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Migration Phases of Icelandic Capelin

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in

Mathematics

by

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October 2023

1. Abstract

Title: Migration Phases of Icelandic Capelin

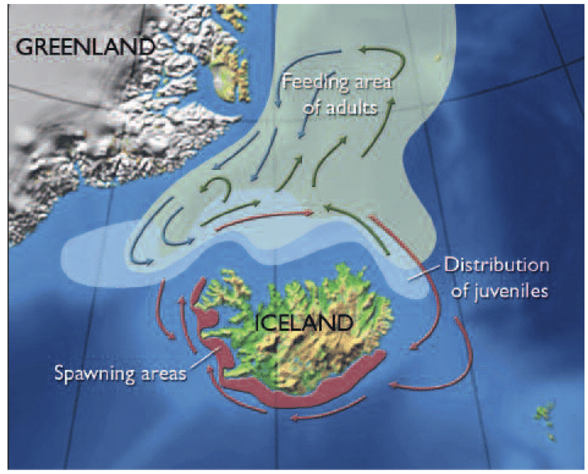
Author: Ilianna Richards

In this paper, we will build mathematical pelagic fish migration models and inspect simulations to discover the behavior for motion amongst a school. I focus on the Icelandic Capelin, and explore all possible migration patterns and their limitations as governed by mathematical equations and not biological observations. We develop a system of ordinary differential equations from a discrete system for the most general motion. First, we will perturb the system to develop a system of stochastic differential equations to study the unique behavior under naturally occurring external forces such as currents or reefs. Then, we will categorize the found transitory and long term behavior of these systems and compare them to the unperturbed solutions. Finally, we will prove that the system exhibits a subcritical pitchfork bifurcation.

Contents

2. Introduction

Climate change and growing global temperatures are affecting the Arctic Ocean currents and, more specifically, the migration patterns of the Capelin (Mallotus villosus), a species of pelagic fish in the sub-arctic seas. The diagram to the right represents the annual migration pattern (Vilhjalmsson, 111). Beginning on the northern side of Iceland, the mature fish swim up the coast of Greenland to Jan Mayen, an island about 600 km north of Iceland, to feed on the zooplankton in early summer. After feeding here, the mature Capelin (2-3 years old) return to the north and northwest of the island to then start their spawning migration to the south of the and western coasts, travelling clockwise most often and with some groups travelling counterclockwise. The majority of the stock spawn in February and March and then dies. The larva drift from the spawning ground and the young Capelin mature off the north-west coast to repeat the process (Einarsson, 5). Although these are the natural migration patterns, there are a number of factors that can affect the specific pathway of the Capelin, including: currents, water temperatures, obstacles, and school speeds.



There have been a number of efforts to model fish school formation and migration. Original models used transitions matrices for prediction and accuracy, while others utilize a biological lens which emphasizes ecological patterns (as in Barbaro, 2). A third approach utilizes transport-diffusion equations to model spatial distribution as a continuum. The model I investigate is an extension of this model and presents a system dependent on the tendency of fish to match direction and speed.

As the fish migrate, they prefer to swim through colder currents. But when the Capelin spawn, they swim through warm water and their roe production increases, and change their speeds and directions in response. It is important that our model captures this change (perturbation) in motion, so we can that we can predict Capelin movement on the feeding ground. I will use an order parameter, that is the signature of the phase of the whole school, to determine how this perturbed system relates to the non perturbed. Then, I will determine, classify, and compare the characteristics of both. I will demonstrate both stable and unstable, migratory solutions, as well as their ordered and disordered phases to prove this system exhibits a subcritical pitchfork bifurcation.

3. Chapter 1: Phases of Order and Disorder

The discrete model we use presents a system that is dependent on the tendency of fish to match direction and speed. In our model, the fish travel in a plane with their own self-propelling force. Consider any j^{th} fish with some speed v_j and some initial direction ϕ_j . Then for any time t and time step Δt, we can calculate the position of $t + \Delta t$ for any j^{th} fish by

$$\begin{pmatrix} x_j(t+\Delta t) \\ y_j(t+\Delta t) \end{pmatrix} = \begin{pmatrix} x_j(t) \\ y_j(t) \end{pmatrix} + v_j(t) \begin{pmatrix} \cos\phi_j(t) \\ \sin\phi_j(t) \end{pmatrix} \Delta t$$

When N fish interact, ϕ and v are updated as follows.

$$\cos(\phi_j(t+\Delta t)) = \frac{1}{N}\Sigma_{k=1}^{N}\cos(\phi_k(t))$$
$$\sin(\phi_j(t+\Delta t)) = \frac{1}{N}\Sigma_{k=1}^{N}\sin(\phi_k(t))$$
$$v_j(t+\Delta t) = \frac{1}{N}\Sigma_{k=1}^{N}v_k(t)$$

This model, referred to as the fish model, is an extension of one originally introduced by Vicsek in 1995 (Vicsek, 12). In the original model, this difference is that Vicsek fixed speed for all molecules. In this model, we allow speed to vary. In this model, the only variables to affect the interaction between these fish are the normalized directional headings, ϕ, and the speed, v. The directional headings and speeds are then averaged across all particles from the previous time step to calculate the directional heading and speed of the next time step.

3.1. **Explain ODE.** This system can be transformed into a system of ordinary differential equations by taking the limit as $\Delta t \to 0^+$. This develops the system of

ODE's (Birnir, 3)

$$\begin{pmatrix} \dot{x_j}(t) \\ \dot{y_j}(t) \end{pmatrix} = \frac{1}{N^2}\ \Sigma_{l=1}^{N} v_l(t)\ \Sigma_{k=1}^{N} \begin{pmatrix} \cos\phi_k(t) \\ \sin\phi_k(t) \end{pmatrix}$$

Or in polar coordinates, these equations can be rewritten as

$$\dot{z_j} = \frac{1}{N^2}\Sigma_{l=1}^{N} v_l(t)\ \Sigma_{k=1}^{N} e^{i\phi_k}$$

Now, suppose that $\alpha > 0$ is the turning rate of how each fish responds to the conditions of surrounding fish, and $\alpha^{-1}\ddot{z_j}$ is the inertia of our school of fish. Then our ODE becomes

$$\ddot{z_j} + \alpha\dot{z_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^{N} v_l(t)\ \Sigma_{k=1}^{N} e^{i\phi_k}$$

Then, by the substitution of $z_k = v_j e^{i\phi_k}$, we can turn this equation into a system of equations for its velocity and directional angle. Thus we now have that

$$\text{(1)} \qquad \dot{v_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^{N} v_l(t)\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j$$

and

$$\text{(2)} \qquad v_j\dot{\phi_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^{N} v_l(t)\ \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j)$$

These are basic differential equations without perturbation.

By similar computations from the polar coordinates of this system, we see

$$\dot{r_j} = v_j\cos(\phi_j - \theta_j)$$

and

$$r_j\dot{\theta_j} = v_j\sin(\phi_j - \theta_j)$$

These two equations govern the positions of the particles in polar coordinates,if given initial conditions $r_k(0)$ and $\theta_k(0)$. We can relate these polar coordinates to

Cartesian coordinates for graphing. We get the system

$$x_j = r_j \cos(\theta_j)$$

$$y_j = r_j \sin(\theta_j)$$

Let $\bar{v} := \frac{1}{N}\Sigma_{i=1}^{N} v_i$ be the average velocity of our system.

It can be obtained by straightforward computations that

$$\frac{1}{N}\Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) = rcos(\psi - \phi_j)$$

As our system tends progresses, it is expected that all of our fish will tend toward the same direction with matching speeds. Thus our system will only be completely ordered if and only if $\phi_j = \psi$ for $t > T$ for some positive time T. Thus, when observing a system of fish interacting, it becomes natural to measure the system based on how much "order" it has.

To get an idea of how the system orders itself, we can determine the average direction ψ and radius r, such that

$$re^{i\psi} = \frac{1}{N}\ \Sigma_{j=1}^{N} e^{i(\phi_j)}$$

This is called the Kuramoto Order Parameter (Birnir, 3)

For any particular time, t, we can evaluate the order of the system from using the direction headings of all fish, so at each time step, we can compute the appropriate $\psi \in [-\pi, \pi]$, average angle at that timestep. Then, from the Kuramoto order parameter

$$re^{i\psi} := \frac{1}{N}\Sigma_{j=1}^{N} e^{i\phi_j}$$

We get that

$$r = \frac{1}{N}\Sigma_{j=1}^{N} \cos(\phi_j - \psi)$$

Clearly, we have that $r \in [0, 1]$. As more of the ϕ_j approach ψ, then we have that more of the $\cos(\phi_j - \psi)$ are approaching 1. The average will approach 1 as more fish align with each other. It is expected that for small perturbation, our systems will approach a migratory solution, and $r \to 1$. The order parameter becomes exactly 1 if and only if every $\phi_j = \psi$.

Taking the derivative, we create the ordinary differential equation

$$\dot{r} + ri\dot{\psi} = \frac{1}{N}\Sigma_{j=1}^{N} i\dot{\phi}_j e^{i(\phi_j - \psi)}$$

we can solve for a system of two ordinary differential equations'

$$\dot{r} = \alpha\bar{v}r\frac{1}{N}\Sigma_{j=1}^{N}\frac{1}{v_j}\sin^2(\psi - \phi_j) \tag{3}$$

$$\dot{\psi} = \alpha\bar{v}\frac{1}{N}\Sigma_{j=1}^{N}\frac{1}{2v_k}\sin(2(\psi - \phi_j)) \tag{4}$$

From all of these equations, it is clear that each fish has an individual directional angle that tends towards the average ψ. This would be what we call the "migratory solution" of the school of fish. In reality, however, it is highly unlikely that each of the thousands of fish are travelling at exactly the same direction and speed. Thus we can add perturbation terms into our equations to represent this.

3.2. **Perturb ODE's.** There are many external factors which could affect a migratory solution, such as currents, food, or turbulence. Thus it becomes natural to question how the system behaves under perturbations. There are two different perturbations possible: deterministic (randomly chosen) angles or white noise (Birnir, 3). The first, deterministic perturbation, is representative of the fish lacking the ability to interpret and match speed and direction of other fish. The second, white noise, represents the environmental factors that could inhibit the fish from being able to align as desired.

3.2.1. *Add Deterministic Perturbations.* We can determine which solutions lie close to stationary solutions by considering a driven version of our system. If we add a driving term to each equation, we can take our system from having a stationary solution to having a migratory solution. Thus, the system will not get stuck at the origin. Let's call these driving terms ν and ω. Then, our system becomes

$$\dot{v_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^N v_l(t)\ \Sigma_{k=1}^N \cos(\phi_k - \phi_j) - \alpha v_j + \nu$$
$$v_j\dot{\phi_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^N v_l(t)\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j) + \omega$$

for small ν and ω. If $\nu = \omega = 0$, we will have an asymptotic stationary solution. See Alethea's paper for further survey of this system (Barbaro, 1).

3.2.2. *Add White Noise.* Let $\omega = 0$ and ν be fixed. We set $\omega = 0$ so that we can see how the system acts with out a driving term forcing the fish to circle. We fix ν so that we can see migratory solutions in addition to stationary solutions.

Consider the two main dynamics equations of $\dot{v}$ and $v\dot{\phi}$. We are curious to examine the dynamics of the system under some perturbations. We can add a deterministic noise to the right hand side of the velocity equation to represent the different rates at which the fish could readjust their velocities to match the school. Similarly, we can add a deterministic noise to the right hand side of the directional equation to represent the external factors that inhibit the fish from matching directions exactly. We shall call these noise parameters σ_v and σ_ϕ , respectively.

This gives us a stochastic system as follows:

$$\dot{v_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^N v_l(t)\ \Sigma_{k=1}^N \cos(\phi_k - \phi_j) - \alpha v_j + \nu + \sigma_v \dot{B_t^j}$$
$$v_j\dot{\phi_j} = \frac{\alpha}{N^2}\Sigma_{l=1}^N v_l(t)\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j) + \sigma_\phi \dot{B_t^j}'$$

where $\dot{B_t^j} \neq \dot{B_t^j}'$ is our Brownian motion.

Let $\bar{v} := \frac{1}{N}\Sigma_{i=1}^{N} v_i$ be the average velocity of our system.

Then we have that these equations become

$$\dot{v_j} = \alpha\bar{v}r\cos(\psi - \phi_j) - \alpha v_j + \nu + \sigma_v \dot{B_t^j}$$
$$v_j\dot{\phi_j} = \alpha\bar{v}r\sin(\psi - \phi_j) + \sigma_\phi \dot{B_t^j}'$$

Now, assuming that v_j is nonzero, we can divide through the previous equation by it and see that

$$\dot{\phi_j} = \alpha\frac{\bar{v}}{v_j}r\sin(\psi - \phi_j) + \frac{\sigma_\phi}{v_j}\dot{B_t^j}'$$

Recall again that

$$re^{i\psi} = \frac{1}{N}\ \Sigma_{j=1}^{N} e^{i(\phi_j)}$$

Thus we have the differential equation

$$\dot{r} + ri\dot{\psi} = \frac{1}{N}\ \Sigma_{j=1}^{N} i\dot{\phi_j}e^{i(\phi_j - \psi)}$$

Thus we have that

$$\begin{aligned}
\dot{r} + ri\dot{\psi} &= \frac{1}{N}\ \Sigma_{j=1}^{N} i\dot{\phi_j}e^{i(\phi_j - \psi)} \\
&= \frac{1}{N}\ \Sigma_{j=1}^{N} [\alpha\frac{\bar{v}}{v_j}r\sin(\psi - \phi_j) + \frac{\sigma_\phi}{v_j}\dot{B_t^j}' ie^{i(\phi_j - \psi)} \\
&= \frac{1}{N}\ \Sigma_{j=1}^{N} [\alpha\frac{\bar{v}}{v_j}r\sin(\psi - \phi_j) + \frac{\sigma_\phi}{v_j}\dot{B_t^j}'][i\cos(\phi_j - \psi) - \sin(\phi_j - \psi)]
\end{aligned}$$

Expanding this, and then splitting into the real an imaginary parts we see that

For r and ψ, these perturbations amount to

$$\dot{r} = \frac{\alpha\bar{v}r}{N}\Sigma_{j=1}^{N}\frac{1}{v_j}\sin^2(\psi - \phi_j) + \frac{1}{N}\Sigma_{j=1}^{N}\frac{\sigma_\phi}{v_j}\dot{B_t^j}' \sin(\psi - \phi_j)$$
$$r\dot{\psi} = \alpha\bar{v}r\frac{1}{N}\Sigma_{j=1}^{N}\frac{1}{2v_j}\sin(2(\psi - \phi_j)) + \frac{1}{N}\Sigma_{j=1}^{N}\frac{\sigma_\phi}{v_j}\dot{B_t^j}' \cos(\psi - \phi_j)$$

Thus far, we have been talking about the deterministic equations of $r, \psi, \phi,$ and v and adding the derivatives of the noise that in reality do not exist. Now we will copy those below instead as stochastic ordinary differential equations, where the noise terms can be properly defined, (1), (2), (3), and (4) become

$$dv_j = (\alpha\bar{v}r\cos(\psi - \phi_j) - \alpha v_j)dt + \nu + \sigma_v dB_t^j \tag{5}$$

$$d\phi_j = (\alpha\frac{\bar{v}}{v_j}r\sin(\psi - \phi_j))dt + \frac{\sigma_\phi}{v_j}dB_t^{j\prime} \tag{6}$$

$$dr = (\alpha\bar{v}r\frac{1}{N}\Sigma_{j=1}^{N}\frac{1}{v_j}\sin^2(\psi - \phi_j))dt + \frac{\sigma_\phi}{N}\Sigma_{j=1}^{N}\frac{1}{v_j}dB_t^{j\prime}\sin(\psi - \phi_j) \tag{7}$$

$$d\psi = (\alpha\bar{v}\frac{1}{N}\Sigma_{j=1}^{N}\frac{1}{2v_j}\sin(2(\psi - \phi_j)))dt + \frac{\sigma_\phi}{rN}\Sigma_{j=1}^{N}\frac{1}{v_j}dB_t^{j\prime}\cos(\psi - \phi_j) \tag{8}$$

This is the complete system of four equations with noise.

3.3. **Definition, Lemmas, and Theorems.** Before performing analysis on this system, we first begin by defining the important terms we will evaluate and presenting simple theorems and lemmas to be used. Some definitions and lemmas I use are with Partial Differential Equations. We can do this because we consider the system of ODEs to be the spatial discretization of a PDE. We pull these definitions and theorems from (Kloeden, 7 and Platen) (Strikwerda, 9)

Definition 3.1. Stability A finite difference scheme $P_{k,h}u_j^n = 0$ for a first order PDE is stable in stability region Λ if there exists an integer N such that for any positive time T, there is a constant C_T such that

$$h\Sigma_{-\infty}^{\infty}|u_j^n|^2 \leq C_T h \Sigma_{m=0}^{N}\Sigma_{-\infty}^{\infty}|u_j^m|^2.$$

The stability region Λ is a region in the $h-k$ plane, defined by a relationship between h and k.

Definition 3.2. Consistency A system is consistent if the errors are of first order or higher in k and h. Moreover, it is consistent specifically of order p if the local truncation error is of $O(h^p)$.

Definition 3.3. Convergence A one step finite difference scheme approximating the solution to a PDE is a convergent scheme if for any solution to the finite difference scheme u_j^n, such that u_j^0 converges to $u_0(x,t)$ as jh converges to x, then u_j^n converges to $u(x,t)$, as (jh, hk) converges to (x,t), for k and h converging to 0.

Theorem 3.1. Lax – Richtmyer Equivalence Theorem

Fundamental Theorem of Numerical Analysis

Given a well-posed linear initial value problem, the finite difference method for the numerical solution of a partial differential equation is consistent and stable if and only if it is convergent.

For a statement and proof see Strikwerda (9).

Lemma 3.1. Let $(B_t)_{t\geq 0}$ be a 1−dimensional Brownian motion and let

$$(X_t)_{t\geq 0}$$

be an Ito process

$$dX_s = b(s)ds + \sigma(s)dB_s$$

. Then we have that

(1) $(dB_t)^2 = dt$

(2) $(dt)^2 = 0$

(3) $dB_t * dt = 0$

Definition 3.4. Euler′s Method Euler's Method, otherwise known as forward Euler's method is an explicit first-order numerical process to solve for an ordinary differential equation with some initial value. Since it is first order, the local error is proportional to the step size. It computes the next step of a discretized scheme based on the previous step, step size, and rate of change of the previous step (Euler-Maruyama Method, 6).

Definition 3.5. Euler Maruyama Method Euler-Maruyama method is a numerical procedure to solve a stochastic differential equation. It becomes an extension of Euler's method for ODE's by introducing a perturbation parameter and Brownian motion to make it a SDE. In our case, our perurbation parameters are σ_ϕ and σ_v.

Lemma 3.2. For all Euler's methods, a solution is said to be consistent if it is of order 1 or higher.

Definition 3.6. Variance $V[y] = E[y^2] - E[y]^2$

3.4. **Stability and Consistency for Euler-Maruyama.** For analysis, we will go back and consider the coupled equations of

$$\dot{v_j} = \frac{\alpha \bar{v}}{N} \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j + \sigma_v \dot{B}_t^j$$
$$v_j \dot{\phi}_j = \frac{\alpha \bar{v}}{N} \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j) + \sigma_\phi \dot{B}_t^{j\prime}$$

It is easier for us to analyze the system from these equations since we would only have to do analysis of 2 equations with 2 parameters, instead of 4 equations with 4 parameters. The remaining two equations are simply deterministic slaves to the two equations above, and do not introduce additional instabilities, so we only need to work for the equations for v_j and ϕ_j, $j = 1, \cdots, N$. The analysis of the r and ψ equations is similar and simpler.

First, to evaluate stability, we will perform a Von Neumann stability Analysis on the Euler Maruyama Method for solving Stochastic differential equation with noise.

In our setting, it is important to establish a distinction between dynamical stability and numerical stability. Dynamical stability is one such that if, for example, we perturb a fish direction by a small enough amount, it will eventually tend back towards its original equilibrium solution. In our system, when we have a disordered state of stationary solutions, it is dynamically unstable, but when we have an ordered migratory solution, it is dynamically stable. Thus if we are given a school with any nonzero random velocity and small perturbations, our scheme will eventually be completely ordered and the direction or each fish will be exactly that of the average (ie all the same).

We are more interested in numerical stability. A partial differential equation is considered stable if the total variation of the expected numerical solution at any fixed time is bounded as the step size tends towards 0. Thus we will aim to calculate the the expected value of the first and second moment of these equations to then determine both the mean and the variance.

Note that our system has two perturbed equations, so we can apply Euler Maruyama's method to solve it. First, we will illustrate the stability analysis by applying it to a simple stochastic ODE. This will make the application to the more complicated system (5)-(8) easier to follow.

Consider the first order stochastic differential equation

$$dy_t = \theta(\mu - y_t)dt + \sigma dB_t + O(dt)^2$$

This basic form of the Euler Maruyama method can be found at (6).

If we discretize this system, we can compute any step based on the previous step. Since our equation above is of first order, there will be some sort of error between our discretized system and the true solution of second order. Thus we get the system as follows:

$$\begin{aligned} y_{i+1} &= y_i + \theta(\mu - y_i)\Delta t + \sigma \Delta B_i + O(\Delta t)^2 \\ &= y_i + \theta(\mu - y_i)\Delta t + \sigma(B_{i+1} - B_i) + O(\Delta t)^2 \\ &= y_i + \theta(\mu - y_i)\Delta t + \sigma B_{i+1} - \sigma B_i + O(\Delta t)^2 \end{aligned}$$

To perform the Von Neumann analysis, we want to compute the expectation of the system and ensure that it's amplification factor is bounded or decreasing, or in other words, we want to show that the error between our expected value and discretized system is $\leq 1 + O(\Delta t)$.

In most cases, I will prove that the amplification factor is ≤ 1. This is obviously less than $1 + O(\Delta t)$, so that still proves stability.

3.4.1. *Stability and Consistency of the scheme for $E[y_{i+1}]$.* To take the expectation, recall that the expected value of brownian motion is 0, so those two terms disappear when taking the expectation of both sides of the equation above. Thus we have that

$$E[y_{i+1}] = E[y_i] + \theta(\mu - E[y_i])\Delta t + O(\Delta t)^2$$

To evaluate our consistency, we again want to ensure that our expected value and variance have order of 1 or higher. Since we are using a version of Euler's method, this ensures that we can use varying time steps, but our system will still evaluate to approximately the solution.

Let us compute the consistency of the expected value now. From the above equation, subtracting both sides by $E[y_i]$ and then dividing by Δt, we get

$$\frac{dE[y]}{dt} + O(\Delta t) = \theta(\mu - E[y_i]) + O(\Delta t)$$

Thus we have the error between any two steps of the expected value of y is of order of Δt. Hence by lemma 3.2, our expected value of our second moment of our solution is consistent. For stability: For simplicity, let $\xi^i = E[y_i]$. Thus we have that

$$\begin{aligned} \xi^{i+1} &= \xi^i + \theta(\mu - \xi^i)\Delta t + O(\Delta t)^2 \\ &= (1 - \theta\mu\Delta t)\xi^i + \theta\mu\Delta t + O(\Delta t)^2 \end{aligned}$$

Note that the $\theta\mu\Delta t$ is our driving term, but has no ξ_i, thus it does not affect the amplification factor, so we can ignore it. Thus, consider

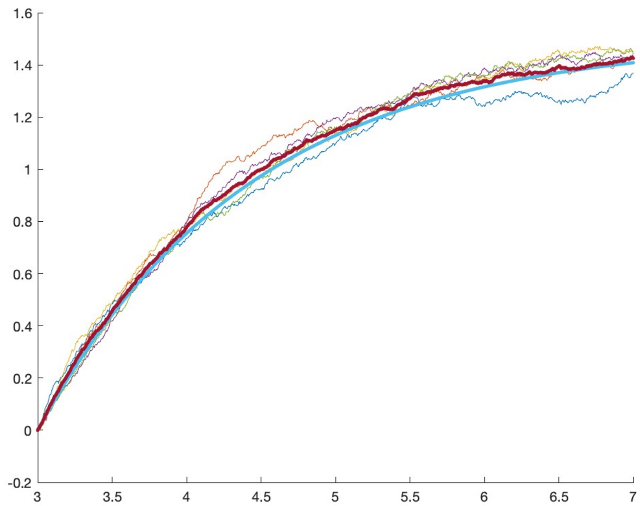
$$\xi^{i+1} = (1 - \theta\mu\Delta t)\xi^i$$

We can easily compute the amplification factor by dividing off both sides by ξ^i. We now have, if our conditions on θ μ and Δt well posed, that

$$\begin{aligned} \xi &= (1 - \theta\mu\Delta t) \\ &\leq 1 \end{aligned}$$

Thus we have that the expectation of our system is bounded. Now, if our variation as well meets this condition, then our system will be stable.

The expectation of the exact solution of y_t is graphed in comparison to 5 runs of this differential equation and their average. It is expected that as we increase the number of runs, our average will approach the true solution. The conditions on our parameters are $\mu = 1.5$, $\theta = 0.7$, and $\sigma = 1$.



3.4.2. *Stability and Consistency of the scheme for* $E[y_{i+1}^2]$. Note that the definition of variance uses the expectation of both the first and second moments of y. We have already proved conditions of $E[y_{i+1}]$, so we want to compute $E[y_{i+1}^2]$.

Since y can only be computed if we know Δy, let's begin there. Consider

$$dy = \theta(\mu - y)dt + \sigma dB_t + O(dt)^2$$

Then we can compute the square such that

$$\begin{aligned}(\Delta y)^2 &= (\theta(\mu - y)\Delta t + \sigma \Delta B_t)^2 + O(\Delta t)^3 \\ &= \theta^2(\mu - y)^2(\Delta t)^2 + 2\sigma(\mu - y)\Delta t \Delta B_t + \sigma^2(\Delta B_t)^2 + O(\Delta t)^3\end{aligned}$$

Recall that $E[\Delta B_t] = 0$ and $E[\Delta B_t^2] = \Delta t$. Thus we have that

$$\begin{aligned}E[(\Delta y)^2] &= E[\theta^2(\mu - y)^2(\Delta t)^2 + 2\sigma(\mu - y)\Delta t \Delta B_t + \sigma^2(\Delta B_t)^2] + O(\Delta t)^3 \\ &= \theta^2(\mu - E[y])^2(\Delta t)^2 + \sigma^2(\Delta t) + O(\Delta t)^3\end{aligned}$$

But since $\Delta y = y_{i+1} - y_i$, we get that $\Delta y^2 = y_{i+1}^2 - 2y_{i+1}y_i + y_i^2$.

Hence we have that

$$E[y_{i+1}^2 - 2y_{i+1}y_i + y_i^2] = E[y_{i+1}^2] - 2E[y_{i+1}]E[y_i] + E[y_i^2]$$
$$= \sigma^2\Delta t + \theta^2(\mu - E[y_i])^2(\Delta t)^2 + O(\Delta t)^3$$

Thus we can rewrite this to solve for $E[y_{i+1}^2]$ such that

$$E[y_{i+1}^2] = 2E[y_{i+1}]E[y_i] - E[y_i^2] + \sigma^2\Delta t + \theta^2(\mu - E[y_i])^2(\Delta t)^2 + O(\Delta t)^3$$

Substitution in our equation for $E[y_{i+1}]$, we get

$$E[y_{i+1}^2] = 2(E[y_i] + \theta(\mu - E[y_i])\Delta t)E[y_i] - E[y_i^2] + \sigma^2\Delta t + \theta^2(\mu - E[y_i])^2(\Delta t)^2 + O(\Delta t)^3$$
$$= -E[y_i^2] + 2(E[y_i] + \theta(\mu - E[y_i])\Delta t)E[y_i] + \sigma^2\Delta t + \theta^2(\mu^2 - 2\mu E[y_i] + E[y_i]^2)(\Delta t)^2 + O(\Delta t)^3$$
$$= -E[y_i^2] + 2E[y_i]^2 + 2\theta E[y_i](\mu - E[y_i]) + \sigma^2)\Delta t + \theta^2(\mu^2 - 2\mu E[y_i] + E[y_i]^2)(\Delta t)^2 + O(\Delta t)^3$$

Hence we have

(9)

$$E[y_{i+1}^2] = -E[y_i^2]+2E[y_i]^2+2\theta E[y_i](\mu-E[y_i])+\sigma^2)\Delta t+\theta^2(\mu^2-2\mu E[y_i]+E[y_i]^2)(\Delta t)^2+O(\Delta t)^3$$

Since the error is of first order and since $E[y_i]$ is consistent, then we have that the scheme for $E[y_i^2]$ is consistent.

For stability, consider equation (4) again. Since $E[y_i]$ is stable, so is $E[y_i]^2$. Further, constants are stable. Recall that smaller order terms do not affect stability. Thus we can ignore the ladder terms since they will not affect the stability of $E[y_{i+1}^2]$. Hence we have that

$$E[y_{i+1}^2] = -E[y_i^2]$$

Hence we can divide both sides by $E[y_i^2]$ and take the absolute value to see that our amplification factor, namely ξ, is

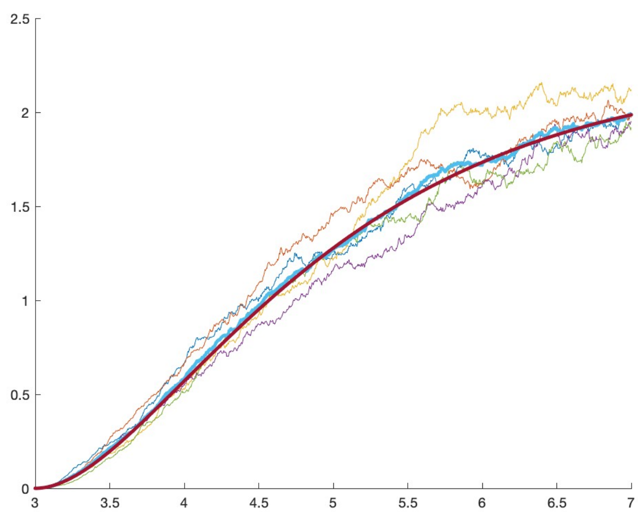
$$\xi = -1$$

$$|\xi| \leq 1$$

Thus the scheme for $E[y_{i+1}^2]$ is stable.

The variation of the exact solution of y_t is dependent on the expectation of the exact solution of y_t^2. This is graphed in comparison to 5 runs computing a specific $y_t^2$and their average.

The conditions on our parameters are $\mu = 1.5$, $\theta = 0.7$, and $\sigma = 0.06$.



3.4.3. *Stability and Consistency of* $V[y_{i+1}]$. Now, to show stability and consistency, note, from above, $\text{Var}[y_{i+1}] = E[y_{i+1}^2] - E[y_{i+1}]^2$

Since the variance is merely the composition of two consistent systems, then the variance is consistent.

This gives us the linear composition of two stable terms. Thus the variance of y_{i+1} must also be stable, but let's still prove it using Von Nuemanns Analysis.

Recall, we have that $E[y_{i+1}] = E[y_i] + \theta(\mu - E[y_i])\Delta t$. This implies that

$$E[y_{i+1}]^2 = E[y_i]^2 + 2E[y_i]\theta(\mu - E[y_i])\Delta t + \theta^2(\mu - E[y_i])^2(\Delta t)^2$$

Additionally

$$E[y_{i+1}^2] = 2E[y_i]^2 - E[y_i^2] + [2\theta(\mu - E[y_i])E[y_i] + \sigma^2]\Delta t + [\theta^2(\mu - E[y_i])^2](\Delta t)^2 + O(\Delta t)^3$$

Then we have that

$$\begin{aligned}
V[y_{i+1}] &= E[y_{i+1}^2] - E[y_{i+1}]^2 \\
&= 2E[y_i]^2 - E[y_i^2] + [2\theta(\mu - E[y_i])E[y_i] + \sigma^2]\Delta t + [\theta^2(\mu - E[y_i])^2](\Delta t)^2 + O(\Delta t)^3 \\
&\quad - (E[y_i]^2 + 2E[y_i]\theta(\mu - E[y_i])\Delta t + \theta^2(\mu - E[y_i])^2(\Delta t)^2) + O(\Delta t)^3 \\
&= 2E[y_i]^2 - E[y_i^2] - E[y_i]^2 + [2\theta(\mu - E[y_i])E[y_i] + \sigma^2 - 2E[y_i]\theta(\mu - E[y_i])]\Delta t \\
&\quad + [\theta^2(\mu - E[y_i])^2 - \theta^2(\mu - E[y_i])^2](\Delta t)^2 + O(\Delta t)^3 \\
&= E[y_i]^2 - E[y_i^2] + [\sigma^2]\Delta t + [0](\Delta t)^2 + O(\Delta t)^3 \\
&= -(E[y_i^2] - E[y_i]^2) + [\sigma^2]\Delta t + O(\Delta t)^3 \\
&= -V[y_i] + [\sigma^2]\Delta t + O(\Delta t)^3
\end{aligned}$$

Thus we have that

$$V[y_{i+1}] = -V[y_i] + [\sigma^2]\Delta t + O(\Delta t)^3 \tag{10}$$

Note that $[\sigma^2]\Delta t + O(\Delta t)^3$ are our driving terms, and have no $V[y_i]$, thus is does not affect the stability of the system. So we can ignore those terms and do analysis on

$$V[y_{i+1}] = -V[y_i]$$

Diving both sides by $V[y_i]$ and take the absolute value to see that our amplification factor, namely ξ, is

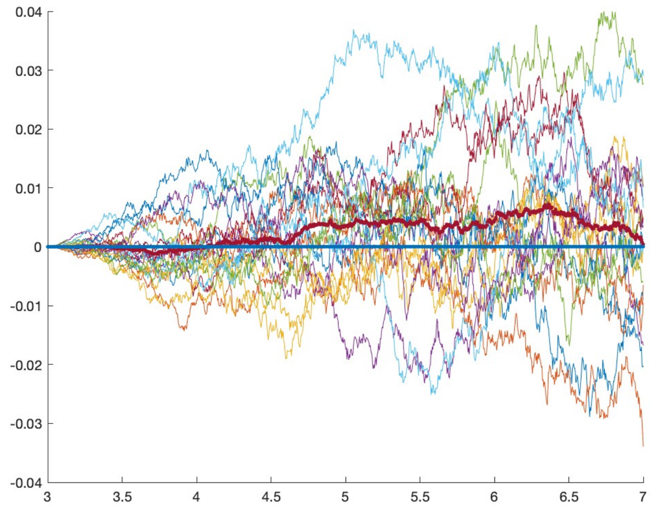
$$\xi = -1$$

$$|\xi| \leq 1$$

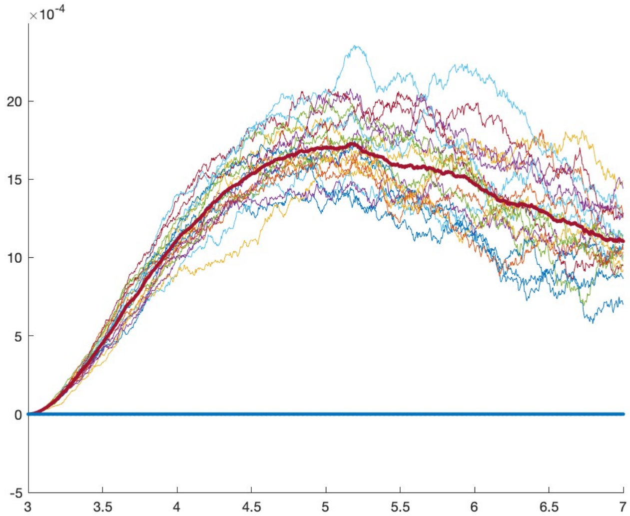
Thus the scheme for $V[y_{i+1}]$ is stable. Hence, by Von Neumann analysis, we have that the variance is stable.

The variation on several runs is now plotted in comparison to the average and the exact expected variation.

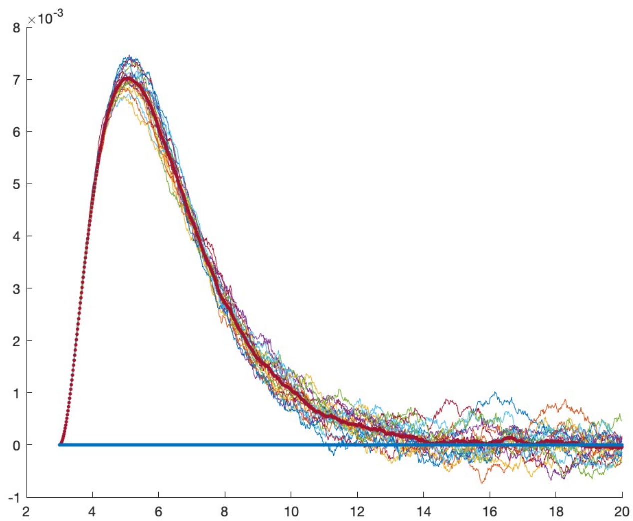
The conditions on our parameters are $\mu = 1.5$, $\theta = 0.7$, and $\sigma = 0.06$.



Now we use smaller $\sigma = 0.0001$.



Now, we use $\sigma = 0.0001$, but lengthen the domain to ensure it settles down.



3.4.4. *Convergence.* By the stability and consistency argument above, by theorem 3.1, our numerical solution is convergent.

3.5. **Results.** We now apply the above analysis to the system (5)-(8), describing the fish dynamics.

3.5.1. *Stability and Convergence Applied to Our System.* Consider the original main equations determining the dynamics of our system.

$$\dot{v_j} = \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N}\cos(\psi - \phi_j) - \alpha v_j + \nu + \sigma_v \dot{B_j}$$
$$v_j\dot{\phi_j} = \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N}\sin(\psi - \phi_j) + \sigma_\phi \dot{B_{j'}}$$

Similar to above, we want to determine the expectation and variance of this system and determine that they are both stable and consistent.

3.5.2. *Stability and Expectation of v.* Beginning with our equation for velocity, we will change it from its deterministic differential equation into its stochastic differential equation. This give us that

$$dv_j = (\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N}\cos(\phi_k - \phi_j) - \alpha v_j)dt + \sigma_v dB_j + O(dt)^2$$

Taking the expectation of both sides, we get that

$$\begin{aligned}
E[\Delta v_j] &= E[(\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N}\cos(\phi_k - \phi_j) - \alpha v_j)\Delta t + \sigma_v \Delta B_j] + O(\Delta t)^2 \\
&= E[(\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N}\cos(\phi_k - \phi_j))\Delta t] - E[(\alpha v_j)\Delta t] + E[\sigma_v \Delta B_j] + \sigma_v dB_j O(\Delta t)^2 \\
&= \frac{\alpha\bar{v}}{N}\Sigma_{k=1}^{N} E[cos(\phi_k - \phi_j)]\Delta t - \alpha E[v_j]\Delta t + \sigma_v E[\Delta B_j] + O(\Delta t)^2 \\
&\leq \frac{\alpha\bar{v}}{N}\Sigma_{k=1}^{N} 1\Delta t - \alpha E[v_j]\Delta t + 0 + O(\Delta t)^2 \\
&= \frac{\alpha\bar{v}}{N} N 1\Delta t - \alpha E[v_j]\Delta t + O(\Delta t)^2 \\
&= \alpha\bar{v}\Delta t - \alpha E[v_j]\Delta t + O(\Delta t)^2
\end{aligned}$$

Then we can rewrite $\Delta v_j = v_{j+1} - v_j$. Thus we get

$$E[\Delta v_{j+1}] - E[\Delta v_j] \leq \alpha\bar{v}\Delta t - \alpha E[v_j]\Delta t + O(\Delta t)^2$$

Hence this becomes

$$E[\Delta v_{j+1}] \leq E[\Delta v_j] + \alpha\bar{v}\Delta t - \alpha E[v_j]\Delta t + O(\Delta t)^2$$
$$= E[v_j](1 - \alpha\Delta t) + \alpha\bar{v}\Delta t + O(\Delta t)^2$$

$$E[\Delta v_{j+1}] \leq E[\Delta v_j] + \alpha\bar{v}\Delta t - \alpha E[v_j]\Delta t$$
$$= E[v_j] - \alpha\Delta t E[v_j] + \alpha\bar{v}\Delta t + O(\Delta t)^2$$
$$= E[v_j] + \alpha\Delta t(\bar{v} - E[v_j]) + O(\Delta t)^2$$

Thus our equation is

$$E[\Delta v_{j+1}] \leq E[v_j] + \alpha\Delta t(\bar{v} - E[v_j]) + O(\Delta t)^2 \tag{11}$$

For consistency, we get that

$$\frac{dE[v_j]}{dt} \leq \alpha(\bar{v} - E[v_j]) + O(\Delta t)$$

Since this is first order, it is consistent.

For stability, again consider (6). We can ignore the ladder terms and begin with

$$E[\Delta v_{j+1}] \leq E[v_j](1 - \alpha\Delta t)$$

This gives us an amplification factor

$$\xi \leq 1 - \alpha\Delta t \leq 1$$

Next, lets calculate the expectation of the second moment for variance purposes. Again, consider

$$dv_j = (\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)dt + \sigma_v dB_j + O(dt)^2$$

Then we have,

$$\begin{aligned}
E[(\Delta v_j)^2] &= E[((\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)\Delta t + \sigma_v \Delta B_j)^2] + O(\Delta t)^3 \\
&= E[(\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)^2(\Delta t)^2 + (\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)\sigma_v \Delta t \Delta B_j + \sigma_v^2(\Delta B \\
&\leq E[(\frac{\alpha\bar{v}}{N} \times N \times 1 - \alpha v_j)^2](\Delta t)^2 E[(\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)\sigma_v \Delta t \Delta B_j] + E[\sigma_v^2 \Delta t]] + O(\\
&= E[(\alpha\bar{v} - \alpha v_j)^2](\Delta t)^2 + 0 + \sigma_v^2 \Delta t + O(\Delta t)^3 \\
&= \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3
\end{aligned}$$

Thus we have that

$$E[(\Delta v_j)^2] \leq \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3$$

Since $\Delta v_j = v_{j+1} - v_j$, we have that $(\Delta v_j)^2 = v_{j+1}^2 - 2v_{j+1}v_j + v_j^2$.

By substitution, this equation becomes

$$E[v_{j+1}^2 - 2v_{j+1}v_j + v_j^2] \leq \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3$$

Note that

$$E[v_{j+1}] = E[v_j + (\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j)dt + \sigma_v dB_j] \leq E[v_j + \alpha(\bar{v} - v_j)\Delta t]$$

Thus, solve for $E[v_{j+1}^2]$ by moving over some terms and substituting, such that

$$
\begin{aligned}
E[v_{j+1}^2] &\leq E[2v_{j+1}v_j] - E[v_j^2] + \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \\
&\leq 2E[(v_j + \alpha(\bar{v} - v_j)\Delta t)v_j] - E[v_j^2] + \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \\
&= 2E[v_j^2 + \alpha v_j(\bar{v} - v_j)\Delta t)] - E[v_j^2] + \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \\
&= 2E[v_j^2] + E[\alpha v_j(\bar{v} - v_j)\Delta t)] - E[v_j^2] + \sigma_v^2 \Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \\
&= E[v_j^2] + (\alpha E[v_j(\bar{v} - v_j)] + \sigma_v^2)\Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3
\end{aligned}
$$

$$
E[v_{j+1}^2] \leq E[v_j^2] + \alpha E[v_j(\bar{v} - v_j) + \sigma_v^2]\Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \tag{12}
$$

For consistency, we have

$$
\frac{dE[v_j]}{dt} \leq \alpha E[v_j(\bar{v} - v_j)] + \sigma_v^2 + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t) + O(\Delta t)^2
$$

This is of second order so we are consistent!

For stability, we can expand this to compute the amplification. We get

$$
\begin{aligned}
E[v_{j+1}^2] &\leq E[v_j^2] + (\alpha E[v_j(\bar{v} - v_j)] + \sigma_v^2)\Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 + O(\Delta t)^3 \\
&= E[v_j^2](1 - \alpha \Delta t + \alpha^2 (\Delta t)^2) + (\alpha \bar{v} + \sigma_v^2)\Delta t + (\alpha^2 \bar{v} - 2\alpha \bar{v} E[v_j])(\Delta t)^2 + O(\Delta t)^3.
\end{aligned}
$$

Since $E[v_j]$ is stable, it doesn't affect our stability, so we can ignore that term and the rest.

Thus we get that the amplification factor

$$
\xi \leq 1 - \alpha \Delta t + \alpha^2 (\Delta t)^3 \leq 1
$$

Hence $E[v_j^2]$ is stable.

To solve for variance, consider the definition 3.6.

Then we get

$$V[v_{j+1}] = E[v_{j+1}^2] - E[v_{j+1}]^2$$
$$\leq E[v_j^2] + (\alpha E[v_j(\bar{v} - v_j)] + \sigma_v^2)\Delta t + \alpha^2 E[(\bar{v} - v_j)^2](\Delta t)^2 - (E[v_j](1 - \alpha\Delta t))^2 + O(\Delta t)^3$$
$$= E[v_j^2] + (\alpha\bar{v}E[v_j] + \sigma_v^2)\Delta t - \alpha E[v_j^2]\Delta t + \alpha^2 E[\bar{v}^2 - 2\bar{v}v_j + v_j^2](\Delta t)^2 + O(\Delta t)^3$$
$$- (E[v_j] + E[v_j]\alpha\Delta t - O(\Delta t)^2)^2$$
$$= E[v_j^2] + (\alpha\bar{v}E[v_j] + \sigma_v^2)\Delta t - \alpha E[v_j^2]\Delta t + \alpha^2\bar{v}^2(\Delta t)^2 - 2\alpha^2\bar{v}E[v_j](\Delta t)^2 + \alpha^2 E[v_j^2](\Delta t)^2 + O(\Delta$$
$$- E[v_j]^2 + 2\alpha E[v_j]^2\Delta t - E[v_j]^2\alpha^2(\Delta t)^2$$
$$= E[v_j^2] - E[v_j]^2 + (\alpha\bar{v}E[v_j] - \alpha E[v_j^2] + 2\alpha E[v_j]^2 + \sigma_v^2)\Delta t$$
$$+ (\alpha^2\bar{v}^2 - 2\alpha^2\bar{v}E[v_j] + \alpha^2 E[v_j^2] - \alpha^2 E[v_j]^2)(\Delta t)^2 + O(\Delta t)^3$$
$$= V[v_j] + (\alpha\bar{v}E[v_j] - \alpha E[v_j^2] + 2\alpha E[v_j]^2 + \sigma_v^2)\Delta t + (\alpha^2\bar{v}^2 - 2\alpha^2\bar{v}E[v_j] + \alpha^2 V[v_j])(\Delta t)^2 + O(\Delta t$$

Hence we have that

(13)

$$V[v_{j+1}] = V[v_j] + (\alpha\bar{v}E[v_j] - \alpha E[v_j^2] + 2\alpha E[v_j]^2 + \sigma_v^2)\Delta t + (\alpha^2\bar{v}^2 - 2\alpha^2\bar{v}E[v_j] + \alpha^2 V[v_j])(\Delta t)^2$$

For consistency, note that

$$\frac{V[v_{j+1}] - V[v_j]}{\Delta t} = (\alpha\bar{v}E[v_j] - \alpha E[v_j^2] + 2\alpha E[v_j]^2 + \sigma_v^2) + (\alpha^2\bar{v}^2 - 2\alpha^2\bar{v}E[v_j] + \alpha^2 V[v_j])(\Delta t) + O(\Delta t)^2$$

Since this is of second order, we are good to go, variance is consistent!

For stability, ignore all terms but the ones with variance. Further, note that higher order terms of Δt are less that the first order of Δt. So we get that

$$V[v_{j+1}] = V[v_j] + \alpha^2(\Delta t)^2 V[v_j]$$

This gives us that our amplification factor

$$\xi = 1 + \alpha^2(\Delta t)^2 \leq 1 + O(\Delta t)$$

Hence we have that $V[v_j]$ is stable.

3.5.3. *Stability and Expectation of $v\dot{\phi}$.* Beginning with our equation for velocity, we will change it from its deterministic differential equation into its stochastic differential equation. This give us that

$$v_j d\phi_j = \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j)dt + \sigma_\phi dB_j + O(dt)^2$$

Then we can take the expectation of both sides and see that

$$\begin{aligned} E[v_j\Delta\phi_j] &= E[\frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j)\Delta t + \sigma_\phi \Delta B_j] + O(\Delta t)^2 \\ &\leq \frac{\alpha\bar{v}}{N} E[\Sigma_{k=1}^{N} 1]\Delta t + \sigma_\phi E[\Delta B_j] + O(\Delta t)^2 \\ &= \alpha\bar{v}\Delta t + O(\Delta t)^2 \end{aligned}$$

Hence we have that

$$E[v_j\Delta\phi_j] \leq \alpha\bar{v}\Delta t + O(\Delta t)^2$$

Since $\Delta\phi = \phi_{j+1} - \phi_j$, we have that

$$E[v_j\phi_{j+1}] \leq E[v_j\phi_j] + \alpha\bar{v}\Delta t + O(\Delta t)^2 \tag{14}$$

For consistency, we get that

$$\frac{dE}{dt} \leq \alpha\bar{v} + O(\Delta t)$$

Since this is of first order, we are consistent!

For stability, this gives us that the amplification factor is

$$\xi \leq 1$$

Thus $E[v_j\phi_{j+1}]$is stable.

Now, lets compute the second moment of $vd\phi$.

Given that $E[v_j\Delta\phi_j] \leq E[\alpha\bar{v}\Delta t + \sigma_\phi\Delta B_j]$, we have that

$$E[(v_j\Delta\phi_j)^2] \leq E[(\alpha\bar{v})^2(\Delta t)^2 + \alpha\bar{v}\Delta t\sigma_\phi\Delta B_j + \sigma_\phi^2(\Delta B_j)^2]$$

Given that $v_j\Delta\phi_j = \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j)\Delta t + \sigma_\phi\Delta B_j$, we have that $(v_j\Delta\phi_j)^2 = (\frac{\alpha\bar{v}}{N})^2\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j)\Sigma_{k=1}^N \sin(\phi_k - \phi_j)(\Delta t)^2 + \sigma_\phi\Delta B_j + \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j)\Delta t\sigma\Delta B_j + \sigma_\phi^2{\Delta B_j}^2$

Taking the expecation of both sides, we see that

$$\begin{aligned} E[(v_j\Delta\phi_j)^2] &\leq E[\frac{(\alpha\bar{v})^2}{N^2} \times N \times N \times (\Delta t)^2] + E[\frac{\alpha\bar{v}}{N} \times N \times \Delta t\sigma_\phi\Delta B_j] + E[\sigma_\phi^2(\Delta B_j)^2] \\ &= \sigma_\phi^2\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \end{aligned}$$

Since $\Delta\phi = \phi_{j+1} - \phi_j$, we have that $(v_j(\Delta\phi))^2 = v_j^2\phi_{j+1}^2 - 2v_j\phi_{j+1}v_j\phi_j + v_j^2\phi_j^2$.

Further, note that

$$\begin{aligned} E[v_j\phi_{j+1}] &= E[v_j\phi_j + v_j d\phi_j] \\ &= E[v_j\phi_j + \frac{\alpha\bar{v}}{N}\ \Sigma_{k=1}^N \sin(\phi_k - \phi_j)dt + \sigma_\phi\Delta B_j] \\ &\leq E[v_j\phi_j + \alpha\bar{v}\Delta t + \sigma_\phi\Delta B_j] \\ &= E[v_j\phi_j + \alpha\bar{v}\Delta t] \end{aligned}$$

Rearranging some expectation terms and substituting this value in, we get that

$$
\begin{aligned}
E[v_j^2\phi_{j+1}^2] &\leq E[2v_j\phi_{j+1}v_j\phi_j] - E[v_j^2\phi_j^2] + \sigma_\phi^2\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \\
&\leq 2E[v_j\phi_j + \alpha\bar{v}\Delta t]E[v_j\phi_j] - E[v_j^2\phi_j^2] + \sigma_\phi^2\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \\
&= 2(E[v_j\phi_j] + \alpha\bar{v}\Delta t)E[v_j\phi_j] - E[v_j^2\phi_j^2] + \sigma_\phi^2\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \\
&= 2E[v_j\phi_j]^2 + 2\alpha\bar{v}\Delta t E[v_j\phi_j] - E[v_j^2\phi_j^2] + \sigma_\phi^2\Delta t + (\alpha\bar{v})^2(\Delta t)^2
\end{aligned}
$$

Hence we have that

$$E[v_j^2\phi_{j+1}^2] \leq -E[v_j^2\phi_j^2] + 2E[v_j\phi_j]^2 + (2\alpha\bar{v}E[v_j\phi_j] + \sigma_\phi^2)\Delta t + (\alpha\bar{v})^2(\Delta t)^2 + O(\Delta t)^3$$

Since the error is of first order and since $E[v_i\phi_{i+1}]$ is consistent, then we have that the scheme for $E[(v_i\phi_{i+1})^2]$ is consistent.

For stability, note none of the ladder terms affect stability, so we have an amplification factor

$$\xi \leq -1$$

$$|\xi| \leq 1$$

Now onto variance!

Recall

$$E[v_j\phi_{j+1}] \leq E[v_j\phi_j] + \alpha\bar{v}\Delta t + O(\Delta t)^2$$

and that

$$E[v_j^2\phi_{j+1}^2] \leq -E[v_j^2\phi_j^2] + 2E[v_j\phi_j]^2 + (2\alpha\bar{v}E[v_j\phi_j] + \sigma_\phi^2)\Delta t + (\alpha\bar{v})^2(\Delta t)^2 + O(\Delta t)^3$$

Hence, from definition 3.6, we get that

$$
\begin{aligned}
V[v_j\phi_{j+1}] &= E[(v_j\phi_j)^2] - E[v_j\phi_j]^2 \\
&\leq -E[v_j^2\phi_j^2] + 2E[v_j\phi_j]^2 + (2\alpha\bar{v}E[v_j\phi_j] + \sigma_\phi^2)\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \\
&\quad - (E[v_j\phi_j] + \alpha\bar{v}\Delta t)^2 + O(\Delta t)^3 \\
&= -E[v_j^2\phi_j^2] + 2E[v_j\phi_j]^2 + (2\alpha\bar{v}E[v_j\phi_j] + \sigma_\phi^2)\Delta t + (\alpha\bar{v})^2(\Delta t)^2 \\
&\quad - E[v_j\phi_j]^2 - 2E[v_j\phi_j]\alpha\bar{v}\Delta t - \alpha^2\bar{v}^2\Delta t^2 + O(\Delta t)^3 \\
&= -E[v_j^2\phi_j^2] + E[v_j\phi_j]^2 + (2\alpha\bar{v}E[v_j\phi_j] + \sigma_\phi^2 - 2E[v_j\phi_j]\alpha\bar{v})\Delta t + ((\alpha\bar{v})^2 - \alpha^2\bar{v}^2)(\Delta t)^2 + O(\Delta t) \\
&= -V[v_j\phi_j] + (\sigma_\phi^2)\Delta t + O(\Delta t)^3
\end{aligned}
$$

Thus we get that

$$
V[v_j\phi_{j+1}] = -V[v_j^2\phi_j^2] + (\sigma_\phi^2)\Delta t + O(\Delta t)^3 \tag{15}
$$

Since the variance is merely the composition of two consistent systems, then the variance is consistent.

For stability, we have the amplification factor

$$
\xi = -1
$$

and so

$$
|\xi| \leq 1
$$

Thus our scheme is stable !

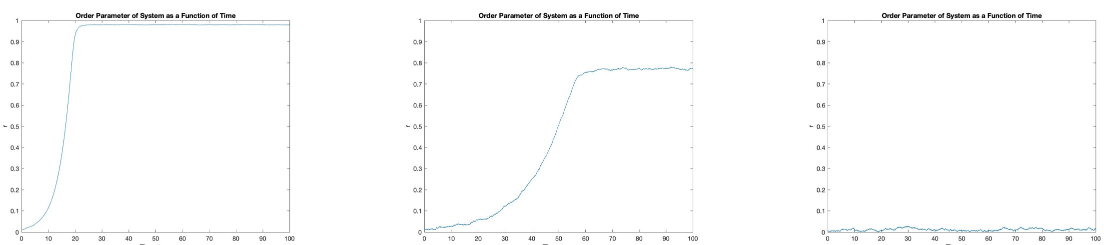
Thus we have proved that our scheme is stable regardless of the perturbation parameter.

Now we see clearly that the perturbation coefficient does not affect the stability of the equation. Thus, we have the perturbed equation for $v_j\phi_{j+1}$ is also stable.

Now we have proved that our system of two equations is stable and consistent. Since our system of four equations is derived by these, this implies that our complete system of four equations is also stable and consistent. Thus, by lemma 3.2, our system (5)-(8) is convergent.

3.5.4. *Motion and tendencies without perturbation.* What we call a *migratory solution* is one such that all the fish will be attracted to each other and then travel in the same direction. Regardless of time step size, our finite difference method will eventually converge to the exact solution $u(x,t)$ as time tends upward.

First, let's look at the system where there is no perturbation in speed. Thus $\sigma_v = 0$.



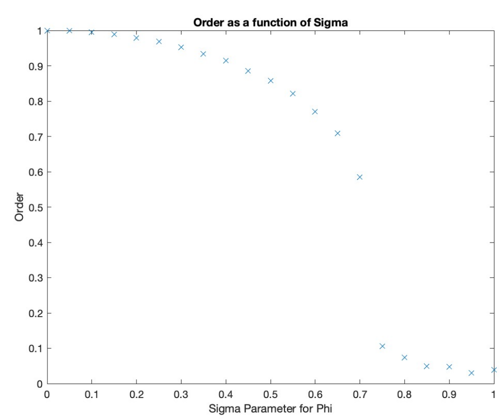
(A) $\sigma_\phi = 0.20$ (B) $\sigma_\phi = 0.60$ (C) $\sigma_\phi = 0.70$

In the first image, the perturbation in our system, $\sigma_\phi = 0.20$ is rather small. Since σ_ϕ was minimal, we approach a true solution rather quickly and directly. But what happens as σ_ϕ grows to 1? When $\sigma_\phi = 0.6$, we see that the system is taking a lot longer to approach a solution, and the curve does not grow as close to 1, but it does eventually tend towards it. In this case, the perturbation becomes too large for the system to completely align as it could before, however it does still have some order. To contrast, for $\sigma_\phi = 0.7$, the system cannot order itself. Thus we expect that between 0.6 and 0.7 is a bifurcation taking the system from order to disorder. Thus we have now determined their are both ordered an disordered phases, and as our σ_ϕ

gets larger, it becomes more difficult for each fish to try to match other directions. We will study exactly where this bifurcation from order to disorder occurs later in this chapter.

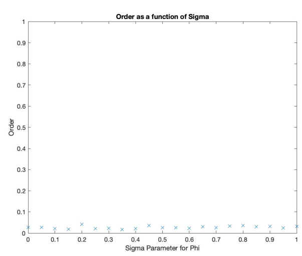
We have already described that an ordered solution is one where all the fish travel with the same speed and in the same direction. What does a disordered phase look like? After the bifurcation point, σ_ϕ becomes large enough that the fish cannot orient together. Here, the fish form separate independently circling families. In these independently circling families, the fish on the edges of the groups may get close enough to another circling solution to then go off and try to match with different fish. Since all of these fish are circling, there comes an infinite sequence of fish being passed off between smaller schools, with each oscillating at different positions, causing our system to have very low order. These circles the fish travel in are the reason that we see oscillations in our images.

3.6. **Bifurcation of the System.** As our equations depend on a σ_ϕ and σ_v adding a small amount of noise to the system, we can explore where our system bifurcates from order to disorder. To begin, let $\sigma_v = 0$. Thus, we will obtain a bifurcation in order parameter by varying $\sigma_\phi \in [0, 1]$. As demonstrated in Birnir et. al (3), if the variance of the noise exceeds a certain boundary, our system will go from order to disorder. Hence, if our σ_ϕ parameter is over a certain number, we do not expect our system to be able to synchronize or be ordered.

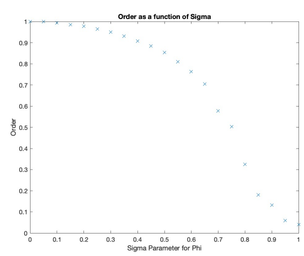


To the right is an image of the bifurcation of the order parameter of the system for σ_ϕ from 0 to 1. This image demonstrates a bifurcation around 0.75. However, depending on the initial conditions our bifurcation point will shift between 0.65 and 0.75, most commonly occurring around $\sigma_\phi = 0.69$.

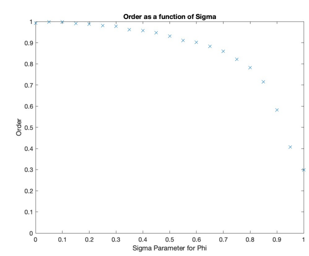
3.6.1. *Internal Turning Rates Characteristics.* Now, let's investigate how the internal turning rate α affect their ability to migrate through different values of σ_ϕ.



(A) $\alpha = 0$



(B) $\alpha = 0.5$



(C) $\alpha = 1$

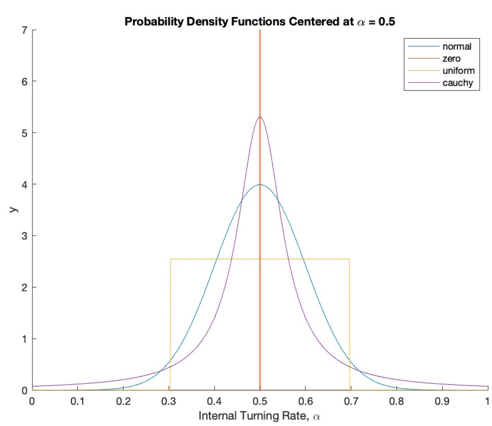
The internal turning rate, α, is the parameter that controls how each fish turns to match the average direction within the system. For internal turning rate constant $\alpha = 0$, $\alpha = 0.5$, and $\alpha = 1.0$, we have a bifurcations above. In the first case, since our turning rate here is zero, it is expected that the fish will not try to line up at all. This is demonstrated through the first plot, since each σ_ϕ step represents an average order of approximately 0.05, which is approaching maximum disorder.

For $\alpha \leq 0.2$, the rate is not quite strong enough to cause the system to eventually align over time. However, when $\alpha \geq 0.3$, we have a better bifurcation. Once α reaches 0.3, the turning rate is strong enough, that fish can push through minor perturbation to align. For example, when $\alpha = 0.5$ we have a nice bifurcation around $\sigma_\phi = 0.95$. Through calculating generating various plots of alpha varying from 0 to 1, we see that the stronger the turning rate, the more noise the system can be perturbed by while still maintaining order. Thus, we see that our α parameter affects how much perturbation a system can take before it becomes disordered. For consistency, we will continue to work under the condition that $\alpha = 0.5$

3.6.2. *Distributions of Internal Turning Rates.* For our survey, we will investigate three different distributions of internal turning rates. The first distribution, we will call the 'zero' distribution. Under this distribution, the internal turning rate α, of all 1000 fish are exactly α. Thus far, we have studied the system where the fish have the same internal turning rate. Thus, if $\alpha = 0.5$, then all 1000 fish have an internal turning rate of exactly $\alpha = 0.5$. Thus the fish will be more likely to travel in a straight line.

In the next chapter, we will begin to study this system where the particles (fish) have a different internal turning rates. This brings us to the next two ways we can distribute the internal turning rates: a uniform or Cauchy distribution

If the distribution is uniform, then the fish turning rates, α have a uniform distribution between $(\alpha - \eta, \alpha + \eta)$ for a particular value of η. In our numerical simulation, $\eta = \pi/16$. This means that if $\alpha = 0.5$, we have that each 1000 of has an equal chance of having an internal turning rate being placed anywhere in the interval for $\alpha \in [\frac{8-\pi}{16}, \frac{8+\pi}{16}]$.

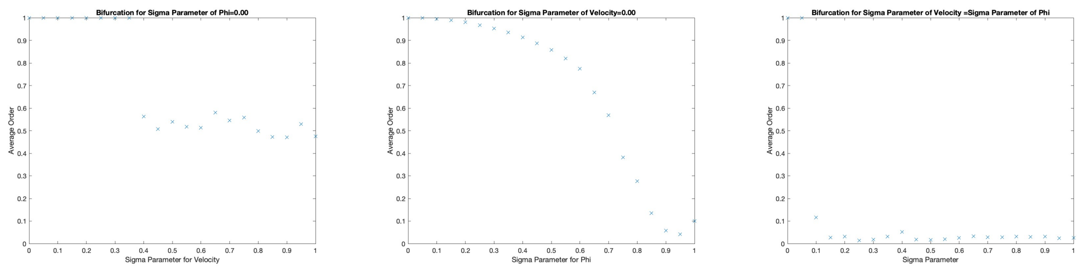


If it is a Cauchy distribution, otherwise known as the Lorentzian distribution when $\gamma = 0.5$, then the fish have individual internal turning rates close to each other determined by a mean, and width γ. In our numerical simulation, the mean is 0.5 and $\gamma = 0.05$. This means that out of the 1000 fish, most of fish are expected to be close to 0.5, with less fish having a turning rate father away from 0.5.

We will refer to these three case as 'zero distribution', 'uniform distribution', and 'Cauchy distribution'. The probability density functions of these distributions are above to the right.

3.6.3. *Bifurcation of Complete Perturbed System.* Now, let's investigate the complete perturbed system. To get an idea of how the movement changes over different values of σ_v and σ_ϕ, we have the bifurcations below. Let's fix σ_ϕ and find the bifurcation in the order parameter for varying σ_v.

For the next computations, we run our program with 200 fish. We set $\alpha = 0.5$ and $\nu = 0.0$.



(A) $\sigma_\phi = 0$ (B) $\sigma_v = 0$ (C) $\sigma_v = \sigma_\phi$

FIGURE 3. Bifurcations for various fixed Parameters

Note again that these bifurcations are from the Kuramoto order parameter, and so

$$r = \frac{1}{N}\,\Sigma_{j=1}^{N} \cos(\phi_j - \psi)$$

Clearly, this is only a function of direction, and not speed, thus we can expect perturbation is direction to affect our order more than perturbation in speed.

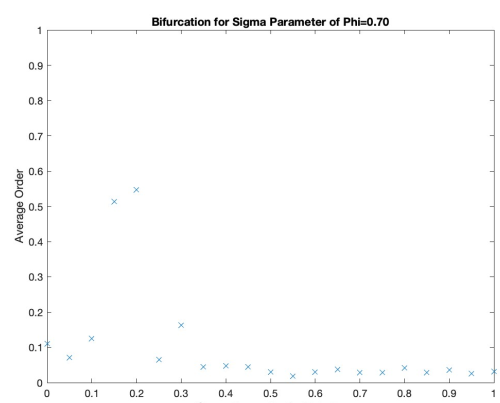
The first of the three images above represents a bifurcation of σ_v when $\sigma_\phi = 0$. In this case, when $\sigma_v < 0.4$, the system can still completely order itself. When $\sigma_v > 0.4$, we can see that our system is semi ordered. For semi ordered motion, the school will still migrate as a whole and tends in an average direction. This average direction, does not stay the same or change as expected when σ_v is large. The fish are all moving together, but there is a lot of chaos within the system. Instead of holding their general position relative to the neighboring fish, perturbations in speed cause the fish to travel independently amongst the school. This randomness in how the fish travel in the school causes the center of mass for the school to travel more randomly then before. However, since the the fish are still travelling together in a group, they maintain semi order. It looks like a school of disordered fish from above, but it is also travelling.

The second of the three images represents a bifurcation of σ_ϕ when $\sigma_v = 0$. This case is the one demonstrated above. Since our order parameter is calculated from ϕ, significant perturbations in ϕ will lead to lower order since ϕ would be farther from its average angle ψ. Thus, it is understandable that as σ_ϕ gets larger, the distance between ψ and ϕ increase, so r decreases.

The third of the three images represents a bifurcation of order when $\sigma_\phi = \sigma_v$. Since each sigma parameter individually attributes to some amount of disorder, the concatenation of the two cause disorder from small σ, $\sigma \leq 0.1$. Once $\sigma \geq 0.1$, there is too much perturbation for the system to be ordered.

If α were to be larger, these boundaries and approximate values where there is shifts from order to disorder will also get larger.

Lets look closer at the above of sigma parameter of $\sigma_\phi = 0.70$.



It is interesting to find that to begin, the fish are unable to order themselves. This is understandable since, in chapter 1, we determined that approximately $\sigma_\phi = 0.70$ will be where the system becomes disordered when there was no perturbation in speed. Now that there is some perturbation in speed, we see that actually helps the group align, and in fact, for $0.1 < \sigma_v < 0.25$, our system has some order and structure. Then, after that cutoff point, the perturbations become too large, and again, the system becomes disordered.

What is most interesting here is that although the group has some direction and order, the fish spread out more with time, as similar to the disorder demonstrated with only a large σ_ϕ and no perturbation in speed.

The disorder in these images is similar to the disorder demonstrated with only a large σ_ϕ and no perturbation in speed. We can see that over time, the group is spreading out. There is a chaotic behavior of each fish here, as many fish are circling around each other and crossing paths.

Originally, the fish try, and successfully match direction and travel, with some perturbation, and travel in one direction. But very quickly the noise becomes to large and then the system becomes disordered. At this point, the center of mass is moving around with the chaos of the system, so it is difficult to predict where the fish will go. This means that as σ gets larger, there will not always necessarily be less order.

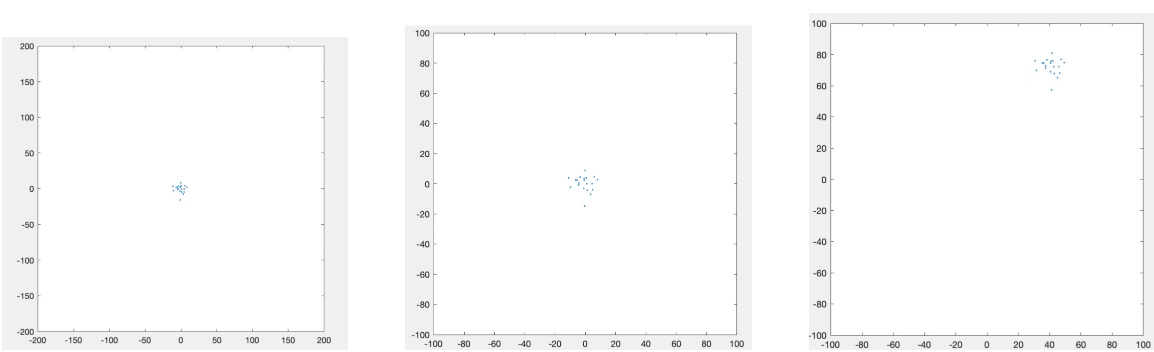
4. Chapter 2: Phases of Movement in Schools of Fish

We aim to classify all the different phases of movement that schools of fish can take determined by the system of equations above. There are several different factors affecting the overall migration pattern of the system: perturbations in direction or speed, and the initial distribution of the position, speed, direction, and internal turning rate of the fish.

4.1. **Non Perturbed System.** To begin, let's look at the case where there is no perturbation in speed or direction. Then we have the unperturbed system as follows:

$$\dot{v_j} = \frac{\alpha \bar{v}}{N}\, \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j$$
$$v_j \dot{\phi_j} = \frac{\alpha \bar{v}}{N}\, \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j)$$

For this theoretical system, we have a migratory solution. For however the fish are distributed, the system will start to align either by spreading the fish out if they are too close together, or they will attract towards each other if they are too far apart. Once they are all in a group, they will migrate in the average direction. The movement is pictured below for when the distribution is 'zero'.

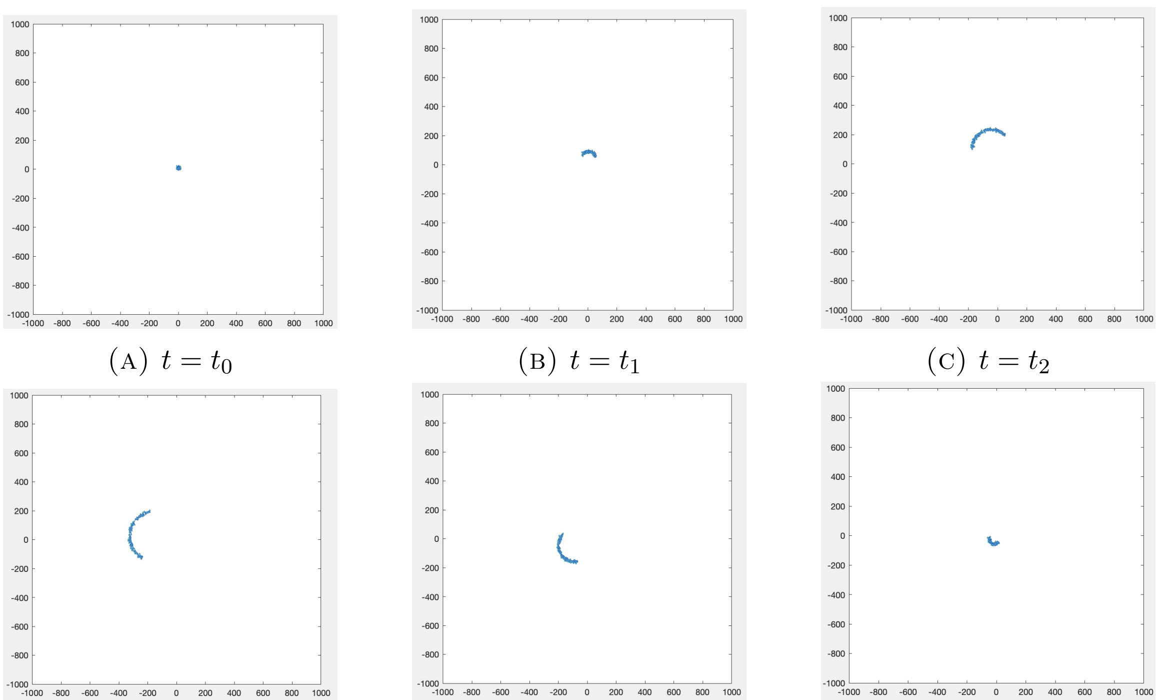


(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$

Figure 4. Three images demonstrating migratory solution

This is the most basic migratory solution. The fish tend towards an average direction that stays constant and continues infinitely in that direction. Since we are not perturbing the system in direction or speed, this the most simple solution.

Now, we are curious to investigate the motion when not all fish have the same internal turning rate. First, let's look at when the system's internal turning rate has a uniform distribution.



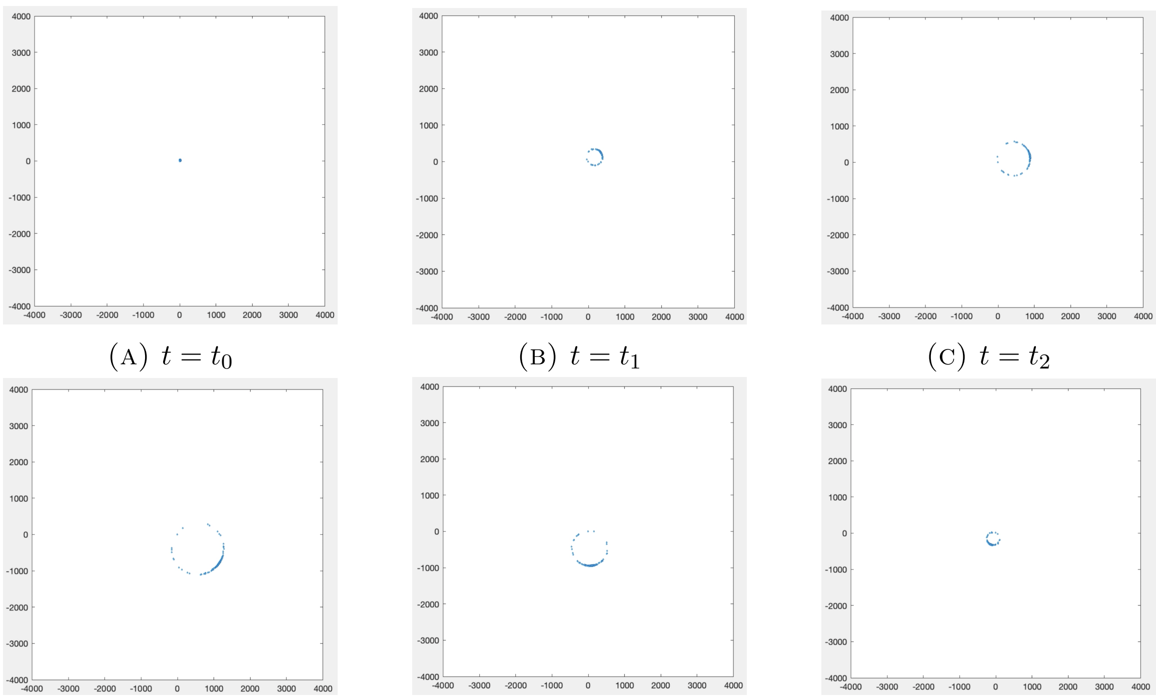
(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$

(D) $t = t_3$ (E) $t = t_4$ (F) $t = t_5$

FIGURE 5. Six images demonstrating migration solution when uniform

With this distribution, the fish will form an arc, similar to that of a flock of birds flying, where the fish travel in the average direction that is changing periodically. Once they hit a certain distance from the origin point, the group slows, and reverses. Then, it starts travelling with the same shape, but opposite direction, shrinking back down to the origin. Over time, this center of mass will move away from the origin in an arc over an ellipse over a period of π, and then shrink along the other half of the ellipse as it shrinks back to the origin. This process repeats for as many timesteps as we have, tracing the same path.

Now, what is the motion when the system's internal turning rate has a cauchy distribution? Then, we have the next motion.



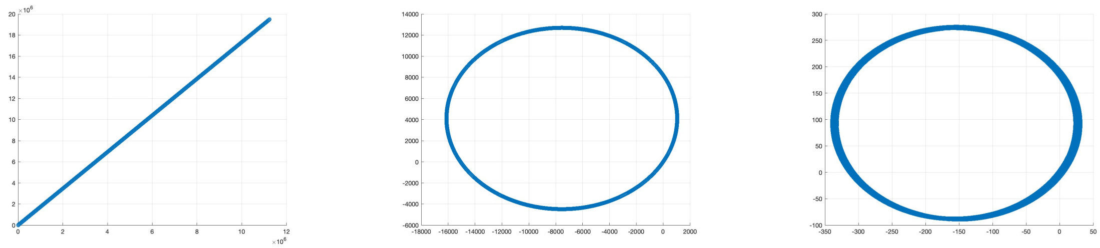
(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$

(D) $t = t_3$ (E) $t = t_4$ (F) $t = t_5$

FIGURE 6. Six images demonstrating migration solution when cauchy

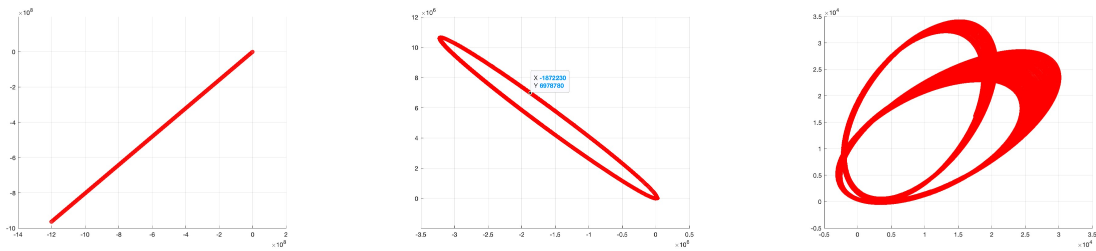
This motion of center of mass is similar as uniform, except instead of just and arc, the group tends to travel as a normally weighted circle of N many particles. The growing and shrinking from the origin is similar.

We can demonstrate the center of mass and variation from the center of mass over time.



(A) *Zero* (B) Uniform (C) Cauchy

FIGURE 7. Center of mass demonstrating 3 migratory solutions



(A) *Zero* (B) Uniform (C) Cauchy

FIGURE 8. Variation from Center of mass demonstrating 3 migratory solutions

These three center off masses represent the three different distributions with no perturbations. For all perturbed solutions with some order, they maintain these general shapes with some variation. Recall that we demonstrated in the previous chapter that that we can understand the system through the expected value (center of mass) and the variation (variation from center of mass).

For 'zero' distribution, we have that the center of mass is going in a straight line as demonstrated above. We also see that our variation is increasing linearly, this means that our fish are spreading out a constant rate.

For 'uniform' distribution, we have a periodic orbit where the fish circle through the origin. The variation is also periodic due to the growing and shrinking motion demonstrated above.

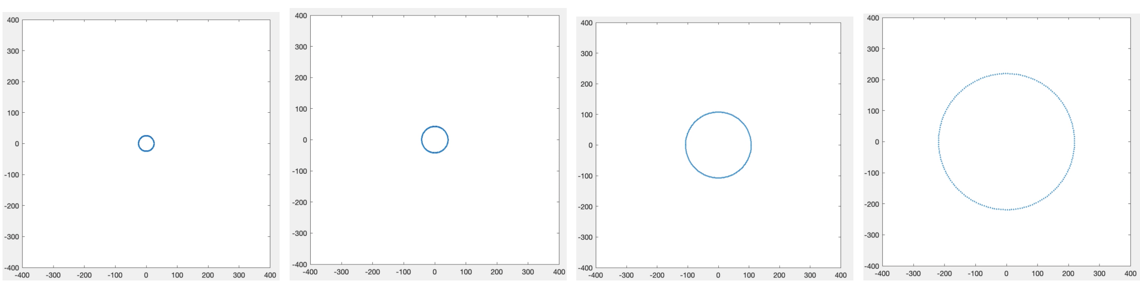
For 'Cauchy' distribution, we have a quasi periodic orbit since the fish are traveling in a circle for the center of mass. However, since our variation has two circles with origins near $(0, 0)$, we know that our system is following a trajectory along a torus. But since a torus is three dimensional, and we are plotting on a two dimensional plane, the paths appear as two circles. The fish follow through these paths infinitely, alternating between the two.

All in all, these results depict the maximum of a linear bound necessary to demonstrate the convergence proved above.

4.1.1. *Past Research.* These migratory phases described are not new. Carolina Trenado found these periodic and quasiperiodic phases discovered. Carolina Trenado found these phases by introducing contrary fish into the system. This is different from our system because we get these phases by perturbing both our speed and direction. See [Yuste, 12] for analysis.

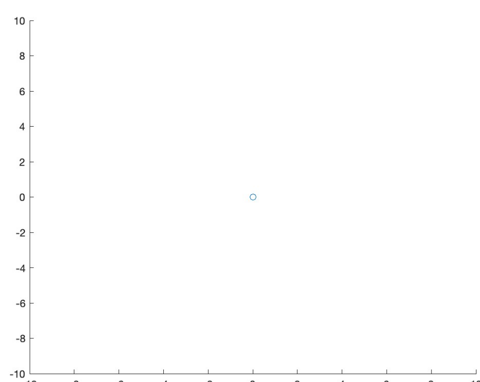
These three migrations are all general solutions to the Stochastic differential equations under particular conditions. However, it is important to highlight all possible solutions. These three above solutions are non stationary solutions, but there is also a stationary solution as described as follows.

4.1.2. *Stationary Solution.* Suppose we have N many fish. Then, set the position of each fish to be the N roots of unity on the unit circle. Set the direction of each fish to be tangent to the circle in the same orientation (clockwise or counterclockwise) for all N fish. Then we have the special stationary solution as demonstrated below. The particles in our case are circling counterclockwise around the origin $(0, 0)$, and spreading out slowly over time (4, Birnir).



(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$ (D) $t = t_3$

FIGURE 9. Four images demonstrating migration pattern for roots of unity



Then, the center of mass is interesting, because the average of the group isn't moving anywhere, since the fish move equally away from the origin in all directions with each step. Thus we have that the center of mass is a single fixed point. If there is a sigma perturbation, the small perturbation pushes the system off of equilibrium, and the stationary solution becomes a migratory solution as defined above. If there is a σ_v perturbation, the solution is still a stationary solution. See for further analysis of this solution (1, Barbaro)

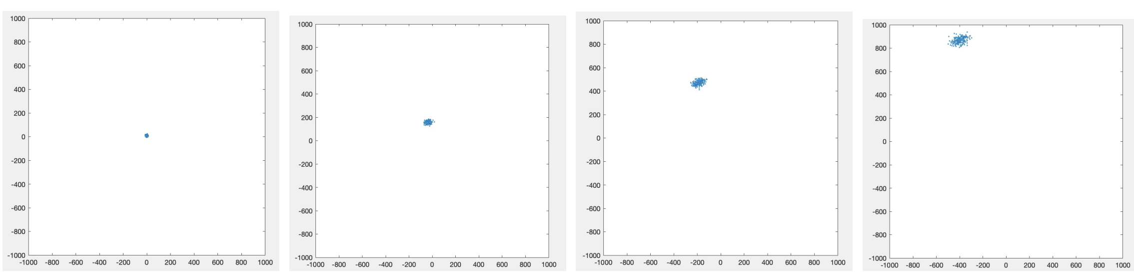
4.2. **Complete Perturbed System.** Next, we want to introduce perturbation into our speed as well as our direction. Let $\nu = 0.2$ be a small acceleration, σ_v, and σ_ϕ be our noise parameter for velocity and direction, respectively. Then we have the stochastic system as follows:

$$\dot{v_j} = \frac{\alpha \bar{v}}{N} \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j + \nu + \sigma_v \dot{B}_j$$
$$v_j \dot{\phi_j} = \frac{\alpha \bar{v}}{N} \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j) + \sigma_\phi \dot{B}_{j'}$$

where $\dot{B}_j \neq \dot{B}_{j'}$ is our Brownian motion.

4.2.1. *Motion of Completely Perturbed System.* The way the school of fish migrates changes depending on the initial distribution of the internal turning of the school. We have already determined this is true if we have an unperturbed system. Let $\nu = 0.2$ $\sigma_v = 0.40$ and $\sigma_\phi = 0.40$ The motion following demonstrates the system these perturbations over the three different distributions of internal turning rates (zero, uniform, cauchy).

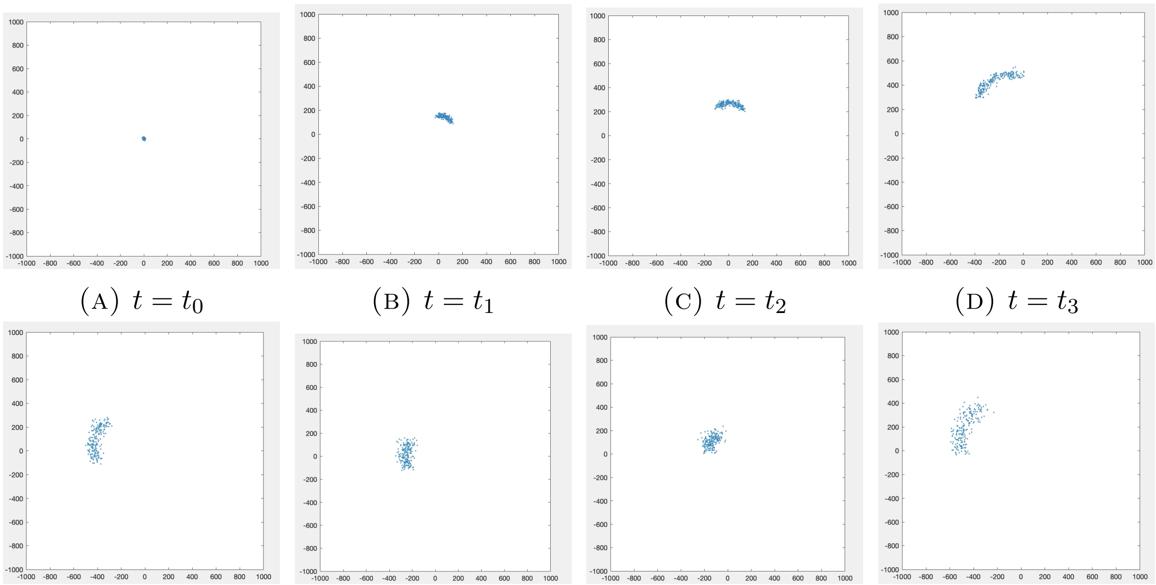
If have a 'zero' distribution, then we have a migratory similar to before, but spreading.



(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$ (D) $t = t_3$

FIGURE 10. Migratory solution from zero distribution

If we are in the uniform case, then we have the motion below:



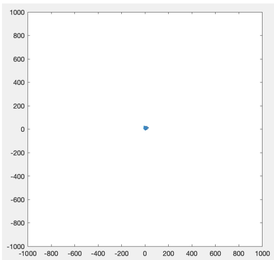
(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$ (D) $t = t_3$

(E) $t = t_4$ (F) $t = t_5$ (G) $t = t_6$ (H) $t = t_7$

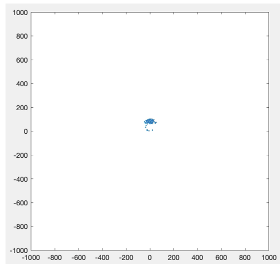
FIGURE 11. Migratory solution from Uniform Distribution

This solution maintains the same motion as the non perturbed system, but with some error, since the group is spreading over time.

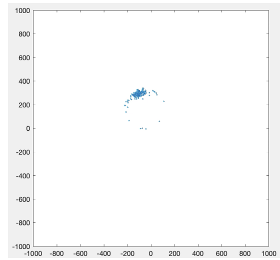
This is similar for when our internal turning rate has a Cauchy distribution.



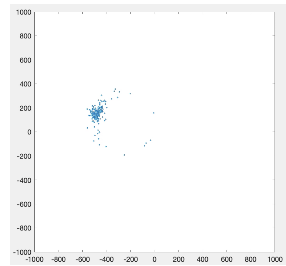
(A) $t = t_0$



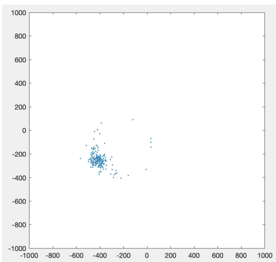
(B) $t = t_1$



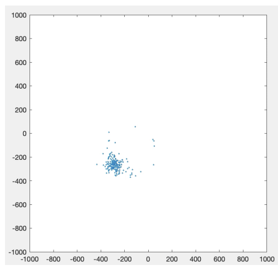
(C) $t = t_2$



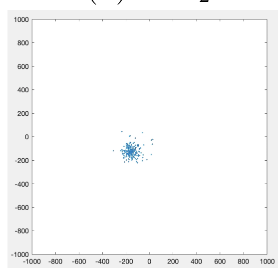
(D) $t = t_3$



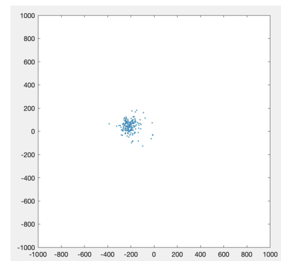
(E) $t = t_4$



(F) $t = t_5$



(G) $t = t_6$



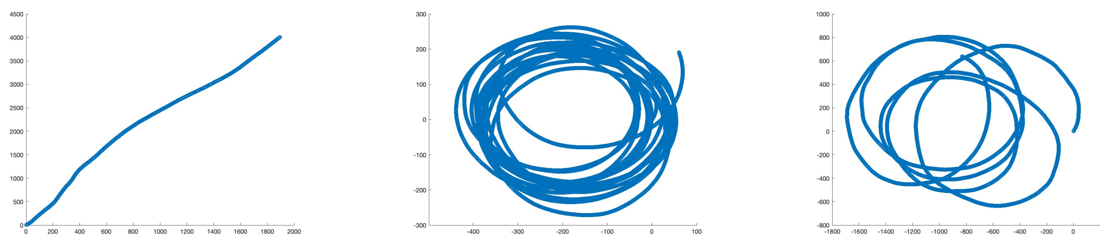
(H) $t = t_7$

FIGURE 12. Migratory solution from Cauchy Distribution

We see through these images the same scattering of fish over time, even though they are still travelling as a group.

4.3. **Expectation and Variation.** In Chapter 1, we proved theoretically that our system of two stochastic differential equations is stable and consistent due to both the expected value and variation being bounded. In this section, I demonstrate how perturbations in these systems affect the expected value and variations and maintain this condition.

Now that we have demonstrated the particular motion of each distribution, we can compute the expected value by plotting the center of mass of the system. Set $\sigma_\phi = 0.20$ and $\sigma_v = 0.20$.



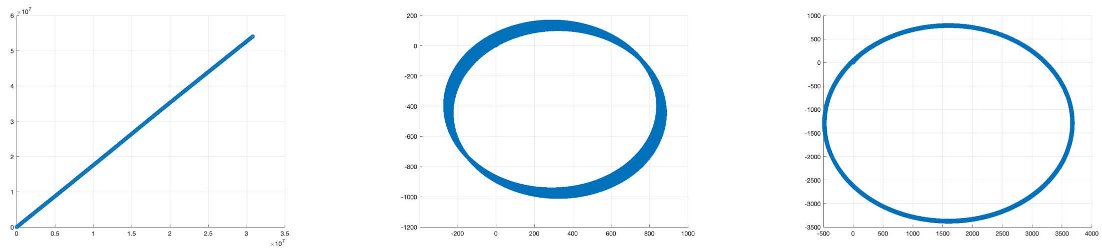
(A) zero (B) uniform (C) cauchy

FIGURE 13. Center of mass for $\sigma_v = 0.20$ and $\sigma_\phi = 0.20$

We can see that these expected values maintain the same general shape as those non perturbed systems above, but with some variation from the exact shape. The center of mass starts at $(0, 0)$. When there is zero distribution, we can see that the migration is not a straight migrating path as before, but is slightly curving. For the uniform and Cauchy distributions, the fish circle around, but do not pass exactly by the same spot again as when there is no perturbation.

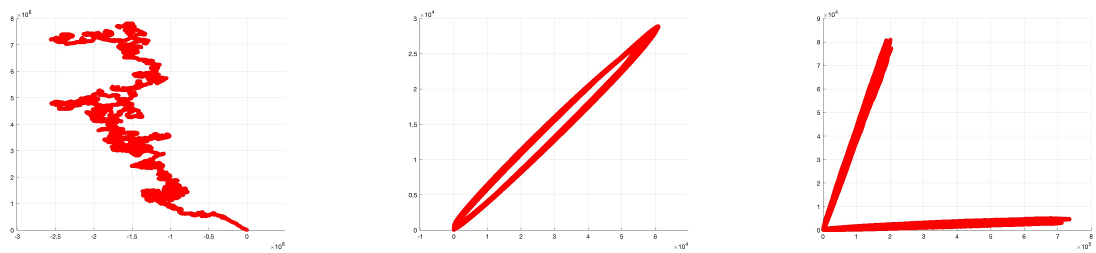
Let's turn down the perturbations to $\sigma_\phi = 0.001$ and $\sigma_v = 0.001$ The center of mass and variation from mass is plotted below for three different distributions.

Now we can see that our center of mass is a lot closer to the non perturbed motion. In 'zero', the small perturbation causes the fish to spread over time. Thus, even though the fish are maintaining a migratory pattern and traveling as a group, the



(A) zero (B) uniform (C) cauchy

FIGURE 14. Center of mass for $\sigma_v = 0.001$ and $\sigma_\phi = 0.001$



(A) zero (B) uniform (C) cauchy

FIGURE 15. Variation from Center of mass for $\sigma_v = 0.001$ and $\sigma_\phi = 0.001$

group is still spreading out, and so the variation of each fish from the expectation of the group is getting larger over time.

Further, for a uniform or Cauchy distribution we can see that our variation is periodic or quasi periodic as well. So in the beginning, the variation is 0 in both the x and y direction, but as time goes on, it gets slightly larger in both directions, until it hits the maximum variation. This is the point where the group has migrated as far as it will from the origin before returning and shrinking back towards the origin. With this shrinking motion, the variation will proceed to go back down in a way exact and opposite to how the variation got larger, thus we see the ellipse of the variation going back toward a small variation. This pattern repeats over time.

When there is a uniform distribution, this motion forms an ellipse. When the distribution is Cauchy, the motion can follow an ellipse as before, or it can travel along a pathway on a torus. The elliptic paths are very narrow for very small σ_v.

This example to the right is a variation from the center of mass for the uniform distribution under these same conditions (The center of mass is as expected). The interesting caveat here is that although we use the same distribution, we sometimes see a solution which travels along a torus, like in the Cauchy distribution. Since the uniform distribution presents us with many different phases, the solution seems to be unstable. We will explore this more later in this chapter.

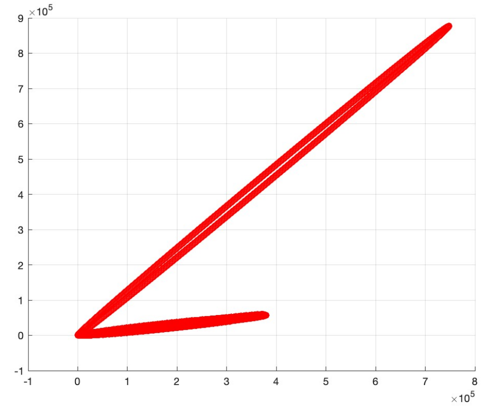
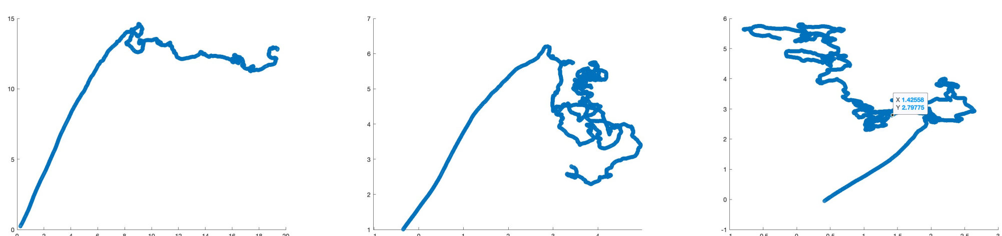


FIGURE 16. One Variation from Center of Mass for Uniform Distributions

4.3.1. *Heteroclincial Phase.* There is also a special heteroclinical phase in which the fish can maintain order for some time, and start to migrate together. The fish travel in a scattered way, but the disorder eventually overcomes the migration and the system becomes disordered.



(A) $\sigma_\phi = \sigma_v = 0.60$ (B) $\sigma_\phi = \sigma_v = 0.70$ (C) $\sigma_\phi = \sigma_v = 0.90$

FIGURE 17. Center of Mass for Heteroclinical Phase when Zero Dist.

We see the fish will travel, and eventually hit a point where it becomes disordered. It is believed that the fish are travelling along a stable manifold towards a fixed point, and once they hit the fixed point, they start travelling along the unstable manifold.

When there is no perturbation, the center of mass will move in a straight line. When perturbed by a small amount, the center of mass moves in particular trajectories: elliptic, on a torus, or straight. When largely perturbed, the center of mass moves in a seemingly random motion around the plane.

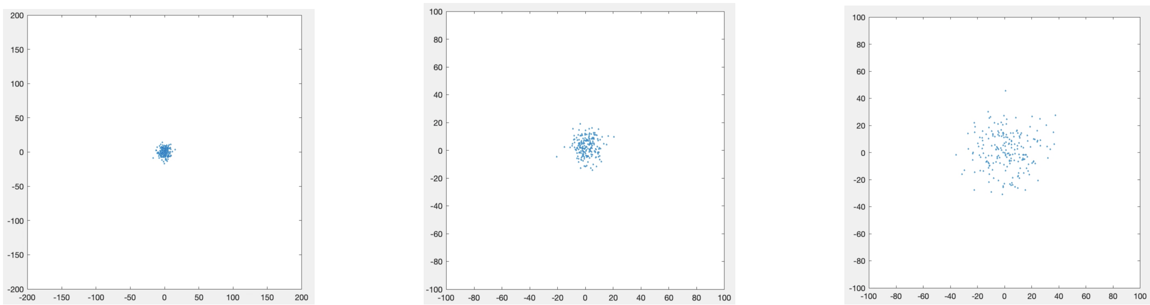
Now that have reviewed the motion of both a nonperturbed system and complete perturbed system, we will investigate to see if any special phases occur when there are only perturbations in either speed or direction. We will demonstrate that when we drive ϕ, only the migratory solution and disordered solution persist. But when we drive only v, then we get special cases around our bifurcation point.

4.4. **System with perturbation only in Sigma.** To begin, set $\sigma_v = 0$. Perturb the system in the direction as in the chapter above by introducing Brownian motion and a perturbation parameter, σ_ϕ, in the directional equation. This gives us the stochastic system as follows:

$$\dot{v_j} = \frac{\alpha\bar{v}}{N} \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j + \nu$$
$$v_j\dot{\phi_j} = \frac{\alpha\bar{v}}{N} \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j) + \sigma_\phi \dot{B_j}$$

Recall that this system has both an ordered, migrating solution, and disordered, scattered solution. For small sigma, we have a migrating solution as one of the three above. As sigma increases, there is more variation in how each individual fish travels.

To see what a disordered solution looks like, set $\sigma_\phi = 1$. As we saw previously, this will cause to much perturbation in the system, the system is not able to synchronize. This is be disordered. Here is the motion. It begins with 200 fish distributed normally on some small square. With the fish heading direction being perturbed by some large amount, the fish become spread out more and more over time, but they still try to maintain together, so they still have a center of mass very close to where they started.



(A) $t = t_0$ (B) $t = t_1$ (C) $t = t_2$

Figure 18. Three images demonstrating disordered solution

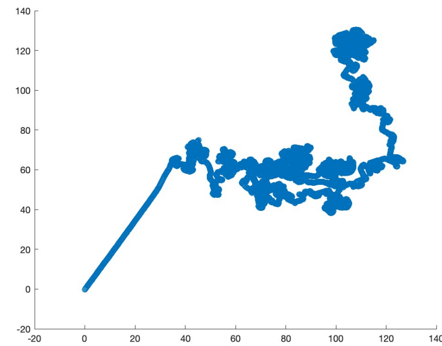
If the internal distribution is zero, then we have a migratory solution for $\sigma_\phi \leq 0.70$ and a disordered solution for $\sigma_\phi \geq 0.70$. Note that this cut off value is approximate, and changes depending on the particular parameters and initial values of the system. If the internal distribution is uniform, we have the migratory solution as with the uniform system above for $\sigma_\phi \leq 0.55$. When $\sigma_\phi \geq 0.55$, we have a disordered solution. If the internal distribution is Cauchy, we have the migratory solution as with the Cauchy system above for $\sigma_\phi \leq 0.70$. When $\sigma_\phi \geq 0.70$, we have a disordered solution.

4.5. **System with perturbation only in Speed.** Now, suppose there is no perturbation is direction, or that $\sigma_\phi = 0$. We can perturb the system in velocity only by adding a perturbation parameter, $\nu = 0.2$ and σ_v, and Brownian motion in the speed equation. This gives us the stochastic system as follows:

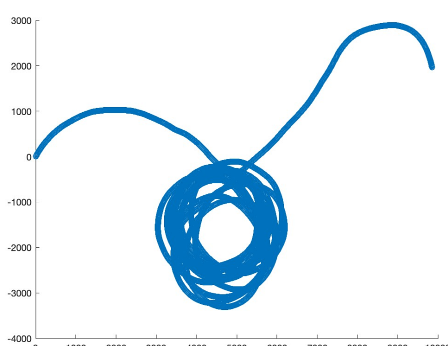
$$\begin{aligned} \dot{v_j} &= \frac{\alpha \bar{v}}{N} \, \Sigma_{k=1}^{N} \cos(\phi_k - \phi_j) - \alpha v_j + \nu + \sigma_v \dot{B}_j \\ v_j \dot{\phi_j} &= \frac{\alpha \bar{v}}{N} \, \Sigma_{k=1}^{N} \sin(\phi_k - \phi_j) \end{aligned}$$

We evaluate this system for various values of σ_v. When we have zero omega distribution, the system will always approach the migratory solution for values $0 \leq \sigma_v \leq 1$. Regardless of how fast the fish are swimming, by re averaging speed every step, system attempts to bring the fish back toward the center of the group.

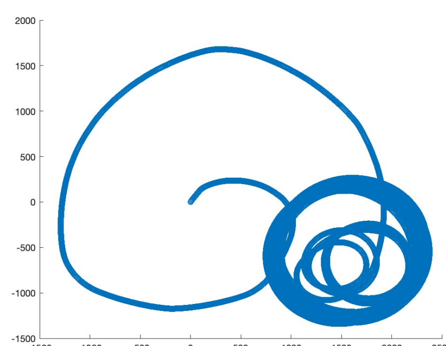
When the system is has an omega distribution that is a uniform distribution, then, for $0 \leq \sigma_v \leq 0.75$, the system approaches a migratory solution. This has the same "circling" motion as the system without perturbation having a uniform distribution. However, as σ_v increases, the tendency of fish to match the speed of it's neighbors is inhibited.The larger the value of σ_ν is, the more the system will spread as it migrates over time. For values of $0.75 \leq \sigma_v \leq 1$, then we have a heteroclinical solution, meaning that the system begins migrating initially, but then pauses and becomes disordered. An example of the heteroclinical solutions' center of mass is to the right.



When the system has a Cauchy distribution, the system approaches a migratory solution for values of $0 \leq \sigma_v \leq 0.75$. For values of $\sigma_v \geq 0.75$ but very close to it, there are some interesting behaviors displayed below. Both demonstrate how the system switches between different phases throughout its migration. To the left, the system begins by traveling in a straight line, circling, then traveling in a straight line again. To the right, the system is quasi periodic, meaning that it switches between two different periods.

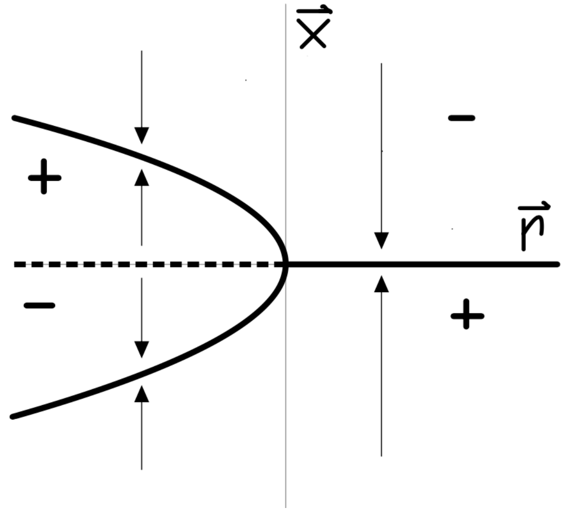


(A) $\sigma_v = 0.75$

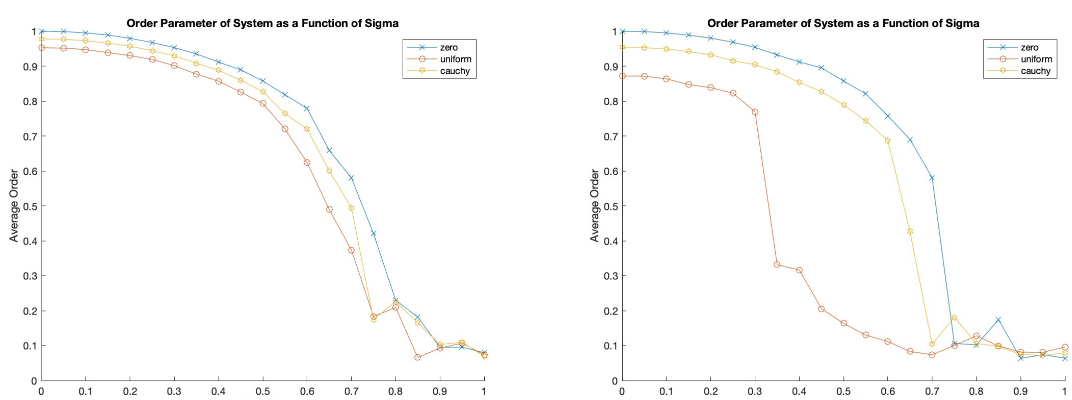


(B) $\sigma_v = 0.77$

4.6. **Pitchfork Bifurcation.** A pitchfork bifurcation is a particular local bifurcation where the system transitions from one fixed point to three fixed points. In the supercritical case, the system transitions from one fixed point to three, whereas in the subcritical case, the transition is vice versa (8, 'pitchfork bifurcation'). The three fixed points has the requirement that one is stable and two are unstable or vise versa. An example of our pitchfork is to the right.



In our case, I believe we have a subcritical pitchfork, since all three different phases end with the same disordered phase. Thus far, we have ran individual simulations solving for the order parameter for particular distribution. But we want to be able to compare all these distributions under a particular set of conditions so we can figure out stability, so let's fix the initial conditions and see how each distribution behaves.



(A) $\nu = 0$

(B) $\nu = 0.2$

We see through these images that the addition of a driving ν term is necessary. If $\nu = 0$, all of our different distributions behave similarly, ie they are ordered and disordered at the same values of σ_ϕ. Thus we have a degenerate solution. However, for larger ν, this driving term causes instability in the case of the uniform distribution while the other two cases are still stable. Thus we know we have a subcritical bifurcation of one unstable branch and two stable branches. Since it is always the case that our zero distribution case has higher order than the case with a Cauchy distribution, then our pitchfork has the case of zero distribution on the top stable branch, Cauchy distribution on the bottom stable branch, and the uniform distribution on the middle unstable branch.

5. Conclusion

Throughout this paper, we have developed a system of stochastic differential equations, and proved that, regardless of initial conditions, any solutions are numerically stable and consistent, and thus convergent. There are many initial conditions that affect the motion and long term trajectory of each solution, including: position, speed, direction, turning rate, and perturbation. We proved there exist stable and unstable migratory and stationary solutions. We find all of these solutions by introducing perturbation into our speed and direction, allowing our speeds to vary, and allowing for different internal turning rates. Reviewing these solutions, we establish numerically that our system exhibits a subcritical bifurcation with two stable and one unstable branch. This resolves numerically a question that has been open for almost thirty years.

6. Citations

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