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Some Computations in the TQFT Defined by a Fusion Category

by

Chan Bae

A dissertation submitted in partial satisfaction of the

requirements for the degree of

Doctor of Philosophy

in

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in the

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of the

University of California, Berkeley

Committee in charge:

Professor Constantin Teleman, Chair

Professor Vera Serganova

Professor Mark Haiman

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Chan Bae

Abstract

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Doctor of Philosophy in Mathematics

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Professor Constantin Teleman, Chair

Following work by Freed and Teleman[15], We provide a detailed construction of finite-dimensional semisimple Hopf algebras using Topological Quantum Field Theory (TQFT). Given a fusion category and an appropriate module category, we show how evaluating an oriented TQFT on appropriate colored bordisms naturally yields a (weak) Hopf algebra and its associated operations. When the module category is equivalent to the category of vector spaces, we recover a genuine Hopf algebra.

To Nayeong

Perhaps you will enjoy reading this

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Reaching further into the past, I would also like to thank my undergraduate advisor Marc Lackenby for being far more generous with his time than he had to be. I believe he is the reason I pursued topology in the first place. I would also like to thank the late Sir Michael Atiyah, who once read an essay I wrote as an undergrad and told me I could become a successful mathematician. I didn't know then just how great his mathematical influence would be on my work.

I would like to thank my parents for encouraging me to follow my dreams and making sure I was always in a position to do just that.

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Introduction

Motivated by advances in the understanding the global aspects of quantum theory ¹, in 1988, Michael Atiyah proposed[1, 2] a set of axioms for Topological Quantum Field Theory (TQFT) following Graeme Segal’s axioms[26] for conformal field theory.

In the Atiyah-Segal framework, an *n-dimensional TQFT* is defined to be a symmetric monoidal functor

$$Z : (\mathbf{Bord}_n^{or}, \sqcup) \rightarrow (\mathbf{Vect}, \otimes)$$

from the category $\mathbf{Bord}_n^{or}$ of *n*-dimensional oriented bordisms with disjoint union, to the category of vector spaces with tensor product.

In 1995, Baez and Dolan [3] extended this definition to define *fully extended* TQFTs, allowing bordisms to be cut up into pieces of arbitrary codimension, and formulated the *cobordism hypothesis*, roughly stating that any fully extended TQFT could be specified by a single object, the image of the point, in the target category.

In [18], Lurie proved the cobordism hypothesis and further extended this framework to allow for *bordisms with singularities*, adding codimension-*k* “domain walls” or “defects”, between bordisms.

Although initially motivated by physics, the machinery of fully extended TQFT is a purely mathematical one. My PhD journey was guided by essentially a single natural question: “What insight does this machinery provide into everyday linear algebra?”. I phrase the question naïvely, as my perspective was, and in many respects continues to be, this naïve. After all, factorizing maps into compositions of simpler maps is an indispensable technique in mathematics. Why shouldn’t we be able to get more out of cutting maps in the new and exciting orthogonal directions?

The structure of this dissertation is as follows. In Chapter 2 we will review the theory of tensor categories necessary for the main result. I also include a new formula (Prop 1.4.0.2) for the separation map of the canonical algebra of a not-necessarily spherical fusion category which may be of some independent interest.

¹In particular Witten’s paper [30].

The main contribution of this dissertation is Chapter 3, a detailed construction of finite-dimensional semisimple weak Hopf algebras using fully extended Topological Quantum Field Theory with defects. This elaborates on the work of Freed and Teleman in [15]. An essentially similar construction was also noted in [25].

Finally, I have found the use of large language models extremely convenient for generating tikz code to draw diagrams. Some diagrams in this dissertation were generated by large language models. Despite significant effort, I have not been able to cajole any model into contributing to the mathematics in this dissertation.

Chapter 1

Tensor Categories

The target 3-category for our TQFT will be the 3-category $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$ of tensor categories, bimodule categories, bimodule functors, and bimodule natural transformations.

We quickly review the theory of tensor categories and fix conventions on the way. An extensive reference on tensor categories is [12]. A less extensive reference is [20]. The ground field will always be $\mathbb{C}$ and all categories will be abelian unless otherwise specified.

Definition 1.0.1. A linear category is a category enriched over the category of vector spaces.

The collection of linear categories form a symmetric monoidal bicategory $\text{Cat}_{\mathbb{C}}$ of $\mathbb{C}$ -linear categories and right exact $\mathbb{C}$ -linear functors.

Definition 1.0.2. A tensor category is a linear monoidal category.

In other words, a tensor category is an “algebra object” in the category of linear categories.¹

Remark. Note that “tensor category” means many things in the literature. We do not assume that a tensor category is rigid (has internal duals), or that the tensor unit is simple. We do assume that tensor functors are right exact.

Example 1. The category **Vect** of finite-dimensional vector spaces with tensor product is the prototypical example of a tensor category.

¹The notation $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$ comes from this point of view. The Morita 3-category of tensor categories is the first in an infinite sequence $\text{Alg}_d(\text{Cat}_{\mathbb{C}})$ of “higher Morita categories” with objects E_d -algebras in $\text{Cat}_{\mathbb{C}}$, 1-morphisms bimodules between E_d -algebras that are themselves E_{d-1} -algebras, and so on. See [17].

Example 2. Let G be a finite group. The category $\mathbf{Vect}[G]$ of finite-dimensional G -graded vector spaces with tensor product defined by $\mathbb{C}_g \otimes \mathbb{C}_h := \mathbb{C}_{gh}$ is a tensor category.

Dualizability

Definition 1.0.3. Let x be an object of a monoidal category $\mathbf{C}$.

An object x^* is a *left dual* of x if there exist maps $\varepsilon : x^* \otimes x \rightarrow 1$ and $\eta : 1 \rightarrow x \otimes x^*$ such that the composite maps

$$\begin{aligned} x &\xrightarrow{\eta \otimes \text{id}_x} x \otimes x^* \otimes x \xrightarrow{\text{id}_x \otimes \varepsilon} x \\ x^* &\xrightarrow{\text{id}_{x^*} \otimes \eta} x^* \otimes x \otimes x^* \xrightarrow{\varepsilon \otimes \text{id}_{x^*}} x^* \end{aligned}$$

are the identity morphisms of x and x^* , respectively.

Similarly, an object *x is a *right dual* of x if there exist maps $\varepsilon : x \otimes {}^*x \rightarrow 1$ and $\eta : 1 \rightarrow {}^*x \otimes x$ such that the composite maps

$$\begin{aligned} x &\xrightarrow{\text{id}_x \otimes \eta} x \otimes {}^*x \otimes x \xrightarrow{\varepsilon \otimes \text{id}_x} x \\ {}^*x &\xrightarrow{\eta \otimes \text{id}_{{}^*x}} {}^*x \otimes x \otimes {}^*x \xrightarrow{\text{id}_{{}^*x} \otimes \varepsilon} {}^*x \end{aligned}$$

are the identity morphisms of x and *x , respectively.

Definition 1.0.4. We say that a monoidal category $\mathbf{C}$ is *rigid* if every object admits left and right duals.

Note that the literature is inconsistent on whether x^* or *x is the left dual. We use the convention used in [12].

Example 3. Let G be a finite group. The tensor category $\text{Rep}(G)$ of finite-dimensional representations of G admits duals. For any representation V of G , the dual representation V^* is both the left and right dual.

Example 4. Consider the category $\text{End}(\mathbf{C})$ of endofunctors of a category $\mathbf{C}$. Composition of functors provides $\text{End}(\mathbf{C})$ with the structure of a monoidal category and left/right duals correspond precisely to left/right adjoints of functors.

Left and right duals are unique up to unique isomorphism. In particular, the unit/counit maps are only defined up to a single choice of scalar. Choosing a unit fixes a counit and vice versa.

Definition 1.0.5. Let x, y be objects of a monoidal category $\mathbf{C}$ with left duals x^*, y^* . For a morphism $f : x \rightarrow y$, we have the *dual morphism* $f^* : y^* \rightarrow x^*$ defined by $f^* := (\epsilon_y \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes f \otimes \text{id}_{x^*}) \circ (\text{id}_{y^*} \otimes \eta_x)$. Similarly if x, y have right duals ${}^*x, {}^*y$ we define ${}^*f := (\text{id}_{{}^*x} \otimes \epsilon_y) \circ (\text{id}_{{}^*x} \otimes f \otimes \text{id}_{{}^*y}) \circ (\eta_x \otimes \text{id}_{{}^*y})$.

Definition 1.0.6. Let $\mathbf{C}$ be a monoidal category. The *opposite category* $\mathbf{C}^{op}$ is the category with the same objects where every morphism has its source and target swapped. The *monoidally opposite category* $\mathbf{C}^{mp}$ has the same underlying category as $\mathbf{C}$ but the monoidal product is defined by $x \otimes_{\mathbf{T}^{mp}} y := y \otimes_{\mathbf{T}} x$. The *monoidally opposite opposite category* $\mathbf{C}^{op,mp}$ has both the morphisms and the monoidal product reversed.

Proposition 1.0.0.1. *Taking duals is a monoidal functor $*$: $\mathbf{C} \rightarrow \mathbf{C}^{op,mp}$.*

To be precise, $*$ is only well-defined up to unique isomorphism, and a choice of unit/counit maps is required to specify a functor on the nose.

The dualization functor $x \mapsto x^*$ provides an equivalence $\mathbf{C}^{op} \rightarrow \mathbf{C}^{mp}$ when it exists.

The Deligne Product of Tensor Categories

Definition 1.0.7. Let $\mathbf{C}, \mathbf{D}$ be linear categories. The *Deligne tensor product* $\mathbf{C} \boxtimes \mathbf{D}$ is a linear category that is defined by the following universal property:

There exists a bifunctor $\boxtimes : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{C} \boxtimes \mathbf{D}$ such that for any linear category $\mathbf{A}$ and any right exact bifunctor $F : \mathbf{C} \times \mathbf{D} \rightarrow \mathbf{A}$, there exists a unique right exact functor $\bar{F} : \mathbf{C} \boxtimes \mathbf{D} \rightarrow \mathbf{A}$ such that $F = \bar{F} \circ \boxtimes$.

Theorem 1.0.1 ([12] Prop. 1.11.2). *The Deligne tensor product $\mathbf{C} \boxtimes \mathbf{D}$ exists.*

Theorem 1.0.2 ([12], Prop. 4.6.1). *Let $\mathbf{T}_1, \mathbf{T}_2$ be tensor categories. Then the Deligne tensor product $\mathbf{T}_1 \boxtimes \mathbf{T}_2$ is a tensor category.*

1.1 Fusion Categories

Definition 1.1.1. We say that a linear category $\mathbf{T}$ is *finite* if

- $\mathbf{T}$ has finite dimensional Hom-spaces.
- Every object has finite length (i.e. has a Jordan-Hölder series of finite length).
- $\mathbf{T}$ has enough projectives.

- $\mathbf{T}$ has finitely many isomorphism classes of simple objects.

The definition is motivated by the following well-known theorem.

Theorem 1.1.1 ([11] Thm 9.6.4). *Any finite k -linear category is equivalent to the category of modules over a finite dimensional k -algebra.*

Definition 1.1.2. We say that a linear category $\mathbf{T}$ is *semisimple* if for any $x \in \mathbf{T}$, the ring $\text{End}(x)$ of endomorphisms is semisimple.

Definition 1.1.3. A fusion category is a finite semisimple rigid tensor category with simple unit.

Example 5. Let H be a finite-dimensional semisimple Hopf algebra. Then $\text{Rep}(H)$ is a fusion category.

In fact, every fusion category arises as the representation category of a *weak* Hopf algebra (Def 1.5.2). We will return to this point.

Pivotal and Spherical Fusion Categories

In an arbitrary monoidal category, left and right duals need not coincide and can be quite different (for example, consider categories of endofunctors!). However, things are simpler in semisimple tensor categories.

Proposition 1.1.1.1 ([12] Prop 4.8.1). *Let $\mathbf{T}$ be a semisimple rigid tensor category. For every object x , we have ${}^*x \cong x^*$.*

Proof. It suffices to take x simple. Then x^* and *x are simple since $\text{End}(x^*) \cong \text{End}(x) \cong \text{End}({}^*x)$ via duality. But $\text{Hom}(x^*, {}^*x) \cong \text{Hom}(1, x \otimes {}^*x) \neq 0$ so ${}^*x \cong x^*$ by Schur's lemma. $\square$

Definition 1.1.4. Let $\mathbf{T}$ be a rigid monoidal category. A *pivotal structure* on $\mathbf{T}$ is a monoidal natural isomorphism $a : \text{id}_{\mathbf{T}} \rightarrow **$.

Definition 1.1.5. Let $\mathbf{T}$ be a rigid monoidal category and x an object of $\mathbf{T}$. The (left) *quantum trace* of a map $a_x : x \rightarrow x^{**}$ is the endomorphism of the unit defined to be the composition

$$1 \xrightarrow{\eta_x} x \otimes x^* \xrightarrow{a_x \otimes \text{id}_{x^*}} x^{**} \otimes x^* \xrightarrow{\epsilon_{x^*}} 1$$

If $\mathbf{T}$ has simple unit, the quantum trace is a scalar.

Let $\mathbf{T}$ be the representation category of a finite-dimensional semisimple Hopf algebra. It is a classical theorem of Radford[24] that the quadruple dual functor is naturally tensor isomorphic to the identity. The theorem holds more generally for all fusion categories.

Theorem 1.1.2 (Theorem 2.6 of [10]). *Let $\mathbf{T}$ be a fusion category. There is a canonical tensor isomorphism $\text{id}_{\mathbf{T}} \rightarrow ** **$.*

For a pivotal fusion category, the pivotal structure also provides a tensor isomorphism $\text{id}_{\mathbf{T}} \rightarrow ** **$.

Definition 1.1.6 (Def 3.5.2 of [7]). Let $\mathbf{T}$ be a fusion category with pivotal structure $a : \text{id}_{\mathbf{T}} \rightarrow **$ and simple unit. We say that $\mathbf{T}$ is *spherical* if $a^2 : \text{id}_{\mathbf{T}} \rightarrow ** **$ is equal to the Radford natural isomorphism $\text{id}_{\mathbf{T}} \rightarrow ** **$.

Notions of Dimension

We recall some notions of dimension in fusion categories and their properties.

By dimension, we mean a choice of scalar d_x for each simple object x_i such that $d_{x_i} d_{x_j} = \sum_k N_{ij}^k d_{x_k}$ where $N_{ij}^k := \dim \text{Hom}(x_k, x_i \otimes x_j)$. That is, we would like dimension to be a ring homomorphism from the Grothendieck ring $\text{Gr}(\mathbf{T})$ to the base field $\mathbb{C}$.

In the presence of a pivotal structure, we have the notion of quantum dimension.

Definition 1.1.7. Let $\mathbf{T}$ be a rigid monoidal category equipped with a pivotal structure a . Let x be an object of $\mathbf{T}$. The *quantum dimension* d_x of x is the quantum trace of a_x .

Proposition 1.1.2.1 ([7] Prop 3.5.4). *Let $\mathbf{T}$ be a pivotal fusion category. Then $\mathbf{T}$ is spherical if and only if $d_x = d_{x^*}$ for all x .²*

In particular, morphisms in a spherical category behave as if they can be drawn and moved around on a sphere, motivating the name.

Without a pivotal structure, *a priori* we cannot define quantum dimensions. The problem is as follows, we have the two pairs of unit/counit maps

$$\begin{aligned} \eta : 1 &\rightarrow x \otimes x^* & \epsilon : x^* \otimes x &\rightarrow 1 \\ n : 1 &\rightarrow {}^*x \otimes x & e : x \otimes {}^*x &\rightarrow 1 \end{aligned}$$

²Note that this is often taken to be the *definition* of sphericity, especially in older literature.

and we would like to define $d_x := \epsilon \circ n$ or $e \circ \eta$. This requires a choice of isomorphism ${}^*x \cong x^*$, which is equivalent to a choice of unit-counit maps $(\eta, \epsilon), (n, e)$. Unfortunately, there is a degree of freedom here, and only the product $d_x^2 := (\epsilon \circ n)(e \circ \eta)$ is well-defined.

Definition 1.1.8. Let $\mathbf{T}$ be a fusion category and x a simple object of $\mathbf{T}$. Fix an object $\bar{x}$ dual to x . The *squared norm* d_x^2 of x is defined to be the scalar $(\epsilon \circ n)(e \circ \eta)$ where (η, ϵ) exhibits $\bar{x}$ as a left dual of x , and (n, e) exhibits $\bar{x}$ as a right dual of x .

The *global dimension* $\dim(\mathbf{T})$ of $\mathbf{T}$ is the sum $\sum_k d_{x_k}^2$ of squared norms of simple objects.

Theorem 1.1.3 (Theorem 2.3 of [10]). *For every simple object x of a fusion category, d_x^2 is real and strictly positive.*

Whenever $\mathbf{T}$ admits a pivotal structure, we have $d_{x_i}^2 = d_{x_i} d_{x_i^*}$, and when $\mathbf{T}$ admits a spherical structure, we have $d_{x_i}^2 = (d_{x_i})^2$. However, in the absence of a pivotal structure, choosing dimensions d_{x_i} to be multiplicative is a difficult cohomological problem.

For every fusion category we may define the Frobenius-Perron dimensions.

Definition 1.1.9. Let $\mathbf{T}$ be a fusion category and x_i the simple objects of $\mathbf{T}$. The *Frobenius-Perron dimensions* $\text{FPdim}(x_i)$ of x_i are the unique positive real numbers satisfying

$$\text{FPdim}(x_i) \text{FPdim}(x_j) = \sum_k N_{ij}^k \text{FPdim}(x_k)$$

for all i, j where $N_{ij}^k := \dim \text{Hom}(x_k, x_i \otimes x_j)$. We extend linearly to define the Frobenius-Perron dimensions for all objects of $\mathbf{T}$.

We also define the Frobenius-Perron dimension $\text{FPdim}(\mathbf{T})$ of the fusion category $\mathbf{T}$ to be the sum $\sum_k \text{FPdim}(x_k)^2$ over all (isomorphism classes of) simple objects x_k of $\mathbf{T}$.

Frobenius-Perron dimensions need not be integers. For example, the noninvertible simple object of a Tambara-Yamagami category [29] over a finite abelian group G has dimension $\sqrt{|G|}$. They are, however, algebraic integers. (Prop. 3.3.4 of [12])

1.2 The Pivotalization of a Fusion Category

This section reformulates some aspects of [4]. Let $\mathbf{T}$ be a fusion category.

Definition 1.2.1. The *pivotalization* $\tilde{\mathbf{T}}$ of $\mathbf{T}$ is the fusion category with objects (x, f) where $f : x \rightarrow x^{**}$ are isomorphisms such that $f^{**} \circ f = \text{id}_x$ (we are identifying the canonically isomorphic x and x^{****}).

Note that for simples x_i , there exist precisely two such isomorphisms $x_i \rightarrow x_i^{**}$. They are related by multiplication by the scalar -1 .

Morphisms $(x, f) \rightarrow (y, g)$ are maps $\phi : x \rightarrow y$ of $\mathbf{T}$ which intertwine f and g . That is, $\phi^{**} \circ f = g^{**} \circ \phi$.

Since the pivotalization $\tilde{\mathbf{T}}$ has a natural pivotal structure, we have no problem defining quantum traces and dimensions.

Definition 1.2.2. Take $(x, f) \in \tilde{\mathbf{T}}$ and $g : x \rightarrow x$. The (left) trace $\text{Tr}(g)$ of g is defined to be the following composition:

$$1 \xrightarrow{\text{coev}_i} x \otimes x^* \xrightarrow{(f \circ g) \otimes \text{id}} x_i^{**} \otimes x^* \xrightarrow{\text{ev}} 1$$

The *dimension* $d(x, f)$ is simply $\text{Tr}(\text{id}_x)$.

In fact, $\tilde{\mathbf{T}}$ is spherical with the above pivotal structure.

In particular,

$$d^2(x, f) = d(x, f)d(*x, *f) = d(x, f)^2$$

Now consider the two linear maps $T_{k,\pm}^{ij} : \Gamma_k^{ij} \rightarrow \Gamma_{k^{**}}^{ij}$ defined by

$$T_{k,\pm}^{ij}(\phi) := (\phi^{**} \circ (f_i \otimes f_j)) \pm (f_k^{**} \circ \phi)$$

Identifying $(-)^{****} \cong \text{id}_{\mathbf{T}}$, we see that

$$T_{k^{**},+}^{ij} \circ T_{k,-}^{ij} = T_{k^{**},-}^{ij} \circ T_{k,+}^{ij} = 0$$

and therefore we have the decomposition

$$\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k) \cong \ker T_{k,+}^{ij} \oplus \ker T_{k,-}^{ij}$$

Remark. There is a natural action of $s\mathbf{V}$ on $\tilde{\mathbf{T}}$ swapping (x, f) with $(x, -f)$. That is, $\tilde{\mathbf{T}}$ can be regarded as a *fusion supercategory*. Then the previous decomposition exhibits $\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k)$ as the internal (to $s\mathbf{V}$, see Def. 1.3.3) hom-space $\underline{\text{Hom}}_{\tilde{\mathbf{T}}}(x_i \otimes x_j, x_k) \cong \text{Hom}_{\tilde{\mathbf{T}}}(x_i \otimes x_j, x_k) \oplus \text{Hom}_{\tilde{\mathbf{T}}}(x_i \otimes x_j, \Pi \otimes x_k)$ (where Π is the odd line).

Definition 1.2.3 (c.f. Def 3.14 of [4]). A *pivotal basis* of $\text{Hom}_{\mathbf{T}}(x_i, x_j \otimes x_k)$ is a basis $\{h_{jk}^{i\alpha}\}$ of $\text{Hom}_{\mathbf{T}}(x_i, x_j \otimes x_k)$ such that each $h_{jk}^{i\alpha}$ is in $\ker T_{k,+}^{ij}$ or $\ker T_{k,-}^{ij}$.

The *pivotal symbols* are the numbers $\epsilon_{jk,\alpha}^i$ such that $\epsilon_{jk,\alpha}^i = 1$ if $h_{jk}^{i\alpha} \in \ker T_{k,+}^{ij}$ and $\epsilon_{jk,\alpha}^i = -1$ if $h_{jk}^{i\alpha} \in \ker T_{k,-}^{ij}$.

Remark. The relationship between Definition 1.2.3 and Def 3.14 of [4] is as follows:

We may choose f_i so that $d(x, f_i) \geq 0$ always. This corresponds to the *root choice* of [4]. Making such a choice of f_i , our notions of pivotal bases and pivotal symbols coincide with those of [4].

Concretely, the pivotal symbols $\epsilon_{jk,\alpha}^i$ keep track of whether the fiber of $h_{jk}^{i\alpha} \in \text{Hom}_{\mathbf{T}}(x_i, x_j x_k)$ in $\tilde{\mathbf{T}}$ lies in the “orientation-preserving”

$$\text{Hom}_{\tilde{\mathbf{T}}}((x_i, f_i), (x_j x_k, f_j f_k)) \cup \text{Hom}_{\tilde{\mathbf{T}}}((x_i, -f_i), (x_j x_k, -f_j f_k))$$

or the “orientation-reversing”

$$\text{Hom}_{\tilde{\mathbf{T}}}((x_i, f_i), (x_j x_k, -f_j f_k)) \cup \text{Hom}_{\tilde{\mathbf{T}}}((x_i, -f_i), (x_j x_k, f_j f_k)).$$

The following is immediate.

Lemma 1.2.0.1 (c.f. Prop 3.17 of [4]). *For simples x_i, x_j of $\mathbf{T}$, we have*

$$d(x_i \otimes x_j, f_i f_j) = \sum_k d(x_k, f_k) \dim \underline{\text{Hom}}_{\tilde{\mathbf{T}}}(x_i \otimes x_j, x_k)$$

Proof.

$$\begin{aligned} d(x_i \otimes x_j, f_i f_j) &= \sum_k d(x_k, f_k) (\dim \text{Hom}((x_i \otimes x_j, f_i f_j), (x_k, f_k)) \\ &\quad + d(x_k, -f_k) \dim \text{Hom}((x_i \otimes x_j, f_i f_j), (x_k, -f_k))) \\ &= \sum_k d(x_k, f_k) (\dim \text{Hom}((x_i \otimes x_j, f_i f_j), (x_k, f_k)) - (\dim \text{Hom}((x_i \otimes x_j, f_i f_j), (x_k, -f_k)))) \end{aligned}$$

□

1.3 Bimodule Categories

As tensor categories categorify rings, (bi)module categories categorify modules.

Definition 1.3.1. Let $\mathbf{T}_1, \mathbf{T}_2$ be tensor categories, a $(\mathbf{T}_1, \mathbf{T}_2)$ -bimodule category is a linear category $\mathbf{M}$ equipped with a tensor functor $\rho : \mathbf{T} \rightarrow \text{End}(\mathbf{M}) \boxtimes \text{End}(\mathbf{M})^{mp}$.

A left $\mathbf{T}$ -module category is a $(\mathbf{T}, \mathbf{Vect})$ -bimodule category and a right $\mathbf{T}$ -module category is a $(\mathbf{Vect}, \mathbf{T})$ -bimodule category.

Example 6. Every tensor category $\mathbf{T}$ is a bimodule category over itself by left and right multiplication. We call this the *regular representation* of $\mathbf{T}$.

Example 7. Let $\mathbf{T}_1, \mathbf{T}_2$ be tensor categories and let $F : \mathbf{T}_1 \rightarrow \mathbf{T}_2$ be a tensor functor. Then $\mathbf{T}_2$ is a left $\mathbf{T}_1$ -module category with $\rho : \mathbf{T}_1 \rightarrow \text{End}(\mathbf{T}_2)$ given by $\rho(x) = x \otimes -$.

If $\mathbf{T}_2 = \mathbf{Vect}$, then we say that F is a *fiber functor*.

Definition 1.3.2. Let $\mathbf{T}$ be a fusion category and $\mathbf{M}$ a left $\mathbf{T}$ -module. Then we define the *dual category* $\mathbf{T}_{\mathbf{M}}^*$ to be the category $\text{End}_{\mathbf{T}}(\mathbf{M}, \mathbf{M})$ of $\mathbf{T}$ -module endofunctors.

Theorem 1.3.1. The dual category $\mathbf{T}_{\mathbf{M}}^*$ is a fusion category.

Internal Homs

We recall the theory of module categories and their internal hom-objects.

Definition 1.3.3. [Internal Homs] Let $\mathbf{T}$ be a finite rigid tensor category and $\mathbf{M}$ a left $\mathbf{T}$ -module category. For objects $m_1, m_2 \in \mathbf{M}$, the functor $\text{Hom}_{\mathbf{M}}(- \otimes m_1, m_2) : \mathbf{T} \rightarrow \mathbf{Vect}$ is left exact and therefore representable.

We define $\underline{\text{Hom}}_{\mathbf{T}}(m_1, m_2) \in \mathbf{T}$ to be the object such that

$$\text{Hom}_{\mathbf{M}}(- \otimes m_1, m_2) \cong \text{Hom}_{\mathbf{T}}(-, \underline{\text{Hom}}_{\mathbf{T}}(m_1, m_2))$$

We will often omit subscripts if they are obvious in context.

Example 8. Let G be a finite group. Then the fusion category $\text{Rep}[G]$ of finite dimensional representations of G acts on $\mathbf{Vect}$ by tensoring with the underlying vector space. The internal hom-object $\underline{\text{Hom}}(\mathbb{C}, \mathbb{C})$ is

$$\bigoplus_i V_i \otimes \text{Hom}_{\mathbf{Vect}}(V_i, \mathbb{C}) \cong \mathbb{C}[G]^*$$

where the V_i range over irreducible representations of G .

The Deligne Product of Bimodule Categories

Let $\mathbf{T}_1, \mathbf{T}_2, \mathbf{T}_3$ be finite tensor categories and let $\mathbf{M}_1$ be a $(\mathbf{T}_1, \mathbf{T}_2)$ -bimodule category, $\mathbf{M}_2$ a $(\mathbf{T}_2, \mathbf{T}_3)$ -bimodule category.

We would like to define a *relative Deligne tensor product* $\mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2$, analogous to the relative tensor product of bimodules.

Definition 1.3.4. A bifunctor $F : \mathbf{M}_1 \times \mathbf{M}_2 \rightarrow \mathbf{A}$ is called $\mathbf{T}_2$ -balanced if it is equipped with a coherent family of natural (in $m \in \mathbf{M}_1, n \in \mathbf{M}_2, x \in \mathbf{T}_2$) isomorphisms

$$F(m \otimes x, n) \cong F(m, x \otimes n)$$

Definition 1.3.5. The *relative Deligne tensor product* $\mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2$ is the $(\mathbf{T}_1, \mathbf{T}_3)$ bimodule category defined by the following universal property:

There exists a bifunctor $\boxtimes_{\mathbf{T}_2} : \mathbf{M}_1 \times \mathbf{M}_2 \rightarrow \mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2$ such that for any linear category $\mathbf{A}$ and any right exact $\mathbf{T}_2$ -balanced bifunctor $F : \mathbf{M}_1 \times \mathbf{M}_2 \rightarrow \mathbf{A}$, there exists a unique right exact functor $\bar{F} : \mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2 \rightarrow \mathbf{A}$ such that $F = \bar{F} \circ \boxtimes_{\mathbf{T}_2}$.

Theorem 1.3.2 ([11] Prop 3.5, [7] Corollary 2.4.11). *The relative Deligne tensor product $\mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2$ exists.*

One concrete construction is $\mathbf{M}_1 \boxtimes_{\mathbf{T}_2} \mathbf{M}_2 := \text{Hom}_{\mathbf{T}_2}(\text{Hom}_{\mathbf{T}_2}(\mathbf{M}_1, \mathbf{T}_2), \mathbf{M}_2)$.

Example 9. For any fusion category $\mathbf{T}$ and left $\mathbf{T}$ -module category $\mathbf{M}$, we have $\mathbf{T} \boxtimes_{\mathbf{T}} \mathbf{M} \cong \mathbf{M}$. Similarly, if $\mathbf{M}$ is a right $\mathbf{T}$ -module category, $\mathbf{M} \boxtimes_{\mathbf{T}} \mathbf{T} \cong \mathbf{M}$.

Note that the relative Deligne tensor product of bimodule categories is not a quotient of the Deligne tensor product as one might naïvely expect. The balancing isomorphisms $F(m \otimes x, n) \cong F(m, x \otimes n)$, which are constraints in the setting of rings and bimodules, become data in the categorified setting. In fact, they may essentially be the *only* data in some cases, as the following example shows.

Example 10. Let G be a finite group. Then the fusion category $\mathbf{Vect}[G]$ of G -graded vector spaces acts on $\mathbf{Vect}$ from the left and right by tensoring with the underlying vector space. The relative Deligne tensor product $\mathbf{Vect} \boxtimes_{\mathbf{Vect}[G]} \mathbf{Vect}$ is equivalent to the category $\text{Rep}(G)$ of finite-dimensional representations of G .

The following example will be important in understanding extended TQFTs defined by a fusion category and defects defined by module categories.

Example 11. Let $\mathbf{T}$ be a fusion category and $\mathbf{M}$ a left $\mathbf{T}$ -module. Then the dual category $\mathbf{T}_{\mathbf{M}}^* = \text{End}_{\mathbf{T}}(\mathbf{M}, \mathbf{M})$ decomposes into $\mathbf{T}_{\mathbf{M}}^* \cong \text{Hom}_{\mathbf{T}}(\mathbf{M}, \mathbf{T}) \boxtimes_{\mathbf{T}} \mathbf{M}$, where $\mathbf{T}$ acts on $\text{Hom}_{\mathbf{T}}(\mathbf{M}, \mathbf{T})$ by postcomposition by right multiplication.³

Algebra Objects in Fusion Categories

Definition 1.3.6. Let $\mathbf{T}$ be a fusion category. We say that an object a of $\mathbf{T}$ is an *algebra object* if it is equipped with an associative *multiplication* $\mu : a \otimes a \rightarrow a$ with *unit map* $u : 1 \rightarrow a$.

Generalizing modules over associative algebras, we may define module objects internal to fusion categories.

Definition 1.3.7. Let $\mathbf{T}$ be a fusion category and a an algebra object of $\mathbf{T}$. An object m of $\mathbf{T}$ equipped eg a map $\rho : m \otimes a \rightarrow m$ is said to be a (*right*) *module* over a if the following diagrams commute.

$$\begin{array}{ccc}
 (m \otimes a) \otimes a & \xrightarrow{\alpha_{m,a,a}} & m \otimes (a \otimes a) \\
 \downarrow \rho \otimes \text{id}_a & & \downarrow \text{id}_m \otimes \mu \\
 m \otimes a & & m \otimes a \\
 \searrow \rho & & \swarrow \rho \\
 & m & \\
 \\
 m \otimes 1 & \xrightarrow{\text{id}_m \otimes u} & m \otimes a \\
 \searrow & & \downarrow \rho \\
 & & m
 \end{array}$$

Related concepts such as left modules, bimodules, morphisms of modules, and the like can all be defined in a straightforward manner.

Definition 1.3.8. Let $\mathbf{T}$ be a fusion category and a an algebra object of $\mathbf{T}$. The right a -module objects of $\mathbf{T}$ with a -module morphisms form a category $\text{Mod}_a(\mathbf{T})$.

³One way to see this is to appeal to Thm 1.3.3 and write $\mathbf{M} \cong \text{Mod}_a(\mathbf{T})$ for some algebra object a of $\mathbf{T}$. Then $\mathbf{T}_{\mathbf{M}}^*$ is equivalent to $\text{Bimod}_a(\mathbf{T})^{mp}$ by Remark 7.12.5 of [12], which in turn is equivalent to ${}_a\text{Mod}(\mathbf{T}) \boxtimes_{\mathbf{T}} \text{Mod}_a(\mathbf{T})$ by Thm 3.3 (2) of [8].

$\text{Mod}_a(\mathbf{T})$ is an abelian category with a natural $\mathbf{T}$ -module category structure. That is, for $x \in \mathbf{T}$ and $m \in \text{Mod}_a(\mathbf{T})$ we have the following a -module structure on $x \otimes m$:

$$(x \otimes m) \otimes a \xrightarrow{\alpha_{x,m,a}} x \otimes (m \otimes a) \xrightarrow{\rho} x \otimes m$$

In fact, it is a fundamental theorem of Ostrik[23] that every $\mathbf{T}$ -module category arises in this way.

Theorem 1.3.3. *Let $\mathbf{T}$ be a fusion category and $\mathbf{M}$ a left $\mathbf{T}$ -module. Then there exists an algebra object a of $\mathbf{T}$ such that $\mathbf{M} \cong \text{Mod}_a(\mathbf{T})$ as $\mathbf{T}$ -module categories.*

If m is a generating object of $\mathbf{M}$ (that is, for any $n \in \mathbf{M}$ there exists $x \in \mathbf{T}$ such that $n \hookrightarrow x \otimes m$), then $a = \underline{\text{Hom}}(m, m)$ is such an algebra object.

Example 12. Consider the regular $\mathbf{T}$ -bimodule category $\mathbf{T}$. We may equivalently consider it a left $\mathbf{T} \boxtimes \mathbf{T}^{mp}$ -module category.

In this case, the internal hom $\underline{\text{Hom}}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}}(1, 1)$ is given by $\bigoplus_i x_i \boxtimes {}^*x_i$, and we have ${}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}} \mathbf{T} \cong \text{Mod}_{\underline{\text{Hom}}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}}(1, 1)}(\mathbf{T} \boxtimes \mathbf{T}^{mp})$.

1.4 A formula for the separation map of the canonical algebra

In this section we present the following explicit formulae for the multiplication and separation of the canonical algebra of a fusion category. In particular, the formula for the separation map is a new generalization of a formula of Müger to the non-spherical case.

We say that an algebra object a is *separable* if there exists a morphism $s : a \rightarrow a \otimes a$ of a -bimodules such that $\mu \circ s = \text{id}_a$. Over $\mathbb{C}$ (or any perfect field), $\text{Mod}_a(\mathbf{T})$ is semisimple if and only if a is separable.

Let $\mathbf{T}$ be a fusion category. It is a bimodule over itself, that is, we have $\mathbf{F} = \mathbf{T} \boxtimes \mathbf{T}^{mp}$ and ${}_{\mathbf{F}} \mathbf{T} \cong \text{Mod}_a(\mathbf{F})$. In this case, we may explicitly write the *canonical algebra* $a = \bigoplus_i x_i \boxtimes {}^*x_i$.

First, observe that

$$\text{Hom}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}}(a \otimes a, a) \cong \bigoplus_{i,j,k} \text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k) \otimes \text{Hom}_{\mathbf{T}}(x_k, x_i \otimes x_j)$$

We pick bases $\{h_{k\alpha}^{ij}\}$ of each $\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k)$. By semisimplicity, we may define *dual bases* $\{h_{ij}^{k\alpha}\}$ by $h_{k\alpha}^{ij} \circ h_{ij}^{k\alpha} = \text{id}_{x_k}$.

Lemma 1.4.0.1 (cf. [9] Section 7). *Take bases $\{h_{k\alpha}^{ij}\}$ of $\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k)$. Then $\mu = \bigoplus_{i,j,k} \sum_{\alpha} h_{k\alpha}^{ij} \boxtimes^* (h_{ij}^{k\alpha}) \in \text{Hom}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}}(a \otimes a, a)$ is the composition of internal homs.*

Proposition 1.4.0.1 (Formula 4.5 of [21]). *Take bases $\{h_{k\alpha}^{ij}\}$ of $\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k)$. If $\mathbf{T}$ is spherical, $s := \frac{1}{\dim \mathbf{T}} \bigoplus_{i,j,k} \frac{d_i d_j}{d_k} \sum_{\alpha} h_{ij}^{k\alpha} \boxtimes^* (h_{k\alpha}^{ij}) \in \text{Hom}(a, a \otimes a)$ is a separation map.*

We may now generalize the previous proposition to the not-necessarily spherical case.

Proposition 1.4.0.2. *Take pivotal bases $\{h_{k\alpha}^{ij}\}$ of $\text{Hom}_{\mathbf{T}}(x_i \otimes x_j, x_k)$.*

$$F(s) = \frac{1}{\dim \tilde{\mathbf{T}}} \bigoplus_{i,j,k} \frac{d_i d_j}{d_k} \sum_{\alpha} \epsilon_{ij}^{k\alpha} h_{ij}^{k\alpha} \boxtimes^* (h_{k\alpha}^{ij}) \in \text{Hom}(a, a \otimes a)$$

is a separation for the algebra a , where i, j, k range over simple objects of $\tilde{\mathbf{T}}$, d_i are the (spherical) dimensions of x_i , and $\epsilon_{ij}^{k\alpha}$ are the pivotal symbols.

Proof. Since $\tilde{\mathbf{T}}$ is spherical, the forgetful functor $F : \tilde{\mathbf{T}} \rightarrow \mathbf{T}$ yields the separation $F(s) : a \rightarrow a \otimes a$ for $\mathbf{T}$. $\square$

1.5 Dualizability in $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$

We quickly state the duality properties of $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$. The main reference for much of this material is [7].

Definition 1.5.1. Tensor categories, bimodule categories, bimodule functors, and bimodule natural transformations form a 3-category $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$. We consider a $(\mathbf{T}_1, \mathbf{T}_2)$ -bimodule to be a morphism $\mathbf{T}_2 \rightarrow \mathbf{T}_1$. The multiplication for objects is the Deligne tensor product. Composition of 1-morphisms is the relative Deligne tensor product.

Theorem 1.5.1. *The dual of a finite tensor category is the opposite category.*

Theorem 1.5.2. *Let $\mathbf{M}$ be a $(\mathbf{T}_1, \mathbf{T}_2)$ -bimodule. The left dual of $\mathbf{M}$ is the category $\text{Hom}_{\mathbf{T}_1}(\mathbf{M}, \mathbf{T}_1)$ of $\mathbf{T}_1$ -module functors $\mathbf{M} \rightarrow \mathbf{T}_1$. Similarly, the right dual is $\text{Hom}_{\mathbf{T}_2}(\mathbf{M}, \mathbf{T}_2)$.*

Remark. There exists a canonical equivalence between the left and right adjoints of 1-morphisms between sufficiently dualizable objects, in particular, if $\mathbf{M}$ a finite bimodule category between finite tensor categories, the left and right duals are equivalent as bimodule categories.

Theorem 1.5.3. *The dual of a bimodule functor $F : \mathbf{M} \rightarrow \mathbf{N}$ is the “conjugate transpose” functor $F^*(n_j) = \oplus_i m_i \otimes \text{Hom}_{\mathbf{N}}(F(m_i), n_j)$ where m_i, n_j are the simple objects of $\mathbf{M}, \mathbf{N}$, respectively.*

Fusion categories classify 3-dimensional fully extended TQFTs with target $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$.

Theorem 1.5.4 ([7] Thm 1,3, [14] Thm B). *Fusion categories are the 3-dualizable objects (up to Morita equivalence) of $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$.*

Reconstruction Theory of Fusion Categories

Every fusion category arises as the category of representations of a weak Hopf Algebra. We present some parts of this reconstruction theory of fusion categories. For a detailed treatment see Section 7.23 of [12] and [22].

Let $\mathbf{T}$ be a fusion category, and $\mathbf{M}$ a faithful semisimple finite left $\mathbf{T}$ -module category. Fix representatives m_i of isomorphism classes of simple objects.

Consider $\text{Fun}(\mathbf{M}, \mathbf{M})$, the semisimple category of exact functors $\mathbf{M} \rightarrow \mathbf{M}$. Composition of functors provides a tensor structure on $\text{Fun}(\mathbf{M}, \mathbf{M})$.

The module structure of $\mathbf{M}$ gives us a tensor functor $\phi : \mathbf{T} \rightarrow \text{Fun}(\mathbf{M}, \mathbf{M})$ by assigning $x \mapsto (x \otimes -)$.

Now, fix representatives $m_1, \dots, m_n$ of isomorphism classes of simple objects in $\mathbf{M}$.

Lemma 1.5.4.1. *Take $R = \text{End}(\oplus m_i) \cong \bigoplus \text{End}(m_i)$. $\mathbf{M}$ is equivalent (as a k -linear category) to ${}_R\text{Mod}$, the category of left R -modules.*

Remark. Since we work only over $k = \mathbb{C}$, $\mathbf{M}$ is simply a direct sum of n copies of $\mathbf{Vect}$.

Then we may make the following identification:

$$\text{Fun}(\mathbf{M}, \mathbf{M}) \cong {}_R\text{Mod}_R$$

$$(B \otimes_R -) \mapsto B$$

where ${}_R\text{Mod}_R$ is the category of R -bimodules.

Composing ϕ with the forgetful functor ${}_R\text{Mod}_R \rightarrow \mathbf{Vect}$ yields a not necessarily monoidal functor $\mathbf{T} \rightarrow \mathbf{Vect}$. Explicitly, we have

$$(\text{forget} \circ \phi)(x) = \bigoplus_{i,j} \text{Hom}_{\mathbf{M}}(m_i, x \otimes m_j) \cong \text{Hom}_{\mathbf{M}}(m, x \otimes m)$$

Defining $\text{End}_{\mathbb{C}}(\phi) := \text{End}(\text{forget} \circ \phi)$, $\text{End}_{\mathbb{C}}(\phi)$ has the structure of a weak Hopf Algebra, and furthermore, T is tensor equivalent to the fusion category of representations of $\text{End}_{\mathbb{C}}(\phi)$. And of course, every fusion category has a faithful semisimple module, namely itself.

Weak Hopf Algebras

Weak Hopf algebras generalize Hopf algebras by relaxing the axioms constraining the unit and counit maps. We recall the definition of a weak Hopf algebra[5]. We use Sweedler's notation for coproducts and leave sums implicit.

Definition 1.5.2. Let W be a finite-dimensional unital associative $\mathbb{C}$ -algebra with multiplication $\mu : W \otimes W \rightarrow W$ and unit $\eta : \mathbb{C} \rightarrow W$. Further let W be a $\mathbb{C}$ -coalgebra with comultiplication $\delta : W \rightarrow W \otimes W$ and counit $\epsilon : W \rightarrow \mathbb{C}$. Finally, let $S : W \rightarrow W$ be a linear map. We say that $(W, \mu, \Delta, \eta, \epsilon, S)$ is a *Weak Hopf Algebra* if it satisfies the following axioms.

Axiom 1 (Multiplicativity of the coproduct).

$$\Delta(xy) = \Delta(x)\Delta(y)$$

for all $x, y \in H$.

Axiom 2 (Weak multiplicativity of the counit).

$$\epsilon(xyz) = \epsilon(xy_{(1)})\epsilon(y_{(2)}z) = \epsilon(xy_{(2)})\epsilon(y_{(1)}z)$$

for all $x, y, z \in H$

Axiom 3 (Weak comultiplicativity of the unit).

$$(\Delta \otimes \text{id})\Delta(1) = (\Delta(1) \otimes 1)(1 \otimes \Delta(1)) = (1 \otimes \Delta(1))(\Delta(1) \otimes 1)$$

Axiom 4 (Antipode).

$$\begin{aligned} x_{(1)}S(x_{(2)}) &= \epsilon(1_{(1)}x)1_{(2)} \\ S(x_{(1)})x_{(2)} &= 1_{(1)}\epsilon(x1_{(2)}) \\ S(x_{(1)})x_{(2)}S(x_{(3)}) &= S(x) \end{aligned}$$

for all $x \in H$.

Example 13. Let G be a finite groupoid, that is, a category with finitely many morphisms, all of which are invertible. Then the groupoid algebra $\mathbb{C}[G]$ with comultiplication defined by $\Delta(g) := g \otimes g$, counit $\epsilon(g) := 1$, and antipode $S(g) := g^{-1}$ fails to be a Hopf algebra, but is a weak Hopf algebra.

Chapter 2

Constructing Hopf Algebras using TQFT

In this chapter, we will use fully-extended TQFTs to construct a (weak) Hopf algebra associated with a fusion category.

Some good expositions of fully extended TQFTs in general are the original Baez-Dolan paper [3], Chapter 5 of [13], and Lurie’s paper [18] on the cobordism hypothesis. For 3-d TQFTs with target $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$, [7] is a detailed reference.

We have seen that any fusion category is equivalent to a representation category of some (non-canonical) semisimple weak Hopf algebra [16, 28]. Freed and Teleman in [15] describe a method for constructing a weak Hopf algebra from the fully extended Topological Quantum Field Theory (TQFT) defined by a spherical fusion category. Essentially the same construction was also noted in [25].

In particular, the weak Hopf algebra operations and the Frobenius algebra structure are exhibited as the result of evaluating the TQFT on appropriate colored bordisms.

In this chapter, we provide a detailed account of the reconstruction via TQFT theory described in [15], and also present a different related construction for the Hopf case. In particular, we provide explicit diagrammatic constructions of the Frobenius-Hopf operations with proofs of the Frobenius/Hopf axioms.

We will work in a 3-category $\mathbf{Bord}^{fr}$ of framed bordisms with defects. While we intend the techniques used in this dissertation to be agnostic to the precise definitions of the bordism categories used, for concreteness we refer to [27] where Stewart provides constructions of categories of bordisms with defects which accommodate every technique used in this dissertation.

Related Work

An essentially similar construction was also noted by Reutter in [25]. This was the first appearance of the basic idea that the author is aware of.

In addition, in [6], Dimofte and Niu define a related construction that they call spark algebras. Spark algebras are more general in that they are not limited to the finite semisimple context, or to 3-dimensional bordisms. Furthermore, [6] does not assume full dualizability and studies properties at varying levels of locality. We restrict ourselves to the fully dualizable (and fully local) setting. Spark algebras are less general in that they come from fiber functors.

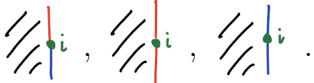
In more detail, the “transversality” condition of [6] says (in the language of this dissertation) that $\mathbf{T} \boxtimes_{\mathbf{T}} \mathbf{M} \cong \mathbf{Vect}$, that is, that $\mathbf{M}$ is given by a fiber functor. The “completeness” condition says that there is a deformation of the sandwich of boundaries that is equivalent to the trivial interface. In our fully local and fully dualizable context of 3-d TQFT with target $\text{Alg}_1(\text{Cat}_{\mathbb{C}})$, this is automatic and implied by the fact that every nonzero module category $\mathbf{M}$ over a fusion category $\mathbf{T}$ induces a Morita equivalence between $\mathbf{T}$ and the dual tensor category $\text{Fun}_{\mathbf{T}}(\mathbf{M}, \mathbf{M})$. In the special case where the union of our assumptions hold (see in particular section 4.2.5 of [6]), spark algebras coincide with our construction.

Preliminaries and Conventions

The main object of this section will be a TQFT $Z_{\mathbf{T}}$ defined by a fusion category $\mathbf{T}$, the regular $\mathbf{T}$ -module category ${}_{\mathbf{T}}\mathbf{T}$, and an indecomposable $\mathbf{T}$ -module category $\mathbf{M}$.

We will define $Z_{\mathbf{T}}(pt_+) = \mathbf{T}$, and we would like to have the module categories ${}_{\mathbf{T}}\mathbf{T}, \mathbf{M}$ be the images of some 1-morphisms. However, there are no 1-bordisms from the empty set to the point. We therefore extend the framed bordism category to include two new 1-bordisms

$$-\bullet, -\bullet : pt_+ \leftarrow \emptyset$$

We also extend it to include families of new 2-bordisms .

or more concisely,

$$\begin{aligned} \bullet_i &: -\bullet \leftarrow -\bullet \\ \bullet_i &: -\bullet \leftarrow -\bullet \\ \bullet_i &: -\bullet \leftarrow -\bullet \end{aligned}$$

where i ranges over the isomorphism classes of simple objects of $\mathbf{M}, \mathbf{T}$, or $\mathbf{T}_{\mathbf{M}}^*$. While we use the same color to denote all three types of new 2-bordisms, it will be clear which one is meant by context.

We simply refer to the new category as $\mathbf{Bord}^{fr}$. As in the case without singularities, we also have a category $\mathbf{Bord}^{or}$ of oriented bordisms with the aforementioned extra bordisms.

Consistent with our convention that a bimodule category ${}_{\mathbf{T}_1}\mathbf{M}_{\mathbf{T}_2}$ is a morphism $\mathbf{T}_2 \rightarrow \mathbf{T}_1$, our bordism pictures should be read from right to left (and up to down) unless otherwise specified.

Since we are interested in oriented theories, we assume that $\mathbf{T}$ is equipped with $\mathrm{SO}(3)$ -invariance data.¹ That is, the action of $\mathrm{SO}(3)$ on framed 3-bordisms (by rotating the framing) translates through the cobordism hypothesis into an action of $\mathrm{SO}(3)$ on fully-dualizable objects of the target category of a fully-extended TQFT. In our context, the target category is $\mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$ and the fully dualizable objects are (up to Morita equivalence) the fusion categories, recall Theorem 1.5.4. Oriented theories therefore come from a fusion category with a homotopy fixed point structure. Note that having $\mathrm{SO}(3)$ -invariance data is a weaker assumption than requiring a spherical structure on $\mathbf{T}$.

Up to a scalar factor which we may freely choose, this automatically induces an $\mathrm{SO}(2)$ -invariance structure on the regular module ${}_{\mathbf{T}}\mathbf{T}$. We further assume that there exists $\mathrm{SO}(2)$ -invariance data for $\mathbf{M}$ compatible with the $\mathrm{SO}(3)$ -invariance data on $\mathbf{T}$.²

In particular, we have identifications $S_{\mathbf{T}\mathbf{T}} \cong \mathrm{id}_{\mathbf{T}\mathbf{T}}$, $S_{\mathbf{T}\mathbf{M}} \cong \mathrm{id}_{\mathbf{T}\mathbf{M}}$ where $S_{\mathbf{T}\mathbf{T}}, S_{\mathbf{T}\mathbf{M}}$ are the Serre functors. Since both module categories are indecomposable, choosing an identification amounts to choosing a single scalar (relative to the $\mathrm{SO}(3)$ -structure on $\mathbf{T}$) that we will choose in the future.

TQFT Computations

By the cobordism hypothesis with singularities, a choice of 2-dualizable left $\mathbf{T}$ -module categories $\mathbf{M}, \mathbf{M}'$ and 1-dualizable module functors $\tilde{r}_i : \mathbf{M}' \rightarrow \mathbf{M}$ defines a fully extended TQFT

$$Z_{\mathbf{T}} : \mathbf{Bord}^{fr} \rightarrow \mathrm{Alg}_1(\mathrm{Cat}_{\mathbb{C}})$$

with $Z_{\mathbf{T}}(pt_+) = \mathbf{T}$, $Z_{\mathbf{T}}(-\bullet) = \mathbf{M}'$, $Z_{\mathbf{T}}(-\circ) = \mathbf{M}$, $Z_{\mathbf{T}}(\bullet_i) = \tilde{r}_i$. Note that $Z_{\mathbf{T}}$ is not to be confused with $\mathcal{Z}(\mathbf{T})$, the Drinfeld centre of $\mathbf{T}$.

By replacing $\mathbf{T}$ with the Morita equivalent category $\mathrm{Fun}_{\mathbf{T}}(\mathbf{M}', \mathbf{M}')$, we may assume that $\mathbf{M}'$ is the regular $\mathbf{T}$ -module category $\mathbf{T}$, and from this point on we will make this assumption.

¹This is possible in all known cases, but it is an open question as to whether all fully extended 3-d framed TFTs admit $\mathrm{SO}(3)$ -invariance data. See also the discussion in Section I.4 of [7]

²I don't know exactly how strong this condition is, but it is at least as strong as requiring sphericity of $\mathbf{T}_{\mathbf{M}}^*$.

Fix representatives m_i of isomorphism classes of simple objects in $\mathbf{M}$, and denote $m = \oplus_i m_i$. We choose the module functors $Z_{\mathbf{T}}(\bullet) = - \otimes m$, $Z_{\mathbf{T}}(\bullet_i) = - \otimes m_i$.

Since we work in the fully-dualizable setting, we will identify left and right adjoints of the 1-morphism $\mathbf{M} : \mathbf{Vect} \rightarrow \mathbf{T}$ and write $\mathbf{M}^\vee$ for the adjoint of $\mathbf{M}$. Similarly, for the module functors $Z_{\mathbf{T}}(\bullet_i) = - \otimes m_i$, the internal hom functor $\underline{\text{Hom}}(m_i, -)$ is an ambidextrous adjoint.

We will work with oriented theories, so we must equip the point with $\text{SO}(3)$ -invariance data, and the red and blue points with $\text{SO}(2)$ -invariance data. $\text{SO}(1)$ is trivial, so there is no additional data to specify for the green points.

Now fix once and for all some choice of $\text{SO}(3)$ -invariance data for $Z_{\mathbf{T}}(pt_+) = \mathbf{T}$, and compatible $\text{SO}(2)$ -invariance data on $\mathbf{M}$.

Our goal is the following theorem.

Theorem 2.0.1. $Z_{\mathbf{T}}$ assigns to  the weak Hopf Algebra

$$W := \text{End}_{\mathbb{C}}(\phi) := (\text{Forget} \circ \phi) \circ (\text{Forget} \circ \phi)^\vee$$

In particular, $\mathbf{T}$ is equivalent as a tensor category to the category of W -modules.

Furthermore,  evaluates to the endomorphism algebra $H := \text{End}_{\mathbf{T}}(z_{\mathbf{M}}(1))$

of a certain algebra object $z_{\mathbf{M}}(1)$ in $\mathbf{T}$. The algebra H has a natural Hopf structure and if $\mathbf{T}$ admits a fiber functor, $\mathbf{T}$ is equivalent as a tensor category to the category of H -modules.

Basic Computations

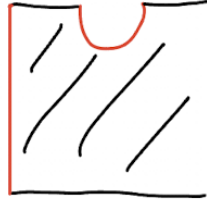
Lemma 2.0.1.1. $Z_{\mathbf{T}}(\bullet_-) = \mathbf{T}_{\mathbf{T}}$ the regular right $\mathbf{T}$ -module, and $Z_{\mathbf{T}}(\bullet_-) = \mathbf{M}^\vee$.

Lemma 2.0.1.2. $Z_{\mathbf{T}}(\bullet\bullet) = \mathbf{T} \boxtimes_{\mathbf{T}} \mathbf{T} \cong \mathbf{T}$, $Z_{\mathbf{T}}(\bullet\bullet) = \mathbf{T} \boxtimes_{\mathbf{T}} \mathbf{M} \cong \mathbf{M}$, and $Z_{\mathbf{T}}(\bullet\bullet) = \mathbf{M}^\vee \boxtimes_{\mathbf{T}} \mathbf{M} \cong \mathbf{T}_{\mathbf{M}}^*$, as $\mathbb{C}$ -linear categories.

Figure 2.1 is the tensor product of $\mathbf{T}$.

Lemma 2.0.1.3. $Z_{\mathbf{T}}$ assigns to  the vector space $A_{m_i} := \text{End}_{\mathbf{M}}(m_i) \cong \mathbb{C}$.

Proof. We read  from top to bottom.


 Figure 2.1: The tensor product in $\mathbf{T}$.

We have: The unit $1 \in T$, the functor $- \otimes m_i : T \rightarrow \mathbf{M}$, its adjoint $(- \otimes m_i)^\vee : \mathbf{M} \rightarrow T$, and finally $\text{Hom}_T(1, -)$.

Putting everything together, we see that the bordism evaluates to

$$\text{Hom}_T(1, (- \otimes m_i)^\vee(1 \otimes m_i)) \cong \text{Hom}_{\mathbf{M}}(m_i, m_i)$$

□

Dimensional Reduction

Dimensional reduction along the three bordisms $\bullet\bullet, \bullet\bullet, \bullet\bullet : \emptyset \rightarrow \emptyset$ defines fully extended 2-d topological field theories $R_{\bullet\bullet}, R_{\bullet\bullet}, R_{\bullet\bullet}$ that assign $\mathbf{T}, \mathbf{M}, \mathbf{T}_{\mathbf{M}}^*$ to the positively oriented point, respectively. Many computations for $Z_{\mathbf{T}}$ are greatly simplified by passing to the reduced theories.

The following analysis is heavily informed by the paper [19] of Moore and Segal, which describes open-closed 2-d theories defined by a Frobenius category.

The main differences stem from the fact that we consider the Frobenius categories arising as $\mathbf{T}$ -module categories. As a result, the tensor structures of $\mathbf{T}$ and $\mathbf{T}_{\mathbf{M}}^*$ provide a natural multiplication of boundary conditions not considered in [19].

Furthermore, we require that the Frobenius structure on the boundary theories be compatible with the $\text{SO}(3)$ -invariance structure on $\mathbf{T}$. This condition fully specifies the Frobenius structure on $\mathbf{M}$.

Definition 2.0.1. Let $\mathbf{M}$ be a $\mathbf{T}$ -module category. A *Frobenius structure* on $\mathbf{M}$ is a nondegenerate trace

$$\theta_{\mathbf{T}_{\mathbf{M}}^*} : \text{End}(\text{id}_{\mathbf{T}_{\mathbf{M}}^*}) \rightarrow \mathbb{C}$$

where $\mathbf{T}_{\mathbf{M}}^*$ is the category $\text{End}_{\mathbf{T}}(\mathbf{M})$ of $\mathbf{T}$ -module endofunctors. Equivalently, it is a choice of automorphism for each simple object of $\mathbf{T}_{\mathbf{M}}^*$.

When $\text{SO}(3)$ -invariance data for $\mathbf{T}$ is fixed, there is a unique (up to a choice of scalar) Frobenius structure on $\mathbf{M}$ compatible with the $\text{SO}(3)$ -invariance data. This Frobenius structure is induced by a relative Frobenius structure.

Definition 2.0.2 (c.f. [15] Section 6.2). A *relative Frobenius structure* on $\mathbf{M}$ is a choice of automorphism of the unit object in $\mathbf{T}_\mathbf{M}^*$.

Remark. Alternatively, equivalent data is a choice of trace of the Frobenius algebra object $z_\mathbf{M}$ of $\mathcal{Z}(\mathbf{T})$, where $z_\mathbf{M} := Z_\mathbf{T}(\bullet \times S^1)$.

Lemma 2.0.1.4. *Let $\lambda \in \text{End}_{\mathbf{T}_\mathbf{M}^*}(1)$ be nonzero. There is a unique Frobenius structure $\theta_{\mathbf{T}_\mathbf{M}^*} : \text{End}(\text{id}_{\mathbf{T}_\mathbf{M}^*}) \rightarrow \mathbb{C}$ such that $\theta_{\mathbf{T}_\mathbf{M}^*}(\lambda) = 1$.*

In other words, a relative Frobenius structure is $\text{SO}(2)$ -invariance data for $\mathbf{M}$ compatible with the $\text{SO}(3)$ -invariance data for $\mathbf{T}$.

As fully extended 2-d theories, the theories $R_{\bullet\bullet}, R_{\bullet\bullet}, R_{\bullet\bullet}$ are fully specified by the Frobenius algebra structures of the images of S^1 .

We have:

$$\begin{aligned} R_{\bullet\bullet}(S^1) &:= Z_\mathbf{T}(\bullet\bullet \times S^1) = \text{End}(\text{id}_\mathbf{T}) \\ R_{\bullet\bullet}(S^1) &:= Z_\mathbf{T}(\bullet\bullet \times S^1) = \text{End}(\text{id}_\mathbf{M}) \\ R_{\bullet\bullet}(S^1) &:= Z_\mathbf{T}(\bullet\bullet \times S^1) = \text{End}(\text{id}_{\mathbf{T}_\mathbf{M}^*}) \end{aligned}$$

as vector spaces.

The Frobenius traces for $R_{\bullet\bullet}(S^1)$ and $R_{\bullet\bullet}(S^1)$ are precisely the relative Frobenius structures on $\mathbf{T}$ and $\mathbf{M}$ chosen earlier.

For $\mathbf{T}$ and $\mathbf{T}_\mathbf{M}^*$, which have tensor structures given by Figure 2.1, $\text{SO}(3)$ invariance on $Z_\mathbf{T}(pt_+) = \mathbf{T}$ and $\text{SO}(2)$ invariance on $Z_\mathbf{T}(\bullet) = {}_\mathbf{T}\mathbf{T}$, $Z_\mathbf{T}(\bullet) = {}_\mathbf{T}\mathbf{M}$ induce spherical structures.

That is,  is a 1-bordism $\emptyset \rightarrow pt_+$ in the dimensional reduction, and the

“green circle”  $\times S^1$ exhibits both the left and right traces of id_{x_i} .

Frobenius Algebras

We first analyze the Frobenius structure of $A_\mathbf{T}$. The case of $A_{\mathbf{T}_\mathbf{M}^*}$ is exactly analogous.

As a vector space, $A_\mathbf{T}$ evaluates to $\oplus \text{End}_\mathbf{T}(x_i)$ where x_i range over representatives of isomorphism classes of simple objects of $\mathbf{T}$.

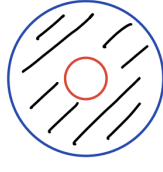
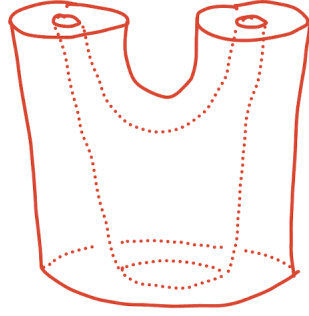
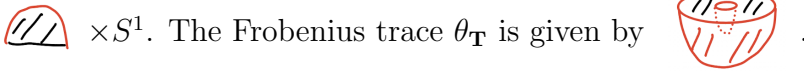

 Figure 2.2: $A_{\mathbf{T}}$

 Figure 2.3: $A_{\mathbf{M}}$

 Figure 2.4: $A_{\mathbf{T}^*}$

 Figure 2.5: The multiplication for $A_{\mathbf{T}}$.

The multiplication is given by the bordism in Figure 2.5. The unit is given by

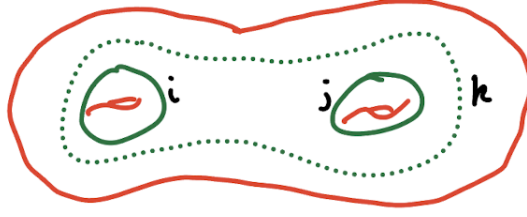
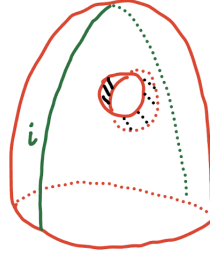


A convenient basis of $A_{\mathbf{T}}$ is given by the bordisms $t_i := \text{diagram} \times S^1$, and reading the bordisms down-to-up gives a dual basis θ_i since $\text{diagram} \times S^1$ evaluates to $\theta_i(t_i) = \dim \text{Hom}(x_i, x_j) = \delta_{ij}$.

Note that $\theta_i(1)$ and $\theta_{\mathbf{T}}(\chi_i)$ are both bordisms diffeomorphic to the 3-ball with red boundary. Therefore

$$\theta_{\mathbf{T}} = \sum_i \theta_i(1)\theta_i, \quad 1_{A_{\mathbf{T}}} = \sum_i \theta_i(1)t_i$$

In particular, $\bullet \bullet \times S^2$ evaluates to $\theta_{\mathbf{T}}(1) = \sum_i \theta_i(1)^2 = \frac{1}{\theta_1(1)^2} \dim \mathbf{T}$. Furthermore $\theta_{\mathbf{T}}(t_i) = \theta_i(1)\theta_i(t_i) = \theta_i(1)$.


 Figure 2.6: $\theta_{\mathbf{T}}(t_i t_j t_k)$

 Figure 2.7: Projection $A_T \rightarrow \text{End}_{\mathbf{T}}(x_i)$

The bordism $i \circledast j \times S^1$ which evaluates to δ_{ij} is diffeomorphic to the bordism evaluating to $\theta_{\mathbf{T}}(\theta_i \theta_j)$.

Now consider the bordism in Figure 2.6 which evaluates to $\theta_{\mathbf{T}}(t_i t_j t_k)$.

We can cut the bordism in such a way that the cross-section is $i \circledast j$ (or i and k), therefore $\theta_{\mathbf{T}}(t_i t_j t_k) = 0$ unless $i = j = k$.

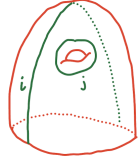
If $i = j = k$, we can evaluate Figure 2.6 using only the Frobenius algebra structure of $i \circledast i$, the result is $\frac{1}{t_i(1)}$.

We therefore have $t_i^2 = \frac{1}{\theta_i(1)} t_i$ and so $\epsilon_i := \theta_i(1) t_i$ form an idempotent basis of $A_{\mathbf{T}}$.

Dimensions

Note that in the dimensional reduction, each $t_i := \text{diagram} \times S^1$ evaluates to the dimension of x_i with respect to the spherical structure of $\mathbf{T}$.

A natural question to ask is what the projection maps $A_{\mathbf{T}} \rightarrow \text{End}_{\mathbf{T}}(x_i)$ are for each x_i . They are given by the bordism of Figure 2.7.

In particular, plugging in t_j into the bordism: , We see that t_j corresponds to $\frac{1}{t_j(1)} \text{id}_{x_j}$.

We can now see the usual properties of dimension in spherical categories.

Define $t_{i \otimes j}$ by  $\times S^1$, and since $\theta_k(t_{i \otimes j}) = N_k^{ij}$, we have the fusion rules $t_{i \otimes j} = \sum_k N_k^{ij} t_k$, as expected.

From evaluating , we have $\theta_{\mathbf{T}}(\chi_{i \otimes j}) = \frac{\theta_i(1)\theta_j(1)}{\theta_1(1)}$. Therefore $[x_i] \mapsto \frac{\theta_i(1)}{\theta_1(1)}$ is a ring homomorphism from the underlying fusion ring of $\mathbf{T}$ to $\mathbb{C}$. Furthermore, $\theta_i(1) = \theta_{i^*}(1)$ by orientability of the theory. In light of this, we will take the relative Frobenius structure on $\mathbf{T}$ such that $\theta_1(1) = 1$ and write $\dim(x_i) := \theta_i(1)$.

Exactly analogous statements are true for $A_{\mathbf{T}_{\mathbf{M}}^*}$. In particular, $\bullet \times S^2$ evaluates to $\theta_{\mathbf{T}_{\mathbf{M}}^*}(1) = \frac{1}{\theta_1(1)^2} \dim \mathbf{T}_{\mathbf{M}}^* = \frac{1}{\theta_1(1)^2} \dim \mathbf{T}$.

The algebra $A_{\mathbf{M}}$

The case of $A_{\mathbf{M}}$ is more complicated. By semisimplicity, $\{\theta_i\}, \{t_i\}$ still form dual bases of $A_{\mathbf{M}}$, and

$$\theta_{\mathbf{M}} = \sum_i \theta_i(1) \theta_i, \quad 1_{A_{\mathbf{M}}} = \sum_i \theta_i(1) t_i$$

We have “dimensions” $\theta_i(1)$ for simple objects m_i of $\mathbf{M}$. We bite the bullet and write $\dim_{\mathbf{M}}(m_i) := \theta_i(1)$.³

The dimensions are compatible with the module structure of $\mathbf{M}$ in the following sense.

³Lemma 2.0.1.5 implies that if $\mathbf{T}$ is a pseudo-unitary fusion category, $\dim_{\mathbf{M}}(m_i)$ is in fact the canonical Frobenius-Perron dimension of m_i . See section 2.5 of [11], Prop. 8.7 of [10].

Lemma 2.0.1.5. *Let x be an object of $\mathbf{T}$ and m_1, m_2 objects of $\mathbf{M}$. Then we have:*

- $\dim_{\mathbf{M}}(x \otimes m) = \dim_{\mathbf{T}}(x) \dim_{\mathbf{M}}(m)$
- $\dim_{\mathbf{T}}(\underline{\text{Hom}}_{\mathbf{T}}(m_1, m_2)) = \dim_{\mathbf{M}}(m_1) \dim_{\mathbf{M}}(m_2)$

Proof. This immediately follows from the fact that  $\times S^1$ evaluates to $x_j \otimes m_i$

and  $\times S^1$ evaluates to $\underline{\text{Hom}}_{\mathbf{T}}(m_j, m_i)$. □

In particular, we have $\dim(m_i)^2 = \dim(\underline{\text{End}}(m_i))$. Taking $\mathbf{M} = \text{Mod}_a(\mathbf{T})$ for some algebra object a of $\mathbf{T}$, we have $\dim_{\mathbf{M}}(m_i)^2 = \frac{\dim_{\mathbf{T}}(m_i)^2}{\dim_{\mathbf{T}}(a)}$. Note that as a is self-dual in $\mathbf{T}$, $\dim_{\mathbf{T}}(a)$ must be real.

Although $\dim_{\mathbf{T}}(m_i)$ and $\dim_{\mathbf{T}}(a)$ is dependent on the choice of $\mathbf{T}$ and a , the TQFT description implies $\dim_{\mathbf{M}}(m_i)$ is a property only of the Morita equivalence class of $\mathbf{T}$.

Evaluating $\bullet \times S^2$ yields $\theta_{\mathbf{M}}(1)$.

Cutting $\bullet \times S^2$ into $(\bullet \times S^2) \circ (-\bullet \times S^2)$, we see that

$$\theta_{\mathbf{M}}(1)^2 = \theta_{\mathbf{T}^*_{\mathbf{M}}}(1) \theta_{\mathbf{T}}(1) = (\dim \mathbf{T})^2 \quad (2.1)$$

since S^2 evaluates to the 1-dimensional space $\text{End}_{\mathcal{Z}(\mathbf{T})}(1)$. As $\theta_{\mathbf{M}}(1) = \sum_i \theta_i(1)^2 = \sum_i \dim_{\mathbf{M}}(m_i)^2$, it must be the positive square root $\dim \mathbf{T}$.

2.1 Reconstruction

Reconstruction of the (Weak) Hopf Algebra

We want to find TQFT pictures that implement a fiber functor (faithful tensor functor) $\text{End}(\mathbf{M}) \rightarrow \mathbf{Vect}$. First we consider the simplest case $\mathbf{M} = \mathbf{Vect}$.

Lemma 2.1.0.1.  evaluates to the module action $\mathbf{T} \boxtimes \mathbf{M} \rightarrow \mathbf{M}$.

Proof. The bordism evaluates to

$$\text{Fun}_{\mathbf{T}}(\mathbf{T}, \mathbf{T}) \boxtimes_{\mathbf{T}} (\mathbf{T} \boxtimes \text{Fun}_{\mathbf{T}}(\mathbf{T}, \mathbf{T})) \boxtimes_{\mathbf{T}} \mathbf{M} \xrightarrow{1 \boxtimes \mathbf{T} \epsilon^R \boxtimes \mathbf{T} 1} \text{Fun}_{\mathbf{T}}(\mathbf{T}, \mathbf{T}) \boxtimes_{\mathbf{T}} \mathbf{T} \boxtimes_{\mathbf{T}} \mathbf{M} \cong \mathbf{M}$$

□

Naturally, bending the bordism yields the associated functor $\mathbf{T} \rightarrow \text{End}(\mathbf{M})$.

Lemma 2.1.0.2.  evaluates to the module action $\mathbf{T} \rightarrow \mathbf{M} \boxtimes \mathbf{M}^L \cong \text{End}(\mathbf{M})$.

In order to land in **Vect**, we cap off the bordism in the simplest possible way.

Lemma 2.1.0.3.  evaluates to $x \mapsto \text{Hom}_{\text{End}(\mathbf{M})}(\text{id}_{\mathbf{M}}, x \otimes -)$.

Proof. This is the composition of the action $\mathbf{T} \rightarrow \text{End}(\mathbf{M})$ with its adjoint, followed by $\text{Hom}_{\mathbf{T}}(1, -)$.

Explicitly, we have

$$\begin{aligned} x \mapsto x \otimes - &\mapsto \bigoplus_{x_i \text{ simple}} x_i \otimes \text{Hom}_{\text{End}(\mathbf{M})}(x_i \otimes -, x \otimes -) \\ &\mapsto \text{Hom}_{\mathbf{T}}(1, 1) \otimes \text{Hom}_{\text{End}(\mathbf{M})}(1 \otimes -, x \otimes -) \cong \text{Hom}_{\text{End}(\mathbf{M})}(\text{id}_{\mathbf{M}}, x \otimes -) \end{aligned}$$

□

Lemma 2.1.0.4. The trace $x \mapsto \text{Hom}_{\text{End}(\mathbf{M})}(\text{id}_{\mathbf{M}}, x \otimes -) : \mathbf{T} \rightarrow \mathbf{Vect}$ is faithful if and only if $\text{Hom}(x, \bigoplus_i \underline{\text{End}}(m_i)) \neq 0$ for all simple $x \in \mathbf{T}$.

Proof. By semisimplicity of $\mathbf{M}$ we may decompose:

$$x \otimes - = \bigoplus_{i,j} \text{Hom}_{\mathbf{M}}(m_j, x \otimes m_i) \otimes m_i \otimes \text{Hom}_{\mathbf{M}}(m_j, -)$$

So we have:

$$\text{Hom}_{\text{End}(\mathbf{M})}(\text{id}_{\mathbf{M}}, x \otimes -) \cong \bigoplus_i \text{Hom}_{\mathbf{M}}(m_i, x \otimes m_i) \otimes \text{Hom}_{\mathbf{M}}(m_i, m_i)^{\otimes 2}$$

and the trace $\text{Hom}_{\text{End}(\mathbf{M})}(\text{id}_{\mathbf{M}}, x \otimes -) : \text{End}(\mathbf{M}) \rightarrow \mathbf{Vect}$ can only be faithful if for all simple $x \in \mathbf{T}$ there exists some simple m such that $\text{Hom}_{\mathbf{M}}(x \otimes m, m) \neq 0$. By semisimplicity of $\mathbf{T}$, this is also sufficient.

□

We note that this is automatically true if $\mathbf{M} \cong \mathbf{Vect}$.

Now we may construct the Hopf algebra.

Lemma 2.1.0.5.  evaluates to a Hopf algebra H such that $\mathbf{T} \cong \text{Rep}(H)$.

Proof. The bordism evaluates to the composition of the fiber functor and its adjoint.

Note that a functor $F : \mathbf{T} \rightarrow \mathbf{Vect}$ has adjoint $F^\vee : \mathbf{Vect} \rightarrow \mathbf{T}$ defined by

$$F^\vee(\mathbb{C}) = \bigoplus_x x \otimes F(x)^*$$

where the direct sum ranges over (isomorphism classes of) simple objects $x \in \mathbf{T}$.

In particular, $F(F^\vee(\mathbb{C})) = \bigoplus_x F(x) \otimes F(x)^* \cong \bigoplus_x \text{End}(F(x)) \cong \text{End}(F)$. $\square$

If $\mathbf{M}$ is not $\mathbf{Vect}$, we need a more complicated bordism.

In this section, we assign to the unlabeled green point $\bullet : -\bullet \leftarrow -\bullet$ the sum $\bigoplus_i m_i$ of all simple objects of $\mathbf{M}$.

Lemma 2.1.0.6. $Z_{\mathbf{T}}$ assigns to  the functor $\mathbf{T} \rightarrow {}_R\text{Mod}_R \rightarrow \mathbf{Vect}$, where R is the $\mathbb{C}$ -algebra $\text{End}_{\mathbf{M}}(m)$.

Proof. First, we evaluate $Z_{\mathbf{T}}$ on 

$$x \in T \mapsto \text{Hom}_T(1, \underline{\text{Hom}}(m, x \otimes m)) \cong \text{Hom}_{\mathbf{M}}(m, x \otimes m)$$

and observe that this functor is exact and faithful. R acts on $\text{Hom}_{\mathbf{M}}(m, x \otimes m)$ by postcomposition on the left and precomposition on the right. $\square$

Corollary 2.1.0.1. $Z_{\mathbf{T}}$ assigns to  the weak Hopf Algebra

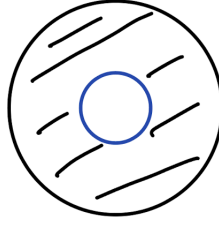
$$W := \text{End}_{\mathbb{C}}(\phi) := (\text{Forget} \circ \phi) \circ (\text{Forget} \circ \phi)^\vee$$

In particular, $\mathbf{T}$ is equivalent as a tensor category to the category of W -modules.

The Object of the Centre Associated with a $\mathbf{T}$ -Module

We take a short detour to reconsider the previous story from the internal tensor-categorical point of view.

Recall that $Z_{\mathbf{T}}$ assigns the Drinfeld Centre $\mathcal{Z}(\mathbf{T}) = \text{Fun}_{\mathbf{T} \boxtimes \mathbf{T}^{mp}}(\mathbf{T}, \mathbf{T})$ to the positively oriented circle. We will discuss the object of $\mathcal{Z}(\mathbf{T})$ that $Z_{\mathbf{T}}$ assigns to the annulus of Figure 2.8 with one boundary component coloured blue.


 Figure 2.8: A bordism evaluating to $z_{\mathbf{M}} \in \mathcal{Z}(\mathbf{T})$

The annulus evaluates to $z_{\mathbf{M}} := \text{ev} \circ \text{ev}^\vee : \mathbf{T} \rightarrow \mathbf{T}$ where $\text{ev} : \mathbf{M} \boxtimes \mathbf{M}^\vee \rightarrow \mathbf{T}$ comes from the adjunction $\mathbf{M} \vdash \mathbf{M}^\vee$. As a $\mathbf{T}$ -bimodule functor, $(\text{ev} \circ \text{ev}^\vee)$ is an object of $\mathcal{Z}(T)$.

Explicitly, we have for $x \in \mathbf{T}$,

$$\begin{aligned} \text{ev}(m \boxtimes F) &= F(m) \\ \text{ev}^\vee(x) &= \bigoplus_{i,j} m_i \boxtimes \underline{\text{Hom}}(m_j, -) \otimes \text{Hom}_{\mathbf{T}}(\underline{\text{Hom}}(m_j, m_i), x) \\ &= \bigoplus_{i,j} m_i \boxtimes \underline{\text{Hom}}(m_j, -) \otimes \text{Hom}_{\mathbf{M}}(m_i, x \otimes m_j) \end{aligned}$$

so

$$z_{\mathbf{M}}(x) = (\text{ev} \circ \text{ev}^\vee)(x) = \bigoplus_{i,j} \underline{\text{Hom}}_{\mathbf{M}}(m_i, m_j) \otimes \text{Hom}_{\mathbf{M}}(m_j, x \otimes m_i)$$

Which is similar to the usual generalized fiber functor $T \rightarrow \text{Bimod}(R)$, except the bimodules are internal to T .

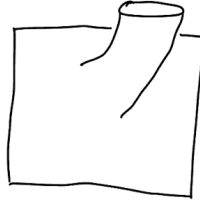
Passing through the forgetful functor $\mathcal{Z}(\mathbf{T}) \rightarrow \mathbf{T}$, $z_{\mathbf{M}} \mapsto z_{\mathbf{M}}(1) \in \mathbf{T}$, we have $z_{\mathbf{M}}(1) = \bigoplus_i \underline{\text{Hom}}_{\mathbf{M}}(m_i, m_i)$.

We now have an alternate description of our previous Hopf algebra pictures.

Recall that the action of $\mathcal{Z}(\mathbf{T})$ on $\mathbf{T}$ has a TQFT description shown in Figure 2.9.

The following computation is immediate.

Lemma 2.1.0.7.  evaluates to $z_{\mathbf{M}}(1)$ and  evaluates to $\text{End}_{\mathbf{T}}(z_{\mathbf{M}}(1))$.


 Figure 2.9: The action $\mathbf{T} \boxtimes \mathcal{Z}(\mathbf{T}) \rightarrow \mathbf{T}$

Evaluating  is also straightforward.

Lemma 2.1.0.8. $Z_{\mathbf{T}}$ assigns to  the vector space

$$W = \bigoplus_{ijkl} W_{ijkl} = \bigoplus_{ijkl} \text{Hom}_{\mathbf{T}}(\underline{\text{Hom}}(m_i, m_j), \underline{\text{Hom}}(m_k, m_l)) \cong \text{End}(a_{\mathbf{M}})$$

Note that $z_{\mathbf{M}}(1) = \bigoplus_i \underline{\text{Hom}}_{\mathbf{M}}(m_i, m_i)$ is not isomorphic to $a_{\mathbf{M}} = \underline{\text{Hom}}_{\mathbf{M}}(\bigoplus m_i, \bigoplus m_i)$ unless $\mathbf{M} \cong \mathbf{Vect}$.

While $\text{End}(a_{\mathbf{M}})$ carries enough information to reconstruct $\mathbf{T}$, $\text{End}(z_{\mathbf{M}}(1))$ in general does not. This is precisely where information is lost and the Hopf algebra version of Tannakian reconstruction fails for fusion categories without a fiber functor to $\mathbf{Vect}$.

Hopf Algebra Operations

Table 2.1 describes how the (weak) Hopf algebra structure can be defined using the TQFT $Z_{\mathbf{T}}$. For H , the outside of each bordism (except the inboundary and outboundary) is coloured red, and the internal blue curves represent hollow “pipes” with the inside coloured blue. Where an internal blue curve ends abruptly, as in the picture for the counit and integral, they represent the pipes ending in a cap, or hemisphere. For W , the bordisms are all solid, and the boundary (except the inboundary and outboundary) is coloured red and blue. One colour (usually red) is shown in the drawings, the complement should be understood to be blue.

We can immediately observe some known facts about finite-dimensional semisimple weak Hopf algebras. For example, the 1-dimensional spaces of left and right

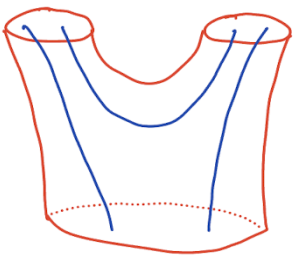
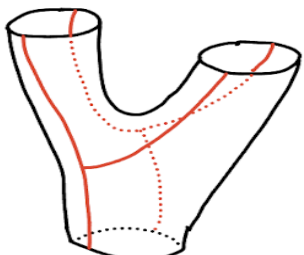
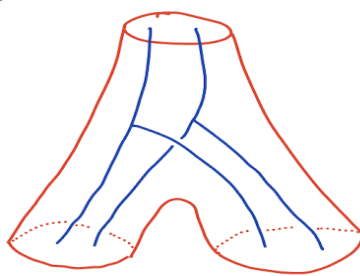
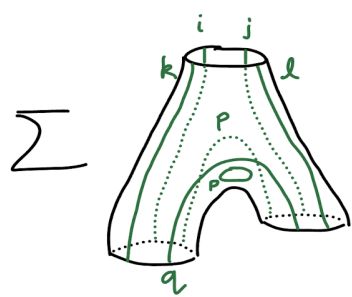
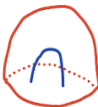
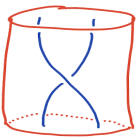
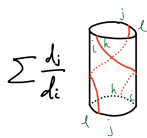
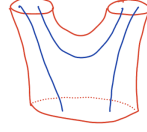
	Bordisms for H	Bordisms for W
Multiplication $\mu : H \otimes H \rightarrow H$		
Comultiplication $\Delta : H \rightarrow H \otimes H$		$\sum$ 
Unit $\eta : \mathbb{C} \rightarrow H$		
Counit $\epsilon : H \rightarrow \mathbb{C}$		
Integral $\int_H : \mathbb{C} \rightarrow H$		
Antipode $S : H \rightarrow H$		$\sum \frac{d_i}{d_i}$ 

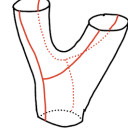
Table 2.1: Summary of Graphical Calculus

integrals must coincide. Also, composing the bordism for the weak Hopf algebra antipode with itself gives a bordism diffeomorphic (relative to the boundary) to the identity, expressing the fact that $S^2 = \text{id}_W$.

Lemma 2.1.0.9. $Z_{\mathbf{T}}$ assigns respectively to



and



the

multiplications $H \otimes H \rightarrow H$ and $W \otimes W \rightarrow W$ given by composition of endomorphisms.

Proof. For the second bordism, cut the bordism as . The first bordism can be cut in much the same way.

In both cases we see that the map is

$$H \otimes H \cong F \circ (F^\vee \circ F) \circ F^\vee \xrightarrow{1 \otimes \epsilon \otimes 1} F \circ F^\vee \cong H$$

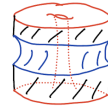
where $\epsilon : F^\vee \circ F \rightarrow \text{id}_T$ is the counit of the adjunction $F^\vee \dashv F$.

□

We will now demonstrate that the graphically defined operations outlined in Table 2.1 are indeed (weak) Hopf algebra structures of H and W .

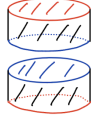
Surgery

Lemma 2.1.0.10. Suppose $\mathbf{M} \cong \mathbf{Vect}$. Then the bordism $b_1 =$



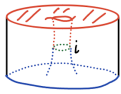
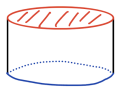
evaluates

to $1 \in V_{\mathbf{M}} \otimes V_{\mathbf{M}} \cong \mathbb{C}$.

In particular, b_1 and $b_2 =$  evaluate to the same element of $V_{\mathbf{M}} \otimes V_{\mathbf{M}}$.

Proof. The first bordism b_1 evaluates to $Z_{\mathbf{T}}(b_1) = \Delta_{\text{Frob}}(1) = 1 \otimes 1$ where Δ_{Frob} is the comultiplication of $V_{\mathbf{M}}$ coming from the Frobenius algebra structure of $V_{\mathbf{M}}$. □

Recall that $1_{A_{\mathbf{T}}} = \sum_i \theta_i(1)t_i$. The following lemma is immediate.

Lemma 2.1.0.11. $\sum_i d_i$  and  evaluate to the same vector in $V_{\mathbf{M}}$.

The Hopf Structure

We now show that the comultiplication, counit, and antipode maps are compatible with the multiplication.

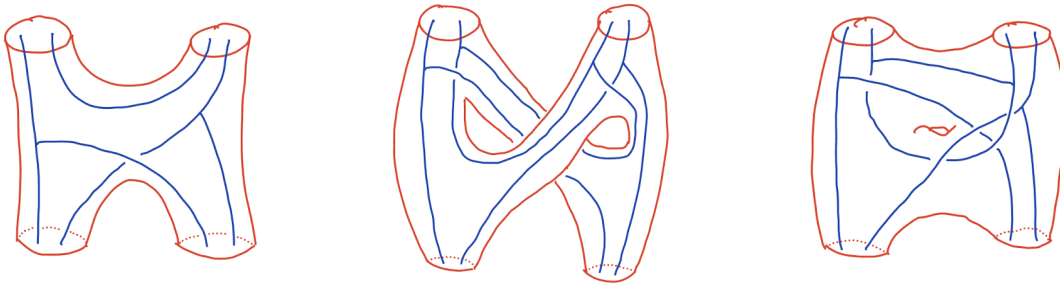


Figure 2.10: $\Delta(xy)$, $\Delta(x)\Delta(y)$, and $\Delta(x)\Delta(y)$ deformed.

First, we show that $\Delta(xy) = \Delta(x)\Delta(y)$ for all $x, y \in H$. This amounts to showing that the first two bordisms of Figure 2.10 evaluate to the same map under $Z_{\mathbf{T}}$.



Figure 2.11: Showing the bialgebra axiom holds.

Deforming the second bordism to the third bordism, we see that it suffices to show that the two pieces shown in Figure 2.11 evaluate to the same linear map.

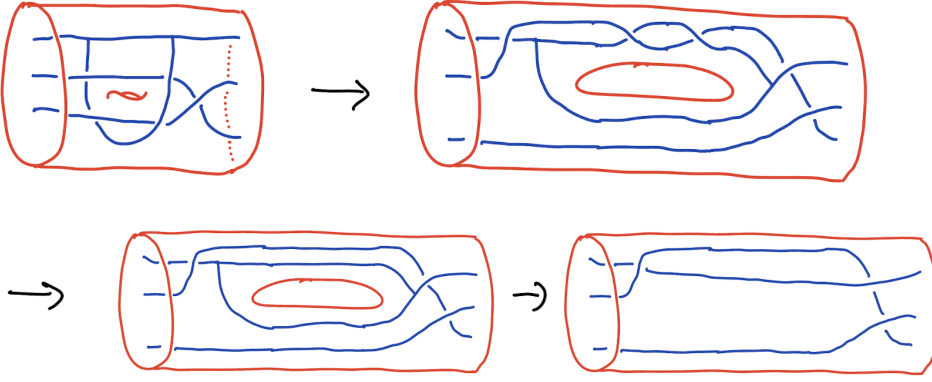


Figure 2.12: Showing the bialgebra axiom holds.

We show the process in figure 2.12. The first step is a simple diffeomorphism. The second step uses the fact that since  has no boundary, any bordism induced by a isotopy from the identity to itself is diffeomorphic⁴ to the trivial bordism. In particular, the double braiding in the top pipe can be removed as  is diffeomorphic rel boundary to the trivial bordism  $\rightarrow$ .

In the third step we cancel the blue loop with the red hole using Lemma 2.1.0.10. The result is related to the left bordism in Figure 2.11 by an isotopy of the identity on , and therefore must evaluate to the same linear map.

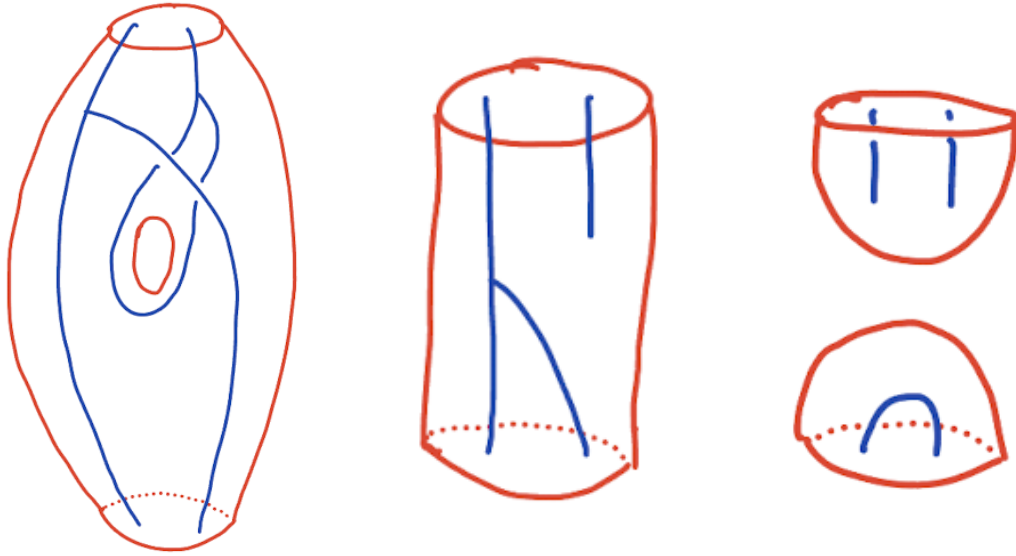
We have shown that H is a bialgebra with our chosen operations. Next, we show that

$S = Z_{\mathbf{T}} \left(\text{Diagram of a cylinder with a double braid} \right)$ is the antipode. It is clear that S is an antihomomorphism. We

must show that $S(h_{(1)})h_{(2)} = \epsilon(h)1 = h_{(1)}S(h_{(2)})$ for all $h \in H$.

To show that $S(h_{(1)})h_{(2)} = \epsilon(h)1$ for all $h \in H$, we take the first bordism of Figure 2.13, and fill in the hole via Lemma 2.1.0.10 to obtain the second bordism.

⁴The diffeomorphism preserves orientation but not necessarily the framed bordism class.


 Figure 2.13: $S(h_{(1)}h_{(2)})$, after removing hole, and $\epsilon(h)1$.

In order to reach the third bordism, we will show that the two bordisms  and  evaluate to the same map $\mathbb{C} \rightarrow H$, namely the Hopf unit. Precomposing both bordisms with the algebra unit of $V_{\mathbf{M}} \cong \mathbb{C}$ (Figure 2.14, the first bordism is immediately diffeomorphic to the bordism expressing the unit of H , and the second bordism is the unit multiplied by the scalar $t(1) = 1$ where $1 \in V_{\mathbf{M}}$ and t is the Frobenius trace.

2.2 The Weak Hopf Algebra

Recall from Lemma 2.1.0.8 the decomposition

$$W = \bigoplus_{ijkl} W_{ijkl}$$

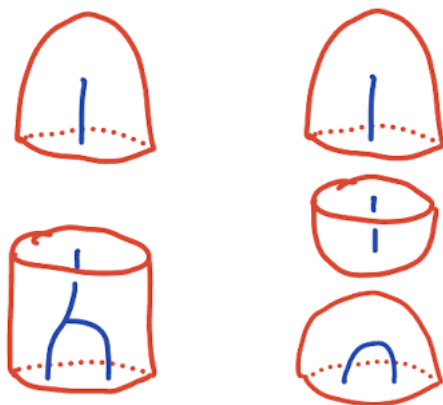
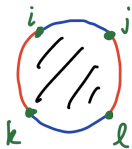


Figure 2.14: The two bordisms  and  evaluate to the same map.

and observe that W_{ijkl} is given by . We may therefore compute that

$$W_{ijkl} = \text{Hom}_{\mathbf{T}}(\underline{\text{Hom}}(m_i, m_j), \underline{\text{Hom}}(m_k, m_l))$$

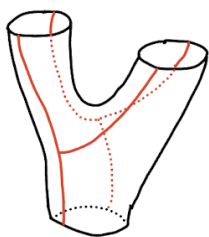


Figure 2.15: Multiplication

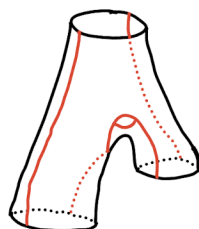
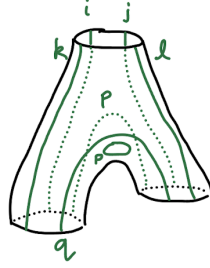


Figure 2.16: Comultiplication


 Figure 2.17: Δ_{ijkl}^{pq} with green defects highlighted

We have natural associative “composition” maps $\mu_{ij i' j'}^{kl} : W_{ijkl} \otimes W_{kl i' j'} \rightarrow W_{ij i' j'}$ (see Figure 2.15 for basic shape), which build the multiplication $\mu : W \otimes W \rightarrow W$.

$$\mu = \sum_{ij i' j'} \bigoplus_{kl} \mu_{ij i' j'}^{kl}$$

Each W_{ijij} is a subalgebra with unit 1_{ij} given by



, and $1 = \sum_{ij} 1_{ij}$ is the

unit for W .

The comultiplication is built up from maps $\Delta_{ijkl}^{pq} : W_{ijkl} \rightarrow W_{ipkq} \otimes W_{pqjl}$, defined by Figure 2.17.

The comultiplication $\Delta : W \rightarrow W \otimes W$ is then

$$\Delta = \sum_{ijkl} \bigoplus_{pq} \Delta_{ijkl}^{pq}$$

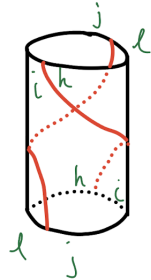
The bordism



defines a map $\epsilon_{ij} : W_{iiij} \rightarrow \mathbb{C}$, and the counit is given by

$$\epsilon = \sum_{i,j} \frac{1}{\dim m_i} \epsilon_{ij},$$

Finally, the bordism



defines a map $S_{ijkl} : W_{ijkl} \rightarrow W_{lkji}$, and the

antipode is given by $S = \sum_{ijkl} \frac{d_j}{d_i} S_{ijkl}$.

The Weak Hopf structure on W

We claim that the previously described operations on  equip W with the structure of a weak Hopf algebra.

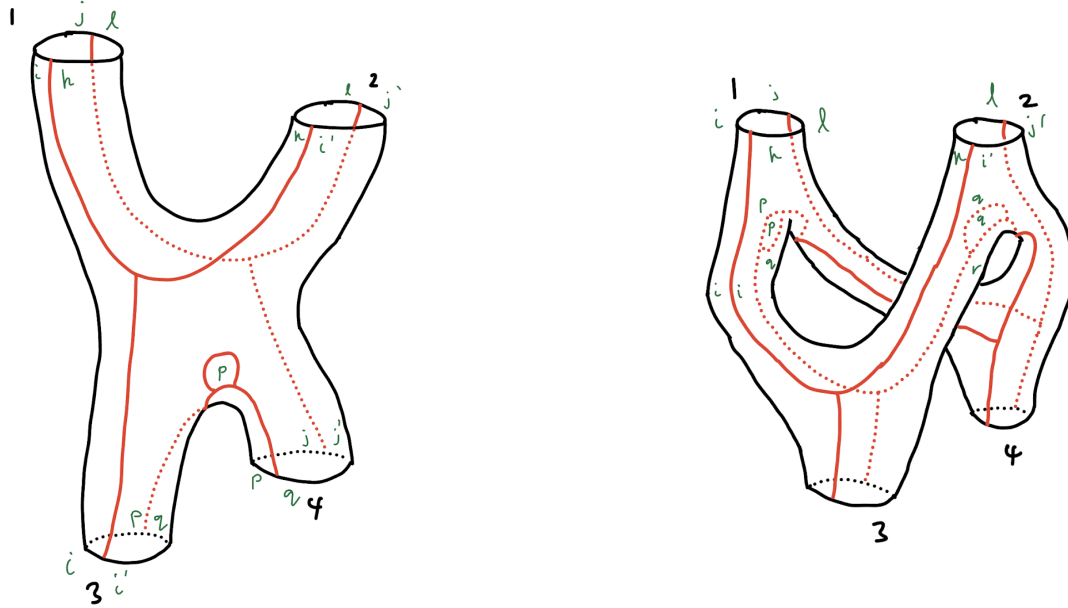


Figure 2.18: $b_1(p, q)$ and $b_2(p, q, r)$.

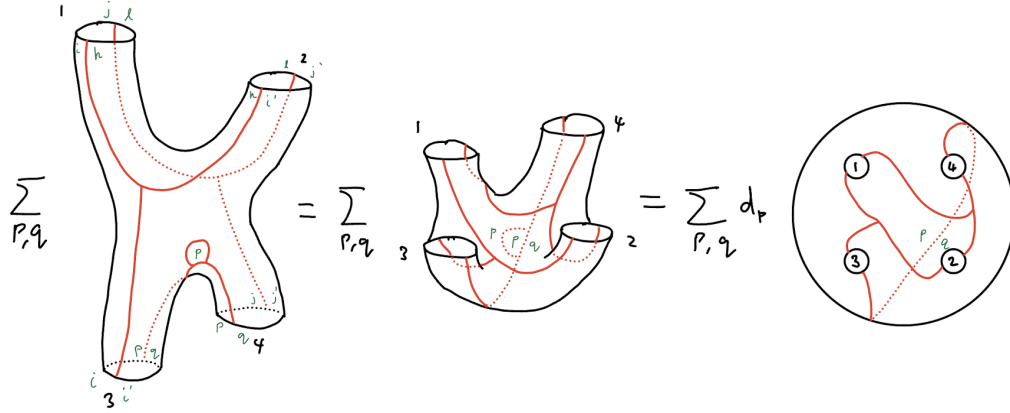
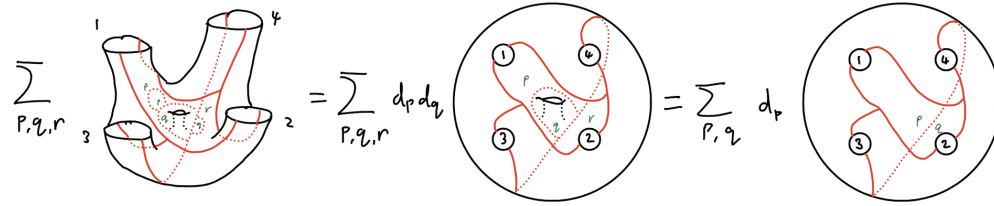
Proof. First we check Axiom 1.

Fix i, j, k, l, i', j' and define the bordisms $b_1(p, q)$ and $b_2(p, q, r)$ as in Figure 2.18.

We wish to show that $\sum_{p,q} Z_{\mathbf{T}}(b_1)$ and $\sum_{p,q,r} Z_{\mathbf{T}}(b_2)$ are equal. We do this in figures 2.19, 2.20.

For Axiom 2, the only surgery needed is to delete blue disks on the boundary to get scalars. Figure 2.21), shows the process.

Axiom 3 is simplest, all three sides of the equation are given by $\sum_{p,p',q,q'} b$ where b is the bordism of Figure 2.22.


 Figure 2.19: Deforming b_1 .

 Figure 2.20: Deforming b_2 .

Finally, for Axiom 4, Figure 2.23 shows from left to right: The bordism evaluating to $\epsilon(1_{(1)}x)1_{(2)}$, the same bordism with the green defects highlighted, the green circle we will surger out highlighted (yellow), the result of the surgery.

Figure 2.24 shows from left to right: The bordism evaluating to $\epsilon(1_{(1)}x)1_{(2)}$, the same bordism with the green defect highlighted, and the same bordism "straightened out".

We see that the final bordisms of Figures 2.23, 2.24 are related by a diffeomorphism that is the identity on the boundaries (a full twist of the bottom disk). We conclude that $\epsilon(1_{(1)}x)1_{(2)} = \epsilon(1_{(1)}x)1_{(2)}$ for all $x \in H$, as we wanted.

Given Axioms 1,2,3 and $\epsilon(1_{(1)}x)1_{(2)} = \epsilon(1_{(1)}x)1_{(2)}$, the final axiom $S(x_{(1)})x_{(2)}S(x_{(3)}) = S(x)$ is equivalent to $S(x)S(y) = S(yx)$ and $\Delta(S(x)) = S(x)_{(2)} \otimes S(x)_{(1)}$. The last two equations are immediate from the two diffeomorphic bordisms of Figure 2.25.

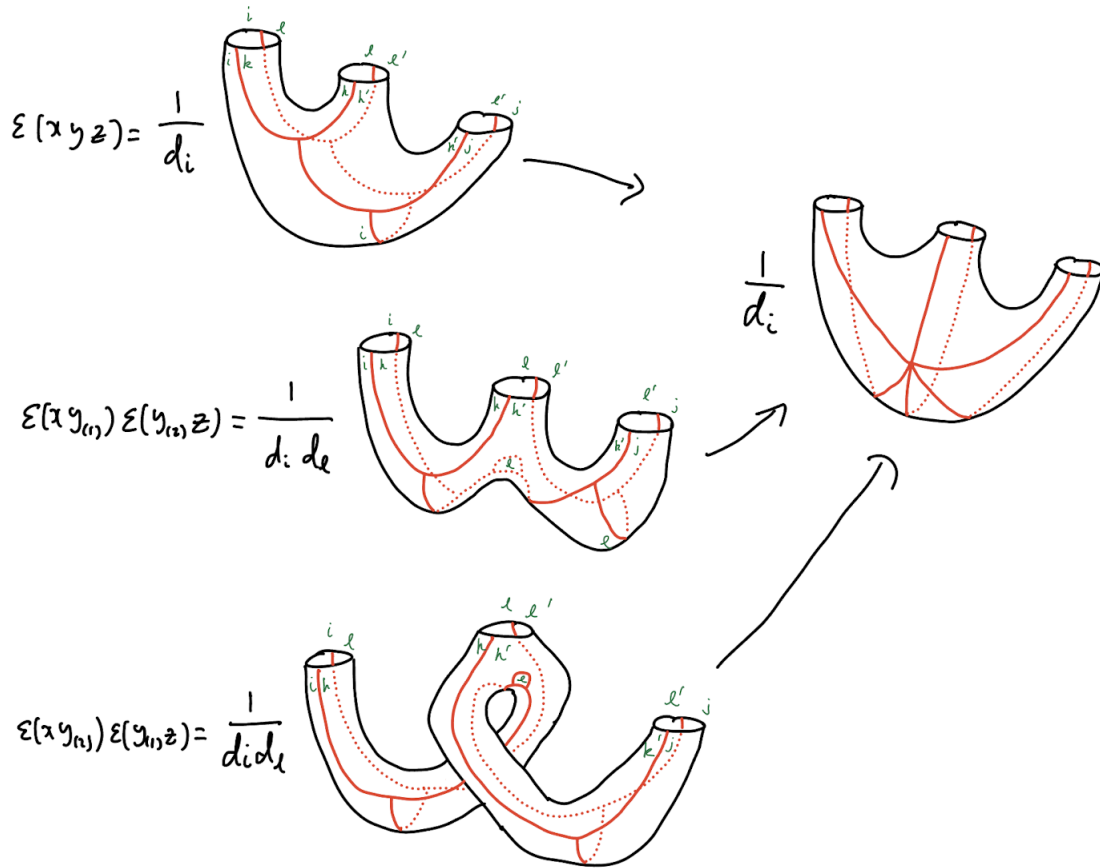
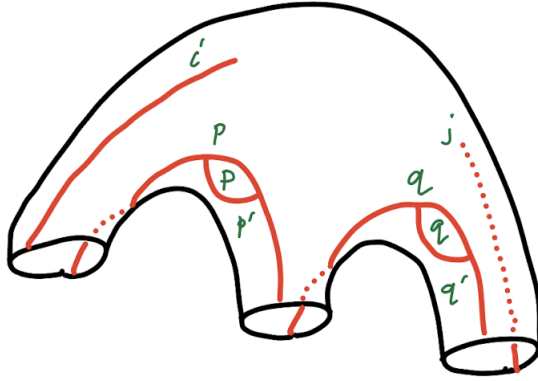
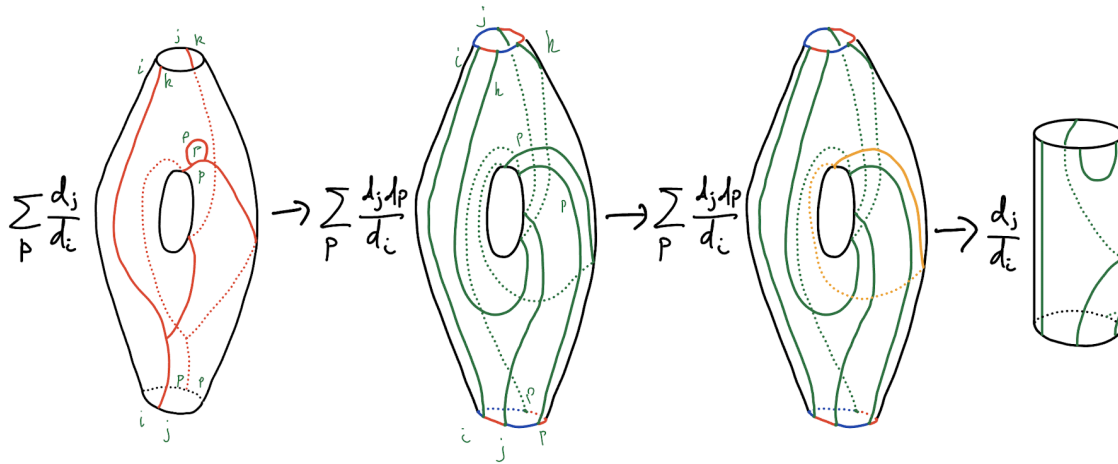
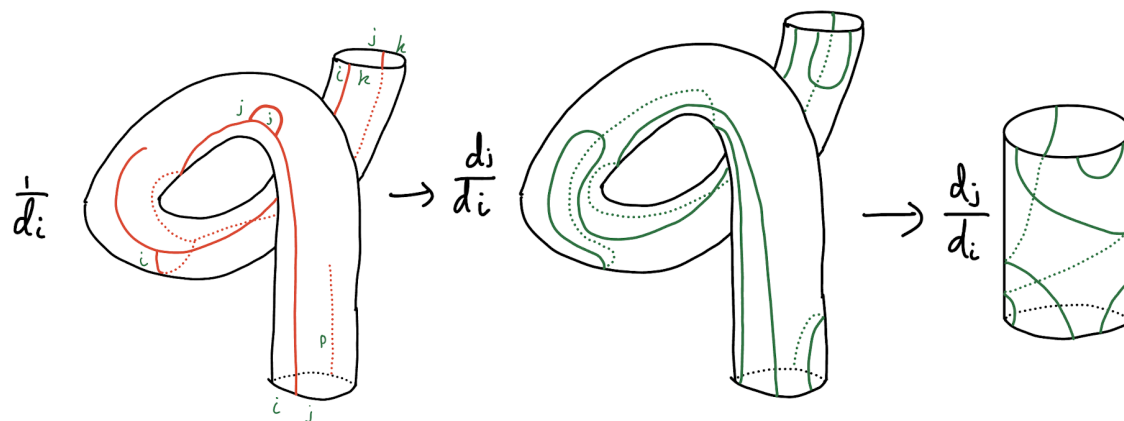
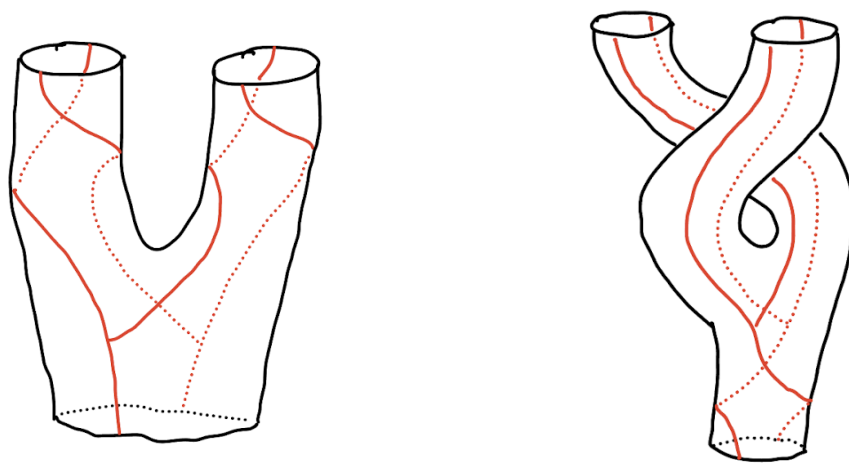


Figure 2.21: Weak multiplicativity of the counit.


 Figure 2.22: Bordism b , for checking the weak comultiplicativity of the unit.

 Figure 2.23: $x_{(1)}S(x_{(2)})$

□


 Figure 2.24: $\epsilon(1_{(1)}x)1_{(2)}$

 Figure 2.25: The diffeomorphic bordisms for $S(x)S(y)$ and $S(yx)$.

Bibliography

- [1] Michael Atiyah. “New invariants of 3- and 4-dimensional manifolds”. en. In: *Proceedings of Symposia in Pure Mathematics*. Ed. by R. Wells. Vol. 48. Providence, Rhode Island: American Mathematical Society, 1988, pp. 285–299. ISBN: 978-0-8218-1482-6. DOI: 10.1090/pspum/048/974342. URL: <http://www.ams.org/pspum/048> (visited on 12/21/2025).
- [2] Michael F Atiyah. “Topological quantum field theory”. en. In: *Publications Mathématiques de l’IHÉS* 68 (1988), pp. 175–186.
- [3] John C. Baez and James Dolan. “Higher-dimensional Algebra and Topological Quantum Field Theory”. In: *Journal of Mathematical Physics* 36.11 (Nov. 1995). arXiv:q-alg/9503002, pp. 6073–6105. ISSN: 0022-2488, 1089-7658. DOI: 10.1063/1.531236. URL: <http://arxiv.org/abs/q-alg/9503002> (visited on 11/22/2025).
- [4] Bruce Bartlett. “Fusion categories via string diagrams”. In: *Communications in Contemporary Mathematics* 18.05 (Oct. 2016), p. 1550080. ISSN: 0219-1997. DOI: 10.1142/S0219199715500807. URL: <https://www.worldscientific.com/doi/10.1142/S0219199715500807> (visited on 03/19/2025).
- [5] G. Bohm, F. Nill, and K. Szlachanyi. *Weak Hopf Algebras I: Integral Theory and C^* -structure*. arXiv:math/9805116. June 1999. DOI: 10.48550/arXiv.math/9805116. URL: <http://arxiv.org/abs/math/9805116> (visited on 03/20/2024).
- [6] Tudor Dimofte and Wenjun Niu. *Tannakian QFT: from spark algebras to quantum groups*. arXiv:2411.04194 [hep-th]. Nov. 2024. DOI: 10.48550/arXiv.2411.04194. URL: <http://arxiv.org/abs/2411.04194> (visited on 08/13/2025).
- [7] Christopher Douglas, Christopher Schommer-Pries, and Noah Snyder. *Dualizable tensor categories*. en. ISBN: 9781470463472 9781470463489 9781470443610 ISSN: 0065-9266, 1947-6221 Issue: 1308 Series: Memoirs of the American Mathematical Society Volume: 268. Nov. 2020. DOI: 10.1090/memo/1308. URL: <https://www.ams.org/memo/1308/> (visited on 05/13/2024).

- [8] Christopher L. Douglas, Christopher Schommer-Pries, and Noah Snyder. “The balanced tensor product of module categories”. In: *Kyoto Journal of Mathematics* 59.1 (Apr. 2019). arXiv:1406.4204 [math]. ISSN: 2156-2261. DOI: 10.1215/21562261-2018-0006. URL: <http://arxiv.org/abs/1406.4204> (visited on 10/18/2025).
- [9] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. *An analogue of Radford’s S^4 formula for finite tensor categories*. arXiv:math/0404504. Apr. 2004. DOI: 10.48550/arXiv.math/0404504. URL: <http://arxiv.org/abs/math/0404504> (visited on 04/10/2025).
- [10] Pavel Etingof, Dmitri Nikshych, and Viktor Ostrik. “On Fusion Categories”. In: *Annals of Mathematics* 162.2 (2005), pp. 581–642. ISSN: 0003-486X. URL: <https://www.jstor.org/stable/20159926> (visited on 03/19/2025).
- [11] Pavel Etingof et al. *Fusion categories and homotopy theory*. arXiv:0909.3140 [math]. Dec. 2009. DOI: 10.48550/arXiv.0909.3140. URL: <http://arxiv.org/abs/0909.3140> (visited on 08/30/2025).
- [12] Pavel Etingof et al. *Tensor categories*. en. Mathematical surveys and monographs volume 205. Providence, Rhode Island: American Mathematical Society, 2015. ISBN: 978-1-4704-3441-0.
- [13] Daniel S. Freed. *Lectures on Field Theory and Topology*. en. Google-Books-ID: 2DurDwAAQBAJ. American Mathematical Soc., Aug. 2019. ISBN: 978-1-4704-5206-3.
- [14] Daniel S. Freed and Constantin Teleman. “Gapped boundary theories in three dimensions”. In: *Communications in Mathematical Physics* 388.2 (Dec. 2021). arXiv:2006.10200 [cond-mat, physics:hep-th, physics:math-ph], pp. 845–892. ISSN: 0010-3616, 1432-0916. DOI: 10.1007/s00220-021-04192-x. URL: <http://arxiv.org/abs/2006.10200> (visited on 11/10/2023).
- [15] Daniel S. Freed and Constantin Teleman. “Topological dualities in the Ising model”. In: *Geometry & Topology* 26.5 (Dec. 2022). arXiv:1806.00008 [cond-mat, physics:hep-th, physics:math-ph], pp. 1907–1984. ISSN: 1364-0380, 1465-3060. DOI: 10.2140/gt.2022.26.1907. URL: <http://arxiv.org/abs/1806.00008> (visited on 12/03/2023).
- [16] Takahiro Hayashi. *A canonical Tannaka duality for finite seimisimple tensor categories*. arXiv:math/9904073. Apr. 1999. DOI: 10.48550/arXiv.math/9904073. URL: <http://arxiv.org/abs/math/9904073> (visited on 03/19/2024).

- [17] Theo Johnson-Freyd and Claudia Scheimbauer. *(Op)lax natural transformations, twisted quantum field theories, and "even higher" Morita categories*. en. Feb. 2015. DOI: 10.1016/j.aim.2016.11.014. URL: <https://arxiv.org/abs/1502.06526v3> (visited on 05/05/2026).
- [18] Jacob Lurie. *On the Classification of Topological Field Theories*. arXiv:0905.0465 [math]. May 2009. DOI: 10.48550/arXiv.0905.0465. URL: <http://arxiv.org/abs/0905.0465> (visited on 11/22/2025).
- [19] Gregory W. Moore and Graeme Segal. *D-branes and K-theory in 2D topological field theory*. arXiv:hep-th/0609042. Sept. 2006. DOI: 10.48550/arXiv.hep-th/0609042. URL: <http://arxiv.org/abs/hep-th/0609042> (visited on 05/27/2025).
- [20] Michael Mueger. *Tensor categories: A selective guided tour*. arXiv:0804.3587 [math]. June 2010. DOI: 10.48550/arXiv.0804.3587. URL: <http://arxiv.org/abs/0804.3587> (visited on 12/30/2025).
- [21] Michael Müger. "From subfactors to categories and topology II: The quantum double of tensor categories and subfactors". In: *Journal of Pure and Applied Algebra* 180.1 (May 2003), pp. 159–219. ISSN: 0022-4049. DOI: 10.1016/S0022-4049(02)00248-7. URL: <https://www.sciencedirect.com/science/article/pii/S0022404902002487> (visited on 12/21/2025).
- [22] Dmitri Nikshych. "Semisimple weak Hopf algebras". In: *Journal of Algebra* 275.2 (May 2004), pp. 639–667. ISSN: 0021-8693. DOI: 10.1016/j.jalgebra.2003.09.025. URL: <https://www.sciencedirect.com/science/article/pii/S0021869303006823> (visited on 05/27/2025).
- [23] Victor Ostrik. "Module categories, weak Hopf algebras and modular invariants". en. In: *Transformation Groups* 8.2 (June 2003), pp. 177–206. ISSN: 1531-586X. DOI: 10.1007/s00031-003-0515-6. URL: <https://doi.org/10.1007/s00031-003-0515-6> (visited on 12/31/2025).
- [24] David E. Radford. "The Order of the Antipode of a Finite Dimensional Hopf Algebra is Finite". In: *American Journal of Mathematics* 98.2 (1976), pp. 333–355. ISSN: 0002-9327. DOI: 10.2307/2373888. URL: <https://www.jstor.org/stable/2373888> (visited on 12/13/2025).
- [25] David Reutter. *Frobenius algebras, Hopf algebras and 3-categories*. en. Perimeter Institute, Canada, Aug. 2017.

- [26] G. B. Segal. “The Definition of Conformal Field Theory”. en. In: *Differential Geometrical Methods in Theoretical Physics*. Ed. by K. Bleuler and M. Werner. Dordrecht: Springer Netherlands, 1988, pp. 165–171. ISBN: 978-94-015-7809-7. DOI: 10.1007/978-94-015-7809-7_9. URL: https://doi.org/10.1007/978-94-015-7809-7_9 (visited on 12/21/2025).
- [27] William Bruce Stewart. “Topological domain walls and relative field theories”. en. In: (Aug. 2024). URL: <https://repositories.lib.utexas.edu/items/dbfc86cd-79bf-4cca-867b-ae9624095740> (visited on 11/22/2025).
- [28] Kornél Szlachányi. “Finite quantum groupoids and inclusions of finite type”. In: *Mathematical physics in mathematics and physics (Siena, 2000)*. Vol. 30. Fields Inst. Commun. Amer. Math. Soc., Providence, RI, 2001, pp. 393–407. ISBN: 978-0-8218-2814-4.
- [29] Daisuke Tambara and Shigeru Yamagami. “Tensor Categories with Fusion Rules of Self-Duality for Finite Abelian Groups”. In: *Journal of Algebra* 209.2 (Nov. 1998), pp. 692–707. ISSN: 0021-8693. DOI: 10.1006/jabr.1998.7558. URL: <https://www.sciencedirect.com/science/article/pii/S0021869398975585> (visited on 04/17/2026).
- [30] Edward Witten. “Supersymmetry and Morse theory”. en. In: *Journal of Differential Geometry* 17.4 (Jan. 1982). ISSN: 0022-040X. DOI: 10.4310/jdg/1214437492. URL: <https://projecteuclid.org/journals/journal-of-differential-geometry/volume-17/issue-4/Supersymmetry-and-Morse-theory/10.4310/jdg/1214437492.full> (visited on 12/21/2025).