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Connected components of attractors and other stable sets

MORRIS W. HIRSCH AND MIKE HURLEY

Summary. We study the action of a continuous map f on a locally connected metric space, with an emphasis on the action induced by f on the space of connected components of an attractor A or a stable set Y of f . The main result for attractors is that, if A is an attractor in the sense of C. Conley, then f permutes the components of A and that the permutation is cyclic if and only if the attractor is indecomposable. This is related to similar results obtained recently by Buescu and Stewart. In the case of an indecomposable stable invariant set Y we show that either the action of f on the components of Y is as described above for the action on the attractor, or else Y has infinitely many components, the action of f on them is a generalized adding machine, so that there are no periodic points in Y . Results along these lines are due to Buescu and Stewart and to Melbourne, Dellnitz and Golubitsky.

1. Introduction and statement of results

An important part of the study of a dynamical system is to attempt to understand the structure of the ‘attractors’ of the system. There have been many definitions given of the term ‘attractor’, for example in [1, 2, 10, 11, 12] (the last two references contain a good summary as of 1985). As one would expect given the variety of definitions, there is no one definition of attractor that is entirely appropriate in all examples, and so it would be a hopeless task to try to classify all attractors. Nonetheless there has been recent progress [1, 10] in understanding the action of the dynamical system on the space of connected components of a compact attractor. The space of components is obtained from A by collapsing each component to a point, and then using the quotient topology. In an obvious way f induces a map on the space of components; abusing notation we will also refer to this induced map as f . The purpose of this note is to combine and generalize results of this type that have appeared in [1, 10]. The general assumptions in [1] are:

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- (1.1) X is a metric space with metric d , and f is a continuous map from X to itself, and A is a nonempty, compact, invariant set for f .
- (1.2) X is locally connected.
- (1.3) X is locally compact.

Buescu and Stewart in [1] show that, if (1.1) through (1.3) hold and if A is asymptotically stable and has a dense orbit, then f cyclically permutes the component of A . In fact, they go further and show that as long as (1.1) through (1.3) hold, if A is merely stable (and contains a dense orbit) then the action of f on the space of components is either cyclic, as in the asymptotically stable case, or else is a generalized adding machine.

The two main results of this paper (theorems A and B) are similar to the above. Throughout the paper we will assume (1.1); most of our results about attractors also require (1.2) and (1.3). The results are as follows; definitions and discussion can be found in the next section.

THEOREM A. *Assume 1.1 through 1.3. If the compact invariant set A is an attractor in the sense of Conley, then A has a finite number of connected components, and these are permuted by f . The permutation is cyclic if and only if the attractor A is indecomposable.*

Because X is locally compact, the condition that A is a Conley attractor is equivalent to the condition that A is asymptotically stable. Thus:

COROLLARY. *Assume 1.1 through 1.3 and that A is asymptotically stable. Then the conclusion of the theorem is valid.*

The same result but with indecomposability replaced by the stronger assumption that the attractor contains a dense orbit is in [1]. A chain transitive attractor is necessarily indecomposable, so we have the following corollary. (Chain transitivity means that there are dense ‘pseudo-orbits’ in A ; it is a natural minimality condition on an attractor, as we will explain.)

COROLLARY. *If A is compact, asymptotically stable, and chain transitive, then f acts as a cyclic permutation on the (finitely many) components of A .*

The corollary extends similar results of Buescu and Stewart, [1], where the stronger assumption of transitivity was used instead of chain transitivity. The

theorem and corollaries were described in [4] by the first author; the current paper supersedes [4], which will not be published.

THEOREM B. *Assume 1.1, 1.2 and suppose that Y is a nonempty, compact, indecomposable, stable invariant set. Then either the conclusion of theorem A holds with Y in place of A , or else Y has infinitely many components and the action of f on the space of components is a generalized adding machine, as described in [1]. In the latter case, there are no periodic points of f in Y .*

Theorem B extends one of the chief results of [1], which was that the conclusion of theorem B holds with the additional hypotheses that Y contains a dense orbit, and that X is locally compact. Melbourne, Dellnitz, and Golubitsky have obtained similar results [10], among them that if A is a stable set which is an omega-limit set of f then A is connected if the restriction of f to A is topologically mixing, and that A has at most k components if it contains a point of period k . An omega-limit set is chain transitive and so is indecomposable, so theorem B applies to such a set A provided that A is compact; the argument of [10] does not require compactness.

When X is not locally compact a Conley attractor is not necessarily stable, so theorem B does not imply a more general version of theorem A, at least not without stronger assumptions on A than that it is a Conley attractor. The situation is discussed in the next-to-last section of the paper.

Examples of stable adding machines can be found in [1].

2. Discussion and definitions

Here are the main assumptions and definitions; throughout this paper we assume condition 1.1; 1.2 (local connectedness) is needed except for the preliminary material in sections 3 and 4, and we will generally assume 1.3 (local compactness) except in sections 6 and 7. (Some of our results do not refer to an invariant set ' A '; in such results the parts of assumption 1.1 referring to A are irrelevant.) Saying that A is *invariant* means that A is nonempty and $f(A) = A$. An invariant set A is said to be (Lyapunov) *stable* if for any neighborhood G of A there is a smaller neighborhood U of A with the property that $f^n(U) \subset G$ for all $n \geq 0$. A is called *asymptotically stable* if it is stable and if in addition there is a neighborhood W of A with the property that the omega-limit set of x is contained in A whenever $x \in W$. (The omega-limit set of x is defined by $\bigcap_{k \geq 0} \overline{\bigcup_{j \geq k} f^j(x)}$.) We say that A is an *attractor in the sense of Conley* (or, more succinctly, a *Conley attractor*) if there is an open neighborhood U of A with the two properties $f(U) \subset U$, and $\bigcap_{n \geq 0} f^n(U) = A$. (The same definition applies even if A is not compact, but we only consider compact attractors here.) In

locally compact spaces there are several equivalent definitions of a compact Conley attractor, among them is asymptotic stability. A number of these equivalent conditions will be described in section 3. At times for brevity we will drop all modifiers and refer to an 'attractor'; throughout this note it is to be understood that the only 'attractors' that are being considered are (nonempty) attractors in the sense of Conley, and that these attractors are compact.

An attractor is *indecomposable* if it is not the disjoint union of two proper, closed subsets that are also attractors. The definition for stable sets is analogous. An equivalent definition is that the corresponding invariant set for the suspension semiflow generated by f is connected. We show that if an attractor either is smallest by inclusion (among attractors) or is chain transitive then it is indecomposable; the same result holds with stable sets in place of attractors. A compact subset $Y \subset X$ is *chain transitive* if for any ordered pair (p, q) of points of Y and any $\varepsilon > 0$ there is a finite sequence $y_0, \dots, y_n$ of points of Y with the properties: $y_0 = p$, $y_n = q$, and $d(f(y_j), y_{j+1}) < \varepsilon$ for all $0 \leq j < n$. Chain transitivity is plainly a weaker condition than transitivity.

When Y is a forward invariant set for f (i.e., when $f(Y) \subset Y$), then the fact that a continuous map sends connected sets to connected sets means that each connected component of Y is mapped into some other connected component of Y . Thus f induces a map on the quotient space of Y obtained by collapsing each connected component to a point. When there is no danger of misunderstanding, we will usually refer to this induced map as f also.

Theorem B refers to (generalized) *adding machines*, which are a class of transitive homeomorphisms of Cantor sets. The simplest adding machine is the binary one, which is defined as follows. Let $X = \{\underline{x}\}$ be the set of all sequences of 0's and 1's, $\underline{x} = (x_1, x_2, \dots)$. A map g from X to itself is obtained by sending each x to the sequence obtained by adding 1 to x_1 , and carrying to the right. The detailed definition of g is

$$\underline{y} = g(\underline{x}) \text{ where } \begin{cases} \text{if } x_n = 1 \text{ for all } n \text{ then } y_n = 0 \text{ for all } n, \\ \text{if } k \geq 1 \text{ satisfies both } x_k = 0 \text{ and } x_n = 1 \text{ for all } n < k, \\ \text{then } y_k = 1, y_n = 0 \text{ when } n < k, \text{ and otherwise } y_n = x_n. \end{cases}$$

A general adding machine is defined similarly, but allows the 'base' $k_n \geq 2$ to vary with n . (In the binary adding machine, $k_n = 2$ for each n .) See [1] for further details. Naturally, the result of theorem B is to be interpreted to mean that under certain circumstances the action of f on the space of components of some compact invariant set is conjugate to an adding machine as described above.

Clearly if f is invertible then our results apply to 'repelling' subsets for f (those that are attracting for f^{-1}). When f is not invertible the dynamics on repelling sets

can be more complicated. For example, there are quadratic polynomials p on whose Julia sets are repelling Cantor sets and the restriction of p to the Julia set is conjugate to a one-sided shift on two symbols, so it is neither cyclic nor an adding machine.

Here is a brief outline of the paper. The next (third) section of the paper describes the two consequences of local compactness that are important for the rest of the paper. The first of the two results is that a compact intersection of nested closed, connected sets is connected, and the second is a nine-fold characterization of a compact Conley attractor. Other than referring to these two results, the arguments in the rest of the paper do not explicitly make use of local compactness.

Section 4 develops some ideas concerning indecomposability for compact attractors and stable invariant sets. The fifth section contains the proof of theorem A and its corollaries, while theorem B is proved in section 6. Section 7 is a brief discussion of what can be salvaged from our arguments about attractors when X is not locally compact.

3. Consequences of local compactness

In our results concerning attractors, we will usually assume that X is locally compact (condition 1.3), even though most of our intermediate results do not require local compactness. We rely on local compactness for two things. First, in a locally compact space, if a nested intersection of closed connected sets has a compact intersection, then the intersection is also connected. This fact is the basic observation that leads to the proof of theorem A. In fact this is the only way that this theorem requires local compactness. The second use of local compactness is to establish a number of different conditions on a compact invariant set that are equivalent to the set being a Conley attractor; this equivalence is used in establishing the corollaries to theorem A that were given in the introduction.

LEMMA 3.1. *Assume 1.1 (X is a metric space), and suppose that C_n are connected subsets of X with $C_{n+1} \subset C_n$ for each n . In addition, suppose that $C \equiv \bigcap_{n>0} C_n$ is nonempty, and that for any neighborhood N of C there is an n with $C_n \subset N$. Then C is connected.*

Proof. Suppose that C is covered by a pair of disjoint neighborhoods U and U' , and let $N \equiv U \cup U'$. By assumption, $C_n \subset N$ for large n , which means either $C \subset C_n \subset U$ or else $C \subset C_n \subset U'$. It follows that C must be disjoint from one of U, U' , and so is connected. $\square$

PROPOSITION 3.2. *Assume 1.1 and 1.3 (X is metric and locally compact), and suppose that C_n are connected subsets of X , nested as in the lemma, and that $C \equiv \bigcap_{n \geq 0} \overline{C_n}$ is nonempty and compact. Then C is connected.*

Proof. We show that the hypotheses of lemma 3.1 are satisfied. Local compactness ensures that any neighborhood of C contains a compact neighborhood of C , so it is enough to consider the case where N is a compact neighborhood of C . The fact that $\bigcap_{n \geq 0} \overline{C_n} = C$ means that for large n , $\overline{C_n}$ will be contained in $N \cup W$ where W is some open neighborhood of infinity (i.e., the complement of W is compact) that is disjoint from N . Because $N \cup W$ is disconnected while $\overline{C_n}$ is connected and meets N , we conclude that $\overline{C_n} \subset N$ for all large n . Now apply lemma 3.1. $\square$

COROLLARY. *Keeping the notation of the proposition, if any of the sets C_n is compact, then C is connected even without assumption 1.3 (local compactness of X).*

Proof. If C_k is compact, then apply the proposition to the nested sequence $C_k \supset C_{k+1} \cdots$ with C_k taking the place of the ambient space X . $\square$

A second useful consequence of local compactness is the following proposition that gives a nine-fold characterization of when a compact invariant set A is an attractor in the sense of Conley. The results of the proposition are known, at least for X compact, but they seem not to have been collected in any one place in the literature; see [1, 2, 3, 5, 6, 11, 12].

PROPOSITION 3.3. *In the following we assume 1.1 and 1.3 (X is locally compact and metric, and A is compact). The following conditions are equivalent.*

- (1) *A is a Conley attractor. This means that there is an open neighborhood U of A with $\overline{f(U)} \subset U$ and $A = \bigcap_{n \geq 0} f^n(U) = \bigcap_{n \geq 0} f^n(\overline{U}) = \bigcap_{n \geq 0} \overline{f^n(U)}$.*
- (2) *If G is any neighborhood of A , then there is an open neighborhood U of A with $U \subset G$, and with U satisfying the same conditions as in (1).*
- (3) *A is an attractor in the sense of [4]: this means that there is a neighborhood U of A as in (1), with the additional property that $\overline{f(U)}$ is compact.*
- (4) *If G is any neighborhood of A , then there is an open neighborhood U of A with $\bigcup_{n \geq 0} f^n(U) \subset G$, and such that the omega-limit set of U is equal to A . (The omega-limit set of U is defined to be $\bigcap_{k \geq 0} \overline{\bigcup_{j \geq k} f^j(U)}$.)*
- (5) *A is stable and is the largest invariant set in some neighborhood of itself.*
- (6) *A is both stable and asymptotically stable.*
- (7) *A is strongly asymptotically stable; this is the same as condition (6) with the additional assumption that the distance from $f^n(x)$ to A goes to 0 for each x in some neighborhood of A .*

- (8) If G is any neighborhood of A , then there is a local Lyapunov function λ for A that is defined on a neighborhood of A in G . This means that there is a neighborhood V_0 of A such that λ is a continuous function defined on $V \equiv \bigcup_{n \geq 0} f^n(V_0) \subset G$, taking on values in $[0, 1]$, and with the properties
- (a) $\lambda^{-1}(0) = A$.
 - (b) If $x \in V - A$ then $\lambda(f(x)) < \lambda(x)$.
- (9) There is a forward invariant open neighborhood U of A with the property that if W is any neighborhood of A then $f^n(U) \subset W$ for all large n .

The proof of the proposition is long, and the arguments are neither difficult nor particularly relevant to the main exposition, so we omit the proof.

4. Indecomposability conditions

In the next few sections we will be showing that certain hypotheses about the dynamics of f restricted to an attractor, or to a stable invariant set, imply minimality for the induced map on the space of components (i.e., that every orbit is dense). A useful way of viewing these assumptions is to think of them as ‘dynamic connectedness’ assumptions, i.e., that an f -invariant set Y is ‘connected’ if we consider not only the topology of Y but also the action of f . Perhaps the simplest way of making sense of this is to actually connect each f -orbit by considering the *suspension semiflow* for f . To define this semiflow, let X' be the quotient space of $X \times [0, 1]$ obtained by identifying $(x, 1)$ with $(f(x), 0)$, and define a semiflow ϕ on X' by the equation

$$\phi(t, (x, s)) = (f^k(x), t + s - k),$$

where k is the positive integer that is defined by $0 \leq t + s - k < 1$. We define an f -invariant set Y to be ‘dynamically connected’, or *indecomposable*, if the corresponding invariant set Y' for the suspension flow is connected.

LEMMA 4.1. Assume 1.1, so that X is metric and A is compact.

- (a) Suppose in addition that X is locally compact and that A is an attractor. A is decomposable if and only if it is the disjoint union of two proper closed subsets that are also attractors.
- (b) Suppose that A is stable. A is decomposable if and only if it is the disjoint union of two proper closed subsets that are also stable.

Proof. Parts (1), (5), and (2) of proposition 3.3 show that with local compactness an attractor is stable, and that a stable isolated compact subset of an attractor is also an attractor, so (a) will follow from (b).

To show (b), assume that $Y = Y_1 \cup Y_2$ where each Y_j is stable, compact, nonempty, and $Y_1 \cap Y_2 = \emptyset$. Then there are disjoint forward invariant neighborhoods U_j of Y_j . By invariance the suspended orbits of points of Y_j will lie in the suspension U'_j of U_j , $j = 1, 2$, which shows that the suspension Y' of Y is disconnected.

Conversely, assume that the suspension Y' of Y is not connected, so there is a continuous surjection $\alpha': Y \rightarrow \{1, 2\}$. Since the semiflow orbit of any point is connected, α' is constant on each orbit, so it induces a continuous surjection $\sigma: Y \rightarrow \{1, 2\}$ that is also constant along orbits. Let $Y_j \equiv \sigma^{-1}(j)$, so that these sets are compact, disjoint, nonempty and forward invariant. Because Y is invariant, the forward invariance of each Y_j implies its invariance. It is easy to check that in this situation the stability of Y implies the stability of each Y_j . $\square$

Our main results will be expressed in terms of indecomposability; we will use the conditions given by the lemma as our working definition of indecomposability, and we will not consider the suspension semiflow again. We also consider several conditions that will imply indecomposability. Simplest of these conditions is 'least by inclusion'. The following fact is an immediate consequence of the last lemma.

COROLLARY. *With assumptions as in the last lemma, if a stable set Y is smallest in the sense that it contains no proper compact subset that is also stable, then Y is indecomposable. Similarly, a smallest attractor is indecomposable.*

Another condition that implies indecomposability is chain transitivity. In order to simplify the preliminary discussion, suppose for the moment that X is compact. An ε -chain for f is a finite sequence $\psi = (x_0, x_1, \dots, x_n)$ with the property that $d(f(x_j), x_{j+1}) < \varepsilon$ for each j ; ψ is said to go from x_0 to x_n . We require that the length, n , of ψ always be at least 1. A point $p \in X$ is called *chain recurrent* for f if for each $\varepsilon > 0$ there is an ε -chain ψ_ε that both begins and ends at p . A subset $Y \subset X$ is called *chain transitive* if for each ordered pair (p, q) of points of Y and each $\varepsilon > 0$ there is an ε -chain $(x_0, \dots, x_n)$ from p to q with each $x_j \in Y$. A chain transitive set is clearly composed of chain recurrent points; in general the converse is not true.

When Y is not compact it is necessary to replace the constants ε in the definition by arbitrary positive functions, but we will only consider compact chain transitive sets in this paper.

Note that any transitive set or any omega-limit set of a point is obviously chain transitive, so an assumption of chain transitivity is weaker than the kinds of conditions that are imposed in [1, 10].

LEMMA 4.2. *A chain transitive compact invariant set Y is indecomposable.*

Proof. This is like the last part of the proof of lemma 4.1. If Y is not indecomposable then there is a continuous f -invariant surjection $\alpha: Y \rightarrow \{1, 2\}$ so that the sets $Y_j = \alpha^{-1}(j)$ are compact, disjoint and forward invariant. By compactness there is an $\varepsilon > 0$ such that $d(y_1, y_2) > \varepsilon$ whenever $y_j \in Y_j$, which, combined with the forward invariance, shows that no ε chain can start in Y_1 and end in Y_2 . $\square$

For compact attractors one can in fact show that an attractor A is smallest if and only if it is chain transitive. This is a consequence of the connection between the collection of attractors of f and the chain recurrent behavior of f . This connection was first noted by Conley for compact X , and has been extended to arbitrary metric spaces; see [2, 3, 6, 7, 8]. Roughly speaking, the connection is that a point q is not chain recurrent if and only if there is an attractor A with a corresponding neighborhood U as in the definition of attractor such that $x \in U - A$.

5. Proof of theorem A

Some basic properties about locally connected spaces will be important in what follows. First, if U is an open subset of a locally connected space, then the connected components of U are open. Second, suppose that U and W are open and that f maps the closure of U into W ; if U^* is a component of U then the closure of $f(U^*)$ is contained in a single component of W . The reason is that connectedness is preserved both by continuous maps and by taking closures.

If Y is a compact invariant set, then an open neighborhood W of Y is called *fitted* if W has finitely many components, each of which meets Y . Local connectedness ensures that any open neighborhood U of the compact invariant set Y contains a subset U' that is fitted, as the next lemma shows.

LEMMA 5.1. *Assume 1.1, 1.2 (X is locally connected and metric), and that Y is a compact, invariant subset of X and that U is an open neighborhood of Y . Then only finitely many components of U meet Y , and the union of these is a fitted open neighborhood $U' \subset U$ of Y . If in addition U is forward invariant, then so is U' , and in*

this case the components of U' are permuted by f (i.e., each component is mapped into another, and each component contains the image of another).

Proof. Since X is locally connected and U is open, each component of U is open; as these components are pairwise disjoint and cover the compact set Y , we see that only a finite number of them actually meet Y . Thus U' is fitted, where U' is the union of the components of U that intersect Y . When U is forward invariant it follows from connectedness that f maps each component of U into another, and since Y is invariant, the same is true of the components of U' , so U' is forward invariant. The invariance of Y also shows that the induced map on the finite set of components of U' is a surjection, and thus a permutation. $\square$

Theorem A is the combination of propositions 5.2 and 5.3.

PROPOSITION 5.2. *Assume 1.1 through 1.3 (X is locally compact, locally connected, and metric), and that A is a compact Conley attractor. Then A consists of a finite number of connected components, and the action of f on the set of components is a permutation.*

Proof. Let U correspond to A in the definition of a Conley attractor, and let U' be the fitted neighborhood of A corresponding to U as in the lemma. Since $f(U) \subset U$, the proof of the lemma and the second remark before the lemma show that if C is any component of U' then $f(C)$ is contained in a single component of U' . In other words, $f(U') \subset U'$, so we could have taken U to be U' in the first place. The lemma shows that f permutes the components $U_1, \dots, U_m$ of U' , so for each positive n , $f^n(U')$ consists of exactly m connected components, one contained in each of the connected components of $f^{n-1}(U')$. In other words, $U' \supset f(U') \supset f^2(U') \supset \dots$ consists of m nested sequences of connected sets. Since $\bigcap f^n(U') = A$, proposition 3.2 shows that A has exactly m connected components (each of the m nested sequences contains points of A so the intersections are nonempty). $\square$

COROLLARY. *Assume 1.1 through 1.3; if A is an asymptotically stable compact invariant set then A has a finite number of components, and these are permuted by f .*

Proof. Combine the proposition and proposition 3.3. $\square$

PROPOSITION 5.3. *Assume 1.1 through 1.3 and suppose that A is a Conley attractor. Then A is indecomposable if and only if f cyclically permutes its components.*

Proof. Let U correspond to A as in the definition of a Conley attractor; as in proposition 5.2 we can assume that U is fitted. Proposition 5.2 shows that each component of U contains exactly one component of A . If f does not cyclically permute all of the components then there are at least two complete cycles of components; let A_1 be the union of the components in the first cycle, and let A_2 be the union of the second. For each A_j the union of the corresponding components of the open set U gives an open set that is mapped into its own interior, and with the intersection of its forward images under f giving A_j . This shows that A is decomposable.

Conversely, if A were decomposable into A' and A'' then the fact that A' and A'' are disjoint means that any component of either of the smaller attractors is a component of A , so that the invariance of A' and of A'' implies that f does not cyclically permute the components of A . $\square$

Because an attractor that is smallest or chain transitive is indecomposable we have:

COROLLARY. *Assume 1.1 through 1.3; if A satisfies any of the 9 conditions of proposition 3.3 and is either a smallest attractor (smallest in the sense of inclusion) or is chain transitive, then f cyclically permutes the connected components of A .*

REMARK. The last corollary is strictly weaker than the proposition. As an example, consider a Morse-Smale diffeomorphism of the plane that contains a sink p and a saddle q , with the unstable manifold of q (other than q itself) contained in the stable manifold of p . Then the union of p and $W^u(q)$ is an indecomposable attractor that is neither smallest nor chain transitive.

The final corollary shows that theorem A holds without local compactness, provided that the definition of attractor is strengthened appropriately. This is the version of theorem A that was described in [4].

COROLLARY. *Assume 1.1, 1.2, and that A satisfies condition (3) of proposition 3.3. Then the conclusion of theorem A holds.*

Proof. The only use of 1.3 (local compactness of X) in the proofs of propositions 5.2 and 5.3 was to ensure that proposition 3.2 could be applied in establishing 5.2. When A satisfies condition 3.3(3) then the corollary to proposition 3.2 can be used in place of the proposition. $\square$

6. Indecomposable stable sets

In this section we show that any indecomposable stable (compact invariant) set either has finitely many components which are cyclically permuted by the action of f , or else the space of components is a Cantor set and the action of f on it is a generalized adding machine. This generalizes theorem 5.1 of [1] by not requiring X to be locally compact, and by assuming indecomposability instead of the existence of dense orbits.

PROPOSITION 6.1. *Assume 1.1, 1.2 (X is locally connected and metric) and that Y is a compact, nonempty, stable invariant subset of X . In addition suppose that U is an open neighborhood of Y and is forward invariant (i.e., $f(U) \subset U$). Then only finitely many components of U intersect Y . Moreover, if Y is indecomposable, then the components of U meeting Y are cyclically permuted by f .*

Proof. The first assertion is just a restatement of lemma 5.1, so we can assume that U has a finite number components, $U_1, \dots, U_m$, that each U_j meets Y , and that these components are permuted by f . If this permutation is not cyclic then we can decompose U as $U' \cup U''$ where U', U'' are disjoint and each is open and forward invariant (and a union of components of U). Now define $Y' = Y \cap U'$ and $Y'' = Y \cap U''$ so that Y', Y'' are disjoint, nonempty, and forward invariant, and each is the union of sets of the form $Y_j \equiv Y \cap U_j$. Since $Y \cap U_j = Y - (\bigcup_{i \neq j} U_i)$, each Y_j is compact, so that the same is true of Y' and Y'' . It follows that Y', Y'' are both stable, so Y is decomposable if the U_j 's are not cyclically permuted by f . $\square$

COROLLARY. *Assume as in the proposition that Y is stable and indecomposable; also assume that Y_0 is an isolated component of Y . Then each component of Y is isolated, and they are permuted cyclically by f .*

Proof. By stability Y has a forward invariant neighborhood contained in any given neighborhood of Y ; in particular we can find a forward invariant open neighborhood U of Y whose component U_0 that contains Y_0 is disjoint from the rest of Y . Let U_j be the component of U containing $f^j(U_0)$; by the proposition there is a positive integer m such that $U_m = U_0$. Similarly, let Y_j be the component of Y that contains $f^j(Y_0)$. The invariance of Y and the fact that the U_j 's are disjoint and are permuted means that $f(Y \cap U_j) = Y \cap U_{j+1}$ for each j . When $j = 0$ we know that $Y_0 = Y \cap U_0$, so we conclude that $f(Y_0) = Y \cap U_1$, so that $Y \cap U_1$ is the single component Y_1 and $f(Y_0) = Y_1$. By induction $f^j(Y_0) = Y \cap U_j = Y_j$ is a single component for each j . Since $Y = Y \cap U$ it follows that $Y = \bigcup_{j=0}^{m-1} Y_j$. $\square$

COROLLARY. *Assume 1.1, 1.2 and suppose that Y is nonempty, compact, stable, and indecomposable. If Y has infinitely many components, then none are isolated, and so the space of components is a Cantor set.*

Proof. It is shown in the proof of theorem 3.1 of [1] that the space of components is compact and metrizable by the usual Hausdorff metric. The space of components is obviously totally disconnected, and the last corollary shows that it is perfect, so it is a Cantor set. $\square$

THEOREM B. *Assume 1.1, 1.2 and suppose that Y is nonempty, compact, stable, and indecomposable. Then either Y has finitely many components which are permuted cyclically by f or else f acts on the space of components as a generalized adding machine. If f has a period k point lying in Y , then Y has m components where m is some divisor of k .*

Proof. If the space of components is finite, then this follows directly from the proposition, for in this case we can choose U so that each component of U contains exactly one component of Y .

In the remaining case, for each $n \geq 1$ we can choose an open set $U = U^n$ as in the proposition, with U^n everywhere within $1/n$ of Y , and with $U^{n+1} \subset U^n$ for each n . It follows from lemma 3.1 that if Y', Y'' are distinct components of Y then for sufficiently large n they will lie in distinct components of U^n . In particular, the number of components of U^n goes to infinity with n . Combining the proposition with remark 5.6 of [1], we complete the proof of the theorem. The basic idea is that, because the components of each U^n are cyclically permuted by f , if U^{n+1} has more components than U^n then the action of f on the components of U^{n+1} is an 'epicycle' upon the action of f on the components of U^n .

The last statement is a consequence of the fact that in the adding machine case, no component of f is ever mapped back to itself, so that f could have no periodic points in Y . $\square$

REMARK. Theorem 3.1 of [1] establishes the result of the corollary given before theorem B when Y is transitive, without assuming that Y is stable or that X is locally connected. Here is an example showing that if transitivity is weakened to chain transitivity, then additional assumptions on X are needed.

EXAMPLE. Here X is a countable subset of the circle with every point but one isolated. Let X be the set of all points of the form $e^{i\theta}$ where either $\theta = 0$ or else θ has the form $\pm\pi/n$, $n \geq 1$. The map f on X is a homeomorphism given by $f(1) = 1$, and any other point of X is mapped counterclockwise around S^1 to the next point

of X . Then $Y = X$ is compact and chain transitive, and since it is the whole space, it is trivially invariant and stable.

7. Components of attractors without local compactness

Without local compactness, conditions like the definition of a Conley attractor or asymptotic stability may not only fail to be equivalent, but they may be satisfied in rather trivial ways.

EXAMPLE. Let X be the set of all doubly infinite sequences of reals chosen from the range $[0, 1]$ that are 0 in all but finitely many coordinates. Give X the uniform metric, and let f be the shift map on X . Then f is an isometry so that the sequence $\underline{0}$ consisting of all 0's is stable and is its own omega-limit set, while the omega-limit set for any other point of X is empty. Hence $\underline{0}$ is asymptotically stable even though nothing but itself is asymptotic to it.

Lemma 3.1 shows that a compact invariant set Y is connected if it is the intersection of a nested sequence of connected sets C_n provided that for each neighborhood N of Y , $C_n \subset N$ for all large n . It follows that, even without local compactness, the proof of theorem A will work for any compact Conley attractor A with the additional property that there is an open neighborhood U of A corresponding to A as in the definition of an attractor and with, for each neighborhood N of A , $f^n(U) \subset N$ for sufficiently large n (i.e., condition (9) of proposition 3.3 is satisfied). As an example of where this sort of argument might apply, see section 2 of [9]. (The corollaries of theorem A also rely on proposition 3.3, so they require X to be locally compact.)

The proof of theorem B works for stable (compact invariant) sets, but it does not carry over for attractors in the general setting (i.e. without local compactness). The reason is that if the ambient space X is not locally compact then a Conley attractor is not necessarily stable. Here is an example.

EXAMPLE. Let $X = \{(k, \underline{z})\}$ where k is either a positive integer or infinity and $\underline{z} = (z_n)$ is a sequence of reals in $[0, 1]$ with $z_n = 0$ if $0 \leq n < k$ and $z_n \neq 0$ for only finitely many n (when $k = \infty$ then $z_n = 0$ for all n). Define a map f from X to itself by $f((k, \underline{z})) = (k + 1, \underline{y})$ where $y_n = \min(1, 2z_n)$ for all $n > k$. If we use the metric

$$d((k, \underline{z}), (p, \underline{w})) = \left| \frac{1}{k} - \frac{1}{p} \right| + \max_{n \geq 0} |z_n - w_n|,$$

then f is continuous, $\cap f^n(X) = (\infty, \underline{0})$ is an attractor, but $(\infty, \underline{0})$ is not stable. (Consider the forward orbit of $(k, \underline{x})$ where $k < \infty$ is large, $x_n > 0$ for some n much larger than k , and otherwise $x_j = 0$.)

In light of these observations it is tempting to strengthen the definition 'attractor' in a non-locally compact space by requiring a stronger convergence property like (3) or (9) of proposition 3.3 (as in the final corollary of section 5). One situation where this would be appropriate is a semilinear second order parabolic equation with space variables in a smooth compact manifold M , standard boundary conditions, and periodic data, say of period 1. Assume that there is a Banach space E of functions on M and a semiflow $= \{S_t\}_{t \geq 0}$ on $E \times Q$, where Q is the circle $R \bmod 1$, with the property that S_t is a compact nonlinear operator for each $t > 0$. For $q \in Q$ let f be the restriction of S_1 to $X \equiv E \times \{q\}$. Assume that S has a compact omega limit set $L \subset E \times Q$ which is an attractor, and set $A = L \cap X$. Then 3.3(3) is satisfied, so the conclusions of Theorem A are applicable to A and f .

Unfortunately, the correspondence between attractors and chain recurrence described at the end of section 4 does not carry over if the definition of attractor is strengthened from 3.3(1) to 3.3(3) or 3.3(9). This can be seen in the example from two paragraphs above. It is easy to see that (3) $\Rightarrow$ (9) $\Rightarrow$ (1) and that (9) implies that A is stable; in the example there is only one Conley attractor, and it is not stable, so the example has no attractors in the sense of either (3) or (9).

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