_1,-\mu_2),(\mu_1,-\mu_2),v,t_1)\,dt_1.$$

Returning to (5.4), we may compute the main term explicitly. We have $J'_{w_l,\mu}(y) = J_{w_l,\mu}(4\pi^2 y)$, so for $\mathrm{Re}(\mu) = \left(-\frac{1}{2}-9\epsilon, \frac{1}{2}+9\epsilon\right)$,

$$\begin{aligned}
&\lim_{y\to 0}|y_1|^{\mu_1}|y_2|^{-\mu_2}J_{w_l,\mu}(4\pi^2 y)\\
&=\lim_{y\to 0}|y_1|^{\mu_1}|y_2|^{-\mu_2}J'_{w_l,\mu}(y)\\
&=\frac{6}{12\pi^6}\left(\prod_{j<k}\frac{(\mu_k-\mu_j)\tan\frac{\pi}{2}(\mu_k-\mu_j)}{\sqrt{9-(\mu_j-\mu_k)^2}}\right)G_b^*(1,2,\mu)\int_{(\mathbb{R}^+)^2}W(t,-\mu,\psi_{11})\\
&\qquad(\pi t_1^{w_l})^{-\mu_1}(\pi t_2^{w_l})^{\mu_2}\lim_{y\to 0}X'_{w_l}((\mu_1,-\mu_2),\mathrm{sign}(y),|y|,t)(\pi t_1)^5(\pi t_2)^3\,dt,
\end{aligned}$$

by dominated convergence. The limit in X'_{w_l} is actually a Whittaker function,

$$\begin{aligned}
\lim_{y\to 0}X'_{w_l}((\mu_1,-\mu_2),\mathrm{sign}(y),|y|,t) &= \int_{\overline{U}_w(\mathbb{R})}\overline{\psi_t}(x')y_1^{*1-\mu_1}y_2^{*1+\mu_2}dx'\\
&=W(I,(\mu_2,\mu_3,\mu_1),\psi_t)\\
&=t_1^{-1+\mu_2}t_2^{-1-\mu_1}W(t,(\mu_2,\mu_3,\mu_1),\psi_{11}),
\end{aligned}$$

again, by dominated convergence. Applying this to the limit of $J_{w_l,\mu}$ gives

$$\lim_{y\to 0} |y_1|^{\mu_1} |y_2|^{-\mu_2} J_{w_l,\mu}(4\pi^2 y)$$
$$= \frac{\pi^{2-\mu_1+\mu_2}}{2} \left(\prod_{j<k} \frac{(\mu_k-\mu_j)\tan\frac{\pi}{2}(\mu_k-\mu_j)}{\sqrt{9-(\mu_j-\mu_k)^2}} \right) G_b^*(1,2,\mu)$$
$$\int_{(\mathbb{R}^+)^2} W(t,-\mu,\psi_{11}) W(t,(\mu_2,\mu_3,\mu_1),\psi_{11}) t_1^4 t_2^2 \, dt$$
$$= \frac{\pi^{-\mu_1+\mu_2}}{16\pi} \left(\prod_{j<k} \frac{\frac{1}{2}(\mu_k-\mu_j)^2}{\sqrt{9-(\mu_j-\mu_k)^2}} \right) G_b^*(1,2,\mu) \frac{\Lambda(\mu)}{\Lambda(\mu_2,\mu_3,\mu_1)}$$
$$= \frac{\pi^{\frac{1}{2}+\mu_2-\mu_3}}{4} \frac{\Gamma\left(\frac{\mu_1-\mu_2}{2}\right)\Gamma\left(\frac{\mu_1-\mu_3}{2}\right)\Gamma\left(\frac{\mu_3-\mu_2}{2}\right)}{\Gamma\left(\frac{1+\mu_2-\mu_3}{2}\right)\Gamma\left(\frac{1+\mu_3-\mu_1}{2}\right)\Gamma\left(\frac{1+\mu_2-\mu_1}{2}\right)} \prod_{j<k} \frac{\frac{1}{2}(\mu_k-\mu_j)^2}{\sqrt{9-(\mu_j-\mu_k)^2}},$$

thus $J_{w_l,\mu}(y) \sim |y_1|^{-\mu_1} |y_2|^{\mu_2} K_{w_l}(\mu)$. This expression then agrees with right hand side of (6.1) over the entire range of holomorphy by analytic continuation and we have Proposition 32. Note that we induced an asymmetry in the original definition of the $J'_{w_l,\mu}$ function, hence the asymmetry here; this is a subtle but important point as it allows us to avoid some symmetry requirements for the test functions of Theorem 33 and Theorem 5.

6.2 Partial Inversion Formula

If we take our test function to be (4.2) then in H_{w_l}, we move $\mathrm{Re}(q) \mapsto \mathrm{Re}(\mu_1,-\mu_2)+\epsilon$, and apply the asymptotics of $J_{w_l,\mu}$ at the double residue $q=(\mu_1,-\mu_2)$ gives Theorem 33 with

$$F_1(\hat{f};y) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(\mu)=\eta} \hat{k}_{(\eta_1,-\eta_2)+\epsilon}(\mu) J_{w_l,\mu}(y) \, d\mu, \tag{6.10}$$

$$F_2(\hat{f};y) = \frac{1}{(2\pi i)^2} \int_{\mathrm{Re}(\mu)=\eta} \int_{\mathrm{Re}(q_2)=-\eta_2+\epsilon} \hat{f}(\mu_1,q_2) \frac{J_{w_l,\mu}(y)}{K_{w_l}(\mu,-q_2)} \tag{6.11}$$
$$\frac{(q_2+\mu_1)(2\mu_1-q_2)(2q_2-\mu_1)}{(\mu_1-\mu_2)(\mu_1-\mu_3)(q_2+\mu_2)(q_2+\mu_3)} \, dq_2 \, d\mu,$$

$$F_3(\hat{f};y) = \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\mu)=\eta}\int_{\mathrm{Re}(q_1)=\eta_1+\epsilon} \hat{f}(q_1,-\mu_2)\frac{J_{w_l,\mu}(y)}{K_{w_l}(q_1,\mu_2)} \tag{6.12}$$
$$\frac{(q_1-\mu_2)\,(2q_1+\mu_2)\,(2\mu_2-q_1)}{(q_1-\mu_1)\,(q_1-\mu_3)\,(\mu_1-\mu_2)\,(\mu_3-\mu_2)}\,dq_1\,d\mu.$$

$$F_{j+3}(\hat{f};y) = \frac{1}{(2\pi i)^2}\int_{\mathrm{Re}(\mu)=\eta} \hat{f}(\mu_1,-\mu_2)\frac{E_{w_l,j}(\mu,y)}{K_{w_l}(\mu)}d\mu. \tag{6.13}$$

It may be possible to study the Kloosterman zeta functions directly by simply not integrating over q in $\hat{k}$; this would require a test function $\hat{k}$ which cancels the intermediate terms in H_{w_l} and J_{w_l} (the terms with a residue at one of q_1 or q_2, but not both, and the terms with a residue at one of s_1 or s_2, but not both, and the term with a residue in t_1).

6.3 Sums of Kloosterman Sums

Let $g(y) = f(Xy_1, Yy_2)$, then the assumption that f have compact support is not strictly necessary, we merely need holomorphy of $\hat{g}$ on $\mathrm{Re}(q_1), \mathrm{Re}(q_2) \in \left(-\frac{1}{2}-\epsilon, -\epsilon\right)$ and the bound

$$\hat{g}(q) \ll \frac{X^{-\mathrm{Re}(q_1)}}{|q_1|^8}\frac{Y^{-\mathrm{Re}(q_2)}}{|q_2|^8},$$

which follows by integration by parts eight times in each y variable. Theorem 5 follows from Theorem 33 by fixing the contours of the error terms and those of the cusp form terms, Eisenstein series terms, and non-long-element Kloosterman sum terms and justifying their absolute convergence in the new locations. Specifically, we want to shift the contours in q as far to the right as possible.

For the cusp form terms in (2.1), we may shift the q contours of $\hat{k}$ up to $\mathfrak{q} = -\frac{5}{14}-\epsilon$ without encountering poles at any of the $q_1-\mu_i$ or $q_2+\mu_i$ terms, thanks to the Kim-Sarnak result. The $K_{w_l}(q_1,-q_2)$ term has poles at $-q_1-q_2=0$, $-2q_1+q_2=0$ and $q_1-2q_2=0$, but we need not encounter these and they are cancelled by the terms in the numerator as well. Now the mean-value estimates of Theorem 23 show that the sum over the cusp forms

j	$\mathfrak{q}$	η	$\mathfrak{u}$	$\mathfrak{r}$	$\mathfrak{t}$
1	$-\epsilon$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\epsilon$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
2	$(-, -\epsilon)$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\epsilon$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
3	$(-\epsilon, -)$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\epsilon$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
4	–	$\left(-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$\left(-\frac{1}{2}, -\right)$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
5	–	$\left(-\frac{1}{2}-3\epsilon, 3\epsilon\right)$	$\left(-, -\frac{1}{2}\right)$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
6	–	$(-3\epsilon, 3\epsilon)$	$-\frac{1}{2}$	$\left(\frac{1}{2}-4\epsilon, 100\epsilon\right)$	ϵ
7	–	$\left(-\epsilon, \frac{1}{2}+3\epsilon\right)$	–	$(-4\epsilon, 100\epsilon)$	$\left(\epsilon, \frac{1}{2}-\epsilon\right)$
8	–	$\left(-\frac{1}{2}-3\epsilon, \epsilon\right)$	–	$(-3\epsilon, 7\epsilon)$	$\left(6\epsilon, \frac{1}{2}-6\epsilon\right)$
9	–	$(-\epsilon, \epsilon)$	–	0	ϵ
10	–	$\left(-\epsilon, \frac{1}{2}+3\epsilon\right)$	–	$(\epsilon, 4\epsilon)$	$\left(-\frac{1}{2}+5\epsilon, 1-10\epsilon\right)$

Table 6.1: Contours for the F_j error terms.

converges absolutely so we have the bound $(XY)^{\frac{5}{14}+\epsilon}$ here.

The terms in (2.1) for both types of Eisenstein series have $\hat{k}$ evaluated at $\mathrm{Re}(\mu) = 0$, as does the trivial Weyl element term, and the sums of Kloosterman sums at the w_4 and w_5 Weyl elements still converge absolutely at $\mathrm{Re}(\mu) = (-\epsilon, \epsilon)$ so for each of these terms we may shift the q contours to $\mathfrak{q} = -2\epsilon$. Again, absolute convergence gives $(XY)^{2\epsilon}$.

In the section on bounds, we will show the F_j error terms of Theorem 33 are all bounded by $(XY)^{10\epsilon}(X^{\frac{1}{2}} + Y^{\frac{1}{2}})$ by taking the contours as in Table 6.1, with $\mathfrak{s} = -\frac{1}{2} - \epsilon$. Lastly, we keep

$$\eta = (-2\epsilon, 0)\,, \qquad \mathfrak{u} = (-\epsilon, \epsilon)\,, \qquad \mathfrak{s} = -\epsilon, \qquad \mathfrak{r} = -\frac{1}{2} + 3\epsilon$$

for $J_{w_4,\mu}$ and

$$\eta = (0, -2\epsilon)\,, \qquad \mathfrak{u} = (\epsilon, -\epsilon)\,, \qquad \mathfrak{s} = -\epsilon, \qquad \mathfrak{r} = -\frac{1}{2} + 3\epsilon$$

for $J_{w_5,\mu}$. Our choice of $\mathfrak{s}$ maintains the absolute convergence of the Kloosterman zeta functions and the exponent on the bounds come from $\mathfrak{q}$. The choice of contours for $J_{w_4,\mu}$ and $J_{w_5,\mu}$ are primarily driven by a desire for convenience in the bounds section.

APPENDIX A

Absolute convergence of the Kloosterman zeta functions

This appendix gives the proof of Proposition 25. For convenience, we set

$$Z_{w_4}(\psi_m, \psi_n, u) = \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_4}(\psi_m, \psi_n, c)|}{c_1 c_2} \left(\frac{c_2^3 \, |n_1|}{|m_1 m_2^2 n_2|} \right)^u$$
$$Z_{w_5}(\psi_m, \psi_n, u) = \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_5}(\psi_m, \psi_n, c)|}{c_1 c_2} \left(\frac{c_1^3 \, |n_2|}{|m_1^2 m_2 n_1|} \right)^u$$
$$Z_{w_l}(\psi_m, \psi_n, u) = \sum_{c_1, c_2 \in \mathbb{N}} \frac{|S_{w_l}(\psi_m, \psi_n, c)|}{c_1 c_2} \left(\frac{c_1^2}{c_2 \, |m_1 n_2|} \right)^{u_1} \left(\frac{c_2^2}{c_1 \, |m_2 n_1|} \right)^{u_2},$$

and assume $u, u_1, u_2 \in \mathbb{R}$.

A.1 The Intermediate Kloosterman Zeta Functions

We first prove Proposition 25 part a; part b will follow by symmetry. Note that the w_4 Kloosterman sum is defined to be zero unless $c_2 | c_1$ and $n_1 c_2^2 = m_2 c_1$, so applying Larsen's bound gives

$$\left| m_1 m_2^2 n_2 \right|^u |n_1|^{-u} Z_{w_4}(\psi_M, \psi_N, u_1, u_2) \leq \sum_{\substack{c_2 | c_1 \\ n_1 c_2^2 = m_2 c_1}} c_1^{\epsilon} c_2^{3u-1} \left(|m_1|, |n_2|, c_2 \right),$$

and the sum is empty unless $\operatorname{sign}(n_1) = \operatorname{sign}(m_2)$.

We decompose the summation conditions as follows: Let $d = (c_1, c_2^2)$, then $c_2 | d$, so let $e = \frac{d}{c_2}$. Then $1 = \left(\frac{c_1}{e c_2}, \frac{c_2}{e} \right)$ and $n_1 \frac{c_2}{e} = n_2 \frac{c_1}{e c_2}$ so let $f = \frac{|n_1|}{c_1/(e c_2)} = \frac{|m_2|}{c_2/e}$. Note that $c_2 = \frac{|m_2| e}{f}$,

$c_1 = \frac{|n_1| e c_2}{f} = \frac{n_1 m_2 e^2}{f^2}$, and so also $f = (|n_1|, |m_2|)$. Applying what we have so far gives

$$\begin{aligned} &|m_1 n_2|^u |m_2|^{1-u-\epsilon} |n_1|^{-u-\epsilon} Z_{w_4}(\psi_M, \psi_N, u_1, u_2) \\ &\leq (|n_1|, |m_2|)^{-3u+1-2\epsilon} \sum_{e \in \mathbb{N}} e^{3u-1+2\epsilon} \left(|m_1|, |n_2|, \frac{|m_2|}{(|n_1|, |m_2|)} e \right). \end{aligned}$$

Let $g = \left(|m_1|, |n_2|, \frac{|m_2|}{(|n_1|,|m_2|)} \right)$, then take $e = hi$ where $h = \left(\frac{|m_2|}{g}, \frac{|n_2|}{g}, e \right)$, then we have

$$\begin{aligned} &|m_1 n_2|^u |m_2|^{1-u-\epsilon} |n_1|^{-u-\epsilon} (|n_1|, |m_2|)^{3u-1+2\epsilon} Z_{w_4}(\psi_M, \psi_N, u_1, u_2) \\ &\leq g \sum_{h | \left(\frac{|m_1|}{g}, \frac{|n_2|}{g} \right)} h^{3u+2\epsilon} \sum_{i \in \mathbb{N}} i^{3u-1+2\epsilon}, \end{aligned}$$

which clearly requires $u < 0$, and in turn allows us to trivially estimate the divisor sum:

$$Z_{w_4}(\psi_m, \psi_n, u) \ll \frac{(|n_1|, |m_2|)^{1-3u-2\epsilon} \left(|m_1|, |n_2|, \frac{|m_2|}{(|n_1|,|m_2|)} \right)}{|m_1 n_2|^{u-\epsilon} |m_2|^{1-u-\epsilon} |n_1|^{-u-\epsilon}}.$$

A.2 The Long Element Kloosterman Zeta Function

Applying Stevens' bound to the partial long-element Kloosterman zeta function gives

$$\begin{aligned} &|m_1 n_2|^{u_1} |m_2 n_1|^{u_2} Z_{w_l}(\psi_m, \psi_n, u) \\ &\leq \sum_{c_1, c_2 \in \mathbb{N}} c_1^{-1/2+2u_1-u_1+\epsilon} c_2^{-1/2+2u_2-u_1+\epsilon} (c_1, c_2)^{1/2} \\ &\qquad \left(|m_1 n_2|, \frac{c_1 c_2}{(c_1, c_2)} \right)^{1/2} \left(|m_2 n_1|, \frac{c_1 c_2}{(c_1, c_2)} \right)^{1/2} \\ &\leq \sum_{c_1, c_2 \in \mathbb{N}} c_1^{-1/2+2u_1-u_2+\epsilon} c_2^{-1/2+2u_2-u_1+\epsilon} (c_1, c_2)^{1/2} \left(|m_1 m_2 n_1 n_2|, \frac{c_1 c_2}{(c_1, c_2)} \right). \end{aligned}$$

Let $c_1 = ab$, $c_2 = bd$, with $(a, d) = 1$. Refining somewhat, we let $a = a_1 a_2$ with $a_2 | D$, $\left(a_1, \frac{D}{a_2} \right) = 1$,$d = d_1 d_2$ with $d_2 | D$, $\left(d_1, \frac{D}{d_2} \right) = 1$, and $b = b_1 b_2$ with $b_2 | \frac{D}{a_2 d_2}$, $\left(b_1, \frac{D}{a_2 b_2 d_2} \right) = 1$, then

$$
\begin{aligned}
&|m_1 n_2|^{u_1} |m_2 n_1|^{u_2} Z_{w_l}(\psi_m, \psi_n, u) \\
&\le \sum_{a_2 b_2 d_2 | D} a_2^{1/2+2u_1-u_2+\epsilon} b_2^{1/2+u_1+u_2+2\epsilon} d_2^{1/2+u_2+\epsilon} \\
&\qquad \sum_{a_1, b_1, d_1 \in \mathbb{N}} a_1^{-1/2+2u_1-u_2+\epsilon} b_1^{-1/2+u_1+u_2+2\epsilon} d_1^{-1/2+2u_2-u_1+\epsilon},
\end{aligned}
$$

The series clearly converge exactly when $\max\{2u_1 - u_2, 2u_2 - u_1\} < -\frac{1}{2}$ (which implies $u_1, u_2 < -\frac{1}{2}$), and the divisor sum is bounded by $d_4(D)$ and can be rolled into the D^ϵ, giving the bound in Proposition 25 part c, and the proposition is complete.

APPENDIX B

Bounds for the Mellin-Barnes Integrals

We are left with two items to prove, which are essentially the same: First, completing Theorem 31 requires justifying the growth hypothesis on $\hat{k}$, in other words, bounding $J_{w,\mu}$, which is also desirable for Theorem 23. Second, the evaluation of the integral transforms, the asymptotics of $E_{w_l,j}$ and F_j given in Proposition 32 and Theorem 33, and the completion of Theorem 5 all require absolute convergence of the Mellin-Barnes integrals.

It is difficult to obtain a general bound for N_{w_l} and J_{w_l} that works for all ranges of the η and $\mathfrak{s}$ parameters, hence it is also difficult to show that these functions converge absolutely over the entire range of holomorphy. Therefore, we will not actually show that these functions are holomorphic over the given ranges. This leads one to question whether it is valid to shift contours as we have freely done; for the skeptical reader, we have a simple justification: Do the shifting before taking the limit in θ back in the original construction. As we have the bound $A'_{w_l} \ll \prod_{j=1}^6 |r_j|^{\mathrm{Re}(r_j)-\frac{1}{2}} \exp\left(\left(\theta - \frac{\pi}{2}\right)|\mathrm{Im}(r_j)|\right)$, both convergence and the validity of the shifts are obvious. Then we only require that the end product converges absolutely at $\theta = \frac{\pi}{2}$, and that is what we will show.

The fundamental asymptotic here is Stirling's formula: For $\mathrm{Re}(z)$ in a compact subset of $\mathbb{R}$ (not containing a pole of the gamma function),

$$|\Gamma(z)| \sim \sqrt{2\pi}\, |z|^{\mathrm{Re}(z)-\frac{1}{2}}\, e^{-\frac{\pi}{2}|\mathrm{Im}(z)|},$$

which leads us to integrals of products in the form

$$\int_{\mathrm{Re}(u)=\mathfrak{u}} \prod_i |a_{i,1}u_1 + \ldots + a_{i,n}u_n + b_{i,1}v_1 + \ldots + b_{i,m}v_m|^{c_i}\, du, \tag{B.1}$$

where $a_{i,j}, \mathfrak{u}_i, c_i \in \mathbb{R}$ are fixed, with $v_i \in \mathbb{C}$ having fixed real part. Note that for a and c non-zero and fixed, $b \in \mathbb{R}$, we have $|a+bi| \asymp |c+bi|$. Provided the exponents are not somehow accumulating on any subspace, we would expect such an integral to converge when $\sum_i c_i < -n-1$, and we give a series of lemmas designed to show that these converge in our situation.

Bounds for integrals of the above type are derived from Hölder's inequality and the following lemma:

Lemma 36. *Suppose $a_1 + a_2 < -1$ with a_1 and a_2 fixed, and $s \geq 0$, then*

$$\int_{-\infty}^{\infty} |1+i(s+t)|^{a_1} |1+i(s-t)|^{a_2} \, dt \ll |1+is|^{\max\{a_1,a_2,a_1+a_2+1\}}.$$

We will occasionally encounter positive exponents in the integrals of type (B.1), but thankfully these always occur in the terms coming from the beta functions, so we have

Lemma 37. *Suppose $\mathrm{Re}(v) = \mathfrak{v}$ with $\mathfrak{v}_1 + \mathfrak{u}$, $\mathfrak{v}_2 - \mathfrak{u}$ not non-positive integers, and $\mathfrak{u}$, $\mathfrak{v}$ fixed, and $p > 0$, then*

$$\left(\int_{\mathrm{Re}(u)=\mathfrak{u}} |B(v_1+u, v_2-u)|^p \, |du| \right)^{\frac{1}{p}} \ll |v_1+v_2|^{\max\left\{\mathfrak{u}-\mathfrak{v}_2, -\mathfrak{u}-\mathfrak{v}_1, \frac{1}{p}-\frac{1}{2}\right\}}.$$

Note that this no longer requires $\mathrm{Re}(v_1 + v_2 - 1) < -1$ as it would if we applied the previous lemma; this is because we are using the exponential decay of the gamma functions.

Increasing in complexity, we have bounds for the G^* function and its residues:

Lemma 38. *(a)*

$$G_b^*(1,2,\mu) \ll \frac{|\mu_1-\mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}} |\mu_1-\mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)-1}{2}} |\mu_3-\mu_2|^{\frac{\mathrm{Re}(\mu_3-\mu_2)-1}{2}}}{|\mu_1-\mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)}{2}} |\mu_1-\mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)}{2}} |\mu_3-\mu_2|^{\frac{\mathrm{Re}(\mu_2-\mu_3)}{2}}},$$

(b) *Suppose* $\mathfrak{u}_2 + \mathrm{Re}(\mu_2), \mathfrak{u}_2 + \mathrm{Re}(\mu_3) > -1$, *then*

$$\int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} |G_l^*(1, u_2, \mu)|\, |du_2| \\ \ll \frac{|\mu_1 - \mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}} |\mu_1 - \mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)-1}{2}} |\mu_2 - \mu_3|^{\mathfrak{u}_2 + \frac{\mathrm{Re}(\mu_2+\mu_3)}{2}}}{|\mu_1 - \mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)}{2}} |\mu_1 - \mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)}{2}} |\mu_3 - \mu_2|^{\frac{\mathrm{Re}(\mu_2-\mu_3)}{2}}},$$

(c) *Suppose* $\mathfrak{u}_1 - \mathrm{Re}(\mu_1), \mathfrak{u}_1 - \mathrm{Re}(\mu_3) > -1$, *then*

$$\int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} |G_r^*(2, u_1, \mu)|\, |du_1| \\ \ll \frac{|\mu_1 - \mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}} |\mu_1 - \mu_3|^{\mathfrak{u}_1 - \frac{\mathrm{Re}(\mu_1+\mu_3)}{2}} |\mu_2 - \mu_3|^{\frac{\mathrm{Re}(\mu_3-\mu_2)-1}{2}}}{|\mu_1 - \mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)}{2}} |\mu_1 - \mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)}{2}} |\mu_3 - \mu_2|^{\frac{\mathrm{Re}(\mu_2-\mu_3)}{2}}},$$

(d) *Suppose* $\mathfrak{u}_1 - \mathrm{Re}(\mu_i), \mathfrak{u}_2 + \mathrm{Re}(\mu_i) > -1 - \epsilon$, *then*

$$\int_{\mathrm{Re}(u)=\mathfrak{u}} |G^*(u, \mu)|\, |du| \\ \ll \frac{\sum_{w \in W} |\mu_3^w - \mu_1^w|^{\frac{\kappa-\delta}{2}+\epsilon} |\mu_3^w - \mu_2^w|^{\frac{\kappa-\delta}{2}+\epsilon} |\mu_2^w - \mu_1^w|^{\delta+\epsilon}}{|\mu_1 - \mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)}{2}} |\mu_1 - \mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)}{2}} |\mu_3 - \mu_2|^{\frac{\mathrm{Re}(\mu_2-\mu_3)}{2}}},$$

where $\kappa = \mathfrak{u}_1 + \mathfrak{u}_2 - \frac{1}{2}$, *and*

$$\delta = \min_i \left\{ \frac{\kappa}{3}, \frac{2\mathfrak{u}_1 + \mathrm{Re}(\mu_i)}{2}, \frac{\mathfrak{u}_1 - \mathrm{Re}(\mu_i)}{2}, \frac{2\mathfrak{u}_2 - \mathrm{Re}(\mu_i)}{2}, \frac{\mathfrak{u}_2 + \mathrm{Re}(\mu_i)}{2} \right\}.$$

We have taken some care to separate the polynomial part of $\Lambda(\mu)$ as it will cancel with that of $\Lambda(-\mu)$. The method of proof here is the same as the previous two lemmas.

The ordering of the integrals of $J_{w,\mu}$ and $E_{w_l,j}$ that we have been using is structured for writing the residues and determining the asymptotics in y; this makes it somewhat more difficult for bounding the result as the quadruple of integrals in s and t do not have sufficient exponential decay and hence need to be treated in the form (B.1). That said, we want to separate the integrals of G^* and its residues, i.e. the u and r integrals for the long element functions, and compute bounds for the integrals of s and t and the three relevent residues in s and t first, the integrals in u and r to be taken later. The contours have generally been chosen so that the s and t integrals are bounded by at most a constant multiple of $|y_1|^{-\mathrm{Re}(s_1)} |y_2|^{-\mathrm{Re}(s_2)}$, so the u and r integrals can be evaluated by the results of the previous section. In actuality, most of terms have some extra decay in u and r, which we ignore for convenience.

Bounding the s and t integrals is highly repetitive and simply involves applying Hölder's inequality many times; as mentioned above, the core of the difficulty lies in the integrals

$$
\begin{aligned}
&M_{w_l,1}(y,r,u,v) := \\
&\int_{\mathrm{Re}(s)=\mathfrak{s}} \int_{\mathrm{Re}(t)=\mathfrak{t}} |y_1|^{-s_1} |y_2|^{-s_2} A_{w_l}(r,s,t,u,v) \Gamma\left(s_1-u_1-t_1\right) \Gamma\left(s_2-u_2\right) \\
&\quad \Gamma\left(1+s_1-r_2\right) \Gamma\left(1+s_2-r_1-t_1-t_2\right) \Gamma\left(t_1\right) \Gamma\left(t_1+t_2\right) \\
&\quad B\left(\frac{u_1-2s_2+r_1+t_2}{2}, \frac{1+u_1-u_2-r_1-t_2}{2}\right) \\
&\quad B\left(\frac{u_1-2s_1+r_2-t_2}{2}, \frac{-u_1+r_1-r_2+t_2}{2}\right) \\
&\quad B\left(\frac{1-2t_1-t_2}{2}, \frac{1-u_1-r_1}{2}\right) \, dt \, ds,
\end{aligned}
$$

$$
\begin{aligned}
&M_{w_l,2}(y,r,u,v) := \\
&|y_2|^{-\mathrm{Re}(u_2)} \int_{\mathrm{Re}(s_1)=\mathfrak{s}_1} \int_{\mathrm{Re}(t)=\mathfrak{t}} |y_1|^{-s_1} A_{w_l}(r,(s_1,u_2),t,u,v) \Gamma\left(s_1-u_1-t_1\right) \\
&\quad \Gamma\left(1+s_1-r_2\right) \Gamma\left(1+u_2-r_1-t_1-t_2\right) \Gamma\left(t_1\right) \Gamma\left(t_1+t_2\right) \\
&\quad B\left(\frac{u_1-2u_2+r_1+t_2}{2}, \frac{1+u_1-u_2-r_1-t_2}{2}\right) \\
&\quad B\left(\frac{u_1-2s_1+r_2-t_2}{2}, \frac{-u_1+r_1-r_2+t_2}{2}\right) \\
&\quad B\left(\frac{1-2t_1-t_2}{2}, \frac{1-u_1-r_1}{2}\right) \, dt \, ds_1,
\end{aligned}
$$

$$M_{w_l,3}(y,r,u,v) :=$$

$$|y_1|^{-\mathrm{Re}(u_1)} \int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} \int_{\mathrm{Re}(t)=\mathfrak{t}} |y_1|^{-t_1} |y_2|^{-s_2} A_{w_l}(r,(u_1+t_1,s_2),t,u,v)\Gamma(s_2-u_2)$$
$$\Gamma(1+u_1-r_2+t_1)\Gamma(1+s_2-r_1-t_1-t_2)\Gamma(t_1)\Gamma(t_1+t_2)$$
$$B\left(\frac{u_1-2s_2+r_1+t_2}{2},\frac{1+u_1-u_2-r_1-t_2}{2}\right)$$
$$B\left(\frac{-u_1+r_2-2t_1-t_2}{2},\frac{-u_1+r_1-r_2+t_2}{2}\right)$$
$$B\left(\frac{1-2t_1-t_2}{2},\frac{1-u_1-r_1}{2}\right)\,dt\,ds_2,$$

$$M_{w_l,4}(y,r,u,v) :=$$

$$|y_1|^{-\mathrm{Re}(u_1)} |y_2|^{-\mathrm{Re}(u_2)} \int_{\mathrm{Re}(t)=\mathfrak{t}} |y_1|^{-t_1} A_{w_l}(r,(u_1+t_1,u_2),t,u,v)$$
$$\Gamma(1+u_1-r_2+t_1)\Gamma(1+u_2-r_1-t_1-t_2)\Gamma(t_1)\Gamma(t_1+t_2)$$
$$B\left(\frac{u_1-2u_2+r_1+t_2}{2},\frac{1+u_1-u_2-r_1-t_2}{2}\right)$$
$$B\left(\frac{-u_1+r_2-2t_1-t_2}{2},\frac{-u_1+r_1-r_2+t_2}{2}\right)$$
$$B\left(\frac{1-2t_1-t_2}{2},\frac{1-u_1-r_1}{2}\right)\,dt,$$

$$M_{w_4}(y,r,u,v) :=$$

$$\int_{\mathrm{Re}(s)=\mathfrak{s}} |y|^{-s}\Gamma(s-u_1)\Gamma(s-u_1-r_1)\Gamma(s-u_1-r_2)A_{w_4}(r,s,u,v)$$
$$B\left(\frac{1+2u_1-2s+r_1+r_2}{2},\frac{-u_1-u_2-r_2}{2}\right)$$
$$B\left(\frac{1+2u_1-2s+r_1}{2},\frac{-2u_1+u_2-r_1}{2}\right)\,ds,$$

$$M_{w_5}(y,r,u,v) = \int_{\mathrm{Re}(s)=\mathfrak{s}} |y|^{-s}\,\Gamma(s-u_2)\,\Gamma(s-u_2-r_1)\,\Gamma(s-u_2-r_2)\,A_{w_5}(r,s,u,v) \, B\left(\frac{1+2u_2-2s+r_1+r_2}{2}, \frac{-u_1-u_2-r_2}{2}\right) B\left(\frac{1+2u_2-2s+r_1}{2}, \frac{u_1-2u_2-r_1}{2}\right) ds.$$

We bound these quantities by taking the absolute value of the integrand, call the resulting integral $|M_{w_l,1}|$, etc., and assume that the real part of each parameter is fixed.

Proposition 39. *For the given parameters and $\delta_1, \delta_2 \in \{0, \epsilon\}$, each of the above functions is bounded by a constant times the expected powers of y:*

(a) $|M_{w_l,1}| \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2}$ *for* $\mathrm{Re}(u) = -\frac{1}{2} - 4\epsilon + \delta$, $\mathrm{Re}(r) = \left(\frac{1}{2} - 4\epsilon, 100\epsilon\right)$, $\mathfrak{s} = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \epsilon$,

(b) $|M_{w_l,1}| \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2}$ *for* $\mathrm{Re}(u) = -\epsilon$, $\mathrm{Re}(r) = 0$, $\mathfrak{s} = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \epsilon$,

(c) $|M_{w_l,2}| \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathrm{Re}(u_2)}$ *for* $\mathrm{Re}(u) = \left(-\epsilon, -\frac{1}{2} - 3\epsilon\right)$, $\mathrm{Re}(r) = (-4\epsilon, 100\epsilon)$, $\mathfrak{s}_1 = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \left(\epsilon, \frac{1}{2} - \epsilon\right)$,

(d) $|M_{w_l,3}| \ll |y_1|^{-\mathfrak{t}_1} |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{s}_2}$ *for* $\mathrm{Re}(u) = \left(-\frac{1}{2} - 3\epsilon, -\epsilon\right)$, $\mathrm{Re}(r) = (-3\epsilon, 7\epsilon)$, $\mathfrak{s}_2 = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \left(6\epsilon, \frac{1}{2} - 6\epsilon\right)$,

(e) $|M_{w_l,4}| \ll |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{t}_1} |y_2|^{-\mathrm{Re}(u_2)}$ *for* $\mathrm{Re}(u) = \left(-\epsilon, -\frac{1}{2} - 3\epsilon\right)$, $\mathrm{Re}(r) = (\epsilon, 4\epsilon)$, $\mathfrak{t} = \left(-\frac{1}{2} + 5\epsilon, 1 - 10\epsilon\right)$,

(f) $|M_{w_4}| \ll |y|^{-\mathfrak{s}}$ *for* $\mathrm{Re}(u) = (-3\epsilon + \delta_1, \epsilon)$, $\mathrm{Re}(r) = -\frac{1}{2} + 3\epsilon$, $\mathfrak{t} = \left(-\frac{1}{2} + 5\epsilon, 1 - 10\epsilon\right)$,

(g) $|M_{w_5}| \ll |y|^{-\mathfrak{s}}$ *for* $\mathrm{Re}(u) = (\epsilon, -3\epsilon + \delta_2)$, $\mathrm{Re}(r) = -\frac{1}{2} + 3\epsilon$, $\mathfrak{s} = -\epsilon$.

As mentioned above, we could do better for most of the terms, but we ignore some decay factors for convenience.

Having these bounds, the two (double) integrals of the G^* functions separate, so we may apply Lemma 4 to obtain Proposition 34, keeping in mind that the worst-case bound comes from having the smallest power on the smallest of $|\mu_i - \mu_j|$, this is done in section B.4.

The content of Theorem 23 is essentially that the spectral side converges absolutely when the trivial element term does; in other words, convergence of the integral

$$\int_{\mathrm{Re}(\mu)=\eta} \left|\hat{k}(\mu)\right| |\mu_1 - \mu_2| \, |\mu_1 - \mu_3| \, |\mu_2 - \mu_3| \, |d\mu| \tag{B.2}$$

for $|\eta_i| \leq \frac{5}{14}$ with $\mathfrak{q} = -\frac{5}{14} - \epsilon$ gives absolute convergence of the spectral side as well as the trivial term. By Proposition 34, increasing the exponent on the $|\mu_i - \mu_j|$ terms to $\frac{13}{8} + 100\epsilon$ gives a more than sufficient condition for convergence of the intermediate and long element terms in Theorem 31, hence justifies our use of (B.2) as the convergence hypothesis on $\hat{k}$ and completes that theorem.

To complete the proof of Theorem 33, we again need to demonstrate that the hypothesis $\hat{f}(q) \ll |q_1 q_2|^{-8}$ is sufficient for absolute convergence of all of the relevant terms. For the spectral side and the trivial term, we start with (B.2) and apply Stirling's formula to K_{w_l} to obtain

$$|K_{w_l}(\mu)| \asymp |\mu_1 - \mu_2|^{\frac{1}{2}+\mathrm{Re}(\mu_1-\mu_2)} \, |\mu_1 - \mu_3|^{\frac{1}{2}+\mathrm{Re}(\mu_1-\mu_3)} \, |\mu_3 - \mu_2|^{\frac{1}{2}+\mathrm{Re}(\mu_3-\mu_2)},$$

for $\mathrm{Re}(\mu)$ constant, so we desire convergence of the integral

$$\begin{aligned} &L_1 := \\ &\int_{\mathrm{Re}(q)=-\frac{5}{14}-\epsilon} |q_1|^{-8} \, |q_2|^{-8} \, |q_1+q_2|^{\frac{31}{14}+2\epsilon} \, |2q_1 - q_2|^{\frac{6}{7}+\epsilon} \, |2q_2 - q_1|^{\frac{6}{7}+\epsilon} \\ &\int_{\mathrm{Re}(\mu)=\eta} \frac{|\mu_1-\mu_2|^{\frac{13}{8}+100\epsilon} \, |\mu_1-\mu_3|^{\frac{13}{8}+100\epsilon} \, |\mu_2-\mu_3|^{\frac{13}{8}+100\epsilon} \, |k_{\mathrm{conv}}(\mu,q)|}{|q_1-\mu_1| \, |q_1-\mu_2| \, |q_1-\mu_3| \, |q_2+\mu_1| \, |q_2+\mu_2| \, |q_2+\mu_3|} \, |d\mu| \, |dq| \, . \end{aligned}$$

From Proposition 34, we note that this also implies absolute convergence of the F_1 error term. To simplify the above integral, we choose

$$k_{\text{conv}}(\mu, q) = \left(\frac{(2+q_1)(2+q_2)(2-q_1+q_2)}{(2+\mu_1)(2+\mu_2)(2+\mu_3)} \right)^{\frac{23}{12}+300\epsilon}. \tag{B.3}$$

We call this function k_{conv} because without it, the above integral would not converge – the total power on μ_1 and μ_2 is not below -1.

Applying the same logic to the remaining error terms, it is sufficient to consider convergence of the integrals

$$\begin{aligned} L_2 := \int_{\text{Re}(\mu)=\left(-\frac{1}{2}-4\epsilon, \frac{1}{2}+\epsilon\right)} \int_{\text{Re}(q_2)=-6\epsilon} & |\mu_1|^{-8} |q_2|^{-8} \\ & \frac{|q_2+\mu_1|^{1+4\epsilon} |2\mu_1 - q_2|^{\frac{3}{2}+5\epsilon} |2q_2 - \mu_1|^{-\epsilon} |k_{\text{conv}}(\mu, q)|}{|q_2+\mu_2| |q_2+\mu_3|} \\ & |\mu_1-\mu_3|^{\frac{1}{2}+100\epsilon} |\mu_2-\mu_1|^{\frac{1}{2}+100\epsilon} |\mu_2-\mu_3|^{\frac{3}{2}+100\epsilon} \, |dq_2| \, |d\mu|, \end{aligned}$$

$$L_3 := \int_{\text{Re}(\mu)=(-\epsilon,\epsilon)} |\mu_1|^{-8} |\mu_2|^{-8} |\mu_1-\mu_2|^{2+100\epsilon} |\mu_1-\mu_3|^{\frac{7}{4}+100\epsilon} |\mu_3-\mu_2|^{\frac{7}{4}+100\epsilon} |d\mu|.$$

Here convergence of the L_2 integral is sufficient to show absolute convergence of F_2 and F_3 (by symmetry), and L_3 gives the absolute convergence of the remaining error terms. The exponents in this last are derived from

$$|K_{w_l}(\mu)|^{-1} \ll |\mu_1-\mu_2|^{\frac{1}{2}} |\mu_1-\mu_3|^{\frac{1}{4}} |\mu_3-\mu_2|^{\frac{1}{4}},$$

$$|E_{w_l,j}| \ll |\mu_1-\mu_2|^{\frac{3}{2}+100\epsilon} |\mu_1-\mu_3|^{\frac{3}{2}+100\epsilon} |\mu_3-\mu_2|^{\frac{3}{2}+100\epsilon}.$$

So we have our final technical requirement for Theorem 33:

Proposition 40. *The integrals L_1, L_2, and L_3 converge.*

B.1 The Beta Function

We will make repeated use of the integrals

$$\int_0^u |1+it|^\alpha \, dt = u \, {}_2F_1\left(\begin{matrix} \frac{1}{2}, -\frac{\alpha}{2}; \\ \frac{3}{2}; \end{matrix} \; -u^2\right) \ll u^{\max\{0,\alpha+1\}} \asymp |1+iu|^{\max\{0,\alpha+1\}}$$

and

$$\int_u^\infty |1+it|^\alpha \, dt \leq u^{\alpha+1} \int_1^\infty t^\alpha dt \ll |1+iu|^{\alpha+1}, \qquad \mathrm{Re}(\alpha) < -1,$$

for $u \geq 1$.

We may obtain the above asymptotic of ${}_2F_1$ from the Barnes integral representation

$${}_2F_1\left(\begin{matrix} a, b; \\ c; \end{matrix} \; z\right) = \frac{\Gamma(c)}{2\pi i \Gamma(a)\Gamma(b)} \int_{-i\infty}^{i\infty} \frac{\Gamma(a+s)\Gamma(b+s)\Gamma(-s)}{\Gamma(c+s)} (-z)^s ds,$$

where the contour is taken to separate the poles of $\Gamma(-s)$ from those of $\Gamma(a+s)$ and $\Gamma(b+s)$. For our first integral, the first poles of $\Gamma(a+s)$ and $\Gamma(b+s)$ that we encounter while shifting the contours to the left are at $-\frac{1}{2}$ and $\frac{\alpha}{2}$, respectively.

Proof of Lemma 36. The result is obvious if $s < 1$, so we assume $s \geq 1$, and split the integral at $-2s$,0, and $2s$, call the resulting integrals I_1, I_2, I_3, and I_4, say. For the first integral, we substitute $t \mapsto -t - 2s$, so it becomes $I_1 = I_{1,a} + I_{1,b}$:

$$\begin{aligned} I_{1,a} &= \int_0^s |1+i(s+t)|^{a_1} |1+i(t+3s)|^{a_2} \, dt \\ &\asymp |1+is|^{a_1+a_2} \int_0^s dt \\ &\ll |1+is|^{a_1+a_2+1}, \\ I_{1,b} &= \int_s^\infty |1+i(s+t)|^{a_1} |1+i(t+3s)|^{a_2} \, dt \\ &\asymp \int_s^\infty |1+it|^{a_1+a_2} \, dt \\ &\ll |1+is|^{a_1+a_2+1}, \end{aligned}$$

similarly $I_4 \ll |1+is|^{a_1+a_2+1}$. For the second, we substitute $t \mapsto -t-s$:

$$\begin{aligned} I_2 &= \int_{-s}^{s} |1+it|^{a_1} \, |1+i(t+2s)|^{a_2} \, dt \\ &\asymp |1+is|^{a_2} \int_{-s}^{s} |1+it|^{a_1} \, dt \\ &\ll |1+is|^{a_2+\max\{0,a_1+1\}} , \end{aligned}$$

similarly, $I_3 \ll |1+is|^{a_1+\max\{0,a_2+1\}}$. □

Using one of the first two terms in the maximum essentially incurs a loss, so we will tend to enforce $a_1, a_2 \geq -1$.

For $\mathrm{Re}(u)$, $\mathrm{Re}(v_1)$, $\mathrm{Re}(v_2)$ fixed, applying the second form of Stirling's formula shows that $B(u-v_1, v_2-u)$ decays exponentially in u unless

$$\max\{\mathrm{Im}(v_1), \mathrm{Im}(v_2)\} > \mathrm{Im}(u) > \min\{\mathrm{Im}(v_1), \mathrm{Im}(v_2)\},$$

and in that case, we have

$$B(v_1+u, v_2-u) \ll \frac{|v_1+u|^{\mathrm{Re}(v_1+u)-\frac{1}{2}} \, |v_2-u|^{\mathrm{Re}(v_2-u)-\frac{1}{2}}}{|v_1+v_2|^{\mathrm{Re}(v_1+v_2)-\frac{1}{2}}},$$

so the proof of Lemma 37 is precisely the same as Lemma 36, without the equivalent of requiring $a_1 + a_2 < -1$.

B.2 The G Function

The bound for the residue $G_b^*(1, 2, \mu)$ in Lemma 4 part a is simply from applying Stirling's formula. The residue $G_l^*(1, u_2, \mu)$ has exponential decay in $\mathrm{Im}(u_2)$ unless $\mathrm{Im}(\mu_2) > \mathrm{Im}(-u_2) > \mathrm{Im}(\mu_3)$ (up to permutation of (μ_2, μ_3), for fixed $\mathrm{Re}(u_2)$), thus it integrates much like a beta function, and the bound in Lemma 4 part b follows by the same logic, similarly for part c.

For Re(u), Re(μ) fixed, applying Stirling's formula gives exponential decay in u_1 or u_2 for G^* unless, up to permutation of μ or $(u_1, -u_2)$,

$$\mathrm{Im}(\mu_1) > \mathrm{Im}(u_1) > \mathrm{Im}(\mu_2) > \mathrm{Im}(-u_2) > \mathrm{Im}(\mu_3),$$

in which case, we have

$$G^*(u,\mu) \ll \frac{|u_1+u_2|^{\frac{1-\mathrm{Re}(u_1+u_2)}{2}} \prod_{i=1}^3 |u_1-\mu_i|^{\frac{\mathrm{Re}(u_1-\mu_i)-1}{2}} |u_2+\mu_i|^{\frac{\mathrm{Re}(u_2+\mu_i)-1}{2}}}{|\mu_1-\mu_2|^{\frac{\mathrm{Re}(\mu_1-\mu_2)}{2}} |\mu_1-\mu_3|^{\frac{\mathrm{Re}(\mu_1-\mu_3)}{2}} |\mu_2-\mu_3|^{\frac{\mathrm{Re}(\mu_2-\mu_3)}{2}}}.$$

For integrals of the G function, it is sufficient to prove the following lemma:

Lemma 41. *Suppose $a_i, b_i > -1-\epsilon$, $a_2+b_2+c > -2-\epsilon$, and $v_1 < v_2 < v_3$ with $v_3 - v_2 > v_2 - v_1$, then*

$$\int_{v_1}^{v_2}\int_{v_2}^{v_3} |1+i(u_1-u_2)|^c \prod_{i=1}^3 |1+i(u_1-v_i)|^{a_i} |1+i(u_2-v_i)|^{b_i}\, du_2\, du_1$$
$$\ll |1+i(v_3-v_1)|^{\frac{\kappa-\delta}{2}+\epsilon} |1+i(v_3-v_2)|^{\frac{\kappa-\delta}{2}+\epsilon} |1+i(v_2-v_1)|^{\delta+\epsilon},$$

where

$$\kappa = a_1+a_2+a_3+b_1+b_2+b_3+c+2,$$
$$\delta = \min\{a_1+a_2+1, a_1+a_2+b_1+c+1\}.$$

Applying this to the G^* function, we have

$$\kappa = \mathfrak{u}_1 + \mathfrak{u}_2 - \frac{1}{2},$$
$$\delta = \min\left\{\frac{\kappa}{3}, \mathfrak{u}_1 + \frac{\mathrm{Re}(\mu_3)}{2}, \frac{\mathfrak{u}_1 - \mathrm{Re}(\mu_2)}{2}\right\}$$

(including the $\frac{\kappa}{3}$ prevents a really good bound from being really bad after permuting μ), and the conditions become

$$
\begin{aligned}
&a_i, b_i > -1-\epsilon \Leftrightarrow \mathfrak{u}_1 - \mathrm{Re}(\mu_i), \mathfrak{u}_2 + \mathrm{Re}(\mu_i) > -1-2\epsilon,\\
&a_2+b_2+c = -\frac{1}{2} > -2-\epsilon.
\end{aligned}
$$

Permuting μ and interchanging $(\mathfrak{u}_1, \mu) \leftrightarrow (\mathfrak{u}_2, -\mu)$ as necessary, we obtain Lemma 4 part d. We do not give the proof that the integral over the region of exponential decay satisfies the same bound, but if one conditions on $v_2 - v_1$ smaller or larger than

$$
\delta = 10\left(|a_1|+|a_2|+|a_3|+|b_1|+|b_2|+|b_3|+|c|+3\right)\log\left|1+i(v_3-v_1)\right|,
$$

the proof in the first case is essentially identical to the case $v_2 - v_1 < 1 \le v_3 - v_2$ below, and in the second case we need only extend the region of integration by $\delta \ll |1+i(v_3-v_1)|^\epsilon$.

Proof of Lemma 41. We will repeatedly use the fact $v_3 - v_1 = (v_3 - v_2) + (v_2 - v_1) \asymp v_3 - v_2$.

If $v_3 - v_2 < 1$, the result is obvious; if $v_2 - v_1 < 1 \le v_3 - v_2$, the integral reduces to

$$
\begin{aligned}
&|1+i(v_3-v_2)|^{a_3} \int_{v_2}^{v_3} |1+i(u_2-v_2)|^{c+b_1+b_2} |1+i(u_2-v_3)|^{b_3}\, du_2\\
&\ll |1+i(v_3-v_2)|^{a_3+\max\{c+b_1+b_2, b_3, c+b_1+b_2+b_3+1\}}\\
&= |1+i(v_3-v_2)|^{a_3+\max\{b_3, c+b_1+b_2+b_3+1\}+\epsilon},
\end{aligned}
$$

since $\delta \le \min\{a_1+a_2+1, a_1+a_2+b_1+b_2+c+2\}$, we have

$$
\kappa - \delta \ge \max\{a_3+b_1+b_2+b_3+c+1, a_3+b_3\},
$$

and the result follows.

Now assume $v_2 - v_1 \ge 1$. We split the integral into three parts: The first with $u_2 > \frac{v_2+v_3}{2}$, the second with $u_1 < \frac{v_1+v_2}{2}$, and the third over the remaining region. For the first integral, we send $u_2 \mapsto u_2 + v_3$, then

$$\int_{-\frac{v_3-v_2}{2}}^{0} |1+i(u_1-v_3-u_2)|^c \, |1+i(u_2+v_3-v_1)|^{b_1}$$
$$|1+i(u_2+v_3-v_2)|^{b_2} \, |1+iu_2|^{b_3} \, du_2$$
$$\asymp |1+i(u_1-v_3)|^c \, |1+i(v_3-v_1)|^{b_1} \, |1+i(v_3-v_2)|^{b_2} \int_{-\frac{v_3-v_2}{2}}^{0} |1+iu_2|^{b_3} \, du_2$$
$$\ll |1+i(u_1-v_3)|^c \, |1+i(v_3-v_1)|^{b_1} \, |1+i(v_3-v_2)|^{b_2+\max\{0,b_3+1\}},$$

since $v_3 - u_1 \geq v_3 - v_2 > v_2 - v_1$. Then

$$I_1 \ll |1+i(v_3-v_1)|^{b_1} \, |1+i(v_3-v_2)|^{b_2+\max\{0,b_3+1\}}$$
$$\int_{v_1}^{v_2} |1+i(u_1-v_1)|^{a_1} \, |1+i(u_1-v_2)|^{a_2} \, |1+i(u_1-v_3)|^{a_3+c} \, du_1,$$

and splitting this integral at $\frac{v_1+v_2}{2}$, we obtain

$$I_{1A} = \int_{0}^{\frac{v_2-v_1}{2}} |1+iu_1|^{a_1} \, |1+i(u_1+v_1-v_2)|^{a_2} \, |1+i(u_1+v_1-v_3)|^{a_3+c} \, du_1$$
$$\ll |1+i(v_3-v_1)|^{a_3+c} \, |1+i(v_1-v_2)|^{a_2+\max\{0,a_1+1\}},$$

and similarly for

$$I_{1B} = \int_{-\frac{v_2-v_1}{2}}^{0} |1+i(u_1+v_2-v_1)|^{a_1} \, |1+iu_1|^{a_2} \, |1+i(u_1+v_2-v_3)|^{a_3+c} \, du_1$$
$$\ll |1+i(v_2-v_1)|^{a_1+\max\{0,a_2+1\}} \, |1+i(v_2-v_3)|^{a_3+c}$$

giving

$$I_1 \ll |1+i(v_3-v_2)|^{a_3+b_1+b_2+b_3+c+1+\epsilon} \, |1+i(v_1-v_2)|^{a_1+a_2+1+\epsilon},$$

using $v_3 - v_1 \asymp v_3 - v_2$ and the hypotheses on a and b.

Similarly, for the second integral, we send $u_1 \mapsto u_1 + v_1$ so

$$\int_0^{\frac{v_2-v_1}{2}} |1+i(u_1+v_1-u_2)|^c \, |1+iu_1|^{a_1} \, |1+i(u_1+v_1-v_2)|^{a_2} \, |1+i(u_1+v_1-v_3)|^{a_3} \, du_1$$
$$\ll |1+i(v_1-u_2)|^c \, |1+i(v_1-v_2)|^{a_2} \, |1+i(v_1-v_3)|^{a_3} \int_0^{\frac{v_2-v_1}{2}} |1+iu_1|^{a_1} \, du_1$$
$$\ll |1+i(v_1-u_2)|^c \, |1+i(v_1-v_2)|^{a_2+\max\{0,a_1+1\}} \, |1+i(v_1-v_3)|^{a_3}$$

since $u_2 - v_1 \geq v_2 - v_1$. Then

$$I_2 \ll |1+i(v_1-v_2)|^{a_2+\max\{0,a_1+1\}} \, |1+i(v_1-v_3)|^{a_3} \int_{v_2}^{v_3} |1+i(u_2-v_1)|^{c+b_1} \, |1+i(u_2-v_2)|^{b_2} \, |1+i(u_2-v_3)|^{b_3} \, du_2,$$

which we split at $\frac{v_2+v_3}{2}$ so

$$I_{2A} = \int_{-\frac{v_3-v_2}{2}}^{0} |1+i(u_2+v_3-v_1)|^{c+b_1} \, |1+i(u_2+v_3-v_2|^{b_2} \, |1+iu_2|^{b_3} \, du_2$$
$$\ll |1+i(v_3-v_1)|^{c+b_1} \, |1+i(v_3-v_2|^{b_2+\max\{0,b_3+1\}},$$

$$I_{2B} = \int_0^{\frac{v_3-v_2}{2}} |1+i(u_2+v_2-v_1)|^{c+b_1} \, |1+iu_2|^{b_2} \, |1+i(u_2+v_2-v_3)|^{b_3} \, du_2$$
$$\ll |1+i(v_2-v_3)|^{b_3} \int_0^{\frac{v_3-v_2}{2}} |1+i(u_2+v_2-v_1)|^{c+b_1} \, |1+iu_2|^{b_2} \, du_2,$$

and we split again at $\frac{v_2-v_1}{2}$, so

$$I_{2Ba} = \int_0^{\frac{v_2-v_1}{2}} |1+i(u_2+v_2-v_1)|^{c+b_1} \, |1+iu_2|^{b_2} \, du_2$$
$$\ll |1+i(v_2-v_1)|^{c+b_1+\max\{0,b_2+1\}},$$

$$
\begin{aligned}
I_{2Ba} &= \int_{\frac{v_2-v_1}{2}}^{\frac{v_3-v_2}{2}} \left|1+i(u_2+v_2-v_1)\right|^{c+b_1} \left|1+iu_2\right|^{b_2} \, du_2 \\
&\ll \left|1+i(v_3-v_2)\right|^{c+b_1+\max\{0,b_2+1\}} .
\end{aligned}
$$

Altogether, this gives

$$
\begin{aligned}
I_2 \ll& \left|1+i(v_2-v_1)\right|^{a_1+a_2+1+\epsilon} \left|1+i(v_3-v_2)\right|^{a_3+b_1+b_2+b_3+c+1+\epsilon} \\
&+ \left|1+i(v_2-v_1)\right|^{a_1+a_2+b_1+b_2+2+2\epsilon} \left|1+i(v_3-v_2)\right|^{a_3+b_3} .
\end{aligned}
$$

Now the third integral becomes

$$
\begin{aligned}
I_3 =& \int_{-\frac{v_2-v_1}{2}}^{0} \int_{0}^{\frac{v_3-v_2}{2}} \left|1+i(u_1-u_2)\right|^{c} \left|1+i(u_1+v_2-v_1)\right|^{a_1} \left|1+iu_1\right|^{a_2} \\
& \left|1+i(u_1+v_2-v_3)\right|^{a_3} \left|1+i(u_2+v_2-v_1)\right|^{b_1} \\
& \left|1+iu_2\right|^{b_2} \left|1+i(u_2+v_2-v_3)\right|^{b_3} \, du_2 \, du_1 \\
\asymp& \left|1+i(v_2-v_1)\right|^{a_1} \left|1+i(v_2-v_3)\right|^{a_3+b_3} \\
& \int_{0}^{\frac{v_2-v_1}{2}} \int_{0}^{\frac{v_3-v_2}{2}} \left|1+i(u_1+u_2)\right|^{c} \left|1+iu_1\right|^{a_2} \\
& \left|1+i(u_2+v_2-v_1)\right|^{b_1} \left|1+iu_2\right|^{b_2} \, du_2 \, du_1,
\end{aligned}
$$

and this we split into three pieces at $u_2 = u_1$ and $u_2 = \frac{v_2-v_1}{2}$:

$$
\begin{aligned}
I_{3a} &\asymp \left|1+i(v_2-v_1)\right|^{b_1} \int_{0}^{\frac{v_2-v_1}{2}} \left|1+iu_1\right|^{a_2+c} \int_{0}^{u_1} \left|1+iu_2\right|^{b_2} \, du_2 \, du_1 \\
&\ll \left|1+i(v_2-v_1)\right|^{b_1} \int_{0}^{\frac{v_2-v_1}{2}} \left|1+iu_1\right|^{a_2+b_2+c+1+\epsilon} \, du_1 \\
&\ll \left|1+i(v_2-v_1)\right|^{a_2+b_1+b_2+c+2+\epsilon} ,
\end{aligned}
$$

$$
\begin{aligned}
I_{3b} \asymp & |1+i(v_2-v_1)|^{b_1} \int_0^{\frac{v_2-v_1}{2}} |1+iu_1|^{a_2} \int_{u_1}^{\frac{v_2-v_1}{2}} |1+iu_2|^{b_2+c} \, du_2 \, du_1 \\
\ll & |1+i(v_2-v_1)|^{b_1} \int_0^{\frac{v_2-v_1}{2}} |1+iu_1|^{a_2} \\
& \quad \left(|1+iu_1|^{b_2+c+1} + |1+i(v_2-v_1)|^{b_2+c+1+\epsilon} \right) \, du_1 \\
\ll & |1+i(v_2-v_1)|^{a_2+b_1+b_2+c+2+\epsilon},
\end{aligned}
$$

by replacing the range of integration in u_2 with $\left(0, \frac{v_2-v_1}{2}\right)$ if $b_2+c+1 > -1$ and (u_1, ∞) otherwise,

$$
\begin{aligned}
I_{3c} \asymp & \int_0^{\frac{v_2-v_1}{2}} |1+iu_1|^{a_2} \, du_1 \int_{\frac{v_2-v_1}{2}}^{\frac{v_3-v_2}{2}} |1+iu_2|^{b_1+b_2+c} \, du_2 \\
\ll & |1+i(v_2-v_1)|^{a_2+1+\epsilon} \\
& \quad \left(|1+i(v_2-v_1)|^{b_1+b_2+c+1} + |1+i(v_3-v_2)|^{b_1+b_2+c+1+\epsilon} \right).
\end{aligned}
$$

Then δ is the minimum exponent of $|1+i(v_2-v_1)|$ and κ is the sum of the two exponents, which is the same in every case; we split the remaining $\kappa - \delta$ evenly between $|v_3 - v_2| \asymp |v_3 - v_1|$.

□

B.3 Proof of Proposition 39

For brevity, we introduce the shorthand

$$
B \begin{pmatrix} a, & b & u \\ & ; & \\ c & & v \end{pmatrix} := |u|^a \, |v|^b \, |u+v|^c \exp -\frac{\pi}{4} \left(|\mathrm{Im}(u)| + |\mathrm{Im}(v)| - |\mathrm{Im}(u+v)| \right),
$$

and supress terms $O\left(\epsilon^2\right)$.

B.3.0.1 (a)

For $\mathrm{Re}(u) = -\frac{1}{2} - 4\epsilon + \delta$, $\mathrm{Re}(r) = \left(\frac{1}{2} - 4\epsilon, 100\epsilon\right)$, $\mathfrak{s} = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \epsilon$,

$$\begin{aligned}
&|M_{w_l,1}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(s)=\mathfrak{s}} \int_{\mathrm{Re}(t)=\mathfrak{t}} |s_1 - u_1 - t_1|^{-\frac{1}{2}+2\epsilon} |s_2 - u_2|^{-\frac{1}{2}+3\epsilon} \\
&\qquad |1 + s_1 - r_2|^{-101\epsilon} |1 + s_2 - r_1 - t_1 - t_2|^{-\frac{1}{2}+\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |t_1 + t_2|^{-\frac{1}{2}+2\epsilon} \\
&\qquad B\begin{pmatrix} -\epsilon, & -\frac{1}{4} + 2\epsilon & ; & u_1 - 2s_2 + r_1 + t_2 \\ & -\frac{1}{4} + \epsilon & & 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} \\
&\qquad B\begin{pmatrix} -\frac{1}{4} + 49\epsilon, & -49\epsilon & ; & u_1 - 2s_1 + r_2 - t_2 \\ & -\frac{1}{4} + \epsilon & & -u_1 + r_1 - r_2 + t_2 \end{pmatrix} \\
&\qquad B\begin{pmatrix} -\epsilon, & 4\epsilon & ; & 1 - 2t_1 - t_2 \\ -\frac{1}{2} - 2\epsilon & & & 1 - u_1 - r_1 \end{pmatrix} |dt|\, |ds|.
\end{aligned}$$

First, apply Hölder in s_1 with exponents $2 - 204\epsilon = \frac{1}{\frac{1}{2}+51\epsilon} + O\left(\epsilon^2\right)$ and $2 + 204\epsilon = \frac{1}{\frac{1}{2}-51\epsilon} + O\left(\epsilon^2\right)$ giving

$$\int_{\mathrm{Re}(s_1)=\mathfrak{s}_1} |s_1 - u_1 - t_1|^{-1+106\epsilon} |1 + s_1 - r_2|^{-202\epsilon} |ds_1| \ll |1 + u_1 - r_2 + t_1|^{-96\epsilon},$$

$$\begin{aligned}
&\int_{\mathrm{Re}(s_1)=\mathfrak{s}_1} |u_1 - 2s_1 + r_2 - t_2|^{-\frac{1}{2}+47\epsilon} |-2s_1 + r_1|^{-\frac{1}{2}-49\epsilon} |ds_1| \\
&\ll |u_1 - r_1 + r_2 - t_2|^{-2\epsilon},
\end{aligned}$$

so

$$
\begin{aligned}
&|M_{w_l,1}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} \int_{\mathrm{Re}(t)=\mathfrak{t}} |1+u_1-r_2+t_1|^{-48\epsilon} |s_2-u_2|^{-\frac{1}{2}+3\epsilon} \\
&\quad |1+s_2-r_1-t_1-t_2|^{-\frac{1}{2}+\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |t_1+t_2|^{-\frac{1}{2}+2\epsilon} |u_1-r_1+r_2-t_2|^{-51\epsilon} \\
&\quad B\begin{pmatrix} -\epsilon, & -\frac{1}{4}+2\epsilon & ; & u_1-2s_2+r_1+t_2 \\ & -\frac{1}{4}+\epsilon & & 1+u_1-u_2-r_1-t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\epsilon, & 4\epsilon & ; & 1-2t_1-t_2 \\ -\frac{1}{2}-2\epsilon & & & 1-u_1-r_1 \end{pmatrix} |dt|\, |ds_2|\,.
\end{aligned}
$$

Now apply Hölder in s_2 with exponents $\frac{4}{3}-4\epsilon$ and $4+36\epsilon$ giving

$$
\begin{aligned}
&\int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} |s_2-u_2|^{-\frac{2}{3}+6\epsilon} |1+s_2-r_1-t_1-t_2|^{-\frac{2}{3}+8\epsilon} |ds_2| \\
&\ll |1+u_2-r_1-t_1-t_2|^{-\frac{1}{3}+14\epsilon},
\end{aligned}
$$

$$
\begin{aligned}
&\int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} |u_1-2s_2+r_1+t_2|^{-4\epsilon} |1+2u_1-u_2-2s_2|^{-1-5\epsilon} |ds_2| \\
&\ll |1+u_1-u_2-r_1-t_2|^{-4\epsilon},
\end{aligned}
$$

so

$$
\begin{aligned}
&|M_{w_l,1}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(t)=\mathfrak{t}} |1+u_1-r_2+t_1|^{-48\epsilon} |1+u_2-r_1-t_1-t_2|^{-\frac{1}{4}+10\epsilon} \\
&\quad |t_1|^{-\frac{1}{2}+\epsilon} |t_1+t_2|^{-\frac{1}{2}+2\epsilon} |u_1-r_1+r_2-t_2|^{-51\epsilon} |1+u_1-u_2-r_1-t_2|^{-\frac{1}{4}+\epsilon} \\
&\quad B\begin{pmatrix} -\epsilon, & 4\epsilon & ; & 1-2t_1-t_2 \\ -\frac{1}{2}-2\epsilon & & & 1-u_1-r_1 \end{pmatrix} |dt|\,.
\end{aligned}
$$

Again, Hölder in t_2 with exponents $2+48\epsilon$, $\frac{1}{52\epsilon}$, and $2+160\epsilon$ giving

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |1+u_2-r_1-t_1-t_2|^{-\frac{1}{2}+8\epsilon} |1+u_1-u_2-r_1-t_2|^{-\frac{1}{2}-10\epsilon} |dt_2| \ll |u_1-2u_2+t_1|^{-2\epsilon},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |u_1-r_1+r_2-t_2|^{-\frac{51}{52}} |2-u_1-r_1-2t_1-t_2|^{-\frac{51}{52}} |dt_2| \ll |2-2u_1-r_2-2t_1|^{-\frac{51}{52}},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |t_1+t_2|^{-1-76\epsilon} |2-u_1-r_1-2t_1-t_2|^{-1+18\epsilon} |dt_2| \ll |2-u_1-r_1-t_1|^{-1+18\epsilon},$$

so

$$|M_{w_l,1}| \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} |1-u_1-r_1|^{4\epsilon} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |1+u_1-r_2+t_1|^{-48\epsilon} |u_1-2u_2+t_1|^{-\epsilon} |2-2u_1-r_2-2t_1|^{-51\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |2-u_1-r_1-t_1|^{-\frac{1}{2}+49\epsilon} |dt_1|.$$

Lastly, we apply Hölder in t_1 with exponents $\frac{1}{46\epsilon}$, $\frac{1}{50\epsilon}$, and $1+96\epsilon$ giving

$$\int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |t_1|^{-\frac{1}{2}-47\epsilon} |2-u_1-r_1-t_1|^{-\frac{1}{2}+\epsilon} |dt_1| \ll |2-u_1-r_1|^{-46\epsilon},$$

so

$$|M_{w_l,1}| \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} |1-u_1-r_1|^{4\epsilon} |2-u_1-r_1|^{-46\epsilon} \ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2}.$$

B.3.0.2 (b)

For $\mathrm{Re}(u) = -\epsilon$, $\mathrm{Re}(r) = 0$, $\mathfrak{s} = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \epsilon$,

$$\begin{aligned}
&|M_{w_l,1}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(s)=\mathfrak{s}} \int_{\mathrm{Re}(t)=\mathfrak{t}} |s_1 - u_1 - t_1|^{-1-\epsilon} |s_2 - u_2|^{-1} \\
&\quad |1 + s_1 - r_2|^{-\epsilon} |1 + s_2 - r_1 - t_1 - t_2|^{-3\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |t_1 + t_2|^{-\frac{1}{2}+2\epsilon} \\
&\quad B\begin{pmatrix} \epsilon, & 0 & ; & u_1 - 2s_2 + r_1 + t_2 \\ & -\frac{1}{2} & & 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} 0, & -\frac{1}{2} + \epsilon & ; & u_1 - 2s_1 + r_2 - t_2 \\ & -\epsilon & & -u_1 + r_1 - r_2 + t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\epsilon, & \epsilon & ; & 1 - 2t_1 - t_2 \\ -\frac{1}{2} + \epsilon & & & 1 - u_1 - r_1 \end{pmatrix} |dt|\, |ds|.
\end{aligned}$$

We apply Hölder in t_2 with exponents $\frac{1}{3\epsilon}$, 2, $\frac{1}{\epsilon}$, $2 + 16\epsilon$, $\frac{1}{\epsilon}$ giving

$$\begin{aligned}
&\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |1 + s_2 - r_1 - t_1 - t_2|^{-1} |-u_1 + r_1 - r_2 + t_2|^{-1} |dt_2| \\
&\ll |1 - u_1 + s_2 - r_2 - t_1|^{-1},
\end{aligned}$$

$$\begin{aligned}
&\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |t_1 + t_2|^{-1+4\epsilon} |1 - u_1 - r_1 - 2t_1 - t_2|^{-1+4\epsilon} |dt_2| \\
&\ll |1 - u_1 - r_1 - t_1|^{-1+4\epsilon},
\end{aligned}$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} 1, & 0 & ; & u_1 - 2s_2 + r_1 + t_2 \\ & -\frac{1}{2\epsilon} & & 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} |dt_2| \ll |1 + 2u_1 - u_2 - 2s_2|^{-\frac{1}{2\epsilon}+2},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} 0, & -1 & ; & u_1-2s_1+r_2-t_2 \\ & -2\epsilon & & -u_1+r_1-r_2+t_2 \end{pmatrix} |dt_2| \ll |-2s_1+r_1|^{-2\epsilon},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} -1, & 1 & ; & 1-2t_1-t_2 \\ & -1 & & 1-u_1-r_1 \end{pmatrix} |dt_2| \ll 1,$$

so

$$\begin{aligned}
&|M_{w_l,1}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(s)=\mathfrak{s}} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |s_1-u_1-t_1|^{-1-\epsilon} |s_2-u_2|^{-1} \\
&\qquad |1+s_1-r_2|^{-\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |1-u_1-r_1-t_1|^{-\frac{1}{2}+2\epsilon} \\
&\qquad |1+2u_1-u_2-2s_2|^{-\frac{1}{2}+2\epsilon} |-2s_1+r_1|^{-\epsilon} |1-u_1+s_2-r_2-t_1|^{-3\epsilon} \, |dt_1| \, |ds| \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} \int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |s_2-u_2|^{-1} |1+u_1-r_2+t_1|^{-\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} \\
&\qquad |1-u_1-r_1-t_1|^{-\frac{1}{2}+2\epsilon} |1+2u_1-u_2-2s_2|^{-\frac{1}{2}+2\epsilon} |-2u_1+r_1-2t_1|^{-\epsilon} \\
&\qquad |1-u_1+s_2-r_2-t_1|^{-3\epsilon} \, |dt_1| \, |ds_2| \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} |1+2u_1-3u_2|^{-\frac{1}{2}+2\epsilon} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |1+u_1-r_2+t_1|^{-\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} \\
&\qquad |1-u_1-r_1-t_1|^{-\frac{1}{2}+2\epsilon} |-2u_1+r_1-2t_1|^{-\epsilon} \\
&\qquad |1-u_1+u_2-r_2-t_1|^{-3\epsilon} \, |dt_1| \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2} |1+2u_1-3u_2|^{-\frac{1}{2}+2\epsilon} \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathfrak{s}_2}.
\end{aligned}$$

The extra decay here comes from the two terms

$$|s_2-u_2|^{-1} \qquad \text{and} \qquad |1+2u_1-u_2-2s_2|^{-\frac{1}{2}},$$

which essentially only contain s_2; we cannot set the contours so that this decay is shifted into the u and r integrals, so we are accepting a loss of $\frac{1}{2}$ by ignoring the term for simplicity.

B.3.0.3 (c)

For $\mathrm{Re}(u) = \left(-\epsilon, -\frac{1}{2} - 3\epsilon\right)$, $\mathrm{Re}(r) = (-4\epsilon, 100\epsilon)$, $\mathfrak{s}_1 = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \left(\epsilon, \frac{1}{2} - \epsilon\right)$,

$$\begin{aligned}
&|M_{w_l,2}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathrm{Re}(u_2)} \int_{\mathrm{Re}(s_1)=\mathfrak{s}_1} \int_{\mathrm{Re}(t)=\mathfrak{t}} |s_1 - u_1 - t_1|^{-1-\epsilon} |1 + s_1 - r_2|^{-101\epsilon} \\
&\quad |1 + u_2 - r_1 - t_1 - t_2|^{-\frac{1}{2}+\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} \\
&\quad B\begin{pmatrix} \frac{1}{4}, \quad 4\epsilon & u_1 - 2u_2 + r_1 + t_2 \\ -\frac{3}{4} - 3\epsilon & 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\frac{1}{4} + 51\epsilon, \quad -\frac{1}{4} - 52\epsilon & u_1 - 2s_1 + r_2 - t_2 \\ \epsilon & -u_1 + r_1 - r_2 + t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\frac{1}{4}, \quad 3\epsilon & 1 - 2t_1 - t_2 \\ -\frac{1}{4} - 2\epsilon & 1 - u_1 - r_1 \end{pmatrix} |dt|\, |ds_1|\,.
\end{aligned}$$

Apply Hölder in t_2 with exponents 4, 4, $\frac{1}{\epsilon}$, $2 + 16\epsilon$, and $\frac{1}{3\epsilon}$ giving

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |1 + u_2 - r_1 - t_1 - t_2|^{-1} |1 - 2t_1 - t_2|^{-1+12\epsilon} |dt_2| \ll |u_2 - r_1 + t_1|^{-1+12\epsilon},$$

$$\begin{aligned}
&\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} |1 + u_2 - r_1 - t_1 - t_2|^{-1+4\epsilon} |2 - u_1 - r_1 - 2t_1 - t_2|^{-1+4\epsilon} |dt_2| \\
&\ll |u_1 + u_2 + t_1|^{-1+4\epsilon},
\end{aligned}$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} \frac{1}{4\epsilon}, \quad 4 & u_1 - 2u_2 + r_1 + t_2 \\ -\frac{3}{4\epsilon} - 3 & 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} |dt_2| \ll |1 + 2u_1 - 3u_2|^{-\frac{1}{2\epsilon}+2},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} -\frac{1}{2}+98\epsilon, & -\frac{1}{2}-108\epsilon & ; & u_1-2s_1+r_2-t_2 \\ & 2\epsilon & & -u_1+r_1-r_2+t_2 \end{pmatrix} |dt_2| \ll |-2s_1+r_1|^{-8\epsilon},$$

$$\int_{\mathrm{Re}(t_2)=\mathfrak{t}_2} B\begin{pmatrix} -1, & 1 & ; & 1-2t_1-t_2 \\ & -1 & & 1-u_1-r_1 \end{pmatrix} |dt_2| \ll 1,$$

so

$$\begin{aligned}
&|M_{w_l,2}| \ll \\
&|y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathrm{Re}(u_2)} |1+2u_1-3u_2|^{-\frac{1}{2}+2\epsilon} \int_{\mathrm{Re}(s_1)=\mathfrak{s}_1} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |s_1-u_1-t_1|^{-1-\epsilon} \\
&\qquad |1+s_1-r_2|^{-101\epsilon} |t_1|^{-\frac{1}{2}+\epsilon} |u_2-r_1+t_1|^{-\frac{1}{4}+3\epsilon} |u_1+u_2+t_1|^{-\frac{1}{4}+\epsilon} \\
&\qquad |-2s_1+r_1|^{-4\epsilon} \; |dt_1| \; |ds_1| \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathrm{Re}(u_2)} |1+2u_1-3u_2|^{-\frac{1}{2}+2\epsilon} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |1+u_1-r_2+t_1|^{-101\epsilon} \\
&\qquad |t_1|^{-\frac{1}{2}+\epsilon} |u_2-r_1+t_1|^{-\frac{1}{4}+3\epsilon} |u_1+u_2+t_1|^{-\frac{1}{4}+\epsilon} |-2u_1+r_1-2t_1|^{-4\epsilon} \; |dt_1| \\
&\ll |y_1|^{-\mathfrak{s}_1} |y_2|^{-\mathrm{Re}(u_2)}.
\end{aligned}$$

B.3.0.4 (d)

For $\mathrm{Re}(u) = \left(-\frac{1}{2} - 3\epsilon, -\epsilon\right)$, $\mathrm{Re}(r) = (-3\epsilon, 7\epsilon)$, $\mathfrak{s}_2 = -\frac{1}{2} - \epsilon$, $\mathfrak{t} = \left(6\epsilon, \frac{1}{2} - 6\epsilon\right)$,

$$\begin{aligned}
&|M_{w_l,3}| \ll \\
&|y_1|^{-\mathfrak{t}_1} |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{s}_2} \int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} \int_{\mathrm{Re}(t)=\mathfrak{t}} |s_2 - u_2|^{-1} \\
&\quad |1 + u_1 - r_2 + t_1|^{-4\epsilon} |1 + s_2 - r_1 - t_1 - t_2|^{-\frac{1}{2}+2\epsilon} |t_1|^{-\frac{1}{2}+6\epsilon} \\
&\quad B\begin{pmatrix} -5\epsilon, & -\frac{1}{2}+4\epsilon & u_1 - 2s_2 + r_1 + t_2 \\ & 2\epsilon & ; \; 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\frac{1}{2}+2\epsilon, & -6\epsilon & -u_1 + r_2 - 2t_1 - t_2 \\ & 5\epsilon & ; \; -u_1 + r_1 - r_2 + t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\frac{1}{4}-3\epsilon, & \frac{1}{4}+3\epsilon & 1 - 2t_1 - t_2 \\ & -\frac{1}{2} & ; \; 1 - u_1 - r_1 \end{pmatrix} |dt| \, |ds_2| \,.
\end{aligned}$$

Apply Hölder in t_2 with exponents $4 + 96\epsilon$, $2 + 32\epsilon$, $\frac{1}{5\epsilon}$, $\frac{1}{6\epsilon}$, and $4 - 48\epsilon$, and terms paired as follows:

$$\begin{aligned}
&|1 + s_2 - r_1 - t_1 - t_2|^{-\frac{1}{2}+2\epsilon} \text{ with } |1 - u_1 - r_1 - 2t_1 - t_2|^{-\frac{1}{4}+3\epsilon} \\
&|1 + u_1 - u_2 - r_1 - t_2|^{-\frac{1}{2}+9\epsilon} \text{ with } |-u_1 + r_2 - 2t_1 - t_2|^{-\frac{1}{2}+8\epsilon} \\
&B\begin{pmatrix} -5\epsilon, & -5\epsilon & u_1 - 2s_2 + r_1 + t_2 \\ & 2\epsilon & ; \; 1 + u_1 - u_2 - r_1 - t_2 \end{pmatrix} \\
&B\begin{pmatrix} -6\epsilon, & -6\epsilon & -u_1 + r_2 - 2t_1 - t_2 \\ & 5\epsilon & ; \; -u_1 + r_1 - r_2 + t_2 \end{pmatrix} \\
&|-u_1 + r_2 - 2t_1 - t_2|^{-\frac{1}{4}-3\epsilon} \text{ with } |1 - u_1 - r_1 - 2t_1 - t_2|^{-\frac{1}{4}-3\epsilon},
\end{aligned}$$

respectively, giving

$$\begin{aligned}
&|M_{w_l,3}| \ll \\
&|y_1|^{-\mathfrak{t}_1} |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{s}_2} \int_{\mathrm{Re}(s_2)=\mathfrak{s}_2} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |s_2 - u_2|^{-1} |1 + u_1 - r_2 + t_1|^{-4\epsilon} \\
&\qquad |t_1|^{-\frac{1}{2}+6\epsilon} |u_1 + s_2 + t_1|^{-\frac{1}{4}+3\epsilon} |1 + 2u_1 - u_2 - r_1 + r_2 + 2t_1|^{-\frac{1}{2}+9\epsilon} \\
&\qquad |1 + 2u_1 - u_2 - 2s_2|^{-3\epsilon} |-2u_1 + r_1 - 2t_1|^{-\epsilon} |dt_1| |ds_2| .
\end{aligned}$$

Here we have lost $\frac{1}{4}$ on the first pair; this is because the total exponent on t_2 is $\frac{5}{4}$ and the most we can accomodate on a single variable using only pairs is 2. Again, there is no choice of contours which does not result in a final bound of positive exponent that is not wasteful.

Applying Hölder in s_2 with exponents $\frac{4}{3}$, $4 + 48\epsilon$, and $\frac{1}{3\epsilon}$ gives

$$\begin{aligned}
&|M_{w_l,3}| \ll \\
&|y_1|^{-\mathfrak{t}_1} |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{s}_2} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |1 + u_1 - r_2 + t_1|^{-4\epsilon} \\
&\qquad |t_1|^{-\frac{1}{2}+6\epsilon} |u_1 + u_2 + t_1|^{-\frac{1}{4}+3\epsilon} |1 + 2u_1 - u_2 - r_1 + r_2 + 2t_1|^{-\frac{1}{2}+9\epsilon} \\
&\qquad |1 + 2u_1 - 2u_2 - u_2|^{-3\epsilon} |-2u_1 + r_1 - 2t_1|^{-\epsilon} |dt_1| \\
&\ll |y_1|^{-\mathfrak{t}_1} |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{s}_2} .
\end{aligned}$$

B.3.0.5 (e)

For $\mathrm{Re}(u) = \left(-\epsilon, -\frac{1}{2} - 3\epsilon\right)$, $\mathrm{Re}(r) = (\epsilon, 4\epsilon)$, $\mathfrak{t} = \left(-\frac{1}{2} + 5\epsilon, 1 - 10\epsilon\right)$,

$$\begin{aligned}
&|M_{w_l,4}| \ll \\
&|y_1|^{-\mathrm{Re}(u_1)-\mathfrak{t}_1} |y_2|^{-\mathrm{Re}(u_2)} \int_{\mathrm{Re}(t)=\mathfrak{t}} |1+u_2-r_1-t_1-t_2|^{-\frac{1}{2}+\epsilon} |t_1|^{-1+5\epsilon} |t_1+t_2|^{-5\epsilon} \\
&\quad B\begin{pmatrix} \frac{1}{2}-2\epsilon, & -\frac{1}{4}+6\epsilon & u_1-2u_2+r_1+t_2 \\ & -\frac{3}{4}-3\epsilon & ; & 1+u_1-u_2-r_1-t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} -\frac{1}{2}+3\epsilon, & -6\epsilon & -u_1+r_2-2t_1-t_2 \\ & 4\epsilon & ; & -u_1+r_1-r_2+t_2 \end{pmatrix} \\
&\quad B\begin{pmatrix} 0, & 0 & 1-2t_1-t_2 \\ & -\frac{1}{2} & ; & 1-u_1-r_1 \end{pmatrix} |dt|.
\end{aligned}$$

Apply Hölder in t_2 with exponents $2+4\epsilon$, $4+96\epsilon$, $\frac{1}{6\epsilon}$, and $4+144\epsilon$ using the pairs

$$\begin{aligned}
&|1+u_2-r_1-t_1-t_2|^{-\frac{1}{2}+\epsilon} \text{ with } |-u_1+r_2-2t_1-t_2|^{-\frac{1}{2}+9\epsilon} \\
&B\begin{pmatrix} \frac{1}{2}-2\epsilon, & -\frac{1}{4}+6\epsilon & u_1-2u_2+r_1+t_2 \\ & -\frac{3}{4}-3\epsilon & ; & 1+u_1-u_2-r_1-t_2 \end{pmatrix} \\
&B\begin{pmatrix} -6\epsilon, & -6\epsilon & -u_1+r_2-2t_1-t_2 \\ & 4\epsilon & ; & -u_1+r_1-r_2+t_2 \end{pmatrix} \\
&|2-u_1-r_1-2t_1-t_2|^{-\frac{1}{2}},
\end{aligned}$$

respectively (ignoring the remaining term), which gives

$$\begin{aligned}
&|M_{w_l,4}| \ll \\
&|y_1|^{-\mathrm{Re}(u_1)-\mathfrak{t}_1} |y_2|^{-\mathrm{Re}(u_2)} \int_{\mathrm{Re}(t_1)=\mathfrak{t}_1} |t_1|^{-1+5\epsilon} |1+u_1+u_2-r_1-r_2+t_1|^{-\frac{1}{2}+9\epsilon} \\
&\quad |1+2u_1-3u_2|^{-\frac{1}{4}-5\epsilon} |-2u_1+r_1-2t_1|^{-2\epsilon} |dt_1| \\
&\ll |y_1|^{-\mathrm{Re}(u_1)-\mathfrak{t}_1} |y_2|^{-\mathrm{Re}(u_2)}.
\end{aligned}$$

B.3.0.6 (f)

For $\mathrm{Re}(u) = (-3\epsilon + \delta_1, \epsilon)$, $\mathrm{Re}(r) = -\frac{1}{2} + 3\epsilon$, $\mathfrak{s} = -\epsilon$,

$$
\begin{aligned}
&|M_{w_4}| \ll \\
&|y|^{-\mathfrak{s}} \int_{\mathrm{Re}(s)=\mathfrak{s}} |s-u_1|^{-\frac{1}{2}+2\epsilon} |s-u_1-r_1|^{-\epsilon} |s-u_1-r_2|^{-\epsilon} \\
&\qquad B\begin{pmatrix} -\frac{1}{2}+2\epsilon, & -\frac{1}{4}+\epsilon & ; & 1+2u_1-2s+r_1+r_2 \\ & \frac{1}{4}+\epsilon & & -u_1-u_2-r_2 \end{pmatrix} \\
&\qquad B\begin{pmatrix} -\frac{1}{4}+\epsilon, & -\frac{1}{4}+3\epsilon & ; & 1+2u_1-2s+r_1 \\ & 0 & & -2u_1+u_2-r_1 \end{pmatrix} |ds|.
\end{aligned}
$$

Applying Hölder on s to separate the first beta function will compensate for the positive exponent, and the remainder may be bounded trivially.

B.3.0.7 (g)

For $\mathrm{Re}(u) = (\epsilon, -3\epsilon + \delta_2)$, $\mathrm{Re}(r) = -\frac{1}{2} + 3\epsilon$, $\mathfrak{s} = -\epsilon$,

$$
\begin{aligned}
&|M_{w_5}| \ll \\
&|y|^{-\mathfrak{s}} \int_{\mathrm{Re}(s)=\mathfrak{s}} |s-u_2|^{-\frac{1}{2}+2\epsilon} |s-u_2-r_1|^{-\epsilon} |s-u_2-r_2|^{-\epsilon} \\
&\qquad B\begin{pmatrix} -\frac{1}{2}+2\epsilon, & -\frac{1}{4}+e & ; & 1+2u_2-2s+r_1+r_2 \\ & \frac{1}{4}+\epsilon & & -u_1-u_2-r_2 \end{pmatrix} \\
&\qquad B\begin{pmatrix} -\frac{1}{4}+\epsilon, & -\frac{1}{4}+3\epsilon & ; & 1+2u_2-2s+r_1 \\ & 0 & & u_1-2u_2-r_1 \end{pmatrix} |ds|.
\end{aligned}
$$

Again, applying Hölder on s to separate the first beta function will compensate for the positive exponent, and the remainder may be bounded trivially.

B.4 Proof of Proposition 34

Using the M functions above, the u and r integrals separate, so we may apply Lemma 4 in u and in r separately: Since $\mathfrak{u}_i \pm \mathrm{Re}(\mu_j), \mathfrak{r}_i \pm \mathrm{Re}(\mu_j) > -1 - \epsilon$, the products

$$\begin{aligned}
&\int_{\mathrm{Re}(u)=\mathfrak{u}} |G^*(u,\mu)|\,|du| \int_{\mathrm{Re}(r)=\mathfrak{r}} |G^*(r,-\mu)|\,|dr|\,,\\
&\int_{\mathrm{Re}(u_2)=\mathfrak{u}_2} |G_l^*(1,u_2,\mu)|\,|du_2| \int_{\mathrm{Re}(r)=\mathfrak{r}} |G^*(r,-\mu)|\,|dr|\,,\\
&\int_{\mathrm{Re}(u_1)=\mathfrak{u}_1} |G_r^*(2,u_1,\mu)|\,|du_1| \int_{\mathrm{Re}(r)=\mathfrak{r}} |G^*(r,-\mu)|\,|dr|\,,\\
&G_b^*(1,2,\mu) \int_{\mathrm{Re}(r)=\mathfrak{r}} |G^*(r,-\mu)|\,|dr|\,,
\end{aligned}$$

are at most

$$\sum_{w\in W} |\mu_3^w - \mu_1^w|^{\frac{\kappa-\delta}{2}+\epsilon} |\mu_3^w - \mu_2^w|^{\frac{\kappa-\delta}{2}+\epsilon} |\mu_2^w - \mu_1^w|^{\delta+\epsilon}\,,$$

where now

$$\begin{aligned}
\delta &= \min\left\{\frac{\kappa}{3}, \delta_u + \delta_r\right\},\\
\delta_r &= \min_i \left\{\frac{2\mathfrak{r}_1 - \mathrm{Re}(\mu_i)+1}{2}, \frac{2\mathfrak{r}_2 + \mathrm{Re}(\mu_i)+1}{2}\right\},
\end{aligned}$$

and the κ and δ_u parameters are given in Table B.1. The bounds may be expressed in this manner because the worst bound happens exactly when the least exponent δ is on the least difference $|\mu_i - \mu_j|$. We have sacrificed some efficiency for symmetry in the bounds for the u residues, but it is unlikely that one could exploit what we lost in any case.

When we apply the above bounds we need to replace $\mathfrak{u}$ and $\mathfrak{r}$ with real part of the arguments of the appropriate G function, which we have done in table Table B.2. For $J_{w_l,\mu}$ the worst bound is $\frac{\kappa-\delta}{2} + \epsilon = \frac{3}{2} + 24\epsilon$, $\delta + \epsilon = -\frac{1}{2} + 44\epsilon$ and for $J_{w_4,\mu}$ and $J_{w_5,\mu}$, the worst bounds are $\frac{\kappa-\delta}{2} + \epsilon = \frac{13}{8} + 3\epsilon$ and $\delta + \epsilon = \frac{3}{4} - 4\epsilon$, so we have Proposition 34.

Form	κ	δ_u
G^*G^*	$\mathfrak{u}_1 + \mathfrak{u}_2 + \mathfrak{r}_1 + \mathfrak{r}_2 - 1$	$\min_i \left\{ \frac{2\mathfrak{u}_1 + \mathrm{Re}(\mu_i)+1}{2}, \frac{2\mathfrak{u}_2 - \mathrm{Re}(\mu_i)+1}{2} \right\}$
$G_l^*G^*$	$\mu_1 + \mathfrak{u}_2 + \mathfrak{r}_1 + \mathfrak{r}_2 - \frac{3}{2}$	$\min \left\{ \frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}, \frac{\mathrm{Re}(\mu_1-\mu_3)-1}{2}, \mathfrak{u}_2 + \frac{\mathrm{Re}(\mu_2+\mu_3)}{2} \right\}$
$G_r^*G^*$	$\mathfrak{u}_1 - \mu_2 + \mathfrak{r}_1 + \mathfrak{r}_2 - \frac{3}{2}$	$\min \left\{ \frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}, \mathfrak{u}_1 - \frac{\mathrm{Re}(\mu_1+\mu_3)}{2}, \frac{\mathrm{Re}(\mu_3-\mu_2)-1}{2} \right\}$
$G_b^*G^*$	$\mu_1 - \mu_2 + \mathfrak{r}_1 + \mathfrak{r}_2 - 2$	$\min \left\{ \frac{\mathrm{Re}(\mu_1-\mu_2)-1}{2}, \frac{\mathrm{Re}(\mu_1-\mu_3)-1}{2}, \frac{\mathrm{Re}(\mu_3-\mu_2)-1}{2} \right\}$

Table B.1: Parameters for bounding the u and r integrals.

	Uses	μ Contours	$\mathfrak{u}$	$\mathfrak{r}$	Form	$\frac{\kappa-\delta}{2}+\epsilon$	$\delta+\epsilon$
$J_{w_l,\mu}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	G^*G^*	$\frac{3}{2}+24\epsilon$	$-\frac{1}{2}+44\epsilon$
$J_{w_l,\mu}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	$G_l^*G^*$	$\frac{11}{8}+23\epsilon$	$-\frac{3}{4}+47\epsilon$
$J_{w_l,\mu}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	$G_r^*G^*$	$\frac{11}{8}+23\epsilon$	$-\frac{3}{4}+47\epsilon$
$J_{w_l,\mu}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	$G_b^*G^*$	$\frac{9}{8}+24\epsilon$	$-\frac{3}{4}+47\epsilon$
$J_{w_4,\mu}$	M_{w_4}	$(-2\epsilon, 0)$	$(-3\epsilon, \epsilon)$	$\left(4-7\epsilon, \frac{3}{2}\right)$	G^*G^*	$\frac{13}{8}+2\epsilon$	$\frac{3}{4}-4\epsilon$
$J_{w_4,\mu}$	M_{w_4}	$(-2\epsilon, 0)$	$(-3\epsilon, \epsilon)$	$\left(4-6\epsilon, \frac{3}{2}\right)$	$G_l^*G^*$	$\frac{13}{8}+2\epsilon$	$\frac{1}{4}$
$J_{w_5,\mu}$	M_{w_5}	$(0, -2\epsilon)$	$(\epsilon, -3\epsilon)$	$\left(4-6\epsilon, \frac{3}{2}-\epsilon\right)$	G^*G^*	$\frac{13}{8}+3\epsilon$	$\frac{3}{4}-4\epsilon$
$J_{w_5,\mu}$	M_{w_5}	$(0, -2\epsilon)$	$(\epsilon, -3\epsilon)$	$\left(4-5\epsilon, \frac{3}{2}\right)$	$G_r^*G^*$	$\frac{13}{8}+3\epsilon$	$\frac{1}{4}+\epsilon$
$E_{w_l,1}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	$G_r^*G^*$	$\frac{11}{8}+23\epsilon$	$-\frac{3}{4}+47\epsilon$
$E_{w_l,2}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	$G_l^*G^*$	$\frac{11}{8}+23\epsilon$	$-\frac{3}{4}+47\epsilon$
$E_{w_l,3}$	$M_{w_l,1}$	$\left(-\frac{1}{2}-3\epsilon, \frac{1}{2}+3\epsilon\right)$	$-\frac{1}{2}-4\epsilon$	$\left(\frac{7}{2}-4\epsilon, 1+100\epsilon\right)$	G^*G^*	$\frac{3}{2}+24\epsilon$	$-\frac{1}{2}+48\epsilon$
$E_{w_l,4}$	$M_{w_l,2}$	$\left(-\epsilon, \frac{1}{2}+3\epsilon\right)$	–	$(3-4\epsilon, 1+100\epsilon)$	$G_b^*G^*$	$1+24\epsilon$	$-\frac{1}{2}+48\epsilon$
$E_{w_l,5}$	$M_{w_l,3}$	$\left(-\frac{1}{2}-3\epsilon, \epsilon\right)$	–	$(3-3\epsilon, 1+7\epsilon)$	$G_b^*G^*$	$1+\epsilon$	$-\frac{1}{2}+2\epsilon$
$E_{w_l,6}$	$M_{w_l,1}$	$(-\epsilon, \epsilon)$	–	$(3, 1)$	$G_b^*G^*$	$1+\epsilon$	0
$E_{w_l,7}$	$M_{w_l,4}$	$\left(-\epsilon, \frac{1}{2}+3\epsilon\right)$	–	$(3+\epsilon, 1+4\epsilon)$	$G_b^*G^*$	$1+3\epsilon$	$-\frac{1}{2}$

Table B.2: Parameters for Proposition 34.

B.5 Proof of Proposition 40

After applying the definition of k_{conv}, we have

$$\begin{aligned}
L_1 = &\int_{\mathrm{Re}(q)=\mathfrak{q}} |q_1|^{\frac{23}{12}+300\epsilon-8} |q_2|^{\frac{23}{12}+300\epsilon-8} |q_2-q_1|^{\frac{23}{12}+300\epsilon} \\
&\quad |q_1+q_2|^{\frac{31}{14}+2\epsilon} |2q_1-q_2|^{\frac{6}{7}+\epsilon} |2q_2-q_1|^{\frac{6}{7}+\epsilon} \\
&\int_{\mathrm{Re}(\mu)=\eta} |\mu_1|^{-\frac{23}{12}-300\epsilon} |\mu_2|^{-\frac{23}{12}-300\epsilon} |\mu_3|^{-\frac{23}{12}-300\epsilon} \\
&\quad \frac{|\mu_1-\mu_2|^{\frac{13}{8}+100\epsilon} |\mu_1-\mu_3|^{\frac{13}{8}+100\epsilon} |\mu_2-\mu_3|^{\frac{13}{8}+100\epsilon}}{|q_1-\mu_1|\,|q_1-\mu_2|\,|q_1-\mu_3|\,|q_2+\mu_1|\,|q_2+\mu_2|\,|q_2+\mu_3|}\ |d\mu|\ |dq| \\
\ll &\int_{\mathrm{Re}(\mu)=\eta} |\mu_1|^{-\frac{2}{3}-100\epsilon} |\mu_2|^{-\frac{2}{3}-100\epsilon} |\mu_3|^{-\frac{2}{3}-100\epsilon} \\
&\int_{\mathrm{Re}(q)=\mathfrak{q}} \frac{|\mu_1-\mu_2|\,|\mu_1-\mu_3|\,|\mu_2-\mu_3|}{|q_1-\mu_1|\,|q_1-\mu_2|\,|q_1-\mu_3|\,|q_2+\mu_1|\,|q_2+\mu_2|\,|q_2+\mu_3|}\ |dq|\ |d\mu|\,.
\end{aligned}$$

using $|q_1+q_2| \ll |q_1|\,|q_2|$, etc. Applying Hölder to the q integrals, gives 6 integrals of the form

$$\left(\int_{\mathrm{Re}(q_i)=\mathfrak{q}_i} (|q_i-\mu_j|\,|q_i-\mu_k|)^{-\frac{3}{2}}\ |dq_i|\right)^{\frac{1}{3}} \ll |\mu_k-\mu_j|^{-\frac{1}{2}}\,,$$

so

$$L_1 \ll \int_{\mathrm{Re}(\mu)=\eta} |\mu_1|^{-\frac{2}{3}-100\epsilon} |\mu_2|^{-\frac{2}{3}-100\epsilon} |\mu_3|^{-\frac{2}{3}-100\epsilon}\ |d\mu| \ll 1.$$

The second integral becomes

$$L_2 = \int_{\mathrm{Re}(\mu)=\eta} \int_{\mathrm{Re}(q_2)=\mathfrak{q}_2} |\mu_1|^{-8} |q_2|^{\frac{23}{12}+300\epsilon-8} |q_2-\mu_1|^{\frac{23}{12}+300\epsilon}$$
$$\frac{|q_2+\mu_1|^{1+4\epsilon} |2\mu_1-q_2|^{\frac{3}{2}+5\epsilon} |2q_2-\mu_1|^{-\epsilon}}{|q_2+\mu_2| |q_2+\mu_3|}$$
$$|\mu_2|^{-\frac{23}{12}-300\epsilon} |\mu_3|^{-\frac{23}{12}-300\epsilon}$$
$$|\mu_1-\mu_3|^{\frac{1}{2}+100\epsilon} |\mu_2-\mu_1|^{\frac{1}{2}+100\epsilon} |\mu_2-\mu_3|^{\frac{3}{2}+100\epsilon} \; |dq_2| \; |d\mu|$$
$$\ll \int_{\mathrm{Re}(\mu)=\eta} |\mu_1|^{-\frac{31}{12}+510\epsilon} |\mu_2|^{-\frac{11}{12}-100\epsilon} |\mu_3|^{-\frac{11}{12}-100\epsilon}$$
$$\int_{\mathrm{Re}(q_2)=\mathfrak{q}_2} \frac{|\mu_2-\mu_3|}{|q_2+\mu_2| |q_2+\mu_3|} \; |dq_2| \; |d\mu|$$
$$\ll 1,$$

and the third is

$$L_3 \ll \int_{\mathrm{Re}(\mu)=\eta} |\mu_1|^{-\frac{5}{2}+300\epsilon} |\mu_2|^{-\frac{5}{2}+300\epsilon} |d\mu| \ll 1.$$

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