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EIKONAL THEORY OF THE TRANSITION TO PHASE INCOHERENCE

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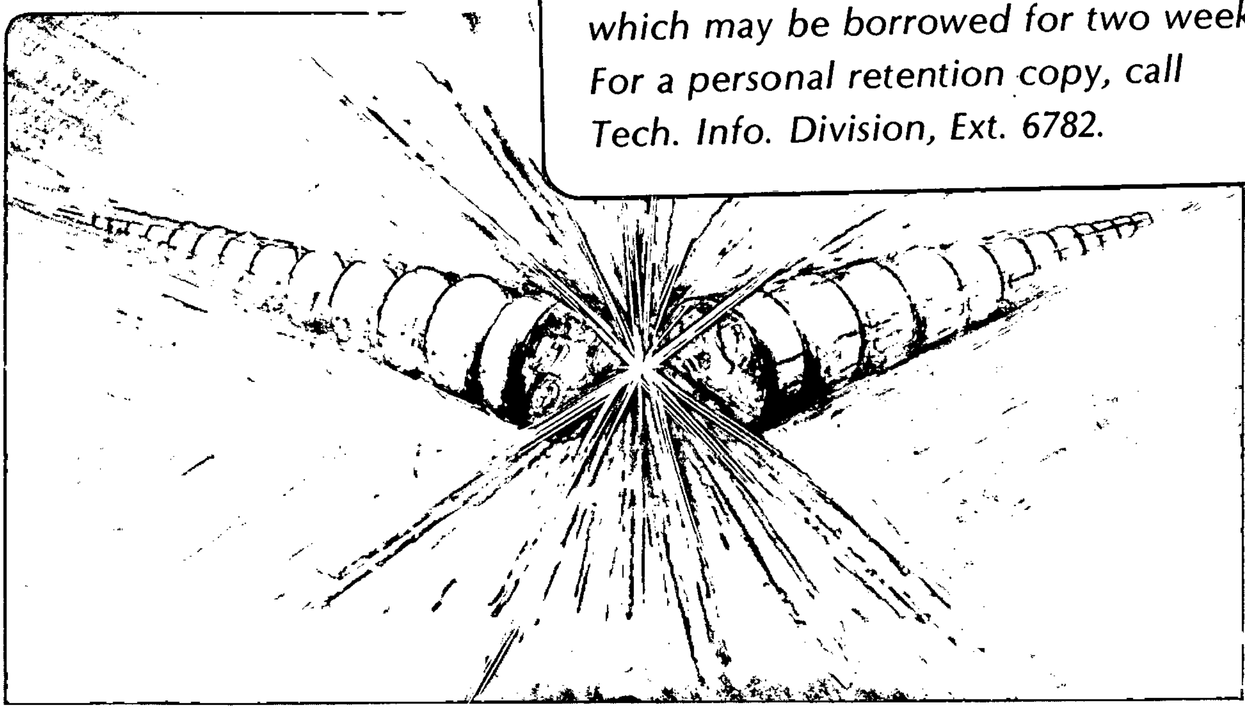

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EIKONAL THEORY OF THE TRANSITION TO PHASE INCOHERENCE*

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When a monochromatic electromagnetic wave propagates through a nonuniform plasma (of n dimensions), its refraction may be studied in terms of its family of rays in 2n-dimensional phase space (k,x). These rays generate an n-dimensional surface, imbedded in the phase space. The wave amplitude and phase are defined on this surface. As the rays twist and separate (from the dynamics of the ray Hamiltonian), the surface develops pleats and becomes convoluted. Projection of the surface onto x-space then yields a multivalued k(x). The local spectral density, as a function of k for given x, exhibits sharp spikes at these k(x), in the ray-optics limit. The next correction yields a finite width to these spikes. As the surface becomes more and more pleated, these spectral peaks overlap; the spectrum changes qualitatively from a line spectrum to a continuous spectrum. Correspondingly, the two-point spatial correlation function loses its long-range order, as the correlation volume contracts. This phenomenon is what we call the transition to incoherence.

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The propagation of a monochromatic electromagnetic wave through a stationary nonuniform plasma is represented by the family of rays $d\underline{x}/dt = \partial\omega/\partial\underline{k}$, $d\underline{k}/dt = -\partial\omega/\partial\underline{x}$. For an incident coherent wave, we use the eikonal form

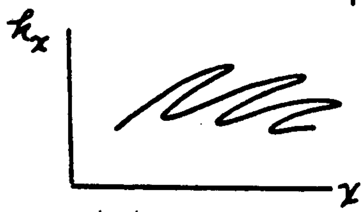
$$\underline{E}(\underline{x},t) = \underline{E}(\underline{x})\, e^{-i\omega_0 t} + c.c., \tag{1}$$

$$\underline{E}(\underline{x}) = \hat{e}(\underline{x})\,\tilde{E}(\underline{x})\, e^{i\Psi(\underline{x})}, \tag{2}$$

in terms of the polarization $\hat{e}(\underline{x})$, amplitude $\tilde{E}(\underline{x})$, and phase function $\Psi(\underline{x})$. The local wave vector is $\underline{k}(\underline{x}) = \nabla\Psi(\underline{x})$. In the 2n-dimensional phase space $(\underline{x}, \underline{k})$, the n-dimensional surface $\underline{k}(\underline{x})$ is termed a "Lagrangian manifold" (1). This surface is generated by the rays; as the rays separate (unstably) or twist (stably), due to refraction by the nonuniform medium, the surface becomes highly convoluted. Thus $\underline{k}(\underline{x})$ becomes multivalued, and a local spectrum arises. As this spectrum becomes continuous, the local spatial correlation becomes short-range, and the wave may be termed incoherent.

For illustrative purposes, let the plasma be two-dimensional: $\underline{x} = (x, y)$, with the spatial variation one-dimensional (in x). Let the phase function $\Psi(\underline{x})$ be specified at $y=0$ (say), so that $k_x(x, y=0) = \partial\Psi(x, y=0)/\partial x$ is known, while $k_y(x, y=0)$ is determined by the dispersion relation $\omega(k_x, k_y, x) \overset{?}{=} \omega_0$. The single-valued curve k_x vs. x is the intersection of the Lagrangian manifold with the plane y=0:

Now, as the rays propagate (in y), they generate a convoluted surface, whose section with the plane $y=y_1$ yields a multi-valued $k_x(x)$:



(At the caustics, a change of representation $\psi(\underset{\sim}{x}) \rightarrow \psi(\underset{\sim}{k})$ allows the eikonal method to be used even in the presence of this folding.) Except at the caustics, Eq. (2) now generalizes to

$$\underset{\sim}{E}(\underset{\sim}{x}) = \sum_j \hat{e}_j(\underset{\sim}{x}) \tilde{E}_j(\underset{\sim}{x}) e^{i\psi^j(\underset{\sim}{x})} \tag{3}$$

with $\underset{\sim}{k}^j(\underset{\sim}{x}) = \nabla\psi^j(\underset{\sim}{x})$ representing the several spectral branches.

To determine the $\underset{\sim}{k}$-spectrum, we introduce the local spatial Fourier transform of the electric field: (2)

$$\underset{\sim}{E}(\underset{\sim}{k},\underset{\sim}{x}) = \int_{\underset{\sim}{y}} e^{-i\underset{\sim}{k}\cdot\underset{\sim}{y}} \underset{\sim}{E}(\underset{\sim}{x}+\underset{\sim}{y})\, w(\underset{\sim}{y}), \tag{4}$$

where $w(\underset{\sim}{y})$ is a window function. A convenient choice is

$$w(\underset{\sim}{y}) = e^{-\frac{1}{2}\underset{\sim}{y}\underset{\sim}{y}:\underset{\approx}{\sigma}^{-2}} \tag{5}$$

where $\underset{\approx}{\sigma}$ is a symmetric matrix, whose eigenvalues are the spatial extent of the window. The local spectral density is defined as

$$I(\underset{\sim}{k},\underset{\sim}{x}) = |\underset{\sim}{E}(\underset{\sim}{k},\underset{\sim}{x})|^2. \tag{6}$$

(With the choice (5), it is the coarse-grained Wigner function.)

To simplify the discussion considerably, we now examine the contribution of only the phase $\psi^j(\underset{\sim}{x})$ of a single branch. Letting $\phi_j(\underset{\sim}{x}) = \exp i\psi^j(\underset{\sim}{x})$, we calculate $\phi_j(\underset{\sim}{k},\underset{\sim}{x})$ from (the analogue of) Eq. (4), and obtain its contribution to the local spectral density

$$I_j(\underset{\sim}{k},\underset{\sim}{x}) \sim |\det(\underset{\approx}{\sigma}^{-2} - i\,\nabla\nabla\psi^j)|^{-1} e^{-\mathrm{Re}\,\underset{\sim}{\kappa}\underset{\sim}{\kappa}:(\underset{\approx}{\sigma}^{-2} - i\,\nabla\nabla\psi^j)^{-1}} \tag{7}$$

where $\underset{\sim}{\kappa} = \underset{\sim}{k} - \underset{\sim}{k}^j(\underset{\sim}{x})$. Next, we minimize the spectral width with respect to $\underset{\approx}{\sigma}$. The appropriate $\underset{\approx}{\sigma}$ is diagonal with respect to the principal axes of $\nabla\nabla\psi^j$, and has components $\sigma_\mu = |\psi^j_{\mu\mu}|^{-\frac{1}{2}}$, where $\{\psi^j_{\mu\mu}\}$ are the eigenvalues of $\nabla\nabla\psi^j$.

With this choice, the spectrum for this one branch is

$$I_j(\underline{k},\underline{x}) \sim \prod_{\mu=1}^{n} |\psi^j_{\mu\mu}|^{-1} e^{-\frac{1}{2}\kappa_\mu^2 |\psi^j_{\mu\mu}|^{-1}} \qquad (8)$$

The spectral width is thus $|\psi^j_{\mu\mu}|^{\frac{1}{2}} = |\partial k_\mu/\partial x_\mu|^{\frac{1}{2}}$, of order $(k/L)^{\frac{1}{2}}$, where L is the scale-length for the nonuniform medium.

Now, with several (N) branches at given $\underline{x}$, we have the possibility of overlap of the corresponding contributions to the total spectral density. The spectral width then is of order $N(k/L)^{\frac{1}{2}}$, and the correlation distance (its reciprocal) of order $(\lambda L)^{\frac{1}{2}}/N$. With this decrease in the spatial phase-correlation, the wave has become incoherent.

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