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Ho, P.-M.

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P.-M. Ho
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Geometrical Aspects of Quantum Spaces [1]

Pei-Ming Ho [2]

Theoretical Physics Group
Lawrence Berkeley Laboratory
University of California
Berkeley, California 94720

(Ph.D. thesis)

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[2]e-mail address: pmho@physics.berkeley.edu

Abstract

Various geometrical aspects of quantum spaces are presented showing the possibility of building physics on quantum spaces.

In the first chapter we will give the motivations for studying noncommutative geometry and also review the definition of a Hopf algebra and some general features of the differential geometry on quantum groups and quantum planes.

In Chapter 2 and Chapter 3 the noncommutative version of differential calculus, integration and complex structure are established for the quantum sphere S_q^2 and the quantum complex projective space $CP_q(N)$, on which there are quantum group symmetries that are represented nonlinearly, and are respected by all the aforementioned structures. The braiding of S_q^2 and $CP_q(N)$ is also described.

In Chapter 4 the quantum projective geometry over the quantum projective space $CP_q(N)$ is developed. Collinearity conditions, coplanarity conditions, intersections and anharmonic ratios will be described.

In Chapter 5 an algebraic formulation of Riemannian geometry on quantum spaces is presented where Riemannian metric, distance, Laplacian, connection, and curvature have their quantum counterparts. This attempt is also extended to complex manifolds. Examples include the quantum sphere, the complex quantum projective space and the two-sheeted space. The quantum group of general coordinate transformations on some quantum spaces is also given.

Brief reviews of Poisson-Lie groups and Lie bialgebras are included in Appendix A and Connes' noncommutative geometry in Appendix B.

Disclaimer

Contents

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Chapter 1

Introduction

In this chapter we give the motivation and background for studying noncommutative geometry and quantum groups.

1.1 Motivation

The purpose of this section is to give the physical and mathematical motivations for working on noncommutative geometry and quantum groups.

1.1.1 Physical Motivation

As early as the advent of the Heisenberg uncertainty relation and the canonical commutation relation $[x,p]=i$ in quantum mechanics, people have speculated that spacetime should not be treated like a classical [1] manifold [1], because the quantum effects of gravitation should impose some lower bound on the measurable distance (the Planck scale). Rather, the coordinates of spacetime are expected to satisfy some nontrivial commutation relations.

If spacetime is quantized, the Lorentz group might as well be quantized, either as the structure group of a vector bundle over spacetime, or as the symmetry group of a macroscopically flat region in spacetime. Similarly, internal symmetry groups might also have to be quantized.

In addition, noncommutative geometry, quantum groups or similar ideas have been applied to other fields of physics such as the quantum Hall effect [2], Wess-

[1] Here and in the following the words "classical" and "quantum" are used to indicate the commutativity and noncommutativity, respectively, of the algebra of functions over the underlying space. Hence not only $SU(2)$, but also E_8 will be called a classical group.

Zumino-Witten models [3, 4], conformal field theory [5, 6], lattice field theory [7, 8, 9], integrable models [10, 11] and string field theory [12].

1.1.2 Mathematical Motivation

A mathematical theorem [13] says that if $\mathcal{A}$ is a commutative C^*-algebra [2] and $\hat{\mathcal{A}}$ the space of maximal ideals of $\mathcal{A}$, then $\mathcal{A}$ is $*$-isomorphic to $C_0(\hat{\mathcal{A}})$ (the set of functions vanishing at infinity) and $\hat{\mathcal{A}}$ is a locally compact Hausdorff space.

Given a manifold, the maximal ideal mentioned above can be identified with the subset of (complex-valued continuous) functions which vanish at a certain point on the manifold. It is an ideal because if any function is multiplied to an element of this subset the product is still in the subset. It is maximal in the sense that if any function not in the subset is adjoined to the subset, their linear span will include the whole algebra $\mathcal{A}$. Conversely, if one is given the algebra of functions on a manifold directly as an algebra without knowing the underlying manifold, the theorem says that one can associate each maximal ideal with a point and identify the space of maximal ideals with the manifold.

The descriptions of functions and points are dual to each other. The evaluation of a function f on the manifold at a point p, $f(p)$, can be thought of as the evaluation of a function p on $\mathcal{A}$ at a "point" f, $p(f)$, i.e., $f(p) = (f,p) = p(f)$. It is expected that properties of a space can be stated in a dual way for the dual space.

The natural question is whether we can generalize classical geometry by considering noncommutative algebras [3]. While in the noncommutative case the concept of points loses its actual meaning, the concept of functions survives (as noncommutative algebra) and the dual description of things becomes the only appropriate choice. In this sense the dual description is more fundamental since it has a wider applicable range.

The effort to restate everything in the dual picture also stimulates a better understanding of the classical objects. Furthermore, it is remarkable that a quantum counterpart can be found for almost every classical notion. Mathematically for such a generalized notion the classical case appears as a very restrictive case among all the noncommutative possibilities. Therefore it would be a great mystery if the physical world (spacetime) is purely classical while all the relevant concepts are naturally embedded in a much more general setting.

[2] A C^*-algebra is a Banach algebra with an involution (a Banach $*$-algebra) satisfying $||x^*x|| = ||x||^2$; a Banach algebra is a complete normed linear space with $||xy|| \leq ||x|| ||y||$.

[3] This question was raised by von Neumann and called a "pointless" geometry.

The Gel'fand-Naimark theorem [13] states that every C^*-algebra is isomorphic and isometric to a Banach algebra of operators on a Hilbert space where the Hermitian conjugation corresponds to the $*$-involution. Hence when one talks about noncommutative algebras it is the safest to restrict oneself to C^*-algebras so that everything can be realized as operators on a Hilbert space. Nevertheless, since the work presented in the following is a kind of piloting work that is aimed at investigating the possibility and difficulty of using noncommutative objects to make physical statements (geometrical statements), we will not be too concerned with the mathematical rigor and allow formal manipulations unless obvious contradiction appears. This has been the attitude of physicists toward mathematics for most of the time.

1.2 Hopf Algebras

In this section we will see that the concept of group, after being rephrased in the dual language, can be generalized to the noncommutative case.

The definition of Hopf algebra [14] is closely related to the definition of groups. It is a generalization of the algebra of functions on a group. While ordinary functions on a group (or any classical manifold) form a commutative algebra, a Hopf algebra is a possibly noncommutative algebra with all essential properties of the algebra of functions on a group except commutativity. Just like in ordinary quantum mechanics the coordinate x and the momentum p satisfying the noncommutative relation $[x,p] = i$ can be viewed as functions on the quantum phase space, elements in a Hopf algebra can be viewed as functions on a quantum group.

The defining properties of a group G are:

1. *Product*: $g = g_1 g_2 \in G$ is defined for all $g_1, g_2 \in G$.

2. *Associativity*: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$.

3. *Unit*: There exists $1_G \in G$ such that $1_G g = g = g 1_G$ for all $g \in G$.

4. *Inverse*: For all $g \in G$ there exists $g^{-1} \in G$ such that $g g^{-1} = 1_G = g^{-1} g$.

All the four properties above can be stated alternatively as properties of the algebra of functions on the group G, which is denoted by $\mathcal{A}$. Since $\mathcal{A}$ is dual to G in the sense that G is also the algebra of functions on $\mathcal{A}$ by the identification $g(f) = f(g)$ for any $g \in G$ and $f \in \mathcal{A}$, the statements for $\mathcal{A}$ are called the dual statements of the corresponding ones for G. Classically there is a multiplication on

$\mathcal{A}$ which is independent of the product defined on G. It is the associative, point-wise multiplication:

$$(f_1 f_2)(g) = f_1(g) f_2(g). \tag{1.1}$$

We also assume the existence of the unit element $1_{\mathcal{A}} \in \mathcal{A}$ so that $1_{\mathcal{A}} f = f = f 1_{\mathcal{A}}$ for any $f \in \mathcal{A}$. In this case the dual statements are the following:

1. *Coproduct*: According to the product $g = g_1 g_2$ on G, any $f \in \mathcal{A}$ can be decomposed into functions of g_1 and functions of g_2 as $f(g) = f(g_1 g_2) = \sum_i f_{1i}(g_1) f_{2i}(g_2)$. This induces the coproduct map $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that $\Delta(f) = \sum_i f_{1i} \otimes f_{2i}$. In Sweedler's notation [14] it is abbreviated as

$$\Delta(f) = f_{(1)} \otimes f_{(2)}. \tag{1.2}$$

Intuitively the coproduct is used to "undo" the product operation in G. As a result of (1.1), the coproduct is an algebra homomorphism, i.e.,

$$\Delta(f_1 f_2) = \Delta(f_1)\Delta(f_2), \tag{1.3}$$

where the multiplication on $\mathcal{A}\otimes\mathcal{A}$ is as usual $(f_1 \otimes f_2)(f_3 \otimes f_4) = (f_1 f_3 \otimes f_2 f_4)$.

2. *Coassociativity*: It is equivalent to say that the product in G is associative or that $f((g_1 g_2) g_3) = f(g_1 (g_2 g_3))$ is true for all functions f. For the latter we apply the coproduct twice to undo the two multiplications involved and derive the coassociativity for the coproduct:

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \tag{1.4}$$

where id is the identity map $id(f) = f$ on $\mathcal{A}$. In Sweedler's notation, it says

$$(a_{(1)})_{(1)} \otimes (a_{(1)})_{(2)} \otimes a_{(2)} = a_{(1)} \otimes (a_{(2)})_{(1)} \otimes (a_{(2)})_{(2)}, \tag{1.5}$$

which will be denoted as $a_{(1)} \otimes a_{(2)} \otimes a_{(3)}$. In general, we denote the result of n consecutive actions of Δ on a as $a_{(1)} \otimes a_{(2)} \otimes \cdots \otimes a_{(n+1)}$.

3. *Counit*: The counit is a linear map $\epsilon : \mathcal{A} \to \boldsymbol{C}$ [4] defined by $\epsilon(f) = f(1_G)$. It undoes the evaluation of a function at the unit of G. It follows from (1.1) that the counit is also an algebra homomorphism:

$$\epsilon(f_1 f_2) = \epsilon(f_1)\epsilon(f_2). \tag{1.6}$$

[4] We will always assume that the field of the algebra is $k = \boldsymbol{C}$.

From the dual statement of the existence of the unit in G, namely $f(1_G g) = f(g) = f(g 1_G)$, we see that

$$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta. \tag{1.7}$$

4. *Coinverse*: The coinverse $S : \mathcal{A} \to \mathcal{A}$, also called the *antipode*, is an anti-automorphism on $\mathcal{A}$ defined by $S(f)(g) = f(g^{-1})$. Because $f(g^{-1}g) = f(1_G) = f(gg^{-1})$ we have

$$m \circ (S \otimes id) \circ \Delta = 1_{\mathcal{A}}\epsilon = m \circ (id \otimes S) \circ \Delta, \tag{1.8}$$

where m is the multiplication map: $m(f_1 \otimes f_2) = f_1 f_2$.

Now the only thing we have to do to give the definition of a Hopf algebra [14] is to write down all the dual statements above which allow a noncommutative multiplication m.

Definition 1 *A Hopf algebra $(\mathcal{A}, m, 1_{\mathcal{A}}, \Delta, \epsilon, S)$ is an associative algebra with the unit element $1_{\mathcal{A}}$ and the following properties:*

$$\Delta(f_1 f_2) = \Delta(f_1)\Delta(f_2), \tag{1.9}$$
$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta, \tag{1.10}$$
$$\epsilon(f_1 f_2) = \epsilon(f_1)\epsilon(f_2), \tag{1.11}$$
$$(\epsilon \otimes id) \circ \Delta = id = (id \otimes \epsilon) \circ \Delta, \tag{1.12}$$
$$m \circ (S \otimes id) \circ \Delta = 1_{\mathcal{A}}\epsilon = m \circ (id \otimes S) \circ \Delta. \tag{1.13}$$

In the definition we left out the requirement that the coinverse is an anti-automorphism because it is a result of the rest [14].

Lemma 1 *For a Hopf algebra,*

$$S(ff') = S(f')S(f). \tag{1.14}$$

Proof In Sweedler's notation,

$$\begin{aligned}
& (ab)_{(1)} \otimes (ab)_{(2)} = a_{(1)}b_{(1)} \otimes a_{(2)}b_{(2)} \\
\Rightarrow\ & \epsilon(ab)_{(1)} \otimes (ab)_{(2)} = \epsilon(a_{(1)})\epsilon(b_{(1)}) \otimes a_{(2)}b_{(2)} \\
\Rightarrow\ & \epsilon(ab)_{(1)} \otimes (ab)_{(2)} = \epsilon(a_{(1)})S(b_{(1)})b_{(2)} \otimes a_{(2)}b_{(3)} \\
\Rightarrow\ & \epsilon(ab)_{(1)} \otimes (ab)_{(2)} = S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)} \otimes a_{(3)}b_{(3)} \\
\Rightarrow\ & \epsilon(ab)_{(1)}S((ab)_{(2)}) = S(b_{(1)})S(a_{(1)})a_{(2)}b_{(2)}S(a_{(3)}b_{(3)}) \\
\Rightarrow\ & S(ab) = S(b_{(1)})S(a_{(1)})\epsilon(a_{(2)})\epsilon(b_{(2)}) \\
\Rightarrow\ & S(ab) = S(b)S(a).
\end{aligned}$$

□

Clearly the algebra of functions on classical groups is a Hopf algebra.

Giving the definition of the Hopf algebra is only the beginning of the story of quantum groups. For an algebra to be the algebra of functions on a quantum group as a deformation of a classical group, it is not sufficient to check that it is a Hopf algebra. One has to check additional properties. For example, the number of independent generators of the algebra should be the same as the classical case. If a classical group is endowed with a $*$-involution, one would hope for the same thing in the quantum case. The involution on a Hopf algebra will make it a $*$-Hopf algebra if the $*$-involution [5] ($* \circ * = id$) is compatible with the Hopf algebra structure:

$$(* \otimes *) \circ \Delta = \Delta \circ *, \tag{1.15}$$

$$* \circ \epsilon = \epsilon \circ *, \tag{1.16}$$

$$* \circ S \circ * = S^{-1}, \tag{1.17}$$

where S^{-1} is the inverse map of S and we have used the same symbol $*$ for the complex conjugation on the left hand side of the second line.

1.2.1 $GL_q(2)$

The simplest example of a matrix quantum group [15] as a Hopf algebra is $GL_q(2)$ [16]. It is a one-parameter deformation of the classical group $GL(2)$. The algebra of functions on $GL(2)$ is generated by the elements $\alpha, \beta, \gamma, \delta$ in the fundamental representation

$$\pi(g) = \begin{pmatrix} \alpha(g) & \beta(g) \\ \gamma(g) & \delta(g) \end{pmatrix} \tag{1.18}$$

for $g \in GL(2)$. Since they are sufficient to distinguish all group members, any function on the group can be written as a function of the generators. In the quantum case the Hopf algebra $\mathcal{A}$ is usually defined to be the set of all polynomials of the generators $\alpha, \beta, \gamma, \delta$, which are now noncommutative objects, e.g., operators on a Hilbert space.

1. *Algebra:*

[5] We will always assume that the $*$-involution reverses the order of a product: $(ab)^* = b^* a^*$. Hence in fact it should be called an anti-involution.

The multiplication in $\mathcal{A}$ is noncommutative and the commutation relations are given compactly as

$$\hat{R}^{ij}_{kl} T^k_m T^l_n = T^i_k T^j_l \hat{R}^{kl}_{mn} \tag{1.19}$$

in terms of the quantum matrix

$$T = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.20}$$

and the $\hat{R}$-matrix

$$\hat{R} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \tag{1.21}$$

where

$$\lambda = q - q^{-1}. \tag{1.22}$$

A very convenient notation is to label each string of contracted indices by a number. For example, in the RTT relation (1.19) there are two strings of indices on both sides. They are $\{i, k, m\}$ and $\{j, l, n\}$, which will be labelled by 1 and 2, respectively. So in this new notation, (1.19) can be written simply as

$$\hat{R}_{12} T_1 T_2 = T_1 T_2 \hat{R}_{12}. \tag{1.23}$$

When the deformation parameter $q \to 1$, the $\hat{R}$-matrix becomes the permutation matrix and the *RTT relation* (1.19) simply says that all generators commute with each other. The form of the RTT relation is therefore a simple generalization of the classical commutation relation.

Explicitly the commutation relations are

$$\alpha\beta = q\beta\alpha, \quad \alpha\gamma = q\gamma\alpha, \tag{1.24}$$

$$\beta\delta = q\delta\beta, \quad \gamma\delta = q\delta\gamma, \tag{1.25}$$

$$\beta\gamma = \gamma\beta, \quad \alpha\delta - \delta\alpha = \lambda\beta\gamma. \tag{1.26}$$

The self-consistency of the commutation relations and the independence among the generators can be checked and thus ensure a correct classical limit for $q \to 1$. In particular, when one commutes a generator through both sides of the RTT relation (1.19), one hopes that no new relation is implied. This is guaranteed by the quantum Yang-Baxter relation

$$\hat{R}^{ij}_{i'j'}\hat{R}^{j'k}_{j''n}\hat{R}^{i'j''}_{lm} = \hat{R}^{jk}_{j'k'}\hat{R}^{ij'}_{lj''}\hat{R}^{j''k'}_{mn}, \tag{1.27}$$

which in our new notation is

$$\hat{R}_{12}\hat{R}_{23}\hat{R}_{12} = \hat{R}_{23}\hat{R}_{12}\hat{R}_{23}. \tag{1.28}$$

2. *Coproduct:*

The coproduct of a generator is defined by the matrix multiplication

$$\begin{pmatrix} \Delta(\alpha) & \Delta(\beta) \\ \Delta(\gamma) & \Delta(\delta) \end{pmatrix} = \begin{pmatrix} \alpha\otimes\alpha+\beta\otimes\gamma & \alpha\otimes\beta+\beta\otimes\delta \\ \gamma\otimes\delta+\delta\otimes\gamma & \gamma\otimes\beta+\delta\otimes\delta \end{pmatrix}, \tag{1.29}$$

which is abbreviated as

$$\Delta(T) = T\dot{\otimes}T, \tag{1.30}$$

where $\dot{\otimes}$ means matrix multiplication by tensor products of the elements. This formula is the same as the classical one. The coproduct of a generator evaluated at $g_1, g_2 \in G$ is therefore $\Delta(T^i_j)(g_1,g_2) = T^i_k(g_1)T^k_j(g_2)$ as one expects. The coproduct of polynomials of the generators is obtained by the linearity of the coproduct

$$\Delta(af+bf') = a\Delta(f)+b\Delta(f') \tag{1.31}$$

for $a,b \in \boldsymbol{C}$ and $f,f' \in \mathcal{A}$ and the property that it is an algebra homomorphism (1.9). Another equivalent way to say that the coproduct is an algebra homomorphism is to say that the algebra is *covariant* under the left transformation

$$T \to T'' = TT' \tag{1.32}$$

or the right transformation

$$T \to T'' = T'T, \tag{1.33}$$

where T' is another quantum matrix satisfying (1.19) whose entries commute with the entries of T. The left- or right-covariance of the algebra means that the commutation relations among the entries of T'' is the same as those for the corresponding entries of T for the left or right transformation (1.32) or (1.33), respectively. In our example it means that T'' also satisfies the RTT

relation (1.19) with T replaced by T'' so that T'' is also a quantum $GL_q(2)$ matrix. This can be easily checked:

$$\begin{aligned}
\hat{R}_{12}T''_1T''_2 &= \hat{R}_{12}T_1T'_1T_2T'_2 \\
&= \hat{R}_{12}T_1T_2T'_1T'_2 \\
&= T_1T_2\hat{R}_{12}T'_1T'_2 \\
&= T_1T_2T'_1T'_2\hat{R}_{12} \\
&= T''_1T''_2\hat{R}_{12}
\end{aligned}$$

for the right transformation and similarly for the left transformation. The algebra (1.25) is said to be *bicovariant* since it is both left and right covariant. Because in the above we did not use the specific expression for $\hat{R}$, the conclusion is applicable for any algebra with commutation relations of the RTT type.

3. *Counit:*

The definition of the counit on generators also coincides with the classical case:

$$\begin{pmatrix} \epsilon(\alpha) & \epsilon(\beta) \\ \epsilon(\gamma) & \epsilon(\delta) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{1.34}$$

In short, $\epsilon(T) = I$, where I is the identity matrix. For the counit to be an algebra homomorphism, we need I to be a quantum matrix. This is true since $I^i_j = \delta^i_j$ and the RTT relation (1.19) is trivially satisfied. (Once again this fact does not depend on the choice of $\hat{R}$.) It follows that one can consistently extend the definition of counit to the whole algebra $\mathcal{A}$ as an algebra homomorphism. It can be checked that (1.12) is also satisfied.

4. *Coinverse:*

The coinverse of the generators is defined so that they form a matrix T^{-1} which satisfies $T^{-1}T = TT^{-1} = I$. It is

$$S(T) = \begin{pmatrix} S(\alpha) & S(\beta) \\ S(\gamma) & S(\delta) \end{pmatrix} = (det_q(T))^{-1}\begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = T^{-1}, \tag{1.35}$$

where $det_q(T) = \alpha\delta - q\beta\gamma$ is called the quantum determinant of T. The quantum determinant is central in $\mathcal{A}$ (it commutes with everything in $\mathcal{A}$) and

$$\Delta(det_q(T)) = det_q(T)\otimes det_q(T). \tag{1.36}$$

It can be shown that T^{-1} is a $GL_{q^{-1}}(2)$ quantum matrix.

This definition of S can be consistently extended to $\mathcal{A}$ such that S is an antihomomorphism (1.14). In addition, (1.13) is satisfied.

This concludes our description of the quantum group $GL_q(2)$ as a Hopf algebra.

Because the quantum determinant is central, it is consistent with the algebra to impose an additional condition

$$det_q(T) = 1. \tag{1.37}$$

This is also compatible with the coproduct since (1.36) implies $\Delta(1) = 1 \otimes 1$, which is always the case as required by (1.9). What we obtain after imposing (1.37) is the deformation of $SL(2)$, naturally named $SL_q(2)$.

A further step can be taken to get $SU_q(2)$. We define the $*$-involution on $SL_q(2)$ for real q by

$$T^\dagger = \begin{pmatrix} \alpha^* & \gamma^* \\ \beta^* & \delta^* \end{pmatrix} = \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix} = T^{-1}. \tag{1.38}$$

It is compatible with the algebra in the sense that the the commutation relations is covariant under the $*$-involution. This is a result of the RTT relation (1.19) and the fact that our $\hat{R}$-matrix is symmetric: $\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}$.

The $*$-involution reverses a product: $(ff')^* = f'^* f^*$ and takes complex conjugation on complex numbers. Therefore it corresponds to the Hermitian conjugation when one realizes the algebra as the algebra of operators on a Hilbert space.

Everything we mentioned in this section can be generalized to $GL_q(N)$, $SL_q(N)$ and $SU_q(N)$ [16]. These and the q-deformation for other classical groups can all be obtained in a systematic way by starting with Poisson-Lie groups, which is briefly described in Appendix 1. For a more detailed discussion see [17].

1.3 Dually Paired Hopf Algebras

Classically the universal enveloping algebra $\mathcal{U}$ is dual to the algebra of functions $\mathcal{A}$ on a group G with respect to the pairing $\langle\cdot,\cdot\rangle : \mathcal{U} \otimes \mathcal{A} \to \boldsymbol{C}$ defined by

$$\langle X, T^i_j \rangle = \pi^i_j(X), \quad X \in \mathcal{U}, \tag{1.39}$$

where π is the representation in which T is represented. (The pairing of the generators can be extended to other elements in $\mathcal{U}$ and $\mathcal{A}$ by using (1.46-1.50) below.)

For example, in the fundamental representation of $SL(2)$ and sl_2, the generators $\{H, X_+, x_-\}$ of $\mathcal{U}$ satisfying $[H, X_\pm] = \pm 2X_\pm$ and $[X_+, X_-] = H$ are represented by

$$\pi(H) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \pi(X_+) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \pi(X_-) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{1.40}$$

and T is represented as a unimodular 2×2 matrix.

Given a basis $\{e_i\}$ for $\mathcal{U}$ one can find its dual basis $\{e^i\}$ for $\mathcal{A}$ such that $\langle e_i, e^j \rangle = \delta^j_i$. As an example, the basis $\{H^l X_+^m X_-^n\}$ for the universal enveloping algebra of sl_2 is dual to the basis $\{(l!m!n!)^{-1}(-\ln\delta)^l(\beta\delta)^m(\gamma\delta^{-1})^n\}$ for the algebra of functions on $SL(2)$, where $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ is an $SL(2)$-matrix. Define the canonical element

$$C = \sum e_i \otimes e^i. \tag{1.41}$$

It can be easily checked that

$$(\pi \otimes id)(C) = T. \tag{1.42}$$

Actually C is the universal expression of T in the sense that it is representation-independent and for any representation π of $\mathcal{U}$ one obtains the representation of the group in terms of the generators of $\mathcal{A}$ by (1.42).

Note that $\mathcal{U}$ is also a Hopf algebra:

$$\Delta(1) = 1 \otimes 1, \quad \Delta(X) = 1 \otimes X + X \otimes 1, \tag{1.43}$$

$$\epsilon(1) = 1, \qquad \epsilon(X) = 0, \tag{1.44}$$

$$S(1) = 1, \qquad S(X) = -X \tag{1.45}$$

for $X = H, X_+, X_-$.

In the quantum case we define a pair of Hopf algebras $(\mathcal{U}, \mathcal{A})$ to be *dual* to each other if there exists a non-degenerate pairing such that

$$\langle XX', f \rangle = \langle X \otimes X', \Delta(f) \rangle, \tag{1.46}$$

$$\langle X, ff' \rangle = \langle \Delta(X), f \otimes f' \rangle, \tag{1.47}$$

$$\langle 1, f \rangle = \epsilon(f), \tag{1.48}$$

$$\langle X, 1 \rangle = \epsilon(X), \tag{1.49}$$

$$\langle S(X), f \rangle = \langle X, S(f) \rangle, \tag{1.50}$$

where we have two sets of maps (Δ, ϵ, S) acting on $X, X' \in \mathcal{U}$ and $f, f' \in \mathcal{A}$, respectively. If they are $*$-Hopf algebras, we require in addition

$$\langle X^*, f \rangle = \langle X, S(f)^* \rangle^*. \tag{1.51}$$

The pairing between $\mathcal{U}$ and $\mathcal{A}$ is defined on generators by (1.39). By requiring that $\mathcal{U}$ and $\mathcal{A}$ are dual one can extend the pairing to the whole algebras $\mathcal{U}$ and $\mathcal{A}$. (Actually the duality of Hopf algebras is so rigid that it can be used to construct the dual of a given Hopf algebra.)

The canonical element [18] can still be defined by (1.41) and (1.42) still holds.

1.3.1 Smash Product

An element in a group acts on the group by left or right multiplication. As a Lie algebra element corresponds to a group element by exponentiation, a Lie algebra element acts on the group as a right or left invariant vector field. The algebra of the vector fields and functions on a group is described in the quantum case by the smash product of $\mathcal{U}$ and $\mathcal{A}$ [19, 20, 21, 22]. The smash product is defined by the original commutation relations in $\mathcal{U}$ and $\mathcal{A}$ together with the additional commutation relations for mixed products:

$$Xf = f_{(1)}\langle X_{(1)}, f_{(2)}\rangle X_{(2)} \tag{1.52}$$

if $\mathcal{U}$ is left-invariant (generating right transformations).

For the group of one-dimensional translations, the coordinate x with $\Delta(x) = 1\otimes x + x\otimes 1$, $\epsilon(x) = 0$, and the Lie algebra generator p with $\Delta(p) = 1\otimes p + p\otimes 1$, $\epsilon(p) = 0$, have the pairing $\langle p, x\rangle = 1$. It is $px = xp + 1$ in the smash product. This enables the identification of p with $\frac{d}{dx}$.

Two kinds of vacua [19, 23] are useful here. Define the vacuum $\rangle_{\mathcal{U}}$ by

$$X\rangle_{\mathcal{U}} = \rangle_{\mathcal{U}}\epsilon(X), \quad X\in\mathcal{U} \tag{1.53}$$

and another vacuum $_{\mathcal{A}}\langle$ by

$$_{\mathcal{A}}\langle f = \epsilon(f)_{\mathcal{A}}\langle, \quad f\in\mathcal{A}. \tag{1.54}$$

It is easy to check that the pairing equals $\langle X, f\rangle = {}_{\mathcal{A}}\langle Xf\rangle_{\mathcal{U}}$. The *action* of $X\in\mathcal{U}$ on $f\in\mathcal{A}$, denoted by $X \triangleright f$, is defined by

$$Xf\rangle_{\mathcal{U}}. \tag{1.55}$$

In our example above, the action of p on x is $(\frac{d}{dx}x) = 1$.

1.4 Quasi-Triangular Hopf Algebras

Sometimes the term "quantum groups" is reserved for "quasi-triangular Hopf algebras". A Hopf algebra is *quasi-triangular* [24] if

$$\sigma\circ\Delta(X) = \mathcal{R}\Delta(X)\mathcal{R}^{-1}, \tag{1.56}$$

where $\sigma(X\otimes Y) = Y\otimes X$, for a certain invertible $\mathcal{R}\in\mathcal{U}\otimes\mathcal{U}$ (called the *universal $\mathcal{R}$-matrix*) satisfying

$$(id\otimes\Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \tag{1.57}$$

$$(\Delta\otimes id)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \tag{1.58}$$

$$(S\otimes id)(\mathcal{R}) = \mathcal{R}^{-1}. \tag{1.59}$$

For a given representation π of $\mathcal{U}$ this R-matrix is given by

$$R^{ij}_{kl} = \langle\mathcal{R}, T^i_k\otimes T^j_l\rangle, \tag{1.60}$$

where T^i_j is the quantum matrix in the same representation.

It follows from (1.56) that the quantum matrix defined in (1.39) satisfies the RTT relation

$$R_{12}T_1T_2 = T_2T_1R_{12}, \tag{1.61}$$

which if written in terms of the $\hat{R}$-matrix $\hat{R}^{ij}_{kl} = R^{ji}_{kl}$ will be (1.19).

The universal $\mathcal{R}$-matrix satisfies the (quantum) Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}. \tag{1.62}$$

The indices 1, 2 and 3 now become the labels for the three copies of $\mathcal{U}$ in the tensor product so that if $\mathcal{R} = r_i\otimes r^i$ then $\mathcal{R}_{12} = r_i\otimes r^i\otimes 1$, $\mathcal{R}_{13} = r_i\otimes 1\otimes r^i$ and $\mathcal{R}_{23} = 1\otimes r_i\otimes r^i$. It follows that the R-matrix in any representation satisfies the same equation.

There is a universal description of the RTT relation (1.61) where T is replaced by C and R is replaced by the universal $\mathcal{R}$-matrix:

$$\mathcal{R}_{12}C_1C_2 = C_2C_1\mathcal{R}_{12}\in\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{A}, \tag{1.63}$$

where $C_1 = e_i\otimes 1\otimes e^i$, $C_2 = 1\otimes e_i\otimes e^i$.

Another form of the RTT relation is

$$\langle\mathcal{R}, f^{(1)}\otimes f'^{(1)}\rangle f^{(2)}f'^{(2)} = f'^{(1)}f^{(1)}\langle\mathcal{R}, f^{(2)}\otimes f'^{(2)}\rangle, \quad f, f'\in\mathcal{A}. \tag{1.64}$$

A more detailed discussion about quasi-triangular Hopf algebras can be found in [25].

1.4.1 An Example: $SL_q(2)$

As an example, the quantum universal enveloping algebra $\mathcal{U}$ for $SL_q(2)$ [24, 26] is generated by H, X_+ and X_- satisfying

$$[H, X_\pm] = \pm 2X_\pm, \tag{1.65}$$

$$[X_+, X_-] = \frac{q^H - q^{-H}}{q - q^{-1}}. \tag{1.66}$$

The fundamental representation for sl_2 (1.40) happens to be also a representation of this algebra although the classical and deformed algebras are apparently different.

$\mathcal{U}$ is a Hopf algebra with the coproduct given by

$$\Delta(1) = 1 \otimes 1, \tag{1.67}$$

$$\Delta(H) = 1 \otimes H + H \otimes 1, \tag{1.68}$$

$$\Delta(X_\pm) = q^{-\frac{1}{2}H} \otimes X_\pm + X_\pm \otimes q^{\frac{1}{2}H}, \tag{1.69}$$

the counit by

$$\epsilon(1) = 1, \tag{1.70}$$

$$\epsilon(H) = \epsilon(X_\pm) = 0 \tag{1.71}$$

and the coinverse by

$$S(1) = 1, \tag{1.72}$$

$$S(H) = -H, \quad S(X_\pm) = -q^{\pm 1} X_\pm. \tag{1.73}$$

In the classical limit $q \to 1$ it is not hard to see that this Hopf algebra structure gives the dual statements of those on the group. While a group element can be written as $g = e^X$ for a Lie algebra element $X = aH + bX_+ + cX_-$, it is up to one's choice whether the group properties (the Hopf algebra structure) is to be stated in terms of a, b, c (functions on the group) or H, X_+, X_-. These two possibilities correspond to the two dually paired Hopf algebras.

By imposing the $*$-involution

$$H^* = H, \quad X_\pm^* = X_\mp \tag{1.74}$$

and $q^* = q$ we obtain the universal enveloping algebra for $SU_q(2)$.

The basis dual to $\{H^l X_+^m X_-^n\}$ is now

$$\{(l![m]_{q^{-1}}![n]_{q^{-1}}!)^{-1}(-\ln\delta)^l(q^{\frac{1}{2}}\beta\delta)^m(q^{-\frac{1}{2}}\gamma\delta^{-1})^n\},$$

where $[m]_{q^{-1}}! = [m]_{q^{-1}}[m-1]_{q^{-1}} \cdots [1]_{q^{-1}}$ and $[m]_{q^{-1}} = \frac{q^{-2m}-1}{q^{-2}-1}$.

We can use (1.41-1.42) to obtain the quantum matrix of $SL_q(2)$ in any representation of $\mathcal{U}$. The $(2j+1)$-dimensional representation of $\mathcal{U}$ for $SL_q(2)$ is well known:

$$\pi_j(H)|j, m\rangle = 2m|j, m\rangle, \tag{1.75}$$

$$\pi_j(X_\pm)|j, m\rangle = ([j \mp m][j \pm m + 1])^{\frac{1}{2}}|j, m \pm 1\rangle, \tag{1.76}$$

where $[n] = \frac{q^n - q^{-n}}{q - q^{-1}}$.

For $SL_q(2)$ the universal R-matrix is

$$\mathcal{R} = \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{\frac{1}{2}(H \otimes H + nH \otimes 1 - n1 \otimes H)} X_+^n \otimes X_-^n, \tag{1.77}$$

where $[n]_q = \frac{q^{2n}-1}{q^2-1}$.

In the fundamental representation (1.40) it is $\langle \mathcal{R}, T_k^i \otimes T_l^j \rangle = q^{-1/2} \hat{R}_{kl}^{ji}$ for the $\hat{R}$-matrix in (1.21).

1.5 Differential Calculus

The differential geometry on a classical manifold can be generalized to the noncommutative case. This is the subject of noncommutative geometry initiated by Connes [2].

1.5.1 Universal Differential Calculus

For any unital, associative algebra $\mathcal{A}$ one can define the exterior derivative $d : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ by [27]

$$df = 1 \otimes f - f \otimes 1. \tag{1.78}$$

The meaning of this expression is the following. When one evaluates df at (x, y), it is $df(x, y) = f(y) - f(x)$, which is the difference between the values of f at the two points x and y. (Note that the tensor product distributes arguments.) If y is very close to x, it coincides with our intuitive notion of the differential df.

The enlarged algebra $\Omega^1_{\mathcal{A}}$ obtained by adjoining the differentials to $\mathcal{A}$ is defined by the left and right multiplications:

$$f df' = f \otimes f' - ff' \otimes 1, \tag{1.79}$$

$$(df)f' = 1 \otimes ff' - f \otimes f'. \tag{1.80}$$

The Leibniz rule

$$d(ff') = (df)f' + f(df') \tag{1.81}$$

follows for $f, f' \in \mathcal{A}$.

Any element in $\Omega^1_{\mathcal{A}}$ can be written in the form $\sum_k f_k df'_k$ for some $f_k, f'_k \in \mathcal{A}$ by using the Leibniz rule. This is the first order universal differential calculus on $\mathcal{A}$. It can be identified with $\mathcal{A}^2 = \{ker(m)\}$, or more explicitly, $\{\sum_i f_i \otimes f'_i \in \mathcal{A} \otimes \mathcal{A} | \sum_i f_i f'_i = 0\}$.

Let

$$d(\sum_i f_i \otimes f'_i) := \sum_i (1 \otimes f_i \otimes f'_i - f_i \otimes 1 \otimes f'_i + f_i \otimes f'_i \otimes 1) \tag{1.82}$$

then the nilpotency of d

$$d^2(f) = 0 \tag{1.83}$$

also follows for $f \in \mathcal{A}$.

One can further extend the definitions of the exterior derivative, the left and right multiplications to differential calculus of all orders such that the Leibniz rule and $d^2 = 0$ hold for differential forms of all degrees. A differential form of degree n will be an element in $\mathcal{A}^{\otimes(n+1)}$.

At a more abstract level the universal differential calculus can be defined simply as a vector space $\Omega(\mathcal{A}) = \{f_0 df_1 df_2 \cdots df_n | f_k \in \mathcal{A}, n \geq 0\}$ [27] equipped with a multiplication defined by juxtaposition modulo the Leibniz rule and the nilpotency of d.

The tensor product representation (1.78-1.80) above is therefore merely a realization of the universal differential calculus.

1.5.2 General Differential Calculus

In this section we will focus on the first order general differential calculus. In general the first order calculus together with the Leibniz rule and $d^2 = 0$ will fix the higher order calculus.

A general differential calculus is defined as the quotient $\Omega(\mathcal{A})/J$ of the universal differential calculus $\Omega(\mathcal{A})$ over an ideal $J \subset \Omega(\mathcal{A})$.

As an example, the classical differential calculus can be identified with $\mathcal{A}^2/J$, where J is the ideal generated by the set $N = \{fdf' - (df')f | f, f' \in \mathcal{A}\}$, i.e., $J = \mathcal{A}N\mathcal{A}$.

Another example is the differential calculus on quantum groups [28, 19, 29, 30, 22, 20]. Given a subset $M \subset ker(\epsilon) \subset \mathcal{A}$ we have a *left-covariant* differential

calculus $\mathcal{A}^2/J$ where [28]

$$J = \{uS(v_{(1)})dv_{(2)}u' | u, u' \in \mathcal{A}, v \in M\}. \tag{1.84}$$

The *left-covariance* [28] of a differential calculus is similar to the left-covariance of the algebra $\mathcal{A}$. Here the left transformation is defined on one-forms by

$$\Delta_L(fdf') = f_{(1)}f'_{(1)} \otimes f_{(2)}df'_{(2)} \tag{1.85}$$

and the left-covariance means

$$\Delta_L(J) \subset \mathcal{A} \otimes J. \tag{1.86}$$

Similarly right-covariance and bicovariance can be defined [28]. An interesting class of bicovariant calculi is considered in [31]. The choice of M is restricted by other requirements like the correct classical limit and so on.

As an example, the left-covariant $3D$ calculus [32] on $SU_q(2)$ is specified by $M = \{\delta + q^2\alpha - (1+q^2), \beta^2, \beta\gamma, \gamma^2, (\alpha-1)\beta, (\alpha-1)\gamma\}$ and J includes elements like $(d\alpha)\delta = q^2\delta(d\alpha)$, etc. It is sometimes more convenient to use the Maurer-Cartan forms

$$T^{-1}dT = \begin{pmatrix} \omega^1 & \omega^0 \\ -q\omega^2 & -q^2\omega^1 \end{pmatrix}, \tag{1.87}$$

where

$$\omega^0 = \delta d\beta - q^{-1}\beta d\delta, \tag{1.88}$$

$$\omega^1 = \delta d\alpha - q^{-1}\beta d\gamma, \tag{1.89}$$

$$\omega^2 = \gamma d\alpha - q^{-1}\alpha d\gamma. \tag{1.90}$$

The commutation relations between functions and forms are

$$\omega^0\alpha = q^{-1}\alpha\omega^0, \quad \omega^0\beta = q\beta\omega^0, \tag{1.91}$$

$$\omega^1\alpha = q^{-2}\alpha\omega^1, \quad \omega^1\beta = q^2\beta\omega^1, \tag{1.92}$$

$$\omega^2\alpha = q^{-1}\alpha\omega^2, \quad \omega^2\beta = q\beta\omega^2 \tag{1.93}$$

and the same formulas with $\alpha \to \gamma$ and $\beta \to \delta$. Since the Maurer-Cartan forms are left-invariant and the commutation relations between functions and forms are the same for $\alpha \to \gamma$ and $\beta \to \delta$, it is easy to see directly that the calculus is left-covariant.

A bicovariant calculus can also be defined for $SU_q(2)$ and is called the $4D$ calculus [28]. However it has four instead of three independent one-forms. This is originated from the fact that although the quantum determinant is central, its differential $d(det_q(T))$ is not central in that calculus.

1.6 Quantum Spaces

Another approach [33] to quantum groups is to derive them as symmetries of quantum spaces. The simplest example is the quantum plane [34]. The coordinates on the quantum plane are x, y with the commutation relation

$$xy = qyx. \tag{1.94}$$

Define the transformation

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \tag{1.95}$$

where $\alpha, \beta, \gamma, \delta$ commute with x, y and their commutation relations are to be determined later. Another way to say it is that we define the left coaction by

$$\Delta_L(x) = \alpha \otimes x + \beta \otimes y, \tag{1.96}$$

$$\Delta_L(y) = \gamma \otimes x + \delta \otimes y. \tag{1.97}$$

Requiring that the algebra of x, y is covariant under the transformation, or equivalently that $\Delta_L(x)\Delta_L(y) = q\Delta_L(y)\Delta_L(x)$, one can derive some commutation relations among $\alpha, \beta, \gamma, \delta$. If we demand in addition the transformation

$$\begin{pmatrix} x & y \end{pmatrix} \to \begin{pmatrix} x & y \end{pmatrix} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{1.98}$$

to be a symmetry of the quantum plane, we arrive at the full set of commutation relations defining a $GL_q(2)$-matrix $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

If we had started with the central extension of the algebra (1.94):

$$xy = qyx + h, \quad h \in \boldsymbol{C}, \tag{1.99}$$

we would have obtained $SL_q(2)$.

In general, for a left coaction $\Delta_L : X \to \mathcal{A} \otimes X$, where X is the algebra of quantum space and $\mathcal{A}$ is the algebra of the quantum group, the coproduct $\Delta : \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ for the quantum group is inferred from the left coaction through

$$(\Delta \otimes id) \circ \Delta_L = (id \otimes \Delta_L) \circ \Delta_L. \tag{1.100}$$

If the algebra $\mathcal{A}$ is determined by the requirement that the coaction Δ_L is an algebra homomorphism, the coproduct for the quantum group defined in this way is guaranteed to be also an algebra homomorphism.

Similarly, since the counit is defined by the assignment of each generator in $\mathcal{A}$ to give the identity transformation and the identity map is always an algebra homomorphism, the counit can always be extended to be an algebra homomorphism. However, there is no general arguments for the existence of the coinverse.

If there is a differential calculus on the quantum space, usually one would require the covariance of the differential calculus under the quantum group transformations as well. The left-coaction of a differential form, for instance, is defined by (1.85).

We always assume the undeformed Leibniz rule for the exterior derivative:

$$d\omega_1\omega_2 = (d\omega_1)\omega_2 \pm \omega_1 d\omega_2, \tag{1.101}$$

where the sign depends on the parity of ω_1. The differential calculus is usually almost fixed by the Leibniz rule, the algebra of functions and the covariance under a transformation. After one obtains the commutation between functions and one-forms, one can use the Leibniz rule and the nilpotency of the exterior derivative

$$d^2 = 0 \tag{1.102}$$

to derive commutation relations between forms.

Furthermore, with the Leibniz rule and the identification $d = \sum_i \omega_i \chi_i$, where $\{\omega_i\}$ is a basis of one-forms and $\{\chi_i\}$ is the dual basis of derivatives, one can derive the commutation relations between functions and derivatives from those between functions and one-forms and vise versa.

In terms of the $\hat{R}$-matrix, (1.94) can be written as

$$x_i x_j = q^{-1} \hat{R}_{ij}^{kl} x_k x_l, \tag{1.103}$$

where $x_1 = x$ and $x_2 = y$. In the classical limit $\hat{R}$ is the permutation matrix and it simply states the commutativity of the algebra. With the RTT relation (1.19) it is easy to see that (1.103) is left-covariant.

The covariant differential calculus is defined by

$$x_i dx_j = q\hat{R}_{ij}^{kl} dx_k x_l. \tag{1.104}$$

It follows from the Leibniz rule and $d^2 = 0$ that

$$dx_i dx_j = -q\hat{R}_{ij}^{kl} dx_k dx_l. \tag{1.105}$$

Eqs.(1.103-1.105) can be directly generalized for the quantum hyperplanes covariant under $GL_q(N)$ [34], $SU_q(N)$ [35] or $SO_q(N)$ [36]. Integrations invariant under coactions can also be defined on quantum groups [15] and quantum spaces [23, 37]. In Chap.2 and Chap.3 we will give interesting examples of quantum spaces with covariant differential calculi and invariant integrals with respect to *nonlinear* quantum group transformations.

Chapter 2

Geometry of the Quantum Sphere S_q^2

2.1 Introduction

Quantum spheres can be defined in any number of dimensions by normalizing a vector of quantum Euclidean space[16]. The differential calculus on quantum Euclidean space[38] induces a calculus on the quantum sphere. The case of two-spheres in three space is special in that there are many more possibilities than the one obtained from the general construction. These have been studied by P. Podleś[39, 40, 41, 42] who has also shown how to define a noncommutative differential calculus on them. In this chapter we study in detail a particular case of Podleś spheres which is one of those special to three space dimensions. In this case the algebra of functions on the sphere is a subalgebra of the algebra of functions on $SU_q(2)$ and the differential calculus on the sphere can be inferred from a differential calculus on $SU_q(2)$. We can also define a stereographic projection [43] and describe the coaction of $SU_q(2)$ on the sphere by fractional transformations on the complex variable in the plane analogous to the classical ones. The quantum sphere appears then as the quantum deformation of the classical two-sphere described as a complex manifold.

All formulas and derivations of the results in this chapter can be easily modified, with a few changes of signs, to describe the quantum unit disk and the coaction of $SU_q(1,1)$ on it, as well as the corresponding invariant anharmonic ratios. This provides a quantum deformation of the Bolyai-Lobachevskiĭ non-Euclidean geometry and of the differential calculus on the Bolyai-Lobachevskiĭ plane. We shall not write here the modified equations appropriate for this case, which can be guessed very easily, but we would like to mention that the commutation relations between the variables z and $\bar{z}$ for the unit disk are appropriate for a representation of z and $\bar{z}$ as bounded operators in a Hilbert space. This is to be contrasted with the case of the quantum sphere where z and $\bar{z}$ must be unbounded operators. In a perfectly analogous way all formulas and derivations for $CP_q(N)$ in the next chapter can be easily modified, with a few changes of sign, to describe a quantum deformation of various higher dimensional non-Euclidean geometries. Again we shall not do this explicitly and leave it as an exercise for the reader. A different deformed algebra of functions on the Bolyai-Lobachevskii plane has been considered in [44].

2.2 The Complex Quantum Manifold S_q^2

In [39] Podleś discovered a family of quantum spheres. They are compact [1] quantum spaces with the quantum symmetry of $SU_q(2)$. That is, the algebra X of functions on the quantum spheres is covariant under an $SU_q(2)$ transformation.

By studying the representations (1.75) and (1.76) of the universal enveloping algebra of $SU_q(2)$ with the coproduct (1.68) and (1.69) as in the classical case, one finds the quantum Clebsch-Gordan coefficients [45] one uses to compose or decompose representations.

A classical sphere can be specified in terms of Cartesian coordinates as $x^2+y^2+z^2=r^2$. The vector $(e_+,e_0,-e_-)=(\frac{1}{\sqrt{2}}(x+iy),z,-\frac{1}{\sqrt{2}}(x-iy))$ transforms as a spin-1 representation under $SU(2)$. In the deformed case we can use the quantum Clebsch-Gordan coefficients to find commutation relations covariant under the linear transformation of the vector $(e_+,e_0,-e_-)=(E_1,E_0,E_{-1})$ as a $j=1$ representation of $SU_q(2)$ [46]. It is

$$\sum_{m'}\begin{bmatrix} 1 & 1 & 1 \\ m' & m-m' & m \end{bmatrix}_q E_{m'}E_{m-m'}=aE_m \tag{2.1}$$

for $a\in \boldsymbol{R}$ and

$$\sum_{m}\begin{bmatrix} 1 & 1 & 0 \\ m & -m & 0 \end{bmatrix}_q E_{m}E_{-m}=-k \tag{2.2}$$

for $k\in \boldsymbol{R}$, $k>0$.

We have in addition to q two free parameters a and k, where k can always be scaled to a fixed number. Only a labels inequivalent quantum spheres. With the $*$-involution $q^*=q$, $e_+^*=e_-$ and $e_0^*=e_0$ ($E_1^*=-E_{-1}$ and $E_0^*=E_0$) it gives a C^*-algebra. It is

$$e_+e_- - e_-e_+ + \lambda e_0^2=\mu e_0, \tag{2.3}$$

$$qe_0e_+ - q^{-1}e_+e_0=\mu e_+, \tag{2.4}$$

$$qe_-e_0 - q^{-1}e_0e_-=\mu e_- \tag{2.5}$$

and

$$e_0^2+qe_-e_+ + q^{-1}e_+e_-=s, \tag{2.6}$$

where $\lambda=q-q^{-1}$, $\mu=[2]^{-1/2}[4]^{1/2}a$ and $s=[3]^{1/2}k>0$ ($[n]=\frac{q^n-q^{-n}}{q-q^{-1}}$).

A particularly interesting case is when this algebra is equivalent to the quotient $SU_q(2)/U(1)$ [47, 48]. The classical $U(1)$ [2] is represented as a subgroup of $SU_q(2)$ by

$$\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \tag{2.7}$$

where $U^*=U^{-1}$. It can be checked that this is an $SU_q(2)$-matrix. $SU_q(2)$ transforms under right multiplication by this matrix as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\to\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\begin{pmatrix} U & 0 \\ 0 & U^{-1} \end{pmatrix}, \tag{2.8}$$

which is again an $SU_q(2)$-matrix by taking U to commute with $\alpha,\beta,\gamma,\delta$.

The algebra of functions X on the quantum sphere $S_q^2=SU_q(2)/U(1)$ is the subalgebra of the algebra of functions on $SU_q(2)$ which is invariant under this $U(1)$ transformation. It is therefore generated by, say, $\alpha\gamma^{-1}$ and $\delta\beta^{-1}$. (The $SU_q(2)$ is three-dimensional and $U(1)$ is one-dimensional, hence the quotient is two-dimensional.) This is also formally equivalent to the algebra of e_+,e_-,e_0 for $\mu=\lambda$ and $s=1$ with the identification $e_0=1+[2]\beta\gamma$, $e_+=q^{-1}[2]^{1/2}\alpha\beta$ and $e_-=-[2]^{1/2}\gamma\delta$. This is the case we will consider in the following. (Strictly speaking, the algebra of functions on $SU_q(2)$ contains only polynomials of the generators $\alpha,\beta,\gamma,\delta$, so $SU_q(2)/U(1)$ is the set of polynomials of $\alpha\beta,\beta\gamma,\gamma\delta$, which is equal to the set of polynomials of e_0,e_+,e_-.)

Let

$$z=\alpha\gamma^{-1} \tag{2.9}$$

and

$$\bar{z}=-\delta\beta^{-1}. \tag{2.10}$$

[1] That is, the algebra of functions is unital.

[2] Since $U(1)$ is one-dimensional there is no commutation relations to deform.

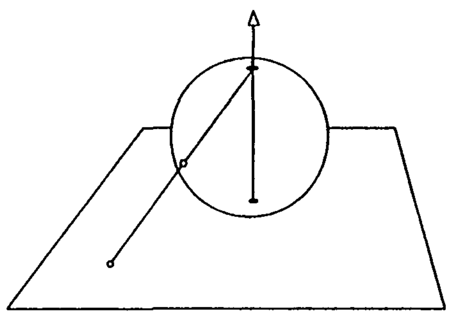

Figure 2.1: stereographic projection

They are the stereographic coordinates. Classically one gets the stereographic coordinates of a point on the sphere by means of a projection from the north pole to the complex plane (Fig.2.1).

Another equivalent description of S_q^2 is $CP_q(1)$, which is obtained from the complex quantum plane with coordinates $\{x, \bar{x}, y, \bar{y}\}$ (see Chap.3) by considering the subalgebra generated by the inhomogeneous coordinates

$$z = xy^{-1}, \quad \bar{z} = \bar{y}^{-1}\bar{x}. \tag{2.11}$$

Since everything about $SU_q(2)$ is well known, we can derive all properties of the sphere from the identification (2.9) and (2.10).

The commutation relation is

$$z\bar{z} = q^{-2}\bar{z}z + q^{-2} - 1, \tag{2.12}$$

or equivalently

$$(1 + z\bar{z}) = q^{-2}(1 + \bar{z}z); \tag{2.13}$$

and the $*$-involution is $z^* = \bar{z}$. Eq.(2.12) differs from the usual quantum plane by an additional inhomogeneous constant term.

The $SU_q(2)$-transformation on $SU_q(2)$ induces rotations on the sphere. In terms of the coordinates $z, \bar{z}$ it is the linear-fractional transformation, abbreviated as the fractional transformation:

$$z \to (az + b)(cz + d)^{-1}, \quad \bar{z} \to -(c - d\bar{z})(a - b\bar{z})^{-1}, \tag{2.14}$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SU_q(2)$ and a, b, c, d commute with z and $\bar{z}$. Eq.(2.12) is covariant under the fractional transformation.

2.3 Patching a Sphere

Classically the stereographic projection maps all points on the sphere except the north pole to the complex plane. The consequence is that the algebra of (smooth) functions on the sphere is a smaller algebra contained in the algebra of (smooth) functions on the plane. Clearly those functions on the plane which do not have a unique finite limit at infinity in all directions are not corresponding to continuous single-valued functions on the sphere. For example, all polynomials of z, $\bar{z}$ are not functions on the sphere.

There are two different ways to cover the whole sphere. One way is to take two complex planes corresponding to the stereographic projections from the north pole and the south pole, respectively. Points in the two patches are identified as the same point on the sphere if their coordinates satisfy, say, $zw = 1$ and $\bar{z}\bar{w} = 1$, where $w, \bar{w}$ are the coordinates on the other patch. While classically a manifold can be understood as a collection of points via patching, the quantum space can only be understood from its algebra of functions. (Sometimes the patching can be described as the intersection of the two algebras on each patch [49], but it obviously does not apply to this case.) The other way is to adjoin the point at infinity to the complex plane and restrict the algebra of functions on the plane to those with a unique finite limit at infinity.

In the deformed case the algebra generated by $(z, \bar{z})$, or any subalgebra of it, does not have the correct classical limit of the algebra of functions on a sphere. However it is known that the C^*-algebra X generated by (e_+, e_0, e_-) does have the correct classical limit and can be taken as the algebra of functions on the quantum sphere [39]. In terms of $(z, \bar{z})$ it is $e_0 = 1 - q[2]\rho^{-1}$, $e_+ = -[2]^{1/2}z\rho^{-1}$ and $e_- = -[2]^{1/2}\rho^{-1}\bar{z}$, where $\rho = 1 + \bar{z}z$. Hence the algebra generated by ρ^{-1}, $z\rho^{-1}$ and $\rho^{-1}\bar{z}$ can also be taken as the algebra X of functions on S_q^2. (We always assume that the algebra is unital, so the unit 1 is always a generator that we do not have to mention explicitly.) It is easy to see that in terms of $w = z^{-1}$ and $\bar{w} = \bar{z}^{-1}$ it is the same algebra generated by ρ_w^{-1}, $w\rho_w^{-1}$ and $\rho_w^{-1}\bar{w}$, where $\rho_w = 1 + \bar{w}w$. It is this algebra that we will consider in the following, e.g., for the integration on S_q^2.

Note that the element ρ^{-1} commutes with $z, \bar{z}$ in a very simple way: $\rho^{-1}z = q^{-2}z\rho^{-1}$ and $\rho^{-1}\bar{z} = q^2\bar{z}\rho^{-1}$. So the appearance of the inverse does not complicate the algebra and one of the benefits of using $(z, \bar{z})$ instead of (e_+, e_0, e_-) is that we need to remember only one simple commutation relation (2.12) instead of four (2.3-2.6).

2.4 Covariant Differential Calculus

The differential calculus [3] for S_q^2 can also be obtained from that for $SU_q(2)$. The $3D$ calculus on $SU_q(2)$ induces a two-dimensional left-covariant calculus on S_q^2, while the $4D$ calculus induces a three-dimensional one. We will consider the former in terms of the stereographic coordinates since it has the right dimension. It is equivalent to the unique left-covariant two-dimensional calculus on S_q^2 obtained by Podleś in [42].

The commutation relations obtained from the $3D$ calculus are

$$zdz = q^{-2}dzz, \quad \bar{z}dz = q^2 dz\bar{z}, \tag{2.15}$$
$$zd\bar{z} = q^{-2}d\bar{z}z, \quad \bar{z}d\bar{z} = q^2 d\bar{z}\bar{z}, \tag{2.16}$$
$$(dz)^2 = (d\bar{z})^2 = 0, \quad dzd\bar{z} = -q^{-2}d\bar{z}dz. \tag{2.17}$$

Derivatives can be defined based on the knowledge about differential forms. When acting on functions, the exterior derivative d is identified with $d = dz\partial + d\bar{z}\bar{\partial}$. The Leibniz rule and $d^2 = 0$ then imply

$$\partial z = 1 + q^{-2}z\partial, \quad \partial\bar{z} = q^2\bar{z}\partial, \tag{2.18}$$
$$\bar{\partial}z = q^{-2}z\bar{\partial}, \quad \bar{\partial}\bar{z} = 1 + q^2\bar{z}\bar{\partial}, \tag{2.19}$$
$$\partial\bar{\partial} = q^{-2}\bar{\partial}\partial. \tag{2.20}$$

When acting on functions, the derivatives $\partial, \bar{\partial}$ can be realized by $q\lambda^{-1}\rho^{-1}\bar{z}$ and $-q\lambda^{-1}\rho^{-1}z$, respectively. (It can be checked that ∂ ($\bar{\partial}$) has the same commutation relations with $z, \bar{z}$ as $q\lambda^{-1}\rho^{-1}\bar{z}$ ($-q\lambda^{-1}\rho^{-1}z$) does.) It turns out that the exterior derivative can be realized by

$$\Xi = q\rho^{-1}(dz\bar{z} - d\bar{z}z), \tag{2.21}$$

as

$$(d\omega) = \lambda^{-1}[\Xi, \omega]_{\pm}, \tag{2.22}$$

where $[\cdot,\cdot]_{\pm}$ is the commutator $(-)$ or anticommutator $(+)$, for even or odd differential forms ω, respectively. (Functions are considered as even forms.)

Eqs.(2.15-2.20) are covariant under the fractional transformation (2.14), which implies

$$dz \to dz(q^{-1}cz + d)^{-1}(cz + d)^{-1}, \tag{2.23}$$
$$\partial \to (cz + d)(q^{-1}cz + d)\partial, \tag{2.24}$$

[3] Podleś studied and classified all left-covariant and bicovariant differential calculi on quantum spheres in [40, 50, 42].

where the second line is obtained by requiring the exterior derivative to be invariant.

The $*$-involution is defined by

$$(dz)^* = d\bar{z}, \tag{2.25}$$
$$\partial^* = -q^{-2}\bar{\partial} + (1 + q^{-2})z\rho^{-1}, \tag{2.26}$$
$$\bar{\partial}^* = -q^2\partial + (1 + q^2)\rho^{-1}\bar{z}, \tag{2.27}$$

where

$$\rho = 1 + \bar{z}z. \tag{2.28}$$

The inhomogeneous pieces on the right hand side of Eqs.(2.26) and (2.27) originate from the nontrivial measure for the integration on the sphere, in contrast with another possible involution which corresponds to a flat complex plane:

$$(dz)^* = d\bar{z}, \tag{2.29}$$
$$\partial^* = -q^2\bar{\partial}, \tag{2.30}$$
$$\bar{\partial}^* = -q^{-2}\partial. \tag{2.31}$$

The latter involution is not covariant under the fractional transformations, but covariant under another transformation:

$$z \to az + m, \tag{2.32}$$
$$\bar{z} \to \bar{a}\bar{z} + \bar{m}, \tag{2.33}$$

where $a, \bar{a}, m, \bar{m}$ commute with $z, \bar{z}$ and

$$a\bar{a} = \bar{a}a, \quad qm\bar{m} - q^{-1}\bar{m}m = \lambda(a\bar{a} - 1), \tag{2.34}$$
$$ma = q^{-2}am, \quad m\bar{a} = q^{-2}\bar{a}m, \tag{2.35}$$
$$\bar{m}a = q^2 a\bar{m}, \quad \bar{m}\bar{a} = q^2\bar{a}\bar{m}. \tag{2.36}$$

The $*$-involution on a, m coincides with the bars. A Hopf algebra structure on the algebra of $a, \bar{a}, m, \bar{m}$ can be easily derived. Its classical limit is the algebra of functions on the group of rotations, scaling and translations on the complex plane. The induced transformations on $dz, d\bar{z}$ and $\partial, \bar{\partial}$ can be derived and the same calculus is also covariant under this transformation.

Back to the quantum sphere. Since we have Eqs.(2.15-2.28) we can forget about how we got them and take them as our definition of the differential calculus on S_q^2. The covariance of the calculus can be checked directly.

There is a symmetry of the calculus

$$z \leftrightarrow \bar{z}, \quad dz \leftrightarrow d\bar{z}, \tag{2.37}$$

$$\partial \leftrightarrow \bar{\partial}, \quad q \leftrightarrow q^{-1}, \tag{2.38}$$

which is induced from a symmetry on S_q^2:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \to \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix}, \quad q \to q^{-1}. \tag{2.39}$$

Now we discuss the complex structure on the differential calculus. The existence of the complex structure relies on the property of our differential calculus that the holomorphic and antiholomorphic parts of the functions and forms are not mixed by the commutation relations. This implies that we can define δ and $\bar{\delta}$ by $d = \delta + \bar{\delta}$ as the decomposition of the exterior derivatives to the holomorphic and antiholomorphic parts. Explicitly,

$$[\delta, z] = dz, \quad [\delta, \bar{z}] = 0, \tag{2.40}$$

$$\left[\bar{\delta}, z\right] = 0, \quad \left[\bar{\delta}, \bar{z}\right] = d\bar{z}, \tag{2.41}$$

$$\delta dz = dz\delta = 0, \quad \bar{\delta} d\bar{z} = d\bar{z}\bar{\delta} = 0, \tag{2.42}$$

$$\{\delta, d\bar{z}\} = 0, \quad \{\bar{\delta}, dz\} = 0, \tag{2.43}$$

$$\delta^2 = \bar{\delta}^2 = 0, \tag{2.44}$$

$$\{\delta, \bar{\delta}\} = 0, \tag{2.45}$$

where $\{\cdot,\cdot\}$, $[\cdot,\cdot]$ are the anticommutator and commutator respectively.

Like the total exterior derivative d, the differentials δ and $\bar{\delta}$ can be realized by $\xi = qdz\rho^{-1}\bar{z}$ and $-\xi^* = -qd\bar{z}\rho^{-1}z$, respectively, as in (2.22). While $\xi^2 = 0 = \xi^{*2}$, it is

$$\Xi^2 = q\lambda d\bar{z}\rho^{-2}dz, \tag{2.46}$$

which is central in the calculus so that $d^2 = 0 = \delta^2 = \bar{\delta}^2$ and $\delta\bar{\delta} + \bar{\delta}\delta = 0$ are correctly realized.

It is interesting that there exist in the calculus elements realizing the differentials d, δ and $\bar{\delta}$, in contrast to Woronowicz' work [32, 28], where an additional one-form $\mathcal{X}$ satisfying $\mathcal{X}^2 = 0$ has to be adjoined to the calculus to realize the differential d. This is also a slight generalization to Connes' formulation where the differential d is realized by an operator F with $F^2 = 1$. Another interesting fact is that Ξ^2 is also invariant and so can be thought of as the Kähler form or the measure on S_q^2.

The coincidence of the close relation between the differential and the Kähler form shows that our quantization is closely attached to the geometry. This interesting story about Ξ is generalized to $CP_q(N)$ in the next chapter.

The action of the universal enveloping algebra on $SU_q(2)$ induces its action on S_q^2. It is more convenient for S_q^2 to use the generators

$$\mathcal{Z}_+ = X_+ q^{H/2}, \tag{2.47}$$

$$\mathcal{Z}_- = q^{H/2} X_- \tag{2.48}$$

and

$$\mathcal{H} = [H]_q = \tfrac{q^{2H}-1}{q^2-1}, \tag{2.49}$$

with

$$\mathcal{Z}_+{}^* = \mathcal{Z}_-, \quad \mathcal{H}^* = \mathcal{H}. \tag{2.50}$$

Their commutation relations are the following:

$$\mathcal{H}\mathcal{Z}_+ - q^4 \mathcal{Z}_+ \mathcal{H} = (1 + q^2)\mathcal{Z}_+, \tag{2.51}$$

$$\mathcal{Z}_- \mathcal{H} - q^4 \mathcal{H}\mathcal{Z}_- = (1 + q^2)\mathcal{Z}_-, \tag{2.52}$$

$$q\mathcal{Z}_+\mathcal{Z}_- - q^{-1}\mathcal{Z}_-\mathcal{Z}_+ = \mathcal{H}. \tag{2.53}$$

They are viewed as right-invariant vector fields on the sphere and they generate infinitesimal fractional transformations (rotations) on the sphere. The actions are given by

$$\mathcal{Z}_+ z = q^2 z \mathcal{Z}_+ + q^{1/2} z^2, \tag{2.54}$$

$$\mathcal{Z}_+ \bar{z} = q^{-2} \bar{z} \mathcal{Z}_+ + q^{-3/2}, \tag{2.55}$$

$$\mathcal{H} z = q^4 z \mathcal{H} + (1 + q^2) z, \tag{2.56}$$

$$\mathcal{H} \bar{z} = q^{-4} \bar{z} \mathcal{H} - q^{-4}(1 + q^2)\bar{z}, \tag{2.57}$$

$$\mathcal{Z}_- z = q^2 z \mathcal{Z}_- - q^{1/2}, \tag{2.58}$$

$$\mathcal{Z}_- \bar{z} = q^{-2} \bar{z} \mathcal{Z}_- - q^{-3/2} \bar{z}^2. \tag{2.59}$$

With the generators viewed as Lie derivatives, their actions can be extended to forms and derivatives by assuming that they commute with the exterior derivative d.

2.5 The Poisson Sphere

The commutation relations of the previous sections give us, in the limit $q \to 1$, a Poisson structure on the sphere. The Poisson Brackets (P.B.s) are obtained as usual as a limit

$$(f,g) = \lim_{h \to 0} \frac{fg \mp gf}{h}, \quad q^2 = e^h = 1 + h + [h^2]. \tag{2.60}$$

where we use + for f, g both odd and − otherwise. For instance, the commutation relation (2.12) gives

$$z\bar{z} = (1-h)\bar{z}z - h + [h^2] \tag{2.61}$$

and therefore [51]

$$(\bar{z}, z) = \rho. \tag{2.62}$$

Similarly one finds

$$(dz, z) = zdz, \quad (d\bar{z}, z) = zd\bar{z}, \tag{2.63}$$

$$(dz, \bar{z}) = -\bar{z}dz, \quad (d\bar{z}, \bar{z}) = -\bar{z}d\bar{z} \tag{2.64}$$

and

$$(d\bar{z}, dz) = d\bar{z}dz. \tag{2.65}$$

In this classical limit functions and forms commute or anticommute according to their even or odd parity, as usual. The P.B. of any quantity with itself vanishes. The P.B. of two even quantities or of an even and an odd quantity is antisymmetric, that of two odd quantities is symmetric. It is

$$d(f,g) = (df, g) \pm (f, dg), \tag{2.66}$$

where the plus (minus) sign applies for even (odd) f. Notice that we have enlarged the concept of Poisson bracket to include differential forms. This is very natural when considering the classical limit of our commutation relations.

In the classical limit, Eq.(2.22) becomes

$$(\Xi, f) = df, \tag{2.67}$$

where

$$\Xi = \xi - \xi^* \tag{2.68}$$

and

$$\xi = dz\bar{z}\rho^{-1}, \quad \xi^* = d\bar{z}z\rho^{-1} \tag{2.69}$$

are ordinary classical differential forms. Now

$$d\Xi = 2d\bar{z}dz\rho^{-2} \tag{2.70}$$

and

$$\Xi^2 = 0. \tag{2.71}$$

As before, the variables z and $\bar{z}$ cover the sphere except for the north pole, while $w = 1/z$ and $\bar{w} = 1/\bar{z}$ miss the south pole. It is

$$(\bar{w}, w) = \bar{w}w(1 + \bar{w}w). \tag{2.72}$$

The Poisson structure is not symmetric between the north and south pole [4]. All P.B.s of regular functions and forms vanish at the north pole $w = \bar{w} = 0$. Therefore, for Eq.(2.67) to be valid, the one-form Ξ must be singular at the north pole. Indeed one finds

$$\xi = \frac{dw\bar{w}}{1+\bar{w}w} - \frac{dw}{w}, \quad \xi^* = \frac{d\bar{w}w}{1+\bar{w}w} - \frac{d\bar{w}}{\bar{w}} \tag{2.73}$$

and

$$\Xi = \frac{wd\bar{w} - \bar{w}dw}{\bar{w}w(1+\bar{w}w)}. \tag{2.74}$$

On the other hand the area two-form

$$d\Xi = 2\frac{d\bar{w}dw}{(1+\bar{w}w)^2} \equiv \Omega \tag{2.75}$$

is regular everywhere on the sphere.

The singularity of Ξ at the north pole is not a real problem if we treat it in the sense of the theory of distributions. Consider a circle C of radius r encircling the origin of the w plane in a counter-clockwise direction and set

$$w = re^{i\theta}, \quad \bar{w} = re^{-i\theta}. \tag{2.76}$$

Using (2.73), we have

$$\int \Xi = \int \frac{\bar{w}dw - wd\bar{w}}{1+\bar{w}w} - 4\pi i. \tag{2.77}$$

As $r \to 0$ the integral in the right hand side tends to zero because the integrand is regular at the origin. The Stokes theorem can be satisfied even at the origin if we modify Eq.(2.75) to read

$$d\Xi = \Omega - 4\pi i\delta(w)\delta(\bar{w})d\bar{w}dw. \tag{2.78}$$

[4] A different deformation of the two-sphere with the symmetry $z \leftrightarrow w = 1/z$ is described in [52]. But its covariant differential calculus is unknown.

It is

$$\int_{S^2} \Omega = 4\pi i \tag{2.79}$$

so that

$$\int_{S^2} d\Xi = 0 \tag{2.80}$$

as it should be for a compact manifold without boundary. Notice that the additional delta function term in (2.78) also has zero P.B.s with all functions and forms as required by consistency.

2.6 Braided Quantum 2-Sphere

In this section we consider the braiding of several copies of S_q^2 [53]. There exists a general formulation [54] for obtaining the braiding of quantum spaces in terms of the universal $\mathcal{R}$-matrix of the quantum group which coacts on the quantum space. Using this formulation the braiding commutation relations are obtained directly for the variables z and $\bar{z}$. (An alternative derivation of the same braiding relations proceeds by first computing the braiding of two copies of the complex quantum plane on which $SU_q(2)$ coacts and then using the expressions of the stereographic variables z and $\bar{z}$ in terms of the coordinates x, y of the quantum plane

$$z = xy^{-1}, \quad \bar{z} = \bar{y}^{-1}\bar{x}. \tag{2.81}$$

Please see Chap.3 for the generalization of this approach to the braiding of $CP_q(N)$.)

The braiding can be extended to the differentials dz and $d\bar{z}$. In Sec.2.7 a braiding property of the $SU_q(2)$ invariant integral on the sphere is given. It is shown that it can be used to compute the integral.

The braiding of two quantum spheres is not symmetric with respect to the exchange of the two spheres. It can be extended to the case of an arbitrary number of spheres given in a certain order.

Classically, a function of k points on a manifold $\mathcal{M}$ is an element in $X^{\otimes k}$ where X is the algebra of functions on the manifold. In the quantum case the analogous object will be a function of $\{x_i^{(a)}\}$, where $i = 1, \cdots, dim(\mathcal{M})$, $a = 1, \cdots, k$ and $x_i^{(a)}$ is the i-th component of the coordinate for the a-th point on the manifold. Because all k points are on the same manifold, the coordinates for all points are transformed simultaneously. For our case of the quantum sphere, this means that all $z^{(a)}, \bar{z}^{(a)}$ are transformed by the same $SU_q(2)$-matrix through (2.14). However, for this transformation to be really a symmetry of the system of k points on the

same sphere, we require that the algebra of $\{z^{(a)}, \bar{z}^{(a)}\}_{a=1}^k$ be covariant under this transformation. This requirement will not be satisfied if one takes the coordinates $z^{(a)}, \bar{z}^{(a)}$ to commute with $z^{(b)}, \bar{z}^{(b)}$ for $a \neq b$ (which is equivalent to saying that a function of k points is an element in $X^{\otimes k}$) because the algebra of $SU_q(2)$ is noncommutative. The correct (covariant) choice is the braided algebra which we are going to show below.

For the case of four braided spheres one can construct a quantum analogue of the classical anharmonic ratio (cross ratio) of four points on a sphere. This quantity, which belongs to the braided algebra of the four spheres, is invariant under the coaction of $SU_q(2)$ as realized by the quantum fractional transformations on the stereographic variables. It commutes with its $*$-conjugate. This is described in Sec.2.8. The existence of the invariant quantum anharmonic ratio seems very remarkable.

Classically, anharmonic ratios are the building blocks of all projective invariants. Here, they open the way to a development of quantum projective geometry (see Chap.4) which may be relevant for the study of nonlinear field theory σ-models.

Let us now look at the general rule of braiding [54] for two ordered (possibly different) quantum spaces X_1, X_2 with (possibly different) left coactions

$$\Delta_L^i : X_i \rightarrow \mathcal{A} \otimes X_i, \quad i = 1, 2, \tag{2.82}$$

defined for the same quantum group $\mathcal{A}$. For $v \in X_1$ and $w \in X_2$ we define the commutation relation in the braiding of X_1 and X_2 by [54]

$$vw = \mathcal{R}(w^{(1')}, v^{(1')}) w^{(2)} v^{(2)}, \tag{2.83}$$

where we use Sweedler's notation

$$\Delta_L^i(f) = f^{(1')} \otimes f^{(2)}. \tag{2.84}$$

(The prime on the upper index is used to indicate that the attached element is not in X_i but in $\mathcal{A}$.) Here $\mathcal{R} \in \mathcal{U} \otimes \mathcal{U}$ is the universal $\mathcal{R}$-matrix for the quantum universal enveloping algebra $\mathcal{U}$ dual to $\mathcal{A}$ with respect to a pairing $\langle \cdot, \cdot \rangle$ and

$$\mathcal{R}(a, b) = \langle \mathcal{R}, a \otimes b \rangle, \tag{2.85}$$

where $\langle X \otimes Y, a \otimes b \rangle = \langle X, a \rangle \langle Y, b \rangle$.

The same formula (2.83) can be used for any number of ordered quantum spaces as long as for each space there exists a left-coaction of $\mathcal{A}$.

The covariance of (2.83) follows (1.64) and (1.100). To check the consistency of (2.83) by braiding, i.e., by permuting three elements into a certain order via two different procedures, we have to use

$$\mathcal{R}(fg,h) = \mathcal{R}(f,h_{(1)})\mathcal{R}(g,h_{(2)}), \tag{2.86}$$

$$\mathcal{R}(f,gh) = \mathcal{R}(f_{(1)},h)\mathcal{R}(f_{(2)},g) \tag{2.87}$$

and

$$\mathcal{R}(1,f) = \mathcal{R}(f,1) = \epsilon(f). \tag{2.88}$$

These are consequences of (1.57-1.58) and general properties of the pairing.

One can extend $SU_q(2)$ by introducing a^{-1}, d^{-1} satisfying

$$aa^{-1} = a^{-1}a = 1, \quad dd^{-1} = d^{-1}d = 1,$$
$$\epsilon(a^{-1}) = \epsilon(d^{-1}) = 1,$$
$$a^{-1*} = d^{-1}, \quad d^{-1*} = a^{-1}$$

and

$$\begin{aligned}\Delta(d^{-1}) &= (c\otimes b + d\otimes d)^{-1}\\ &= \sum_{n=0}^{\infty}(-1)^n(d^{-1}c)^n d^{-1}\otimes(d^{-1}b)^n d^{-1}.\end{aligned}$$

The transformation for z and the braided copy z' can then be written as

$$z \to \sum_{n=0}^{n=\infty} f_n z^n, \quad z' \to \sum_{n=0}^{n=\infty} f_n z'^n,$$

where

$$\begin{aligned}f_0 &= bd^{-1},\\ f_n &= (-1)^{n-1}d^{-2}(cd^{-1})^{n-1}, \quad n \geq 1.\end{aligned}$$

Consider the algebra $\mathcal{A}$ generated by $\{1,z\}$ and its braided copy $\mathcal{A}'$ generated by $\{1,z'\}$. Eq.(2.83) gives

$$zz' = \sum_{n,m=0}^{\infty}\mathcal{R}(f_m,f_n)z'^m z^n. \tag{2.89}$$

To calculate $\mathcal{R}(f_m,f_n)$, we notice that, for example, (2.86-2.88) give

$$\mathcal{R}(a,a_{(1)})\mathcal{R}(a^{-1},a_{(2)}) = \mathcal{R}(a,a)\mathcal{R}(a^{-1},a) = 1$$

and so

$$\mathcal{R}(a^{-1},T) = \begin{pmatrix} q^{-1/2} & 0\\ 0 & q^{1/2}\end{pmatrix}, \quad \mathcal{R}(T,a^{-1}) = \begin{pmatrix} q^{-1/2} & 0\\ 0 & q^{1/2}\end{pmatrix},$$

$$\mathcal{R}(d^{-1},T) = \begin{pmatrix} q^{1/2} & 0\\ 0 & q^{-1/2}\end{pmatrix}, \quad \mathcal{R}(T,d^{-1}) = \begin{pmatrix} q^{1/2} & 0\\ 0 & q^{-1/2}\end{pmatrix}.$$

It is not hard to prove that for any functions f, g of $a^{\pm1}, b, c, d^{\pm1}$,

$$\mathcal{R}(bf,g) = 0, \quad \mathcal{R}(f,cg) = 0,$$
$$\mathcal{R}(cf,g) = 0, \quad \text{if } g \text{ has no } b,$$
$$\mathcal{R}(g,bf) = 0, \quad \text{if } g \text{ has no } c$$

and

$$\mathcal{R}(d^{\pm1}, f(a,b,c,d)) = f(q^{\mp1/2},0,0,q^{\pm1/2}), \tag{2.90}$$

$$\mathcal{R}(f(a,b,c,d),d^{\pm1}) = f(q^{\mp1/2},0,0,q^{\pm1/2}) \tag{2.91}$$

together with (2.90),(2.91) with $d^{\pm1}$ replaced by $a^{\mp1}$.

One then gets

$$\begin{aligned}\mathcal{R}(f_1,f_1) &= q^2,\\ \mathcal{R}(f_2,f_0) &= -\lambda q,\\ \mathcal{R}(f_m,f_n) &= 0, \quad \text{all other } n,m.\end{aligned}$$

Therefore

$$zz' = q^2 z'z - \lambda q z'^2. \tag{2.92}$$

Similarly, we consider the braiding between $\mathcal{A} = \langle\{1,z\}\rangle$ and $\bar{\mathcal{A}} = \langle\{1,\bar{z}\}\rangle$. It is

$$\bar{z} \to \sum_{n=0}^{\infty} g_n \bar{z}^n,$$

$$\begin{aligned}g_0 &= -ca^{-1},\\ g_n &= q^{-2(n-1)}(ba^{-1})^{n-1}a^{-2}, \quad n \geq 1\end{aligned}$$

and

$$\begin{aligned}\mathcal{R}(g_1,f_1) &= q^{-2},\\ \mathcal{R}(g_0,f_0) &= -\lambda q^{-1},\\ \mathcal{R}(g_m,f_n) &= 0, \quad \text{all other } n,m.\end{aligned}$$

Therefore
$$z\bar{z} = \sum_{n,m=0}^{\infty} \mathcal{R}(g_m, f_n)\bar{z}^m z^n \tag{2.93}$$
gives
$$z\bar{z} = q^{-2}\bar{z}z - \lambda q^{-1}. \tag{2.94}$$
Hence we have re-discovered the commutation relation between z and $\bar{z}$ for S_q^2 as a braiding relation.

Since $\bar{z}'$ transforms like $\bar{z}$ it follows immediately that the braiding between $\mathcal{A}$ and $\bar{\mathcal{A}}'$ is given by
$$z\bar{z}' = q^{-2}\bar{z}'z - \lambda q^{-1}. \tag{2.95}$$

For consistency with the $*$-involution of the braided algebra the braiding order of $z, \bar{z}, z'$ and $\bar{z}'$ has to be $z < z' < \bar{z}' < \bar{z}$ after we have fixed $z < z'$ and $z < \bar{z}$ as in (2.89) and (2.93). It is crucial that we braid separately $\mathcal{A} = \langle\{1, z\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$, and $\bar{\mathcal{A}} = \langle\{1, \bar{z}\}\rangle$ with $\mathcal{A}'$ and $\bar{\mathcal{A}}'$ instead of simply braiding the whole algebra $\langle\{1, z, \bar{z}\}\rangle$ with $\langle\{1, z', \bar{z}'\}\rangle$. Otherwise we will not be able to have the usual properties of the $*$-involution (e.g.$(f(z)g(z'))^* = g(z')^* f(z)^*$) for the braiding relations.

The differential calculus can also be defined on the braided spheres by imposing the Leibniz rule on the exterior derivatives d and d' so that d' acts on z' and $\bar{z}'$ in the same way d acts on z and $\bar{z}$, and
$$dz' = z'd, \quad d\bar{z}' = \bar{z}'d,$$
$$d'z = zd', \quad d'\bar{z} = \bar{z}d'$$
together with
$$dd' = -d'd.$$
Then (2.92),(2.95) and their $*$-involution will imply commutation relations between functions and forms of different copies of the sphere. As a consequence, the area element of the second copy $K' = dz'd\bar{z}'(1 + \bar{z}'z')^{-2}$ is central in the whole braided algebra, while $K = dzd\bar{z}(1 + \bar{z}z)^{-2}$ is only central in the original copy $(z, \bar{z})$.

2.7 Integration

The integration $\langle\cdot\rangle : X \to \boldsymbol{C}$ on the quantum sphere is defined by the requirement that it is invariant under the $SU_q(2)$-transformation
$$\Delta_L(z) = (a \otimes z + b \otimes 1)(c \otimes z + d \otimes 1)^{-1}, \tag{2.96}$$

$$\Delta_L(\bar{z}) = -(c \otimes 1 - d \otimes \bar{z})(a \otimes 1 - b \otimes \bar{z})^{-1}. \tag{2.97}$$
namely,
$$f^{(1')}\langle f^{(2)}\rangle_{S_q^2} = 1\langle f\rangle_{S_q^2} \tag{2.98}$$
for $\Delta_L(f) = f^{(1')} \otimes f^{(2)}$ as in Sweedler's notation. This is equivalent to
$$\langle \mathcal{O}f(z, \bar{z})\rangle_{S_q^2} = \epsilon(\mathcal{O})\langle f(z, \bar{z})\rangle_{S_q^2}, \quad \mathcal{O} \in \mathcal{U} \tag{2.99}$$
or
$$\langle \mathcal{O}f(z, \bar{z})\rangle_{S_q^2} = 0, \quad \mathcal{O} = \mathcal{Z}_+, \mathcal{Z}_-, \mathcal{H}. \tag{2.100}$$

The invariance of the integration is sufficient to fix itself up to a normalization, which can be taken as $\langle 1\rangle_{S_q^2} = 1$. Like everything else, the integration on S_q^2 can be induced from the integration on $SU_q(2)$ as
$$\langle f(z, \bar{z})\rangle_{S_q^2} = \langle f(\alpha\gamma^{-1}, -\delta\beta^{-1})\rangle_{SU_q(2)}. \tag{2.101}$$
The invariance of $\langle\cdot\rangle_{SU_q(2)}$ implies the invariance of $\langle\cdot\rangle_{S_q^2}$ under the $SU_q(2)$ transformations.

It is instructive to see other ways to derive the integration on S_q^2. One way to do it is to consider integrable functions f, for example ρ^{-l} for a positive integer l, in (2.100), which will then give recursive relations for $\langle\rho^{-n}\rangle_{S_q^2}$. It can easily be solved
$$\langle\rho^{-l}\rangle_{S_q^2} = \frac{1}{[l+1]_q}\langle 1\rangle_{S_q^2}, \quad l \geq 0. \tag{2.102}$$

Another way is to use the cyclic property [5] which follows (2.98) and properties of the quantum group. Without going into details we give the cyclic property:
$$< f(z, \bar{z})g(z, \bar{z}) >=< g(z, \bar{z})f(q^{-2}z, q^2\bar{z}) > . \tag{2.103}$$

It is also possible to look at the integration as the integration of forms $\int : \Omega^2(X) \to \boldsymbol{C}$, where $\Omega^2(X)$ is the second order differential calculus on X. Then the requirement is the Stokes theorem
$$\int d\omega = 0. \tag{2.104}$$

[5] Similar cyclic properties have been found by H. Steinacker[55] for integrals over higher dimensional quantum spheres in quantum Euclidean spaces.

This condition is manifestly invariant under the transformation and it is also strong enough to determine this integration [6] up to normalization. The relation between these two integrations is given by the measure μ

$$\int \mu f = \langle f \rangle_{S_q^2}, \quad f \in X, \tag{2.105}$$

up to normalizations. If the measure is invariant, the integration $\langle\cdot\rangle_{S_q^2}$ obtained from $\int$ in this way will also be invariant. In our case the measure is proportional to Ξ^2, or to $d\bar{z}\rho^{-2}dz$, which is invariant.

With a different measure one obtains from $\int$ an invariant integration on another space. For the same algebra X with the $*$-involution given by (2.29-2.31), the integration on X should be that for a flat quantum plane:

$$\langle f \rangle_{C^2} = \int \frac{dz d\bar{z}}{2\pi i} f, \quad f \in X. \tag{2.106}$$

The Stokes theorem implies that it is invariant under translation:

$$\langle \partial f \rangle_{C^2} = \langle \bar{\partial} f \rangle_{C^2} = 0 \tag{2.107}$$

for integrable functions f.

Therefore the integration on the sphere and that on the plane is related to each other by

$$\langle f \rangle_{S_q^2} = \langle \rho^{-2} f \rangle_{C^2} \tag{2.108}$$

up to normalizations.

It is interesting that there exists another way, which has no classical analogue, to determine the integration. Consider braided copies of X. Because z' and $\bar{z}'$ are always on the same side of the variables z and $\bar{z}$ in the braiding order ($z < z' < \bar{z}' < \bar{z}$), the integration on $z', \bar{z}'$, has the following property: if

$$f(z', \bar{z}') g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) f_i(z', \bar{z}'), \tag{2.109}$$

then

$$\langle f(z', \bar{z}') \rangle_{S_q^2} g(z, \bar{z}) = \sum_i g_i(z, \bar{z}) \langle f_i(z', \bar{z}') \rangle_{S_q^2}. \tag{2.110}$$

This can be shown using (2.83) and the invariance of the integral under the $SU_q(2)$ coaction. However,

$$f(z', \bar{z}') \langle g(z, \bar{z}) \rangle_{S_q^2} \neq \sum_i \langle g_i(z, \bar{z}) \rangle_{S_q^2} f_i(z', \bar{z}'). \tag{2.111}$$

[6] One has to fix the set of integrable functions in advance.

The above property (2.110) can be used to derive explicit integral rules. For example, consider the case of $f(z', \bar{z}') = \bar{z}'\rho'^{-n}$, where $\rho' = 1 + \bar{z}'z'$ and $g(z, \bar{z}) = z$. Since

$$\bar{z}'\rho'^{-n} z = q^2 z \bar{z}' \rho'^{-n} + q^{1-2n}\lambda([n+1]_q - [n]_q \rho')\rho'^{-n}, \quad n \geq 0,$$

where $[n]_q = \frac{q^{2n}-1}{q^2-1}$, using (2.110) and $< \bar{z}'\rho'^{-n} >= 0$ we get the recursion relation:

$$[n+1]_q < \rho'^{-n} >= [n]_q < \rho'^{-(n-1)} >, \quad n \geq 1, \tag{2.112}$$

which leads to (2.102).

2.8 Anharmonic Ratios

Let us first review the classical case. The coordinates x, y on a plane transform as

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{2.113}$$

by an $SU(2)$ matrix $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The determinant-like object $xy' - yx'$ defined for x, y together with the coordinates on a second plane x', y' is invariant under the $SU(2)$ transformation. On each plane we define $z = x/y$ so that

$$z - z' = y^{-1}(xy' - yx')y'^{-1}.$$

It now follows that with x_i, y_i for $i = 1, 2, 3, 4$ as coordinates on four copies of the planes,

$$\begin{aligned} &(z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1} \\ = \ &(x_1 y_2 - y_1 x_2)(x_4 y_2 - y_4 x_2)^{-1}(x_4 y_3 - y_4 x_3)(x_1 y_3 - y_1 x_3)^{-1} \end{aligned}$$

is invariant because all the factors y_i^{-1} cancel and only the invariant parts $(x_i y_j - y_i x_j)$ survive. Therefore the anharmonic ratio is invariant.

Permuting the indices in the above expression we may get other anharmonic ratios, but they are all functions of the one above. For example,

$$(z_2 - z_3)(z_2 - z_4)^{-1}(z_1 - z_4)(z_3 - z_1)^{-1} = (z_2 - z_1)(z_2 - z_4)^{-1}(z_3 - z_4)(z_3 - z_1)^{-1} - 1.$$

The coordinates of the $SU_q(2)$ covariant quantum plane obey

$$xy = qyx,$$

an equation covariant under the transformation (2.113) with T now being an $SU_q(2)$ matrix. As explained in Sec.2.6, braided quantum planes can be introduced by using (2.83). Let $\mathcal{V}$ be the i-th copy and $\mathcal{W}$ be the j-th one, then we have for $i < j$,

$$\begin{aligned} x_i y_j &= q y_j x_i + q\lambda x_j y_i, \\ x_i x_j &= q^2 x_j x_i, \\ y_i y_j &= q^2 y_j y_i, \\ y_i x_j &= q x_j y_i. \end{aligned}$$

In the deformed case we have to be more careful about the ordering. Let the deformed determinant-like object be

$$(ij) = x_i y_j - q y_i x_j,$$

which is invariant under the $SU_q(2)$ transformation, and let

$$[ij] = z_i - z_j = q^{-1} y_i^{-1} (ij) y_j^{-1},$$

where $z_i = x_i y_i^{-1}$.

Using the relations

$$\begin{aligned} y_i(ij) &= q(ij)y_i, \\ (ij)y_j &= q y_j (ij) \end{aligned}$$

for $i < j$ and

$$\begin{aligned} y_i(jk) &= q^3 (jk) y_i, \\ (ij) y_k &= q^3 y_k (ij) \end{aligned}$$

for $i < j < k$, we can see that, for example,

$$A = [12][24]^{-1}[34][13]^{-1}$$

is again invariant. Similarly, $B = [12][23]^{-1}[34][14]^{-1}$ as well as a number of others are invariant.

To find out whether these invariants are independent of one another, we now discuss the algebra of the $[ij]$'s.

Because $[ij] = [ik] + [kj]$ and $[ij] = -[ji]$ the algebra of $[ij]$ for $i, j = 1, 2, 3, 4$ is generated by only three elements $[12], [23], [34]$. It is easy to prove that

$$[ij][kl] = q^2 [kl][ij]$$

if $i < j \leq k < l$.

It follows that we have

$$[ij][ik][jk] = q^4 [jk][ik][ij] \tag{2.114}$$

for $i < j < k$, and

$$[12][34] + [14][23] = [12][24] + [24][23].$$

Using these relations we can check the dependency between the different anharmonic ratios. For example, let $C = [13][23]^{-1}[24][14]^{-1}$, and $D = [14][13]^{-1}[23][24]^{-1}$, both invariant, then

$$\begin{aligned} B^{-1}AC &= [14][34]^{-1}[23][24]^{-1}([34][23]^{-1}[24])[14]^{-1} \\ &= [14][34]^{-1}[23][24]^{-1}([24][23]^{-1}[34])[14]^{-1} \\ &= 1, \end{aligned}$$

where we used the relation $[34][23]^{-1}[24] = [24][23]^{-1}[34]$ which follows from (2.114), and

$$\begin{aligned} q^2 B - D^{-1} &= ([12][34][23]^{-1} - [24][23]^{-1}[13])[14]^{-1} \\ &= ([12][34][23]^{-1} - [24][23]^{-1}([12] + [23]))[14]^{-1} \\ &= ([12][34] - [12][24] - [24][23])[23]^{-1}[14]^{-1} \\ &= (-[14][23])[23]^{-1}[14]^{-1} \\ &= -1. \end{aligned}$$

In this manner it can be checked that all products of four terms $[ij]$, $[kl]$, $[mn]^{-1}$, $[pr]^{-1}$ in arbitrary order, which are invariant, are functions of only one invariant, say, A. Namely, all invariants are related and just like in the classical case, there is only one independent anharmonic ratio. It can be checked that the anharmonic ratio commutes with all the $\bar{z}_i$'s and so commutes with its $*$-complex conjugate, which is also an invariant.

Chapter 3

Geometry of the Quantum Complex Projective Space $CP_q(N)$

In the previous chapter the quantum sphere was described as a complex quantum manifold. The braiding of several copies of the quantum sphere was introduced and quantum anharmonic ratios (cross ratios) of four points on the sphere were defined which are invariant under the fractional transformation which describes the coaction of the quantum group $SU_q(2)$ on the complex coordinates $z, \bar{z}$ on the quantum sphere. In this and the next chapters we will extend these results on the one-dimensional complex projective space $CP_q(1) \sim S_q^2$ to higher dimensions. In this chapter we will define the quantum projective space $CP_q(N)$ [56] in terms of both homogeneous and inhomogeneous complex coordinates and study the differential calculus on it. $CP_q(N)$ is shown to be the quantum deformation of a Kähler manifold with the Fubini-Study metric. In Sec. 3.2 we consider the Poisson limit. Then, in Sec. 3.3 we introduce the braiding of several copies of $CP_q(N)$. In the next chapter we will study the projective geometry on $CP_q(N)$.

The algebra of functions on complex projective space has been considered by a number of authors, see for example [57], [58] and [59]. What we have shown here is that a rich construction of differential geometry and projective geometry can be carried out on this space. It is not hard to extend most of the results of this chapter to the case of quantum Grassmannian manifolds [60].

3.1 $CP_q(N)$ as a Complex Manifold

3.1.1 $SU_q(N+1)$ Covariant Complex Quantum Space

For completeness, we list here the formulas we shall need to construct the complex projective space. Remember that the $SU_q(N+1)$ symmetry can be represented [61] on the complex quantum space C_q^{N+1} with coordinates $x_i, \bar{x}^i, i = 0, 1, \ldots, N$ which satisfy the relations

$$x_i x_j = q^{-1} \hat{R}^{kl}_{ij} x_k x_l, \tag{3.1}$$

$$\bar{x}^i x_j = q (\hat{R}^{-1})^{ik}_{jl} x_k \bar{x}^l \tag{3.2}$$

and

$$\bar{x}^i \bar{x}^j = q^{-1} \hat{R}^{ji}_{lk} \bar{x}^k \bar{x}^l. \tag{3.3}$$

Here q is a real number, $\hat{R}^{kl}_{ij}$ is the $GL_q(N+1)$ $\hat{R}$-matrix [16] with indices running from 0 to N, and $\bar{x}^i = x_i^*$ is the $*$-conjugate of x_i. The Hermitian length

$$L = x_i \bar{x}^i \tag{3.4}$$

is real and central. The $\hat{R}$-matrix satisfies the characteristic equation

$$(\hat{R} - q)(\hat{R} + q^{-1}) = 0. \tag{3.5}$$

Derivatives $D^i, \bar{D}_i$ can be introduced (the usual symbols $\partial^a, \bar{\partial}_b$ are reserved below for the derivatives on $CP_q(N)$) which satisfy

$$D^i x_j = \delta^i_j + q \hat{R}^{ik}_{jl} x_k D^l, \qquad D^i \bar{x}^j = q (\hat{R}^{-1})^{ji}_{lk} \bar{x}^k D^l, \tag{3.6}$$

$$\bar{D}_i \bar{x}^j = \delta^i_j + q^{-1} (\hat{R}^{-1})^{lj}_{ki} \bar{x}^k \bar{D}_l, \quad \bar{D}_i x_j = q^{-1} \tilde{\Phi}^{lk}_{ji} x_k \bar{D}_l \tag{3.7}$$

and

$$D^i D^j = q^{-1} \hat{R}^{ji}_{lk} D^k D^l, \tag{3.8}$$

$$D^i \bar{D}_j = q^{-1} \tilde{\Phi}^{ki}_{lj} \bar{D}_k D^l, \tag{3.9}$$

$$\bar{D}_i \bar{D}_j = q^{-1} \hat{R}^{kl}_{ij} \bar{D}_k \bar{D}_l. \tag{3.10}$$

Here we have defined

$$\tilde{\Phi}^{ij}_{kl} = \hat{R}^{ji}_{lk} q^{2(i-l)} = \hat{R}^{ji}_{lk} q^{2(k-j)}, \tag{3.11}$$

which satisfies

$$\tilde{\Phi}^{ri}_{sj} (\hat{R}^{-1})^{jk}_{il} = (\hat{R}^{-1})^{ri}_{sj} \tilde{\Phi}^{jk}_{il} = \delta^r_l \delta^k_s \tag{3.12}$$

and (sum over the index k)

$$\hat{\Phi}^{ik}_{jk} = \delta^i_j q^{2i+1}, \tag{3.13}$$

$$\hat{\Phi}^{ki}_{kj} = \delta^i_j q^{2(N-i)+1}. \tag{3.14}$$

Using

$$\hat{R}^{ij}_{kl}(q^{-1}) = (\hat{R}^{-1})^{ji}_{lk}(q) \tag{3.15}$$

and

$$\hat{R}^{ij}_{kl} = \hat{R}^{kl}_{ij}, \tag{3.16}$$

one can show that there is a symmetry of this algebra:

$$q \to q^{-1}, \tag{3.17}$$

$$x_i \to kq^{-2i}\bar{x}^i, \tag{3.18}$$

$$\bar{x}^i \to l x_i, \tag{3.19}$$

$$D^i \to k^{-1}q^{2i}\bar{D}_i \tag{3.20}$$

and

$$\bar{D}_i \to l^{-1}D^i, \tag{3.21}$$

where k and l are arbitrary constants. Exchanging the barred and unbarred quantities in (3.17-3.21), we get another symmetry which is related to the inverse of this one.

Using the fact that L commutes with $x_i, \bar{x}^i$, a $*$-involution can be defined for D^i

$$(D^i)^* = -q^{-2i'}L^n\bar{D}_iL^{-n}, \tag{3.22}$$

where

$$i' = N - i + 1 \tag{3.23}$$

for any real number n. The $*$-involutions corresponding to different n's are related to one another by the symmetry of conjugation by L

$$a \to L^m a L^{-m}, \tag{3.24}$$

where a can be any function or derivative and m is the difference in the n's.

The differentials $\xi_i = dx_i, \bar{\xi}^i = (\xi_i)^*$ satisfy

$$x_i\xi_j = q\hat{R}^{kl}_{ij}\xi_k x_l, \tag{3.25}$$

$$\bar{x}^i\xi_j = q(\hat{R}^{-1})^{ik}_{jl}\xi_k\bar{x}^l \tag{3.26}$$

and

$$\xi_i\xi_j = -q\hat{R}^{kl}_{ij}\xi_k\xi_l, \tag{3.27}$$

$$\bar{\xi}^i\xi_j = -q(\hat{R}^{-1})^{ik}_{jl}\xi_k\bar{\xi}^l. \tag{3.28}$$

All the above relations are covariant under the transformation

$$x_i \to x_jT^j_i, \qquad \bar{x}^i \to (T^{-1})^i_j\bar{x}^j, \tag{3.29}$$

$$D^i \to (T^{-1})^i_jD^j, \quad \bar{D}_i \to \bar{D}_jq^{2i'}T^j_iq^{-2j'}, \tag{3.30}$$

$$\xi_i \to \xi_jT^j_i, \qquad \bar{\xi}^i \to (T^{-1})^i_j\bar{\xi}^j, \tag{3.31}$$

where $T^i_j \in SU_q(N+1)$.

The holomorphic and antiholomorphic differentials $\delta, \bar{\delta}$ satisfy the undeformed Leibniz rule, $\delta^2 = \bar{\delta}^2 = 0$ and $\bar{\delta}x_j = x_j\bar{\delta}$ etc.

3.1.2 Algebra and Calculus on $CP_q(N)$

Define for $a = 1, ..., N$, [1]

$$z_a = x_0^{-1}x_a, \quad \bar{z}^a = \bar{x}^a(\bar{x}^0)^{-1}. \tag{3.32}$$

Since

$$x_0x_a = qx_ax_0, \quad x_0\bar{x}^0 = \bar{x}^0x_0 \tag{3.33}$$

and

$$x_0\bar{x}^a = q^{-1}\bar{x}^ax_0, \tag{3.34}$$

it follows from (3.1) and (3.2) that [2]

$$z_az_b = q^{-1}\hat{R}^{ce}_{ab}z_cz_e, \tag{3.35}$$

$$\bar{z}^az_b = q^{-1}(\hat{R}^{-1})^{ac}_{be}z_c\bar{z}^e - \lambda q^{-1}\delta^a_b, \tag{3.36}$$

where $\hat{R}^{ac}_{be}$ is the $GL_q(N)$ $\hat{R}$-matrix with indices running from 1 to N and $\lambda = q - 1/q$.

Since

$$dz_a = x_0^{-1}(\xi_a - \xi_0z_a), \quad d\bar{z}^a = (\bar{\xi}^a - \bar{z}^a\bar{\xi}^0)(\bar{x}^0)^{-1} \tag{3.37}$$

[1] The letters a, b, c, e etc. run from 1 to N, while i, j, k, l run from 0 to N.

[2] Due to our conventions in this chapter, to compare the $N = 1$ case with the results in Chap.2, one has to make a change of variables $z \to \bar{z}$.

and

$$x_0\xi_0 = q^2\xi_0 x_0, \quad x_0\bar{\xi}^0 = \bar{\xi}^0 x_0, \tag{3.38}$$

it follows from (3.25) and (3.26) that

$$z_a dz_b = q\hat{R}^{ce}_{ab} dz_c z_e, \tag{3.39}$$

$$\bar{z}^a dz_b = q^{-1}(\hat{R}^{-1})^{ac}_{be} dz_c \bar{z}^e, \tag{3.40}$$

$$dz_a dz_b = -q\hat{R}^{ce}_{ab} dz_c dz_e \tag{3.41}$$

and

$$d\bar{z}^a dz_b = -q^{-1}(\hat{R}^{-1})^{ac}_{be} dz_c d\bar{z}^e. \tag{3.42}$$

The derivatives $\partial^a, \bar{\partial}_a$ are defined by requiring $\delta \equiv dz_a\partial^a$ and $\bar{\delta} \equiv d\bar{z}^a\bar{\partial}_a$ to be exterior differentiations. It follows from (3.39) and (3.40) that

$$\partial^a z_b = \delta^a_b + q\hat{R}^{ac}_{be} z_c \partial^e, \tag{3.43}$$

$$\partial^a \bar{z}^b = q^{-1}(\hat{R}^{-1})^{ba}_{ec} \bar{z}^c \partial^e, \tag{3.44}$$

$$\bar{\partial}_a z_b = q\Phi^{ec}_{ba} z_c \bar{\partial}_e, \tag{3.45}$$

$$\bar{\partial}_a \bar{z}^b = \delta^b_a + q^{-1}(\hat{R}^{-1})^{eb}_{ca} \bar{z}^c \bar{\partial}_e, \tag{3.46}$$

$$\partial^b \partial^a = q^{-1}\hat{R}^{ab}_{ce} \partial^e \partial^c \tag{3.47}$$

and

$$\partial^a \bar{\partial}_b = q\Phi^{ca}_{eb} \bar{\partial}_c \partial^e, \tag{3.48}$$

where the Φ matrix is defined by

$$\Phi^{ca}_{db} = \hat{R}^{ac}_{bd} q^{2(c-b)} = \hat{R}^{ac}_{bd} q^{2(d-a)}. \tag{3.49}$$

Similarly as in the case of quantum spaces the algebra of the differential calculus on $CP_q(N)$ has the symmetry:

$$q \to q^{-1}, \tag{3.50}$$

$$z_a \to rq^{-2a}\bar{z}^a, \tag{3.51}$$

$$\bar{z}^a \to sz_a, \tag{3.52}$$

$$\partial^a \to r^{-1}q^{2a}\bar{\partial}_a \tag{3.53}$$

and

$$\bar{\partial}_a \to s^{-1}\partial^a, \tag{3.54}$$

where $rs = q^2$. Again we also have another symmetry by exchanging the barred and unbarred quantities in the above.

Also the $*$-involutions

$$z_a^* = \bar{z}^a, \tag{3.55}$$

$$dz_a^* = d\bar{z}^a \tag{3.56}$$

and

$$\partial^{a*} = -q^{2n-2a'}\rho^n \bar{\partial}_a \rho^{-n}, \tag{3.57}$$

where

$$a' = N - a + 1 \tag{3.58}$$

and

$$\rho = 1 + \sum_{a=1}^{N} z_a \bar{z}^a, \tag{3.59}$$

can be defined for any n. Corresponding to different n's they are related with one another by the symmetry of conjugation by ρ to some powers followed by a rescaling by appropriate powers of q.

In particular, the choice $n = N + 1$ gives the $*$-involution which has the correct classical limit of Hermitian conjugation with the standard measure $\rho^{-(N+1)}$ of $CP(N)$.

The transformation (3.29) induces a transformation on $CP_q(N)$

$$z_a \to (T^0_0 + z_b T^b_0)^{-1}(T^0_a + z_c T^c_a). \tag{3.60}$$

One can then calculate how the differentials transform

$$dz_a \to dz_b M^b_a, \quad d\bar{z}^a \to (M^\dagger)^a_b d\bar{z}^b, \tag{3.61}$$

where M^b_a is a matrix of functions in z_a with coefficients in $SU_q(N+1)$ and $(M^\dagger)^a_b \equiv (M^b_a)^*$. Since $\delta, \bar{\delta}$ are invariant, it follows the transformation on the derivatives

$$\partial^a \to (M^{-1})^a_b \partial^b, \quad (\partial^a)^* \to (\partial^b)^*((M^\dagger)^{-1})^b_a. \tag{3.62}$$

The covariance of the $CP_q(N)$ relations under the transformation (3.60), (3.61) and (3.62) follows directly from the covariance in C^{N+1}_q.

3.1.3 Kähler Two-Form

Similar to the case of a quantum sphere, the quantization of the differential calculus on $CP_q(N)$ is closely connected to the geometry.

The exterior derivatives can be realized by an element already in the calculus:

$$\delta\omega = \lambda^{-1}\left[\xi,\omega\right]_{\pm}, \quad \bar{\delta}\omega = -\lambda^{-1}\left[\xi^{*},\omega\right]_{\pm}, \tag{3.63}$$

where

$$\xi = -q^{-1}\delta\rho\rho^{-1} \tag{3.64}$$

and the sign is + (−) if ω is odd (even).

In the same way

$$\Xi = \xi - \xi^{*} \tag{3.65}$$

realizes $d = \delta + \bar{\delta}$.

Consistent with the nilpotency of $\delta, \bar{\delta}$ and d, we have $\xi^2 = \xi^{*2} = 0$ and Ξ^2 central. It is easy to see that all the quantities Ξ^2, $\delta\xi^{*}$ and $\bar{\delta}\xi$ differ from one another only by numerical factors. This shows that the object

$$K = \delta\xi^{*} = \bar{\delta}\xi \tag{3.66}$$

is central and satisfies $\delta K = \bar{\delta}K = 0$. Moreover, it can be checked that it is also invariant. We will call this the Kähler two-form.

Motivated by the classical role played by a Kähler form, we define the metric $g_{a\bar{b}}$ on $CP_q(N)$ by

$$K = dz_a g_{a\bar{b}} d\bar{z}^b. \tag{3.67}$$

It is

$$g_{a\bar{b}} = q^{-1}\rho^{-2}(\rho\delta_{ab} - q^2\bar{z}^a z_b), \tag{3.68}$$

which in the classical limit ($q \to 1$) is exactly the Fubini-Study metric. The inverse satisfying

$$g^{\bar{b}c}g_{c\bar{a}} = g_{a\bar{c}}g^{\bar{c}b} = \delta_{ab} \tag{3.69}$$

can also be found:

$$g^{\bar{b}c} = q\rho(\delta_{bc} + \bar{z}^b z_c). \tag{3.70}$$

Using the metric (3.68) we can define another basis of one-forms $\{\xi^a, \xi_a\}$ by

$$\xi^a = dz_a, \quad \xi_a = g_{a\bar{b}} d\bar{z}^b, \tag{3.71}$$

so that $K = \xi^a\xi_a$. The commutation relations in this basis are:

$$\xi^a\xi^b = -q\hat{R}^{ab}_{cd}\xi^c\xi^d, \tag{3.72}$$

$$\xi_a\xi^b = -q^{-1}(\hat{R}^{-1})^{bd}_{ac}\xi^c\xi_d, \tag{3.73}$$

$$\xi_a\xi_b = -q\hat{R}^{dc}_{ba}\xi_c\xi_d, \tag{3.74}$$

which is the same as the commutation relations (3.41-3.42) for dz_a and $d\bar{z}^a$ under the replacement $d\bar{z}^a \leftrightarrow \xi_a = g_{a\bar{b}}d\bar{z}^b$ ($dz_a = \xi^a$ by definition).

As in the classical case, since the Kähler form K is invariant, one can take K^N as the volume form (measure), which is found to be proportional to

$$\rho^{-(N+1)}d\bar{z}^N \cdots d\bar{z}^1 dz_1 \cdots dz_N. \tag{3.75}$$

Similar constructions exist also for the complex quantum space C_q^{N+1}. A general sufficient condition for the existence of one-forms realizing the holomorphic and antiholomorphic exterior derivatives on a complex quantum manifold is presented in [56, 60], where the central invariant Kähler form also exists. Whenever such a Kähler form exists, one can use it to define the metric and a Hodge $*$ map. It will be shown in Chap. 5 that one can use these data to define the connection, curvature two-form, Ricci tensor and scalar curvature.

The integration on $CP_q(N)$ [56, 60] can be obtained in analogous ways as for the sphere in Sec.2.7. The discussion on the patching of $CP_q(N)$ is similar to that of S_q^2. Here an appropriate definition of the algebra should be made such that it is equivalent to $SU_q(N+1)/U_q(N)$ [57].

3.2 Poisson Structures on $CP(N)$

The commutation relations in the previous sections give us, in the limit $q \to 1$, a Poisson structure on $CP(N)$. As usual, the Poisson Brackets (P.B.s) are obtained as the limit

$$(f,g) = \lim_{h\to 0}\frac{fg \mp gf}{h}, \quad q = e^h = 1 + h + [h^2]. \tag{3.76}$$

It is straightforward to find

$$(z_a, z_b) = z_a z_b, \qquad a < b, \tag{3.77}$$

$$(z_a, \bar{z}^b) = \begin{cases} z_a\bar{z}^b, & a \neq b \\ 2(1 + \sum_{c=1}^{a} z_c\bar{z}^c), & a = b \end{cases} \tag{3.78}$$

$$(z_a, dz_b) = \begin{cases} z_a dz_b + 2 z_b dz_a, & a < b \\ 2 z_a dz_a, & a = b \ , \\ z_a dz_b, & a > b \end{cases} \tag{3.79}$$

$$(\bar{z}^a, dz_b) = \begin{cases} -\bar{z}^a dz_b, & a \neq b \\ -2 \sum_{c=1}^{a} \bar{z}^c dz_c, & a = b \end{cases} \tag{3.80}$$

and those following from the $*$-involution, which satisfies

$$(f, g)^* = (g^*, f^*). \tag{3.81}$$

The P.B. of two differential forms f and g of degrees m and n respectively satisfies

$$(f, g) = (-1)^{mn+1} (g, f). \tag{3.82}$$

The exterior derivatives $\delta, \bar{\delta}, d$ act on the P.B.s distributively, for example

$$d(f, g) = (df, g) \pm (f, dg), \tag{3.83}$$

where the plus (minus) sign applies for even (odd) f. Notice that we have extended the concept of Poisson Bracket to include differential forms.

The Fubini-Study Kähler form

$$K = dz_a g^{a\bar{b}} d\bar{z}^b \tag{3.84}$$

has vanishing Poisson bracket with all functions and forms and, naturally, it is closed.

3.3 Braided $CP_q(N)$

The braiding of the quantum plane C_q^{N+1} induces a braiding of $CP_q(N)$.

Let the first copy of the quantum plane be denoted by $x_i, \bar{x}^i$ and the second by $x'_i, \bar{x}'^i$. A consistent and covariant choice of the commutation relations between them is

$$x_i x'_j = q^\alpha \hat{R}^{kl}_{ij} x'_k x_l, \tag{3.85}$$

$$\bar{x}^i x'_j = q^\beta (\hat{R}^{-1})^{ik}_{jl} x'_k \bar{x}^l \tag{3.86}$$

and their $*$-involutions for arbitrary numbers α, β. If we choose $\alpha = -\beta$ then the Hermitian length L will remain central, $Lf' = f'L$ for any function f' of $x', \bar{x}'$. However, L' does not commute with $x, \bar{x}$.

Assuming that the exterior derivatives of the two copies satisfy the Leibniz rule

$$\delta' f = \pm f \delta', \quad \bar{\delta}' f = \pm f \bar{\delta}'. \tag{3.87}$$

$$\delta f' = \pm f' \delta, \quad \bar{\delta} f' = \pm f' \bar{\delta}, \tag{3.88}$$

where the plus (minus) signs apply for even (odd) f and f', and

$$\delta\delta' = -\delta'\delta, \quad \delta\bar{\delta}' = -\bar{\delta}'\delta, \tag{3.89}$$

$$\bar{\delta}\delta' = -\delta'\bar{\delta}, \quad \bar{\delta}\bar{\delta}' = -\bar{\delta}'\bar{\delta}, \tag{3.90}$$

we obtain the commutation relations between functions and forms of different copies by letting $\delta, \bar{\delta}, \delta'$ and $\bar{\delta}'$ act on (3.85) and (3.86). As usual, the commutation relations between derivatives and functions of different copies can also be derived from the commutation relations between differential forms and functions using the Leibniz rule of the exterior derivatives and the identifications $\delta = dx_i D^i, \bar{\delta} = d\bar{x}^i \bar{D}_i$ for both copies.

From the above we derive the braiding relations of two braided copies of $CP_q(N)$ in terms of the inhomogeneous coordinates. They are independent of the particular choice of α and β. We have

$$z_a z'_b = q \hat{R}^{ce}_{ab} (z'_c - q^{-1} \lambda z_c) z_e, \tag{3.91}$$

$$\bar{z}'^a z_b = q^{-1} (\hat{R}^{-1})^{ac}_{be} z_c \bar{z}'^e - q^{-1} \lambda \delta^a_b \tag{3.92}$$

and their $*$-involutions as well as the commutation relations between functions and forms of different copies following the assumption that their exterior derivatives anticommute.

The braiding can be extended to an arbitrary number of ordered copies of $CP_q(N)$.

Chapter 4

Quantum Projective Geometry

We will show in this chapter that many concepts of projective geometry have an analogue in the deformed case. We shall study the collinearity conditions in Sec.4.1, the deformed anharmonic ratios (cross ratios) in Sec.4.2, the coplanarity conditions in Sec.4.3. In Sec.4.4 we will show that the anharmonic ratios are the building blocks of other invariants.

4.1 Collinearity Condition

Classically the collinearity conditions for m distinct points in $CP(N)$ can be given in terms of the inhomogeneous coordinates $\{z_a^A | A = 1, 2, \cdots, m; a = 1, 2, \cdots, N\}$ as

$$(z_a^A - z_a^B)(z_a^C - z_a^D)^{-1} = (z_b^A - z_b^B)(z_b^C - z_b^D)^{-1}, \tag{4.1}$$

where $A \neq B, C \neq D = 1, \cdots, m$ and $a, b = 1, \cdots, N$.

In the deformed case, the coordinates $\{z_a^A\}$ of m points must be braided for the commutation relations to be covariant, namely,

$$z_a^A z_b^B = q\hat{R}^{ce}_{ab}(z_c^B - q^{-1}\lambda z_c^A) z_e^A, \quad A \leq B, \tag{4.2}$$

as an extension of (3.91). Eq.(3.92) can also be generalized in the same way, but we shall not need it in this section. This braiding has the interesting property that the algebra of $CP_q(N)$ is *self-braided*, that is, (4.2) allows the choice $A = B$. This property makes it possible to talk about the coincidence of points. Actually, the whole differential calculus for braided $CP_q(N)$ described in Sec.3.3 has this property.

Another interesting fact about this braiding is that for a fixed index a the commutation relation is identical to that for braided S_q^2

$$z_a^A z_a^B = q^2 z_a^B z_a^A - q\lambda z_a^A z_a^A, \quad A \leq B. \tag{4.3}$$

Since there is no algebraic way to say that two "points" are distinct in the deformed case, the collinearity conditions should avoid using expressions like $(z_a^A - z_a^B)^{-1}$, which are ill defined. Denote

$$[AB]_a = z_a^A - z_a^B. \tag{4.4}$$

The collinearity conditions in the deformed case can be formulated as

$$[AB]_a[CD]_b = q^2[CD]_a[AB]_b, \quad \forall a, b \tag{4.5}$$

and $A < B \leq C < D$. By (4.2) this equation is formally equivalent to the quantum counterpart of (4.1):

$$[AB]_a[CD]_a^{-1} = [AB]_b[CD]_b^{-1}, \tag{4.6}$$

where the ordering of A, B, C, D is arbitrary. The advantage of this formulation is that (4.5) is a quadratic polynomial condition and polynomials are well defined in the braided algebra.

Therefore the algebra Q of functions of m collinear points is the quotient of the algebra $\mathcal{A}$ of m braided copies of $CP_q(N)$ over the ideal $I = \{f\alpha g : \forall f, g \in \mathcal{A}; \forall \alpha \in CC\}$ generated by α which stands for the collinearity conditions (4.5), i.e., $\alpha \in CC = \{[AB]_a[CD]_b - q^2[CD]_a[AB]_b : A < B \leq C < D\}$.

Two requirements have to be checked for this definition $Q = \mathcal{A}/I$ to make sense. The first one is that for any $f \in \mathcal{A}$ and $\alpha \in CC$,

$$f\alpha = \sum_i \alpha_i f_i, \quad \forall f \in \mathcal{A}, \tag{4.7}$$

for some $f_i \in \mathcal{A}$ and $\alpha_i \in CC$. This condition ensures that the ideal I generated by the collinearity conditions is not "larger" than what we want, as compared with the classical case.

In fact, it is sufficient (for formal manipulations, at least) to consider only $B = C = m - 1, D = m$ in either (4.5) or (4.6). That is, we need only two points to fix a line.

We now check that (4.7) is satisfied. Obviously we only have to consider the cases $f = z_c^E$, for arbitrary E and c. Let $\alpha(AB)_{ab} = [AB]_a[CD]_b - q^2[CD]_a[AB]_b$ for $C = m - 1$ and $D = m$. Using (4.2) one finds, after considerable algebra, for $B \leq A < C < D$,

$$z_a^B \alpha(AC)_{bc} = q^2 \hat{R}^{he}_{ab}\hat{R}^{fg}_{ec}\alpha(AC)_{hf} z_g^B. \tag{4.8}$$

For $A \leq B \leq C < D$, one finds similarly

$$z_a^B \alpha(AC)_{bc} = q^2 \hat{R}^{he}_{ab}\hat{R}^{fg}_{ec}(\alpha(AC)_{hf} z_g^B + q^{-1}\lambda\alpha(AB)_{hf}[AB]_g). \tag{4.9}$$

Hence (4.7) is proven for $B \leq C$. Using

$$[CD]_a \alpha(AC)_{bc} = (\hat{R}^{-1})^{he}_{ab} (\hat{R}^{-1})^{fg}_{ec} \alpha(AC)_{hf} [CD]_g \tag{4.10}$$

for $B = D$ and

$$[BD]_a \alpha(AC)_{bc} = q^{-2} (\hat{R}^{-1})^{he}_{ab} (\hat{R}^{-1})^{fg}_{ec} \alpha(AC)_{hf} [BD]_g \tag{4.11}$$

for $B > D$, together with the above two equations we immediately see that (4.7) is satisfied for $f = z^B_a$ with $B \geq D$. Therefore the first requirement is satisfied.

The second requirement is the invariance of I under the fractional transformation (3.60). While this can be directly checked for (4.5), it is equivalent but simpler to consider another expression of the collinearity conditions:

$$[AB]^{-1}_a [AB]_b = [CD]^{-1}_a [CD]_b, \tag{4.12}$$

where the ordering of A, B, C, D is arbitrary. Again we only have to consider the independent cases of $B < A = C = m - 1$ and $D = m$. It can be shown that the fractional transformation has

$$[AB]_a \to U(B)^{-1} [AB]_b M^b_a(A) V(A)^{-1}, \tag{4.13}$$

where $U(B) = T^0_0 + z^B_e T^e_0$, $V(A) = T^0_0 + q z^A_f T^f_0$ and

$$M^b_a(A) = (T^b_a T^0_0 - q^{-1} T^b_0 T^0_a) + q z^A_c (T^b_a T^c_0 - q^{-1} T^b_0 T^c_a). \tag{4.14}$$

So

$$\begin{aligned} & [AB]^{-1}_a [AB]_b \to \\ & V(A)([AB]_c M^c_a)^{-1}([AB]_e M^e_b) V(A)^{-1} \\ = \; & V(A)([AB]_c [AC]^{-1}_c [AC]_c M^c_a)^{-1}([AB]_e [AC]^{-1}_e [AC]_e M^e_b) V(A)^{-1} \\ = \; & V(A)([AC]_c M^c_a)^{-1}([AB]_g [AC]^{-1}_g)^{-1}([AB]_s [AC]^{-1}_s)([AC]_e M^e_b) V(A)^{-1} \\ = \; & V(A)([AC]_c M^c_a)^{-1}([AC]_e M^e_b) V(A)^{-1}, \end{aligned} \tag{4.15}$$

(where we used (4.6) for the second equality) which equals the transformation of $[AC]^{-1}_a [AC]_b$. This means that the relations $[AB]^{-1}_a [AB]_b - [AC]^{-1}_a [AC]_b = 0$ are preserved by the transformation.

4.2 Anharmonic Ratios

Classically the anharmonic ratio of four collinear points is an invariant of the projective mappings, which are the linear transformations of the homogeneous coordinates. In the deformed case, the homogeneous coordinates are the coordinates x_i of the $GL_q(N+1)$-covariant quantum space, and the linear transformations are the $GL_q(N+1)$ transformations which induce the fractional transformations (3.60) on the coordinates z_a of the projective space $CP_q(N)$.

We consider the following anharmonic ratio of $CP_q(N)$ for four collinear points $\{z^A_a | A = 1,2,3,4\}$

$$[A1]_a [A4]^{-1}_a [B4]_a [B1]^{-1}_a, \tag{4.16}$$

where $A, B = 2, 3$. We wish to show that it is invariant. Using (4.13) and denoting $\tau(A) = [1A]_a [14]^{-1}_a$, which is independent of the index a according to the collinearity condition, we get

$$[AB]_a \to U(B)^{-1} (\tau(A) - \tau(B)) P_a(A) V(A)^{-1}, \tag{4.17}$$

where $P_a(A) = -[14]_b M^b_a(A)$. Then the anharmonic ratio (4.16) transforms as

$$\begin{aligned} [A1]_a [A4]^{-1}_a [B4]_a [B1]^{-1}_a & \to U(1)^{-1} \tau(A) (1 - \tau(A))^{-1} (1 - \tau(B)) \tau(B)^{-1} U(1) \\ & = \tau(A)(1 - \tau(A))^{-1}(1 - \tau(B)) \tau(B)^{-1} \\ & = [A1]_a [A4]^{-1}_a [B4]_a [B1]^{-1}_a, \end{aligned} \tag{4.18}$$

where we have used $z^1_a \tau(A) = \tau(A) z^1_a$ for any $A \geq 1$, which is true because we can represent $\tau(A)$ as $[1A]_a [14]^{-1}_a$ with the same index a and then use $z^1_a [AB]_a = q^2 [AB]_a z^1_a$.

Because of the nice property (4.3), we can use the results about the anharmonic ratios of S^2_q (which is a special case of $CP_q(N)$ with $N = 1$ but no collinearity condition is needed there) in Sec.2.8. Note that all the invariants as functions of z^A_a for a fixed a in $CP_q(N)$ are also invariants as functions of $z^A = z^A_a$ in S^2_q. The reason is the following. Consider the matrix T^a_b defined by

$$T^0_0 = \alpha, \quad T^0_a = \beta, \tag{4.19}$$

$$T^a_0 = \gamma, \quad T^a_a = \delta, \tag{4.20}$$

where $\alpha, \beta, \gamma, \delta$ are components of an $SU_q(2)$-matrix, $T^b_b = 1$ for all $b \neq 0, a$ and all other components vanishing. It is a $GL_q(N+1)$-matrix, but the transformation

(3.60) of z_a^A by this matrix is the fractional transformation on S_q^2 with coordinate $z^A = z_a^A$.

Therefore, by simply dropping the subscript a, the anharmonic ratio (4.16) becomes an anharmonic ratio of S_q^2. On the other hand, since all other anharmonic ratios of S_q^2 are functions of only one of them, their corresponding anharmonic ratios of $CP_q(N)$ (by putting in the subscript a) would be functions of (4.16) and hence are invariant. Therefore we have established the fact that all invariant anharmonic ratios of $CP_q(N)$ are functions of only one of them.

4.3 Coplanarity Condition

In the above we have seen that the collinearity condition is imposed by taking the quotient of the algebra of inhomogeneous coordinates over an ideal generated by (4.5). This is analogous to what one does to obtain S_q^2 as the quotient of $SU_q(2)$ over $U(1)$ in Sec.2.2. In general, by taking the quotient of the algebra over an ideal we obtain the algebra for a submanifold. However, there is another way to formulate the collinearity condition. It will allow easy generalizations to the coplanarity conditions.

4.3.1 Homogeneous Coordinates

Consider in terms of homogeneous coordinates two points x^A and x^B defining a line. Classically a point on this line can be given as

$$x_i^C = \mu_A x_i^A + \mu_B x_i^B, \quad i = 0, \cdots N, \tag{4.21}$$

for some μ_A and μ_B.

In the quantum case, we interpret μ_A and μ_B as operators corresponding to the "observables" [1] when one measures the location of a point on this line.

The commutation relations between μ_A, μ_B and those between the μ's and x^A, x^B are to be determined by the requirement that x^C defined in (4.21) shall satisfy the correct commutation relations among different components of x^C and those with x^A, x^B according to (3.1) and (3.85).

The transformation of x^C should be the same as x^A and x^B, namely it transforms linearly under $GL_q(N+1)$. This is achieved by simply assigning μ_A and μ_B to be invariant.

[1] Strictly speaking only the ratio of μ_A and μ_B can be an observable in the projective space.

The fact that one can adjoin μ_A and μ_B in a consistent and covariant way is almost equivalent to the consistency and covariance of the collinearity condition (4.5) above. The latter can be obtained from the former. From (4.21) we have

$$z_i^C = \sigma_A z_i^A + \sigma_B z_i^B, \tag{4.22}$$

where $\sigma_A = (x_0^C)^{-1}\mu_A x_0^A$ and similarly for σ_B. Since $x_0^C = \mu_A x_0^A + \mu_B x_0^B$, it follows that $\sigma_A + \sigma_B = 1$. Choose a certain number for the index i in (4.22) one can solve for σ_A (or σ_B) in terms of the z's. Choosing another number for i gives another solution. Equate the two solutions; one gets the collinearity condition (4.5).

The same procedure can be carried out for not only lines but also planes, or higher dimensional subspaces spanned by a finite number of points.

Consider the r-dimensional subspace spanned by $r+1$ points $\{x^A\}_{A\in I}$ where I is a set of $r+1$ different numbers. Define

$$x^B = \sum_{A\in I} \mu_A^B x^A \tag{4.23}$$

for some $B > I$, namely, $B > A$ for all $A \in I$. Here and later we shall skip the lower indices when no confusion can arise.

The requirement $B > I$ comes from our convention that for all the coplanar points (all points lying on the same hyperplane) we label them in a certain order and we always choose the first $r+1$ of them as the reference points and all others are referred to by their corresponding μ's.

To get the commutation relations for the μ's, we impose appropriate commutation relations for x^B.

First, according to (3.85), for x^B to commute correctly with x^A, $A \in I$, we let

$$x^A \mu_C^B = q^{2\alpha} \mu_C^B x^A, \quad C < B, \tag{4.24}$$

$$x^A \mu_A^B = q^{\alpha}(q\mu_A^B x^A + \lambda \textstyle\sum_{C<A} \mu_C^B x^C), \tag{4.25}$$

$$x^A \mu_C^B = \mu_C^B x^A, \quad C > B \tag{4.26}$$

for $C \in I$.

It turns out that (4.26) is also sufficient to guarantee that x^B commutes with x^A correctly for $A < I$, i.e. A is smaller than any number in I. For $A > B$, we find

$$x^A \mu_C^B = \mu_C^B x^A. \tag{4.27}$$

However it is not possible to do this for other possibilities of A. Hence we are allowed to have additional points which are not on the same hyperplane with x^B but they have to be labelled outside the range reserved for the hyperplane.

For two points x^B and x^C with $I < B < C$ on the same hyperplane to commute correctly with each other we need

$$\mu_D^B \mu_E^C = q^{2\alpha} \mu_E^C \mu_D^B, \quad D < E, \tag{4.28}$$

$$\mu_D^B \mu_D^C = q^{\alpha+1} \mu_D^C \mu_D^B, \tag{4.29}$$

$$\mu_D^B \mu_E^C = \mu_E^C \mu_D^B + q^\alpha \lambda \mu_D^C \mu_E^B, \quad D > E. \tag{4.30}$$

The last requirement is for the components of x^B to commute correctly among themselves. It is

$$\mu_C^B \mu_D^B = q^{\alpha+1} \mu_D^B \mu_C^B, \quad C < D. \tag{4.31}$$

This concludes our attempt to adjoin the μ's to the algebra of $\{x^A\}_{A \in I}$. Similarly the $*$-involution of the μ's can also be adjoined together with the μ's to the algebra of $\{x^A, \bar{x}^A\}_{A \in I}$.

Since the labels $A, B, C, \cdots$ only need to be ordered, they can be real numbers and so we can have uncountably many points on a hyperplane. Let S be the set of all labels of points on a hyperplane. In the above we took the smallest $r+1$ numbers in S as the set of labels I for the reference points of the hyperplane and the algebra of functions of all the points $\{x^A\}_{A \in S}$ is generated by $\{x^A\}_{A \in I}$ and $\{\mu_A^B\}_{A \in I}^{B \in S \setminus I}$.

Had we made a different choice of I, we would get another algebra which in general involves more complicated commutation relations for the μ variables. But it is simply a change of variables.

Another choice of I that would lead to a simpler algebra is to take I to be the largest $r+1$ numbers in S. Let us use ν instead of μ for this choice. Eq.(4.23) is now replaced by

$$x^B = \sum_{A \in I} \nu_A^B x^A \tag{4.32}$$

for $B \in S$, $B < I$. The algebra for this case can be obtained from (4.24-4.31) by replacing μ by ν, "$<$" by "$>$" and "$>$" by "$<$".

4.3.2 Inhomogeneous Coordinates

In this subsection we will get the coplanarity condition as a generalization of the collinearity condition (4.5).

From the previous subsection, for $r+1$ points spanning an r-dimensional hyperplane, we have

$$z^B = \sum_{A \in I} \sigma_A^B z^A,$$

where $\sigma_A^B = (x_0^B)^{-1} \nu_A^B x_0^A$ and $\sum_{A \in I} \sigma_A^B = 1$. By a change of variables for σ_A^B, and letting $I = \{1, 2, \cdots, r, r+1\}$, $B = 0$, it is

$$[01]_i = \sum_{j=1}^{r} \tau_j [j(j+1)]_i, \tag{4.33}$$

where $[AB]_i = z_i^A - z_i^B$ and the τ's are independent linear combinations of the σ's.

Choose a set K of r different integers from $1, 2, \cdots, N$. Consider the r equations (4.33) for $i \in K$. Let $K = \{\alpha_1, \alpha_2, \cdots, \alpha_r\}$, $M_i^j = [j(j+1)]_{\alpha_i}$ and $M_i^0 = [01]_{\alpha_i}$. Then

$$\tau_j = M_i^0 (M^{-1})_j^i, \quad j = 1, 2, \cdots, r, \tag{4.34}$$

where M^{-1} is the inverse matrix of $(M_j^i)_{i,j=1}^r$.

To find the inverse matrix of M, we note that the commutation relation between two entries of M is almost the same as that for a $GL_q(r)$-matrix. Let $\{g^i; i = 1, 2, \cdots, r\}$ be defined as a quantum hyperplane:

$$g^i g^j = q g^j g^i, \quad i < j \tag{4.35}$$

and T be a $GL_q(r)$-matrix which commutes with all the g^i's. Then the commutation relations for M_j^i is identical to those for $g^i T_j^i$ (i not summed over). With the help of this identification

$$M_j^i = g^i T_j^i, \tag{4.36}$$

we find

$$(M^{-1})_j^i = (g^j)^{-1} (T^{-1})_j^i. \tag{4.37}$$

It is known [16] that

$$(T^{-1})_j^i = (-q)^{i-j} \sum_{\sigma \in P} (-q)^{l(\sigma)} T_{\sigma_1}^1 \cdots T_{\sigma_{j-1}}^{j-1} T_{\sigma_{j+1}}^{j+1} \cdots T_{\sigma_r}^r (det_q(T))^{-1},$$

where P is the permutation group of $r-1$ objects, $l(\sigma)$ is the length of a permutation σ and $det_q(T)$ is the quantum determinant

$$det_q(T) = \epsilon_{i_1 i_2 \cdots i_r} T_{i_1}^1 \cdots T_{i_r}^r,$$

where

$$\epsilon_{\sigma_1 \sigma_2 \cdots \sigma_r} = (-q)^{l(\sigma)}$$

for σ being a permutation of r objects and 0 otherwise.

Even though M is not a $GL_q(r)$-matrix we define

$$det_q(M) = \epsilon_{i_1 \cdots i_r} M_{i_1}^1 \cdots M_{i_r}^r.$$

M^{-1} is then found to be

$$(M^{-1})^i_j = (-1)^{j-1}\epsilon_{ii_2\cdots i_r} M^1_{i_2}\cdots M^{j-1}_{i_j} M^{j+1}_{i_{j+1}}\cdots M^r_{i_r}(det_q(M))^{-1}.$$

Hence by (4.34)

$$(-1)^{j-1}\tau_j = det_q(M(j))(det_q(M(0)))^{-1},$$

where

$$det_q(M(j)) = \epsilon_{i_1\cdots i_r} M^0_{i_1} M^1_{i_2}\cdots M^{j-1}_{i_j} M^{j+1}_{i_{j+1}}\cdots M^r_{i_r}$$

(so that $det_q(M(0)) = det_q(M)$).

Since this solution of τ is independent of the choice of K, by choosing another set K' we have another matrix M' and $(-1)^{j-1}\tau_j = det_q(M'(j))(det_q(M'(0)))^{-1}$. Therefore we get the coplanarity condition

$$det_q(M(j))(det_q(M(0)))^{-1} = det_q(M'(j))(det_q(M'(0)))^{-1} \tag{4.38}$$

for all $j = 1,\cdots,r$ and any two sets of indices K and K'. This is obviously equivalent to

$$det_q(M(j))(det_q(M(k)))^{-1} = det_q(M'(j))(det_q(M'(k)))^{-1} \tag{4.39}$$

for all $j,k = 0,\cdots,r$.

If $N \geq 2r$ then one can choose $K < K'$, i.e., any element in K is smaller than any element in K', then one can show that

$$det_q(M(0))det_q(M'(0)) = q^r det_q(M'(0))det_q(M(0))$$

and a polynomial type of coplanarity condition is available:

$$det_q(M(j))det_q(M'(k)) = q^r det_q(M'(j))det_q(M(k)). \tag{4.40}$$

The algebra of functions of $r+1$ coplanar points is therefore the quotient of the algebra generated by $\{z^A\}^r_{A=0}$ over the ideal generated by (4.40).

4.4 Other Invariants

The anharmonic ratios are important because they are the building blocks of invariants in classical projective geometry. For example, in the N-dimensional classical case for given $2(N+1)$ points with homogeneous coordinates $\{x^A_i\}$, inhomogeneous

coordinates $\{z^A_a\}$ where $A = 1,\cdots,2(N+1)$, $i = 0,1,\cdots,N$ and $a = 1,\cdots,N$, we can construct an invariant

$$I = \frac{det(x^1,x^2,\cdots,x^N,x^{N+1})det(x^{N+2},x^{N+3},\cdots,x^{2(N+1)})}{det(x^1,x^2,\cdots,x^N,x^{N+2})det(x^{N+1},x^{N+3},\cdots,x^{2(N+1)})}, \tag{4.41}$$

where $det(x^{A_0},\cdots,x^{A_N})$ is the determinant of the matrix $M^i_j = x^{A_i}_j$, $i,j = 0,\cdots,N$, which equals the determinant of the matrix

$$\begin{pmatrix} 1 & \cdots & 1 \\ z^{A_0}_1 & \cdots & z^{A_N}_1 \\ \vdots & \ddots & \vdots \\ z^{A_0}_N & \cdots & z^{A_N}_N \end{pmatrix} \tag{4.42}$$

multiplied by the factor $x^{A_0}_0\cdots x^{A_N}_0$, which cancels between the numerator and denominator of I. It can be shown that this invariant I is in fact the anharmonic ratio of four points z, z', z^{N+1}, z^{N+2}, where z (z') is the intersection of the line fixed by z^{N+1}, z^{N+2} with the $(N-1)$-dimensional subspace fixed by $z^1,\cdots,z^N$ ($z^{N+3},\cdots,z^{2(N+1)}$).

For the case of $N = 2$ (see Fig.4.1), I is the ratio of the areas of four triangles:

$$I = \frac{\Delta_{123}\Delta_{456}}{\Delta_{124}\Delta_{356}}, \tag{4.43}$$

which is easily found to be

$$I = \frac{\overline{A3}\,\overline{B4}}{\overline{A4}\,\overline{B3}}, \tag{4.44}$$

the anharmonic ratio of the four points $A, B, 3, 4$.

It is remarkable that all this can also be done in the quantum case. We have already formulated the condition for N points to share an $(N-1)$-dimensional subspace and we will construct an invariant I_q using the quantum determinant. Furthermore, we will show how to describe the intersection between subspaces of arbitrary dimension spanned by given points. It can be shown that the invariant I_q is indeed an anharmonic ratio in the same sense as the classical case. These are the topics of this section.

First, the intersection of two hyperplanes is represented by

$$x^A = \sum_{B\in I}\mu^A_B x^B = \sum_{C\in I'}\nu^A_C x^C,$$

where the ordering is $I < A < I'$. For x^A to have the commutation relation (3.1), we let

$$\mu^A_B\nu^A_C = q^{3\alpha-1}\nu^A_C\mu^A_B$$

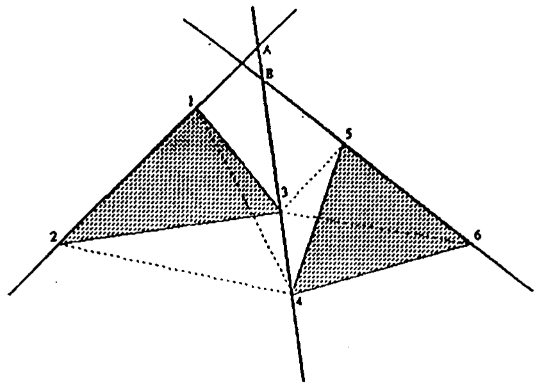

Figure 4.1: The invariant I as a cross ratio of $A, B, 3, 4$.

for $B \in I$ and $C \in I'$.

The quantum determinant of the matrix $(x_j^{A_i})_{i,j=0}^{N}$, where there are $N+1$ vectors labelled by $A_0, \cdots, A_N$, is defined by

$$det_q(x^{A_0}, \cdots, x^{A_N}) = \epsilon_{i_0 i_1 \cdots i_N} x_{i_0}^{A_0} x_{i_1}^{A_1} \cdots x_{i_N}^{A_N}.$$

Under a simultaneous transformation $x^{A_i} \to T x^{A_i}$ for all i, where elements of T commute with components of x, the determinant is transformed by

$$det_q(x) \to det_q(T) det_q(x).$$

If one interchanges two neighboring vectors in the matrix, since $(I + q\hat{R})^{ij}_{kl}\epsilon_{\cdots kl \cdots} = 0$, $det_q(x)$ changes by

$$det_q(\cdots, x^A, x^B, \cdots) = -q^{\pm(\alpha-1)} det_q(\cdots, x^B, x^A, \cdots)$$

for $A < B$ $(+)$ or $A > B$ $(-)$. Note that $det_q(x)$ vanishes if $A = B$.

The invariant I_q of $2N+2$ points is defined to be

$$I_q = (1, 2, \cdots, N, N+1)(1, 2, \cdots, N, N+2)^{-1} \times$$
$$\times (N+2, N+3, \cdots, 2N+1, 2N+2)(N+1, N+3, \cdots, 2N+1, 2N+2)^{-1},$$

where we denote $(A_0, \cdots, A_N) = det_q(x^{A_0}, \cdots, x^{A_N})$

Consider the line fixed by x^{N+1} and x^{N+2}. It has an intersection with the hyperplane spanned by $x^1, \cdots, x^N$ at

$$x^A = \nu_{N+1} x^{N+1} + \nu_{N+2} x^{N+2} = \sum_{i=1}^{N} \mu_i x^i,$$

$N < A < N+1$. Similarly it has an intersection with the hyperplane spanned by $x^{N+3}, \cdots, x^{2N+2}$ at

$$x^B = \mu_{N+1} x^{N+1} + \mu_{N+2} x^{N+2} = \sum_{i=N+3}^{2N+2} \nu_i x^i,$$

$N+2 < B < N+3$.

Now

$$\begin{aligned}
(1, \cdots, N+1) &= det_q(\mu_1^{-1}\mu_1 x^1, x^2, \cdots, x^{N+1}) \\
&= \mu_1^{-1} det_q(\sum_{i=1}^{N} \mu_i x^i, x^2, \cdots, x^{N+1}) \\
&= \mu_1^{-1} det_q(\nu_{N+1} x^{N+1} + \nu_{N+2} x^{N+2}, x^2, \cdots, x^{N+1}) \\
&= \mu_1^{-1} det_q(\nu_{N+2} x^{N+2}, x^2, \cdots, x^{N+1}) \\
&= \mu_1^{-1} \nu_{N+2}(N+2, 2, \cdots, N+1).
\end{aligned}$$

Similarly,

$$(1, \cdots, N, N+2) = -q^{\alpha-1}\mu_1^{-1}\nu_{N+1}(N+2, 2, \cdots, N+1),$$
$$(N+2, N+3, \cdots, 2N+2) = (-q^{\alpha-1})^N \nu_{2N+2}^{-1}\mu^{N+1}(N+1, \cdots, 2N+1)$$

and

$$(N+1, N+3, \cdots, 2N+2) = (-q^{\alpha-1})^{N-1}\nu_{2N+2}^{-1}\mu^{N+2}(N+1, \cdots, 2N+1). \quad (4.45)$$

Therefore,

$$I_q = \nu_{N+2}\nu_{N+1}^{-1}\mu_{N+1}\mu_{N+2}^{-1}.$$

Note that

$$[A(N+1)]^{-1}[A(N+2)] = -(x_0^{N+2})^{-1}\nu_{N+2}^{-1}\nu_{N+1}x_0^{N+1}$$

and similarly

$$[B(N+1)]^{-1}[B(N+2)] = -(x_0^{N+2})^{-1}\mu_{N+2}^{-1}\mu_{N+1}x_0^{N+1}.$$

Thus

$$I_q = [A(N+2)]^{-1}[A(N+1)][B(N+1)]^{-1}[B(N+2)],$$

where we have used

$$x_0^{N+1}\mu_{N+1} = q^{\alpha-1}\mu_{N+1}x_0^{N+1},$$
$$x_0^{N+1}\mu_{N+2} = q^{-2\alpha}\mu_{N+2}x_0^{N+1}$$

and the same for the ν's. Hence I_q is indeed the anharmonic ratio for the four points $A, B, (N+1), (N+2)$.

Chapter 5

Quantum Riemannian Geometry

An algebraic formulation of Riemannian geometry on quantum spaces [62] is presented, where Riemannian metric, distance, Laplacian, connection, and curvature have their counterparts if the space is equipped with a $*$-involution and a Hodge $*$ map. This description is also extended to complex manifolds. Examples include the quantum sphere, the complex quantum projective space and the two-sheeted space. In addition to the possibility of applying it to describe physics at the Planck scale, this formulation can be used for Kaluza-Klein theories to build models with the extra dimensions corresponding to quantum spaces.

5.1 Introduction

In [63] Chamseddine, Felder and Fröhlich developed the notions of Riemannian metric, connection and curvature in the framework of the non-commutative geometry of Connes [2]. In their formulation the Hilbert space representation is a prerequisite. The purpose of this chapter is to propose a purely algebraic formulation of Riemannian geometry on quantum spaces. It is hoped that it will be suitable for physicists to build physical models. The question of mathematical rigor is left for future study.

In Sec.5.2 we describe this algebraic formulation of Riemannian geometry on quantum spaces and complex quantum manifolds. It is applied to the quantum sphere S_q^2 (Chap.2) in Sec.5.3. In particular the explicit expression for the quantum distance is worked out. A comment on its relation to Connes' work is made. The complex projective space $CP_q(N)$ (Chap.3) is considered in Sec.5.4, and the two-sheeted space [63] in Sec.5.5.

5.2 Riemannian Structure on Quantum Spaces

In Sec.5.2.1 we give the convention and assumption about the differential calculus for this chapter.

5.2.1 Differential Calculus on Quantum Spaces

A quantum space is specified by a unital, associative, non-commutative $*$-algebra $\mathcal{A}$ generated by $\{1, x^a, \mathcal{I}_i\}$ over the field $k = \boldsymbol{C}$, where the x^a's are the coordinates on the quantum space, and the $\mathcal{I}_i$'s are non-commutative constants, including for example the generators of the algebra of functions of the quantum group which specifies the quantum symmetry on the quantum space. $\mathcal{A}$ is called the algebra of functions on the quantum space.

To talk about differential geometry on the quantum space, we should have $\mathcal{A}$ extended to the differential calculus $\Omega(\mathcal{A})$ generated by $\{1, x^a, \xi^a, \chi_a, \mathcal{I}_i\}$, where the commutation relations among the generators are given so that one knows how to rewrite a product of elements in $\Omega(\mathcal{A})$ in any preferred order of the elements. (But the commutation relations between ξ^a and χ_b are not necessary.) The ξ^a's are differential one-forms and the χ_a's are the derivations dual to them so that the exterior derivative is $d = \xi^a\chi_a$ when acting on functions. The $*$-involution on functions is also extended to all elements in $\Omega(\mathcal{A})$ and it always reverses the ordering of a product. All constants, namely the unity or $\mathcal{I}_i$'s, should vanish under d.

All the commutation relations should never mix terms of differential forms of different degrees so that $\Omega(\mathcal{A})$ is graded. The action of d is $(da) = [d, a] = da - ad$ for even a (including elements in $\mathcal{A}$) and $(da) = \{d, a\} = da + ad$ for odd a. The Leibniz rule follows from this definition. We take the convention that $(da)^* = d(a^*)$ for even a and $(da)^* = -d(a^*)$ for odd a. That is, d is anti-self-conjugate: $d^* = -d$. We also require the nilpotency of d, namely, $dd \equiv 0$.

5.2.2 Riemannian Metric, Vector Fields and Tensor Fields

A general coordinate transformation is specified by $x^a \to x'^a$, where $x'^a = x'^a(x)$ are elements in $\mathcal{A}$. (Einstein's summation convention applies to the whole chapter unless otherwise stated.) This transformation induces the transformation of ξ^a. For example, if

$$\xi^a = \sum f_i^a(x) dg_i^a(x), \tag{5.1}$$

then

$$\xi^a \to \xi'^a = \sum f_i^a(x') dg_i^a(x'),$$

where 'a' is not summed over. Re-expressing ξ'^a in terms of ξ^a and using commutation relations between x^a and ξ^a one can re-write the formula above as

$$\xi^a \to \xi'^a = \xi^b M_b{}^a(x)$$

for a certain matrix $M_b{}^a$ of elements in $\mathcal{A}$.

Since the transformation is not supposed to change the exterior derivative, i.e., $d = \xi^a\chi_a \to d' = \xi'^a\chi'_a = d$, so $\chi_a \to M^{-1}{}_a{}^b\chi_b$.

A Riemannian metric $g^{ab}(x)$ is an invertible matrix of elements in $\mathcal{A}$ which transforms like a rank-two tensor (to be defined later):

$$g^{ab} \to g'^{ab} = M_c^{*a} g^{cd} M_d{}^b,$$

where $M_c^{*a} = (M_c^a)^*$, and is also Hermitian-symmetric:

$$(g^{ab})^* = g^{ba}. \tag{5.2}$$

Note that this symmetry is preserved by the transformation. In the classical case there is no need of the $*$-involution (complex conjugation) in (5.2) if all coordinates are real. But if one is allowed to use complex coordinates, for example, $(x, y) \to (x + iy, x - iy)$, then (5.2) is a reality condition for the Riemannian manifold. The existence of the inverse of g^{ab} is assumed and it is denoted as g_{ab} so that $g^{ab}g_{bc} = \delta_c^a = g_{cb}g^{ba}$. The transformation of g_{ab} follows this definition.

A covariant vector field is a set of elements $\{\alpha_a\}$ in $\mathcal{A}$ which transform like $\{\chi_a\}$:

$$\alpha_a \to M_a^{-1b}\alpha_b.$$

Similarly, $\{\beta^a\}$ is called a contra-variant vector field if it transforms like $\{\xi^a\}$:

$$\beta^a \to \beta^b M_b{}^a.$$

Note that $\alpha = \xi^a\alpha_a$ and $\beta = \beta^a\chi_a$ are both invariant: $\alpha \to \alpha$, $\beta \to \beta$, so that we can simply use α, β to denote the vector fields $\{\alpha_a\}$ and $\{\beta^a\}$ in a coordinate-independent way.

Similarly we can define rank-two tensors of different types according to their transformations:

$$\alpha^{ab} \to M_c^{*a}\alpha^{cd}M_d{}^b,$$
$$\alpha_a{}^b \to M_a^{-1c}\alpha_c{}^d M_d{}^b,$$
$$\alpha^a{}_b \to M_c^{*a}\alpha^c{}_d(M_b^{-1d})^*,$$
$$\alpha_{ab} \to M_a^{-1c}\alpha_{cd}(M_b^{-1d})^*,$$

where $M_c^{*a} = (M_c{}^a)^*$. Just like in the commutative case, the positions of indices of a tensor tell you the way it transforms. In these formulas, the ordinary contraction of indices is of this type: $\searrow$, When indices are contracted as $\nearrow$, $*$-involution is involved.

Furthermore the Riemannian metric g^{ab} can be used to raise or lower indices:

$$\alpha^a = \alpha_b^* g^{ba}, \tag{5.3}$$

$$\alpha_a = g_{ab}(\alpha^b)^*, \tag{5.4}$$

$$\alpha^{ab} = g^{ac}\alpha_c{}^b, \tag{5.5}$$

$$\alpha_{ab} = \alpha_a{}^c g_{cb}, \tag{5.6}$$

and so on. Because of eq.(5.2), if we raise and then lower an index we will get back to the original object. It can also be checked that $(\alpha^a)^*\beta^b$ and $\alpha_a\beta^b$ are tensors if α and β are vectors. The contraction of indices of two tensors can make a new tensor. But sometimes the contraction of indices has to be accompanied by appropriate $*$-involution.

As $\alpha^a\beta_a$ is invariant for any two vector fields α and β, we can always use the Riemannian metric to define the inner product $\langle\cdot,\cdot\rangle$ between vector fields α, β as in the classical case. For example,

$$\langle\alpha,\beta\rangle = \langle\xi^a\alpha_a,\xi^b\beta_b\rangle = \alpha_a^*\langle\xi^a,\xi^b\rangle\beta_b = \alpha_a^* g^{ab}\beta_b,$$

$$\langle\alpha,\beta\rangle = \langle\alpha^a\chi_a,\beta^b\chi_b\rangle = \alpha^a\langle\chi_a,\chi_b\rangle\beta^{b*} = \alpha^a g_{ab}\beta^{b*}.$$

In both cases, $\langle\alpha,\beta\rangle = \langle\beta,\alpha\rangle^*$.

The magnitude of a vector $|\alpha|^2 = \langle\alpha,\alpha\rangle$ is real: $|\alpha|^{2*} = |\alpha|^2$, due to eq.(5.2).

The invariant operator

$$\nabla^2 = \chi^a\chi_a = \chi_a^* g^{ab}\chi_b \tag{5.7}$$

is called the Laplacian. It can be used to define the equation of motion for a scalar field Φ with mass m as

$$(\nabla^2 + m^2)\Phi = 0.$$

The non-commutativity forbids any tensor of rank higher than two. Therefore physical laws, if written as equations of motion, can only be written in terms of scalars, vectors, and rank-two tensors. Fortunately all major classical physical laws are governed by tensorial equations of rank less than or equal to two. On the other hand if the physical laws are written in terms of differential forms, there is no limit to the degree of the forms.

Given a Hilbert space representation of the algebra $\mathcal{A}$ on $\mathcal{H}$, one can define the "distance" between two states s, s' (as generalized points) as [2]

$$D(s,s') = sup\{|s(f) - s'(f)| : \left\||df|^2\right\| \leq 1, f \in \mathcal{A}\}. \tag{5.8}$$

The definition of $|df|^2$ is based on the metric g^{ab} and therefore the metric possesses the classical geometrical meaning. The norm $\|\cdot\|$ is defined by

$$\|f\|^2 = sup\{\frac{\langle\phi|f^*f|\phi\rangle}{\langle\phi|\phi\rangle} : |\phi\rangle \in \mathcal{H}\}.$$

An algebraic version of quantum distance can also be given without mentioning Hilbert space representations. An example is given in Sec.5.3.

5.2.3 Connection

As defined in [63], a connection ∇ acts on $f \in \mathcal{A}$ as d: $\nabla f = df$, on ξ^a by the connection one-forms $\omega_b{}^a$:

$$\nabla\xi^a = \xi^b \otimes_{\mathcal{A}} \omega_b{}^a \tag{5.9}$$

and on a one-form $\alpha = \xi^a\alpha_a$ by the Leibniz rule:

$$\begin{aligned}\nabla\alpha &= (\nabla\xi^a)\alpha_a - \xi^a \otimes_{\mathcal{A}} (d\alpha_a)\\ &= -\xi^a \otimes_{\mathcal{A}} (\nabla\alpha)_a,\end{aligned}$$

where $(\nabla\alpha)_a = d\alpha_a - \omega_a{}^b\alpha_b$. The tensor product $\otimes_{\mathcal{A}}$ is defined so that $\alpha f \otimes_{\mathcal{A}} \beta = \alpha \otimes_{\mathcal{A}} f\beta$ if $f \in \mathcal{A}$.

For $(\nabla\alpha)_a$ to be a covariant vector the connection one-form has the transformation

$$\omega_a{}^b \rightarrow M_a^{-1c}\omega_c{}^d M_d{}^b - M_a^{-1c} dM_c{}^b. \tag{5.10}$$

For the Leibniz rule to hold on the inner product:

$$d\langle\alpha,\beta\rangle = \nabla\langle\alpha,\beta\rangle = (\nabla\alpha)^a\beta_a + \alpha^a(\nabla\beta)_a,$$

we define

$$(\nabla\alpha)^a = d\alpha^a + \alpha^b\omega_b{}^a,$$

which also ensures that $(\nabla\alpha)^a$ is a contra-variant vector.

The covariant derivation of α in the direction of the vector field β is

$$(\nabla_\beta\alpha)^a = \langle\beta,(\nabla\alpha)^a\rangle.$$

The equation of geodesic flows is therefore

$$(\nabla_\alpha \alpha)^a = 0.$$

To define the action of ∇ on rank-two tensor fields we consider the scalar $f = \alpha_a^* \gamma^{ab} \beta_b$. Because f is a scalar field, we have the equation: $\nabla f = df$. By the undeformed Leibniz rules of ∇, we should have for the left hand side (omitting the symbols $\otimes_{\mathcal{A}}$)

$$\begin{aligned}\nabla(\alpha_a^* \gamma^{ab} \beta_b) &= (\nabla\alpha)_a^* \gamma^{ab}\beta_b + \alpha_a^*(\nabla\gamma)^{ab}\beta_b + \alpha_a^*\gamma^{ab}(\nabla\beta)_b \\ &= (d\alpha_a - \omega_a{}^c\alpha_c)^*\gamma^{ab}\beta_b + \alpha_a^*(\nabla\gamma)^{ab}\beta_b + \alpha_a^*\gamma^{ab}(d\beta_b - \omega_b{}^c\beta_c),\end{aligned}$$

and for the right hand side

$$d(\alpha_a^*\gamma^{ab}\beta_b) = d(\alpha_a^*)\gamma^{ab}\beta_b + \alpha_a^* d\gamma^{ab}\beta_b + \alpha_a^*\gamma^{ab}d\beta_b.$$

Identifying them we find

$$(\nabla\gamma)^{ab} = d\gamma^{ab} + \gamma^{ac}\omega_c{}^b + (\omega_c{}^a)^*\gamma^{cb},$$

which also ensures $(\nabla\gamma)^{ab}$ to be a rank-two tensor.

Suppose one has the physical law $(\nabla\alpha)^a = \beta^a$, it is equivalent to $(\nabla\alpha)_a = \beta_a$ if

$$(\nabla g)^{ab} = dg^{ab} + \omega^{ab} + (\omega^{ba})^* = 0, \tag{5.11}$$

which is called the *metricity condition*.

If $dg^{ab} = 0$, we have from the metricity condition $(\omega^{ab})^* = -\omega^{ba}$, where $\omega^{ab} = g^{ac}\omega_c{}^b$.

The torsion T^a is the covariant vector defined by [63]

$$T^a = (d - m \circ \nabla)\xi^a = d\xi^a - \xi^b\omega_b{}^a,$$

where m is the multiplication map $m(\alpha \otimes \beta) = \alpha\beta$.

In the classical case Eq.(5.11) and

$$T^a = 0 \tag{5.12}$$

plus the reality conditions imply that the connection one-form $\omega_a{}^b$ is uniquely fixed by the metric g^{ab}. The general expression for the analogous reality condition in the quantum case is so far unknown. The difficulty is that the general transformation will spoil the reality of a non-invariant quantity. One has to invent appropriate conditions for each particular case according to its algebraic properties.

For the quantum complex Hermitian manifold defined later, the situation is much simpler. Just like its classical counterpart, Eq.(5.11) alone determines the connection uniquely.

5.2.4 Curvature

The curvature two-form is a rank-two tensor defined by

$$R_a{}^b = d\omega_a{}^b - \omega_a{}^c\omega_c{}^b. \tag{5.13}$$

Using (5.11), one can show that [1]

$$(R_{ab})^* = R_{ba}.$$

It is easy to check that the Bianchi identity and consistency condition are satisfied:

$$\begin{aligned}dR_a{}^b - \omega_a{}^c R_c{}^b + R_a{}^c\omega_c{}^b = 0,\\ dT^a = \xi^b R_b{}^a.\end{aligned}$$

Classically, in order to have the scalar curvature and Ricci tensor one usually just strips the differential forms from the curvature two-form $R_a{}^b$ to get $R_a{}^b{}_{cd}$ and then contracts b, d for Ricci tensor, and contracts in addition a, c for the scalar curvature. However in the deformed case this kind of operation is not covariant under general transformations.

Another more elegant way of defining the classical scalar curvature and the Ricci tensor is to use the Hodge $*$. In the quantum case, the Hodge $*$ is required to satisfy

$$*(f\alpha g) = f(*\alpha)g, \quad \forall f, g \in \mathcal{A}, \alpha \in \Omega(\mathcal{A}) \tag{5.14}$$

and

$$(*\alpha)^* = *(\alpha^*), \quad \forall \alpha \in \Omega(\mathcal{A}), \tag{5.15}$$

so that the scalar curvature defined as

$$\mathcal{R} = (-1)^{D+1} * (\xi^a(*R_a{}^b)\xi_b) \tag{5.16}$$

is invariant under general transformations (D is the dimension of the space) and real ($\mathcal{R}^* = \mathcal{R}$). The integral

$$\int \xi^a(*R_a{}^b)\xi_b \tag{5.17}$$

is a candidate for the action of a gravitational theory on the quantum space.

[1] Note that the $*$-involution reverses the order of everything in a product, including one-forms, which may cause sign changes. In the classical case since R_{ab} is a two-form, $(R_{ab})^* = -\overline{R_{ab}} = R_{ba}$, where $\overline{R_{ab}}$ is the complex conjugate of R_{ab}.

Similarly one can try to define the Ricci tensor as

$$\mathcal{R}_a{}^b = *((*R_a{}^c)\xi_c\xi^b).$$

There are, however, many other inequivalent expressions that are covariant under the general transformations. For example, it is equally justified to define the Ricci tensor as

$$\mathcal{R}_a{}^b = *(\xi_a\xi^c(*R_c{}^b)).$$

This ambiguity in the Ricci tensor makes the scalar curvature better suited for physical applications.

One can define the operator $\delta = -*d*$ for a quantum space, and naturally one will define the Laplacian by $\nabla^2 = -(d+\delta)^2$, which is equivalent to $-\delta d$ when acting only on functions if $\delta\delta = 0$. In such cases the metric is determined by the Hodge $*$ according to (5.7).

5.2.5 Complex Manifolds

We define complex quantum manifolds (more precisely, Hermitian manifolds) to be an associative $*$-algebra $\mathcal{A}$ generated by $\{1, z^a, \bar{z}^{\bar{a}}\}$ together with its differential calculus $\Omega(\mathcal{A})$ generated by $\{1, z^a, \bar{z}^{\bar{a}}, dz^a, d\bar{z}^{\bar{a}}, \partial_a, \bar{\partial}_{\bar{a}}\}$ with the following properties:

1. $d = \delta + \bar{\delta}$, where $\delta = dz^a\partial_a$ and $\bar{\delta} = \bar{\partial}_{\bar{a}}d\bar{z}^{\bar{a}}$ [2] (when acting on functions) with $\delta\delta = \bar{\delta}\bar{\delta} = 0$ and $\delta\bar{\delta} = -\bar{\delta}\delta$. δ and $\bar{\delta}$ should observe the Leibniz rule separately, and $(\delta\alpha)^* = (-1)^p\bar{\delta}(\alpha^*)$ for any form α of degree p.

2. The generators of the algebra are divided into the holomorphic part $\{z^a\}$ and the anti-holomorphic part $\{\bar{z}^{\bar{a}} = (z^a)^*\}$. The one-forms $\{dz^a\}$ and $\{d\bar{z}^{\bar{a}}\}$ are all independent.

3. Denote $\Omega^+(\mathcal{A}) = \{dz^a\alpha_a : \alpha_a \in \mathcal{A}\}$ and $\Omega^-(\mathcal{A}) = \{\alpha_{\bar{a}}d\bar{z}^{\bar{a}} : \alpha_{\bar{a}} \in \mathcal{A}\}$. The commutation relations between one-forms and functions in $\mathcal{A}$ satisfy $\Omega^\pm(\mathcal{A})\mathcal{A} = \mathcal{A}\Omega^\pm(\mathcal{A})$. That is, $\Omega^\pm(\mathcal{A})$ do not get mixed by the commutation relations.

4. The metric $g^{\bar{a}b} = (g^{\bar{b}a})^*$ is given. The connection one-forms $\omega_a{}^b$ and $\omega_{\bar{a}}{}^{\bar{b}} = (\omega_a{}^b)^*$ are assumed to be $\omega_a{}^b \in \Omega^+(\mathcal{A})$ and $\omega_{\bar{a}}{}^{\bar{b}} \in \Omega^-(\mathcal{A})$. The Leibniz rule holds for the connection ∇ and we have $\nabla dz^a = dz^b \otimes_{\mathcal{A}} \omega_b{}^a$ and $\nabla d\bar{z}^{\bar{a}} = -d\bar{z}^{\bar{b}} \otimes_{\mathcal{A}} \omega_{\bar{b}}{}^{\bar{a}}$.

[2] Obviously here δ is the holomorphic part of d rather than $-*d*$ defined in the previous section.

The coordinate transformations are restricted to the holomorphic transformations only. Holomorphic transformations are defined as those which map z^a to holomorphic functions $f^a(z)$ and $\bar{z}^{\bar{a}} = (z^a)^*$ to $(f^a(z))^*$. The properties of a complex manifold imply that this transformation induces the map $dz^a \to dz^b M_b{}^a$ where $M_a{}^b$ is holomorphic, namely, $\bar{\delta}M_a{}^b = 0$. Similarly, we have $d\bar{z}^{\bar{a}} \to (dz^b M_b{}^a)^* = (M_a{}^b)^* d\bar{z}^{\bar{b}}$ where $(M_a{}^b)^*$ is anti-holomorphic, $\delta(M_a{}^b)^* = 0$.

All the formulas we had before for Riemannian manifolds can be easily modified for a complex manifold with the understanding that the indices are only summed over the holomorphic or anti-holomorphic part.

With the fourth property of a complex manifold, Eq.(5.11) says

$$dg^{\bar{a}b} + \omega_{\bar{c}}{}^{\bar{a}}g^{\bar{c}b} + g^{\bar{a}c}\omega_c{}^b = 0.$$

Since $d = \delta + \bar{\delta}$ we can separate the equation into $\Omega^+(\mathcal{A})$ and $\Omega^-(\mathcal{A})$, and so the connection can be directly solved:

$$\omega_a{}^b = -g_{a\bar{c}}(\delta g^{\bar{c}b}). \tag{5.18}$$

Only the holomorphic transformations will be consistent with this solution. (That is, the transformation of the connection induced from the transformation of the metric by these expressions will be the same as (5.10) only for holomorphic transformations.)

From Eq.(5.13) the curvature two-form can now be expressed directly in terms of the metric as

$$R_a{}^b = \bar{\delta}\omega_a{}^b = -\bar{\delta}(g_{a\bar{c}}\delta g^{\bar{c}b}). \tag{5.19}$$

The curvature two-form gives the scalar curvature according to (5.16) with the indices restricted to the holomorphic or anti-holomorphic part. The scalar curvature for a complex quantum manifold is

$$\mathcal{R} = (-1)^{D+1} * (dz^a(*R_{ab})d\bar{z}^{\bar{b}}). \tag{5.20}$$

The definition of the Ricci tensor is not unique.

The condition (5.14) for the Hodge $*$ can be weakened to be no more than an ordering prescription:

$$*(f(z)\alpha g(\bar{z})) = f(z)(*\alpha)g(\bar{z}).$$

Due to the extensive use of the $*$-involution in the quantum case, we see that the complex structure helps to admit a Riemannian structure.

5.3 Quantum Sphere S_q^2

The Riemannian structure on a quantum space with a quantum group symmetry can be required to respect its quantum symmetry.

In this section we consider the quantum sphere described in Chap.2 as an example of both Riemannian manifolds and complex manifolds.

5.3.1 S_q^2 as a Complex Manifold

In this subsection we will treat S_q^2 as a complex quantum manifold.

Metric And Connection

The notation itself suggests that we take $\Omega^+(X) = Xdz$ and $\Omega^-(X) = Xd\bar{z}$, where X denotes the algebra of functions on S_q^2. Let the only possible value for an index be 0, that is, $z^0 = z$ and $\bar{z}^{\bar{0}} = \bar{z}$. Also denote $g = g^{\bar{0}0}$ and $g^{-1} = g_{0\bar{0}}$. To define the metric g we note that for S_q^2 the Laplacian, the $SU_q(2)$-invariant differential operator of order two, is

$$\nabla^2 = -c(1+\bar{z}z)^2\bar{\partial}\partial, \tag{5.21}$$

where c is an arbitrary real number. On the other hand, the Laplacian of S_q^2 as a complex manifold, the holomorphic-transformation independent differential operator of order two, is $\partial^* g\partial$. Equating these two expressions one gets

$$g = q^2c(1+\bar{z}z)^2.$$

Because the factor $(1+\bar{z}z)$ will appear frequently, we shall denote it in the following by

$$\rho = 1+\bar{z}z.$$

Classically any two-dimensional complex manifold is also a Kähler manifold and one can locally find a Kähler potential. Analogy can be made here. Define the Kähler form to be $K = dzg^{-1}d\bar{z}$. (It plays a special role in the differential calculus as described in Chap.2.) Obviously $dK = 0$ for the same reason as in the classical case. The Kähler potential V defined by $\delta\bar{\delta}V = K$ therefore exists. One can solve for V in terms of the q-deformed log:

$$\begin{aligned} V &= q^{-4}c^{-1}\sum_{n=0}^{\infty}\log_{q^{-1}}(1-\rho^{-1}) \\ &= q^{-4}c^{-1}\sum_{n=0}^{\infty}\frac{\rho^{-(n+1)}}{[n+1]_{q^{-1}}}, \end{aligned}$$

where $[n]_q = \frac{q^{2n}-1}{q^2-1}$. In fact, the Kähler form $dzg^{-1}d\bar{z}$ is also the volume form (up to normalization).

Using (5.18) and (5.19), we can immediately find the connection form and the curvature two-form:

$$\omega_0{}^0 = -q^2(1+q^2)dz\rho^{-1}\bar{z}, \tag{5.22}$$

$$R_0{}^0 = q^4(1+q^2)dzd\bar{z}\rho^{-2}. \tag{5.23}$$

It is easy to see that the torsion is zero in this case.

We define the vielbeins on S_q^2 as

$$e = \rho^{-1}dz, \quad \bar{e} = \rho^{-1}d\bar{z}. \tag{5.24}$$

The commutation relations between $e, \bar{e}$ and $z, \bar{z}$ are simply classical:

$$ez = ze, \quad e\bar{z} = \bar{z}e,$$

$$\bar{e}z = z\bar{e}, \quad \bar{e}\bar{z} = \bar{z}\bar{e}.$$

The Hodge $*$ map satisfying (5.14), (5.15) and $*^2 = \pm$ as in the classical case is given by

$$\begin{aligned} (*e) &= ie, \quad (*\bar{e}) = -i\bar{e}, \\ (*1) &= ic'^{-1}e\bar{e}, \quad (*e\bar{e}) = -ic', \end{aligned} \tag{5.25}$$

where c' is a constant, and the action of Hodge $*$ on any form follows (5.14).

Now we can define $\delta = -*d*$ and $\nabla^2 = -\frac{1}{2}(d+\delta)^2$. [3] When acting on a function f,

$$\begin{aligned} \nabla^2 f &= \frac{1}{2}*d*df \\ &= -c'\rho^2\bar{\partial}\partial f. \end{aligned}$$

Hence c' should be identified with c because of (5.21). This identification further justifies our Hodge $*$ structure.

The scalar curvature is found to be

$$\mathcal{R} = cq^2(1+q^2). \tag{5.26}$$

The Ricci tensor defined by (5.2.4) is

$$\mathcal{R}_0{}^0 = cq^4(1+q^2). \tag{5.27}$$

[3] Here we have a factor of $\frac{1}{2}$ because we want to identify ∇^2 with (5.21), which is only the holomorphic part in $-(d+\delta)^2$.

5.3.2 S_q^2 As a Riemannian Manifold

In this section we treat S_q^2 as a (real) Riemannian manifold. The difference between Riemannian and complex manifolds is that the latter is not invariant under general coordinate transformations. When indices are contracted for a complex manifold they are summed over only half (the holomorphic part) of the possible values if viewed as a Riemannian manifold.

Assuming that $g^{00} = g^{\bar{0}\bar{0}}$ in the Riemannian case (since we had $g^{\bar{0}0} = g^{0\bar{0}}$ in the complex case), we get from (5.21)

$$g^{00} = g^{\bar{0}\bar{0}} = \frac{c}{1+q^{-2}}\rho^2.$$

Note that since the normalization of g is changed from Sec.(5.3.1), the parameter c' used in (5.25) should be changed accordingly.

The equation $\nabla g^{00} = 0$ is identical to the one solved earlier for S_q^2 as a complex manifold and we assign ${\omega_0}^0$ to be the same as (5.22). Let

$${\omega_{\bar{0}}}^{\bar{0}} = -(1+q^{-2})d\bar{z}\rho^{-1}z,$$

which would be the complex manifold connection form ${\omega_0}^0$ had we labelled the coordinates the opposite way: $\bar{z}^0 = z$, and $z^0 = \bar{z}$. It satisfies $\nabla g^{\bar{0}\bar{0}} = 0$.

Similarly, the curvature two-form ${R_0}^0$ is given by (5.23) and ${R_{\bar{0}}}^{\bar{0}}$ is

$${R_{\bar{0}}}^{\bar{0}} = \delta {\omega_{\bar{0}}}^{\bar{0}} = (1+q^{-2})d\bar{z}dz\rho^{-2}.$$

A straightforward calculation shows

$$\begin{aligned}
{\mathcal{R}_0}^0 &= \tfrac{cq^4}{4}(1+q^2)^2, \\
{\mathcal{R}_{\bar{0}}}^{\bar{0}} &= \tfrac{cq^{-2}}{4}(1+q^2)^2, \\
\mathcal{R} &= \tfrac{c}{4}(1+q^2)^3.
\end{aligned}$$

This is different from (5.26) and (5.27) by constant factors. The reason is that the Riemannian structure of S_q^2 as a Riemannian manifold is concerned with the general transformations and that of S_q^2 as a complex manifold is concerned only with the holomorphic transformations. Unlike the situation in the classical case, with only the holomorphic description of the complex manifold S_q^2 one will never be able to know its Riemannian structure due to the non-commutativity of the algebra.

Distances on S_q^2

Now we consider the "distance" between "points" on S_q^2. As mentioned earlier. (5.8) can be used to define a number called the *distance* between any two states.

Before finding the distance we display a representation of S_q^2 which shows clearly the correspondence between states and points. The basis of the Hilbert space is labelled as $|k,\theta\rangle$ for $k = 0,1,2,\cdots,\infty$ and $\theta \in [0,2\pi)$. (It is an irreducible representation given in [39] for S_q^2 supplemented with an additional label θ. It is also equivalent by Fourier transform to an irreducible representation given in [32] for $SU_q(2)$.) The algebra is represented in the following way:

$$\begin{aligned}
z|k,\theta\rangle &= e^{i\theta}(q^{-2k}-1)^{1/2}|k-1,\theta\rangle, \\
\bar{z}|k,\theta\rangle &= e^{-i\theta}(q^{-2(k+1)}-1)^{1/2}|k+1,\theta\rangle.
\end{aligned}$$

So we have

$$\rho|k,\theta\rangle = q^{-2k}|k,\theta\rangle.$$

Roughly speaking, θ corresponds to the azimuthal angel on S_q^2 and $\frac{q^{2k}-1}{q^{2k}+1}$ corresponds to the cosine of the polar angle.

However, for what follows it is not necessary to specify the representation. The only thing we need is $Sp(\rho)$, the spectrum of ρ, which follows the commutation relations

$$z\rho = q^{-2}\rho z, \quad \bar{z}\rho = q^2\rho\bar{z}$$

and $\rho = 1 + \bar{z}z \geq 1$. It is easy to see that $Sp(\rho) = \{q^{-2k} : k = 0,1,2,\cdots\}$. Hence in the following θ is interpreted as the collection of all parameters except k, which labels the eigenvalue of ρ.

Now we consider the distance between the two states $|k,\theta\rangle$ and $|k',\theta\rangle$. In the classical limit, it is just the radius ($\frac{1}{2}$) times the difference in their polar angles:

$$D(p,p') = |F(z(p),\bar{z}(p)) - F(z(p'),\bar{z}(p'))|,$$

where $F(z,\bar{z}) = \frac{1}{2}\cos^{-1}\left(\frac{z\bar{z}-1}{z\bar{z}+1}\right) = \sin^{-1}(\rho^{-1/2})$. For convenience we shall suppress the index θ of a state from now on. It is fixed for all considerations below.

Given the distance function F, we can always decompose F as $F = f(\rho)+h(z,\bar{z})$, where $h = \sum_{n=1}^{\infty} f_n(\rho)z^n + g_n(\rho)\bar{z}^n$. Since $\langle k|h|k\rangle = 0$ for all k, if F gives the distance between states $|k\rangle$ and $|k'\rangle$ then f does, too. But we have to check that the magnitude of df is not larger than 1. Note that

$$|dF|^2 = |df|^2 + |dh|^2 + (CT).$$

where the cross-terms are

$$(CT) = (\partial f)^* g(\partial h) + (\bar{\partial} f)^* g(\bar{\partial} h) + (\partial h)^* g(\partial f) + (\bar{\partial} h)^* g(\bar{\partial} f).$$

Since $\langle p|(CT)|p\rangle = 0$, we have

$$\langle p|\,|df|^2\,|p\rangle \leq \langle p|\,|dF|^2\,|p\rangle \leq \left\||dF|^2\right\| \leq 1, \quad \forall p,$$

which implies that $\||df|^2\| \leq 1$. Hence we only have to consider functions of ρ for our purpose.

Assume that the distance function is $F(\rho)$ with $|dF|^2 = 1$. Because $(\partial\rho^n) = q\lambda^{-1}\rho^{-1}((q^2\rho)^n - \rho^n)\bar{z}$, where $\lambda = q - q^{-1}$, we have

$$(\partial F(\rho)) = q\lambda^{-1}\rho^{-1}(F(q^2\rho) - F(\rho))\bar{z}. \tag{5.28}$$

Similarly,

$$(\bar{\partial} F(\rho)) = -q\lambda^{-1}\rho^{-1}(F(q^{-2}\rho) - F(\rho))z. \tag{5.29}$$

Therefore,

$$\begin{aligned} |dF|^2 &= (\partial F)^* g^{00}(\partial F) + (\bar{\partial} F)^* g^{\bar{0}\bar{0}}(\bar{\partial} F) \\ &= \frac{cq^4}{\lambda^2(1+q^{-2})}(W(q^{-2}\rho) + W(\rho)), \end{aligned}$$

where $W(\rho) = (\rho - 1)(F(q^2\rho) - F(\rho))^2$. $|dF|^2 = 1$ implies that $W(\rho)$ can only be the constant $\frac{1}{2}c^{-1}q^{-4}\lambda^2(1 + q^{-2})$. Hence,

$$F(q^2\rho) - F(\rho) = -\lambda\left(\frac{2cq^4(\rho - 1)}{1 + q^{-2}}\right)^{-1/2} \tag{5.30}$$

and so F can be solved as a power series expansion:

$$F(\rho) = -\left(\frac{1+q^{-2}}{2cq^2}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2[n+1/2]_{q^{-1}}}\rho^{-n-1/2}.$$

This is not the only solution of (5.30). Any function $f(\rho)$ satisfying $f(q^2\rho) = f(\rho)$ can be added to it and (5.30) still holds. However, due to the structure of $Sp(\rho)$, such functions will not contribute to the distance between $|k\rangle$ and $|k'\rangle$. For $q = 1$, $c = 4$, this solution is the power series expansion of $-\sin^{-1}(\rho^{-1/2})$.

It remains to argue that. that the assumption $|dF|^2 = 1$ is correct. Consider a function $f(\rho)$ with $|df|^2 < 1$. Then $F' = f + \epsilon F$ has $|dF'|^2 < 1$ for $|\epsilon|$ sufficiently small. And an appropriate phase of ϵ can make $|F'(k) - F'(k')| > |f(k) - f(k')|$. So any f with $|df|^2 < 1$ is not the distance function.

Therefore we have the following lemma:

Lemma 2 *The distance (5.8) between the states $|m,\theta\rangle$ and $|n,\theta\rangle$ is equal to*

$$\left|F(q^{-2m}) - F(q^{-2n})\right|.$$

The distance between the north pole and the south pole on S_q^2, for example, can be expressed as

$$F(\infty) - F(1) = \left(\frac{1+q^{-2}}{2cq^2}\right)^{1/2}\sum_{n=0}^{\infty}\frac{(2n)!}{(2^n n!)^2[n+1/2]_{q^{-1}}},$$

which can be called the deformed $\pi/2$.

The distance between any two points can be obtained, by using the quantum group symmetry of $SU_q(2)$, from the distance between the north pole ($z = \infty$) and an arbitrary point, which we have just calculated above.

Using the commutation relations of $z, \bar{z}$, one can check that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} z\rho^{-1/2} & -q\rho^{-1/2} \\ \rho^{-1/2} & \rho^{-1/2}\bar{z} \end{pmatrix} \tag{5.31}$$

is an $SU_q(2)$-matrix. This matrix transforms the north pole ($z = \infty$) to z. It also transforms the point

$$z'' = (\delta z' - q^{-1}\beta)(-q\gamma z' + \alpha)^{-1}$$

to z'. The quantum group symmetry tells us that the distance between z and z' is the same as the distance between the north pole and z'', which is a function of z, z'. Therefore we have

Proposition 1 *The distance between $(z,\bar{z})$ and $(z',\bar{z}')$ on S_q^2 is $|F(\rho'')|$, where $\rho'' = (1 + z''\bar{z}'') = (1 + z\bar{z})(1 + z'\bar{z}')(z - z')^{-1}(\bar{z} - \bar{z}')^{-1}$.*

Note that as they are the coordinates of two points on the same sphere, the commutation relation between z and z' should be that of the standard braiding (2.92):

$$zz' = q^2 z'z - q\lambda z'^2,$$

which is covariant under simultaneous $SU_q(2)$ transformations on z and z'. This implies that z and z'' simply commute with each other. (The braiding is also formally satisfied by (∞, z''). Divide the braiding relation on both sides by z we get $z'z^{-1} = q^2 z^{-1}z' - q\lambda z^{-1}z'^2 z^{-1}$ which is satisfied by $(z, z') = (\infty, z'')$ but not by $(z, z') = (z'', \infty)$.)

A state $|s\rangle$ in the Hilbert space representation of the braided algebra generated by $\{z, \bar{z}, z', \bar{z}'\}$ corresponds to two "points" on S_q^2. So the distance between them is $\langle s|F(\rho'')|s\rangle$. This is a modification of Connes' formula (5.8) required by the braiding.

Relation to Connes' Formulation

Here we make a comment on the relation of our work to Connes' quantum Riemannian geometry [2] by re-formulating the quantum sphere in a way as close to his as possible. A sketch of Connes' calculus is in Appendix 2.

To do so we consider the Hilbert space realization of $\Omega(X)$. The Hilbert space representation presented here is composed of two parts. The first part $\{|\psi\rangle\}$ is the Hilbert space representing the algebra generated by $z, \bar{z}$. An example is given earlier in this section. Another example is the GNS construction using the integration $\langle \cdot \rangle_{S^2_q}$. The second part V is a vector space of, say, two-component column vectors representing the differential forms. The differential calculus can then be represented in terms of the representation π of S^2_q as (for $v \in V$)

$$\pi(dz)|\psi\rangle \otimes v = \sqrt{c(1+q^2)}\pi(\rho)|\psi\rangle \otimes \tau v,$$
$$\pi(d\bar{z})|\psi\rangle \otimes v = \sqrt{c(1+q^2)}\pi(\rho)|\psi\rangle \otimes \tau^\dagger v,$$

where $\tau = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $\tau^\dagger = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and they satisfy $q\tau\tau^\dagger + q^{-1}\tau^\dagger\tau = \mathcal{I}$ for $\mathcal{I} = \begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$. The vielbeins e and $\bar{e}$ (5.24) are represented by the γ-matrices τ and $\tau^\dagger$, which satisfy a deformed Clifford algebra. The column v is used to specify the direction of a cotangent vector at a "point" on S^2_q.

Let the Dirac operator be

$$\mathcal{D} = k\begin{pmatrix} i & \bar{z} \\ -z & -i \end{pmatrix},$$

where $k = q\lambda^{-1}\sqrt{c(1+q^2)}$. It is chosen such that $dz = [D, z]$ and $d\bar{z} = [D, \bar{z}]$. The goal is that the exterior derivative is realized by D.

Since D^2 is not central, $(d^2\alpha) = [D^2, \alpha]$ for a form α is non-zero. The nilpotency is achieved by taking the quotient of the algebra over the ideal called the auxiliary fields. They are the differential forms $\{a[D^2, b]c : a, b, c \in \mathcal{A}\}$. For our case the auxiliary fields are found to be

$$a\begin{pmatrix} q & 0 \\ 0 & q^{-1} \end{pmatrix}$$

for all functions a in S^2_q.

The calculus is Z_2-graded by

$$\gamma = k^{-2}(dzd\bar{z} - d\bar{z}dz)\rho^{-2} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The $SU_q(2)$-invariant integration on two-forms can be defined by the trace

$$\int \alpha = Tr(\gamma\alpha|\mathcal{D}|^{-2}), \tag{5.32}$$

where Tr is the trace over the extended Hilbert space $\{|\psi\rangle \otimes v\}$. Although this formula resembles that of Connes for 2-dimensional integration [2], according to him the power of $|\mathcal{D}|^{-1}$ in the trace is determined by the spectrum of $\mathcal{D}$. In the classical case one gets the classical dimension, but we get zero in this particular case. Therefore unlike Connes' integration, (5.32) does not have the cyclic property $\int \alpha\beta = \int \beta\alpha$. Nevertheless, it can be directly checked that the integration satisfies the consistency condition

$$\int Aux = 0 \tag{5.33}$$

for auxiliary fields, and the Stokes theorem

$$\int d\alpha = 0 \tag{5.34}$$

for any one-form α. The Stokes theorem can be used to derive recursion relations for the integration of two-forms. Equations (5.33) and (5.34) vanish already on the trace over the 2×2 matrices and hence remain so for any representation $|\psi\rangle$ of the algebra for S^2_q. They determine up to normalization the integration of two-forms. Hence it agrees with the integration introduced in Sec.2.7.

5.4 Complex Quantum Projective Space $CP_q(N)$

The results in Sec.5.3 can be generalized to the quantum projective spaces $CP_q(N)$ described in Chap.3. In Sec.5.4.1 we consider the Hodge $*$ map in a more general setting. In Sec.5.4.2 we find the Riemannian structure on $CP_q(N)$.

5.4.1 Construction of the Hodge $*$ Map

A prerequisite of the Riemannian structure is the Hodge $*$ map. In general, if there exists for a complex quantum manifold a Kähler form $K = dz^a g_{a\bar{b}} d\bar{z}^b$ which is real and central, a Hodge $*$ satisfying (5.14) and (5.15) can always be constructed if there also exists a basis of one-forms, denoted dz^a, $d\bar{z}^{\bar{b}}$ below, such that $\epsilon_{a_1\cdots a_N}$ defined by

$$dz^{a_1}\cdots dz^{a_p}=\epsilon_{a_1\cdots a_N}dz^{a_1}\cdots dz^{a_N}$$

is central.

Let

$$*(dz^{a_1}\cdots dz^{a_p})=dz^{a_1}\cdots dz^{a_p}K^{N-p},$$
$$*(d\bar{z}^{\bar{a}_r}\cdots d\bar{z}^{\bar{a}_1})=K^{N-r}d\bar{z}^{\bar{a}_r}\cdots d\bar{z}^{\bar{a}_1}.$$

Since K is central, the property (5.14) is satisfied. Since K is real, (5.15) is also satisfied. Now we consider the Hodge $*$ map of a differential form which is not purely holomorphic or antiholomorphic. The idea is to "patch" the holomorphic part and the antiholomorphic part together.

Denote

$$\xi_a=g_{a\bar{b}}d\bar{z}^{\bar{b}}.$$

Because $K=dz^a\xi_a$ is central,

$$K^p=(dz^{a_1}\cdots dz^{a_p})(\xi_{a_p}\cdots\xi_{a_1}).$$

So we have

$$*(dz^{a_1}\cdots dz^{a_p})=(dz^{a_1}\cdots dz^{a_N})(\epsilon_{a_1\cdots a_N}\xi_{a_N}\cdots\xi_{a_{p+1}}).$$

Similarly,

$$*(d\bar{z}^{\bar{a}_r}\cdots d\bar{z}^{\bar{a}_1})=(\epsilon_{a_1\cdots a_N}\eta_{\bar{a}_{r+1}}\cdots\eta_{\bar{a}_N})(d\bar{z}^{\bar{N}}\cdots d\bar{z}^{\bar{1}}),$$

where

$$\eta_{\bar{a}}=(\xi_a)^*=dz^b g_{b\bar{a}}.$$

Let $\mu(z,\bar{z})$ be the real function defined by the volume form

$$K^N=dz^1\cdots dz^N\mu(z,\bar{z})d\bar{z}^{\bar{N}}\cdots d\bar{z}^{\bar{1}}. \tag{5.35}$$

Then we define

$$*(d\bar{z}^{\bar{b}_1}\cdots d\bar{z}^{\bar{b}_r}dz^{a_1}\cdots dz^{a_p})=(\epsilon_{b_1\cdots b_N}\eta_{\bar{b}_{r+1}}\cdots\eta_{\bar{b}_N})\mu^{-1}(\epsilon_{a_1\cdots a_N}\xi_{a_N}\cdots\xi_{a_{p+1}}). \tag{5.36}$$

Roughly speaking, we put the Hodge $*$ of the antiholomorphic part and that of the holomorphic part together, and then take out from the middle the volume form (5.35). It can be shown that the properties (5.14) and (5.15) hold.

The commutativity of the Hodge $*$ with all functions (5.14) is not necessary for the invariance of the scalar curvature. As mentioned in Sec.5.2.5, simply a prescription of ordering $*(f(z)\alpha g(\bar{z}))=f(z)(*\alpha)g(\bar{z})$ is sufficient. Its significance is that any other prescription of ordering, say, $*(f(\bar{z})\alpha g(z))=f(\bar{z})(*\alpha)g(z)$, gives the same result.

This construction of the Hodge $*$ map is, however, not unique. When one patches the holomorphic and antiholomorphic parts as in (5.36), one can choose to put the holomorphic part before or after the antiholomorphic part. They are in general inequivalent. There is also the freedom to normalize (5.36) by different constant factors for different pairs of (p,r).

5.4.2 Riemannian Structure on $CP_q(N)$

The algebra of $CP_q(N)$ (see Chap.3) is given by the commutation relations

$$z^a z^b=q^{-1}\hat{R}^{ab}_{cd}z^c z^d,$$
$$\bar{z}^{\bar{a}}z^b=q^{-1}(\hat{R}^{-1})^{bd}_{ac}z^c\bar{z}^d-q^{-1}\lambda\delta^b_a,$$
$$z^a dz^b=q\hat{R}^{ab}_{cd}dz^c z^d,$$
$$\bar{z}^{\bar{a}}dz^b=q^{-1}(\hat{R}^{-1})^{bd}_{ac}dz^c\bar{z}^d,$$

where $\hat{R}$ is the $\hat{R}$-matrix of $GL_q(N)$ [16]. The $*$-involution is $z^{a*}=\bar{z}^{\bar{a}}$.

The Kähler form $K=dz^a g_{a\bar{b}}d\bar{z}^{\bar{b}}$ for $CP_q(N)$ is given by the deformed Fubini-Study metric (3.68)

$$g_{a\bar{b}}=q^{-1}\rho^{-2}(\rho\delta_{ab}-q^2\bar{z}^a z^b),$$

where $\rho=1+\sum_{a=1}^N z^a\bar{z}^{\bar{a}}$. The inverse of the metric is $g^{\bar{a}b}=q\rho(\delta_{ab}+\bar{z}^{\bar{a}}z^b)$. This Kähler form is not only real and central, but also invariant under the quantum group transformation

$$z^a\to(T^0_0+z^bT^0_b)^{-1}(T^a_0+z^cT^a_c),$$

where T^b_a is an $SU_q(N+1)$-matrix. Consequently its corresponding Hodge $*$ defined as above is also covariant under this quantum group transformation.

The deformed Fubini-Study metric implies that the connection one-form is

$$\omega_a{}^b=C^{bd}_{ac}\bar{z}^{\bar{c}}\rho^{-1}dz^d,$$

where $C^{bd}_{ac}=\delta_{ac}\delta_{bd}+q^{N-d}\delta_{ab}\delta_{cd}$. The curvature two-form is

$$R_a{}^b=-C^{bd}_{ac}g_{c\bar{e}}d\bar{z}^{\bar{e}}dz^d.$$

The scalar curvature and Ricci tensor are, up to normalizations,

$$\mathcal{R} \propto 1, \quad \mathcal{R}_a{}^b \propto \delta_a^b. \tag{5.37}$$

As in the classical case, this result can also be obtained by arguments based on the quantum group symmetry.

5.5 Two-Sheeted Space

Using the algebraic formulation of Riemannian geometry, we reproduce in this section the theory of gravity for the two-sheeted space which was first described in [63]. In that paper the Riemannian geometry on quantum spaces is formulated in terms of Connes' non-commutative geometry.

The two-sheeted space is the product of a classical 4-dimensional manifold $\mathcal{M}_4$ and a space of two discrete points Z_2. Denote the two points in Z_2 as a and b. The algebra of functions on Z_2 is generated by 1 and e, where $1(a) = 1(b) = 1$ and $e(a) = -e(b) = 1$. It follows that e is real and

$$e^2 = 1. \tag{5.38}$$

Acting with the exterior derivative on (5.38), we find

$$ede = -dee.$$

We also define $dede = 0$. In addition, e commutes with the coordinates $\{x^\mu\}$ on $\mathcal{M}_4$ and $\{dx^\mu\}$, and de commutes with $\{x^\mu\}$ and anti-commutes with $\{dx^\mu\}$.

To obtain the results in [63] we assume that the vielbeins and connection one-forms can be written as

$$E^a = dx^\mu e_\mu^a, \quad E^5 = de\lambda$$

and

$$\Omega^{AB} = dx^\mu(\omega_\mu^{AB} + ev_\mu^{AB}) + de(l^{AB} + ek^{AB}),$$

where e_μ^a, λ, ω_μ^{AB}, v_μ^{AB}, l^{AB} and k^{AB} are all real functions of x. The indices A, B take values in $\{1,2,3,4,5\}$, where $\{1,2,3,4\}$ correspond to dx^μ or E^a, and $\{5\}$ corresponds to de or E^5.

The Hodge $*$ map defined on E^A is the classical one. For example, $*(E^A E^B) = \frac{1}{3!}\epsilon^{ABCDF} E^C E^D E^F$. This map does not have the property (5.14), hence the Lagrangian (5.17) is invariant only under the coordinate transformation restricted to

$\mathcal{M}_4$, i.e., $x^\mu \to x'^\mu(x), e \to e$. This is one of the essential differences between our formulation and that of [63].

The integration over the whole space can be decomposed into the usual integration over the four-dimensional manifold followed by the integration

$$\int_{Z_2} E^5(a + be) = a$$

for arbitrary numbers a, b. (The requirement that $\int_{Z_2} dee$ vanishes implies the cyclic property of Connes' integration in this case: $\int \alpha\beta = \int \beta\alpha$.)

Using the metricity condition (5.11) and the torsion-free condition (5.12) one can partially solve for the connection. But many components of the connection are still free. They should be viewed as independent fields. It turns out that they are not dynamical fields because in the Lagrangian (the scalar curvature) they do not have time derivatives. Their equations of motion are simply constraints which are solved by their vanishing.

The action for gravity on this two-sheeted space defined by

$$I = \int_{\mathcal{M}_4 \times Z_2} E^A(*R^{AB})E^B$$

is, after taking out all non-dynamical fields, the same as [63]

$$I = -\int_{\mathcal{M}_4} (\mathcal{R}_4 - 2\lambda^{-1}\nabla_\mu\partial^\mu\lambda)\sqrt{g_4}d^4x,$$

where $\mathcal{R}_4$ is the usual scalar curvature of $\mathcal{M}_4$, ∇_μ is the usual covariant derivative on $\mathcal{M}_4$ and g_4 is the determinant of the metric on $\mathcal{M}_4$. The only new dynamical field introduced by the Z_2 structure in spacetime is $\lambda(x)$. By changing variables $\lambda = exp(\sigma)$ [63] we get

$$I = -\int_{\mathcal{M}_4} (\mathcal{R}_4 - 2\partial_\mu\sigma\partial^\mu\sigma)\sqrt{g_4}d^4x.$$

5.6 General Coordinate Transformations

The general coordinate transformations (GCTs) mentioned above will not be a physical symmetry unless the algebra X on the quantum space and its differential calculus are covariant. In this section we consider the quantum group of general coordinate transformations as a symmetry of a quantum space. The Riemannian structure defined in this chapter applied to this quantum space will be invariant under the coaction of the GCT group.

Consider the N-dimensional quantum space:

$$x_i x_j = q x_j x_i, \quad i < j, \quad i,j = 1,2,\cdots,N. \tag{5.39}$$

which is the same as the $GL_q(N)$-covariant quantum hyperplane, but we will consider a differential calculus not covariant under $GL_q(N)$ transformations. It is

$$x_i dx_j = q dx_j x_i, \quad i < j, \quad i,j = 1,2,\cdots,N, \tag{5.40}$$

$$dx_i x_j = q x_j dx_i, \quad i < j, \quad i,j = 1,2,\cdots,N \tag{5.41}$$

and

$$x_i dx_i = dx_i x_i, \quad i = 1,2,\cdots,N. \tag{5.42}$$

This can be called the *ordering deformation*. A consistent $*$-involution can be defined by $q^* = q^{-1}$ and $x_i^* = x_i$.

Introduce the notation

$$\underline{x}^{\underline{m}} = x_1^{m_1} x_2^{m_2} \cdots x_N^{m_N}, \tag{5.43}$$

where $\underline{m} = (m_1, m_2, \cdots, m_N)$ and the m_i's are non-negative integers. Let

$$[\underline{m}, \underline{n}] = \sum_{i<j} (m_i n_j - m_j n_i), \tag{5.44}$$

then

$$\underline{x}^{\underline{m}} \underline{x}^{\underline{n}} = q^{[\underline{m},\underline{n}]} \underline{x}^{\underline{n}} \underline{x}^{\underline{m}}. \tag{5.45}$$

Define the general coordinate transformation as a left coaction $\Delta_L : X \to \mathcal{A} \otimes X$, where X is the algebra of the quantum space and $\mathcal{A}$ is the algebra of the quantum group of GCTs, by

$$\Delta_L(x_i) = \sum_{\underline{m}} a_{i\underline{m}} \otimes \underline{x}^{\underline{m}}. \tag{5.46}$$

In order to fix the algebra $\mathcal{A}$ we introduce braided copies of X, labelled by a, with coordinates $\{x_i^{(a)}\}_{i=1}^N$:

$$x_i^{(a)} x_j^{(b)} = q x_j^{(b)} x_i^{(a)}, \quad i < j, \quad i,j = 1,2,\cdots,N, \tag{5.47}$$

$$x_i^{(a)} x_i^{(b)} = x_i^{(b)} x_i^{(a)}, \quad i = 1,2,\cdots,N \tag{5.48}$$

for arbitrary a, b. For the algebra of two or more braided copies of X to be covariant under the left coaction of $\mathcal{A}$, $\mathcal{A}$ is completely determined:

$$a_{i\underline{m}} a_{j\underline{n}} = q^{1-[\underline{m},\underline{n}]} a_{j\underline{n}} a_{i\underline{m}}, \quad i < j, \tag{5.49}$$

$$a_{i\underline{m}} a_{i\underline{n}} = q^{-[\underline{m},\underline{n}]} a_{i\underline{n}} a_{i\underline{m}} \tag{5.50}$$

for $i,j = 1,2,\cdots,N$.

Define the coproduct on $\mathcal{A}$ by

$$(\Delta \otimes id) \circ \Delta_L = (id \otimes \Delta_L) \circ \Delta_L. \tag{5.51}$$

The coproduct of the generators $a_{i\underline{m}}$ is an infinite power series, so it is not acceptable to check directly whether the coproduct is a coassociative algebra homomorphism. [4] However, following the general arguments in Sec.1.6, the coproduct is a homomorphism because $\mathcal{A}$ is completely fixed by the coaction. Moreover, we have

Lemma 3 *The coproduct on $\mathcal{A}$ defined by (5.51) is coassociative:*

$$(\Delta \otimes id) \circ \Delta = (id \otimes \Delta) \circ \Delta. \tag{5.52}$$

Proof

$$\begin{aligned}
& (((\Delta \otimes id) \circ \Delta) \otimes id) \circ \Delta_L \\
= \; & (\Delta \otimes id \otimes id) \circ (\Delta \otimes id) \circ \Delta_L \\
= \; & (\Delta \otimes id \otimes id) \circ (id \otimes \Delta_L) \circ \Delta_L \\
= \; & (\Delta \otimes \Delta_L) \circ \Delta_L \\
= \; & (id \otimes id \otimes \Delta_L) \circ (\Delta \otimes id) \circ \Delta_L \\
= \; & (id \otimes id \otimes \Delta_L) \circ (id \otimes \Delta_L) \circ \Delta_L \\
= \; & (id \otimes ((id \otimes \Delta_L) \circ \Delta_L)) \circ \Delta_L \\
= \; & (id \otimes ((\Delta \otimes id) \circ \Delta_L)) \circ \Delta_L \\
= \; & (id \otimes \Delta \otimes id) \circ (id \otimes \Delta_L) \circ \Delta_L \\
= \; & (id \otimes \Delta \otimes id) \circ (\Delta \otimes id) \circ \Delta_L \\
= \; & (((id \otimes \Delta) \circ \Delta) \otimes id) \circ \Delta_L.
\end{aligned}$$

Therefore

$$(((\Delta \otimes id) \circ \Delta) \otimes id) \circ \Delta_L(x_i) = (((id \otimes \Delta) \circ \Delta) \otimes id) \circ \Delta_L(x_i).$$

Using (5.46) we find for the coefficient of $\underline{x}^{\underline{m}}$

$$(\Delta \otimes id) \circ \Delta(a_{i\underline{m}}) = (id \otimes \Delta) \circ \Delta(a_{i\underline{m}}).$$

[4] The author is indebted to P. Podleś for pointing out this problem and the importance to introduce the braided copies so that, as to be shown right away, it can still be proven that the coproduct is a coassociative algebra homomorphism.

Then by extending the coproduct as an algebra homomorphism to the full algebra $\mathcal{A}$, the coassociativity on $\mathcal{A}$ is preserved. $\square$

The counit is defined as an algebra homomorphism by

$$\epsilon(a_{i\underline{m}}) = \begin{cases} 1 & \text{if} \quad m_j = \delta^i_j \\ 0 & \text{otherwise} \end{cases} \tag{5.53}$$

and $\epsilon(1) = 1$.

The only thing we are missing for $\mathcal{A}$ to be a Hopf algebra is the coinverse. Even classically it is not possible to find explicit expressions for the coinverse of the generators of $\mathcal{A}$. For this problem we resort to another description of the GCT group. The idea is that we take a few subgroups of the GCT group that generate the whole GCT group by iteration.

To illustrate the idea, we consider the two-dimensional case. Consider two global covariant transformations of X. The first is the linear transformation and translation [5]

$$\rho_1(x_i) = M_i^j \otimes x_j + V_i \otimes 1, \tag{5.54}$$

where $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ and $V = \begin{pmatrix} E \\ F \end{pmatrix}$. The algebra for M and V is given by

$$AB = q^{-1}BA, \quad AC = qCA, \tag{5.55}$$
$$AF = qFA, \quad BC = q^2 CB, \tag{5.56}$$
$$BD = qDB, \quad BF = qFB, \tag{5.57}$$
$$CD = q^{-1}DC, \quad CE = q^{-1}EC, \tag{5.58}$$
$$DE = q^{-1}ED, \quad EF = qFE \tag{5.59}$$

and all other commutation relations are trivial, e.g., $AD = DA$. This group of transformations is a subgroup of the GCT group. The commutation relations can be easily derived from (5.49-5.50).

The second transformation is the fractional transformation

$$\rho_2(x_i) = (c_i + d_i x_i)(a_i + b_i x_i)^{-1} \tag{5.60}$$

with

$$b_1 b_2 = q^{-1} b_2 b_1, \quad c_1 c_2 = q c_2 c_1, \tag{5.61}$$
$$b_1 d_2 = q^{-1} d_2 b_1, \quad d_1 b_2 = q^{-1} b_2 d_1, \tag{5.62}$$
$$c_1 d_2 = q d_2 c_1, \quad d_1 c_2 = q c_2 d_1 \tag{5.63}$$

[5] For the following discussion it is not necessary to include the translation.

(all other relations are trivial). Similarly, this is a subgroup of the GCT group and the algebra can be derived from (5.49-5.50) [6]. It appears to be a braiding of two classical groups.

It is straightforward to find the Hopf algebras corresponding to these two transformations and we will not write them out explicitly. Let us just denote them as $H_i = (\Delta_i, \epsilon_i, S_i)$ for $i = 1, 2$. (For the n-dimensional case, we replace M by an $n \times n$-matrix and V by an n-vector. ρ_2 remains the same for $i = 1, 2, \cdots, n$. Their Hopf algebras can be worked out similarly.)

Given a sequence $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_K)$ of length K with $\alpha_i = 1, 2$, one can define the the coaction

$$\rho_\alpha = (id^{\otimes(K-1)} \otimes \rho_{\alpha_K}) \circ (id^{\otimes(K-2)} \otimes \rho_{\alpha_{K-1}}) \circ \cdots (id \otimes \rho_{\alpha_2}) \circ \rho_{\alpha_1}. \tag{5.64}$$

This coaction corresponds to a group of transformations which is a series of consecutive transformations of ρ_1 and ρ_2. It has a Hopf algebra structure $H_\alpha = (\Delta_\alpha, \epsilon_\alpha, S_\alpha)$ inherited from H_1, H_2:

$$\Delta_\alpha = \Delta_{\alpha_1} \otimes \Delta_{\alpha_2} \otimes \cdots \otimes \Delta_{\alpha_K}, \tag{5.65}$$
$$\epsilon_\alpha = \epsilon_{\alpha_1} \otimes \epsilon_{\alpha_2} \otimes \cdots \otimes \epsilon_{\alpha_K}, \tag{5.66}$$
$$S_\alpha = S_{\alpha_1} \otimes S_{\alpha_2} \otimes \cdots \otimes S_{\alpha_K}. \tag{5.67}$$

Classically it can be shown that alternating transformations of the first and second type generate the GCTs. In the quantum case we can take the definition of the group of GCTs as $H_{\alpha(N)}$ in the limit of $N \to \infty$ for $\alpha(N) = (1, 2, 1, 2, 1, 2, \cdots)$ being a sequence of length N of alternating 1's and 2's. Therefore although the quantum GCT is strictly speaking not a Hopf algebra, it is the limit of a series of Hopf algebras.

5.7 Conclusion

In the investigation of quantum spaces presented in this thesis we found that abundant geometrical structures (symmetry, differential calculus, complex structure, projective geometry and Riemannian geometry) can be extended to the noncommutative case without losing the classical geometrical intuition. This indicates that the notion of geometry is naturally embedded in a much larger setting than the classical

[6] By expanding (5.60) in power series of x_i one can use (5.49-5.50) to obtain commutation relations among $b_i a_i^{-1}$, $c_i a_i^{-1}$ and $d_i a_i^{-1}$. Choosing a_i to be central we get (5.61-5.63).

picture. The natural question is then whether physics has the same generalization as its mathematical tools.

Quantum mechanics of many different particles [48] is relatively well understood. For classical gauge field theories on quantum spaces, while a description is established for a quantum principal bundle [47] where the fiber is a quantum group and the base is a quantum space, it is still unclear what the dynamics of the gauge fields is except for particular simple examples. An important physical model built upon a quantum space (the two-sheeted space) where its classical dynamics is clear is the Connes-Lott model [64]. It is remarkable that the Higgs field arises there naturally in Connes' formulation.

Many attempts to formulate quantum mechanics of identical particles [65] and quantum field theory [66, 67] on quantum spaces have been made, but their general formulation for generic quantum spaces is still unclear. However it is expected that new regularization methods may arise as a result of the deformation of spacetime [68, 67].

On the other hand, noncommutative geometry may also play a role in string theory [69]. At least intuitively any theory incorporating quantum effects of gravity should exhibit some fuzziness in spacetime, and therefore may be related to noncommutative geometry. The question of interest is how to utilize noncommutative geometry to understand better the mystery of physics at the Planck scale. It will be an exciting moment when the answer emerges.

Appendix A

Poisson-Lie Groups and Lie Bialgebras

In this Appendix we briefly introduce the relation among quantum groups, Poisson-Lie groups and Lie bialgebras. For a more detailed and advanced discussion see [17].

A.1 Poisson-Lie Groups

The covariance of the algebra of a quantum group implies that it is a Poisson-Lie group in the Poisson limit, and the consistency of the quantum algebra corresponds to the Jacobi identity of the Poisson structure.

As an example we consider quantum groups with the RTT relation

$$R_{12}T_1T_2 = T_2T_1R_{12}. \tag{A.1}$$

Let $q = 1 + h + \mathcal{O}(h^2)$ and expand everything in powers of h. The R-matrix is

$$R = I + hr + \mathcal{O}(h^2), \tag{A.2}$$

where I is the identity matrix $I^{ij}_{kl} = \delta^i_k\delta^j_l$ and r is called the classical r-matrix. From the RTT relation,

$$T_1T_2 - T_2T_1 = -h(r_{12}T_1T_2 - T_2T_1r_{12}) + \mathcal{O}(h^2), \tag{A.3}$$

which in the limit of $h \to 0$ gives the Poisson Bracket (PB)

$$(T_1, T_2) = \lim_{h\to 0}\frac{T_1T_2 - T_2T_1}{-h} = [r_{12}, T_1T_2]. \tag{A.4}$$

This Poisson structure makes the classical group a *Poisson-Lie group*, which simply means that the group multiplication is a Poisson map. The Poisson structure on two copies of the group is defined by

$$(f_1 \otimes f_1', f_2 \otimes f_2')_{G\times G} = (f_1, f_2) \otimes f_1' f_2' + f_1 f_2 \otimes (f_1', f_2'). \tag{A.5}$$

Using the group multiplication $\cdot : G \times G \to G$, functions on $G \times G$ correspond to functions on G by the coproduct

$$\Delta(T^i_j) = T^i_k \otimes T^k_j. \tag{A.6}$$

Hence the statement that the group multiplication is a Poisson map means that

$$(f_{(1)} \otimes f_{(2)}, f'_{(1)} \otimes f'_{(2)})_{G\times G} = (f, f')_{(1)} \otimes (f, f')_{(2)} \tag{A.7}$$

in Sweedler's notation. For $f = T_1$ and $f' = T_2$, it is

$$(T_1 \otimes T_1, T_2 \otimes T_2)_{G\times G} = [r_{12}, T_1 T_2 \otimes T_1 T_2]. \tag{A.8}$$

Eq.(A.8) can be easily checked by using (A.4) for any r-matrix.

On the other hand, for (A.4) to be a Poisson structure one has to check the Jacobi identity. As one might expect, this is guaranteed by the Yang-Baxter equation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}, \tag{A.9}$$

which contains at its second order in h the classical Yang-Baxter equation

$$[r_{12}, r_{13}] + [r_{13}, r_{23}] + [r_{12}, r_{23}] = 0. \tag{A.10}$$

(The zeroth and first order terms are trivial.)

In general, since the coproduct is an algebra homomorphism in a Hopf algebra, the Poisson limit of the quantum group is a Poisson-Lie group. While the Poisson structure comes as a result of the commutation relations, its Jacobi identity is guaranteed by the consistency of the algebra.

A.2 Lie Bialgebras

Up to the usual global issue, Poisson-Lie groups are equivalent to Lie bialgebras. A *Lie bialgebra* is a Lie algebra g equipped with the additional structure $\delta : g \to g \wedge g$ satisfying

$$(\delta \wedge id) \circ \delta = 0, \tag{A.11}$$

$$\delta([x, y]) = [\delta(x), y] + [x, \delta(y)]. \tag{A.12}$$

Given a Poisson-Lie group G, one can define a Lie algebraic structure on g^*, which is the dual of $g = Lie(G)$, according to

$$[(df_1)_e, (df_2)_e] = (d(f_1, f_2))_e, \tag{A.13}$$

where we have identified g^* with the cotangent space at the unit $e \in G$. The Jacobi identity of the PB implies the Jacobi identity for the Lie bracket.

The δ map for g can be obtained from this Lie bracket on g^*:

$$\langle X, [(df_1)_e, (df_2)_e] \rangle = \langle \delta(X), (df_1)_e \wedge (df_2)_e \rangle, \tag{A.14}$$

where $\langle \cdot, \cdot \rangle$ is the pairing between the tangent space and cotangent space at $e \in G$. In the above we have identified g with the tangent space at e.

The Jacobi identity for the Lie bracket on g^* implies (A.11), and the fact that the group multiplication is a Poisson map implies (A.12). Hence we see that Lie bialgebra is simply the tangent/cotangent space of a Poisson-Lie group.

Now consider again the quantum groups with RTT relations. How do we describe their classical limit in terms of Lie-bialgebras? They are actually quasi-triangular Lie bialgebras. Let r also denote the classical universal r-matrix which is an element in $g \otimes g$ as the second order term of the universal $\mathcal{R}$-matrix in the h expansion. A Lie bialgebra (g, δ) is *quasi-triangular* if

1. $I = r + \sigma(r)$, where $\sigma(X \otimes Y) = Y \otimes X$, is adjoint invariant.
2. $\delta(X) = -ad_X(r)$, where $ad_X(Y \otimes Z) = [X, Y] \otimes Z + Y \otimes [X, Z]$.
3. r satisfies $(\delta \otimes id)(r) = [r_{13}, r_{23}]$, and $(id \otimes \delta)(r) = [r_{13}, r_{12}]$.

The first property comes from the antisymmetry of a P.B. $(T_1, T_2) = -(T_2, T_1)$. The second states how r determines δ as described above and the third ensures the classical Yang-Baxter equation.

The quasi-triangular Lie bialgebra is *factorizable* if in addition I is a non-degenerate bilinear form on g^*. This means that one can invert the relation

$$\langle X, \cdot \rangle = \langle I, \omega \otimes \cdot \rangle \tag{A.15}$$

for a given $X \in g$ to solve for a unique $\omega \in g^*$. Therefore one can decompose any $X \in g$ uniquely into $X = X_1 + X_2$ with $\langle X_1, \cdot \rangle = \langle r, \omega \otimes \cdot \rangle$ and $\langle X_2, \cdot \rangle = \langle r, \cdot \otimes \omega \rangle$.

The universal $\mathcal{R}$-matrix in Sec.1.2.1 for $SL_q(2)$ has the corresponding universal r-matrix

$$r = \frac{1}{4} H \otimes H + X_+ \otimes X_-. \tag{A.16}$$

For $SL_q(N)$ it is

$$r = \frac{1}{2}\sum_{i,j=1}^{N-1}(A^{-1})_{ij}H_i \otimes H_j + \sum_{i<j} e_{ij} \otimes e_{ji}, \tag{A.17}$$

where A is the Cartan matrix and e_{ij} is represented as $\pi(e_{ij})^k_l = \delta^k_i \delta^j_l$ in the defining representation. The factorization corresponds to upper and lower triangular decompositions.

Appendix B

Connes' Differential Calculus

In this Appendix we give a sketch of Connes' noncommutative geometry [2]. (See [70] for a brief introduction of its application to particle physics.)

B.1 Differential Calculus

Denote the algebra of functions by $\mathcal{A}$ and the universal differential calculus by $\Omega(A)$. In Connes' calculus, we also have the Leibniz rule and the nilpotency of the exterior derivative:

$$d(ab) = (da)b + (-1)^{deg(a)}a(db), \tag{B.1}$$

$$d^2 = 0. \tag{B.2}$$

The first observation is that if one defines differential forms by a certain operator F:

$$da = [F, a]_{\pm}, \tag{B.3}$$

where $[\cdot,\cdot]_{\pm}$ is the commutator for a of even degrees and anti-commutator for odd degrees, then the Leibniz rule is guaranteed. But to have $d^2 = 0$ we need

$$\begin{aligned} d^2a &= d[F,a] && \text{(B.4)}\\ &= \{F,[F,a]\} && \text{(B.5)}\\ &= [F^2,a] && \text{(B.6)}\\ &= 0, && \text{(B.7)} \end{aligned}$$

which is guaranteed if F^2 is a constant. If the constant is non-zero, one can always rescale it to be 1.

Connes defines the differential calculus on an (involutive) algebra $\mathcal{A}$ to be given by a representation π of A on a Hilbert space $\mathcal{H}$ and an operator F with the properties

$$F^* = F, \tag{B.8}$$

$$F^2 = 1 \tag{B.9}$$

and $[F, \pi(a)]$ compact for any $a \in \mathcal{A}$. This is called an odd Fredholm module. An even Fredholm module is given by an odd Fredholm module with a $Z/2$ grading operator γ on $\mathcal{H}$ such that

$$\gamma^* = \gamma, \tag{B.10}$$

$$\gamma^2 = 1, \tag{B.11}$$

$$\gamma\pi(a) := \pi(a)\gamma, \quad \forall a \in \mathcal{A}, \tag{B.12}$$

$$\gamma F = -F\gamma. \tag{B.13}$$

The information about the metric on a manifold is not encoded in the calculus defined above, but there is a way to generalize the approach above to include the data of the metric in the calculus. To do so, instead of using F we use a self-adjoint operator $\mathcal{D}$, which we call the Dirac operator, on that Hilbert space $\mathcal{H}$ in the following way [2]. First extend the domain of π to the universal differential calculus $\Omega(\mathcal{A})$ of $\mathcal{A}$ by

$$\pi(a_0(da_1)...(da_n)) = \pi(a_0)[\mathcal{D}, \pi(a_1)]...[\mathcal{D}, \pi(a_n)]. \tag{B.14}$$

There will be elements in $\Omega(\mathcal{A})$ which vanish under π, so that elements in $\Omega(\mathcal{A})$ which differ from each other by those elements will have the same representation. Therefore it is natural to define the general differential calculus $\Omega_{\mathcal{D}}(\mathcal{A})$ on $\mathcal{A}$ to be

$$\Omega_{\mathcal{D}}(\mathcal{A}) = \pi(\Omega(\mathcal{A}))/Aux, \tag{B.15}$$

where $Aux = (ker\pi + d(ker\pi))$. Since Aux (called auxiliary fields) is a two-sided ideal, it is easy to see that this defines unambiguously a consistent differential calculus on $\mathcal{A}$.

B.2 Integration

Consider the case where the differential calculus is specified by F. The integration of a differential form a over $\mathcal{A}$ is defined by

$$\int a = Tr(a) \tag{B.16}$$

for the odd case and

$$\int a = Tr(\gamma a) \tag{B.17}$$

for the even case. It can be shown that the Stoke's theorem holds:

$$\int da = 0. \tag{B.18}$$

In the case when the differential calculus is defined by the Dirac operator $\mathcal{D}$, the integration is given by

$$\int a = Tr_\omega(\pi(a)|\mathcal{D}|^{-d}), \tag{B.19}$$

where Tr_ω is the Dixmier trace.

B.3 An Example

The simplest example is the two-point set Z_2. A function on this space is represented by a diagonal 2 by 2 matrix of complex numbers. The Dirac operator can be defined by

$$\mathcal{D} = \begin{pmatrix} 0 & M^* \\ M & 0 \end{pmatrix}, \tag{B.20}$$

where M is an arbitrary complex number.

Now we can say that the algebra of functions is generated by $\{1, e\}$, [1] where 1 is the unit matrix and

$$e = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{B.21}$$

Define $de = [\mathcal{D}, e]$, then we have the commutation relations

$$ee = e, \tag{B.22}$$

$$ede = de(1 - e), \tag{B.23}$$

$$dede = M^*M \tag{B.24}$$

and the $*$-structure $e^* = e$, $(de)^* = -de$.

Because $ee = e$, the most general function of e is just $f(e) = a + be$ for arbitrary complex numbers a, b. So any linear functional is specified by two numbers u, v as $L(f) = ua + vb$. The integration of a one-form $f(e)de$ according to Connes' formula should be $Tr(f(e)de|\mathcal{D}|^{-1})$, which vanishes. To get nonzero integration, one has to consider the integration over a zero-form or a two-form. For either case the result is proportional to the trace of the 2×2 matrix representation of the integrand.

[1] Notice that this basis is different from the one used in Sec.5.5.

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Ernest Orlando Lawrence Berkeley National Laboratory
One Cyclotron Road | Berkeley, California 94720

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