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Abstract

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Decoupling a second-order linear dynamical system requires that one develop a transformation that simultaneously diagonalizes the coefficient matrices that define the system in terms of its distribution of inertia and viscoelasticity. A traditional approach to decoupling a viscously damped system uses the eigenvectors of the corresponding undamped system to diagonalize the mass, damping, and stiffness matrices through a real congruence transformation in the configuration space, a process known as classical modal analysis. However, it is well known that classical modal analysis fails to decouple a linear dynamical system if its damping matrix does not satisfy a commutativity relationship involving the system matrices. Such a system is said to be non-classically damped. We demonstrate that it is possible to decouple any non-classically damped system in the configuration and state spaces through generally time-dependent transformations constructed using spectral data obtained from the solution of a quadratic eigenvalue problem.

When a non-classically damped system has complex but non-defective eigenvalues, the effect of non-classical damping is that it introduces constant phase shifts among the components of the system's free response. Decoupling of free vibration in the configuration space is achieved through a real, linear, time-shifting transformation that eliminates these phase differences, yielding classical modes of vibration. This decoupling transformation, referred to as phase synchronization, preserves both the eigenvalues and their multiplicities. When cast in a state space form, the transformation between the coupled and decoupled systems is real, linear, but time-invariant. Through the concept of real quadratic conjugation, we illustrate that there is no fundamental difference in the representation of the free response of a system with complex eigenvalues and one with real eigenvalues, and thus systems with non-defective real eigenvalues can also be decoupled by phase synchronization. When phase synchronization is extended to forced systems, the decoupling transformation in both the configuration and state spaces is nonlinear and depends continuously on the applied excitation.

If a non-classically damped system is defective, it may only be partially decoupled if one insists on preserving the geometric multiplicities of the defective eigenvalues. We present the first systematic effort to decouple defective systems in free or forced vibration by not demanding invariance of the geometric multiplicities. In the course of this development, the notion of critical damping in multi-degree-of-freedom systems is clarified and expanded. It is shown that the decoupling of defective systems is a rather delicate procedure that depends on the multiplicities of the system eigenvalues. A generalized state space-based decoupling transformation is developed that relates the response of any non-classically damped system to that of its decoupled form. In principle, one could extract from the state space a decoupling transformation in the configuration space, but it generally does not have an explicit form. The decoupling transformation in both the configuration and state spaces is real and time-dependent. Several numerical examples are provided to illustrate the theoretical developments.

Contents

List of Select Symbols

$\mathbf{0}$	zero vector
c	number of non-defective complex conjugate pairs of eigenvalues
$\mathbf{c}$	vector of eigensolution coefficients
$\mathbf{f}(t)$	forcing vector of the coupled system
$\mathbf{g}(t)$	forcing vector of the decoupled system
i	imaginary unit ($\mathrm{i} = \sqrt{-1}$)
m_k	algebraic multiplicity of the kth eigenvalue
n	number of degrees of freedom
n_c	number of defective complex conjugate pairs of eigenvalues
n_r	number of defective real eigenvalues
$\mathbf{p}(t)$	response vector of the decoupled system
$p_k(t)$	kth independent coordinate of the decoupled system
r	number of non-defective real quadratic conjugate pairs of eigenvalues
$\mathbf{s}_k(t)$	kth damped mode
t	time
$\mathbf{u}_k$	kth eigenvector of the undamped system
$\mathbf{v}_k$	eigenvector of the kth distinct eigenvalue
$\mathbf{v}_j^k$	jth eigenvector or generalized eigenvector of the kth repeated eigenvalue
$\mathbf{x}(t)$	response vector of the coupled system
$x_k(t)$	kth generalized coordinate of the coupled system
$\mathbf{z}_k$	kth eigenvector of the classically damped system obtained by phase synchronization
$\mathbf{C}$	viscous damping matrix of the coupled system

$\mathbf{D}$	viscous damping matrix of the decoupled system
$\mathbf{I}_m$	identity matrix of order m
$\mathbf{J}_p$	Jordan matrix of the decoupled system
$\mathbf{J}_x$	Jordan matrix of the coupled system
$\mathbf{K}$	stiffness matrix of the coupled system
$\mathbf{M}$	mass (inertia) matrix of the coupled system
$\mathbf{O}$	zero matrix
$\mathbf{U}$	modal matrix of the undamped system
$\mathbf{V}_p$	matrix of pairing vectors for the decoupled system
$\mathbf{V}_x$	matrix of eigenvectors and generalized eigenvectors of the coupled system
α_k	real part of the kth eigenvalue
λ_k	kth eigenvalue ($\lambda_k = \alpha_k + \mathrm{i}\omega_k$)
ρ_k	geometric multiplicity of the kth eigenvalue
φ_{kj}	phase shift in the jth component of the kth damped mode due to non-classical damping
ω_k	imaginary part of the kth eigenvalue
$\mathbf{\Lambda}$	diagonal matrix of eigenvalues
$\mathbf{\Omega}$	stiffness matrix of the decoupled system
$\dot{\mathbf{A}}(t)$	first time derivative of the time-dependent matrix $\mathbf{A}(t)$
$\ddot{\mathbf{A}}(t)$	second time derivative of the time-dependent matrix $\mathbf{A}(t)$
$\overline{\mathbf{A}}$	complex conjugate of the matrix $\mathbf{A}$
$\widetilde{\mathbf{A}}$	real quadratic conjugate of the matrix $\mathbf{A}$
$\widehat{\mathbf{A}}$	generalized conjugate of the matrix $\mathbf{A}$
$\mathbf{A}^\mathrm{T}$	transpose of the matrix $\mathbf{A}$
$\mathbf{a} \cdot \mathbf{b}$	$\mathbf{a}^\mathrm{T}\mathbf{b}$ for column vectors $\mathbf{a}$ and $\mathbf{b}$ of identical length
$\mathrm{e}^{\mathbf{A}}$	exponential function for the matrix $\mathbf{A}$
$\mathbf{A} \oplus \mathbf{B}$	direct sum (block-diagonal matrix) of the matrices $\mathbf{A}$ and $\mathbf{B}$
$\bigoplus_{k=1}^{m} \mathbf{A}_k$	direct sum (block-diagonal matrix) of the m matrices $\mathbf{A}_k$ ($k = 1, 2, \ldots, m$)

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Chapter 1

Introduction

Coordinate coupling in linear dynamical systems under viscous damping has long been viewed as an undesired phenomenon with respect to system analysis in both practice and theoretical pursuits. Consequently, the decoupling of dynamical systems is a subject with a long history that attracts much attention from researchers even to this day. The problem of decoupling a linear dynamical system is concerned with developing a transformation that simultaneously diagonalizes the coefficient matrices that define the system in terms of its distribution of inertia and viscoelasticity. The equation of motion of an n-degree-of-freedom linear dynamical system subject to viscous damping and external forcing has the matrix-vector representation

$$\mathbf{M}\ddot{\mathbf{x}}(t)+\mathbf{C}\dot{\mathbf{x}}(t)+\mathbf{K}\mathbf{x}(t)=\mathbf{f}(t)\ , \tag{1.1}$$

where the real and order n coefficient matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ are positive definite (i.e., rigid body modes have been removed) and correspond to the system inertia, damping, and elasticity, respectively. The real n-dimensional column vectors $\mathbf{x}(t)$ and $\mathbf{f}(t)$ denote, respectively, the generalized coordinates and external excitation.

The individual equations comprising system (1.1) are often coupled since the mass, damping, and stiffness matrices are, in general, not diagonal. Coupling is not an inherent property of a system but rather depends on the choice of generalized coordinates. This dissertation addresses the issue of devising a general methodology for decoupling by which any system of the form (1.1) is transformed into, and its response exactly recovered from the solution of,

$$\ddot{\mathbf{p}}(t)+\mathbf{D}\dot{\mathbf{p}}(t)+\mathbf{\Omega}\,\mathbf{p}(t)=\mathbf{g}(t)\ , \tag{1.2}$$

for which the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ are real, diagonal, of order n, and represent the damping and elasticity properties, respectively, of the decoupled system. Moreover, the real n-long column vectors $\mathbf{p}(t)$ and $\mathbf{g}(t)$ denote the decoupled (or modal) coordinates and applied excitation, respectively. We begin with a survey of classical techniques and methods proposed in the literature for analyzing the response of a linear dynamical system of the type (1.1).

1.1 Classical modal analysis

It is well known that a class of systems of the form (1.1) can be decoupled by a congruence transformation in the n-dimensional configuration space using the eigenvectors of the undamped system (e.g., see [1–3]). Associated with the undamped form of system (1.1) is the generalized eigenvalue problem (e.g., see [1])

$$\lambda \mathbf{M}\mathbf{u} = \mathbf{K}\mathbf{u} \tag{1.3}$$

that, because the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$ are real and positive definite, generates n real and positive eigenvalues λ_k ($k = 1, 2, \ldots, n$) with corresponding real eigenvectors $\mathbf{u}_k$ that are orthogonal with respect to $\mathbf{M}$ and $\mathbf{K}$. It is customary to normalize the eigenvectors in accordance with $\mathbf{u}_j \cdot (\mathbf{M}\mathbf{u}_k) = \delta_{jk}$ ($j = 1, 2, \ldots, n$), where $\mathbf{a} \cdot \mathbf{b} = \mathbf{a}^{\mathrm{T}}\mathbf{b}$ for any vectors $\mathbf{a}$ and $\mathbf{b}$, and δ_{jk} denotes the Kronecker delta. Upon arranging the normalized eigenvectors in a modal matrix $\mathbf{U}$ and defining a linear time-invariant coordinate transformation

$$\mathbf{x}(t) = \mathbf{U}\mathbf{p}(t)\ , \qquad \mathbf{U} = \left[\ \mathbf{u}_1 \ \middle|\ \cdots \ \middle|\ \mathbf{u}_n \ \right]\ , \tag{1.4}$$

application of transformation (1.4) converts system (1.1) into the form (1.2) with

$$\mathbf{D} = \mathbf{U}^{\mathrm{T}}\mathbf{C}\mathbf{U}\ , \qquad \boldsymbol{\Omega} = \mathbf{U}^{\mathrm{T}}\mathbf{K}\mathbf{U} = \bigoplus_{k=1}^{n} \lambda_k\ , \qquad \mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\mathbf{f}(t)\ . \tag{1.5}$$

If the matrix $\mathbf{D}$ is diagonal upon congruence transformation, then system (1.1) has been decoupled by a procedure referred to as classical modal analysis. Consequently, a system (1.1) for which $\mathbf{D}$ is diagonal is said to be classically damped. Since the coordinate transformation (1.4) is real, the modal response $\mathbf{p}(t)$ can be identified with a displacement, and hence classical modal analysis is amenable to physical interpretation. It was known to Lord Rayleigh [4] in the late 19th century that a sufficient condition for classical damping is that the damping matrix $\mathbf{C}$ of system (1.1) be a linear combination of the mass matrix $\mathbf{M}$ and stiffness matrix $\mathbf{K}$. It was not until 1965 that a necessary and sufficient condition for classical damping was provided by Caughey and O'Kelly [5]: system (1.1) is classically damped if and only if its coefficient matrices satisfy

$$\mathbf{C}\mathbf{M}^{-1}\mathbf{K} = \mathbf{K}\mathbf{M}^{-1}\mathbf{C}\ . \tag{1.6}$$

Practically speaking, condition (1.6) implies a fairly uniform distribution of energy dissipation in a system [6, 7], but there is no reason for this to be true for any given system. For example, systems incorporating base isolation or involving soil-structure or fluid-structure interaction are not appropriately characterized as classically damped [6–9]. The conclusion is that it is generally not possible to decouple system (1.1) by classical modal analysis.

1.2 Complex modal analysis

Of course, one may also consider decoupling in the $2n$-dimensional state space. In view of the inadequacy of classical modal analysis, a procedure known as complex modal analysis was developed in the mid 20th century to decouple non-classically damped systems of the form (1.1) in the state space via complex congruence transformation (e.g., see [10–12]). However, complex modal analysis is limited by the requirement that system (1.1) be non-defective (i.e., every eigenvalue of system (1.1) has an eigenvector), and there is a clear numerical disadvantage to analysis in the $2n$-dimensional state space over the n-dimensional configuration space. Moreover, unlike classical modal analysis, complex modal analysis provides little in the way of physical insight since the complex congruence transformation involved generally makes it impossible to relate the $2n$ state variables to corresponding displacements and velocities.

1.3 Approximate methods for non-classically damped systems

Efforts by engineers and researchers in the mid 20th century to the present day to analyze non-classically damped systems while avoiding state space techniques, but exploiting the ease and physical insight of classical modal analysis, have led to the development of numerous schemes for quantifying the degree of coordinate coupling, through so-called indices of coupling or non-proportionality (e.g., see [13–24]), and for approximating the system response via analysis in the configuration space. For example, Knowles has proposed replacing the original system matrices with other simultaneously diagonalizable matrices such that the difference between the two systems, in the sense of a matrix norm, is minimized [25]. However, this does not imply that the incurred error in the system response is minimized. A more common procedure is to replace the matrix $\mathbf{D}$ by some equivalent diagonal form and then proceed with classical modal analysis as usual (e.g., see [26–30]). The simplest and most popular approach involves neglecting the off-diagonal elements of $\mathbf{D}$. This technique is commonly justified so long as $\mathbf{D}$ is diagonally dominant [1, 2] or the natural frequencies are sufficiently far apart [31]. However, even when either or both of these conditions are satisfied, significant errors in the system response may be incurred by neglecting the off-diagonal elements of $\mathbf{D}$ because of the response's dependence on other factors, such as the type of forcing and its distribution [16, 32]. In response, various iterative techniques aimed at increasing the accuracy of solution have been developed (e.g., see [33–36]).

1.4 Exact methods for non-classically damped systems

In 2002, Garvey et al. [37, 38] introduced the notion of structure-preserving transformations and illustrated how they may be used to exactly decouple non-defective linear dynamical systems of the form (1.1). A structure-preserving transformation is defined as a real equivalence transformation $(\mathbf{U}_\ell, \mathbf{U}_r)$ such that

$$\mathbf{U}_\ell^{\mathrm{T}} \begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \mathbf{U}_r = \begin{bmatrix} \mathbf{C}_0 & \mathbf{M}_0 \\ \mathbf{M}_0 & \mathbf{O} \end{bmatrix} , \tag{1.7}$$

$$\mathbf{U}_\ell^{\mathrm{T}} \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \mathbf{U}_r = \begin{bmatrix} \mathbf{K}_0 & \mathbf{O} \\ \mathbf{O} & -\mathbf{M}_0 \end{bmatrix} , \tag{1.8}$$

$$\mathbf{U}_\ell^{\mathrm{T}} \begin{bmatrix} \mathbf{O} & \mathbf{K} \\ \mathbf{K} & \mathbf{C} \end{bmatrix} \mathbf{U}_r = \begin{bmatrix} \mathbf{O} & \mathbf{K}_0 \\ \mathbf{K}_0 & \mathbf{C}_0 \end{bmatrix} , \tag{1.9}$$

where the order $2n$ left and right transformation matrices $\mathbf{U}_\ell$ and $\mathbf{U}_r$, respectively, are invertible and $\mathbf{O}$ denotes the zero matrix of order n. A structure-preserving transformation is said to be diagonalizing if the real and order n matrices $\mathbf{M}_0$, $\mathbf{C}_0$, and $\mathbf{K}_0$ are diagonal. A notable feature of a structure-preserving transformation is that it preserves the eigenvalues of system (1.1) and their multiplicities (i.e., the transformation is strictly isospectral).

How does a structure-preserving transformation decouple the equation of motion (1.1)? Begin by casting Eq. (1.1) in a symmetric state space realization, say,

$$\begin{bmatrix} \mathbf{C} & \mathbf{M} \\ \mathbf{M} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{O} \\ \mathbf{O} & -\mathbf{M} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} , \tag{1.10}$$

where the applied excitation $\mathbf{f}(t) = \mathbf{0}$ for convenience. Define a linear time-invariant coordinate transformation

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{U}_r \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} , \tag{1.11}$$

for which the order $2n$ transformation matrix $\mathbf{U}_r$ is real and nonsingular. Apply transformation (1.11) to the state space representation (1.10), and then premultiply the resulting equation by a real, order $2n$, and invertible matrix $\mathbf{U}_\ell^{\mathrm{T}}$. If the equivalence transformation $(\mathbf{U}_\ell, \mathbf{U}_r)$ is structure-preserving and diagonalizing, then Eq. (1.10) becomes

$$\begin{bmatrix} \mathbf{C}_0 & \mathbf{M}_0 \\ \mathbf{M}_0 & \mathbf{O} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{K}_0 & \mathbf{O} \\ \mathbf{O} & -\mathbf{M}_0 \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix} , \tag{1.12}$$

the upper half of which yields the decoupled equation of motion

$$\mathbf{M}_0\,\ddot{\mathbf{p}}(t)+\mathbf{C}_0\,\dot{\mathbf{p}}(t)+\mathbf{K}_0\,\mathbf{p}(t)=\mathbf{0}\ . \tag{1.13}$$

While structure-preserving transformations are certainly very powerful in principle, the various algorithms developed to generate these transformations (if they exist at all) are fairly convoluted and can be quite restrictive [37, 39, 40]. Moreover, because structure-preserving transformations are strictly isospectral, their application to defective systems (i.e., those systems with eigenvalues which do not have corresponding eigenvectors) is very limited (e.g., see [41, 42]).

1.5 Goals and motivation for exact decoupling

To be clear, our goal here is to develop an exact decoupling transformation for system (1.1) that is real, that preserves the eigenvalues of the system, and which (if possible) exists in both the configuration and state spaces. The reasons for these conditions are as follows. First, if the decoupling transformation is real, then the solution $\mathbf{p}(t)$ of the decoupled system (1.2) and its time derivative $\dot{\mathbf{p}}(t)$ can be associated with a physical system displacement and velocity, respectively, making interpretation of system behavior more manageable than when using complex modal analysis. Second, if the eigenvalues of system (1.1) are unaltered, then the decoupled system (1.2) exhibits the same fundamental characteristics as the coupled system (i.e., the coupled and decoupled systems are fundamentally related). Lastly, it is often the case that we are interested in the system response $\mathbf{x}(t)$ only and not its corresponding velocity $\dot{\mathbf{x}}(t)$ as well, and thus it is ideal to be able to decouple in the configuration space to minimize computational effort. Of course, if desired, a decoupling transformation in the configuration space may be cast in a state space form.

On a final note, the following work on decoupling any linear dynamical system of the form (1.1) is motivated by both practical and academic reasons. From a practical standpoint, decoupling facilitates system analysis and design by revealing characteristic system behaviors whose importance with respect to a desired design criterion can be evaluated. Systems derived from practical applications are almost always non-defective since it is rare to obtain exactly repeated eigenvalues, and thus the treatment of defective systems here is more of academic interest. While it is quite common in the literature to avoid the issue of systems with defective eigenvalues, recent efforts have revealed that defective systems may be more common in practice than originally thought [43, 44], and hence greater attention to the analysis of defective systems seems justified.

Chapter 2

Decoupling of Non-Defective Systems in Free Motion

In this chapter, we devise a procedure by which the unforced form of a non-defective system (1.1) (i.e., $\mathbf{f}(t) = \mathbf{0}$) is decoupled into the form (1.2) with $\mathbf{g}(t) = \mathbf{0}$ and the free response $\mathbf{x}(t)$ is recovered exactly from the decoupled system response $\mathbf{p}(t)$. We begin by discussing the solutions of the quadratic eigenvalue problem for non-defective systems in Section 2.1. We next address the issue of decoupling by first considering systems with all complex eigenvalues in Section 2.2, and we follow this with a treatment of systems with all real eigenvalues in Section 2.3. Decoupling systems with mixed eigenvalues is briefly summarized in Section 2.4, and the relationship between phase synchronization and structure-preserving transformations is explored in Section 2.5. We conclude the chapter by illustrating in Section 2.6 how the decoupling methodology developed herein is a direct generalization of classical modal analysis. Much of the presentation given here is based on [45–47], but some topics (such as eigenvalue indexing) are not discussed for the sake of generality, while others (such as real quadratic conjugation) are expanded upon for clarity.

2.1 The quadratic eigenvalue problem

Assume that $\mathbf{x}(t) = \mathbf{v}\,e^{\lambda t}$ is a solution to the homogeneous (i.e., unforced) form of the equation of motion (1.1), where $\mathbf{v}$ is an n-long vector of unspecified constants and λ is an undetermined scalar parameter. Consequently, associated with system (1.1) is the quadratic eigenvalue problem (e.g., see [48–50])

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{v} = \mathbf{0}\,, \tag{2.1}$$

the solution of which yields $2n$ complex or real eigenvalues λ_j $(j = 1, 2, \ldots, 2n)$ and up to n linearly independent eigenvectors $\mathbf{v}_j$. When system (1.1) is non-defective and simple, the $2n$ eigenvalues λ_j are distinct and may be divided into two categories: $2c$ complex and $2r = 2(n-c)$ real eigenvalues. Because the coefficient matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$ of system (1.1) are real, the $2c$ complex eigenvalues and their associated eigenvectors necessarily form c complex conjugate pairs. Some of the eigenvalues of a non-defective system (1.1) may be repeated so long as those eigenvalues that are repeated possess a full set of corresponding linearly independent eigenvectors. In this case, system (1.1) is said to be semi-simple. In either case, the free response $\mathbf{x}(t)$ of a non-defective system (1.1) may be cast in the form

$$\mathbf{x}(t) = \sum_{j=1}^{2n} c_j \mathbf{v}_j \, \mathrm{e}^{\lambda_j t} \;, \tag{2.2}$$

for which the $2n$ eigensolution coefficients c_j are determined from the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$.

2.2 Complex eigenvalues

Suppose that every eigenvalue of system (1.1) is complex. Write the n eigenvalues λ_k $(k = 1, 2, \ldots, n)$ (that constitute the n complex conjugate pairs) in the rectangular form $\lambda_k = \alpha_k + \mathrm{i}\omega_k$, where the parameters $\alpha_k < 0$ and $\omega_k > 0$ are real. The associated eigenvectors $\mathbf{v}_k$ are also complex, and it is convenient to express their elements v_{kl} $(l = 1, 2, \ldots, n)$ in polar form:

$$\mathbf{v}_k = \begin{bmatrix} v_{k1} & \cdots & v_{kn} \end{bmatrix}^{\mathrm{T}} = \begin{bmatrix} r_{k1}\mathrm{e}^{-\mathrm{i}\varphi_{k1}} & \cdots & r_{kn}\mathrm{e}^{-\mathrm{i}\varphi_{kn}} \end{bmatrix}^{\mathrm{T}} \;, \tag{2.3}$$

for which the coefficients r_{kl} and phase angles φ_{kl} are real. It is also convenient to normalize the eigenvectors $\mathbf{v}_k$ and their complex conjugates $\overline{\mathbf{v}}_k$ in accordance with

$$2\lambda_k \mathbf{v}_k \cdot (\mathbf{M}\,\mathbf{v}_k) + \mathbf{v}_k \cdot (\mathbf{C}\,\mathbf{v}_k) = \lambda_k - \overline{\lambda}_k \;, \tag{2.4}$$
$$2\overline{\lambda}_k \overline{\mathbf{v}}_k \cdot (\mathbf{M}\,\overline{\mathbf{v}}_k) + \overline{\mathbf{v}}_k \cdot (\mathbf{C}\,\overline{\mathbf{v}}_k) = \overline{\lambda}_k - \lambda_k \;, \tag{2.5}$$

which reduce to normalization with respect to the mass matrix $\mathbf{M}$ in the event that system (1.1) is either undamped or classically damped [51].

2.2.1 Underdamped modes of real vibration

Since the $2n$ complex eigensolutions of system (1.1) occur as n complex conjugate pairs, the system's free response $\mathbf{x}(t)$ has the representation

$$\mathbf{x}(t) = \sum_{k=1}^{n} \left(c_k \mathbf{v}_k \, \mathrm{e}^{\lambda_k t} + \overline{c}_k \overline{\mathbf{v}}_k \, \mathrm{e}^{\overline{\lambda}_k t} \right) = \sum_{k=1}^{n} \mathbf{s}_k(t) \;. \tag{2.6}$$

By expressing the complex coefficients c_k in polar form as $2c_k = A_k e^{-i\theta_k}$, with the coefficients A_k and phase angles θ_k real, each vector $\mathbf{s}_k(t)$ may be written as

$$\mathbf{s}_k(t) = c_k \mathbf{v}_k e^{\lambda_k t} + \bar{c}_k \bar{\mathbf{v}}_k e^{\bar{\lambda}_k t} = A_k e^{\alpha_k t} \begin{bmatrix} r_{k1} \cos(\omega_k t - \theta_k - \varphi_{k1}) \\ \vdots \\ r_{kn} \cos(\omega_k t - \theta_k - \varphi_{kn}) \end{bmatrix} = \begin{bmatrix} s_{k1}(t) \\ \vdots \\ s_{kn}(t) \end{bmatrix}. \tag{2.7}$$

We define $\mathbf{s}_k(t)$ as an underdamped mode of real vibration (or simply, an underdamped mode) since every component $s_{kl}(t)$ executes physically excitable oscillatory decay at a real characteristic exponential decay rate α_k and real damped frequency ω_k. Moreover, every mode evolves independently of one another, with any particular mode $\mathbf{s}_k(t)$ capable of being independently excited by the initial conditions

$$\mathbf{x}(0) = A_k \begin{bmatrix} r_{k1} \cos(\theta_k + \varphi_{k1}) \\ \vdots \\ r_{kn} \cos(\theta_k + \varphi_{kn}) \end{bmatrix}, \tag{2.8}$$

$$\dot{\mathbf{x}}(0) = \alpha_k A_k \begin{bmatrix} r_{k1} \cos(\theta_k + \varphi_{k1}) \\ \vdots \\ r_{kn} \cos(\theta_k + \varphi_{kn}) \end{bmatrix} + \omega_k A_k \begin{bmatrix} r_{k1} \sin(\theta_k + \varphi_{k1}) \\ \vdots \\ r_{kn} \sin(\theta_k + \varphi_{kn}) \end{bmatrix} \tag{2.9}$$

upon prescribing arbitrary values for A_k and θ_k.

It is interesting to note that each modal component $s_{kl}(t)$ is separated by a constant phase shift φ_{kl}. This observation physically manifests itself as a modal response in which all system components pass through their respective equilibria at different times, with the relative time shifts among components being constant. When a system is classically damped, all components execute synchronous motion when vibrating in a mode, passing through their respective equilibria at the same instant. Thus, the effect of non-classical damping is that it introduces relative phase drifts among system components in all modes of vibration. Should system (1.1) be classically damped, then the phase shifts $\varphi_{kl} = 0$ and the eigenvectors $\mathbf{v}_k$ coincide with the natural modes $\mathbf{u}_k$ of the undamped system (assuming the eigenvectors have been normalized in accordance with Eqs. (2.4) and (2.5)).

2.2.2 The mechanics of phase synchronization

To decouple a non-classically damped system (1.1), we must transform the damped modes $\mathbf{s}_k(t)$ so that the modal responses of all system components are either in phase or out of phase, which is characteristic of a classically damped system. To do so, we will need to eliminate the relative phase shifts φ_{kl} introduced by non-classical damping, which may be accomplished by appropriately time shifting each modal component $s_{kl}(t)$. We refer to this time-shifting transformation as phase synchronization. Suppose we shift each modal component $s_{kl}(t)$ by φ_{kl}/ω_k amount of time into another function $y_{kl}(t)$:

$$\mathbf{y}_k(t) = \begin{bmatrix} y_{k1}(t) \\ \vdots \\ y_{kn}(t) \end{bmatrix} = \begin{bmatrix} s_{k1}(t + \frac{\varphi_{k1}}{\omega_k}) \\ \vdots \\ s_{kn}(t + \frac{\varphi_{kn}}{\omega_k}) \end{bmatrix} = A_k \mathrm{e}^{\alpha_k t} \cos(\omega_k t - \theta_k) \begin{bmatrix} r_{k1} \mathrm{e}^{\frac{\alpha_k \varphi_{k1}}{\omega_k}} \\ \vdots \\ r_{kn} \mathrm{e}^{\frac{\alpha_k \varphi_{kn}}{\omega_k}} \end{bmatrix} . \tag{2.10}$$

If we take

$$p_k(t) = A_k \mathrm{e}^{\alpha_k t} \cos(\omega_k t - \theta_k) , \qquad \mathbf{z}_k = \begin{bmatrix} z_{k1} \\ \vdots \\ z_{kn} \end{bmatrix} = \begin{bmatrix} r_{k1} \mathrm{e}^{\frac{\alpha_k \varphi_{k1}}{\omega_k}} \\ \vdots \\ r_{kn} \mathrm{e}^{\frac{\alpha_k \varphi_{kn}}{\omega_k}} \end{bmatrix} , \tag{2.11}$$

it becomes clear that $\mathbf{y}_k(t) = p_k(t)\,\mathbf{z}_k$ represents a damped mode of vibration for a classically damped system whose kth modal response, characterized by oscillatory decay, is given by $p_k(t)$ and has corresponding natural mode $\mathbf{z}_k$. The form of this classically damped system (i.e., its inertia, damping, and stiffness matrices) is irrelevant to the decoupling procedure itself, but, in principle, we may recover the system matrices, if desired, by an inverse congruence transformation using a modal matrix $\mathbf{Z}$ whose columns are the natural modes $\mathbf{z}_k$. It can be verified that the response $p_k(t)$ for each decoupled coordinate satisfies the equation of motion

$$\ddot{p}_k(t) - (\lambda_k + \overline{\lambda}_k)\dot{p}_k(t) + \lambda_k \overline{\lambda}_k p_k(t) = 0 , \tag{2.12}$$

and thus, upon comparing Eq. (2.12) to the decoupled system (1.2) with $\mathbf{g}(t) = \mathbf{0}$, we observe that the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ have the structures

$$\mathbf{D} = -(\mathbf{\Lambda} + \overline{\mathbf{\Lambda}}) , \qquad \mathbf{\Omega} = \mathbf{\Lambda}\overline{\mathbf{\Lambda}} , \tag{2.13}$$

where $\mathbf{\Lambda}$ is an order n matrix of the eigenvalues λ_k on the diagonal:

$$\mathbf{\Lambda} = \bigoplus_{k=1}^{n} \lambda_k . \tag{2.14}$$

Moreover, we conclude that phase synchronization constitutes a strictly isospectral transformation between the non-classically damped system (1.1) and its decoupled form (1.2) since the eigenvalues and their multiplicities are preserved. Preservation of the system eigenvalues is important since it implies that there exists a fundamental similarity between the original and decoupled systems.

2.2.3 *A configuration space representation of decoupling*

The system modes $\mathbf{s}_k(t)$, in terms of the modal responses $p_k(t)$ and natural modes $\mathbf{z}_k$, may be recovered from the transformed modes $\mathbf{y}_k(t)$ by inverting the time-shifting transformation:

$$\mathbf{s}_k(t) = \begin{bmatrix} y_{k1}(t - \frac{\varphi_{k1}}{\omega_k}) \\ \vdots \\ y_{kn}(t - \frac{\varphi_{kn}}{\omega_k}) \end{bmatrix} = \begin{bmatrix} p_k(t - \frac{\varphi_{k1}}{\omega_k})\, z_{k1} \\ \vdots \\ p_k(t - \frac{\varphi_{kn}}{\omega_k})\, z_{kn} \end{bmatrix} = \bigoplus_{l=1}^{n} p_k(t - \tfrac{\varphi_{kl}}{\omega_k})\, \mathbf{z}_k \,. \tag{2.15}$$

Since the free response $\mathbf{x}(t)$ is a linear superposition of the modes $\mathbf{s}_k(t)$,

$$\mathbf{x}(t) = \sum_{k=1}^{n} \bigoplus_{l=1}^{n} p_k(t - \tfrac{\varphi_{kl}}{\omega_k})\, \mathbf{z}_k \,. \tag{2.16}$$

Thus, we have shown that a non-classically damped system (1.1) with complex eigenvalues may be decoupled into and retrieved from system (1.2) by a linear time-shifting transformation in the n-dimensional configuration space.

It is interesting to note that the decoupled solutions $p_k(t)$ are not connected to the system free response $\mathbf{x}(t)$ at the same instant in time, which is a direct consequence of the time shifting resulting from phase synchronization. Unfortunately, this disconnect in time is an inconvenience when determining the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ in terms of the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. In addition, it is desirable to avoid extracting the phase shifts φ_{kl} and constructing the modal vectors $\mathbf{z}_k$ after having to solve the system's quadratic eigenvalue problem. For these reasons and for subsequent developments, it is more convenient to consider decoupling in the $2n$-dimensional state space.

2.2.4 *A state space representation of decoupling*

Based on the paired eigensolution summation representation of the free response $\mathbf{x}(t)$ in Eq. (2.6), we may cast $\mathbf{x}(t)$ in the matrix-vector form

$$\mathbf{x}(t) = \mathbf{V}\mathrm{e}^{\mathbf{\Lambda}t}\mathbf{c} + \overline{\mathbf{V}}\mathrm{e}^{\overline{\mathbf{\Lambda}}t}\overline{\mathbf{c}} \,, \tag{2.17}$$

where $\mathbf{V}$ is an order n matrix whose columns are the system eigenvectors $\mathbf{v}_k$, and $\mathbf{c}$ is an n-dimensional vector of the corresponding eigensolution coefficients c_k:

$$\mathbf{V} = \begin{bmatrix} \mathbf{v}_1 \,\big|\, \cdots \,\big|\, \mathbf{v}_n \end{bmatrix} , \qquad \mathbf{c} = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix}^{\mathrm{T}} . \tag{2.18}$$

Writing Eq. (2.17) and its derivative as a state equation gives us

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V\Lambda}} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{\Lambda}t}\mathbf{c} \\ \mathrm{e}^{\overline{\mathbf{\Lambda}}t}\overline{\mathbf{c}} \end{bmatrix} . \tag{2.19}$$

Since $2c_k = A_k \mathrm{e}^{-\mathrm{i}\theta_k}$, Eqs. (2.10) and (2.11) imply that phase synchronization does not disturb the eigensolution coefficients c_k, and hence the decoupled solutions $p_k(t)$ have the equivalent representations

$$p_k(t) = A_k \mathrm{e}^{\alpha_k t} \cos(\omega_k t - \theta_k) = c_k \mathrm{e}^{\lambda_k t} + \overline{c}_k \mathrm{e}^{\overline{\lambda}_k t} . \tag{2.20}$$

Consequently, the modal response $\mathbf{p}(t)$ may be expressed in matrix-vector form as

$$\mathbf{p}(t) = \mathrm{e}^{\mathbf{\Lambda}t}\mathbf{c} + \mathrm{e}^{\overline{\mathbf{\Lambda}}t}\overline{\mathbf{c}} . \tag{2.21}$$

Casting Eq. (2.21) and its derivative in the form of a state equation,

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{\Lambda}t}\mathbf{c} \\ \mathrm{e}^{\overline{\mathbf{\Lambda}}t}\overline{\mathbf{c}} \end{bmatrix} , \tag{2.22}$$

where $\mathbf{I}$ is the identity matrix of order n, unless otherwise indicated by a subscript. Combining Eqs. (2.19) and (2.22) yields the state space representation of the free response $\mathbf{x}(t)$ for system (1.1):

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} . \tag{2.23}$$

Curiously, while the decoupling transformation in the configuration space is linear and time-shifting in nature, decoupling in the state space is achieved through a linear time-invariant transformation. It should be noted that, while some of the matrices in Eq. (2.23) contain complex elements, the overall transformation is real. Since the original and decoupled systems are connected at the same instant in time, representation (2.23) is convenient for calculating the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ given the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. Indeed, by inverting Eq. (2.23) and setting $t = 0$, we obtain the initial conditions for the decoupled system:

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V\Lambda} & \overline{\mathbf{V\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} . \tag{2.24}$$

Upon determining $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$, the free response $p_k(t)$ for each decoupled coordinate may be evaluated exactly as

$$p_k(t) = \mathrm{e}^{\alpha_k t} \left[p_k(0) \cos \omega_k t + \left(\frac{\dot{p}_k(0) - \alpha_k p_k(0)}{\omega_k} \right) \sin \omega_k t \right] . \tag{2.25}$$

In addition to streamlining the calculation of the modal initial conditions, the state space representation (2.23) has the advantage of simplifying the decoupling transformation by eliminating time shifting of each of the decoupled solutions $p_k(t)$. Moreover, it is interesting to note that, because the $2n$ complex eigenvalues necessarily form n pairs of complex conjugates, the order $2n$ transformation matrices of Eq. (2.23) are partitioned into blocks of size n. Consequently, it is possible to extract a concise analytical expression for the free response $\mathbf{x}(t)$ only from transformation (2.23), making it unnecessary to evaluate the larger state equation. Isolating the upper half of Eq. (2.23) yields

$$\mathbf{x}(t) = \mathbf{T}_1 \, \mathbf{p}(t) + \mathbf{T}_2 \, \dot{\mathbf{p}}(t) \ , \tag{2.26}$$

where the real coefficient matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are given by

$$\mathbf{T}_1 = (\mathbf{V}\overline{\mathbf{\Lambda}} - \overline{\mathbf{V}}\mathbf{\Lambda})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} \ , \qquad \mathbf{T}_2 = (\overline{\mathbf{V}} - \mathbf{V})(\overline{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} \ . \tag{2.27}$$

Unlike the transformation of Eq. (2.16), recovering the free response $\mathbf{x}(t)$ via Eq. (2.26) requires not only the modal displacements $p_k(t)$, but the corresponding velocities $\dot{p}_k(t)$ as well, which can be obtained exactly from Eq. (2.25). If we consider the operator $\mathcal{L} = \mathbf{T}_1 + \mathbf{T}_2 \, \mathrm{d}/\mathrm{d}t$, then transformation (2.26) represents a linear mapping between $\mathbf{x}(t)$ and $\mathbf{p}(t)$: $\mathbf{x}(t) = \mathcal{L}\,\mathbf{p}(t)$.

To summarize, if the coupled n-degree-of-freedom system (1.1) is non-defective and possesses complex eigenvalues only (but not necessarily distinct), it can be decoupled by phase synchronization into a set of n independent, underdamped oscillators with initial conditions given by Eq. (2.24). Upon solution of the decoupled system (1.2), the free response $\mathbf{x}(t)$ may be recovered through either transformation (2.16) or (2.26). All parameters required for decoupling are obtained by solving the quadratic eigenvalue problem (2.1).

2.3 Real eigenvalues

Now suppose that all eigenvalues of system (1.1) are real, and thus the associated eigenvectors may be taken as real. Since energy is dissipated due to viscous damping, every eigenvalue is also negative. Consequently, the free response $\mathbf{x}(t)$ is described by pure exponential decay. For the case when the eigenvalues of system (1.1) are complex, we have demonstrated how the order n equation of motion (1.1) can be decoupled into a system of n independent oscillators of the form (1.2) by synchronizing the components of the non-classically damped modes of vibration to yield classically damped modes (i.e., via phase synchronization). When system (1.1) possesses all real eigenvalues, it is not immediately clear if or how the non-classically damped system may be decoupled. However, we will show that, with some new terminology and slight modification, phase synchronization can also be used to decouple a non-oscillatory system (1.1).

2.3.1 The concept of real quadratic conjugation

Consider a quadratic equation $P(c) = 0$ with real coefficients that has roots c_1 and c_2. Assuming c_1 and c_2 are distinct, there are two possibilities regarding the nature of the roots: either c_1 and c_2 are complex conjugates, or both roots are real. If c_1 and c_2 are real, we may regard them as being real quadratic conjugates. Making use of complex notation, express the first real root $c_1 = c$ in rectangular form: $c = a + \mathrm{i}b$, where a is real and b is imaginary. The second real root $c_2 = \widetilde{c}$ is interpreted as the real quadratic conjugate of the first: $\widetilde{c} = a - \mathrm{i}b$. By simple algebraic manipulation, it can be verified that the rectangular components a and b are given by

$$a = \frac{1}{2}\left(\widetilde{c} + c\right) , \qquad b = \frac{\mathrm{i}}{2}\left(\widetilde{c} - c\right) . \tag{2.28}$$

It is also possible to write c and its real quadratic conjugate $\widetilde{c}$ in polar form: $c = r\mathrm{e}^{-\mathrm{i}\theta}$, and hence $\widetilde{c} = r\mathrm{e}^{\mathrm{i}\theta}$, where the coefficient r satisfies

$$r^2 = c\widetilde{c} \tag{2.29}$$

and may either be real or imaginary, depending on the signs of c and $\widetilde{c}$. Likewise, the signs of the real quadratic conjugate roots dictate if the phase angle θ is imaginary or complex:

$$\theta = \begin{cases} \dfrac{\mathrm{i}}{2} \ln\left(\dfrac{c}{\widetilde{c}}\right) & \text{for } \dfrac{c}{\widetilde{c}} > 0 , \\[2ex] -\dfrac{\pi}{2} + \dfrac{\mathrm{i}}{2} \ln\left|\dfrac{c}{\widetilde{c}}\right| & \text{for } \dfrac{c}{\widetilde{c}} < 0 . \end{cases} \tag{2.30}$$

Limiting cases (such as when $c = 0$ or $\widetilde{c} = 0$) should be interpreted appropriately.

While a complex number has a unique complex conjugate, the same is not true of real quadratic conjugation. To illustrate this point, let $P(c) = 0$ be a fourth order polynomial equation with distinct real roots c_i $(i = 1, 2, 3, 4)$. One may assign (c_1, c_2) and (c_3, c_4) as real quadratic conjugate pairs. Associated with these pairs are two quadratic polynomials $P_{12}(c)$ and $P_{34}(c)$, respectively, whose product necessarily yields the original fourth order polynomial: $P(c) = P_{12}(c)P_{34}(c)$. However, we may just as well take (c_1, c_3) and (c_2, c_4) to be real quadratic conjugate pairs, and the product of the associated quadratic polynomials $P_{13}(c)$ and $P_{24}(c)$, respectively, must also generate $P(c)$. Thus, the same fourth order polynomial $P(c)$ may be factored into different sets of real quadratic conjugate pairs. In general, for an order $2n$ polynomial equation with all real roots, there are $(2n)!/(2^n n!)$ different ways to pair the roots as real quadratic conjugates.

2.3.2 *Pure exponential decay as imaginary vibration*

Consider a single-degree-of-freedom system with displacement $x(t)$ that is overdamped and in free motion. By assuming a solution of the form $x(t) = c\mathrm{e}^{\lambda t}$, where c and λ are scalar parameters to be determined, we obtain a quadratic characteristic equation whose roots yield the system eigenvalues λ_1 and λ_2, which are distinct, real, and negative. Consequently, the system's free response $x(t)$ takes the form

$$x(t) = c_1\mathrm{e}^{\lambda_1 t} + c_2\mathrm{e}^{\lambda_2 t} , \tag{2.31}$$

for which the coefficients c_1 and c_2, obtained by applying initial conditions, must be real. In light of our previous discussion on real quadratic conjugation, express the real eigenvalues in complex notation such that $\lambda = \lambda_1 = \alpha + \mathrm{i}\omega$ and $\widetilde{\lambda} = \lambda_2 = \alpha - \mathrm{i}\omega$, where α and ω are calculated as

$$\alpha = \frac{1}{2}(\widetilde{\lambda} + \lambda) , \qquad \omega = \frac{\mathrm{i}}{2}(\widetilde{\lambda} - \lambda) , \tag{2.32}$$

with $\lambda < \widetilde{\lambda} < 0$ so that ω is a positive imaginary number by convention. In addition, pair the coefficients of Eq. (2.31) as real quadratic conjugates according to $c = c_1$ and $\widetilde{c} = c_2$, and write the coefficient c in polar form: $2c = A\mathrm{e}^{-\mathrm{i}\theta}$. In doing so, the overdamped system's non-oscillatory free response $x(t)$ has the equivalent representations

$$x(t) = c\mathrm{e}^{\lambda t} + \widetilde{c}\mathrm{e}^{\widetilde{\lambda} t} = A\mathrm{e}^{\alpha t}\cos(\omega t - \theta) . \tag{2.33}$$

It is implied by Eq. (2.33) that pure exponential decay may be thought of as exponentially decaying oscillation with real decay rate α at an imaginary damped frequency ω. While it may be disconcerting that, depending on the signs of the coefficients c and $\widetilde{c}$, the amplitude A of "oscillation" may possibly be imaginary and the phase angle θ will either be imaginary or complex, the combined effect is such that the system response $x(t)$ will always be real. Thus, using the concept of real quadratic conjugation, we have shown that the free response of an overdamped system is functionally identical to that of an underdamped system vibrating at an imaginary frequency. It is this result that sets the stage for extending phase synchronization to decoupling non-oscillatory systems.

2.3.3 *Overdamped modes of imaginary vibration*

It is possible to extend our previous discussion on imaginary vibration of an overdamped single-degree-of-freedom oscillator to higher dimensional systems. If the $2n$ eigensolutions of the order n non-defective system (1.1) are real, they may be grouped into n real quadratic conjugate pairs so that the system's free response $\mathbf{x}(t)$ can be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{n}\left(c_k\mathbf{v}_k\mathrm{e}^{\lambda_k t} + \widetilde{c}_k\widetilde{\mathbf{v}}_k\mathrm{e}^{\widetilde{\lambda}_k t}\right) = \sum_{k=1}^{n}\mathbf{s}_k(t) . \tag{2.34}$$

As in the single-degree-of-freedom case, we shall express the eigenvalues λ_k and coefficients c_k in rectangular and polar form, respectively: $\lambda_k = \alpha_k + \mathrm{i}\omega_k$, where the real decay rate α_k and imaginary damped frequency ω_k are determined from Eq. (2.32), and $2c_k = A_k \mathrm{e}^{-\mathrm{i}\theta_k}$. Paralleling the case when the system eigenvalues are complex, normalize the eigenvectors $\mathbf{v}_k$ and their real quadratic conjugates $\widetilde{\mathbf{v}}_k$ according to

$$2\lambda_k \mathbf{v}_k \cdot (\mathbf{M}\,\mathbf{v}_k) + \mathbf{v}_k \cdot (\mathbf{C}\,\mathbf{v}_k) = \lambda_k - \widetilde{\lambda}_k \,, \tag{2.35}$$

$$2\widetilde{\lambda}_k \widetilde{\mathbf{v}}_k \cdot (\mathbf{M}\,\widetilde{\mathbf{v}}_k) + \widetilde{\mathbf{v}}_k \cdot (\mathbf{C}\,\widetilde{\mathbf{v}}_k) = \widetilde{\lambda}_k - \lambda_k \,. \tag{2.36}$$

Should system (1.1) be undamped or classically damped, Eqs. (2.35) and (2.36) reduce to normalization with respect to the mass matrix $\mathbf{M}$. In addition, if the real eigenvector components are cast in the polar form $v_{kl} = r_{kl} \mathrm{e}^{-\mathrm{i}\varphi_{kl}}$, as in Eq. (2.3), then each vector $\mathbf{s}_k(t)$ has the representation

$$\mathbf{s}_k(t) = c_k \mathbf{v}_k \mathrm{e}^{\lambda_k t} + \widetilde{c}_k \widetilde{\mathbf{v}}_k \mathrm{e}^{\widetilde{\lambda}_k t} = A_k \mathrm{e}^{\alpha_k t} \begin{bmatrix} r_{k1} \cos(\omega_k t - \theta_k - \varphi_{k1}) \\ \vdots \\ r_{kn} \cos(\omega_k t - \theta_k - \varphi_{kn}) \end{bmatrix} . \tag{2.37}$$

We refer to $\mathbf{s}_k(t)$ as an overdamped mode of imaginary vibration (or simply, an overdamped mode) since every component exhibits oscillatory decay at a real exponential decay rate α_k and imaginary damped frequency ω_k. While the parameters A_k, r_{kl}, θ_k, and φ_{kl} may not be real because of real quadratic conjugation, every overdamped mode $\mathbf{s}_k(t)$ is necessarily real. Moreover, the evolution of any particular mode is independent of the others, and any mode $\mathbf{s}_k(t)$ may be independently excited by the real initial conditions (2.8) and (2.9).

2.3.4 Phase synchronization of imaginary vibration

As we have demonstrated, from a mathematical standpoint, there need not be a distinction between the functional representation of oscillatory and non-oscillatory motions since pure exponential decay may be thought of as oscillatory decay at an imaginary damped frequency. Consequently, decoupling of a non-classically damped system (1.1) with all real eigenvalues is treated in essentially the same manner as when the system eigenvalues are all complex, with some minor differences here and there. For example, decoupling of non-oscillatory systems in the configuration space is still achieved via phase synchronization, but the associated time shifts may no longer be real.

So long as complex conjugation is replaced with real quadratic conjugation, the decoupling transformation developed in Section 2.2 for oscillatory systems is directly applicable to non-oscillatory systems. Specifically, arrange the n real eigenvalues (that constitute the n real quadratic conjugate pairs) and their associated eigenvectors in the order n matrices $\mathbf{\Lambda}$ and $\mathbf{V}$, respectively, in accordance with Eqs. (2.14) and (2.18), and let $\widetilde{\mathbf{\Lambda}}$ and $\widetilde{\mathbf{V}}$ be the corresponding matrices of assigned real quadratic conjugates. Phase synchronization of the overdamped modes of imaginary vibration yields coefficient matrices for the decoupled system (1.2) given by $\mathbf{D} = -(\mathbf{\Lambda} + \widetilde{\mathbf{\Lambda}})$ and $\mathbf{\Omega} = \mathbf{\Lambda}\widetilde{\mathbf{\Lambda}}$, and the free response $\mathbf{x}(t)$ of the original system may be recovered from the modal free response $\mathbf{p}(t)$ in the state space by

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \widetilde{\mathbf{V}} \\ \mathbf{V\Lambda} & \widetilde{\mathbf{V\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \widetilde{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} . \tag{2.38}$$

The response $p_k(t)$ for each decoupled coordinate is evaluated exactly as

$$p_k(t) = \left(\frac{\widetilde{\lambda}_k p_k(0) - \dot{p}_k(0)}{\widetilde{\lambda}_k - \lambda_k} \right) \mathrm{e}^{\lambda_k t} - \left(\frac{\lambda_k p_k(0) - \dot{p}_k(0)}{\widetilde{\lambda}_k - \lambda_k} \right) \mathrm{e}^{\widetilde{\lambda}_k t} \tag{2.39}$$

upon determining the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$:

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \widetilde{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \widetilde{\mathbf{V}} \\ \mathbf{V\Lambda} & \widetilde{\mathbf{V\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} . \tag{2.40}$$

By isolating the upper half of the state equation (2.38), we observe that the free response $\mathbf{x}(t)$ may still be obtained directly via Eq. (2.26), but the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are now given by

$$\mathbf{T}_1 = (\mathbf{V}\widetilde{\mathbf{\Lambda}} - \widetilde{\mathbf{V}}\mathbf{\Lambda})(\widetilde{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} , \qquad \mathbf{T}_2 = (\widetilde{\mathbf{V}} - \mathbf{V})(\widetilde{\mathbf{\Lambda}} - \mathbf{\Lambda})^{-1} . \tag{2.41}$$

In summary, if the order n non-classically damped system (1.1) is non-defective and has all real eigenvalues that are distinct, then, using the concept of real quadratic conjugation, it may be decoupled by phase synchronization into a set of n independent, overdamped oscillators with initial conditions (2.40). After solving the decoupled system (1.2), transformation (2.26) may be used to recover the free response $\mathbf{x}(t)$. All parameters required for decoupling are obtained through solution of the quadratic eigenvalue problem (2.1). Because of the non-uniqueness associated with real quadratic conjugation, there are $(2n)!/(2^n n!)$ different forms of the decoupled system (1.2), but the decoupling transformation (2.26) will of course yield the same free response $\mathbf{x}(t)$ regardless of the chosen pairing scheme.

Should some eigenvalues be repeated, Eq. (2.26) remains a valid decoupling transformation so long as the repeated eigenvalues are not paired as real quadratic conjugates. The reason for this is clear upon inspecting Eq. (2.39) – pairing (non-defective) repeated real eigenvalues implies that the associated modal solutions $p_k(t)$ are undefined. Moreover, the pairing of repeated real eigenvalues as real quadratic conjugates implies that the corresponding degree of freedom is critically damped, which cannot be the case when system (1.1) is non-defective. The issue of critical damping will be discussed when the decoupling of defective systems is addressed in Chapter 4.

2.4 Mixed eigenvalues

In general, the eigenspectrum of system (1.1) consists of some combination of complex and real eigenvalues. Suppose $2c$ of the system eigenvalues are complex and the remaining $2r = 2(n-c)$ eigenvalues are real. The $2c$ complex eigenvalues necessarily form c complex conjugate pairs, and the $2r$ real eigenvalues may be grouped into r real quadratic conjugate pairs. Based on the methodologies developed previously for decoupling systems with all complex or all real eigenvalues, an extension to systems with mixed eigenvalues is straightforward. The order n matrix $\mathbf{\Lambda}$ of Eq. (2.14) is now composed of some arrangement of the c complex and r real eigenvalues (that constitute the c complex conjugate and r real quadratic conjugate pairs, respectively) on the diagonal. While the particular arrangement of eigenvalues in $\mathbf{\Lambda}$ is not important, it may be convenient to, say, partition $\mathbf{\Lambda}$ so that the c complex eigenvalues are followed by the r real eigenvalues, or vice versa. Normalize the eigenvectors and their conjugates according to, respectively,

$$2\lambda_k \mathbf{v}_k \cdot (\mathbf{M}\,\mathbf{v}_k) + \mathbf{v}_k \cdot (\mathbf{C}\,\mathbf{v}_k) = \lambda_k - \hat{\lambda}_k \ , \tag{2.42}$$

$$2\hat{\lambda}_k \hat{\mathbf{v}}_k \cdot (\mathbf{M}\,\hat{\mathbf{v}}_k) + \hat{\mathbf{v}}_k \cdot (\mathbf{C}\,\hat{\mathbf{v}}_k) = \hat{\lambda}_k - \lambda_k \ , \tag{2.43}$$

in which case the eigenvectors are normalized with respect to the mass matrix $\mathbf{M}$ if it so happens that system (1.1) is undamped or classically damped. In normalization (2.43), the ornamenting hat denotes either complex or real quadratic conjugation, whichever is appropriate. However the eigenvalues are arranged in $\mathbf{\Lambda}$, the corresponding order n matrix $\mathbf{V}$ of eigenvectors, defined in Eq. (2.18), is constructed to be conformable to $\mathbf{\Lambda}$. The structure of the conjugate matrices $\widehat{\mathbf{\Lambda}}$ and $\widehat{\mathbf{V}}$ is dictated by the choice of pairing scheme for the real eigenvalues. Analogous to Eqs. (2.6) and (2.34), the system's free response $\mathbf{x}(t)$ can be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{n} \left(c_k \mathbf{v}_k \, \mathrm{e}^{\lambda_k t} + \hat{c}_k \hat{\mathbf{v}}_k \, \mathrm{e}^{\hat{\lambda}_k t} \right) = \sum_{k=1}^{n} \mathbf{s}_k(t) \ . \tag{2.44}$$

Simultaneous application of phase synchronization to the underdamped and overdamped modes $\mathbf{s}_k(t)$ reveals that the decoupled system (1.2) has as its coefficient matrices

$$\mathbf{D} = -(\mathbf{\Lambda} + \widehat{\mathbf{\Lambda}}) \ , \qquad \mathbf{\Omega} = \mathbf{\Lambda}\widehat{\mathbf{\Lambda}} \ , \tag{2.45}$$

and that the general form of the response $p_k(t)$ for each decoupled coordinate is

$$p_k(t) = c_k \mathrm{e}^{\lambda_k t} + \hat{c}_k \mathrm{e}^{\hat{\lambda}_k t} \ . \tag{2.46}$$

The free response $\mathbf{x}(t)$ of system (1.1) is related to the modal free response $\mathbf{p}(t)$ by the state equation

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V}\boldsymbol{\Lambda} & \widehat{\mathbf{V}\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \widehat{\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} , \tag{2.47}$$

and hence the modal initial conditions are determined from the system initial conditions according to

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \widehat{\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V}\boldsymbol{\Lambda} & \widehat{\mathbf{V}\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} . \tag{2.48}$$

Extracting the upper half of the order $2n$ state equation (2.47) yields the order n transformation (2.26) that gives the free response $\mathbf{x}(t)$ directly. Of course, the coefficient matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ must be modified slightly to account for conjugation of both complex and real eigenvalues and eigenvectors:

$$\mathbf{T}_1 = (\mathbf{V}\widehat{\boldsymbol{\Lambda}} - \widehat{\mathbf{V}}\boldsymbol{\Lambda})(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} , \qquad \mathbf{T}_2 = (\widehat{\mathbf{V}} - \mathbf{V})(\widehat{\boldsymbol{\Lambda}} - \boldsymbol{\Lambda})^{-1} . \tag{2.49}$$

Thus, when the coupled n-degree-of-freedom system (1.1) is non-defective and possesses mixed eigenvalues, it may be decoupled by phase synchronization into c underdamped and $r = n - c$ overdamped, independent oscillators with initial conditions governed by Eq. (2.48). Upon solution of the decoupled system (1.2), the free response $\mathbf{x}(t)$ is obtained via transformation (2.26). All parameters required for decoupling are obtained by solving the quadratic eigenvalue problem (2.1). Assuming the real eigenvalues are distinct, there are $(2r)!/(2^r r!)$ possible forms of the decoupled system (1.2) that depend on the choice of pairing scheme. Transformation (2.26) still holds when some of the eigenvalues, complex or real, are repeated, as long as the repeated real eigenvalues are not paired as real quadratic conjugates to avoid introducing critical damping where it does not exist.

2.5 Phase synchronization and structure-preserving transformations

We now demonstrate how phase synchronization, when cast in a state space form, generates a diagonalizing structure-preserving transformation. For the sake of generality, suppose system (1.1) possesses mixed eigenvalues, and further assume that the corresponding eigenvectors are normalized in accordance with Eqs. (2.42) and (2.43). We have shown that decoupling via phase synchronization yields the linear time-invariant coordinate transformation (2.47) in the state space. It can be readily verified that by applying the coordinate transformation (2.47) to the symmetric state space realization (1.10) and then premultiplying the resulting equation by $\mathbf{S}^{\mathrm{T}}$, we obtain the state equation

$$\begin{bmatrix} \mathbf{D} & \mathbf{I} \\ \mathbf{I} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{\Omega} & \mathbf{O} \\ \mathbf{O} & -\mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{2.50}$$

the upper half of which contains the decoupled equation of motion (1.2) with $\mathbf{g}(t) = \mathbf{0}$. Comparing the state equation (2.50) with Eq. (1.12), it is clear that the state space formulation of phase synchronization coincides with a diagonalizing structure-preserving transformation with $\mathbf{U}_\ell = \mathbf{U}_r = \mathbf{S}$ and coefficient matrices $\mathbf{M}_0 = \mathbf{I}$, $\mathbf{C}_0 = \mathbf{D}$, and $\mathbf{K}_0 = \mathbf{\Omega}$. The advantage of the decoupling procedure described herein is that generating the order $2n$ transformation matrix $\mathbf{S}$ is far simpler than constructing a diagonalizing structure-preserving transformation $(\mathbf{U}_\ell, \mathbf{U}_r)$ by the algorithms detailed in [37, 39].

2.6 Reduction to classical modal analysis

The decoupling procedure developed herein represents a direct generalization of classical modal analysis. First, consider the case in which the eigenvalues of system (1.1) are all complex and the associated eigenvectors $\mathbf{v}_k$ are normalized in accordance with Eqs. (2.4) and (2.5). In the event that system (1.1) is classically damped, the eigenvectors $\mathbf{v}_k = \overline{\mathbf{v}}_k$ coincide with the classical normal modes $\mathbf{u}_k$ of the undamped system. As a result, the matrix of eigenvectors $\mathbf{V} = \overline{\mathbf{V}} = \mathbf{U}$, and hence the transformation matrices $\mathbf{T}_1 = \mathbf{U}$ and $\mathbf{T}_2 = \mathbf{O}$. Consequently, Eq. (2.26) simplifies to the classical modal transformation $\mathbf{x}(t) = \mathbf{U}\,\mathbf{p}(t)$.

Now suppose system (1.1) has all real eigenvalues and the corresponding eigenvectors are normalized using Eqs. (2.35) and (2.36). Should system (1.1) be classically damped, then the set of system eigenvectors $\mathbf{v}_k$ and $\widetilde{\mathbf{v}}_k$ coincides with the set of classical normal modes $\mathbf{u}_k$. Among the $(2n)!/(2^n n!)$ different ways to pair the real eigenvalues, there is a particular pairing scheme for which $\mathbf{v}_k = \widetilde{\mathbf{v}}_k = \mathbf{u}_k$, and hence reduction to classical modal analysis for a non-oscillatory system with this choice of pairing follows in the same manner as for an oscillatory system. Because this pairing scheme may not be the one chosen for decoupling, reduction to classical modal analysis for a non-oscillatory system is not as clean as for an oscillatory system, for which there is a natural pairing scheme.

Finally, for the general case of mixed eigenvalues (i.e., $2c$ are complex and $2r = 2(n-c)$ are real), it is obvious that reduction to classical modal analysis is achieved for the particular pairing scheme (of the $(2r)!/(2^r r!)$ different ways to pair the real eigenvalues) that results in $\mathbf{V} = \widehat{\mathbf{V}} = \mathbf{U}$, where it is assumed that the eigenvectors have been normalized according to Eqs. (2.42) and (2.43). Should the eigenvectors not be normalized as such, reduction to classical modal analysis is still achieved, albeit with $\mathbf{V} \neq \mathbf{U}$, since eigenvectors are unique up to a multiplicative constant.

2.7 An illustrative example

Here we provide a numerical example that illustrates the decoupling procedure for an unforced non-defective system of the form (1.1). We focus on a system with all real eigenvalues to highlight the non-uniqueness associated with real quadratic conjugation. In addition, without loss of generality, we take the mass matrix $\mathbf{M} = \mathbf{I}$ in the following example (and in subsequent examples) for convenience since we may always convert a system

$$\mathbf{M}_1 \ddot{\mathbf{x}}_1(t) + \mathbf{C}_1 \dot{\mathbf{x}}_1(t) + \mathbf{K}_1 \mathbf{x}_1(t) = \mathbf{f}_1(t) \; , \tag{2.51}$$

where the coefficient matrices $\mathbf{M}_1$, $\mathbf{C}_1$, and $\mathbf{K}_1$ are positive definite, into system (1.1) with $\mathbf{M} = \mathbf{I}$ through a coordinate transformation $\mathbf{x}_1(t) = \mathbf{M}_1^{-\frac{1}{2}} \mathbf{x}(t)$, followed by multiplication on the left by $\mathbf{M}_1^{-\frac{1}{2}}$:

$$\mathbf{C} = \mathbf{M}_1^{-\frac{1}{2}} \mathbf{C}_1 \mathbf{M}_1^{-\frac{1}{2}} \; , \qquad \mathbf{K} = \mathbf{M}_1^{-\frac{1}{2}} \mathbf{K}_1 \mathbf{M}_1^{-\frac{1}{2}} \; , \qquad \mathbf{f}(t) = \mathbf{M}_1^{-\frac{1}{2}} \mathbf{f}_1(t) \; . \tag{2.52}$$

Additional examples of decoupling non-defective systems in free motion can be found in [45–47].

Example 1

Consider a non-classically damped, 2-degree-of-freedom system with mass matrix $\mathbf{M} = \mathbf{I}_2$ and for which the damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{C} = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} , \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} . \tag{2.53}$$

The initial conditions are prescribed as $\mathbf{x}(0) = [1, -1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [1, 1]^{\mathrm{T}}$. Solving the associated quadratic eigenvalue problem, we find that the system's eigenvalues are all real and distinct (i.e., the system is non-defective):

$$\lambda_1 = -3.73 \; , \qquad \lambda_2 = -2 \; , \qquad \lambda_3 = -1 \; , \qquad \lambda_4 = -0.27 \; . \tag{2.54}$$

Of the $(2 \cdot 2)!/(2^2 \cdot 2!) = 3$ possible ways to pair the eigenvalues λ_i $(i = 1, 2, 3, 4)$ as real quadratic conjugates, suppose we assign (λ_1, λ_3) and (λ_2, λ_4) as conjugate pairs. The associated eigenvectors $\mathbf{v}_i$, normalized in accordance with Eqs. (2.42) and (2.43) for the chosen pairing scheme, are given by

$$\mathbf{v}_1 = \begin{bmatrix} -0.58 \\ 0.79 \end{bmatrix} , \qquad \mathbf{v}_2 = \begin{bmatrix} 0.76 \\ 0.76 \end{bmatrix} , \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 1.17 \end{bmatrix} , \qquad \mathbf{v}_4 = \begin{bmatrix} 0.89 \\ 0.33 \end{bmatrix} . \tag{2.55}$$

For our choice of real quadratic conjugate pairs, Eqs. (2.14) and (2.18) imply that

$$\mathbf{\Lambda} = \begin{bmatrix} -3.73 & 0 \\ 0 & -2 \end{bmatrix}, \qquad \mathbf{V} = \left[\begin{array}{c|c} -0.58 & 0.76 \\ 0.79 & 0.76 \end{array}\right], \tag{2.56}$$

whose corresponding conjugates are

$$\widehat{\mathbf{\Lambda}} = \begin{bmatrix} -1 & 0 \\ 0 & -0.27 \end{bmatrix}, \qquad \widehat{\mathbf{V}} = \left[\begin{array}{c|c} 0 & 0.89 \\ 1.17 & 0.33 \end{array}\right]. \tag{2.57}$$

From Eq. (2.49),

$$\mathbf{T}_1 = \begin{bmatrix} 0.21 & 0.91 \\ 1.31 & 0.26 \end{bmatrix}, \qquad \mathbf{T}_2 = \begin{bmatrix} 0.21 & 0.07 \\ 0.14 & -0.25 \end{bmatrix}. \tag{2.58}$$

By Eqs. (2.45), (2.56), and (2.57), the coefficient matrices for the decoupled system are

$$\mathbf{D} = \begin{bmatrix} 4.73 & 0 \\ 0 & 2.27 \end{bmatrix}, \qquad \mathbf{\Omega} = \begin{bmatrix} 3.73 & 0 \\ 0 & 0.54 \end{bmatrix}, \tag{2.59}$$

which describe a set of independent, overdamped degrees of freedom $p_j(t)$ $(j = 1, 2)$ with viscous damping factors $\zeta_1 = 1.22$ and $\zeta_2 = 1.55$, respectively. Using Eq. (2.48), the initial conditions for the decoupled system are $\mathbf{p}(0) = [-0.94, 1.29]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [-0.01, 0.41]^{\mathrm{T}}$. The solution $\mathbf{p}(t)$ of the decoupled system defined by Eq. (2.59) is illustrated in Fig. 1(a), and the system response $\mathbf{x}(t)$ obtained via transformation (2.26) is shown in Fig. 1(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (2.26) are indeed the same. Should we have decided to let the real quadratic conjugate pairs be (λ_1, λ_2) and (λ_3, λ_4), then the overdamped decoupled system would have the form

$$\mathbf{D} = \begin{bmatrix} 5.73 & 0 \\ 0 & 1.27 \end{bmatrix}, \qquad \mathbf{\Omega} = \begin{bmatrix} 7.46 & 0 \\ 0 & 0.27 \end{bmatrix}, \tag{2.60}$$

where the corresponding viscous damping factors for $p_j(t)$ are $\zeta_1 = 1.05$ and $\zeta_2 = 1.22$, respectively. Finally, if instead (λ_1, λ_4) and (λ_2, λ_3) are assigned as conjugate pairs, then

$$\mathbf{D} = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}, \qquad \mathbf{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \tag{2.61}$$

for which the associated viscous damping factors are $\zeta_1 = 2$ and $\zeta_2 = 1.06$. It is straightforward to show that, with appropriate modifications to the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$, the solutions of the alternative decoupled system representations (2.60) and (2.61) lead to the system response $\mathbf{x}(t)$ depicted in Fig. 1(b), as should be the case.

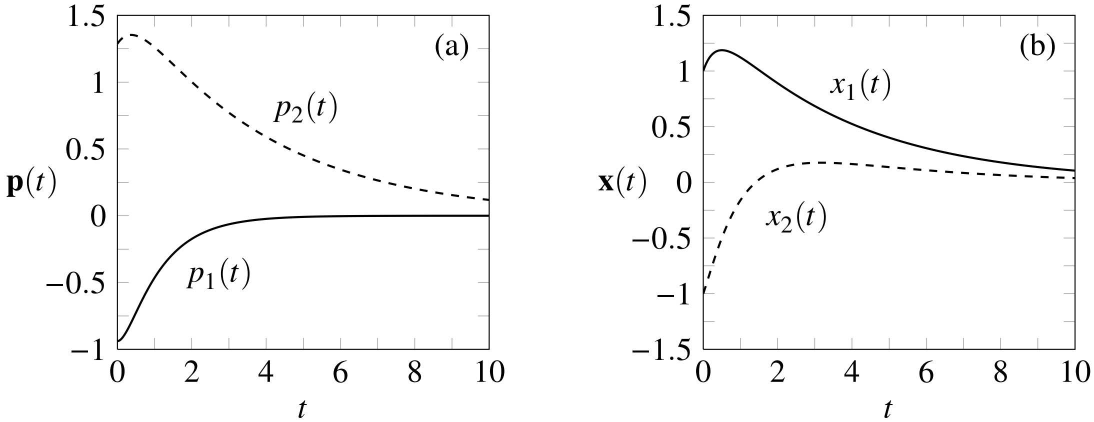

Fig. 1 Free response of Example 1. (a) Decoupled solutions $p_j(t)$ $(j = 1, 2)$. (b) System responses $x_j(t)$.

Chapter 3

Decoupling of Non-Defective Systems in Forced Motion

This chapter is concerned with decoupling a forced non-defective system (1.1) into the form (1.2) and exactly recovering the forced response $\mathbf{x}(t)$ from the decoupled system response $\mathbf{p}(t)$. For the sake of generality, we consider the case in which system (1.1) has mixed eigenvalues. Additionally, while it is possible to approach decoupling in the n-dimensional configuration space, it is more convenient to perform manipulations in the $2n$-dimensional state space. We begin in Section 3.1 by establishing a state space representation of system (1.1) and postulating an associated solution based on the free response. Next, the relationship between the applied excitation $\mathbf{f}(t)$ and modal forcing $\mathbf{g}(t)$ is explored in Section 3.2, and a transformation that recovers the forced response $\mathbf{x}(t)$ exactly is presented in Section 3.3. The chapter closes with a discussion in Section 3.4 demonstrating how the decoupling procedure developed here generalizes classical modal analysis. The methodology presented here is largely based on [46], but we perform manipulations in the state space in a different and more generalized manner.

3.1 State space formulation

Suppose a non-defective system (1.1) possesses a mixed eigenspectrum and is acted upon by an arbitrary external excitation $\mathbf{f}(t)$. It is assumed that the eigenvectors are normalized in accordance with Eqs. (2.42) and (2.43). We may cast system (1.1) in the nonsymmetric state space realization (e.g., see [52])

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{f}(t) \; . \tag{3.1}$$

It is well known that the application of external forcing does not alter the fundamental structure of the decoupled system for the case of classical damping (e.g., see [1–3]). Thus, it is reasonable to postulate that system (1.1) is decoupled into the form (1.2) such that the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ are the same as those in the case of free motion under initial excitation (i.e., $\mathbf{D}$ and $\boldsymbol{\Omega}$ are as defined in Eq. (2.45)). Based on this assumption and the free response's state space representation (2.47), define a linear time-invariant coordinate transformation

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V}\boldsymbol{\Lambda} & \widehat{\mathbf{V}\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \widehat{\boldsymbol{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} = \mathbf{S} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} \; , \tag{3.2}$$

for which $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are n-dimensional vectors whose relationships to the forced modal response $\mathbf{p}(t)$ and its associated velocity $\dot{\mathbf{p}}(t)$ are to be determined. It also remains to be seen how the modal excitation $\mathbf{g}(t)$ is related to the applied forcing $\mathbf{f}(t)$.

3.2 Transformation of the applied forcing

Inserting transformation (3.2) into the first-order formulation (3.1) and premultiplying the resulting state equation by $\mathbf{S}^{-1}$, we obtain

$$\begin{bmatrix} \dot{\mathbf{p}}_1(t) \\ \dot{\mathbf{p}}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\boldsymbol{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{T}_2^{\mathrm{T}} \\ \mathbf{T}_1^{\mathrm{T}} - \mathbf{D}\,\mathbf{T}_2^{\mathrm{T}} \end{bmatrix} \mathbf{f}(t) \; , \tag{3.3}$$

where the matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are as given in Eq. (2.49). The upper and lower halves of state equation (3.3) are, respectively,

$$\dot{\mathbf{p}}_1(t) - \mathbf{p}_2(t) = \mathbf{T}_2^{\mathrm{T}}\,\mathbf{f}(t) \; , \tag{3.4}$$

$$\dot{\mathbf{p}}_2(t) + \mathbf{D}\,\mathbf{p}_2(t) + \boldsymbol{\Omega}\,\mathbf{p}_1(t) = (\mathbf{T}_1^{\mathrm{T}} - \mathbf{D}\,\mathbf{T}_2^{\mathrm{T}})\,\mathbf{f}(t) \; . \tag{3.5}$$

Using Eq. (3.4) to eliminate $\mathbf{p}_2(t)$ from Eq. (3.5) yields an equation of motion in $\mathbf{p}_1(t)$:

$$\ddot{\mathbf{p}}_1(t) + \mathbf{D}\,\dot{\mathbf{p}}_1(t) + \boldsymbol{\Omega}\,\mathbf{p}_1(t) = \mathbf{T}_1^{\mathrm{T}}\,\mathbf{f}(t) + \mathbf{T}_2^{\mathrm{T}}\,\dot{\mathbf{f}}(t) \; . \tag{3.6}$$

Comparing Eq. (3.6) to the decoupled system (1.2), it becomes clear that $\mathbf{p}_1(t)$ corresponds to the modal displacement $\mathbf{p}(t)$, and thus the modal forcing $\mathbf{g}(t)$ is related to the applied excitation $\mathbf{f}(t)$ by

$$\mathbf{g}(t) = \mathbf{T}_1^{\mathrm{T}}\,\mathbf{f}(t) + \mathbf{T}_2^{\mathrm{T}}\,\dot{\mathbf{f}}(t)\ . \tag{3.7}$$

While continuous differentiability of the driving force $\mathbf{f}(t)$ seems to be implied by Eq. (3.7), this constraint may be relaxed by treating the derivative $\dot{\mathbf{f}}(t)$ in the sense of distributions (e.g., the derivative of a unit step is a Dirac delta; see [53]).

3.3 Recovering the forced response

Combining Eqs. (3.2) and (3.4) with $\mathbf{p}_1(t) = \mathbf{p}(t)$, we have that the forced response $\mathbf{x}(t)$ of a non-defective system (1.1) with mixed eigenvalues is related to the modal response $\mathbf{p}(t)$ by the state equation

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V\Lambda} & \widehat{\mathbf{V\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \widehat{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) - \mathbf{T}_2^{\mathrm{T}}\,\mathbf{f}(t) \end{bmatrix}\ , \tag{3.8}$$

and hence the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ are calculated from the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ according to

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \widehat{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \widehat{\mathbf{V}} \\ \mathbf{V\Lambda} & \widehat{\mathbf{V\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{T}_2^{\mathrm{T}}\,\mathbf{f}(0) \end{bmatrix}\ . \tag{3.9}$$

As in the case of free vibration, it is possible to extract an analytical expression from the state space transformation (3.8) that recovers the forced response $\mathbf{x}(t)$ directly. Isolating the upper half of Eq. (3.8) gives

$$\mathbf{x}(t) = \mathbf{T}_1\,\mathbf{p}(t) + \mathbf{T}_2\,\dot{\mathbf{p}}(t) - \mathbf{T}_2\mathbf{T}_2^{\mathrm{T}}\,\mathbf{f}(t)\ . \tag{3.10}$$

It is interesting to note that transformation (3.10) depends continuously on the driving force $\mathbf{f}(t)$ and constitutes a nonlinear mapping between $\mathbf{x}(t)$ and $\mathbf{p}(t)$.

In summary, upon solving the quadratic eigenvalue problem (2.1) and evaluating the decoupled system (1.2) subject to the excitation (3.7) and initial conditions (3.9), the forced response $\mathbf{x}(t)$ of any non-defective system (1.1) can be determined exactly from the modal response $\mathbf{p}(t)$ via transformation (3.10). Simplification of Eq. (3.10) to the free response transformation (2.26) when $\mathbf{f}(t) = \mathbf{0}$ is obvious. A flowchart outlining the general procedure for decoupling any non-defective system (1.1) and determining its response (free or forced) is illustrated in Figure 2.

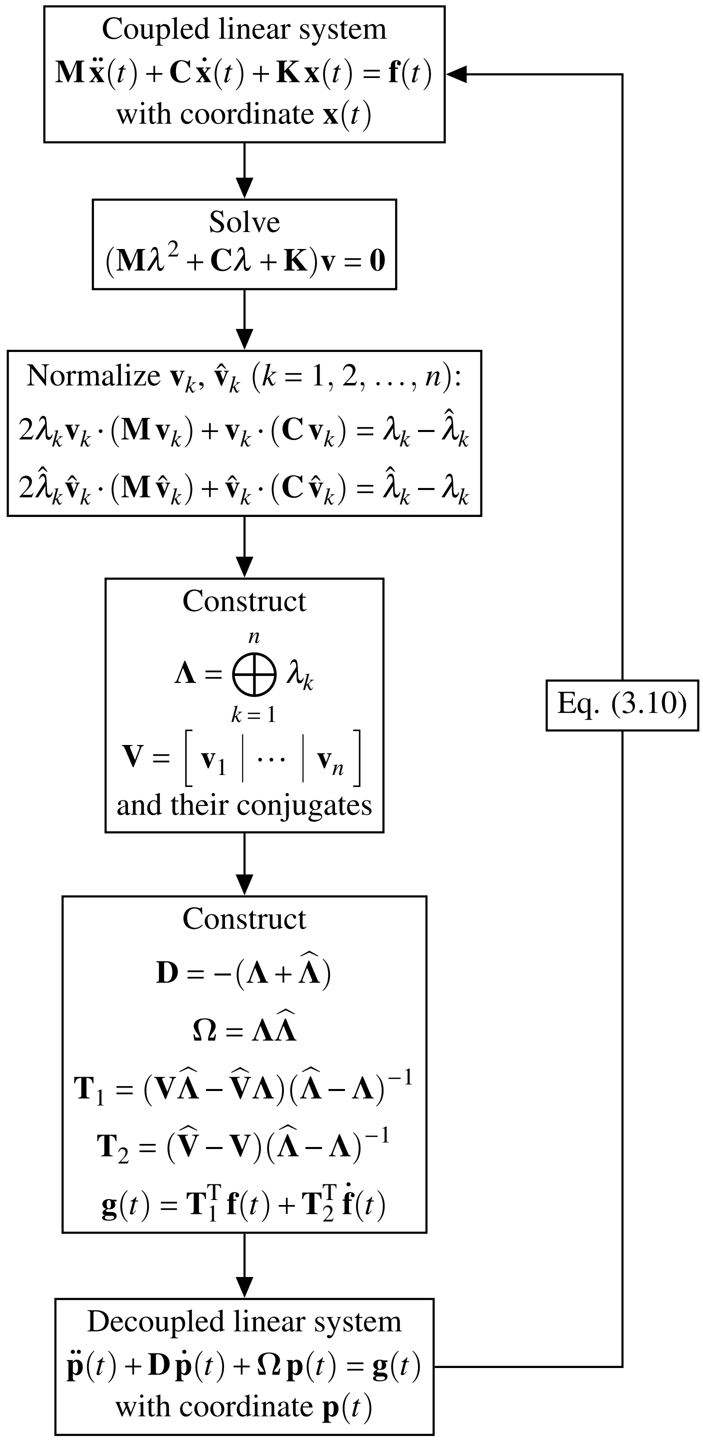

Fig. 2 Flowchart for decoupling and response calculation of any non-defective linear dynamical system in free or forced motion. This flowchart is based on Figure 1 in [46].

3.4 Reduction to classical modal analysis

We now demonstrate how the decoupling methodology developed in this chapter represents a direct generalization of classical modal analysis. Suppose the eigenvectors of system (1.1) are normalized in accordance with Eqs. (2.42) and (2.43). Should system (1.1) be classically damped, then among the $(2r)!/(2^r r!)$ different ways to pair the real eigenvalues, there is a particular pairing scheme such that the matrix $\mathbf{V}$ of eigenvectors and its conjugate coincide with the modal matrix $\mathbf{U}$ containing the eigenvectors of the undamped system: $\mathbf{V} = \widehat{\mathbf{V}} = \mathbf{U}$. Consequently, the transformation matrices $\mathbf{T}_1 = \mathbf{U}$ and $\mathbf{T}_2 = \mathbf{O}$, and hence Eqs. (3.8) and (3.7) reduce to the coordinate transformation $\mathbf{x}(t) = \mathbf{U}\,\mathbf{p}(t)$ and excitation $\mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\,\mathbf{f}(t)$, respectively, that are indicative of classical modal analysis. Of course, reduction to classical modal analysis occurs regardless of eigenvector normalization, with multiplicative constants appearing here and there if normalizations (2.42) and (2.43) are not used.

3.5 Efficiency of the decoupling algorithm

The utility of an algorithm is often based not only on how well it performs a desired task, but also on how efficiently it does so. To motivate the acceptance and widespread use of the decoupling algorithm illustrated in Figure 2, its efficiency should be examined and compared to that of direct numerical integration of system (1.1). Here we assume that the system response $\mathbf{x}(t)$ and applied forcing $\mathbf{f}(t)$ are sufficiently smooth. Counting the number of floating point operations (flops) executed by an algorithm is one way of measuring its performance. For direct numerical integration, typically the n-degree-of-freedom system (1.1) is transformed into the first-order form (3.1) and then discretized into a difference equation with m time steps, from which the response $\mathbf{x}(t)$ can be solved for through recursion (e.g., see [3, 52, 54]). An estimate of the flop count for this procedure is [55–57]

$$N_1 = 160n^3 + 16mn^2\,, \tag{3.11}$$

where $m \gg n$ in general. Using the decoupling algorithm outlined in Figure 2, the quadratic eigenvalue problem (2.1) is solved, the decoupled system (1.2) is then constructed, the response of each independent subsystem is obtained through direct numerical integration via discretization, and finally the system response $\mathbf{x}(t)$ is recovered from transformation (3.10). An estimate of the flop count for the decoupling algorithm is [55–57]

$$N_2 = 213n^3 + (10m+4)n^2 + 16mn\,. \tag{3.12}$$

Figure 3 illustrates how the flop count estimates N_1 and N_2 scale with the size of system (1.1) for $m = 10^6$ time instants. As shown, solution of system (1.1) using the decoupling algorithm in Figure 2 is more efficient than by direct numerical integration, and this is true so long as there are approximately $m > 4000$ time instants.

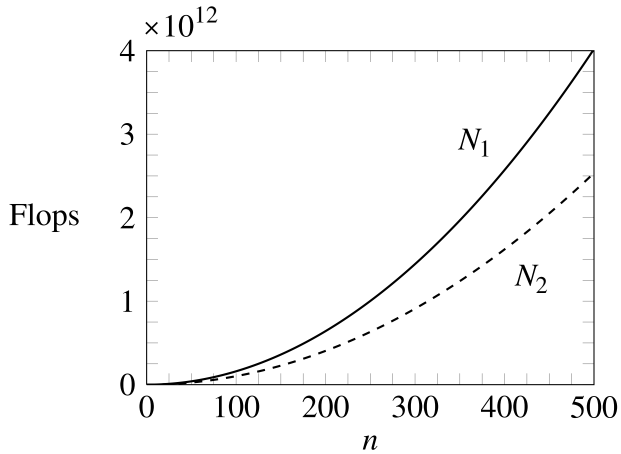

Fig. 3 Comparison of the estimated flops for solution of a non-defective system (1.1) with n degrees of freedom using direct numerical integration via discretization (N_1) and using the decoupling algorithm (N_2) illustrated in Figure 2. The number of time instants is $m = 10^6$. This diagram is adapted from Figure 2 in [46].

3.6 An illustrative example

Here we provide a numerical example illustrating the decoupling procedure for a forced non-defective system of the type (1.1). We focus on a system that has complex and real eigenvalues to demonstrate the method by which a system with mixed damping characteristics is decoupled. Refer to [46] for additional examples of decoupling non-defective systems in forced motion.

Example 2

Suppose a non-classically damped, 2-degree-of-freedom system with mass matrix $\mathbf{M} = \mathbf{I}_2$ has damping matrix $\mathbf{C}$, stiffness matrix $\mathbf{K}$, and excitation $\mathbf{f}(t)$ given by

$$\mathbf{C} = \begin{bmatrix} 3 & -1 \\ -1 & 4 \end{bmatrix} , \qquad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix} , \qquad \mathbf{f}(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \sin t \ . \tag{3.13}$$

Take as initial conditions $\mathbf{x}(0) = [1, 0]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [1, -1]^{\mathrm{T}}$. Solution of the associated quadratic eigenvalue problem reveals that the system is non-defective and has mixed eigenvalues:

$$\lambda_1 = \overline{\lambda}_2 = -1.5 + \mathrm{i}\,0.87 \ , \qquad \lambda_3 = -3 \ , \qquad \lambda_4 = -1 \ , \tag{3.14}$$

for which the corresponding eigenvectors $\mathbf{v}_i$ ($i = 1, 2, 3, 4$), normalized according to Eqs. (2.42) and (2.43), are

$$\mathbf{v}_1 = \overline{\mathbf{v}}_2 = \begin{bmatrix} 0.76\,\mathrm{e}^{-\mathrm{i}\,45^\circ} \\ -0.76\,\mathrm{e}^{-\mathrm{i}\,165^\circ} \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0.82 \\ -0.82 \end{bmatrix}, \qquad \mathbf{v}_4 = \begin{bmatrix} 1.41 \\ 0 \end{bmatrix}. \tag{3.15}$$

By Eqs. (2.14) and (2.18),

$$\mathbf{\Lambda} = \begin{bmatrix} -1.5 + \mathrm{i}\,0.87 & 0 \\ 0 & -3 \end{bmatrix}, \qquad \mathbf{V} = \left[\begin{array}{c|c} 0.76\,\mathrm{e}^{-\mathrm{i}\,45^\circ} & 0.82 \\ -0.76\,\mathrm{e}^{-\mathrm{i}\,165^\circ} & -0.82 \end{array}\right], \tag{3.16}$$

where the associated conjugates are given by

$$\widehat{\mathbf{\Lambda}} = \begin{bmatrix} -1.5 - \mathrm{i}\,0.87 & 0 \\ 0 & -1 \end{bmatrix}, \qquad \widehat{\mathbf{V}} = \left[\begin{array}{c|c} 0.76\,\mathrm{e}^{\mathrm{i}\,45^\circ} & 1.41 \\ -0.76\,\mathrm{e}^{\mathrm{i}\,165^\circ} & 0 \end{array}\right]. \tag{3.17}$$

It follows from Eq. (2.49) that

$$\mathbf{T}_1 = \begin{bmatrix} -0.39 & 1.71 \\ 1.07 & 0.41 \end{bmatrix}, \qquad \mathbf{T}_2 = \begin{bmatrix} -0.62 & 0.30 \\ 0.23 & -0.41 \end{bmatrix}. \tag{3.18}$$

From Eqs. (2.45), (3.16), and (3.17), the decoupled system's coefficient matrices are

$$\mathbf{D} = \begin{bmatrix} 3 & 0 \\ 0 & 4 \end{bmatrix}, \qquad \mathbf{\Omega} = 3\,\mathbf{I}_2\,, \tag{3.19}$$

implying that the degree of freedom $p_1(t)$ is underdamped with viscous damping factor $\zeta_1 = 0.87$, while $p_2(t)$ is overdamped with $\zeta_2 = 1.15$. The corresponding initial conditions are $\mathbf{p}(0) = [-1.47, 0.90]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [2.54, 1.55]^{\mathrm{T}}$ from Eq. (3.9). The modal excitation $\mathbf{g}(t)$ determined from Eq. (3.7) is depicted in Fig. 4. The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 5(a), and the system response $\mathbf{x}(t)$ recovered from transformation (3.10) is shown in Fig. 5(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (3.10) are indeed the same.

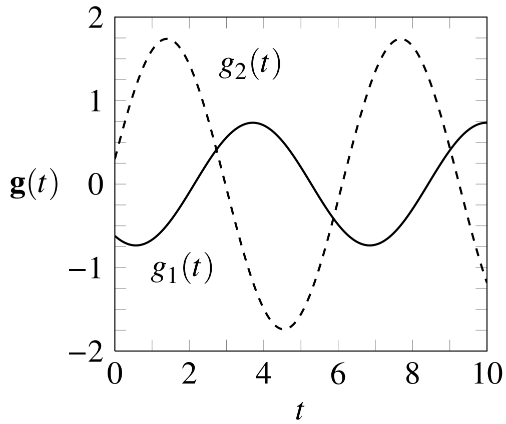

Fig. 4 Modal excitation components $g_j(t)$ ($j = 1, 2$) for Example 2.

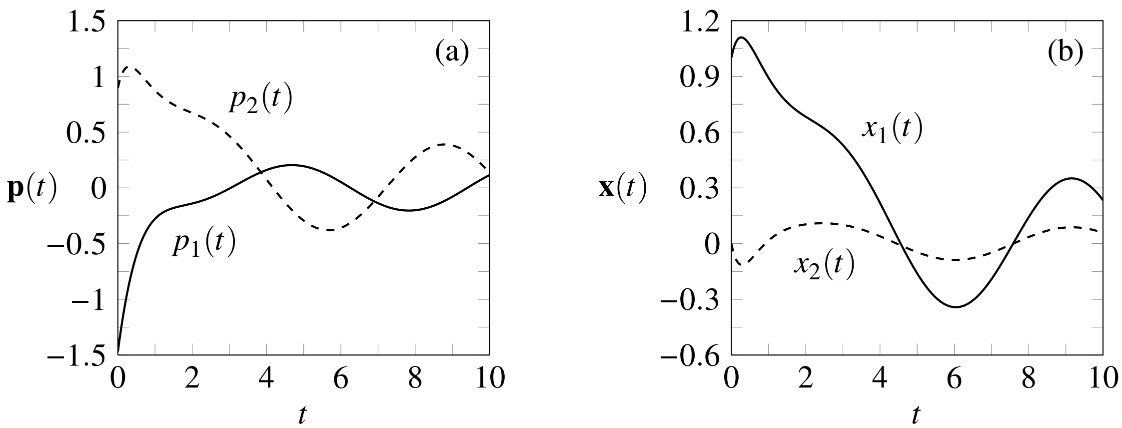

Fig. 5 Forced response of Example 2. (a) Decoupled solutions $p_j(t)$ ($j = 1, 2$). (b) System responses $x_j(t)$.

Chapter 4

Decoupling of Defective Systems in Free Motion

In this chapter, we present a method by which an unforced defective system (1.1) (i.e., $\mathbf{f}(t) = \mathbf{0}$) is decoupled into the form (1.2) with $\mathbf{g}(t) = \mathbf{0}$ and the free response $\mathbf{x}(t)$ is recovered exactly from the decoupled system response $\mathbf{p}(t)$. We begin with a discussion of the quadratic eigenvalue problem for defective systems in Section 4.1. Next, we develop a generalized decoupling transformation in Section 4.2 via analysis in state space, and the relationship between the state space representations of the original and decoupled systems is briefly explored in Section 4.3. To streamline the introduction of new details, an assumption regarding the defective eigenvalues of system (1.1) is made in Section 4.4 for convenience. We illustrate how the generalized decoupling procedure outlined in Section 4.2 is applied to defective systems with complex eigenvalues in Section 4.5, and we follow this with an application to the case of defective real eigenvalues in Section 4.6. While the decoupling of systems in free motion with defective complex eigenvalues was briefly touched upon in [45], we provide an alternative and more in-depth analysis here. We discuss in Section 4.7 how the decoupling procedure is affected when the constraint imposed in Section 4.4 is relaxed. We conclude the chapter by demonstrating in Section 4.8 how the decoupling methodology described here reduces to classical modal analysis when system (1.1) is classically damped.

4.1 The quadratic eigenvalue problem

In the event that system (1.1) is defective, solution of the quadratic eigenvalue problem (2.1) reveals that some of the eigenvalues are repeated and do not have associated with them a full complement of linearly independent eigenvectors. Such eigenvalues are termed defective. As an example, an eigenvalue that is repeated more than n times is necessarily defective. Let λ_k denote the kth eigenvalue of system (1.1), and suppose it is defective. The number of times the defective eigenvalue λ_k is repeated is referred to as its algebraic multiplicity m_k. Considering the quadratic matrix pencil $\mathbf{Q}(\lambda) = \mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K}$, the geometric multiplicity ρ_k of the defective eigenvalue λ_k is given by the dimension of the null space of $\mathbf{Q}(\lambda_k)$, and hence it is equivalent to the number of linearly independent eigenvectors associated with λ_k. For obvious reasons, the geometric multiplicity cannot exceed either the algebraic multiplicity m_k or the system order n, whichever is smaller: $\rho_k \leqslant \min(m_k, n)$. The associated ρ_k eigenvectors $\mathbf{v}_j^k$ ($j = 1, 2, \ldots, \rho_k$) are supplemented with $m_k - \rho_k$ generalized eigenvectors to form a complete set of vectors for λ_k, where the final eigenvector $\mathbf{v}_{\rho_k}^k$ and the generalized eigenvectors constitute a Jordan chain of length $m_k - \rho_k + 1$ characterized by the recursive scheme (see [50])

$$
\begin{aligned}
\mathbf{0} &= \mathbf{Q}(\lambda_k)\,\mathbf{v}_{\rho_k}^k\ , \\
\mathbf{0} &= \mathbf{Q}(\lambda_k)\,\mathbf{v}_{\rho_k+1}^k + \mathbf{Q}'(\lambda_k)\,\mathbf{v}_{\rho_k}^k\ , \\
\mathbf{0} &= \mathbf{Q}(\lambda_k)\,\mathbf{v}_{\rho_k+2}^k + \mathbf{Q}'(\lambda_k)\,\mathbf{v}_{\rho_k+1}^k + \frac{1}{2}\mathbf{Q}''(\lambda_k)\,\mathbf{v}_{\rho_k}^k\ , \\
&\ \ \vdots \\
\mathbf{0} &= \mathbf{Q}(\lambda_k)\,\mathbf{v}_{m_k}^k + \mathbf{Q}'(\lambda_k)\,\mathbf{v}_{m_k-1}^k + \frac{1}{2}\mathbf{Q}''(\lambda_k)\,\mathbf{v}_{m_k-2}^k\ .
\end{aligned}
\tag{4.1}
$$

In the above sequence, $\mathbf{Q}'(\lambda_k)$ and $\mathbf{Q}''(\lambda_k)$ denote the first and second derivatives of $\mathbf{Q}(\lambda)$, respectively, with respect to λ and evaluated at the defective eigenvalue λ_k:

$$\mathbf{Q}'(\lambda_k) = \left.\frac{\mathrm{d}\mathbf{Q}(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda\,=\,\lambda_k} = 2\mathbf{M}\lambda_k + \mathbf{C}\ , \tag{4.2}$$

$$\mathbf{Q}''(\lambda_k) = \left.\frac{\mathrm{d}^2\mathbf{Q}(\lambda)}{\mathrm{d}\lambda^2}\right|_{\lambda\,=\,\lambda_k} = 2\mathbf{M}\ . \tag{4.3}$$

Regardless of the nature of the system's eigenspectrum, the free response $\mathbf{x}(t)$ of system (1.1) may be cast in the general matrix-vector form (e.g., see [48, 49])

$$\mathbf{x}(t) = \mathbf{V}_x\,\mathrm{e}^{\mathbf{J}_x t}\mathbf{c}\ , \tag{4.4}$$

for which $\mathbf{J}_x$ is an order $2n$ Jordan matrix with the system eigenvalues on the diagonal, $\mathbf{V}_x$ is an $n \times 2n$ matrix of the associated eigenvectors and generalized eigenvectors, and $\mathbf{c}$ is a

$2n$-dimensional vector of coefficients determined from the initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$. Writing Eq. (4.4) and its derivative in the form of a state equation,

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} \mathrm{e}^{\mathbf{J}_x t} \mathbf{c} = \mathbf{S}_x \mathrm{e}^{\mathbf{J}_x t} \mathbf{c} \,, \tag{4.5}$$

where the order $2n$ matrix $\mathbf{S}_x$ is invertible since $\mathbf{J}_x$ and $\mathbf{V}_x$ constitute a Jordan pair. If system (1.1) is non-defective, the Jordan matrix $\mathbf{J}_x$ is diagonal and every eigenvalue (repeated or not) has a corresponding eigenvector, and thus Eq. (4.4) reduces to the eigensolution summation (2.2). However, when system (1.1) is defective, the Jordan decomposition (4.4) is the only available representation of the free response $\mathbf{x}(t)$.

4.2 A generalized state space representation

Defective or not, we assume that system (1.1) may be decoupled into the form (1.2) such that the diagonal coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ are always given by Eq. (2.45), where the arrangement of eigenvalues in the diagonal matrix $\boldsymbol{\Lambda}$ and its conjugate $\widehat{\boldsymbol{\Lambda}}$ is dependent on the choice of eigensolution pairing. We have demonstrated previously that it is possible to decouple a non-defective system (1.1) while preserving both the algebraic and geometric multiplicities of the associated eigenvalues (i.e., the decoupling transformation is strictly isospectral). If one insists that geometric multiplicities be preserved, then a defective system (1.1) may only be partially decoupled [58]. Our main goal is to demonstrate how defective systems can be decoupled, at any cost, by not requiring geometric multiplicities to be preserved.

Since phase synchronization for a non-defective system (1.1) does not disturb the eigensolution coefficients, we shall construct the decoupling transformation for a defective system (1.1) so that it also preserves these coefficients. Consequently, the modal response $\mathbf{p}(t)$ may be cast in a matrix-vector form analogous to Eq. (4.4):

$$\mathbf{p}(t) = \mathbf{V}_p \mathrm{e}^{\mathbf{J}_p t} \mathbf{c} \,. \tag{4.6}$$

Here, $\mathbf{J}_p$ is an order $2n$ Jordan matrix consisting of the system eigenvalues, but it may not necessarily be the same as $\mathbf{J}_x$. The matrix $\mathbf{J}_p$ has the same diagonal elements (i.e., the eigenvalues) as $\mathbf{J}_x$, but the off-diagonal elements are generally different since we do not impose preservation of geometric multiplicities. The matrix $\mathbf{V}_p$ is an $n \times 2n$ matrix of vectors that essentially pairs the eigensolutions according to a specified pairing scheme. Expressing Eq. (4.6) and its derivative as a state equation,

$$\begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} \mathrm{e}^{\mathbf{J}_p t} \mathbf{c} = \mathbf{S}_p \mathrm{e}^{\mathbf{J}_p t} \mathbf{c} \,, \tag{4.7}$$

where the order $2n$ matrix $\mathbf{S}_p$ will always be invertible so long as eigenvalues are paired to preserve the second-order structure of the equation of motion (1.1). Combining Eqs. (4.5) and (4.7), the response $\mathbf{x}(t)$ of system (1.1) is recovered from the modal response $\mathbf{p}(t)$ in the state space according to

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} \mathrm{e}^{\mathbf{J}_x t} \, \mathrm{e}^{-\mathbf{J}_p t} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S}_x \, \mathrm{e}^{\mathbf{J}_x t} \, \mathrm{e}^{-\mathbf{J}_p t} \, \mathbf{S}_p^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} , \tag{4.8}$$

and thus the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ are connected to the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} . \tag{4.9}$$

A generalized procedure for decoupling system (1.1) into the form (1.2) and recovering the response $\mathbf{x}(t)$ from the decoupled solution $\mathbf{p}(t)$ via state space transformation is summarized as follows. (i) Solve the quadratic eigenvalue problem (2.1) to determine the system eigenvalues and eigenvectors. If system (1.1) is defective, additional analysis is required to obtain generalized eigenvectors as needed. (ii) Based on the nature of the system's eigenspectrum, specify an appropriate eigenvalue pairing scheme, and arrange the eigenvalues accordingly in the diagonal matrix $\mathbf{\Lambda}$ and its conjugate $\widehat{\mathbf{\Lambda}}$. (iii) Construct the matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ from the system eigenvalues, eigenvectors, and generalized eigenvectors. (iv) Form the matrices $\mathbf{J}_p$ and $\mathbf{V}_p$ such that the solution $\mathbf{p}(t)$ of the decoupled system (1.2) satisfies the specified pairing scheme. (v) Obtain the initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ for the decoupled system via transformation (4.9). (vi) Construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ in accordance with Eq. (2.45), and then determine the response $\mathbf{p}(t)$ of the decoupled system. (vii) Retrieve the response $\mathbf{x}(t)$ from the decoupled solution $\mathbf{p}(t)$ using transformation (4.8).

The heart of the decoupling procedure described herein lies in steps (ii) through (iv). To illustrate, we shall demonstrate how the above procedure relates to the decoupling of a non-defective system (1.1) with mixed eigenvalues ($2c$ complex and the remaining $2r$ real) as presented in Chapter 2. Because the eigenvalues are generally semi-simple (i.e., they may be repeated but have corresponding eigenvectors), they may be paired as complex and real quadratic conjugates to yield c underdamped and r overdamped (not necessarily distinct) decoupled degrees of freedom, respectively. Consequently, the c complex and r real eigenvalues and eigenvectors (that constitute the c complex conjugate and r real quadratic conjugate pairs, respectively) may be arranged in some order in the matrices $\mathbf{\Lambda}$ and $\mathbf{V}$, respectively, in accordance with Eqs. (2.14) and (2.18). Comparing the summation representation of the response $\mathbf{x}(t)$ given in Eq. (2.44) to the general form (4.4), it is clear that

$$\mathbf{J}_x = \mathbf{\Lambda} \oplus \widehat{\mathbf{\Lambda}} \, , \qquad \mathbf{V}_x = \left[\mathbf{V} \;\middle|\; \widehat{\mathbf{V}} \right] . \tag{4.10}$$

The decoupled solutions $p_k(t)$ given by Eq. (2.46) imply that

$$\mathbf{J}_p = \mathbf{J}_x \,, \qquad \mathbf{V}_p = \left[\mathbf{I} \,\middle|\, \mathbf{I}\right] \tag{4.11}$$

in Eq. (4.6). As a result, Eqs. (4.8) and (4.9) reduce to the familiar transformations (2.47) and (2.48), respectively, for a non-defective system (1.1). There currently does not exist criteria for eigenvalue pairing when system (1.1) is defective. Put another way, it is not clear how to form $\mathbf{\Lambda}$ and its conjugate $\widehat{\mathbf{\Lambda}}$ when the eigenvalues are defective in order to construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ that define the decoupled system (1.2). This chapter presents a solution to this problem.

4.3 State transformation of the equation of motion

Suppose we cast an unforced system (1.1) and its decoupled form (1.2), respectively, in the nonsymmetric state space realizations

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{M}^{-1}\mathbf{K} & -\mathbf{M}^{-1}\mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} \,, \tag{4.12}$$

$$\begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\mathbf{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} \,. \tag{4.13}$$

Since the eigenvalues of system (1.1) and their algebraic multiplicities are preserved during decoupling, the nonsymmetric order $2n$ matrices $\mathbf{A}$ and $\mathbf{B}$ have the same eigenvalues (i.e., they are isospectral), but the corresponding geometric multiplicities may be different (that is, $\mathbf{A}$ and $\mathbf{B}$ are not strictly isospectral), which is reflected in the associated Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$, respectively.

In the general case of a defective system (1.1), what is the real transformation that converts its state space representation (4.12) into Eq. (4.13), from which the decoupled system (1.2) is extracted? In other words, how are the matrices $\mathbf{A}$ and $\mathbf{B}$ related? To answer this question, begin by writing the general state space transformation (4.8) in the more compact form

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{S}_x \, e^{\mathbf{J}_x t} \, e^{-\mathbf{J}_p t} \, \mathbf{S}_p^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} = \mathbf{S}(t) \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} \,. \tag{4.14}$$

Substitute the state vector (4.14) and its derivative into Eq. (4.12) and then premultiply the resulting equation by $\mathbf{S}^{-1}(t)$ to obtain

$$\begin{bmatrix} \dot{\mathbf{p}}(t) \\ \ddot{\mathbf{p}}(t) \end{bmatrix} = \left(\mathbf{S}^{-1}(t)\,\mathbf{A}\,\mathbf{S}(t) - \mathbf{S}^{-1}(t)\,\dot{\mathbf{S}}(t) \right) \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} . \tag{4.15}$$

Comparing Eqs. (4.13) and (4.15) reveals that the constant matrices **A** and **B** are related by the time-dependent transformation

$$\mathbf{B} = \mathbf{S}^{-1}(t)\,\mathbf{A}\,\mathbf{S}(t) - \mathbf{S}^{-1}(t)\,\dot{\mathbf{S}}(t) . \tag{4.16}$$

Time variation in Eq. (4.16) is a direct result of not requiring that geometric multiplicities be preserved during decoupling. In other words, transformation (4.16) is time-varying when the Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$ are not identical. In the event that $\mathbf{J}_x = \mathbf{J}_p$, the matrix $\mathbf{S}(t) = \mathbf{S}$ is independent of time t, and hence Eq. (4.16) reduces to the similarity transformation $\mathbf{B} = \mathbf{S}^{-1}\mathbf{A}\,\mathbf{S}$. One situation for which this reduction occurs is when system (1.1) is non-defective, in which case the constant matrix **S** is given by the order $2n$ transformation in state equation (2.47). This similarity transformation was implied in Eqs. (3.1)-(3.3) when decoupling of non-defective systems in forced vibration was addressed.

4.4 Restriction on the geometric multiplicity

To streamline our subsequent discussion of decoupling defective systems, it will be assumed that every defective eigenvalue λ_k of system (1.1) has unit geometric multiplicity (i.e., $\rho_k = 1$), and hence the length of its associated Jordan chain is given by the algebraic multiplicity m_k. In the case of defective complex eigenvalues, relaxation of this constraint leads to a trivial modification of the decoupling procedure. When system (1.1) possesses defective real eigenvalues, however, the situation is more delicate and requires some discussion.

4.5 Complex eigenvalues

Suppose that solution of the quadratic eigenvalue problem (2.1) reveals that the $2n$ eigenvalues of system (1.1) are complex and defective. As the system matrices **M**, **C**, and **K** are real, these eigenvalues necessarily form n_c complex conjugate pairs, where $2n_c < 2n$. Let m_k ($k = 1, 2, \ldots, n_c$) be the algebraic multiplicity of the defective complex eigenvalue λ_k. Then m_k is also the algebraic multiplicity of the complex conjugate eigenvalue $\overline{\lambda}_k$ so that $m_1 + \cdots + m_{n_c} = n$. Under the assumption of unit geometric multiplicity, each defective eigenvalue λ_k has a single (complex) eigenvector $\mathbf{v}_1^k$ associated with it. Let $\mathbf{J}_k$ be a Jordan block of order m_k formed from the defective eigenvalue λ_k:

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k & 1 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix} . \tag{4.17}$$

In addition, construct an $n \times m_k$ matrix $\mathbf{V}_k$ whose columns contain the single eigenvector $\mathbf{v}_1^k$ and the $m_k - 1$ generalized eigenvectors that constitute the m_k-long Jordan chain associated with λ_k, and take $\mathbf{c}_k$ to be an m_k-dimensional vector consisting of the complex coefficients c_j^k ($j = 1, 2, \ldots, m_k$) for the corresponding eigensolutions:

$$\mathbf{V}_k = \left[\, \mathbf{v}_1^k \, \middle| \, \cdots \, \middle| \, \mathbf{v}_{m_k}^k \, \right] , \qquad \mathbf{c}_k = \left[c_1^k \quad \cdots \quad c_{m_k}^k \right]^{\mathrm{T}} . \tag{4.18}$$

Expressing the general form (4.4) of the free response $\mathbf{x}(t)$ in terms of the Jordan pairs $(\mathbf{J}_k, \mathbf{V}_k)$ and their complex conjugates,

$$\mathbf{x}(t) = \sum_{k=1}^{n_c} \left(\mathbf{V}_k \, \mathrm{e}^{\mathbf{J}_k t} \mathbf{c}_k + \overline{\mathbf{V}}_k \, \mathrm{e}^{\overline{\mathbf{J}}_k t} \overline{\mathbf{c}}_k \right) . \tag{4.19}$$

If system (1.1) is non-defective and it possesses a complex eigenvalue λ_k repeated m_k times, then associated with λ_k are m_k independent underdamped oscillators that execute oscillatory free motion, governed by the equation of motion (1.2), with identical decay rate α_k and damped frequency ω_k. The initial conditions, however, are generally different because the eigensolution coefficients c_k are typically not the same. In progressing to defective oscillatory systems, each defective complex eigenvalue λ_k with algebraic multiplicity m_k yields m_k identical and independent underdamped systems subject to different sets of initial conditions upon decoupling. Specifically, every decoupled coordinate $p_j^k(t)$ ($j = 1, 2, \ldots, m_k$) corresponding to the defective eigenvalue λ_k satisfies the same equation of motion

$$\ddot{p}_j^k(t) - (\lambda_k + \overline{\lambda}_k)\dot{p}_j^k(t) + \lambda_k \overline{\lambda}_k p_j^k(t) = 0 , \tag{4.20}$$

for which the solution $p_j^k(t)$ is of the form

$$p_j^k(t) = c_j^k \mathrm{e}^{\lambda_k t} + \overline{c}_j^k \mathrm{e}^{\overline{\lambda}_k t} . \tag{4.21}$$

The decoupled equation of motion (4.20) implies that the n_c eigenvalues (that constitute the n_c complex conjugate pairs) may be arranged in the diagonal matrix $\mathbf{\Lambda}$ according to

$$\mathbf{\Lambda} = \bigoplus_{k=1}^{n_c} \mathbf{\Lambda}_k \,, \qquad \mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{m_k} \,. \tag{4.22}$$

The (complex) conjugate of $\mathbf{\Lambda}$ immediately follows. Given Eq. (4.22) and its conjugate, we may then construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ in accordance with Eq. (2.45) to establish the decoupled system (1.2). Equation (4.22) permits us to write the Jordan block (4.17) in the form $\mathbf{J}_k = \mathbf{\Lambda}_k + \mathbf{N}_k$, where $\mathbf{N}_k$ is an order m_k nilpotent matrix of 1's on the superdiagonal. Comparing Eqs. (4.4) and (4.19) reveals that

$$\mathbf{J}_x = \mathbf{J} \oplus \overline{\mathbf{J}} \,, \qquad \mathbf{V}_x = \left[\mathbf{V} \;\middle|\; \overline{\mathbf{V}} \right] , \tag{4.23}$$

where the order n block diagonal matrix $\mathbf{J}$ and the $n \times n$ matrix $\mathbf{V}$ of eigenvectors and generalized eigenvectors are given by, respectively,

$$\mathbf{J} = \bigoplus_{k=1}^{n_c} \mathbf{J}_k = \mathbf{\Lambda} + \mathbf{N} \,, \qquad \mathbf{V} = \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_{n_c} \end{bmatrix} . \tag{4.24}$$

Based on the arrangement of eigenvalues in $\mathbf{\Lambda}$, the modal response vector $\mathbf{p}(t)$ has the conformable structure

$$\mathbf{p}(t) = \begin{bmatrix} \mathbf{p}_1^{\mathrm{T}}(t) & \cdots & \mathbf{p}_{n_c}^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} , \qquad \mathbf{p}_k(t) = \begin{bmatrix} p_1^k(t) & \cdots & p_{m_k}^k(t) \end{bmatrix}^{\mathrm{T}} , \tag{4.25}$$

from which we deduce that

$$\mathbf{J}_p = \mathbf{\Lambda} \oplus \overline{\mathbf{\Lambda}} \,, \qquad \mathbf{V}_p = \left[\mathbf{I} \;\middle|\; \mathbf{I} \right] \tag{4.26}$$

in order to satisfy the response (4.21) for each decoupled coordinate $p_j^k(t)$. In addition, it can be shown, by exploiting the commutativity of $\mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{m_k}$ and $\mathbf{N}_k$ in multiplication, that $\mathrm{e}^{\mathbf{J}_x t} \mathrm{e}^{-\mathbf{J}_p t} = \mathrm{e}^{\mathbf{N}t} \oplus \mathrm{e}^{\mathbf{N}t}$. Consequently, the state space formulation (4.8) for defective oscillatory motion has the equivalent time-dependent representations

$$\begin{aligned} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} &= \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) \end{bmatrix} . \end{aligned} \tag{4.27}$$

The modal and system initial conditions are related by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} . \tag{4.28}$$

As with non-defective systems, it is possible to isolate an order n transformation from the state space representation (4.27) to directly give the response $\mathbf{x}(t)$:

$$\mathbf{x}(t) = \mathbf{T}_1 \, \mathrm{e}^{\mathbf{N}t} \, \mathbf{p}(t) + \mathbf{T}_2 \, \mathrm{e}^{\mathbf{N}t} \, \dot{\mathbf{p}}(t) \; , \tag{4.29}$$

where the transformation matrices $\mathbf{T}_1$ and $\mathbf{T}_2$ are as defined in Eq. (2.27). Thus, upon calculating the modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ from Eq. (4.28), the free response $p_j^k(t)$ for each decoupled coordinate may be evaluated exactly as

$$p_j^k(t) = \mathrm{e}^{\alpha_k t} \left[p_j^k(0) \cos \omega_k t + \left(\frac{\dot{p}_j^k(0) - \alpha_k p_j^k(0)}{\omega_k} \right) \sin \omega_k t \right] , \tag{4.30}$$

from which the response $\mathbf{x}(t)$ of system (1.1) can be obtained directly via transformation (4.29). In addition to the quadratic eigenvalue problem (2.1), it becomes necessary to evaluate the recursive scheme (4.1) to obtain all parameters required for retrieving the free response $\mathbf{x}(t)$ from Eq. (4.29).

In summary, if the coupled n-degree-of-freedom system (1.1) is defective with n_c pairs of complex conjugate eigenvalues of respective algebraic multiplicity m_k, then it may be decoupled into n underdamped degrees of freedom corresponding to n_c collections of m_k identical, underdamped single-degree-of-freedom oscillators subject to different initial conditions. In the event that system (1.1) is non-defective, the nilpotent matrix $\mathbf{N} = \mathbf{O}$, and thus the Jordan matrix $\mathbf{J} = \mathbf{\Lambda}$. As a result, Eqs. (4.27)-(4.29) reduce to transformations (2.23), (2.24), and (2.26), respectively, associated with non-defective complex eigenvalues, as expected.

The partitioning of the order $2n$ transformation matrices in state equation (4.27) into blocks of size n makes it convenient to look further into the mechanics of transformation (4.16) and how it relates the state matrices $\mathbf{A}$ and $\mathbf{B}$ of the nonsymmetric state space realizations (4.12) and (4.13), respectively. Based on Eq. (4.27), the matrix $\mathbf{S}(t)$ and its derivative are given by, respectively,

$$\mathbf{S}(t) = \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix}^{-1} \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} = \mathbf{S}_0 \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} , \tag{4.31}$$

$$\dot{\mathbf{S}}(t) = \mathbf{S}_0 \begin{bmatrix} \mathbf{N} & \mathbf{O} \\ \mathbf{O} & \mathbf{N} \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} . \tag{4.32}$$

Inserting Eqs. (4.31) and (4.32) into transformation (4.16) yields

$$\mathbf{B} = \begin{bmatrix} \mathrm{e}^{-\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{-\mathbf{N}t} \end{bmatrix} \left(\mathbf{S}_0^{-1}\mathbf{A}\,\mathbf{S}_0 - \begin{bmatrix} \mathbf{N} & \mathbf{O} \\ \mathbf{O} & \mathbf{N} \end{bmatrix} \right) \begin{bmatrix} \mathrm{e}^{\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{\mathbf{N}t} \end{bmatrix} , \tag{4.33}$$

where it can be verified that

$$\mathbf{S}_0^{-1}\mathbf{A}\,\mathbf{S}_0 = \begin{bmatrix} \mathbf{N} & \mathbf{I} \\ -\boldsymbol{\Omega} & \mathbf{N}-\mathbf{D} \end{bmatrix} . \tag{4.34}$$

Finally, by exploiting the commutativity in blockwise multiplication of the diagonal coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ with the matrix exponential $\mathrm{e}^{\mathbf{N}t}$, it is straightforward to show that the resulting matrix $\mathbf{B}$ is indeed the order $2n$ state matrix in Eq. (4.13).

4.6 Real eigenvalues

Suppose that solution of the quadratic eigenvalue problem (2.1) for system (1.1) yields real eigenvalues λ_k that are repeated and defective. By Theorem 4.2 in [48], every eigenvector $\mathbf{v}_j^k$ associated with a defective real eigenvalue λ_k satisfies

$$2\mathbf{v}_j^k \cdot (\mathbf{M}\,\mathbf{v}_j^k)\lambda_k + \mathbf{v}_j^k \cdot (\mathbf{C}\,\mathbf{v}_j^k) = 0 . \tag{4.35}$$

However, the quadratic eigenvalue problem (2.1) implies that

$$\lambda_k = -\frac{\mathbf{v}_j^k \cdot (\mathbf{C}\,\mathbf{v}_j^k)}{2\mathbf{v}_j^k \cdot (\mathbf{M}\,\mathbf{v}_j^k)} \pm \mathrm{i}\,\frac{\sqrt{4(\mathbf{v}_j^k \cdot (\mathbf{M}\,\mathbf{v}_j^k))(\mathbf{v}_j^k \cdot (\mathbf{K}\,\mathbf{v}_j^k)) - (\mathbf{v}_j^k \cdot (\mathbf{C}\,\mathbf{v}_j^k))^2}}{2\mathbf{v}_j^k \cdot (\mathbf{M}\,\mathbf{v}_j^k)} . \tag{4.36}$$

Equations (4.35) and (4.36) are simultaneously satisfied when the discriminant of Eq. (4.36) vanishes, resulting in

$$\lambda_k = -\sqrt{\frac{\mathbf{v}_j^k \cdot (\mathbf{K}\,\mathbf{v}_j^k)}{\mathbf{v}_j^k \cdot (\mathbf{M}\,\mathbf{v}_j^k)}} \pm \mathrm{i}\,0 = \alpha_k \pm \mathrm{i}\,0 . \tag{4.37}$$

Thus, defective real eigenvalues are such that $\omega_k = 0$, implying that these eigenvalues correspond to the boundary between real vibration (complex eigenvalues) and imaginary vibration (non-defective real eigenvalues). Consequently, we associate defective real eigenvalues with critical damping.

Critical damping in multi-degree-of-freedom systems, classically damped or not, has been discussed extensively in the literature (e.g., see [59–64]). The common thread in these works is that every defective real eigenvalue λ_k giving rise to a critically damped degree of freedom has algebraic multiplicity $m_k = 2$. However, there is generally no reason why the algebraic multiplicities of the defective real eigenvalues should be limited in this way for a non-classically damped system. We place no such restriction on the algebraic multiplicities. By a thorough examination, we shall demonstrate that decoupling of systems with defective real eigenvalues is a delicate procedure that depends on the multiplicities of these eigenvalues.

4.6.1 Critical damping in a single-degree-of-freedom system

Recall that a critically damped single-degree-of-freedom oscillator in free vibration with natural frequency ω possesses a repeated and defective eigenvalue $\lambda = -\omega$ and is governed by the equation of motion (e.g., see [3, 65])

$$\ddot{x}(t) + 2\omega\dot{x}(t) + \omega^2 x(t) = 0 \,. \tag{4.38}$$

Expressing Eq. (4.38) and its derivative in a nonsymmetric first-order form,

$$\begin{bmatrix} \dot{x}(t) \\ \ddot{x}(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & -2\omega \end{bmatrix} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} \,. \tag{4.39}$$

The solution of the state equation (4.39) is given by

$$\begin{bmatrix} x(t) \\ \dot{x}(t) \end{bmatrix} = \mathbf{V}\mathrm{e}^{\mathbf{J}t}\mathbf{c} \,, \tag{4.40}$$

where $\mathbf{V}$ contains the eigenvector and generalized eigenvector of the state matrix $\mathbf{A}$, $\mathbf{J}$ is the Jordan normal form of $\mathbf{A}$ (i.e., $\mathbf{J} = \mathbf{V}^{-1}\mathbf{A}\,\mathbf{V}$), and $\mathbf{c}$ is a vector of real coefficients determined by the initial conditions $x(0)$ and $\dot{x}(0)$:

$$\mathbf{V} = \left[\begin{array}{c|c} 1 & 0 \\ -\omega & 1 \end{array}\right] , \qquad \mathbf{J} = \begin{bmatrix} -\omega & 1 \\ 0 & -\omega \end{bmatrix} , \qquad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \,. \tag{4.41}$$

Extracting the first row of Eq. (4.40), the solution $x(t)$ of the critically damped system (4.38) has the equivalent representations

$$x(t) = \left[\begin{array}{c|c} 1 & 0 \end{array}\right] \mathrm{e}^{\mathbf{J}t}\mathbf{c} = (c_1 + c_2 t)\,\mathrm{e}^{-\omega t} \,. \tag{4.42}$$

Upon applying initial conditions,

$$x(t) = [x(0) + (\dot{x}(0) + \omega x(0))\,t]\,\mathrm{e}^{-\omega t} \,. \tag{4.43}$$

4.6.2 *Even algebraic multiplicity*

Consider the case in which system (1.1) possesses $n_r < 2n$ defective real eigenvalues λ_k ($k = 1, 2, \dots, n_r$), each of even algebraic multiplicity m_k and unit geometric multiplicity, such that $m_1 + \cdots + m_{n_r} = 2n$. Construct an order m_k Jordan block $\mathbf{J}_k$, an $n \times m_k$ matrix $\mathbf{V}_k$ of the single eigenvector $\mathbf{v}_1^k$ and the $m_k - 1$ generalized eigenvectors, and an m_k-long vector $\mathbf{c}_k$ of eigensolution coefficients according to Eqs. (4.17) and (4.18), respectively. In this case, the free response $\mathbf{x}(t)$ may be expressed as

$$\mathbf{x}(t) = \sum_{k=1}^{n_r} \mathbf{V}_k \, \mathrm{e}^{\mathbf{J}_k t} \mathbf{c}_k \ . \tag{4.44}$$

Since defective real eigenvalues in multi-degree-of-freedom systems correspond to critical damping, we seek decoupled degrees of freedom governed by an equation of motion of the form (4.38). Specifically, much like the case of defective complex conjugate eigenvalues, associated with each defective real eigenvalue λ_k of even algebraic multiplicity m_k are $\overline{m}_k = m_k/2$ identical, critically damped single-degree-of-freedom systems described by

$$\ddot{p}_j^k(t) - 2\lambda_k \dot{p}_j^k(t) + \lambda_k^2 p_j^k(t) = 0 \tag{4.45}$$

that are subject to different initial conditions $p_j^k(0)$ and $\dot{p}_j^k(0)$ ($j = 1, 2, \dots, \overline{m}_k$). It follows from the decoupled equation of motion (4.45) that the n_r eigenvalues may be arranged in $\mathbf{\Lambda}$ as

$$\mathbf{\Lambda} = \bigoplus_{k=1}^{n_r} \mathbf{\Lambda}_k \ , \qquad \mathbf{\Lambda}_k = \lambda_k \mathbf{I}_{\overline{m}_k} \ , \tag{4.46}$$

where $\mathbf{\Lambda}$ is its own conjugate since the eigenvalues are identical in the case of critical damping. The decoupled system (1.2) is subsequently established by using Eq. (4.46) to form the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ according to Eq. (2.45). Next, from Eq. (4.44), we deduce that

$$\mathbf{J}_x = \bigoplus_{k=1}^{n_r} \mathbf{J}_k \ , \qquad \mathbf{V}_x = \begin{bmatrix} \mathbf{V}_1 & \cdots & \mathbf{V}_{n_r} \end{bmatrix} \ . \tag{4.47}$$

As an extension of the single-degree-of-freedom response (4.42), the $\overline{m}_k$ decoupled solutions $p_j^k(t)$ associated with the equation of motion (4.45) for the defective eigenvalue λ_k

may be arranged in a column vector as

$$\mathbf{p}_k(t) = \begin{bmatrix} p_1^k(t) \\ p_2^k(t) \\ \vdots \\ p_{\overline{m}_k}^k(t) \end{bmatrix} = \begin{bmatrix} (c_1^k + c_2^k t)\,\mathrm{e}^{\lambda_k t} \\ (c_3^k + c_4^k t)\,\mathrm{e}^{\lambda_k t} \\ \vdots \\ (c_{m_k-1}^k + c_{m_k}^k t)\,\mathrm{e}^{\lambda_k t} \end{bmatrix} . \tag{4.48}$$

Analogous to Eq. (4.42), we may cast Eq. (4.48) in the matrix-vector form

$$\mathbf{p}_k(t) = \mathbf{E}_k\,\mathrm{e}^{\mathbf{\Gamma}_k t}\mathbf{c}_k \ , \tag{4.49}$$

for which the order m_k Jordan matrix $\mathbf{\Gamma}_k$ and the $\overline{m}_k \times m_k$ matrix $\mathbf{E}_k$ have the block diagonal structures

$$\mathbf{\Gamma}_k = \bigoplus_{j=1}^{\overline{m}_k} \begin{bmatrix} \lambda_k & 1 \\ 0 & \lambda_k \end{bmatrix} , \qquad \mathbf{E}_k = \bigoplus_{j=1}^{\overline{m}_k} \left[\, 1 \;\middle|\; 0 \,\right] . \tag{4.50}$$

Finally, with the modal response $\mathbf{p}(t)$ given by Eq. (4.25) (with $n_c = n_r$) so as to be conformable to $\mathbf{\Lambda}$, it follows that

$$\mathbf{J}_p = \bigoplus_{k=1}^{n_r} \mathbf{\Gamma}_k \ , \qquad \mathbf{V}_p = \bigoplus_{k=1}^{n_r} \mathbf{E}_k \ . \tag{4.51}$$

The free response $\mathbf{x}(t)$ of system (1.1) may then be obtained via Eq. (4.8) upon solution of the decoupled system, where the modal initial conditions are given by Eq. (4.9) and the response $p_j^k(t)$ of each critically damped degree of freedom is

$$p_j^k(t) = \left[p_j^k(0) + \left(\dot{p}_j^k(0) - \lambda_k p_j^k(0)\right) t\right] \mathrm{e}^{\lambda_k t} \ . \tag{4.52}$$

Unlike the cases in which system (1.1) is non-defective or has defective complex eigenvalues only, the order $2n$ transformation matrices $\mathbf{S}_x$ and $\mathbf{S}_p$ generally do not simplify into neatly partitioned matrices with order n blocks when the eigenvalues are real and defective, and the same is true of the matrix exponential product $\mathrm{e}^{\mathbf{J}_x t}\mathrm{e}^{-\mathbf{J}_p t}$. Consequently, it is generally not possible to extract an order n transformation from the state equation (4.8) in a concise form, such as in Eqs. (2.26) and (4.29), that recovers the free response $\mathbf{x}(t)$ directly. However, in principle, one may isolate such a transformation from Eq. (4.8) using a computer algebra system upon calculating $\mathbf{S}_x$ and $\mathbf{S}_p$.

In the special case when every defective real eigenvalue λ_k has algebraic multiplicity $m_k = 2$, the Jordan matrices $\mathbf{J}_p$ and $\mathbf{J}_x$ are identical, and hence both algebraic and geometric multiplicities are preserved during decoupling. As a result, transformation (4.8) is no longer time-dependent because the matrix exponential product $e^{\mathbf{J}_x t} e^{-\mathbf{J}_p t}$ simplifies to an order $2n$ identity matrix: $\mathbf{S}(t) = \mathbf{S} = \mathbf{S}_x \mathbf{S}_p^{-1}$. Moreover, when system (1.1) and its decoupled form (1.2) are cast in the state space according to Eqs. (4.12) and (4.13), respectively, the relationship (4.16) between the state matrices $\mathbf{A}$ and $\mathbf{B}$ reduces to a similarity transformation.

To summarize, when the coupled n-degree-of-freedom system (1.1) has n_r defective real eigenvalues λ_k with even algebraic multiplicities m_k, it may be decoupled into n degrees of freedom that are critically damped. Associated with each of the n_r defective eigenvalues are $m_k/2$ identical, critically damped systems subject to different initial conditions.

4.6.3 Odd algebraic multiplicity with an unpaired distinct eigenvalue

Let the n-degree-of-freedom system (1.1) have all real eigenvalues, of which $2r + 1$ are distinct. By the method of phase synchronization, $2r$ of these eigenvalues are paired as real quadratic conjugates to generate r decoupled systems that are overdamped. What then is to be done with the unpaired distinct eigenvalue? We shall refer to this eigenvalue as a free eigenvalue and denote it by λ_f. Since $2r + 1$ is odd, then $2(n - r) - 1$ is also odd. Assuming each defective eigenvalue has unit geometric multiplicity, it is implied that, even if some of the defective eigenvalues are of even algebraic multiplicity, at least one eigenvalue will necessarily have odd algebraic multiplicity. In order to decouple a system of this nature, we will need to introduce an additional, and rather peculiar, criterion for the pairing of eigenvalues.

To streamline our presentation, we consider the case when $r = 0$ so that system (1.1) has only one distinct eigenvalue, which is necessarily a free eigenvalue λ_f. In addition, we assume that the remaining defective real eigenvalues are identical, i.e., there is a single defective real eigenvalue λ_1 of algebraic multiplicity $m_1 = 2n - 1$. The corresponding free response $\mathbf{x}(t)$ is given by

$$\mathbf{x}(t) = \mathbf{V}_1 e^{\mathbf{J}_1 t} \mathbf{c}_1 + \mathbf{v}_f e^{\lambda_f t} c_f \,, \tag{4.53}$$

where the order m_1 Jordan block $\mathbf{J}_1$, the $n \times m_1$ matrix $\mathbf{V}_1$ of vectors, and the m_1-dimensional vector $\mathbf{c}_1$ of eigensolution coefficients are as detailed in Eqs. (4.17) and (4.18), respectively. In Eq. (4.53), $\mathbf{v}_f$ and c_f denote the eigenvector and eigensolution coefficient, respectively, associated with the free eigenvalue λ_f. Let $\hat{m}_1 = m_1 - 1 = 2(n - 1)$. Similar to the case of even algebraic multiplicity, the single defective real eigenvalue λ_1 of odd algebraic multiplicity $m_1 = 2n - 1$ has associated with it $\overline{m}_1 = \hat{m}_1/2 = n - 1$ decoupled degrees of freedom $p_j^1(t)$ $(j = 1, 2, \ldots, \overline{m}_1)$ that are critically damped and governed by the equation of motion (4.45) with $k = 1$. Since system (1.1) is to be decoupled into n independent subsystems, what is the remaining degree of freedom and its corresponding equation of motion?

The free eigenvalue λ_f remains unpaired, and because the defective eigenvalue λ_1 is the only other eigenvalue, we must conclude that it is necessary to pair λ_1 and λ_f in order to completely decouple system (1.1). Treating the eigenvalues λ_1 and λ_f as real quadratic conjugates, this unusual pairing implies that the remaining degree of freedom $p^*(t)$ is effectively overdamped and governed by the equation of motion

$$\ddot{p}^*(t) - (\lambda_1 + \lambda_f)\dot{p}^*(t) + \lambda_1\lambda_f p^*(t) = 0 \, . \tag{4.54}$$

Consequently, by Eqs. (4.45) (with $k = 1$) and (4.54), the eigenvalues may be arranged in $\mathbf{\Lambda}$ and its conjugate $\widehat{\mathbf{\Lambda}}$ according to

$$\mathbf{\Lambda} = \lambda_1 \mathbf{I}_{\overline{m}_1} \oplus \lambda_1 \, , \qquad \widehat{\mathbf{\Lambda}} = \lambda_1 \mathbf{I}_{\overline{m}_1} \oplus \lambda_f \, , \tag{4.55}$$

from which the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ defining the decoupled system (1.2) may be determined via Eq. (2.45). A comparison of Eqs. (4.6) and (4.53) reveals that

$$\mathbf{J}_x = \mathbf{J}_1 \oplus \lambda_f \, , \qquad \mathbf{V}_x = \left[\, \mathbf{V}_1 \, \middle| \, \mathbf{v}_f \, \right] \, . \tag{4.56}$$

With the assistance of Eqs. (4.48)-(4.50), it is straightforward to show that the responses $p_j^1(t)$ for the $\overline{m}_1$ critically damped degrees of freedom have the equivalent representations

$$\mathbf{p}_1(t) = \begin{bmatrix} p_1^1(t) \\ p_2^1(t) \\ \vdots \\ p_{\overline{m}_1}^1(t) \end{bmatrix} = \begin{bmatrix} (c_1^1 + c_2^1 t)\mathrm{e}^{\lambda_1 t} \\ (c_3^1 + c_4^1 t)\mathrm{e}^{\lambda_1 t} \\ \vdots \\ (c_{\hat{m}_1 - 1}^1 + c_{\hat{m}_1}^1 t)\mathrm{e}^{\lambda_1 t} \end{bmatrix} = \mathbf{E}_1 \, \mathrm{e}^{\mathbf{\Gamma}_1 t} \mathbf{c}_1 \, , \tag{4.57}$$

for which the matrices $\mathbf{\Gamma}_1$ and $\mathbf{E}_1$ are

$$\mathbf{\Gamma}_1 = \bigoplus_{j=1}^{\overline{m}_1} \begin{bmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{bmatrix} \, , \qquad \mathbf{E}_1 = \bigoplus_{j=1}^{\overline{m}_1} \left[\, 1 \, \middle| \, 0 \, \right] \, . \tag{4.58}$$

The response $p^*(t)$ for the overdamped degree of freedom corresponding to the equation of motion (4.54) is given by

$$p^*(t) = c_f \, \mathrm{e}^{\lambda_f t} + c_{m_1}^1 \mathrm{e}^{\lambda_1 t} \, . \tag{4.59}$$

It is important to note that the eigensolution coefficient associated with the defective eigenvalue λ_1 is the coefficient $c_{m_1}^1$ that remains after having established $\overline{m}_1$ critically damped systems. This ensures that the initial conditions for $p^*(t)$ are independent of those for the critically damped solutions $p_j^1(t)$. When initial conditions are applied to Eq. (4.59), the response $p^*(t)$ has the form

$$p^*(t) = \left(\frac{\lambda_f p^*(0) - \dot{p}^*(0)}{\lambda_f - \lambda_1} \right) \mathrm{e}^{\lambda_1 t} - \left(\frac{\lambda_1 p^*(0) - \dot{p}^*(0)}{\lambda_f - \lambda_1} \right) \mathrm{e}^{\lambda_f t} \, . \tag{4.60}$$

Structuring the modal response vector as

$$\mathbf{p}(t) = \left[\mathbf{p}_1^{\mathrm{T}}(t) \quad p^*(t) \right]^{\mathrm{T}} \tag{4.61}$$

to be conformable to $\mathbf{\Lambda}$, it is implied by Eqs. (4.57) and (4.59) that

$$\mathbf{J}_p = \mathbf{\Gamma}_1 \oplus \lambda_1 \oplus \lambda_f \, , \qquad \mathbf{V}_p = \mathbf{E}_1 \oplus \left[\, 1 \; \middle| \; 1 \, \right] \, . \tag{4.62}$$

We are now in a position to solve the decoupled system and then retrieve the free response $\mathbf{x}(t)$ of system (1.1) via transformation (4.8).

In summary, if the coupled n-degree-of-freedom system (1.1) possesses a free eigenvalue λ_f and a single defective real eigenvalue λ_1 with odd algebraic multiplicity $m_1 = 2n-1$ (i.e., $r = 0$), then the system may be decoupled into $(m_1 - 1)/2 = n - 1$ identical, independent critically damped oscillators subject to different initial conditions and an overdamped degree of freedom formed by pairing the defective eigenvalue λ_1 with λ_f. In the general case when $r > 0$, the $2r$ distinct real eigenvalues of system (1.1) are arranged into r real quadratic conjugate pairs, and thus the matrices $\mathbf{\Lambda}$, $\widehat{\mathbf{\Lambda}}$, $\mathbf{J}_x$, $\mathbf{V}_x$, $\mathbf{J}_p$, and $\mathbf{V}_p$ are constructed blockwise by a direct sum of Eqs. (4.55), (4.56), and (4.62) with their respective non-defective counterparts. Lastly, since the pairing of non-defective real eigenvalues as real quadratic conjugates is not unique, it should be no surprise that the free eigenvalue λ_f is also not unique, and hence the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ of the decoupled system may take on a variety of forms.

Is it possible to decouple a system of the type described in this section by leaving the free eigenvalue λ_f unpaired and have Eq. (4.8) remain a valid transformation to determine the system's free response $\mathbf{x}(t)$? A short answer to this question is "no" because the second-order structure inherent in transformation (4.8), based on eigenvalue pairing, would otherwise be violated. To elaborate upon this, replace the matrix $\left[1 \,\middle|\, 1\right]$ in $\mathbf{V}_p$ from Eq. (4.62) with $\left[0 \,\middle|\, 1\right]$ so that the free eigenvalue λ_f is no longer paired with the defective real eigenvalue λ_1. It can be verified that the order $2n$ transformation matrix $\mathbf{S}_p$ is singular when $\mathbf{V}_p$ has this structure, and hence the free response $\mathbf{x}(t)$ of system (1.1) may not be recovered from Eq. (4.8). Indeed, since it is implied from Eq. (4.59) that the decoupled solution $p^*(t)$ would correspond to a massless first-order system in this case, system (1.1) is decoupled not into the form (1.2), but rather

$$\mathbf{A}_2\, \ddot{\mathbf{p}}(t) + \mathbf{A}_1\, \dot{\mathbf{p}}(t) + \mathbf{A}_0\, \mathbf{p}(t) = \mathbf{0} \, , \tag{4.63}$$

$$\mathbf{A}_2 = \mathbf{I}_{\overline{m}_1} \oplus 0 \, , \qquad \mathbf{A}_1 = -(2\lambda_1\, \mathbf{I}_{\overline{m}_1} \oplus \lambda_f) \, , \qquad \mathbf{A}_0 = (\lambda_1)^2\, \mathbf{I}_{\overline{m}_1} \oplus \lambda_f^2 \, . \tag{4.64}$$

From Eq. (4.64), the leading coefficient matrix $\mathbf{A}_2$ is singular, and hence the decoupled system (4.63) has an infinite eigenvalue (e.g., see pg. 255 of [50]). Since the decoupling transformation is eigenvalue-preserving, this implies that the coupled system (1.1) must also have an infinite eigenvalue, but we know that is not the case because the mass matrix

$\mathbf{M}$ is positive definite. Thus, it is not surprising that the transformation matrix $\mathbf{S}_p$ is singular in this situation because of an apparent disconnect between the spectrum of the coupled system (1.1) and that of the decoupled form (4.63).

4.6.4 Odd algebraic multiplicity

Suppose system (1.1) possesses an even number $n_r < 2n$ of defective real eigenvalues λ_k ($k = 1, 2, \ldots, n_r$), each of odd algebraic multiplicity m_k and unit geometric multiplicity, such that $m_1 + \cdots + m_{n_r} = 2n$. We wish to show that a system of the type described here may be decoupled in a manner similar to that in Section 4.6.3, namely by pairing particular eigenvalues in an appropriate (though somewhat peculiar) manner.

To streamline our discussion, we shall consider the case when system (1.1) has $n_r = 2$ defective eigenvalues λ_1 and λ_2 with algebraic multiplicities m_1 and m_2, respectively, where $m_1 \neq m_2$ in general. The free response $\mathbf{x}(t)$ may be cast in the form of Eq. (4.44) with $n_r = 2$ and where $\mathbf{J}_k$ $(k = 1, 2)$, $\mathbf{V}_k$, and $\mathbf{c}_k$ have the structure (4.17) and (4.18), respectively. Since the defective real eigenvalues λ_1 and λ_2 have odd algebraic multiplicity, decoupling begins in essentially the same way as in the procedure outlined in Section 4.6.3. That is to say, associated with each defective eigenvalue λ_k are $\overline{m}_k = \hat{m}_k/2 = (m_k - 1)/2$ critically damped, decoupled coordinates $p_j^k(t)$ ($j = 1, 2, \ldots, \overline{m}_k$) governed by the equation of motion (4.45). The independent critically damped systems (4.45) corresponding to λ_1 and λ_2 account for $\overline{m}_1 + \overline{m}_2 = n - 1$ degrees of freedom of the decoupled system (1.2). What about the remaining degree of freedom?

Because the defective eigenvalues λ_1 and λ_2 are the only eigenvalues, we have no choice but to pair them if complete decoupling is to be achieved. As $\lambda_1 \neq \lambda_2$, treat the eigenvalues as real quadratic conjugates so that their pairing generates an effectively overdamped degree of freedom $p^*(t)$ governed by the equation of motion

$$\ddot{p}^*(t) - (\lambda_1 + \lambda_2)\dot{p}^*(t) + \lambda_1\lambda_2 p^*(t) = 0 \,. \tag{4.65}$$

It follows from Eqs. (4.45) and (4.65) that we may form $\mathbf{\Lambda}$ and its conjugate $\widehat{\mathbf{\Lambda}}$ as

$$\mathbf{\Lambda} = \lambda_1 \mathbf{I}_{\overline{m}_1} \oplus \lambda_1 \oplus \lambda_2 \mathbf{I}_{\overline{m}_2} \,, \qquad \widehat{\mathbf{\Lambda}} = \lambda_1 \mathbf{I}_{\overline{m}_1} \oplus \lambda_2 \oplus \lambda_2 \mathbf{I}_{\overline{m}_2} \,, \tag{4.66}$$

which then allows us to construct the coefficient matrices $\mathbf{D}$ and $\mathbf{\Omega}$ that define the decoupled system (1.2) using Eq. (2.45). The free response $\mathbf{x}(t)$ is governed by Eq. (4.44) with $n_r = 2$, and thus the matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ are formed according to Eq. (4.47) with $n_r = 2$. Analogous to Eqs. (4.48) and (4.49), the $\overline{m}_k$ critically damped responses $p_j^k(t)$ corresponding to each defective eigenvalue λ_k may be cast in the compact matrix-vector form

$$\mathbf{p}_k(t) = \begin{bmatrix} p_1^k(t) \\ p_2^k(t) \\ \vdots \\ p_{\overline{m}_k}^k(t) \end{bmatrix} = \begin{bmatrix} (c_1^k + c_2^k t)\mathrm{e}^{\lambda_k t} \\ (c_3^k + c_4^k t)\mathrm{e}^{\lambda_k t} \\ \vdots \\ (c_{\hat{m}_k - 1}^k + c_{\hat{m}_k}^k t)\mathrm{e}^{\lambda_k t} \end{bmatrix} = \mathbf{E}_k \, \mathrm{e}^{\mathbf{\Gamma}_k t} \mathbf{c}_k \ , \tag{4.67}$$

where the matrices $\mathbf{\Gamma}_k$ and $\mathbf{E}_k$ are as given in Eq. (4.50). The solution $p^*(t)$ of the over-damped system governed by the equation of motion (4.65) is of the form

$$p^*(t) = c_{m_1}^1 \mathrm{e}^{\lambda_1 t} + c_{m_2}^2 \mathrm{e}^{\lambda_2 t} \ . \tag{4.68}$$

The eigensolution coefficients $c_{m_1}^1$ and $c_{m_2}^2$ associated with the defective eigenvalues λ_1 and λ_2, respectively, are those coefficients that do not correspond to the $\overline{m}_1 + \overline{m}_2$ critically damped systems, and thus the initial conditions for $p^*(t)$ are independent of those for the critically damped solutions $p_j^k(t)$. Upon applying initial conditions to Eq. (4.68),

$$p^*(t) = \left(\frac{\lambda_2 p^*(0) - \dot{p}^*(0)}{\lambda_2 - \lambda_1} \right) \mathrm{e}^{\lambda_1 t} - \left(\frac{\lambda_1 p^*(0) - \dot{p}^*(0)}{\lambda_2 - \lambda_1} \right) \mathrm{e}^{\lambda_2 t} \ . \tag{4.69}$$

Based on the structure of $\mathbf{\Lambda}$, take the modal response vector $\mathbf{p}(t)$ as

$$\mathbf{p}(t) = \begin{bmatrix} \mathbf{p}_1^{\mathrm{T}}(t) & p^*(t) & \mathbf{p}_2^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} \ , \tag{4.70}$$

and hence Eqs. (4.67) and (4.70) imply that

$$\mathbf{J}_p = \mathbf{\Gamma}_1 \oplus \lambda_1 \oplus \mathbf{\Gamma}_2 \oplus \lambda_2 \ , \qquad \mathbf{V}_p = \left[\, \mathbf{E}_1 \oplus 1 \oplus \mathbf{E}_2 \; \middle| \; \mathbf{e}_{\overline{m}_1+1}^{\overline{m}} \, \right] \ , \tag{4.71}$$

where $\mathbf{e}_i^l$ denotes a unit vector of length l that has 1 as its ith element and 0 for the rest, and $\overline{m} = (m_1 + m_2)/2$. The decoupled system can now be readily solved, and the free response $\mathbf{x}(t)$ of system (1.1) may be recovered from transformation (4.8).

To summarize, if the coupled system (1.1) has two defective real eigenvalues λ_1 and λ_2 with odd algebraic multiplicities m_1 and m_2, respectively, it may be decoupled into $(m_1 - 1)/2$ and $(m_2 - 1)/2$ identical, critically damped degrees of freedom with different initial conditions and one degree of freedom that is overdamped, formed by pairing the defective eigenvalues λ_1 and λ_2. In the general case of even $n_r < 2n$, the matrices $\mathbf{\Lambda}$, $\widehat{\mathbf{\Lambda}}$, $\mathbf{J}_p$, and $\mathbf{V}_p$ are constructed by a direct sum of $n_r/2$ sub-matrices of the form (4.66) and (4.71), respectively. Likewise, the modal response vector $\mathbf{p}(t)$ is partitioned conformably with $\mathbf{\Lambda}$ into $n_r/2$ sub-vectors with the structure (4.70). The matrices $\mathbf{J}_x$ and $\mathbf{V}_x$ are still given by Eq. (4.47). The decoupled system is not unique in this case since the pairing of the defective eigenvalues as real quadratic conjugates is not unique.

4.7 Relaxation of the unit geometric multiplicity constraint

How are the decoupling procedures outlined in the previous sections affected when the unit geometric multiplicity restriction on the defective eigenvalues is lifted? In the case of complex eigenvalues, easing of this constraint is a non-issue since the complex eigenvalues and their associated eigenvectors must always occur in complex conjugate pairs for real system matrices $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{K}$. Indeed, transformations (4.27) and (4.29) remain valid when some defective complex eigenvalues λ_k have $\rho_k > 1$ eigenvectors so long as the order m_k Jordan block $\mathbf{J}_k$ given in Eq. (4.17) is replaced with

$$\mathbf{J}_k = \begin{bmatrix} \lambda_k \mathbf{I}_{\rho_k - 1} & 0 & 0 & \cdots & 0 \\ 0 & \lambda_k & 1 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \lambda_k & 1 \\ 0 & \cdots & 0 & 0 & \lambda_k \end{bmatrix} = \lambda_k \mathbf{I}_{m_k} + \mathbf{N}_k = \mathbf{\Lambda}_k + \mathbf{N}_k \; . \tag{4.72}$$

Note that the nilpotent matrix $\mathbf{N}_k$ is redefined by Eq. (4.72).

When there are defective real eigenvalues λ_k with geometric multiplicity $\rho_k > 1$, decoupling becomes an even more delicate task than described earlier, and this statement is best demonstrated through a simple example. Suppose system (1.1) has a lone distinct real eigenvalue, necessarily making it a free eigenvalue λ_f, and a single defective real eigenvalue λ_1 of odd algebraic multiplicity m_1 and geometric multiplicity $\rho_1 = 2$ (i.e., the defective eigenvalue has two associated eigenvectors). In this case, the methodology presented in Section 4.6.3 fails to decouple system (1.1) since the first eigenvector $\mathbf{v}_1^1$ for λ_1 does not initiate a Jordan chain, and hence its corresponding eigensolution should be treated as if it has been generated from a distinct eigenvalue. Consequently, the defective eigenvalue λ_1, with associated eigenvector $\mathbf{v}_1^1$, and the free eigenvalue λ_f are paired as real quadratic conjugates to yield an overdamped degree of freedom upon decoupling. Contrast this with the case of unit geometric multiplicity: an overdamped degree of freedom is formed from pairing λ_1 and λ_f, but associated with λ_1 is the generalized eigenvector $\mathbf{v}_{m_1}^1$. Lastly, since the Jordan chain for λ_1 is of even length $m_1 - 1$, the decoupling procedure in Section 4.6.2 may be utilized to generate $(m_1 - 1)/2$ identical, critically damped degrees of freedom.

Of course, there are many other combinations of eigenvalues and their multiplicities to consider, but this example sufficiently illustrates the cautious manner in which decoupling should be approached for systems with defective real eigenvalues of arbitrary (yet permissible) geometric multiplicity. Moreover, the point to take from this example is that, with careful attention to the pairing of eigenvalues, decoupling is still possible using the basic procedures outlined in this chapter.

4.8 Reduction to classical modal analysis

As one would expect, transformation (4.8) reduces to classical modal analysis in the event that system (1.1) is classically damped. Because classical modal analysis constitutes a strictly isospectral transformation, the Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$ are identical, simplifying the matrix exponential product $\mathrm{e}^{\mathbf{J}_x t}\mathrm{e}^{-\mathbf{J}_p t}$ to an order $2n$ identity matrix, which leaves the transformation $\mathbf{S}_x\mathbf{S}_p^{-1}$. If the system eigenvectors are normalized in accordance with Eqs. (2.42) and (2.43), then there exists a particular pairing of the non-defective real eigenvalues such that $\mathbf{S}_x\mathbf{S}_p^{-1} = \mathbf{U} \oplus \mathbf{U}$, and thus the upper half of the state equation (4.8) yields the classical modal transformation $\mathbf{x}(t) = \mathbf{U}\,\mathbf{p}(t)$. Multiplicative constants appear here and there if normalizations (2.42) and (2.43) are not used. Moreover, the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ defined by Eq. (2.45) are the same as those given by classical modal analysis.

4.9 Illustrative examples

The following four numerical examples illustrate how unforced defective systems of the form (1.1) are decoupled by the methodologies provided in this chapter. The first example (Example 3) demonstrates the decoupling procedure for a system with defective complex conjugate eigenvalues, while the remaining examples focus on decoupling systems with defective real eigenvalues. The defective eigenvalues of the first three examples are all of unit geometric multiplicity. The last example (Example 6) serves to clarify how a system that has a defective (real) eigenvalue with geometric multiplicity greater than unity may be decoupled.

Example 3

A non-classically damped, 2-degree-of-freedom system has mass matrix $\mathbf{M} = \mathbf{I}_2$, and its damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}. \tag{4.73}$$

The initial conditions are prescribed as $\mathbf{x}(0) = [-1,\, 1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [-2,\, 2]^{\mathrm{T}}$. Solving the associated quadratic eigenvalue problem, we find that there is $n_c = 1$ pair of defective complex conjugate eigenvalues with algebraic multiplicity $m_1 = 2$ and unit geometric multiplicity:

$$\lambda_1 = -1 + \mathrm{i}\sqrt{2}\,, \qquad \mathbf{v}_1^1 = \begin{bmatrix} -\mathrm{i}\sqrt{2} \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2^1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}. \tag{4.74}$$

The sole eigenvector $\mathbf{v}_1^1$ has not been subjected to any normalization scheme. Since the eigenvalues are all complex and defective, the system may be decoupled by the methodology presented in Section 4.5. From Eq. (4.22), we have

$$\mathbf{\Lambda} = (-1 + \mathrm{i}\sqrt{2})\,\mathbf{I}_2\ . \tag{4.75}$$

By Eqs. (4.17), (4.18), and (4.24),

$$\mathbf{J} = \begin{bmatrix} -1+\mathrm{i}\sqrt{2} & 1 \\ 0 & -1+\mathrm{i}\sqrt{2} \end{bmatrix}, \qquad \mathbf{V} = \left[\begin{array}{c|c} -\mathrm{i}\sqrt{2} & 3 \\ 1 & 0 \end{array}\right], \qquad \mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}. \tag{4.76}$$

Evaluating Eq. (2.27) yields

$$\mathbf{T}_1 = \begin{bmatrix} -1 & 3 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{T}_2 = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}. \tag{4.77}$$

Using Eqs. (2.45) and (4.75) to construct the decoupled system's coefficient matrices,

$$\mathbf{D} = 2\,\mathbf{I}_2\ , \qquad \mathbf{\Omega} = 3\,\mathbf{I}_2\ . \tag{4.78}$$

Equation (4.78) implies that the decoupled system consists of $m_1 = 2$ identical, underdamped single-degree-of-freedom oscillators with viscous damping factor $\zeta_1 = \zeta_2 = 0.58$. However, the initial conditions for each decoupled coordinate $p_j^1(t) = p_j(t)$ $(j = 1, 2)$ are different: $\mathbf{p}(0) = [1, 0.5]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [1.5, -3]^{\mathrm{T}}$ from Eq. (4.28). The decoupled solution $\mathbf{p}(t)$ is illustrated in Fig. 6(a), and the system free response $\mathbf{x}(t)$ obtained via transformation (4.29) is shown in Fig. 6(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (4.29) are indeed the same.

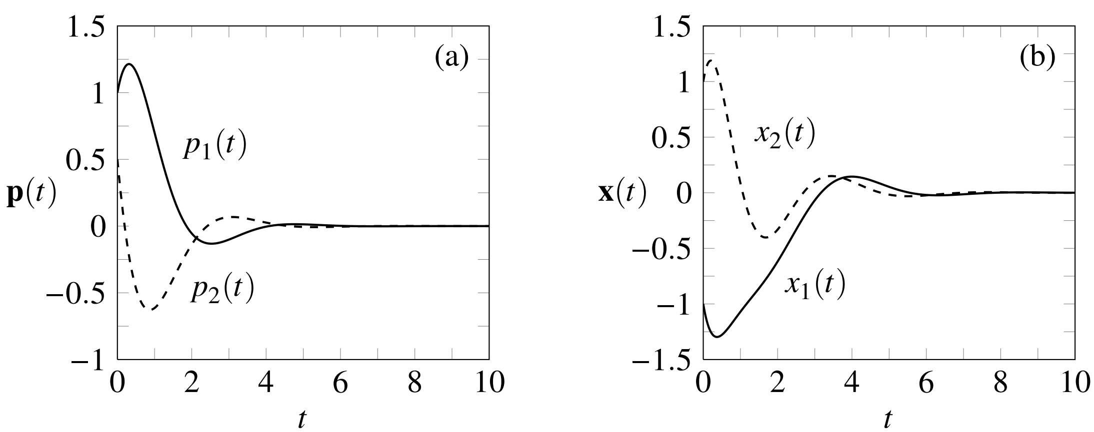

Fig. 6 Free response of Example 3. (a) Decoupled solutions $p_j(t)$ $(j = 1, 2)$. (b) System responses $x_j(t)$.

Example 4

Consider a non-classically damped, 2-degree-of-freedom system with mass matrix $\mathbf{M} = \mathbf{I}_2$ and for which the damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} , \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} . \tag{4.79}$$

Let the initial conditions be $\mathbf{x}(0) = [1, -1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [-2, -1]^{\mathrm{T}}$. Solution of the associated quadratic eigenvalue problem reveals that the system possesses a single eigenvalue, a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 4$ and unit geometric multiplicity. The corresponding eigenvector $\mathbf{v}_1^1$ and generalized eigenvectors $\mathbf{v}_i^1$ $(i = 2, 3, 4)$ are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \qquad \mathbf{v}_2^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} , \qquad \mathbf{v}_3^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} , \qquad \mathbf{v}_4^1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} . \tag{4.80}$$

The sole eigenvector $\mathbf{v}_1^1$ has not been subjected to any normalization scheme. Since the algebraic multiplicity of the single eigenvalue λ_1 is even, we follow the decoupling procedure presented in Section 4.6.2 with $n_r = 1$. As $\overline{m}_1 = 2$, Eq. (4.46) yields

$$\boldsymbol{\Lambda} = \widehat{\boldsymbol{\Lambda}} = -\mathbf{I}_2 . \tag{4.81}$$

By Eqs. (4.17), (4.18), and (4.47),

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \qquad \mathbf{V}_x = \left[\begin{array}{c|c|c|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & -1 \end{array}\right] , \tag{4.82}$$

and it follows from Eqs. (4.50) and (4.51) that

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \qquad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \tag{4.83}$$

From Eqs. (2.45) and (4.81), the coefficient matrices of the decoupled system are

$$\mathbf{D} = 2\mathbf{I}_2 , \qquad \boldsymbol{\Omega} = \mathbf{I}_2 . \tag{4.84}$$

Thus, the decoupled system is composed of $\overline{m}_1 = 2$ identical, critically damped single-degree-of-freedom oscillators. The initial conditions for the decoupled coordinates $p_j^1(t) = p_j(t)$ $(j = 1, 2)$ are $\mathbf{p}(0) = [0, -2]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [1, 2]^{\mathrm{T}}$ from Eq. (4.9). The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 7(a), and the system free response $\mathbf{x}(t)$ recovered from transformation (4.8) is shown in Fig. 7(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (4.8) are indeed the same.

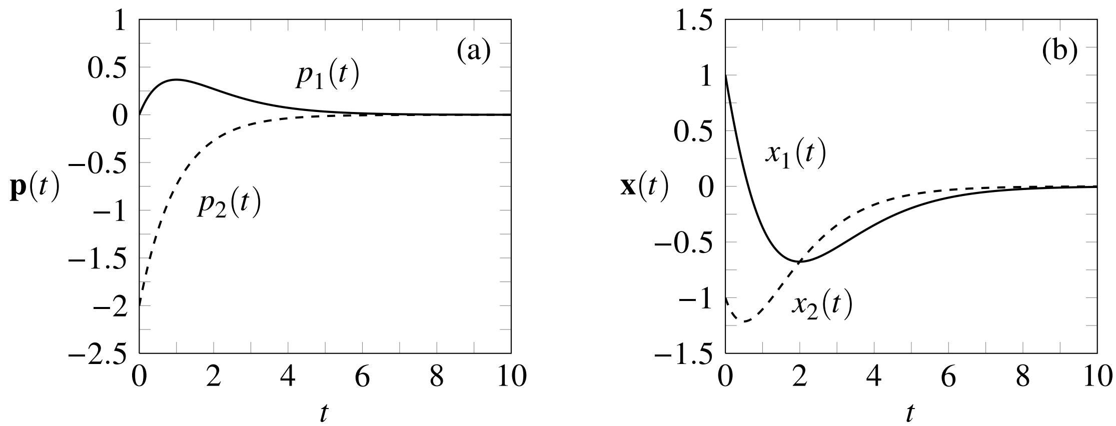

Fig. 7 Free response of Example 4. (a) Decoupled solutions $p_j(t)$ $(j = 1, 2)$. (b) System responses $x_j(t)$.

Example 5

Suppose a non-classically damped, 2-degree-of-freedom system with mass matrix $\mathbf{M} = \mathbf{I}_2$ has damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}, \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}. \tag{4.85}$$

The initial conditions are prescribed as $\mathbf{x}(0) = [-1, 1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [2, 1]^{\mathrm{T}}$. Solving the associated quadratic eigenvalue problem, we find that the system has a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 3$ and unit geometric multiplicity whose non-normalized eigenvector $\mathbf{v}_1^1$ and generalized eigenvectors $\mathbf{v}_i^1$ $(i = 2, 3)$ are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2^1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3^1 = \begin{bmatrix} 0.5 \\ -1 \end{bmatrix}. \tag{4.86}$$

The remaining eigenvalue is necessarily real and unpaired, and hence it is a free eigenvalue: $\lambda_f = -2$, $\mathbf{v}_f = [1, -1]^{\mathrm{T}}$. This system may be decoupled using the procedure outlined in

Section 4.6.3. Since $\hat{m}_1 = 2$ and $\overline{m}_1 = 1$, Eq. (4.55) implies that

$$\mathbf{\Lambda} = -\mathbf{I}_2 , \qquad \widehat{\mathbf{\Lambda}} = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} . \tag{4.87}$$

From Eqs. (4.17), (4.18), and (4.56), we have

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} , \qquad \mathbf{V}_x = \left[\begin{array}{c|c|c|c} 1 & 0 & 0.5 & 1 \\ 0 & 1 & -1 & -1 \end{array}\right] , \tag{4.88}$$

while Eqs. (4.58) and (4.62) yield

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} , \qquad \mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} . \tag{4.89}$$

By Eqs. (2.45) and (4.87), the decoupled system's coefficient matrices are

$$\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} , \qquad \mathbf{\Omega} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} . \tag{4.90}$$

The first row of the decoupled system defined by Eq. (4.90) represents the $\overline{m}_1 = 1$ critically damped degree of freedom $p_1^1(t) = p_1(t)$ associated with the defective eigenvalue λ_1, while the second row corresponds to the overdamped degree of freedom $p^*(t) = p_2(t)$ generated by pairing λ_1 with the free eigenvalue λ_f. Using Eq. (4.9), the modal initial conditions are $\mathbf{p}(0) = [-3, 2]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [6, -4]^{\mathrm{T}}$. The decoupled solution $\mathbf{p}(t)$ is illustrated in Fig. 8(a), and the system free response $\mathbf{x}(t)$ obtained via transformation (4.8) is shown in Fig. 8(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (4.8) are indeed the same.

Example 6

A non-classically damped, 3-degree-of-freedom system has mass matrix $\mathbf{M} = \mathbf{I}_3$, and its damping matrix $\mathbf{C}$ and stiffness matrix $\mathbf{K}$ are given by

$$\mathbf{C} = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} , \qquad \mathbf{K} = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} . \tag{4.91}$$

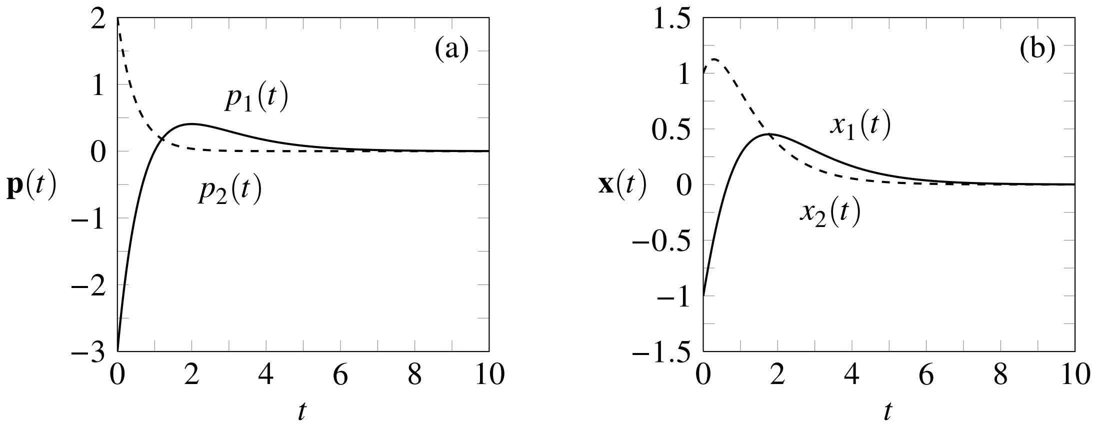

Fig. 8 Free response of Example 5. (a) Decoupled solutions $p_j(t)$ $(j = 1, 2)$. (b) System responses $x_j(t)$.

Take as initial conditions $\mathbf{x}(0) = [1, 0, -1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [2, 1, 0]^{\mathrm{T}}$. Solution of the associated quadratic eigenvalue problem reveals that the system possesses a defective real eigenvalue $\lambda_1 = -1$ with algebraic multiplicity $m_1 = 4$ and geometric multiplicity $\rho_1 = 2$. The corresponding eigenvectors $\mathbf{v}_i^1$ $(i = 1, 2)$ and generalized eigenvectors $\mathbf{v}_j^1$ $(j = 3, 4)$ are

$$\mathbf{v}_1^1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} , \quad \mathbf{v}_2^1 = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} , \quad \mathbf{v}_3^1 = \begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix} , \quad \mathbf{v}_4^1 = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} . \tag{4.92}$$

The remaining eigenvalues are real and distinct:

$$\lambda_2 = -3.41 , \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -0.41 \\ 0.17 \end{bmatrix} , \quad \lambda_3 = -0.59 , \quad \mathbf{v}_3 = \begin{bmatrix} 0.17 \\ 0.41 \\ 1 \end{bmatrix} . \tag{4.93}$$

The eigenvectors $\mathbf{v}_i^1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ have not been subjected to any normalization scheme. To decouple this system with mixed defective and non-defective real eigenvalues, the matrices $\mathbf{J}_x$, $\mathbf{V}_x$, etc., will need to be constructed blockwise via direct sum using the matrix structures (4.10) and (4.11) at the end of Section 4.2 for non-defective eigenvalues and those presented in Section 4.6 for defective real eigenvalues. However, while the algebraic multiplicity of the defective real eigenvalue λ_1 is even, we may not employ the decoupling methodology of Section 4.6.2 because λ_1 has more than one associated eigenvector. Instead, proceed with decoupling as follows. Pair the defective eigenvalue λ_1 with corresponding eigenvector $\mathbf{v}_1^1$ and, say, the distinct eigenvalue λ_3 as real quadratic conjugates so as to yield an overdamped degree of freedom. Of course, one may pair λ_1 with λ_2 instead, if preferred.

The remaining distinct eigenvalue λ_2 necessarily becomes a free eigenvalue (i.e., $\lambda_f = \lambda_2$). Moreover, the Jordan chain for λ_1 is of odd length $m_1 - 1 = 3$, and thus we may use the procedure outlined in Section 4.6.3 to generate $(m_1 - 2)/2 = 1$ critically damped degree of freedom associated with λ_1 and an overdamped degree of freedom by pairing λ_1 with λ_2. To implement this, use Eqs. (2.14) and (4.55) to construct

$$\mathbf{\Lambda} = -\mathbf{I}_3 , \qquad \widehat{\mathbf{\Lambda}} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -3.41 & 0 \\ 0 & 0 & -0.59 \end{bmatrix} . \tag{4.94}$$

With the assistance of Eqs. (4.10) and (4.56),

$$\mathbf{J}_x = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.59 \end{bmatrix} , \tag{4.95}$$

$$\mathbf{V}_x = \left[\begin{array}{c|c|c|c|c|c} 0 & 1 & 0 & 1 & 1 & 0.17 \\ 0 & 2 & -1 & -0.41 & 0 & 0.41 \\ 2 & -2 & 0 & 0.17 & 0 & 1 \end{array}\right] . \tag{4.96}$$

In addition, Eqs. (4.11), (4.58), and (4.62) imply that

$$\mathbf{J}_p = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3.41 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.59 \end{bmatrix} , \tag{4.97}$$

$$\mathbf{V}_p = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} . \tag{4.98}$$

From Eqs. (2.45) and (4.94), the decoupled system's coefficient matrices are given by

$$\mathbf{D} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4.41 & 0 \\ 0 & 0 & 1.59 \end{bmatrix}, \qquad \boldsymbol{\Omega} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3.41 & 0 \\ 0 & 0 & 0.59 \end{bmatrix}. \tag{4.99}$$

To summarize, the decoupled system is composed of (i) a single critically damped degree of freedom $p_1^1(t) = p_1(t)$ (corresponding to the first row of Eq. (4.99)) associated with the defective eigenvalue λ_1; (ii) an overdamped degree of freedom $p^*(t) = p_2(t)$ (second row of Eq. (4.99)) generated by pairing λ_1 with the non-defective eigenvalue λ_2; and (iii) an overdamped degree of freedom $p_3(t)$ (third row of Eq. (4.99)) obtained by pairing λ_1, with associated eigenvector $\mathbf{v}_1^1$, and the non-defective eigenvalue λ_3. From Eq. (4.9), the modal initial conditions are $\mathbf{p}(0) = [0.5, 0.15, 0.65]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [0, 1.91, -1]^{\mathrm{T}}$. The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 9(a), and the system response $\mathbf{x}(t)$ recovered from transformation (4.8) is shown in Fig. 9(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (4.8) are indeed the same.

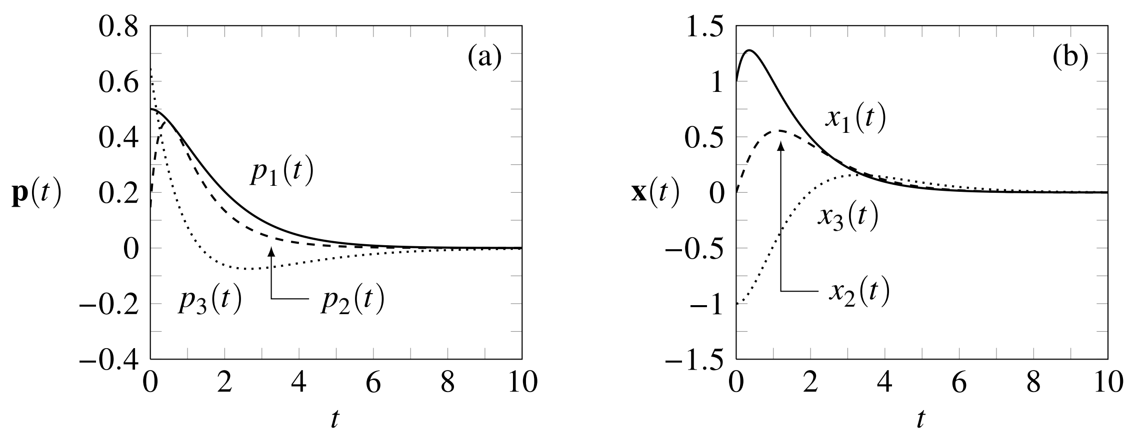

Fig. 9 Free response of Example 6. (a) Decoupled solutions $p_j(t)$ $(j = 1, 2, 3)$. (b) System responses $x_j(t)$.

Chapter 5

Decoupling of Defective Systems in Forced Motion

This chapter is concerned with (i) decoupling a forced defective system (1.1) into the form (1.2) and (ii) recovering the forced response $\mathbf{x}(t)$ exactly from the decoupled system response $\mathbf{p}(t)$. We begin in Section 5.1 by developing a generalized decoupling transformation in the state space that relates the decoupled solution $\mathbf{p}(t)$ to the response $\mathbf{x}(t)$ of any system of the type (1.1). The relationship between the applied excitation $\mathbf{f}(t)$ and the modal forcing $\mathbf{g}(t)$ is also established. We show in Section 5.2 that an explicit decoupling transformation exists in the configuration space when the eigenvalues are complex. The chapter closes by illustrating in Section 5.3 how the methods presented here reduce to classical modal analysis if the coupled system (1.1) is classically damped.

5.1 A generalized state space representation

In general, the spectrum of a defective system (1.1) may contain any combination of complex conjugate and real eigenvalues that are defective or not. With the assistance of previous work on forced non-defective systems and defective systems in free motion, decoupling of forced vibration in this general setting is a fairly straightforward task. Begin by casting the equation of motion (1.1) in the first-order formulation

$$\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{A} \begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} + \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{f}(t) \ , \tag{5.1}$$

where the order $2n$ state matrix $\mathbf{A}$ is the same nonsymmetric matrix defined in Eq. (4.12). As before, we assume that the coefficient matrices $\mathbf{D}$ and $\boldsymbol{\Omega}$ of the decoupled system (1.2) in free vibration remain unchanged upon application of an excitation $\mathbf{f}(t)$. Consequently, we define a generalized coordinate transformation for forced vibration based on the free response's time-dependent decoupling transformation (4.14):

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \mathbf{S}_x \, e^{\mathbf{J}_x t} \, e^{-\mathbf{J}_p t} \, \mathbf{S}_p^{-1} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} = \mathbf{S}(t) \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} . \tag{5.2}$$

It remains to be seen how the transformed coordinates $\mathbf{p}_1(t)$ and $\mathbf{p}_2(t)$ are related to the response $\mathbf{p}(t)$ of the decoupled system (1.2) and its corresponding velocity $\dot{\mathbf{p}}(t)$. In the usual manner, we next apply the coordinate transformation (5.2) to the first-order equation (5.1) and premultiply the resulting state equation by $\mathbf{S}^{-1}(t)$, which gives

$$\begin{bmatrix} \dot{\mathbf{p}}_1(t) \\ \dot{\mathbf{p}}_2(t) \end{bmatrix} = \mathbf{B} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} + \mathbf{S}^{-1}(t) \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{f}(t) = \begin{bmatrix} \mathbf{O} & \mathbf{I} \\ -\boldsymbol{\Omega} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{p}_1(t) \\ \mathbf{p}_2(t) \end{bmatrix} + \begin{bmatrix} \mathbf{g}_1(t) \\ \mathbf{g}_2(t) \end{bmatrix} , \tag{5.3}$$

where the state matrix $\mathbf{B}$ is related to $\mathbf{A}$ by the time-varying transformation (4.16) and the transformed excitations $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ are determined from

$$\begin{bmatrix} \mathbf{g}_1(t) \\ \mathbf{g}_2(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} e^{\mathbf{J}_p t} \, e^{-\mathbf{J}_x t} \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{f}(t) . \tag{5.4}$$

The upper and lower halves of state equation (5.3) are, respectively,

$$\dot{\mathbf{p}}_1(t) - \mathbf{p}_2(t) = \mathbf{g}_1(t) , \tag{5.5}$$

$$\dot{\mathbf{p}}_2(t) + \mathbf{D}\,\mathbf{p}_2(t) + \boldsymbol{\Omega}\,\mathbf{p}_1(t) = \mathbf{g}_2(t) . \tag{5.6}$$

By eliminating $\mathbf{p}_2(t)$ from Eq. (5.6) using Eq. (5.5), we obtain

$$\ddot{\mathbf{p}}_1(t) + \mathbf{D}\,\dot{\mathbf{p}}_1(t) + \boldsymbol{\Omega}\,\mathbf{p}_1(t) = \mathbf{D}\,\mathbf{g}_1(t) + \dot{\mathbf{g}}_1(t) + \mathbf{g}_2(t) , \tag{5.7}$$

which, by comparing it to system (1.2), implies that $\mathbf{p}_1(t)$ can be identified with the decoupled response $\mathbf{p}(t)$ and the modal excitation $\mathbf{g}(t)$ is given by

$$\mathbf{g}(t) = \mathbf{D}\,\mathbf{g}_1(t) + \dot{\mathbf{g}}_1(t) + \mathbf{g}_2(t) . \tag{5.8}$$

Lastly, setting $\mathbf{p}_1(t) = \mathbf{p}(t)$ in Eq. (5.5) and combining it with Eq. (5.2) yields the generalized state space transformation that relates the decoupled solution $\mathbf{p}(t)$ to the system response $\mathbf{x}(t)$ in forced vibration:

$$\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix} \mathrm{e}^{\mathbf{J}_x t} \, \mathrm{e}^{-\mathbf{J}_p t} \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{p}(t) \\ \dot{\mathbf{p}}(t) - \mathbf{g}_1(t) \end{bmatrix} . \tag{5.9}$$

The modal initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ are connected to the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ by the state equation

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{V}_p \\ \mathbf{V}_p \mathbf{J}_p \end{bmatrix} \begin{bmatrix} \mathbf{V}_x \\ \mathbf{V}_x \mathbf{J}_x \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_1(0) \end{bmatrix} . \tag{5.10}$$

Thus, whether the eigenvalues of system (1.1) are defective or non-defective, complex or real, the system's forced response $\mathbf{x}(t)$ may be recovered via transformation (5.9) upon solution of the decoupled system (1.2) with excitation (5.8) and initial conditions (5.10). Differences between defective and non-defective systems and in how they are decoupled arise in the structures of the matrices $\mathbf{\Lambda}$, $\widehat{\mathbf{\Lambda}}$, $\mathbf{J}_x$, $\mathbf{V}_x$, $\mathbf{J}_p$, and $\mathbf{V}_p$, but the overall transformation in (5.9) is real regardless of the nature of the system's spectrum. In principle, one could extract the upper half of the state equation (5.9) to obtain a time-dependent decoupling transformation for $\mathbf{x}(t)$ itself in the n-dimensional configuration space, but there is generally no concise analytical form. A flowchart outlining the general procedure for decoupling any system of the form (1.1) and determining its response (free or forced) is illustrated in Fig. 10.

5.2 Complex eigenvalues

When the eigenvalues of system (1.1) are complex and defective, a decoupling transformation in the configuration space exists in a concise analytical form. By comparing the generalized decoupling transformation (5.9) for forced motion with its unforced counterpart (4.8), it is not difficult to see that one obtains state equation (5.9) by replacing the modal velocity $\mathbf{p}(t)$ in transformation (4.8) with $\dot{\mathbf{p}}(t) - \mathbf{g}_1(t)$. Making this substitution in Eq. (4.29) yields the configuration space transformation

$$\mathbf{x}(t) = \mathbf{T}_1 \mathrm{e}^{\mathbf{N}t} \mathbf{p}(t) + \mathbf{T}_2 \mathrm{e}^{\mathbf{N}t} \dot{\mathbf{p}}(t) - \mathbf{T}_2 \mathrm{e}^{\mathbf{N}t} \mathbf{g}_1(t) , \tag{5.11}$$

where, with the aid of the time-dependent transformation $\mathbf{S}(t)$ in Eq. (4.31) and the state equation (5.3), it is straightforward to show that

$$\begin{bmatrix} \mathbf{g}_1(t) \\ \mathbf{g}_2(t) \end{bmatrix} = \begin{bmatrix} \mathrm{e}^{-\mathbf{N}t} & \mathbf{O} \\ \mathbf{O} & \mathrm{e}^{-\mathbf{N}t} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \mathbf{\Lambda} & \overline{\mathbf{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{VJ} & \overline{\mathbf{VJ}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{O} \\ \mathbf{M}^{-1} \end{bmatrix} \mathbf{f}(t) . \tag{5.12}$$

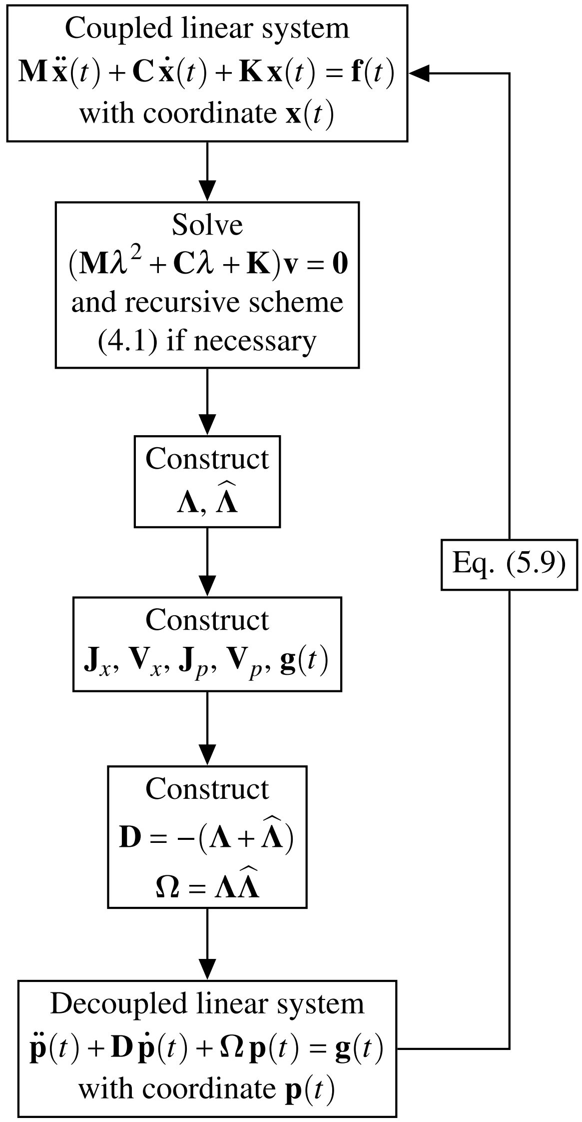

Fig. 10 Flowchart for decoupling and response calculation of any linear dynamical system in free or forced motion.

Moreover, by swapping $\mathbf{p}(0)$ in transformation (4.28) for $\dot{\mathbf{p}}(0)-\mathbf{g}_1(0)$, we find that the initial conditions $\mathbf{p}(0)$ and $\dot{\mathbf{p}}(0)$ for the forced decoupled system are related to the system initial conditions $\mathbf{x}(0)$ and $\dot{\mathbf{x}}(0)$ by

$$\begin{bmatrix} \mathbf{p}(0) \\ \dot{\mathbf{p}}(0) \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{I} \\ \boldsymbol{\Lambda} & \overline{\boldsymbol{\Lambda}} \end{bmatrix} \begin{bmatrix} \mathbf{V} & \overline{\mathbf{V}} \\ \mathbf{V}\mathbf{J} & \overline{\mathbf{V}\mathbf{J}} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(0) \\ \dot{\mathbf{x}}(0) \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{g}_1(0) \end{bmatrix} . \tag{5.13}$$

In the event that the eigenvalues of system (1.1) are complex but non-defective, the nilpotent matrix $\mathbf{N} = \mathbf{O}$, and hence the Jordan matrix $\mathbf{J} = \mathbf{\Lambda}$. In this case, assuming the eigenvectors and their complex conjugates have been normalized in accordance with Eqs. (2.4) and (2.5), respectively, it can be shown that Eq. (5.12) for the excitations $\mathbf{g}_1(t)$ and $\mathbf{g}_2(t)$ simplifies to yield

$$\mathbf{g}_1(t) = \mathbf{T}_2^{\mathrm{T}}\mathbf{f}(t) , \qquad \mathbf{g}_2(t) = (\mathbf{T}_1^{\mathrm{T}} - \mathbf{D}\,\mathbf{T}_2^{\mathrm{T}})\,\mathbf{f}(t) . \tag{5.14}$$

As a result, the configuration space decoupling transformation (5.11) and modal forcing (5.8) reduce to the corresponding equations (3.10) and (3.7), respectively, for a non-defective system, as expected.

5.3 Reduction to classical modal analysis

Finally, we demonstrate how the decoupling transformation (5.9) is a direct generalization of classical modal analysis. Should system (1.1) be classically damped, the Jordan matrices $\mathbf{J}_x$ and $\mathbf{J}_p$ are identical because classical modal analysis preserves both the eigenvalues and their multiplicities. Consequently, the time-dependent coordinate transformation $\mathbf{S}(t)$ in Eq. (5.2) reduces to the time-invariant form $\mathbf{S}(t) = \mathbf{S} = \mathbf{S}_x \mathbf{S}_p^{-1}$. If the eigenvectors are normalized according to Eqs. (2.42) and (2.43), then there is a certain pairing of the non-defective real eigenvalues for which $\mathbf{S} = \mathbf{U} \oplus \mathbf{U}$, implying that the excitations $\mathbf{g}_1(t) = \mathbf{0}$ and $\mathbf{g}_2(t) = (\mathbf{M}\,\mathbf{U})^{-1}\,\mathbf{f}(t) = \mathbf{U}^{\mathrm{T}}\,\mathbf{f}(t)$ in Eq. (5.4). It follows that Eq. (5.8) yields the modal forcing $\mathbf{g}(t) = \mathbf{U}^{\mathrm{T}}\,\mathbf{f}(t)$ indicative of classical modal analysis. Furthermore, the upper half of the state equation (5.9) reduces to the classical modal transformation $\mathbf{x}(t) = \mathbf{U}\,\mathbf{p}(t)$. If normalizations (2.42) and (2.43) are not used, multiplicative constants may appear in certain equations.

5.4 An illustrative example

Here we provide a numerical example that demonstrates the decoupling procedure for a forced defective system of the type (1.1). We focus on a system with a non-defective complex conjugate eigenvalue and a defective real eigenvalue to illustrate how one would decouple a defective system with mixed damping characteristics.

Example 7

Consider a non-classically damped, 2-degree-of-freedom system with mass matrix $\mathbf{M} = \mathbf{I}_2$ and damping matrix $\mathbf{C}$, stiffness matrix $\mathbf{K}$, and excitation $\mathbf{f}(t)$ given by

$$\mathbf{C} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} , \qquad \mathbf{K} = \begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix} , \qquad \mathbf{f}(t) = \begin{bmatrix} 0.2\cos t \\ -0.7\sin t \end{bmatrix} . \tag{5.15}$$

Let the initial conditions be $\mathbf{x}(0) = [-1,\, 1]^{\mathrm{T}}$ and $\dot{\mathbf{x}}(0) = [2,\, -1]^{\mathrm{T}}$. Solving the associated quadratic eigenvalue problem, we find that the system has a pair of non-defective complex conjugate eigenvalues, $\lambda_1 = \overline{\lambda}_2 = -1+\mathrm{i}$, and a defective real eigenvalue $\lambda_3 = -1$ with algebraic multiplicity $m_3 = 2$ and unit geometric multiplicity:

$$\mathbf{v}_1 = \overline{\mathbf{v}}_2 = \begin{bmatrix} 1 \\ \mathrm{i} \end{bmatrix} , \qquad \mathbf{v}_1^3 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} , \qquad \mathbf{v}_2^3 = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix} . \tag{5.16}$$

The eigenvectors $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_1^3$ have not been subjected to any normalization scheme. Because the system possesses both non-defective and defective eigenvalues, like in Example 4.9, the decoupling transformation involves a combination of the matrix structures given at the end of Section 4.2 for non-defective eigenvalues and those in Section 4.6.2 for defective real eigenvalues with even algebraic multiplicity. From Eqs. (2.14) and (4.46) with $n_r = 1$, we have

$$\mathbf{\Lambda} = \begin{bmatrix} -1+\mathrm{i} & 0 \\ 0 & -1 \end{bmatrix} , \qquad \widehat{\mathbf{\Lambda}} = \begin{bmatrix} -1-\mathrm{i} & 0 \\ 0 & -1 \end{bmatrix} . \tag{5.17}$$

Combining Eqs. (4.10) and (4.47) yields

$$\mathbf{J}_x = \begin{bmatrix} -1+\mathrm{i} & 0 & 0 & 0 \\ 0 & -1-\mathrm{i} & 0 & 0 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & -1 \end{bmatrix} , \qquad \mathbf{V}_x = \left[\begin{array}{c|c|c|c} 1 & 1 & 1 & 1 \\ \mathrm{i} & -\mathrm{i} & 0 & 0.5 \end{array}\right] . \tag{5.18}$$

Likewise, a direct sum of Eq. (4.11) with Eqs. (4.50) and (4.51) gives

$$\mathbf{J}_p = \mathbf{J}_x \,, \qquad \mathbf{V}_p = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} . \tag{5.19}$$

By Eqs. (2.45) and (5.17), the coefficient matrices for the decoupled system are

$$\mathbf{D} = 2\mathbf{I}_2 \,, \qquad \mathbf{\Omega} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} . \tag{5.20}$$

The first row of the decoupled system defined by Eq. (5.20) corresponds to the underdamped degree of freedom $p_1(t)$ associated with the non-defective complex eigenvalue λ_1 and its conjugate, whereas the second row represents the critically damped degree of freedom $p_1^3(t) = p_2(t)$ arising from the defective real eigenvalue λ_3. The excitation $\mathbf{g}_1(t)$ determined from Eq. (5.4) and the modal forcing $\mathbf{g}(t)$ given by Eq. (5.8) are depicted in Figs. 11(a) and 11(b), respectively. Using Eq. (5.10), the modal initial conditions are

$\mathbf{p}(0) = [0, -1]^{\mathrm{T}}$ and $\dot{\mathbf{p}}(0) = [1, 0.6]^{\mathrm{T}}$. The response $\mathbf{p}(t)$ of the decoupled system is illustrated in Fig. 12(a), and the system response $\mathbf{x}(t)$ obtained via transformation (5.9) is shown in Fig. 12(b). It can be verified that the solution by direct numerical integration of the original system and that obtained by Eq. (5.9) are indeed the same.

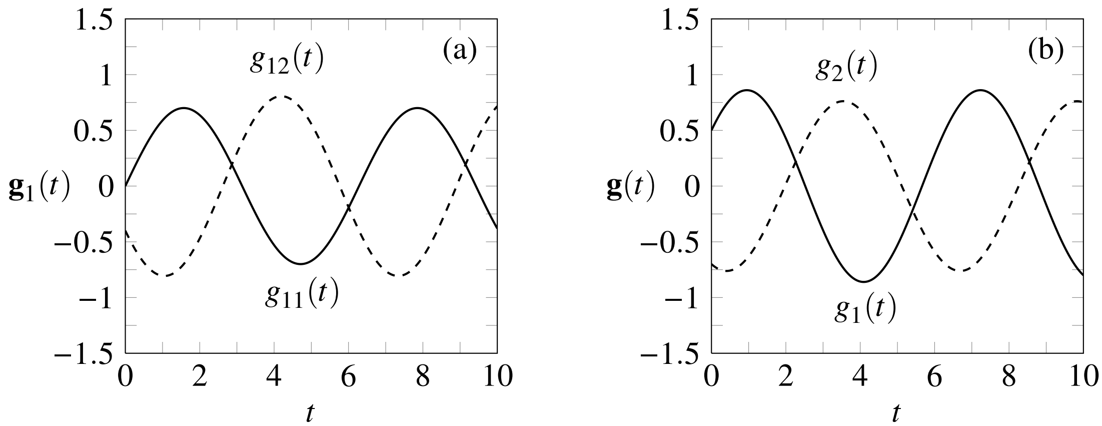

Fig. 11 Transformed excitations for Example 7. (a) Excitation components $g_{1j}(t)$ ($j = 1, 2$). (b) Modal forcing components $g_j(t)$.

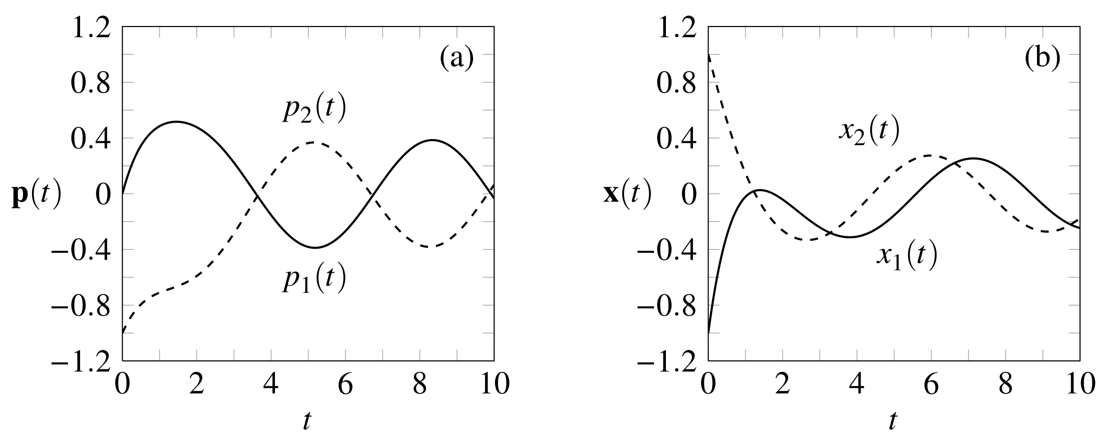

Fig. 12 Forced response of Example 7. (a) Decoupled solutions $p_j(t)$ ($j = 1, 2$). (b) System responses $x_j(t)$.

Chapter 6

Closing Comments

We have demonstrated how classical modal analysis may be extended to decouple any second-order linear dynamical system (1.1). By formulating a decoupling transformation in a general framework, a complete solution to the problem of decoupling linear dynamical systems, defective or not, in free or forced motion has been provided. Major results presented in this dissertation are summarized in the following statements:

1. Any non-defective system can be decoupled via phase synchronization, which generates a real, nonlinear, and time-dependent decoupling transformation in both the configuration and state spaces that preserves the system eigenvalues and their multiplicities. In the case of free vibration, the decoupling transformation becomes linear and time-shifting in the configuration space, while it reduces to a linear but time-invariant transformation in the state space.

2. Non-defective complex conjugate eigenvalues are associated with underdamped degrees of freedom. Pairing of non-defective real eigenvalues as real quadratic conjugates generates overdamped degrees of freedom. Non-uniqueness in pairing the real eigenvalues results in a set of admissible decoupled systems.

3. Any system with defective eigenvalues can be decoupled if one does not insist on preserving geometric multiplicities. In contrast with non-defective systems, the decoupling transformation for defective systems in free vibration is time-dependent in both the configuration and state spaces.

4. Associated with defective complex conjugate eigenvalues are underdamped degrees of freedom. Defective real eigenvalues correspond to critical damping. Depending on the multiplicities of these eigenvalues, decoupling yields some combination of critically damped degrees of freedom and overdamped degrees of freedom, the latter resulting from a necessary pairing of either defective and non-defective eigenvalues, or two distinct defective eigenvalues.

5. All parameters required for generating the decoupled system (1.2) and evaluating the general decoupling transformation (5.9) are obtained by solving the quadratic eigenvalue problem (2.1), determining generalized eigenvectors from the recursive scheme (4.1) when eigenvalues are defective, and applying an admissible eigenvalue pairing scheme as described herein.

6. Decoupling transformation (5.9) is a direct generalization of classical modal analysis.

As system decoupling plays a fundamental role in linear vibratory analysis, it is our hope that the information presented in this dissertation will facilitate numerical and qualitative analyses of linear dynamical systems, as well as inspire both theorists and practitioners to pursue further research in oft-slighted, but mathematically fascinating, defective systems.

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