Generalized Instrumental Variables

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Abstract

This paper concerns the assessment of direct causal effects from a combination of: (i) non-experimental data, and (ii) qualitative domain knowledge. Domain knowledge is encoded in the form of a directed acyclic graph (DAG), in which all interactions are assumed linear, and some variables are presumed to be unobserved. We provide a generalization of the well-known method of Instrumental Variables, which makes allows its application to models with few conditional independeces.

1 Introduction

This paper explores the feasibility of inferring linear cause-effect relationships from various combinations of data and theoretical assumptions. The assumptions are represented in the form of an acyclic causal diagram which contains both arrows and bi-directed arcs [9, 10]. The arrows represent the potential existence of direct causal relationships between the corresponding variables, and the bi-directed arcs represent spurious correlations due to unmeasured common causes. All interactions among variables are assumed to be linear. Our task is to decide whether the assumptions represented in the diagram are sufficient for assessing the strength of causal effects from non-experimental data, and, if sufficiency is proven, to express the target causal effect in terms of estimable quantities.

This decision problem has been tackled in the past half century, primarily by econometricians and social scientists, under the rubric "The Identification Problem" [6] – it is still unsolved. Certain restricted classes of models are nevertheless known to be identifiable, and these are often assumed by social scientists as a matter of convenience or convention [5]. A hierarchy of three such classes is given in [7]: (1) no bidirected arcs, (2) bidirected arcs restricted to root variables, and (3) bidirected arcs restricted to variables that are not connected through directed paths.

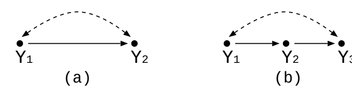

Figure 1: (a) a "bow-pattern", and (b) a bow-free model

Recently, [4] have shown that the identification of the entire model is ensured if variables standing in direct causal relationship (i.e., variables connected by arrows in the diagram) do not have correlated errors; no restrictions need to be imposed on errors associated with indirect causes. This class of models was called "bow-free", since their associated causal diagrams are free of any "bow pattern" [10] (see Figure 1).

Most existing conditions for Identification in general models are based on the concept of Instrumental Variables (IV) [11], [2]. IV methods take advantage of conditional independence relations implied by the model to prove the Identification of specific causal-effects. When the model is not rich in conditional independences, these methods are not much informative. In [3], we proposed a new graphical criterion for Identification which does not make direct use of conditional independence, and thus can be successfully applied to models in which IV methods would fail.

In this paper, we provide an important generalization of the method of Instrumental Variables that reduces the impact of the independence relations implied by the model on the performance of the method.

2 Linear Models and Identification

A linear model for a set of random variables $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ is defined by a set of equations of the form

$$Y_j = \sum_i c_{ji} Y_i + e_j \qquad , j = 1, \ldots, n$$

where the error terms e_j are assumed to have normal distribution with zero mean, and variance/covariance matrix Ψ, $[\Psi_{ij}] = Cov(e_i, e_j)$.

$Z = e_1$
$W = e_2$
$X = aZ + e_3$
$Y = bW + cX + e_4$
$Cov(e_1, e_2) = \alpha \neq 0$
$Cov(e_2, e_3) = \beta \neq 0$
$Cov(e_3, e_4) = \gamma \neq 0$

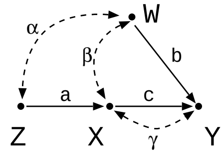

Figure 2: A simple linear model and its causal diagram

An equation $Y = \beta X + e$ encodes two distinct assumptions: (1) the possible existence of (direct) causal influence of X on Y; and, (2) the absence of causal influence on Y of any variable that does not appear on the right-hand side of the equation. The parameter β quantifies the (direct) causal effect of X on Y. That is, the equation claims that a unit increase in X would result in β units increase of Y, assuming that everything else remains the same. The variable e is called an "error" or "disturbance"; it represents unobserved background factors that the modeler decides to keep unexplained.

The equations and the pairs of error-terms (e_i, e_j) with non-zero correlation define the structure of the model. The model structure can be represented by a directed graph, called causal diagram, in which the set of nodes is defined by the variables $Y_1, \ldots, Y_n$, and there is a directed edge from Y_i to Y_j if the coefficient of Y_i in the equation for Y_j is different from zero. Additionally, if error-terms e_i and e_j have non-zero correlation, we add a (dashed) bidirected edge between Y_i and Y_j. Figure 2 shows a model with the respective causal diagram.

In this work, we consider only recursive models, that is, $c_{ji} = 0$ for $i \geq j$. The structural parameters of the model, denoted by θ, are the coefficients c_{ij}, and the non-zero entries of the error covariance matrix Ψ.

Fixing the model structure and assigning values to the parameters θ, the model determines a unique covariance matrix Σ over the observed variables $\{Y_1, \ldots, Y_n\}$, given by (see [1], page 85)

$$\Sigma(\theta) = (I - C)^{-1}\Psi\left[(I - C)^{-1}\right]^T \qquad (1)$$

where C is the matrix of coefficients c_{ji}.

Conversely, in the Identification problem, after fixing the structure of the model, one attempts to solve for θ in terms of the observed covariance Σ. This is not always possible. In some cases, no parametrization of the model could be compatible with a given Σ. In other cases, the structure of the model may permit several distinct solutions for the parameters. In these cases, the model is called *nonidentified.*

However, even if the model is nonidentifiable, some parameters may be uniquely determined from the given assumptions and data. Whenever this is the case, the specific parameters are *identified.*

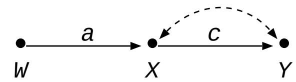

Figure 3: Typical Instrumental Variable

Finally, since the conditions we seek involve the structure of the model alone, and do not depend on the numerical values of parameters θ, we insist only on having identification almost everywhere, allowing few pathological exceptions. The concept of identification almost everywhere is formalized in section 5.

3 Graph Background

Definition 1 *A* path *in a graph is a sequence of edges (directed or bidirected) such that each edge starts in the node ending the preceding edge. A* directed path *is a path composed only by directed edges, all oriented in the same direction. Node X is a* descendent *of node Y if there is a directed path from Y to X. Node Z is a collider in a path p if there is a pair of consecutive edges in p such that both edges are oriented toward Z (e.g.,$\ldots \rightarrow Z \leftarrow \ldots$).*

Let p be a path between X and Y, and let Z be an intermediate variable in p. We denote the subpath of p consisting of the edges between X and Z by $p[X \sim Z]$.

Definition 2 (*d-separation*)
A set of nodes $\mathbf{Z}$ d-separates *X from Y in a graph, if Z* blocks *every path between X and Y. A path p is blocked by a set $\mathbf{Z}$ (possibly empty) if one of the following holds:*

(i) p contains at least one non-collider that is in $\mathbf{Z}$;

(ii) p contains at least one collider that is outside $\mathbf{Z}$ and has no descendant in $\mathbf{Z}$.

4 Instrumental Variable Methods

The traditional definition qualifies a variable Z as instrumental, relative to a cause X and effect Y if [10]:

1. Z is independent of all error terms that have an influence on Y that is not mediated by X;
2. Z is not independent of X.

The intuition behind this definition is that all correlation between Z and Y must be intermediated by X. If we can find Z with these properties, then the causal effect of X on Y, denoted by c, is identified and given by $c = \sigma_{ZY}/\sigma_{ZX}$.

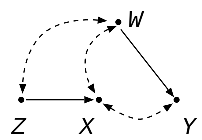

Figure 4: IV Examples

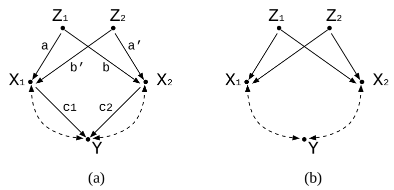

Figure 5: Simultaneous use of two IVs

Figure 3 shows a typical example of an instrumental variable. It is easy to verify that variable Z satisfy properties (1) and (2) in this model.

A generalization of the IV method is offered through the use of conditional IV's. A conditional IV is a variable Z that may not have properties (1) and (2), but there is a conditioning set $\mathbf{W}$ which makes it happens. When such pair $(Z, \mathbf{W})$ is found, the causal effect of X on Y is identified and given by $c = \sigma_{ZY.\mathbf{W}} / \sigma_{ZX.\mathbf{W}}$.

[11] provides the following equivalent graphical criterion for conditional IV's, based on the concept of d-separation:

1. $\mathbf{W}$ contains only non-descendents of Y;
2. $\mathbf{W}$ d-separates Z from Y in the subgraph G_c obtained by removing edge $X \to Y$ from G;
3. $\mathbf{W}$ does not d-separate Z from X in G_c.

As an example of the application of this criterion, Figure 4 show the graph obtained by removing edge $X \to Y$ from the model of Figure 2. After conditioning on variable W, Z becomes d-separated from Y but not from X. Thus, parameter c is identified.

However, although the method of conditional IV's is very useful, it cannot be applied to a simple model like the one in Figure ($5a$). In this case, variables Z_1 and Z_2 do not qualify as IV's with respect to either c_1 or c_2. Also, there is no conditioning set which makes it happens. Therefore, the conditional IV method fails, despite the fact that the model is completely identified.

Figure ($5b$) shows the graph obtained by removing edges $X_1 \to Y$ and $X_2 \to Y$ from the model. Note that in this graph, Z_1 and Z_2 satisfy the graphical conditions for a conditional IV. Intuitively, if we could use both $Z1$ and Z_2 together as instrumental variables, we would be able to identify parameters c_1 and c_2.

Next theorem states the main result of this paper, which extends the method of conditional IV's to allow the use of multiple instrumental variables to obtain the simultaneous identification of a subset of parameters of the model.

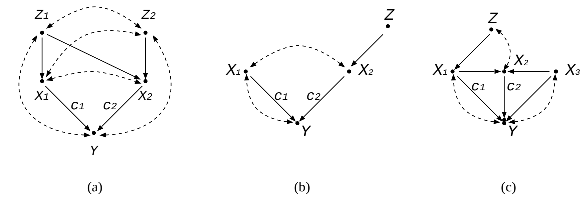

Figure 6: More examples of new criterion

Theorem 1 *Fix a variable Y, and consider the edges $X_1 \to Y, \ldots, X_n \to Y$, in a causal graph G. Assume that we can find triples $(Z_1, \mathbf{W}_1, p_1)$, ..., $(Z_n, \mathbf{W}_n, p_n)$, such that:*

(i) For $i = 1, \ldots, n$, Z_i and the elements of $\mathbf{W}_i$ are non-descendents of Y; and p_i is an unblocked path between Z_i and Y including edge $X_i \to Y$.

(ii) Let $\overline{G}$ be the causal graph obtained from G by deleting edges $X_1 \to Y, \ldots, X_n \to Y$. Then, $\mathbf{W}_i$ d-separates Z_i from Y in $\overline{G}$;

(iii) $\mathbf{W}_i$ does not block path p_i in G, that is, no variable in p_i belongs to $\mathbf{W}_i$.

(iv) For $1 \le i < j \le n$, variable Z_j does not appear in path p_i, and if paths p_i and p_j have a common variable V, then both $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V.

Then, the parameters of edges $X_1 \to Y, \ldots, X_n \to Y$ are identified almost everywhere, and can be computed by solving a system of linear equations.

Figure 6 shows more examples in which the method of conditional IV's fails and our new criterion is able to prove the identification of parameters c_i's. In particular, model (a) is a bow-free model, and thus is completely identifiable. Model (b) illustrates an interesting case in which variable X_2 is used as the instrumental variable for $X_1 \to Y$, while Z is the instrumental variable for $X_2 \to Y$. Finally, in model (c) we have an example in which the parameter of edge $X_3 \to Y$ is nonidentifiable, and still the method can prove the identification of c_1 and c_2.

5 Preliminary Results

5.1 Identification Almost Everywhere

Let h denote the total number of parameters in model G. Then, each vector $\theta \in \Re^h$ defines a parametrization of the model. For each parametrization θ, model G generates a unique covariance matrix $\Sigma(\theta)$. Let $\theta(\lambda_1, \ldots, \lambda_n)$ denote the vector of values assigned by θ to parameters $\lambda_1, \ldots, \lambda_n$.

Parameters $\lambda_1, \ldots, \lambda_n$, are identified almost everywhere if $\Sigma(\theta) = \Sigma(\theta')$ implies $\theta(\lambda_1, \ldots, \lambda_n) = \theta'(\lambda_1, \ldots, \lambda_n)$, except when θ resides on a set of Lebesgue measure zero.

5.2 Wright's Method of Path Coefficients

Here, we describe an important result introduced by Sewall Wright [12], which is extensively explored in the proof.

Given variables X and Y in a recursive linear model, the correlation coefficient of X and Y, denoted ρ_{XY}, can be expressed as a polynomial on the parameters of the model. More precisely,

$$\sigma_{Z,Y} = \sum_{\text{paths } p_l} T(p_l) \tag{2}$$

where term $T(p_l)$ represents the multiplication of the parameters of edges along path p_l, and the summation ranges over all unblocked paths between X and Y. We refer to Eq.(2) as Wright's Equation for X and Y.

Wright's method of path coefficients [12] consists in forming Eq.(2) for each pair of variables in the model, and solving for the parameters in terms of the correlations among the variables. Whenever there is a unique solution for a parameter λ, this parameter is identified.

We can use this method to study the identification of the parameters in the model of Figure 5. From the equations for ρ_{Y_1,Y_5} and ρ_{Y_2,Y_5} we can see that parameters c_1 and c_2 are identified if and only if $Det \begin{bmatrix} a & a' \\ b & b' \end{bmatrix} \neq 0$

5.3 Partial Correlation Lemma

Next lemma provides a convenient expression for the partial correlation coefficient of Y_1 and Y_2, given $Y_3, \ldots, Y_n$, denoted $\rho_{12.3\ldots n}$. The proof of the lemma is given in the appendix.

Lemma 1 *The partial correlation $\rho_{12.3\ldots n}$ can be expressed as the ratio:*

$$\rho_{12.3\ldots n} = \frac{\phi(1, 2, \ldots, n)}{\psi(1, 3, \ldots, n) \cdot \psi(2, 3, \ldots, n)} \tag{3}$$

where ϕ and psi are functions of the correlations among $Y_1, Y_2, \ldots, Y_n$, satisfying the following conditions:

(i) $\phi(1, 2, \ldots, n) = \phi(2, 1, \ldots, n)$.

(ii) $\phi(1, 2, \ldots, n)$ *is linear on the correlations* $\rho_{12}, \rho_{32}, \ldots, \rho_{n2}$, *with no constant term.*

(iii) The coefficients of $\rho_{12}, \rho_{32}, \ldots, \rho_{n2}$, *in* $\phi(1, 2, \ldots, n)$ *are polynomials on the correlations among the variables* $Y_1, Y_3, \ldots, Y_n$. *Moreover, the coefficient of* ρ_{12} *has the constant term equal to 1, and the coefficients of* $\rho_{32}, \ldots, \rho_{n2}$, *are linear on the correlations* $\rho_{13}, \rho_{14}, \ldots, \rho_{1n}$, *with no constant term.*

(iv) $(\psi(i_1, \ldots, i_{n-1}))^2$, *is a polynomial on the correlations among the variables* $Y_{i_1}, \ldots, Y_{i_{n-1}}$, *with constant term equal to 1.*

5.4 Path Lemmas

The following lemmas explore some consequences of the conditions in Theorem 1.

Lemma 2 *W.l.o.g., we may assume that, for $1 \leq i < j \leq n$, paths p_i and p_j do not have any common variable other than (possibly) Z_i.*

Proof: Assume that paths p_i and p_j have some variables in common distinct from Z_i. Let V be the closest variable to X_i in path p_i which also belongs to path p_j.

We show that after replacing triple $(Z_i, \mathbf{W}_i, p_i)$ by triple $(V, \mathbf{W}_i, p_i[V \sim X_i])$, the conditions of Theorem 1 still hold.

It follows from condition (iv) that subpath $p_i[V \sim Y]$ must point to V. Since p_i is unblocked, subpath $p_i[Z_i \sim V]$ must be a directed path from V to Z_i.

Now, variable V cannot be a descendent of Y, because $p_i[Z_i \sim V]$ is a directed path from V to Z_i, and Z_i is a non-descendent of Y. Thus, condition (i) still holds.

Consider the causal graph $\overline{G}$. Assume that there exists a path p between V and Y witnessing that $\mathbf{W}_i$ does not d-separate V from Y in $\overline{G}$. Since $p_i[Z_i \sim V]$ is a directed path from V to Z_i, we can always find another path witnessing that $\mathbf{W}_i$ does not d-separate Z_i from Y in $\overline{G}$ (for example, if p and $p_i[Z_i \sim V]$ do not have any variable in common other than V, then we can just take their concatenation). But this is a contradiction, thus condition (ii) still holds.

It is easy to see that condition (iii) holds. Condition (iv) follows from the fact that $p_i[V \sim Y]$ and $p_j[Z_j \sim V]$ point to V. □

In the following, we assume that the conditions of lemma 2 hold.

Lemma 3 *For all $1 \leq i \leq n$, there exists no unblocked path between Z_i and Y, different from p_i, which includes edge $X_i \rightarrow Y$ and is composed only by edges from $p_1, \ldots, p_i$.*

Proof: Let p be an unblocked path between Z_i and Y, different from p_i, and assume that p is composed only by edges from $p_1, \ldots, p_i$.

According to condition (iv), if Z_i appears in some path p_j, with $j \neq i$, then it must be that $j > i$. Thus, p must start with some edges of p_i.

Since p is different from p_i, it must contain at least one edge from $p_1, \dots, p_{i-1}$. Let (V_1, V_2) denote the first edge in p which does not belong to p_i.

From lemma 2, it follows that variable V_1 must be a Z_k for some $k < i$, and by condition (iv), both subpath $p[Z_i \sim V_1]$ and edge (V_1, V_2) must point to V_1. But this implies that p is blocked by V_1, which contradicts our assumptions. $\square$

The proofs for the next two lemmas are very similar to the previous one, and so are omitted.

Lemma 4 *For all $1 \leq i \leq n$, there is no unblocked path between Z_i and some W_{i_j} composed only by edges from $p_1, \dots, p_i$.*

Lemma 5 *For all $1 \leq i \leq n$, there is no unblocked path between Z_i and Y including edge $X_j \rightarrow Y$, with $j < i$, composed only by edges from $p_1, \dots, p_i$.*

6 Proof of Theorem 1

6.1 Notation

Fix a variable Y in the model. Let $\mathbf{X} = \{X_1, \dots, X_k\}$ be the set of all non-descendents of Y which are connected to Y by an edge (directed, bidirected, or both). Define the following set of edges incoming Y:

$$Inc(Y) = \{(X_i, Y) : X_i \in \mathbf{X}\}$$

Note that for some $X_i \in \mathbf{X}$ there may be more than one edge between X_i and Y (one directed and one bidirected). Thus, $|Inc(Y)| \geq |\mathbf{X}|$. Let $\lambda_1, \dots, \lambda_m$, $m \geq k$, denote the parameters of the edges in $Inc(Y)$.

It follows that edges $X_1 \rightarrow Y, \dots, X_n \rightarrow Y$, belong to $Inc(Y)$, because $X_1, \dots, X_n$, are clearly non-descendents of Y. W.l.o.g., let λ_i be the parameter of edge $X_i \rightarrow Y$, $1 \leq i \leq n$, and let $\lambda_{n+1}, \dots, \lambda_m$ be the parameters of the remaining edges in $Inc(Y)$.

Let Z be any non-descendent of Y. Wright's equation for the pair (Z, Y), is given by

$$\sigma_{Z,Y} = \sum_{\text{paths } p_l} T(p_l) \tag{4}$$

where each term $T(p_l)$ corresponds to an unblocked path between Z and Y. Next lemma proves a property of such paths.

Lemma 6 *Let Y be a variable in a recursive model, and let Z be a non-descendent of Y. Then, any unblocked path between Z and Y must include exactly one edge from $Inc(Y)$.*

Lemma 6 allows us to write Eq. (4) as

$$\sigma_{Z,Y} = \sum_{j=1}^{m} a_j \cdot \lambda_j \tag{5}$$

$$\sigma_{24} = \beta\lambda_1 + a\alpha\lambda_1 + \lambda_2$$
$$\sigma_{34} = \lambda_1 + \beta\lambda_2 + a\alpha\lambda_2$$

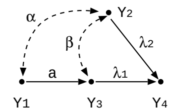

Figure 7: Wright's equations

Thus, the correlation between Z and Y can be expressed as a linear function of the parameters $\lambda_1, \dots, \lambda_m$, with no constant term. Figure 7 shows an example of those equations for a simple model.

6.2 Basic Linear Equations

Consider a triple $(Z_i, \mathbf{W}_i, p_i)$, and let $\mathbf{W}_i = \{W_{i_1}, \dots, W_{i_k}\}$ [1]. From lemma 1, we can express the partial correlation of Z_i and Y given $\mathbf{W}_i$ as:

$$\rho_{Z_i Y . \mathbf{W}_i} = \frac{\phi_i(Z_i, Y, W_{i_1}, \dots, W_{i_k})}{\psi_i(Z_i, W_{i_1}, \dots, W_{i_k}) \cdot \psi_i(Y, W_{i_1}, \dots, W_{i_k})} \tag{6}$$

where function ϕ_i is linear on the correlations $\rho_{Z_i Y}$, $\rho_{W_{i_1} Y}$, $\dots$, $\rho_{W_{i_k} Y}$, and ψ_i is a function of the correlations among the variables given as arguments. We abbreviate $\phi_i(Z_i, Y, W_{i_1}, \dots, W_{i_{k_i}})$ by $\phi_i(Z_i, Y, \mathbf{W}_i)$, and $\psi_i(V, W_{i_1}, \dots, W_{i_k})$ by $\psi_i(V, \mathbf{W}_i)$.

We have seen that the correlations $\rho_{Z_i Y}$, $\rho_{W_{i_1} Y}$, $\dots$, $\rho_{W_{i_k} Y}$, can be expressed as linear functions of the parameters $\lambda_1, \dots, \lambda_m$. Since ϕ_i is linear on these correlations, it follows that we can express ϕ_i as a linear function of the parameters $\lambda_1, \dots, \lambda_m$.

Formally, by lemma 1, $\phi_i(Z_i, Y, \mathbf{W}_i)$ can be written as:

$$\begin{aligned} \phi_i(Z_i, Y, \mathbf{W}_i) = \ & b_{i_0} \rho_{Z_i Y} + b_{i_1} \rho_{W_{i_1} Y} + \\ & \dots + b_{i_k} \rho_{W_{i_k} Y} \end{aligned} \tag{7}$$

Also, for each $V_j \in \{Z_i\} \cup \mathbf{W}_i$, we can write:

$$\rho_{V_j Y} = a_{i_j 1} \lambda_1 + \dots + a_{i_j m} \lambda_m \tag{8}$$

Replacing each correlation in Eq.(7) by the expression given by Eq. (8), we obtain

$$\phi_i(Z_i, Y, \mathbf{W}_i) = q_{i1} \lambda_1 + \dots + q_{im} \lambda_m \tag{9}$$

where the coefficients q_{il}'s are given by:

$$q_{il} = \sum_{j=0}^{k} b_{i_j} a_{i_j l} \qquad , l = 1, \dots, m \tag{10}$$

Lemma 7 *The coefficients $q_{i,n+1}, \dots, q_{im}$ in Eq. (9) are identically zero.*

[1] To simplify the notation, we assume that $|\mathbf{W}_i| = k$, for $i = 1, \dots, n$

Proof: The fact that $\mathbf{W}_i$ d-separates Z_i from Y in $\overline{G}$, implies that $\rho_{Z_iY \cdot \mathbf{W}_i} = 0$ in any probability distribution compatible with $\overline{G}$ ([10], pg. 142). Thus, $\phi_i(Z_i, Y, \mathbf{W}_i)$ must vanish when evaluated in $\overline{G}$. But this implies that the coefficient of each of the λ_i's in Eq. (9) must be identically zero.

Now, we show that the only difference between evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on the causal graphs $\overline{G}$ and G, consists on the coefficients of parameters $\lambda_1, \ldots, \lambda_n$.

First, observe that coefficients $b_{i_0}, \ldots, b_{i_k}$ are polynomials on the correlations among the variables $Z_i, W_{i_1}, \ldots, W_{i_k}$. Thus, they only depend on the unblocked paths between such variables in the causal graph. However, the insertion of edges $X_1 \to Y, \ldots, X_n \to Y$, in $\overline{G}$ does not create any new unblocked path between any pair of $Z_i, W_{i_1}, \ldots, W_{i_k}$ (and obviously does not eliminate any existing one). Hence, the coefficients $b_{i_0}, \ldots, b_{i_k}$ have exactly the same value in the evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on $\overline{G}$ and G.

Now, let λ_l be such that $l > n$, and let $V_j \in \{Z_i\} \cup \mathbf{W}_i$. Note that the insertion of edges $X_1 \to Y, \ldots, X_n \to Y$, in $\overline{G}$ does not create any new unblocked path between V_j and Y including the edge whose parameter is λ_l (and does not eliminate any existing one). Hence, coefficients a_{i_jl}, $j = 0, \ldots, k$, have exactly the same value on $\overline{G}$ and G.

From the two previous facts, we conclude that, for $l > n$, the coefficient of λ_l in the evaluations of $\phi_i(Z_i, Y, \mathbf{W}_i)$ on $\overline{G}$ and G have exactly the same value, namely zero. Next, we argue that $\phi_i(Z_i, Y, \mathbf{W}_i)$ does not vanish when evaluated on G.

Finally, let λ_l be such that $l \leq n$, and let $V_j \in \{Z_i\} \cup \mathbf{W}_i$. Note that there is no unblocked path between V_j and Y in $\overline{G}$ including edge $X_l \to Y$, because this edge does not exist in $\overline{G}$. Hence, the coefficient of λ_l in the expression for the correlation ρ_{V_jY} on $\overline{G}$ must be zero.

On the other hand, the coefficient of λ_l in the same expression on G is not necessarily zero. In fact, it follows from the conditions on Theorem 1 that, for $l = i$, the coefficient of λ_i contains the term $T(p_i)$. □.

From lemma 7, we get that $\phi_i(Z_i, Y, \mathbf{W}_i)$ is a linear function only on the parameters $\lambda_1, \ldots, \lambda_n$.

6.3 System of Equations Φ

Rewriting Eq.(6) for each triple $(Z_i, \mathbf{W}_i, p_i)$, we obtain the following system of linear equations on the parameters $\lambda_1, \ldots, \lambda_n$:

$$\Phi = \begin{cases} \phi_1(Z_1, Y, \mathbf{W}_1) = & \rho_{Z_1Y \cdot \mathbf{W}_1} \\ & \cdot \psi_1(Z_1, \mathbf{W}_1) \cdot \psi_1(Y, \mathbf{W}_1) \\ \ldots & \\ \phi_n(Z_n, Y, \mathbf{W}_n) = & \rho_{Z_nY \cdot \mathbf{W}_n} \\ & \cdot \psi_n(Z_n, \mathbf{W}_n) \cdot \psi_n(Y, \mathbf{W}_n) \end{cases}$$

where the terms on the right-hand side can be computed from the correlations among the variables $Y, Z_i, W_{i_1}, \ldots, W_{i_k}$, estimated from data.

Our goal is to show that Φ can be solved uniquely for the λ_i's, and so prove the identification of $\lambda_1, \ldots, \lambda_n$. Next lemma proves an important result in this direction. Let Q denote the matrix of coefficients of Φ.

Lemma 8 *$Det(Q)$ is a non-trivial polynomial on the parameters of the model.*

Proof: From Eq.(10), we get that each entry q_{il} of Q is given by

$$q_{il} = \sum_{j=0}^{k} b_{i_j} \cdot a_{i_jl}$$

where b_{i_j} is the coefficient of $\rho_{W_{i_j}Y}$ (or ρ_{Z_iY}, if $j = 0$), in the linear expression for $\phi_i(Z_i, Y, \mathbf{W}_i)$ in terms of correlations (see Eq.(7)); and a_{i_jl} is the coefficient of λ_l in the expression for the correlation $\rho_{W_{i_j}Y}$ in terms of the parameters $\lambda_1, \ldots, \lambda_m$ (see Eq.(8)).

From property (iii) of lemma 1, we get that b_{i_0} has constant term equal to 1. Thus, we can write $b_{i_0} = 1 + \hat{b}_{i_0}$, where $\hat{b}_{i_0}$ represent the remaining terms of b_{i_0}.

Also, from condition (i) of Theorem 1, it follows that a_{i_0i} contains term $T(p_i)$. Thus, we can write $a_{i_0i} = T(p_i) + \hat{a}_{i_0i}$, where $\hat{a}_{i_0i}$ represents all the remaining terms of a_{i_0i}.

Hence, a diagonal entry q_{ii} of Q, can be written as

$$q_{ii} = T(p_i)[1 + \hat{b}_{i_0}] \; + \; \hat{a}_{i_0i} \cdot b_{i_0} \; + \; \sum_{j=1}^{k} b_{i_j} \cdot a_{i_ji} \quad (11)$$

Now, the determinant of Q is defined as the weighted sum, for all permutations π of $\langle 1, \ldots, n \rangle$, of the product of the entries selected by π (entry q_{il} is selected by permutation π if the i^{th} element of π is l), where the weights are 1 or (-1), depending on the parity of the permutation. Then, it is easy to see that the term

$$T^* = \prod_{j=1}^{n} T(p_j)$$

appears in the product of permutation $\pi = \langle 1, \ldots, n \rangle$, which selects all the diagonal entries of Q.

We prove that $det(Q)$ does not vanish by showing that T^* appears only once in the product of permutation $\langle 1, \ldots, n \rangle$, and that T^* does not appear in the product of any other permutation.

Before proving those facts, note that, from the conditions of lemma 2, for $1 \leq i < j \leq n$, paths p_i and p_j

have no edge in common. Thus, every factor of T^* is distinct from each other.

Proposition: Term T^* appears only once in the product of permutation $\langle 1, \ldots, n \rangle$.

Proof: Let τ be a term in the product of permutation $\langle 1, \ldots, n \rangle$. Then, τ has one factor corresponding to each diagonal entry of Q.

A diagonal entry q_{ii} of Q can be expressed as a sum of three terms (see Eq.(11)).

Let i be such that for all $l > i$, the factor of τ corresponding to entry q_{ll} comes from the first term of q_{ll} (i.e., $T(p_l)[1 + \hat{b}_{l_0}]$).

Assume that the factor of τ corresponding to entry q_{ii} comes from the second term of q_{ii} (i.e., $\hat{a}_{i_0 i} \cdot b_{i_0}$). Recall that each term in $\hat{a}_{i_0 i}$ corresponds to an unblocked path between Z_i and Y, different from p_i, including edge $X_i \rightarrow Y$. However, from lemma 3, any such path must include either an edge which does not belong to any of $p_1, \ldots, p_n$, or an edge which appears in some of $p_{i+1}, \ldots, p_n$. In the first case, it is easy to see that τ must have a factor which does not appear in T^*. In the second, the parameter of an edge of some p_l, $l > i$, must appear twice as a factor of τ, while it appears only once in T^*. Hence, τ and T^* are distinct terms.

Now, assume that the factor of τ corresponding to entry q_{ii} comes from the third term of q_{ii} (i.e., $\sum_{j=1}^{k} b_{i_j} \cdot a_{i_j i}$). Recall that b_{i_j} is the coefficient of $\rho_{W_{i_j} Y}$ in the expression for $\phi_i(Z_i, Y, \mathbf{W}_i)$. From property (iii) of lemma 1, b_{i_j} is a linear function on the correlations $\rho_{Z_i W_{i_1}}, \ldots, \rho_{Z_i W_{i_k}}$, with no constant term. Moreover, correlation $\rho_{Z_i W_{i_l}}$ can be expressed as a sum of terms corresponding to unblocked paths between Z_i and W_{i_l}. Thus, every term in b_{i_j} has the term of an unblocked path between Zi and some W_{i_l} as a factor. By lemma 4, we get that any such path must include either an edge that does not belong to any of $p_1, \ldots, p_n$, or an edge which appears in some of $p_{i+1}, \ldots, p_n$. As above, in both cases τ and T^* must be distinct terms.

After eliminating all those terms from consideration, the remaining terms in the product of $\langle 1, \ldots, n \rangle$ are given by the expression:

$$T^* \cdot \prod_{i=1}^{n} (1 + \hat{b}_{i_0})$$

Since $\hat{b}_{i_0}$ is a polynomial on the correlations among variables $W_{i_1}, \ldots, W_{i_k}$, with no constant term, it follows that T^* appears only once in this expression. □

Proposition: Term T^* does not appear in the product of any permutation other than $\langle 1, \ldots, n \rangle$.

Proof: Let π be a permutation different from $\langle 1, \ldots, n \rangle$, and let τ be a term in the product of π.

Let i be such that, for all $l > i$, π selects the diagonal entry in the row l of Q. As before, for $l > i$, if the factor of τ corresponding to entry q_{ll} does not come from the first term of q_{ll} (i.e., $T(p_l)[1 + \hat{b}_{l_0}]$), then τ must be different from T^*. So, we assume that this is the case.

Assume that π does not select the diagonal entry q_{ii} of Q. Then, π must select some entry q_{il}, with $l < i$. Entry q_{il} can be written as:

$$q_{il} = b_{i_0} a_{i_0 l} + \sum_{j=1}^{k_i} b_{i_j} a_{i_j l}$$

Assume that the factor of τ corresponding to entry q_{il} comes from term $b_{i_0} \cdot a_{i_0 l}$. Recall that each term in $a_{i_0 l}$ corresponds to an unblocked path between Z_i and Y including edge $X_l \rightarrow Y$. Thus, in this case, lemma 5 implies that τ and T^* are distinct terms.

Now, assume that the factor of τ corresponding to entry q_{il} comes from term $\sum_{j=1}^{k} b_{i_j} a_{i_j l}$. Then, by the same argument as in the previous proof, terms τ and T^* are distinct. □

Hence, term T^* is not cancelled out and the lemma holds. □

6.4 Identification of $\lambda_1, \ldots, \lambda_n$

Lemma 8 gives that $det(Q)$ is a non-trivial polynomial on the parameters of the model. Thus, $det(Q)$ only vanishes on the roots of this polynomial. However, [8] has shown that the set of roots of a polynomial has Lebesgue measure zero. Thus, system Φ has unique solution almost everywhere.

It just remains to show that we can estimate the entries of the matrix of coefficients of system Φ from data.

Let us examine again an entry q_{il} of matrix Q:

$$q_{il} = \sum_{j=0}^{k} b_{i_j} \cdot a_{i_j l}$$

From condition (iii) of lemma 1, the factors b_{i_j} in the expression above are polynomials on the correlations among the variables $Z_i, W_{i_1}, \ldots, W_{i_k}$, and thus can be estimated from data.

Now, recall that $a_{i_0 l}$ is given by the sum of terms corresponding to each unblocked path between Z_i and Y including edge $X_l \rightarrow Y$. Precisely, for each term t in $a_{i_0 l}$, there is an unblocked path p between Z_i and Y including edge $X_l \rightarrow Y$, such that t is the product of the parameters of the edges along p, except for λ_l.

However, notice that for each unblocked path between Z_i and Y including edge $X_l \rightarrow Y$, we can obtain an unblocked path between Z_i and X_l, by removing edge $X_l \rightarrow Y$. On the other hand, for each unblocked path between Z_i and X_l we can obtain an unblocked path between Z_i and Y, by extending it with edge $X_l \rightarrow Y$.

Thus, factor $a_{i_0 l}$ is nothing else but $\rho_{Z_i X_l}$. It is easy to see that the same argument holds for $a_{i_j l}$ with $j > 0$. Thus, $a_{i_j l} = \rho_{W_{i_j} X_l}$, $j = 0, \dots, k$.

Hence, each entry of matrix Q can be estimated from data, and we can solve the system of equations Φ to obtain the parameters $\lambda_1, \dots, \lambda_n$.

7 Conclusion

In this paper, we presented a generalization of the method of Instrumental Variables. The main advantage of our method over traditional IV approaches, is that it is less sensitive to the set of conditional independences implied by the model. The method, however, does not solve the Identification problem. But, it illustrates a new approach to the problem which seems powerful enough to achieve this goal.

Appendix

Proof of Lemma 1: Functions $\phi(1, \dots, n)$ and $\psi(i_1, \dots, i_{n-1})$ are defined recursively. For $n = 3$,

$$\begin{cases} \phi^3(1,2,3) & = \rho_{12} - \rho_{13}\rho_{23} \\ \psi^2(i_1, i_2) & = \sqrt{(1 - \rho^2_{i_1,i_2})} \end{cases}$$

For $n > 3$, we have

$$\begin{cases} \phi^n(1,\dots,n) = & \left(\psi^{n-2}(n,3,\dots,n-1)\right)^4 \\ & \cdot \phi^{n-1}(1,2,3,\dots,n-1) \\ & - \left(\psi^{n-2}(n,3,\dots,n-1)\right)^2 \\ & \cdot \phi^{n-1}(1,n,3,\dots,n-1) \\ & \cdot \phi^{n-1}(2,n,3,\dots,n-1) \\ \psi^{n-1}(i_1,\dots,i_{n-1}) = & \Big[\Big(\psi^{n-2}(i_1,i_2,\dots,i_{n-2}) \\ & \cdot \psi^{n-2}(i_{n-1},i_2,\dots,i_{n-2}) \Big)^2 \\ & - \left(\phi^{n-1}(i_1,i_{n-1},i_2,\dots,i_{n-2})\right)^2 \Big]^{\frac{1}{2}} \end{cases}$$

Using induction and the recursive definition of $\rho_{12.3\dots n}$, it is easy to check that:

$$\rho_{12.3\dots N} = \frac{\phi^N(1,2,\dots,N)}{\psi^{N-1}(1,N,3,\dots,N-1)\cdot\psi^{N-2}(N,3,\dots,N-1)}$$

Now, we prove that functions ϕ^n and ψ_{n-1} as defined satisfy the properties $(i) - (iv)$. This is clearly the case for $n = 3$. Now, assume that the properties are satisfied for all $n < N$.

Property (i) follows from the definition of $\phi^N(1, \dots, N)$ and the assumption that it holds for $\phi^{N-1}(1, \dots, N-1)$.

Now, $\phi^{N-1}(1, \dots, N-1)$ is linear on the correlations $\rho_{12}, \dots, \rho_{N-1,2}$. Since $\phi^{N-1}(2, N, 3, \dots, N-1)$ is equal to $\phi^{N-1}(N, 2, 3, \dots, N-1)$, it is linear on the correlations $\rho_{32}, \dots, \rho_{N,2}$. Thus, $\phi^N(1, \dots, N)$ is linear on $\rho_{12}, \rho_{32}, \dots, \rho_{N,2}$, with no constant term, and property (ii) holds.

Terms $\left(\psi^{N-2}(N, 3, \dots, N-1)\right)^2$ and $\phi^{N-1}(1, N, 3, \dots, N-1)$ are polynomials on the correlations among the variables $1, 3, \dots, N$. Thus, the first part of property (iii) holds. For the second part, note that correlation ρ_{12} only appears in the first term of $\phi^N(1, \dots, N)$, and by the inductive hypothesis $\left(\psi^{N-2}(N, 3, \dots, N-1)\right)^4$ has constant term equal to 1. Also, since $\phi^N(1, 2, 3, \dots, N) = \phi^N(2, 1, 3, \dots, N)$ and the later one is linear on the correlations $\rho_{12}, \rho_{13}, \dots, \rho_{1N}$, we must have that the coefficients of $\phi^N(1, 2, \dots, N)$ must be linear on these correlations. Hence, property (iv) holds.

Finally, for property (iv), we note that by the inductive hypothesis, the first term of $\left(\psi^{N-2}(N, 3, \dots, N-1)\right)^2$ has constant term equal to 1, and the second term has no constant term. Thus, property (iv) holds. □

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