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by

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Professor Ian Agol, Chair
Professor Prasad Raghavendra
Professor David Nadler

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Abstract

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This thesis consists of two projects related to groups acting on metric spaces of non-positive curvature.

In the first project, we show that relatively hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes are virtually special, provided the peripheral subgroups are virtually special in a way that is compatible with the cubulation. This extends Agol's result for cubulated hyperbolic groups (which led to the proof of the Virtual Haken Conjecture), and applies to a wide range of peripheral subgroups. In particular, we deduce virtual specialness for properly and cocompactly cubulated groups that are hyperbolic relative to virtually abelian groups. As another consequence, by using a theorem of Martin and Steenbock we obtain virtual specialness for $C'(1/6)$-small cancellation quotients of free products of virtually special groups. For the proof of our main result, we prove a relative version of Wise's quasiconvex hierarchy theorem.

In our second project, we study the metric and topological properties of the space $\mathscr{D}_\Gamma$ of metric structures on the non-elementary hyperbolic group Γ, which parametrizes geometric actions of Γ on Gromov hyperbolic spaces. This space contains the Teichmüller space when Γ is a surface group and the Culler-Vogtmann outer space when Γ is a free group. Equipped with a natural metric reminiscent of Thurston's metric on Teichmüller space, we prove that $\mathscr{D}_\Gamma$ is unbounded, contractible and separable, and that $\mathrm{Out}(\Gamma)$ acts metrically properly by isometries on it. If we restrict to the subspace $\mathscr{D}_\Gamma^\delta$ of the points represented by actions on δ-hyperbolic spaces with exponential growth rate 1, we prove that it is either empty or proper, and that the Bowen-Margulis map from $\mathscr{D}_\Gamma^\delta$ into the space $\mathbb{P}\mathcal{C}urr(\Gamma)$ of projective geodesic currents on Γ is continuous. By finding an $\mathrm{Out}(\Gamma)$-invariant geodesic bicombing for $\mathscr{D}_\Gamma$ we also construct a boundary for this space, which parametrizes improper actions of Γ on hyperbolic spaces. As a corollary of this construction, we deduce continuous extension of translation length functions to the space of geodesic currents, which we use to disprove a conjecture of Bonahon about small actions of hyperbolic groups on $\mathbb{R}$-trees.

Para Marcia, Camila, y Rosa

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Chapter 1

Introduction

Geometric group theory is the study of groups via their isometric actions on metric spaces. The goal is to recover algebraic properties of the group from geometric properties of the spaces it acts on. For example, the behavior of the fundamental group of a closed Riemannian manifold -which acts isometrically on the universal cover- is influenced by curvature, and this is particularly explicit in the presence of non-positive (sectional) curvature. On the combinatorial side, a lot can be deduced about a group once we know it acts interestingly on a tree. Following this philosophy, we can look for isometric actions of groups on spaces satisfying some metric version of *non-positive curvature*, which is the main theme of this thesis.

In his seminal paper in 1987 [Gro87], Gromov introduced and popularized many notions of non-positive curvature for metric spaces and groups. Among them, the class of *δ-hyperbolic spaces* (Section2.3) plays a central role, and since then there has been a very active and fruitful branch in geometric group theory that studies groups via their isometric actions on δ-hyperbolic spaces. As an illustration of this, Gromov also defined the class of (*word*) *hyperbolic groups* (Section 2.4); those finitely generated groups whose Cayley graphs are δ-hyperbolic. These groups generalize finitely generated non-abelian free groups and fundamental groups of closed hyperbolic manifolds and appear naturally in low-dimensional topology, geometric topology, representation theory, and combinatorial group theory. If we allow non-hyperbolicity in some isolated portions of the Cayley graphs we obtain *relatively hyperbolic groups* (Section 2.5), generalizing free products and fundamental groups of cusped hyperbolic manifolds.

In the same article, Gromov introduced CAT(0) *cube complexes* (Section 2.7), merely as examples of combinatorial spaces that have local and global non-positively curved behavior. Since then, groups acting on these spaces have become objects of interest, and they have been protagonists of some of the most relevant advances in 3-manifold topology in the last 15 years. Building on Haglund-Wise's theory of *special cube complexes and groups* [HW08] (Section 2.8), and the work of Sageev, Groves-Manning, Kahn-Markovic, Wise, and many others, Agol solved the Virtual Haken and Virtual Fibered Conjectures [Ago13].

This thesis is devoted to the study of hyperbolic and relatively hyperbolic groups, as well as CAT(0) cube complexes and groups acting on them. Our work can be described in terms of 2 independent projects: *relatively hyperbolic groups acting on* CAT(0) *cube complexes*, and *deformation spaces for geometric actions of hyperbolic groups*.

1.1 Cubulated relatively hyperbolic groups

CAT(0) cube complexes are a particular kind of non-positively curved complexes built by gluing Euclidean cubes. These spaces are higher-dimensional versions of simplicial trees, and studying groups acting on them can be seen as a generalized Bass-Serre theory. Groups acting geometrically on CAT(0) cube complexes (sometimes referred to as cubulable groups) are better understood than arbitrary CAT(0) groups: they are bi-automatic, satisfy the Tits alternative, and have the Haagerup property (so they do not have property (T)). On the other hand, since the work of Sageev, Wise, and others, many groups have been shown to be cubulable, including small cancellation groups, most 3-manifold groups, limit groups, many Coxeter groups, and hyperbolic free-by-cyclic groups.

Inside cubulable groups, we have the family of *virtually (cocompact) special groups*, introduced by Haglund and Wise. These groups have finite index subgroups embedding nicely into right-angled Artin groups, and so they inherit some of their properties. In particular, virtually special groups are residually finite, large, linear over $\mathbb{Z}$, satisfy the strong Atiyah conjecture [Sch14, Thm. 1.2], and have many separable subgroups [HW08]. Our first project is motivated by the following theorem of Agol, which was the last step in the proofs of the Virtual Haken and Virtual Fibered Conjectures.

Theorem 1.1.1 (Agol [Ago13, Thm. 1.1])**.** *Let Γ be a hyperbolic group acting properly and cocompactly on a* CAT(0) *cube complex X. Then Γ has a finite index subgroup Γ' acting freely on X such that $\Gamma' \backslash X$ is special.*

The assumption of hyperbolicity in the theorem above is in some sense necessary, since there are examples of infinite simple groups acting properly and cocompactly on products of trees [BM00]. The goal of this first project is to extend Agol's result to relatively hyperbolic groups, but even in this case, we must impose some extra assumptions. If a relatively hyperbolic group is virtually special, then all its peripheral subgroups are also virtually special, so this is a necessary condition. Our main result asserts that this condition is (almost) sufficient, and it will be published in [Ore20]. Groves and Manning [GM22] have also proven it independently via different methods.

Theorem 1.1.2 (Reyes [Ore20, Thm. 1.2], Groves-Manning [GM22, Thm. A])**.** *Let Γ be a group acting properly and cocompactly on the* CAT(0) *cube complex X, and suppose Γ is hyperbolic relative to* compatible *virtually special subgroups. Then there exists a finite index subgroup $\Gamma' < \Gamma$ acting freely on X such that $\Gamma' \backslash X$ is a special cube complex.*

The compatibility condition in the theorem above relates the virtually special cubulations of the peripheral subgroups and the cubulation X of Γ. Its precise definition is given in

Section 3.1. This condition is satisfied when Γ is cubulated and hyperbolic relative to virtually abelian subgroups, and so Theorem 1.1.2 applies for these groups.

Corollary 1.1.3. *If Γ acts properly and cocompactly on a* CAT(0) *cube complex and is hyperbolic relative to virtually abelian subgroups, then Γ is virtually compact special.*

The preceding corollary recovers some remarkable results, such as virtual specialness of cusped hyperbolic 3-manifolds and of limit groups [Wis21, Sec. 17 & Sec. 18] due to Wise (see also [CF19] and [GM21]). Another consequence of our main result depends on the combination theorem for cubulations in small cancellation theory due to Martin and Steenbock [MS17].

Theorem 1.1.4 (Martin-Steenbock [MS17, Thm. 1.1])**.** *Let F be the free product of finitely many groups $\Gamma_1, \dots, \Gamma_r$, and assume each Γ_i acts properly and cocompactly on the* CAT(0) *cube complex X_i. If Γ is a quotient of F by a finite set of relators that satisfies the classical $C'(1/6)$-small cancellation condition over F, then Γ acts properly and cocompactly on a* CAT(0) *cube complex X.*
Moreover, this complex is constructed in such a way that for each i, there is a Γ_i-equivariant combinatorial isometric embedding $\dot{X}_i \hookrightarrow X$, where $\dot{X}_i$ is the cubical barycentric subdivision of X_i.

The "moreover" part of the previous theorem is implicit in the construction of X, see [MS17, Rmk. 3.43] (cf. [JW22, Cor. 4.5]). The group Γ is hyperbolic relative to the free factors $\Gamma_1, \dots, \Gamma_r$ [Osi06, p. 2 Ex. (II)], and since a compact non-positively curved complex is virtually special if and only if its cubical barycentric division is virtually special (Corollary 3.2.2), the existence of equivariant isometric embeddings $\dot{X}_i \hookrightarrow X$ imply that the cubulation (Γ, X) satisfies the compatibility condition provided each of the cubulations (Γ_i, X_i) is virtually special. Hence Theorem 1.1.2 implies the following combination result.

Corollary 1.1.5. *Let F be the free product of finitely many virtually compact special groups. If Γ is a quotient of F by a finite set of relators satisfying the classical $C'(1/6)$-small cancellation condition over F, then Γ is also virtually compact special.*

We prove Theorem 1.1.2 with an adaptation of Agol's methods as follows.

1.1.1 A cubulated malnormal hierarchy

One of the main tools in Agol's proof of Theorem 1.1.1 is Wise's quasiconvex hierarchy theorem (Theorem 2.8.6), which says that a hyperbolic group is virtually special if and only if it can be obtained from finite groups after a finite sequence of (virtual) amalgamations or HNN extensions over quasiconvex subgroups. Our proof of Theorem 1.1.2 follows Agol's approach, and for that we require a notion of hierarchy that is appropriate to relatively hyperbolic groups.

Definition 1.1.6. Let $\mathcal{CMVH}$ denote the smallest class of cubulated and relatively hyperbolic groups (Γ, X) (here Γ acts properly and cocompactly on the CAT(0) cube complex X) relative to compatible virtually special subgroups, that is closed under the following operations:

1. $(\{o\}, X) \in \mathcal{CMVH}$ for any finite CAT(0) cube complex X, where $\{o\}$ is the trivial group.

2. If Γ splits as a finite graph of groups $(G, \mathcal{G})$ satisfying

   - each edge/vertex group is convex in (Γ, X);
   - if v is a vertex of G then the collection $\mathcal{A}_v := \{\Gamma_e \colon e \text{ an edge attached to } v\}$ is *relatively malnormal* in Γ_v; and,
   - if Γ_v is a vertex group, then it has a convex core $X_v \subset X$ with $(\Gamma_v, X_v) \in \mathcal{CMVH}$,

   then $(\Gamma, X) \in \mathcal{CMVH}$.

3. If $H < \Gamma$ with $|\Gamma : H| < \infty$ and $(H, X) \in \mathcal{CMVH}$, then $(\Gamma, X) \in \mathcal{CMVH}$.

The notation $\mathcal{CMVH}$ is meant to be an abbreviation for "cubulated (relatively) malnormal virtual hierarchy", and the main concepts involved in this definition are explained in Chapter 2. Note that the compatibility between the peripheral structure on Γ and the splitting $(G, \mathcal{G})$ in item (2) above is only reflected in the relative quasiconvexity of edge/vertex subgroups (which is implied by convexity, see Theorem 2.5.7) and the relative malnormality of edge groups in the vertex groups. This seems to be a weak assumption when we compare, for instance, with the quasiconvex, malnormal, and $\mathcal{P}$-fully elliptic hierarchy that requires fully relative quasiconvexity of edge/vertex groups [Ein19, Sec. 3]. In Section 3.4 we prove a relative version of Wise's quasiconvex hierarchy theorem.

Theorem 1.1.7 (Reyes, Relative quasiconvex hierarchy theorem [Ore20, Thm. 1.8])**.** *If $(\Gamma, X) \in \mathcal{CMVH}$ then there is a finite index subgroup $\Gamma' < \Gamma$ acting freely on X such that $\Gamma' \backslash X$ is special.*

To prove this result we use group theoretical *Dehn filling*, introduced independently by Groves-Manning [GM08] and Osin [Osi07] (Subsection 2.5.2). Dehn filling is used to find plenty of hyperbolic quotients for relatively hyperbolic groups splitting as in item (2) of Definition 1.1.6. We prove that relative malnormality is promoted to almost malnormality for sufficiently long Dehn fillings (Theorem 3.3.2), inducing splittings for these quotients. By Wise's quasiconvex hierarchy theorem, these hyperbolic quotients turn out to be virtually special, and hence many double cosets of relatively quasiconvex subgroups are separable (Proposition 3.4.6). This allows us to apply Haglund-Wise's double coset criterion, from which we deduce virtual specialness.

The second main ingredient in Agol's proof of Theorem 1.1.1 is the "coloring" trick to produce a hierarchy for a cubulated hyperbolic group. By adapting this argument to the relatively hyperbolic setting, we prove the theorem below, which together with Theorem 1.1.7 and Corollary 3.2.2 implies Theorem 1.1.2.

Theorem 1.1.8 (Reyes [Ore20, Thm. 1.9])**.** *If (Γ, X) is cubulated and hyperbolic relative to compatible virtually special subgroups, then $(\Gamma, \dot{X}) \in \mathcal{CMVH}$, where $\dot{X}$ is the cubical barycentric subdivision of X.*

While the theorem above is enough to deduce our main result, virtual compact specialness for relatively hyperbolic groups implies the existence of strong virtual hierarchies, as was proven by Einstein [Ein19, Thm. 1]. In consequence, Theorem 1.1.2 implies the following.

Corollary 1.1.9. *Let $(\Gamma, \mathcal{P})$ be a cubulated relatively hyperbolic group with compatible virtually special subgroups. Then there exists a finite index subgroup $\Gamma_0 < \Gamma$ with induced relatively hyperbolic structure $(\Gamma_0, \mathcal{P}_0)$ so that Γ_0 has a quasiconvex, malnormal and fully $\mathcal{P}_0$-elliptic hierarchy terminating in groups isomorphic to elements of $\mathcal{P}_0$.*

The proof of Theorem 1.1.8 follows the exact same steps as that one of Theorem 1.1.1. For (Γ, X) as in the statement, we use the main theorem in the Appendix of [Ago13] and Einstein's malnormal special quotient theorem for relatively hyperbolic groups [Ein19] to find an infinite-sheeted cover $\mathcal{X} \to \Gamma\backslash\dot{X}$, which is a non-positively curved cube complex with finite embedded walls and "large injectivity radius" (Theorem 3.5.4). This cover is used as the model to construct our hierarchy. We use the coloring trick to find a convenient Γ-invariant coloring of the walls of $\mathcal{X}$ by finitely many colors, labeled $1, 2 \dots, k$. The hierarchy is described as a sequence $\mathcal{V}_{k+1}, \mathcal{V}_k, \dots, \mathcal{V}_0$, where each $\mathcal{V}_j$ is a set of locally convex subcomplexes of the cubical barycentric subdivision of $\mathcal{X}$, probably with multiplicities (Definition 3.7.2). The sets in $\mathcal{V}_{k+1}$ are cubical neighborhoods of vertices of $\mathcal{X}$, and at each step, we glue finite covers of sets in $\mathcal{V}_j$ to produce the sets in $\mathcal{V}_{j-1}$. The gluing is along walls with color j, and induces a splitting of the fundamental groups of subcomplexes in $\mathcal{V}_j$ satisfying item (2) in the definition of $\mathcal{CMVH}$ (Proposition 3.8.12). At the final step, the subcomplexes in $\mathcal{V}_0$ are finite-sheeted covers of $\Gamma\backslash\dot{X}$, whose fundamental groups belong to $\mathcal{CMVH}$.

1.2 Metric structures on hyperbolic groups

The second project deals with deformation spaces for hyperbolic groups. If Γ is the fundamental group of a closed hyperbolic surface, the *Teichmüller space* of Γ, denoted by $\mathscr{T}_\Gamma$, can be described as the set of all the proper and cocompact actions of Γ on the hyperbolic plane, up to Γ-equivariant isometry (Example 4.3.3). Similarly, if Γ is a finitely generated free group, we can consider *Culler-Vogtmann's outer space* $\mathscr{CV}_\Gamma$, which encodes the geometric actions of Γ on trees (Example 4.3.6). Both Teichmüller and outer spaces have been studied extensively, and nowadays they (and their generalizations) are standard tools in the understanding of mapping class groups, outer automorphism groups of free groups and related notions in group theory and low-dimensional topology [Bes02; DH18; FM12; GH21; Ota98; Wie18].

The situation differs dramatically in higher dimensions, since by Mostow rigidity, if Γ is the fundamental group of a closed aspherical manifold M of dimension $n \geq 3$, then there exists at most one hyperbolic structure on M. This gets worse in dimensions $n \geq 5$, as there are examples of such M admitting no negatively curved Riemannian metrics [DJ91, Thm. 5c.1 and Rmk. on p. 386].

Besides examples coming from geometric topology or representation theory, finding interesting deformation spaces for arbitrary hyperbolic groups is in general complicated, and most constructions rely on the existence of non-trivial isometric actions on $\mathbb{R}$-trees (see e.g. [Cla05; GL07; Pau88]). However, since Gromov hyperbolicity is a metric property, it is natural to consider structures associated to interesting isometric actions of Γ on hyperbolic metric spaces, under the appropriate equivalence relation. This coarse-geometric perspective was adopted in [Fur02] by Furman, where he considered the space of all hyperbolic, left-invariant metrics on Γ that are quasi-isometric to a word metric with respect to a finite and symmetric generating set of Γ. Since we also want to include hyperbolic groups with torsion, it is more convenient to consider pseudo metrics instead of metrics.

Definition 1.2.1. For a hyperbolic group Γ, we let $\mathcal{D}_\Gamma$ denote the space of all hyperbolic, left-invariant pseudo metrics on Γ that are quasi-isometric to a word metric with respect to a finite, generating set.

The relevant object is then the quotient of $\mathcal{D}_\Gamma$ under rough similarity, where two pseudo metrics d, d' on a set X are *roughly similar* if there exist $k, A > 0$ such that $|d(x,y) - k \cdot d'(x,y)| \leq A$ for all $x, y \in X$.

Definition 1.2.2 (Space of metric structures, Furman [Fur02, §1])**.** Let Γ be a non-elementary hyperbolic group. The *space of metric structures* on Γ is $\mathscr{D}_\Gamma$, the set of equivalence classes of pseudo metrics in $\mathcal{D}_\Gamma$, where two pseudo metrics are in the same class if they are roughly similar. Points in $\mathscr{D}_\Gamma$ will be called *metric structures*, and we denote by $[d]$ the metric structure induced by $d \in \mathcal{D}_\Gamma$.

Equivalently, the points in $\mathscr{D}_\Gamma$ are represented by proper, cobounded, and isometric actions of Γ on geodesic hyperbolic spaces, where two such actions represent the same point in $\mathscr{D}_\Gamma$ if and only if there exists a Γ-equivariant rough similarity between the underlying spaces (Lemma 4.3.2). The space $\mathscr{D}_\Gamma$ was defined by Furman in 2002 [Fur02] and has appeared intermittently in the literature. We can further equip $\mathscr{D}_\Gamma$ with a distance, which was also considered by Fricke and Furman [FF22].

Definition 1.2.3 (Metric on $\mathscr{D}_\Gamma$)**.** Given $\rho = [d], \rho = [d_*] \in \mathscr{D}_\Gamma$, we define

$$\Lambda(\rho, \rho_*) := \inf\left\{\lambda_1\lambda_2 \colon \exists A \geq 0 \text{ s.t. } \frac{1}{\lambda_1}d - A \leq d_* \leq \lambda_2 d + A\right\},$$

and

$$\Delta(\rho, \rho_*) := \log \Lambda(\rho, \rho_*).$$

It is clear that Λ and Δ are independent of the representatives d and d_*, and that Δ is non-negative, symmetric, and satisfies the triangle inequality. It turns out that Δ is indeed a metric, and so we can study $\mathscr{D}_\Gamma$ from a geometric perspective, similar to what happens with Teichmüller and outer spaces. In fact, when Γ is a surface group, the restriction of Δ to $\mathscr{T}_\Gamma$ coincides with the (symmetrized) Thurston's metric [Thu98], and hence the natural map $\mathscr{T}_\Gamma \to \mathscr{D}_\Gamma$ is a continuous injection (Remark 4.3.26). The work of Francaviglia and Martino [FM11] implies the same result in the case Γ is a free group, so that the map $\mathscr{CV}_\Gamma \to \mathscr{D}_\Gamma$ is also continuous and injective. As we will see, many other geometric structures and deformation spaces induce subsets of metric structures, such as quasi-Fuchsian spaces, negatively curved Riemannian metrics, Hitchin components (and more generally, spaces of Anosov representations), geodesic currents, cubulations, random walks, etc.

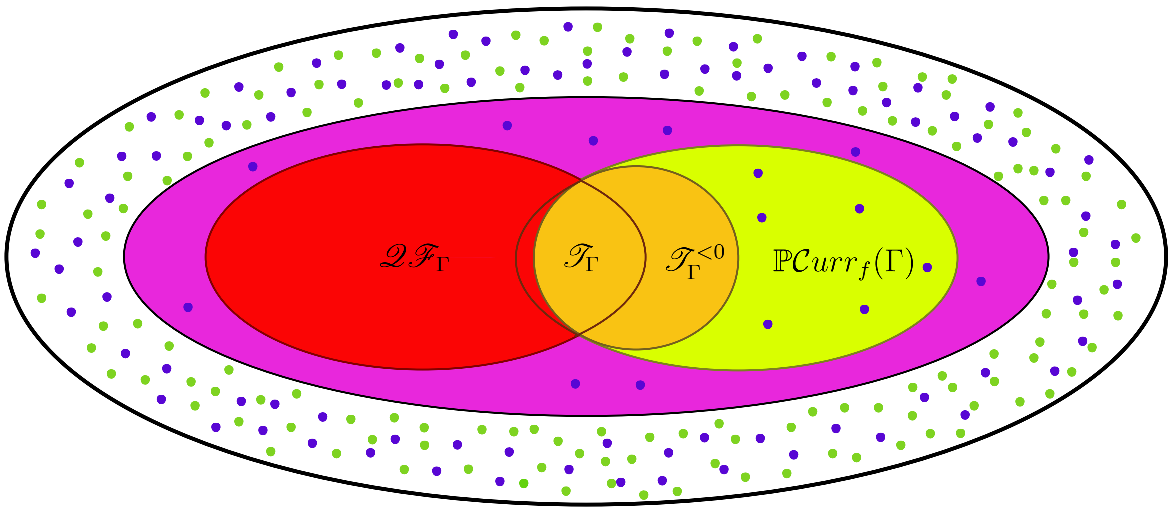

Figure 1.1: $\mathscr{D}_\Gamma$ for Γ a surface group. $\mathscr{QF}_\Gamma$ is the quasi-Fuchsian space, $\mathscr{T}_\Gamma$ is the Teichmüller space, $\mathscr{T}_\Gamma^{<0}$ is the space of isotopy classes of marked negatively curved Riemannian metrics on a closed surface with fundamental group Γ, and $\mathbb{P}\mathcal{C}urr_f(\Gamma)$ is the space of (projective) filling geodesic currents on Γ. The purple bubble represents the closure in $(\mathscr{D}_\Gamma, \Delta)$ of the space of metric structures induced by geometric actions of Γ on CAT(0) cube complexes considered with the combinatorial distance. Blue dots represent metric structures induced by word metrics, and green dots represent metric structures induced by Green metrics

Even though the space of metric structure has been present for some time, not much is known about its global properties. The goal of this project is to study the geometry and topology of $(\mathscr{D}_\Gamma, \Delta)$, in a similar spirit to what has been done for Teichmüller spaces, outer spaces, and other deformation spaces associated to hyperbolic groups. We expect the space of metric structures to be a natural framework to understand classical deformation spaces from a common perspective, and to develop enough tools to translate techniques and results from one deformation space to another. Now we state our results, which are part of the published paper [Ore23] and the preprint [CR22] which is joint work with Stephen Cantrell.

1.2.1 Topological properties of $\mathscr{D}_\Gamma$ and the action of $\mathrm{Out}(\Gamma)$

A desirable property for a deformation space is contractibility, which is classical for Teichmüller spaces, outer spaces, quasi-Fuchsian spaces, and Hitchin components. Our first result is that the space of metric structures also satisfies this property.

Theorem 1.2.4 (Reyes [Ore23, Thm. 1.3])**.** *If Γ is non-elementary and hyperbolic, then the metric space $(\mathscr{D}_\Gamma, \Delta)$ is unbounded, contractible, and separable.*

In similar settings, contractibility is in general not immediate, but in our case it will follow easily from the fact that $\mathcal{D}_\Gamma$ is closed under the addition of pseudo metrics (Corollary 4.2.10). Unboundedness is interesting since it holds even when $\mathrm{Out}(\Gamma)$ is finite. We exhibit an unbounded sequence in $\mathscr{D}_\Gamma$ by finding metrics that (almost) kill arbitrarily large powers of some infinite order element of Γ, with some similarity to group theoretical Dehn filling. Separability follows by verifying that the set of metric structures induced by word metrics on Γ is dense in $\mathscr{D}_\Gamma$ (Lemma 4.2.12).

Another common feature of the classical deformation spaces we have mentioned is properness: closed balls are compact. The space of metric structures is quite huge, so to obtain properness we must look at smaller subsets. A natural way to construct them is by considering metric structures represented by pseudo metrics with "bounded geometry". This is what we do in the next definitions, which depend on concepts that we discuss in Chapter 2.

Definition 1.2.5. For $\delta, \alpha \geq 0$ we define:

- $\mathscr{D}_\Gamma^\delta$ as the space of all metric structures of the form $\rho = [d]$, where $d \in \mathcal{D}_\Gamma$ is a δ-hyperbolic pseudo metric with exponential growth rate 1; and
- $\mathscr{D}_\Gamma^{\delta,\alpha}$ as the space of all metric structures of the form $\rho = [d]$, where d is a δ-hyperbolic and α-roughly geodesic pseudo metric with exponential growth rate 1.

By applying a Bochi-type inequality for sets of isometries of hyperbolic spaces due to Breuillard and Fujiwara [BF21, Thm. 1.4], we produce appropriate pseudo metrics representing metric structures for any bounded subspace of $\mathscr{D}_\Gamma^{\delta,\alpha}$. As a consequence of this, we get the following result, which gives a unified proof that Teichmüller and outer spaces are proper. Indeed, the Teichmüller space is contained in $\mathscr{D}_\Gamma^{\log 2}$ for Γ a surface group, and the outer space coincides with $\mathscr{D}_\Gamma^0$ when Γ is a free group.

Theorem 1.2.6 (Reyes [Ore23, Thm. 1.5])**.** *For any $\delta, \alpha \geq 0$, the subsets $\mathscr{D}_\Gamma^\delta$ and $\mathscr{D}_\Gamma^{\delta,\alpha}$ are either empty or proper subspaces of $\mathscr{D}_\Gamma$.*

We also study the natural action of $\mathrm{Out}(\Gamma)$ on $\mathscr{D}_\Gamma$ induced by pullback. This action is isometric and proper, extending the cases of mapping class group acting on Teichmüller spaces and Hitchin components and $\mathrm{Out}(\Gamma)$ acting on the outer space for Γ a free group. When Γ is torsion-free, we apply a recent finiteness theorem due to Besson, Courtois, Gallot and Sambusetti [Bes+21, Thm. 1.4] to interpret each $\mathscr{D}_\Gamma^{\delta,\alpha}$ as a "thick part" for this action, as in the case of Teichmüller and outer spaces.

Theorem 1.2.7 (Reyes [Ore23, Thm. 1.6 & Thm. 1.7])**.** *The action of* $\mathrm{Out}(\Gamma)$ *on* $\mathscr{D}_\Gamma$ *is isometrically proper. Moreover, if* Γ *is torsion-free, then the action of* $\mathrm{Out}(\Gamma)$ *on each* $\mathscr{D}_\Gamma^{\delta,\alpha}$ *is proper and cocompact, provided* $\mathscr{D}_\Gamma^{\delta,\alpha}$ *is non-empty.*

1.2.2 Continuity of the Bowen-Margulis currents

In the late 80s, Bonahon introduced the space $\mathcal{C}urr(\Gamma)$ of *geodesic currents* for Γ a surface group [Bon88], and later for arbitrary hyperbolic groups [Bon91]. This space consists of all the Γ-invariant Radon measures on the double Gromov boundary of Γ, and it can be thought of as a completion of the set of conjugacy classes in Γ. Furman [Fur02] then used quasiconformal measures to define the *Bowen-Margulis map* $BM\colon \mathscr{D}_\Gamma \to \mathbb{P}\mathcal{C}urr(\Gamma)$ into the space of projective geodesic currents (Subsection 4.3.2). This map generalizes the invariant measures maximizing the entropy for geodesic flows of closed negatively curved manifolds, and Furman also showed that BM is injective [Fur02, Thm. 4.1]. When restricted to $\mathscr{D}_\Gamma^{\delta}$, we prove that the Bowen-Margulis map is continuous.

Theorem 1.2.8 (Reyes [Ore23, Thm. 1.8])**.** *For any* $\delta \geq 0$ *such that* $\mathscr{D}_\Gamma^{\delta}$ *is non-empty, the map* $BM : \mathscr{D}_\Gamma^{\delta} \to \mathbb{P}\mathcal{C}urr(\Gamma)$ *is continuous.*

The main ingredients in the proof of this result are Theorem 1.2.6 and a description of Bowen-Margulis currents in terms of quasiconformal measures on the Gromov boundary of Γ. We emphasize the difference between our argument with previous instances of this result. For (homothety classes of) marked negatively curved Riemannian metrics on closed manifolds, continuity was shown by using that the geodesic flows on the unit tangent bundles are Anosov [Kat+89]. On the other hand, the continuity of BM on outer spaces was deduced from the explicit computation of the Bowen-Margulis currents and the fact that the Gromov boundaries of non-abelian free groups are Cantor sets [KN07, Thm. A].

1.2.3 Geodesics and the Manhattan boundary

The (symmetrized) Thurston's metric on outer space is not geodesic [FM11, Sec. 6], and to the best of the author's knowledge, is it unknown whether this phenomenon occurs for Teichmüller spaces. Surprisingly, $(\mathscr{D}_\Gamma, \Delta)$ is geodesic in a very canonical way. In joint work with Stephen Cantrell, we construct a geodesic bicombing on $\mathscr{D}_\Gamma$ consisting of *bi-infinite* geodesics.

Theorem 1.2.9 (Cantrell–Reyes [CR22, Thm. 1.2 & Thm. 4.10])**.** *The space* $(\mathscr{D}_\Gamma, \Delta)$ *is geodesic. Moreover, for any two distinct points* $\rho, \rho_* \in \mathscr{D}_\Gamma$ *there exists an unparametrized, oriented, bi-infinite geodesic* $\sigma_\bullet^{\rho_*/\rho} : \mathbb{R} \to$ *such that* $\sigma_0^{\rho_*/\rho} = \rho$ *and* $\sigma_{\Delta(\rho,\rho_*)}^{\rho_*/\rho} = \rho_*$*, and such that the assignment* $\sigma : (\rho, \rho_*) \mapsto \sigma_\bullet^{\rho_*/\rho}$ *is continuous,* $\mathrm{Out}(\Gamma)$*-invariant, and consistent.*

We also show that the points at infinity of the geodesics in the bicombing σ can be represented by (rough similarity classes of) appropriate pseudo metrics on Γ, which we collect into the *Manhattan boundary.*

Definition 1.2.10. Let $\overline{\mathcal{D}}_\Gamma$ be the set of all the unbounded left-invariant pseudo metrics d on Γ such that there exist $\lambda > 0$ and $d_0 \in \mathcal{D}_\Gamma$ such that

$$(x|y)_{o,d} \leq \lambda (x|y)_{o,d_0} + \lambda \tag{1.1}$$

for all $x, y \in \Gamma$, where $(\cdot|\cdot)_{\cdot,d'}$ denotes the Gromov product for the pseudo metric d'. We also set $\partial_M \mathcal{D}_\Gamma := \overline{\mathcal{D}}_\Gamma \backslash \mathcal{D}_\Gamma$.

Definition 1.2.11 (Manhattan boundary)**.** The *Manhattan boundary* of $\mathscr{D}_\Gamma$ is $\partial_M \mathscr{D}_\Gamma$, the quotient of $\partial_M \mathcal{D}_\Gamma$ under the equivalence relation of rough similarity. Its elements are called *boundary metric structures.* The *closure* of $\mathscr{D}_\Gamma$ is $\overline{\mathscr{D}}_\Gamma := \mathscr{D}_\Gamma \cup \partial_M \mathscr{D}_\Gamma$.

It might not be evident at first, but the inequality (1.1) is equivalent to d satisfying an appropriate generalization of *bounded backtracking*, which has been studied for actions on $\mathbb{R}$-trees (Lemma 4.7.1). Indeed, we prove that several interesting actions of Γ on hyperbolic spaces induce elements in $\partial_M \mathscr{D}_\Gamma$. In particular, by item (3) below we obtain that $\partial_M \mathscr{D}_\Gamma$ is an extension of the Thurston boundary for Teichmüller spaces (Corollary 4.7.13) and the Culler-Vogtmann boundary for outer spaces (Corollary 4.7.7).

Theorem 1.2.12 (Cantrell–Reyes [CR22, Thm. 1.6])**.** *The following actions induce points in* $\overline{\mathscr{D}}_\Gamma$*.*

1. *Natural actions on coned-off Cayley graphs for finite, symmetric generating sets, where we cone-off a finite number of quasiconvex subgroups of infinite index.*

2. *Non-trivial Bass-Serre tree actions with quasiconvex edge stabilizers of infinite index. More generally, cocompact actions on* $\mathrm{CAT}(0)$ *cube complexes with quasiconvex wall stabilizers and without global fixed points.*

3. *Small actions on* $\mathbb{R}$*-trees, when* Γ *is a surface group or a free group.*

In the case of surface groups, we can say something stronger since we can embed the space $\mathbb{P}\mathcal{C}urr(\Gamma)$ of projective geodesic currents into $\overline{\mathscr{D}}_\Gamma$ (Corollary 4.7.13). This is done by analyzing the pseudo metric d_μ on the hyperbolic plane for a non-zero geodesic current μ, defined by Burger, Iozzi, Parreau and Pozzetti [Bur+21, Sec. 4]. This embedding is consistent with the extension of the symmetrized Thurston's metric defined on the space

$\mathbb{P}\mathcal{C}urr_f(\Gamma)$ of projective filling currents, studied recently by Sapir [Sap22]. In another recent paper [DM22], Martínez-Granado and De Rosa discuss the pseudo metrics d_μ in more detail.

In the case of free groups, item (3) above follows since small actions of free groups on $\mathbb{R}$-trees have bounded backtracking, which was proven by Guirardel [Gui98, Cor. 2]. In a forthcoming work [KM], Kapovich and Martínez-Granado show that for freely indecomposable hyperbolic groups, small actions on $\mathbb{R}$-trees have bounded backtracking, and hence induce pseudo metrics in $\overline{\mathcal{D}}_\Gamma$ (Proposition 4.7.6).

1.2.4 Continuous extensions of translation lengths and a conjecture of Bonahon

We also study the translation length functions for points in $\overline{\mathscr{D}}_\Gamma$. Each left-invariant pseudo metric d on Γ induces a *(stable) translation length function* $\ell_d : \Gamma \to \mathbb{R}$ (Section 2.2). When Γ is torsion-free, we prove that for pseudo metrics representing points in $\overline{\mathscr{D}}_\Gamma$, the translation length function continuously extends to a function on $\mathcal{C}urr(\Gamma)$. In 1988, Bonahon conjectured that the only isometric actions of a hyperbolic group Γ on $\mathbb{R}$-trees whose translation length function continuously extends to $\mathcal{C}urr(\Gamma)$ are the ones that are small [Bon91, p. 164]. However, according to item (1) in Theorem 1.2.12, such a continuous extension exists for every Bass-Serre tree action with quasiconvex edge stabilizers. Since this occurs for fundamental groups of some closed hyperbolic 3-manifolds, we answer in the negative to Bonahon's conjecture.

Theorem 1.2.13 (Cantrell–Reyes [CR22, Thm. 1.8])**.** *There are hyperbolic groups Γ that act isometrically on $\mathbb{R}$-trees so that the action is not small and the corresponding translation length function extends continuously to $\mathcal{C}urr(\Gamma)$.*

Organization of the thesis

Chapter 2 introduces the required preliminary material about δ-hyperbolic spaces, hyperbolic and relatively hyperbolic groups, subgroup separability, CAT(0) cube complexes and cubulated groups, and virtually special cube complexes and groups. This chapter is mostly expository, with a focus on the definitions and examples, so only a few proofs are provided. Even though some familiarity with geometric group theory and non-positive curvature is desirable, this chapter is intended to be accessible to a first-year graduate student.

We continue with Chapter 3, in which we discuss cubulated relatively hyperbolic groups and virtual specialness. The goal is to prove Theorem 1.1.2 (Theorem 3.1.2), by adapting Agol's proof of Theorem 1.1.1. In the first sections we introduce compatible virtually special peripheral subgroups, explain how to deduce Corollaries 1.1.3 and 1.1.5, and prove functoriality of the canonical completion/retraction of special cube complexes. Then we move on to examine group theoretical Dehn filling in more detail, where we prove some results about (relative) malnormality and weak separation of double cosets of relatively quasiconvex subgroups. With these results at hand we can prove Theorem 1.1.7 (Theorem 3.4.5). The rest of the chapter consists of the proof of Theorem 1.1.8 (Theorem 3.5.1), which closely follows Agol's approach in the hyperbolic case.

In Chapter 4 we explore the space of metric structures on hyperbolic groups. The first sections focus on building the required machinery to study this space, as well as introducing relevant and motivational examples. Then we analyse the geometry and topology of this space, as well as some functions associated to it. Here is where we prove Theorem 1.2.4 (Propositions 4.4.1, 4.4.3 and 4.4.13), Theorem 1.2.6 (Theorem 4.4.22), Theorem 1.2.7 (Theorems 4.4.29 and 4.4.30), Theorem 1.2.9 (Theorems 4.4.4 and 4.4.16), and Theorem 1.2.8 (Theorem 4.5.1). These results are also used to study the Manhattan boundary of the space of metric structures. We exhibit many examples of boundary metric structures, and in particular, we prove Theorem 1.2.12 (Corollaries 4.7.7 and 4.7.13, and Propositions 4.7.17 and 4.7.18). Finally, we discuss the continuous extension of the stable translation length function to the space of geodesic currents, which is used to prove Theorem 1.2.13 (Theorem 4.8.4).

At the end of the thesis, Chapter 5 collects some questions and discusses some possible future directions for the projects of this work.

Chapter 2

Preliminaries

We begin with a review of the material that will be used throughout this thesis. Except for Lemma 2.3.10 and Proposition 2.7.9, all the results in this chapter have already appeared in the literature, and we will be explicit when slightly different statements are needed. Most of the results are stated without proof, and examples are included. The interested reader may want to consult the suggested references.

2.1 Pseudo metric spaces and quasi-isometric maps

We start by discussing pseudo metric spaces and the maps between them that will be of our interest.

A *pseudo metric* on a set X is a function $d : X \times X \to \mathbb{R}_{\geq 0}$ satisfying the following for all $x, y, z \in X$:

1. $d(x, x) = 0$;
2. $d(x, y) = d(y, x)$; and,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

A *pseudo metric space* (X, d) is a set X with a pseudo metric d on X. If in addition d satisfies that $d(x, y) = 0$ implies $x = y$, then d is called a *metric* and (X, d) is a *metric space*. If (X, d) is a pseudo metric space, $A \subset X$ is non-empty and $R \geq 0$, the *R-neighborhood* of A is the set $N_R(A) = N_{R,d}(A)$ consisting of all points $x \in X$ such that $d(x, a) \leq R$ for some $a \in A$. Also, two subsets $A, B \subset X$ are *C-Hausdorff close* ($C \geq 0$) if $A \subset N_C(B)$ and $B \subset N_C(A)$. The *diameter* of $A \subset X$ is $\operatorname{diam}(A) = \operatorname{diam}_d(A) = \sup\{d(a, b) \colon a, b \in A\} \in [0, \infty]$.

Given a group Γ, there are two relevant ways to construct pseudo metrics on Γ. In the sequel, all the group actions are considered by the left, and o always denotes the identity group element.

Example 2.1.1 (Orbit pseudo metrics)**.** Suppose that Γ acts on the pseudo metric space (X, d) and let $w \in X$ be a base-point. We define a pseudo metric $d_X^w = d_{(X,d)}^w$ on Γ according to

$$d_X^w(x, y) := d(xw, yw) \text{ for } x, y \in \Gamma.$$

We call this the *orbit pseudo metric* induced by this action (with respect to w).

Example 2.1.2 (Cayley graphs and word metrics)**.** Let $S \subset \Gamma$ be a generating set. From this data we define the *Cayley graph* $\mathrm{Cay}(\Gamma, S)$ as the directed graph whose vertex set is Γ, and where we attach an edge from x to xs for all $x \in \Gamma$ and $s \in S$. The *word length* with respect to S is the function $|\cdot|_S : \Gamma \to \mathbb{Z}_{\geq 0}$ that assigns to $x \in \Gamma$ the minimal number of edges in a directed edge path in $\mathrm{Cay}(\Gamma, S)$ from o to x. The *word metric* with respect to Γ is defined as

$$d_S(x, y) := |x^{-1}y|_S \text{ for } x, y \in \Gamma.$$

Note that d_S is not necessarily symmetric. However, this property is satisfied when S is *symmetric*, meaning that $s \in S$ if and only if $s^{-1} \in S$. Under this assumption, $d_S = d^o_{\mathrm{Cay}(\Gamma,S)}$, where $o \in \Gamma$ is seen as a vertex in $\mathrm{Cay}(\Gamma, S)$ and $\mathrm{Cay}(\Gamma, S)$ is endowed the path metric so that each edge has length 1.

We will consider different classes of maps between pseudo metric spaces.

Definition 2.1.3. Given two pseudo metric spaces (X, d_X) and (Y, d_Y) and constants $\lambda_1, \lambda_2 > 0$ and $\varepsilon \geq 0$, a map $F : X \to Y$ is a $(\lambda_1, \lambda_2, \varepsilon)$-*quasi-isometric embedding* if for any $x, y \in X$ we have

$$\frac{1}{\lambda_1} d_X(x, y) - \varepsilon \leq d_Y(Fx, Fy) \leq \lambda_2 d_X(x, y) + \varepsilon.$$

We say that F is a $(\lambda_1, \lambda_2, \varepsilon)$-*quasi-isometry* if in addition there is some $C \geq 0$ such that $N_C(F(X)) = Y$.

We distinguish three subclasses of quasi-isometric embeddings:

- *Rough similarities*: $(\lambda, \lambda^{-1}, \varepsilon)$-quasi-isometries for $\lambda > 0, \varepsilon \geq 0$;
- *Rough isometries*: $(1, 1, \varepsilon)$-quasi-isometries for $\varepsilon \geq 0$;
- *Isometric embeddings*: $(1, 1, 0)$-quasi-isometric embeddings.

We say that two pseudo metrics d, d_* on a set X are *quasi-isometric* (resp. *roughly similar* or *roughly isometric*) through the identity if the identity map $\mathrm{Id} : (X, d) \to (X, d_*)$ is a quasi-isometry (resp. rough similarity or rough isometry). In most situations, we will simply say that d and d_* are quasi-isometric (resp. roughly similar or roughly isometric). Similarly, we can talk of d and d_* being quasi-isometric, $(\lambda_1, \lambda_2, \varepsilon)$-quasi-isometric, etc.

Example 2.1.4 (Finitely generated groups)**.** Let Γ be a finitely generated group, and let $S, T \subset \Gamma$ be two finite, symmetric generating sets. Then d_S and d_T are quasi-isometric. Therefore, we can talk of the *quasi-isometry type* of a finitely generated group.

Similarly, if $F : X \to \Gamma$ is a map from the pseudo metric space (X, d) into the finitely generated group Γ, we say that F is a quasi-isometric embedding/quasi-isometry if it satisfies that property when Γ is equipped with some word metric with respect to a finite and symmetric generating set. The same considerations apply to maps of the form $\Gamma \to X$ or $H \to \Gamma$, where H is also a finitely generated group.

We will also need some metric version of connectedness for pseudo metric spaces. A pseudo metric space is *geodesic* if every two points can be joined by a continuous arc isometric to an interval of length equal to the distance between the points. Such arcs will be called *geodesic segments*. In the discrete setting, a $(\lambda_1, \lambda_2, \varepsilon)$-*quasigeodesic* in a pseudo metric space (X, d) is a $(\lambda_1, \lambda_2, \varepsilon)$-quasi-isometric embedding $\gamma : \{0, \ldots, n\} \to X$, where n is a non-negative integer and $\{0, \ldots, n\}$ is endowed with its usual distance induced by $\mathbb{Z}$. An α-*rough geodesic* is a $(1, 1, \alpha)$-quasigeodesic, and a rough geodesic is an α-rough geodesic for some α. Sometimes we will work with pseudo metrics on countable sets, so they cannot be geodesic (unless they have diameter 0). However, finding enough quasigeodesics or rough geodesics will suffice.

Definition 2.1.5. The pseudo metric space (X, d) is $(\lambda_1, \lambda_2, \varepsilon)$-*quasigeodesic* (resp. α-*roughly geodesic*) if for any two points $x, y \in X$ there is a $(\lambda_1, \lambda_2, \varepsilon)$-quasigeodesic (resp. α-rough geodesic) $\gamma : \{0, \ldots, n\} \to X$ with $\gamma_0 = x$ and $\gamma_n = y$.

More generally, we will say that a function ψ on $X \times X$ is α-*roughly geodesic* if for any $x, y \in X$ there exists a sequence $x = x_0, \ldots, x_n = y$ such that

$$j - i - \alpha \leq \psi(x_i, x_j) \leq j - i + \alpha \tag{2.1}$$

for all $0 \leq i \leq j \leq n$. In that case, we will say that the sequence $x_0, \ldots, x_n$ is an (α, ψ)-*rough geodesic*. Similarly, we can talk about functions on $X \times X$ being roughy geodesic/quasi-geodesic, and of pairs of functions on $X \times X$ being quasi-isometric/roughly similar/roughly isometric.

2.2 Isometric group actions

Now we turn our attention to group actions on pseudo metric spaces. Given a group Γ, let $\mathbf{conj}_\Gamma$ be the set of conjugacy classes of elements in Γ. Also, let $\mathbf{conj}'_\Gamma$ be the set of conjugacy classes of non-torsion elements of Γ. If there is no ambiguity, we will simply denote these sets by $\mathbf{conj}$ and $\mathbf{conj}'$ respectively. We also use $[x]$ to denote the conjugacy class of $x \in \Gamma$.

Let Γ act on the pseudo metric space (X, d). We will be most interested when this action is *isometric* (or by isometries), meaning that $d(gu, gv) = d(u, v)$ for all $x \in \Gamma$ and $u, v \in X$. If Γ is finitely generated and acts isometrically on the pseudo metric space (X, d), we say that this action is *geometric* if it is *metrically proper* and *cobounded*. Metric properness means that if $B \subset X$ is bounded then there are at most finitely many $x \in \Gamma$ such that $xB \cap B \neq \emptyset$, and coboundedness means that there is a bounded set $B \subset \Gamma$ such that $\Gamma \cdot B = X$.

Remark 2.2.1. The notion of geometric action used here differs from the one that is usually considered, where cocompactness is required (see e.g. [Kap23]).

A key property of geometric actions is that the orbit maps define quasi-isometric maps, so that the large-scale geometry of the group can be recovered from that of the space it is acting on. This is the content of the Milnor-Schwarz lemma.

Lemma 2.2.2 (Milnor-Schwarz Lemma)**.** *Let Γ be a group acting geometrically on the roughly geodesic metric space (X, d).*

1. *If the action is cocompact, then Γ is finitely generated and for any finite, symmetric generating set $S \subset \Gamma$ and $w \in X$, the map $\Gamma \mapsto X$, $x \to xw$ is a quasi-isometry between (Γ, d_S) and (X, d).*

2. *If Γ is finitely generated and the action is cobounded, then for any finite, symmetric generating set $S \subset \Gamma$ and $w \in X$, the map $\Gamma \mapsto X$, $x \to xw$ is a quasi-isometry between (Γ, d_S) and (X, d).*

Note that the quasi-isometries given by the lemma above are Γ-*equivariant* since for all $x, y, g \in \Gamma$ and $w \in X$ we have $d^w_X(gx, gy) = d^w_X(x, y)$.

Isometric group actions induce rough isometry classes of pseudo metrics of the groups acting on. That is, if Γ acts isometrically on the pseudo metric space (X, d), then for all $u, v \in X$ the orbit pseudo metrics d^u_X and d^v_X on Γ satisfy $|d^u_X - d^v_X| \leq 2d(u, v)$. In this case, all the pseudo metrics d^u_X are *left-invariant*, meaning that the left action of Γ on (Γ, d^u_X) is isometric.

For invariant pseudo metrics on groups, we get useful invariants by analyzing the rates at which powers of group elements escape from the identity.

Definition 2.2.3. Given a left-invariant pseudo metric d on Γ, its *(stable) translation length function* is the map $\ell_d : \mathbf{conj} \to \mathbb{R}$ given by the formula

$$\ell_d[x] = \lim_{n \to \infty} \frac{d(o, x^n)}{n}, \text{ for } [x] \in \mathbf{conj}.$$

The stable translation length is well-defined by subadditivity, and the function ℓ_d is sometimes called the *marked length spectrum* of d. Similarly, if f is an isometry of a pseudo metric space (X, d), we use $\ell_d[f]$ to denote the stable translation length of $1 \in \mathbb{Z}$, when $\mathbb{Z}$ is equipped with the pseudo metric $d^w_X(m, n) = d(f^m w, f^n w)$ for some (any) $w \in X$.

Note that if d and d_* are left-invariant pseudo metrics on Γ that are roughly isometric, then $\ell_d = \ell_{d_*}$. Therefore, we can define the stable translation length function of an isometric group action as the translation length function of any of its orbit pseudo metrics. If d, d_* are left-invariant pseudo metrics on Γ that are quasi-isometric, then ℓ_d and ℓ_{d_*} do not necessarily coincide. However, in this case we have that for any $[x] \in \mathbf{conj}$, $\ell_d[x] > 0$ if and only if $\ell_{d_*}[x] > 0$. Note that $\ell_d[x] > 0$ implies $[x] \in \mathbf{conj}'$.

We also need a way of measuring the size of an invariant pseudo metric on a countable group. We do this by looking at the evolution of the cardinalities of balls around the identity.

Definition 2.2.4. Given a left-invariant pseudo metric d on Γ, the *exponential growth rate* of d is defined by

$$v_d = \limsup_{n\to\infty} \frac{1}{n} \log \#\{x \in \Gamma : d(o,x) \leq n\} \in [0,\infty].$$

As before, we can also define the exponential growth rate of an isometric action as the exponential growth rate of any of its orbit pseudo metrics. The property of having finite (resp. positive) exponential growth rate is a quasi-isometric invariant among left-invariant pseudo metrics. For example, if Γ is finitely generated and S is a finite, symmetric generating subset, then the exponential growth rate $v_S := v_{d_S}$ is always finite.

2.3 Gromov hyperbolicity

Gromov hyperbolic spaces are a central topic in this thesis, and in this section we introduce them in the slightly more general setting of pseudo metric spaces. For a more detailed approach to Gromov hyperbolic spaces, we invite the reader to check the references [CDP90; DSU17; GH90].

2.3.1 Hyperbolicity and its consequences

Let (X,d) be a pseudo metric space and $w \in X$ a base-point. The *Gromov product* on (X,d) based at w is the function $(\cdot|\cdot)_{w,d} : X \times X \to \mathbb{R}$ given by

$$(x|y)_{w,d} := \frac{d(x,w) + d(w,y) - d(x,y)}{2}.$$

If there is no ambiguity in the pseudo metric d, we just write $(x|y)_w$ instead of $(x|y)_{w,d}$.

Definition 2.3.1. The pseudo metric space (X,d) is *δ-hyperbolic* ($\delta \geq 0$) if for all $x,y,z,w \in X$ the following inequality is satisfied:

$$(x|z)_w \geq \min\{(x|y)_w, (y|z)_w\} - \delta. \tag{2.2}$$

A pseudo metric space is *(Gromov) hyperbolic* if it is δ-hyperbolic for some $\delta \geq 0$. Similarly, we say that a pseudo metric d on a space X is *hyperbolic* (resp. *δ-hyperbolic*) if (X,d) is a hyperbolic (resp. δ-hyperbolic) pseudo metric space.

Example 2.3.2 (Elementary examples)**.** Trivial examples of hyperbolic spaces include bounded pseudo metric spaces, and the real line. Such hyperbolic spaces are sometimes called *elementary*.

Example 2.3.3 ($\mathbb{R}$-trees)**.** An $\mathbb{R}$-*tree* is a geodesic metric space such that every pair of points is joined by a unique embedded arc. This forces all $\mathbb{R}$-trees to be 0-hyperbolic. In fact, $\mathbb{R}$-trees are precisely the geodesic 0-hyperbolic spaces. Familiar examples are simplicial trees equipped with the graph metric, and more generally, metric trees.

Example 2.3.4 (Real hyperbolic spaces)**.** Given $n \geq 2$, let $(\mathbb{H}^n, d_{\mathbb{H}^n})$ be the unique simply connected, n-dimensional complete Riemannian manifold with constant negative sectional curvature equal to -1, given by the Killing–Hopf theorem. These are the *real hyperbolic spaces*, and they are δ-hyperbolic with $\delta = \log 2$ [NŠ16, Cor. 5.4]. Often, we will work with the 2-dimensional real hyperbolic space $\mathbb{H}^2$, which we will simply call *hyperbolic plane*. Different descriptions and models for $\mathbb{H}^n$ are given in [BH99, Ch. I.2].

Example 2.3.5 (Pinched negatively curved manifolds)**.** Generalizing the example above, let $(M, \mathfrak{g})$ be any simply connected, Riemannian manifold with sectional curvatures at most $-a < 0$. If $d_{\mathfrak{g}}$ denotes the length metric on M induced by $\mathfrak{g}$, then $(M, d_{\mathfrak{g}})$ is δ-hyperbolic for δ depending only on a [GH90, Ch. 3].

Non-Example 2.3.6 (Euclidean spaces)**.** The Euclidean plane $\mathbb{R}^2$ with its standard metric is not δ-hyperbolic for any δ, by the existence of homotheties. In consequence, if a pseudo metric space contains an isometrically embedded copy of $\mathbb{R}^2$ then it cannot be hyperbolic.

Among geodesic metric spaces, hyperbolicity is equivalent to having uniformly thin triangles. That is, if (X, d) is a geodesic metric space, then it is δ-hyperbolic for some $\delta \geq 0$ if and only if for any geodesic triangle in X with vertices x, y, z and geodesic segments $[x, y], [y, z], [y, z]$, then $[x, z] \subset N_{\delta'}([x, y] \cup [y, z])$, for δ' depending only on δ [GH90, Ch. 2, Prop. 21]. Triangles satisfying the condition above will be called δ'-*slim*. In the roughly geodesic setting, hyperbolicity implies the existence of quasi-centers. Given points x, y, z in a pseudo metric space (X, d), a κ-*quasi-center* of this triple is a point $p \in X$ such that

$$\max\{(x|y)_{p,d}, (y|z)_{p,d}, (z|x)_{p,d}\} \leq \kappa.$$

We say that p is a (κ, d)-quasi-center if we want to make explicit that p is a κ-quasi-center with respect to the pseudo metric d. If (X, d) is δ-hyperbolic and α-roughly geodesic, then any triple of points in (X, d) has a κ-quasi-center, where κ only depends on δ and α [GH90, Ch. 2, Prop. 21].

Sometimes we want to conclude properties for hyperbolic spaces as if they were geodesic or roughly geodesic. A way of doing this is by considering the *injective hull* functor [Lan13], which allows us to isometrically embed metric spaces into geodesic metric spaces in a way that is equivariant with respect to the isometry groups. More precisely, given a metric space (X, d) we can construct another metric space $(\hat{X}, \hat{d})$, the *injective hull of* (X, d) and an isometric embedding $i : (X, d) \hookrightarrow (\hat{X}, \hat{d})$. The properties that we need are that $(\hat{X}, \hat{d})$ is geodesic, δ-hyperbolic if (X, d) is δ-hyperbolic [Lan13, Prop. 1.3], and that any isometry of X extends uniquely under i to an isometry of $\hat{X}$ (this follows from functoriality [Lan13, Prop. 3.7]). In addition, from the proof of [Lan13, Prop. 1.3] we see that if X is roughly geodesic, then $i(X)$ is coarsely dense in $\hat{X}$. If (X, d) is a pseudo metric space, we consider the injective hull of its metric identification. We summarize these properties into the following proposition.

Proposition 2.3.7 (Lang [Lan13]). *If (X, d) is a δ-hyperbolic pseudo metric space, then its injective hull $(\hat{X}, \hat{d})$ is δ-hyperbolic and geodesic and there is an isometric embedding $i : (X, d) \to (\hat{X}, \hat{d})$. In addition, every isometry of (X, d) extends uniquely to an isometry of $(\hat{X}, \hat{d})$, and the map i is equivariant with respect to the group of isometries of (X, d). Moreover, if (X, d) is α-roughly geodesic then $N_{\delta+4\alpha+1}(i(X)) = \hat{X}$.*

One of the main properties of hyperbolic spaces is the Morse lemma, which says that uniform quasigeodesic with the same endpoints are uniformly close. Since we do not assume that hyperbolic spaces are geodesic, the version we will consider is the following (cf. [Väi05, Thm. 3.7]).

Lemma 2.3.8 (Morse Lemma). *For any $\delta, \varepsilon \geq 0$ and $\lambda_1, \lambda_2 > 0$ there exists some $C = C(\delta, \lambda_1, \lambda_2, \varepsilon) \geq 0$ such that the following holds. Suppose (X, d) is a δ-hyperbolic pseudo metric space and γ_1, γ_2 are $(\lambda_1.\lambda_2, \varepsilon)$-quasigeodesics in (X, d) joining the same pair of points. Then the images of γ_1 and γ_2 are C-Hausdorff close in (X, d).*

A classical consequence of this lemma is that hyperbolicity is preserved under quasi-isometries among roughly hyperbolic spaces.

Corollary 2.3.9. *For any $\delta, \alpha, \varepsilon \geq 0$ and $\lambda_1, \lambda_2 > 0$ there exists some $\delta' = \delta'(\delta, \alpha, \lambda_1, \lambda_2, \varepsilon) \geq 0$ such that the following holds. Suppose $F : (X, d_X) \to (Y, d_Y)$ is $(\lambda_1.\lambda_2, \varepsilon)$-quasi-isometric embedding, that (X, d_X) and (Y, d_Y) are α-roughly geodesic and that (Y, d_Y) is δ-hyperbolic. Then (X, d_X) is δ'-hyperbolic.*

The injective hull functor can be used together with the Morse Lemma 2.3.8 to prove that hyperbolic quasigeodesic spaces are actually roughly geodesic.

Lemma 2.3.10. *For any $\lambda_1, \lambda_2 > 0$ and $\varepsilon, \delta \geq 0$ there exists $\alpha = \alpha(\delta, \lambda_1, \lambda_2, \varepsilon) \geq 0$ such that if (X, d) is a $(\lambda_1, \lambda_2, \varepsilon)$-quasigeodesic and δ-hyperbolic pseudo metric space, then (X, d) is α-roughly geodesic.*

Proof. Without loss of generality, we can assume that (X, d) is a metric space. By Proposition 2.3.7, we consider (X, d) as a subset of its injective hull $(\hat{X}, \hat{d})$, which is δ-hyperbolic and geodesic. Let $\gamma : \{0, \dots, n\} \to X$ be a $(\lambda_1, \lambda_2, \varepsilon)$-quasigeodesic, and apply the Morse Lemma 2.3.8 to obtain a constant $C = C(\delta, \lambda_1, \lambda_2, \varepsilon) \geq 0$ and a geodesic segment $\gamma' : [0, d(\gamma(0), \gamma(n))] \to \hat{X}$ joining $\gamma(0)$ and $\gamma(n)$ such that $\gamma(\{0, \dots, n\})$ and $\gamma'([0, d(\gamma(0), \gamma(n))])$ are C-Hausdorff close. If $j \in [0, d(\gamma(0), \gamma(n))]$ is an integer, then there is some $i_j \in \{0, \dots, n\}$ such that $\hat{d}(\gamma'(j), \gamma(i_j)) \leq C$, for which we can assume $i_0 = 0$. In this way, if we define $m = \lfloor \hat{d}(\gamma(0), \gamma(n)) \rfloor$, $x_j = \gamma(i_j)$ for $0 \leq j \leq m$ and $x_{m+1} = \gamma(n)$, then the sequence $x_0, x_1, \dots, x_m, x_{m+1}$ is a $(2C + 1)$-rough geodesic in (X, d) joining $\gamma(0)$ and $\gamma(n)$. As every pair of points in X can be joined by a $(\lambda_1, \lambda_2, \varepsilon)$-quasigeodesic, the conclusion follows by taking $\alpha = 2C(\delta, \lambda_1, \lambda_2, \varepsilon) + 1$. □

We mention one last consequence of the Morse Lemma 2.3.8, which is that among hyperbolic and roughly geodesic pseudo metric spaces, quasi-isometries are also close to preserving Gromov products. The following proposition appears in [GH90, Ch. 5, Prop. 15 (i)] in the case of geodesic metric spaces, and it can be easily adapted to the roughly geodesic setting by applying Proposition 2.3.7.

Proposition 2.3.11. *For all $\alpha, \delta, \varepsilon \geq 0$ and $\lambda_1, \lambda_2 > 0$ there exists $A = A(\alpha, \delta, \lambda_1, \lambda_2, \varepsilon) \geq 0$ such that the following holds. Let $(X, d_X), (Y, d_Y)$ be δ-hyperbolic and α-roughly geodesic pseudo metric spaces, and $F : X \to Y$ a $(\lambda_1, \lambda_2, \varepsilon)$-quasi-isometric embedding. Then for all $x, y, w \in X$:*

$$\frac{1}{\lambda_1}(x|y)_{w,d_X} - A \leq (Fx|Fy)_{Fw,d_Y} \leq \lambda_2(x|y)_{w,d_X} + A.$$

2.3.2 The Gromov boundary

If (X, d) is a δ-hyperbolic pseudo metric space and $w \in X$, there is a well-defined *Gromov boundary* $\partial X = \partial(X, d)$ consisting of the equivalence classes of sequences $(x_n)_n$ in X such that $(x_m|x_n)_w \to \infty$ as $m, n \to \infty$, where two sequences $(x_n)_n$ and $(y_n)_n$ represent the same point at infinity if $(x_n|y_n)_w \to \infty$. δ-hyperbolicity implies that the Gromov boundary is independent of the base-point w. The Gromov product can be extended to $X \cup \partial X$ via

$$(p|q)_w := \inf\{\liminf_{n\to\infty} (p_n|q_n)_w \colon p_n \to p, q_n \to q\},$$

where for $r \in \partial X$ the notation $r_n \to r$ means that the sequence $(r_n)_n$ represents r. In this way, by (2.2) we get that for all $p, q, r \in X \cup \partial X$ and $w \in X$:

$$(p|r)_w \geq \min\{(p|q)_w, (q|r)_w\} - \delta. \tag{2.3}$$

We can also define the *Busemann function* on $X \times X \times \partial X$ according to

$$\beta_d(x, y; p) := \sup\{\limsup_{n\to\infty} (d(x, p_n) - d(y, p_n)) \colon p_n \to p\} = d(x, y) - 2(x|p)_{y,d} \tag{2.4}$$

for $x, y \in X$, $p \in \partial X$. δ-hyperbolicity implies that for all $x, y \in X$ and $p \in \partial X$ we have

$$|\beta_d(x, y; p) + \beta_d(y, x; p)| \leq 4\delta.$$

In addition, for all $\varepsilon, \delta > 0$ satisfying $0 < \varepsilon\delta < \log 2$ and $w \in X$ there is a metric $\varrho = \varrho_{\varepsilon,w}$ on ∂X such that for all $p, q \in \partial X$

$$(2\varepsilon\delta)^{-1} e^{-\varepsilon(p|q)_{w,d}} \leq \varrho_{\varepsilon,w}(p, q) \leq e^{-\varepsilon(p|q)_{w,d}}, \tag{2.5}$$

see e.g. [Sch06, Thm. 1.2]. These are called *visual metrics* on ∂X, and induce a canonical topology on ∂X [GH90, Ch. 7, Prop. 10].

Example 2.3.12. Let $\mathbb{H}^n$ be the real hyperbolic space of dimension n. By using the Poincaré sphere model, it can be shown that $\partial\mathbb{H}^n$ is homeomorphic to a sphere of dimension $n-1$.

Example 2.3.13. If T is a locally finite metric tree, then ∂T is homeomorphic to a Cantor set.

By applying Proposition 2.3.11 it can be shown that among roughly geodesic hyperbolic spaces, the Gromov boundary is a quasi-isometric invariant (see [GH90, Ch. 7, §4, Prop. 14]).

Proposition 2.3.14. *Let (X, d_X) and (Y, d_Y) be hyperbolic and roughly geodesic pseudo metric spaces. Then any quasi-isometric embedding $F : X \to Y$ naturally extends to a continuous injective map $\overline{F} : \partial X \to \partial Y$. In particular, if F is a quasi-isometry, then $\overline{F}$ is a homeomorphism.*

2.3.3 Isometric actions on hyperbolic spaces

By Proposition 2.3.14, every isometry in a hyperbolic pseudo metric space extends to a homeomorphism of its Gromov boundary. This allows us to give a dynamical classification of isometries of hyperbolic spaces. If (X, d) is hyperbolic, then an isometry f of (X, d) is either:

- *Elliptic*: if it has bounded orbits.
- *Parabolic*: if it has unbounded orbits and it has a unique fixed point in ∂X.
- *Loxodromic*: if it has unbounded orbits and fixes two distinct points in ∂X.

If f is an isometry of (X, d) with unbounded orbits, then it is loxodromic if and only if $\ell_d[f] > 0$, in which case the orbit map $\mathbb{Z} \to X$, $n \mapsto f^n w$ is a quasi-isometric embedding for any $w \in X$. Similarly, if d is a left-invariant and hyperbolic pseudo metric on the group Γ, then we can talk about elements of Γ acting *elliptically/parabolically/loxodromically* on (Γ, d).

As in the case of a single isometry, the behavior of a finite set of isometries of a hyperbolic space is restricted by the stable translation lengths of the elements in the semigroup they generate. A modern version of this is the following Bochi-type inequality due to Breuillard and Fujiwara [BF21]. They proved this result when (X, d) is metric and geodesic, but by using Proposition 2.3.7 we can also deduce it when (X, d) is just pseudo metric and roughly geodesic.

Theorem 2.3.15 (Breuillard–Fujiwara [BF21, Thm. 1.4])**.** *There exists a universal constant $K > 0$ such that the following holds. Suppose (X, d) is a δ-hyperbolic, α-roughly geodesic pseudo metric space and S is a finite set of isometries of (X, d). Then*

$$\inf_{x\in X} \max_{s\in S} d(sx, x) \leq \frac{1}{2} \max_{s_1,s_2\in S} \ell_d[s_1 s_2] + K\delta + 4\alpha + 1.$$

2.4 Hyperbolic groups

Since hyperbolicity is a quasi-isometric invariant (Corollary 2.3.9), it is natural to look at finitely generated groups whose Cayley graphs (for finite generating sets) are hyperbolic. In this way we obtain the class of hyperbolic groups, introduced by Gromov [Gro87] in 1987. Since then, it has become one of the most studied classes of groups in geometric group theory. Standard references about hyperbolic groups include [CDP90; GH90].

Definition 2.4.1. A finitely generated group Γ is *hyperbolic* if for some finite, symmetric generating set $S \subset \Gamma$ the metric space (Γ, d_S) is hyperbolic. A hyperbolic group is *non-elementary* if it is not virtually cyclic.

Equivalently, by the Milnor-Schwarz Lemma 2.2.2 and Corollary 2.3.9, a finitely generated group is hyperbolic if and only if it admits a geometric action on a hyperbolic, roughly geodesic metric space. If Γ is hyperbolic then (Γ, d_S) is hyperbolic for any finite, symmetric generating set $S \subset \Gamma$ since any two such metrics are quasi-isometric. Also, by Proposition 2.3.11, the space $\partial\Gamma := \partial(\Gamma, d_S)$ is independent of the choice of S, as well as its topology. We call this the *Gromov boundary* of Γ, which is a compact metrizable space when Γ is infinite. Indeed, a hyperbolic group is non-elementary if and only if its Gromov boundary is infinite.

Since the left-action of Γ on (Γ, d_S) is by isometries, there exists a natural topological action of Γ on $\partial\Gamma$. If $x \in \Gamma$ is a non-torsion element, then the homomorphism $\mathbb{Z} \to (\Gamma, d_S)$ given by $n \to x^n$ is a quasi-isometric embedding, and hence x is loxodromic and fixes precisely two points in $\partial\Gamma$ [BH99, Cor. III.Γ.3.10].

Example 2.4.2 (Elementary hyperbolic groups)**.** Trivial examples of hyperbolic groups include finite groups and groups that are virtually $\mathbb{Z}$. These groups are called *elementary*.

Example 2.4.3 (Free groups)**.** Let S be any set and consider the topological graph G_S given by a single vertex v, where we attach a loop to this vertex for each element of S. The *free group* generated by S is $F(S)$, the fundamental group of G_S based at v. The group $F(S)$ acts by Deck transformations on the universal cover of T_S of G_S, which is a tree (hence 0-hyperbolic) when endowed with the length metric that assigns length 1 to each edge. Equivalently, we can consider $S \cup S^{-1}$ as a generating set for $F(S)$ and whose word metric d_S is hyperbolic. When S is finite, $F(S)$ is finitely generated and hence $F(S)$ is hyperbolic. The *rank* of $F(S)$ is the cardinality of S, and we usually denote $F_n = F(\{1, 2, \dots, n\})$. Usually, by a *free group* we will mean a finitely generated free group of rank at least 2.

Example 2.4.4 (Fundamental groups of negatively curved manifolds)**.** Let $(M, \mathfrak{g})$ be a closed Riemannian manifold of negative sectional curvature. Then $\Gamma = \pi_1(M)$ acts properly discontinuously and cocompactly by Deck transformations on the universal cover $\widetilde{M}$ of M. If $\widetilde{\mathfrak{g}}$ is the lift of $\mathfrak{g}$ to $\widetilde{M}$, then this action is by isometries on $(\widetilde{M}, d_{\widetilde{\mathfrak{g}}})$, hence geometric. Hyperbolicity of $(\widetilde{M}, d_{\widetilde{\mathfrak{g}}})$ combined with the Milnor-Schwarz Lemma 2.2.2 and Corollary 2.3.9 imply that Γ is hyperbolic. When M is 2-dimensional (i.e. a surface) and orientable, we call Γ a *hyperbolic surface group*, or simply a *surface group*.

Example 2.4.5 (Small cancellation groups)**.** Let $\Gamma = \langle S \mid R \rangle$ be a group presentation where $R \subset F(S)$ is a set of freely reduced and cyclically reduced words in the free group $F(S)$ such that R is symmetric and closed under taking cyclic permutations. A *piece* with respect to this presentation is a non-trivial freely reduced word $u \in F(S)$ such that there exist two distinct elements $r_1, r_2 \in R$ that have u as a maximal common initial segment.

Given $0 < \lambda < 1$, the presentation above is said to satisfy the $C'(\lambda)$ *small cancellation condition* if whenever u is a piece with respect to the presentation and u is a subword of some $r \in R$, then $|u|_S < \lambda |r|_S$. Finitely generated groups with a presentation satisfying the $C'(1/6)$ small cancellation condition are hyperbolic [GH90, Appendix, Thm. 36]. Examples of these groups include the 1-relator groups $\Gamma = \langle S \mid r^n \rangle$ where S is finite, $r \in F(S)$ is a non-trivial cyclically reduced word which is not a proper power in $F(S)$, and $n \geq 6$. Indeed, it can be shown that the 1-relator group with torsion $\Gamma = \langle S \mid r^n \rangle$ is hyperbolic for *every* $n \geq 2$ [New68].

Example 2.4.6 (Free products of hyperbolic groups)**.** If Γ_1 and Γ_2 are hyperbolic groups, then the free product $\Gamma_1 * \Gamma_2$ is hyperbolic. The same holds for some amalgamated free products of free groups (Theorem 2.4.19).

Example 2.4.7 (Non-linear hyperbolic groups)**.** A group Γ is *linear* over the field k if there exists a finite-dimensional vector space V over k and a monomorphism $\Gamma \hookrightarrow \mathrm{GL}(V)$. A group is *non-linear* if it is not linear over any field. In [Kap05], Kapovich produced non-linear hyperbolic groups as quotients of certain *superrigid* cocompact lattices in rank-1 Lie groups. Recent examples of non-linear hyperbolic groups have been constructed as HNN extensions of such lattices [CST19].

Example 2.4.8 (Hyperbolic groups with property (T))**.** A countable discrete group Γ has *property (T)* if every isometric action of Γ on a real Hilbert space has a fixed point. Examples of hyperbolic groups with property (T) are lattices in $\mathrm{Sp}(1, n)$, the isometry groups of quaternionic hyperbolic spaces. Indeed, every discrete group with property (T) is a quotient of a torsion-free hyperbolic group with property (T) [Cor05].

Example 2.4.9 (Exotic hyperbolic fundamental groups of manifolds)**.** Examples of hyperbolic groups can be constructed via *hyperbolization* procedures. For example, in [DJ91, Thm. 5c.1 and Rmk. on p. 386] Davis and Januszkiewicz constructed for $n \geq 5$, a closed, aspherical manifold with hyperbolic fundamental group, but admitting no negatively curved Riemannian metric.

Example 2.4.10 (Random groups)**.** Let $m \geq 2$ be an integer and consider the free group $F_m = F(S_m)$, so that S_m has m elements. Let $0 \leq d \leq 1$ be the *density parameter*, and let ℓ be a (large) length. We choose $(2m-1)^{d\ell}$ times (rounded to the nearest integer) at random a reduced word of length ℓ in the letters $S_m \cup S_m^{-1}$ uniformly among all such words, and let R be the set of words so obtained. A *random group at density d and length ℓ* is the group with presentation $G = \langle S_m \mid R \rangle$. A property occurs *with overwhelming probability* in this model if its probability of occurrence tends exponentially to 1 as $\ell \to \infty$. This model of random groups was introduced by Gromov, who proved what if $d < 1/2$ then with overwhelming probability, a random group at density d is infinite, hyperbolic, and torsion-free, and that if $d > 1/2$ then with overwhelming probability, a random group at density d is either trivial or $\mathbb{Z}/2\mathbb{Z}$ [Gro93]. Results of similar nature hold for other models of random groups.

2.4.1 Strongly Markov structures

Hyperbolic groups are combinatorially well-behaved, and an illustration of this is that they are strongly Markov: each hyperbolic group Γ and generating S can be represented by a finite directed graph $\mathcal{G}$ called the *Cannon graph* [Can84]. The graph $\mathcal{G}$ comes equipped with a labeling π that assigns a generator (in S) to each directed edge of $\mathcal{G}$ and this labeling has a variety of useful properties. The triple $(\mathcal{G}, \pi, S)$ is referred to as a *strongly Markov structure* on Γ. For a precise definition and list of properties for strongly Markov structures, see [CT21]. The main property of the Cannon graph is that there is an initial vertex $*$ such that the labeling map π induces a bijection between the elements in Γ and the finite paths in $\mathcal{G}$ starting at $*$. More precisely, if $(e_1, \ldots, e_n)$ is a path determined by consecutive edges $e_1, \ldots, e_n$ in $\mathcal{G}$ then the map

$$(e_1, \ldots, e_n) \mapsto \pi(e_1)\pi(e_2)\cdots\pi(e_n)$$

defines a bijection between the finite paths starting at $*$ and the elements of Γ. Furthermore, this bijection preserves path/word length, i.e. this map sends paths of length n to elements of word length n with respect to S. For our purposes, we need to understand the properties of certain components within strongly Markov structures.

Definition 2.4.11. Let $\mathcal{G}$ be a finite directed graph. A (connected) *component* in $\mathcal{G}$ is a maximal subgraph $\mathcal{C}$ of $\mathcal{G}$ such that for any two vertices in $\mathcal{C}$ we can find a loop in $\mathcal{C}$ that visits both of these vertices.

Each Cannon graph $\mathcal{G}$ can be described by a $0-1$ matrix A called the *transition matrix.* This matrix is $n \times n$, where n is the number of vertices in $\mathcal{G}$ and each row/column in A corresponds to a vertex. The (i, j)th entry of A is 1 if and only if there is a directed edge going from the ith to the jth vertex. Likewise, each connected component within $\mathcal{G}$ can be described by a submatrix within the transition matrix for $\mathcal{G}$. We call a component $\mathcal{C}$ *maximal* if the number of paths of length n belonging to $\mathcal{C}$ has the same exponential growth rate as that of the metric d_S, i.e. the number of paths of length n living entirely in $\mathcal{C}$ has exponential growth rate v_S. This is the same as saying that the submatrix that describes the component $\mathcal{C}$ has spectral radius e^{v_S}. Each strongly Markov structure necessarily has (possibly multiple) maximal components. Suppose we have a strongly Markov structure $(\mathcal{G}, \pi, S)$ on Γ and that we have fixed a maximal component $\mathcal{C}$. We define

$$\Gamma_{\mathcal{C}} := \{x \in \Gamma : \ x = \pi(e_1)\cdots\pi(e_n) \text{ for some finite path } (e_1, \ldots, e_n) \text{ living in } \mathcal{C}\}.$$

Intuitively, $\Gamma_{\mathcal{C}}$ contains all group elements that can be seen as a finite path in $\mathcal{C}$. The following result, which is a consequence of a combinatorial property known as *growth quasi-tightness*, is crucial to our work.

Proposition 2.4.12. *Suppose we have a strongly Markov structure $(\mathcal{G}, \pi, S)$ on a hyperbolic group Γ and that we have fixed a maximal component $\mathcal{C}$. Then there exists a finite set $B \subset \Gamma$ such that $B\Gamma_{\mathcal{C}}B = \Gamma$.*

Intuitively this result says that maximal components see all group elements up to thickening by a uniformly bounded amount. This result was originally observed within the proof Lemma 4.6 in [GMM18]. See [CT21] for a more detailed discussion.

2.4.2 Quasiconvex subgroups

If (X, d) is a geodesic metric space, a subset $A \subset X$ is λ-*quasiconvex* if any geodesic segment in X with endpoints in A lies in the λ-neighborhood of A. A subset is quasiconvex if it is λ-quasiconvex for some λ.

Definition 2.4.13. A subgroup H of the hyperbolic group Γ is *quasiconvex* if for some finite, symmetric generating set $S \subset \Gamma$, H is a quasiconvex subset of $\mathrm{Cay}(\Gamma, S)$.

By the Morse Lemma 2.3.8, the definition of quasiconvexity is independent of the chosen finite generating set S. A related notion is the concept of undistorted subgroup.

Definition 2.4.14. A finitely generated subgroup H of the finitely generated group Γ is *undistorted* if the inclusion $H \hookrightarrow \Gamma$ is a quasi-isometric embedding for any choices of finite generating subsets of H and Γ.

Among hyperbolic groups, quasiconvex subgroups are the same as undistorted subgroups [BH99, Cor. III.Γ.3.6], from which we get that cyclic and finite index subgroups of hyperbolic groups are quasiconvex. Also, the intersection of finitely many quasiconvex subgroups is quasiconvex [BH99, Prop. III.Γ.3.9], quasiconvex subgroups of hyperbolic groups are also hyperbolic [BH99, Prop. III.Γ.3.7], and being a quasiconvex subgroup is a transitive relation among hyperbolic groups.

Example 2.4.15. Every finitely generated subgroup of a finitely generated free group is quasiconvex. Similarly, it can be proven that every finitely generated subgroup of a surface group is quasiconvex (see. e.g. [Git97]).

Example 2.4.16. Let M, N be closed negatively curved Riemannian manifolds such that there is a convex, totally geodesic immersion $M \to N$. Then the group $H = \pi_1(M)$ injects as a quasiconvex subgroup of $\Gamma = \pi_1(N)$.

Example 2.4.17. A remarkable result of Kahn and Markovic asserts that if Γ is the fundamental group of a closed hyperbolic 3-manifold (i.e. a torsion-free group acting geometrically on the 3-dimensional hyperbolic space $\mathbb{H}^3$), then it contains many quasiconvex surfaces subgroups [KM12], solving the Surface Subgroup Conjecture in the affirmative. Quasiconvex surface subgroups were also found for cocompact lattices in rank-1 Lie groups [Ham15].

Non-Example 2.4.18. Quasiconvex subgroups of hyperbolic groups have *finite height*. That is, if $H < \Gamma$ is a quasiconvex subgroup, then there is some n such that for any distinct left cosets $g_1H, \dots, g_nH$ with $g_i \in \Gamma$, the intersection $g_1Hg_1{}^{-1} \cap \dots \cap g_nHg_n{}^{-1}$ is finite [Git+98, Main Thm.]. A generalization of this property is given in Corollary 3.3.11. In particular, if $H \trianglelefteq \Gamma$ is an infinite normal subgroup of infinite index, then H is not quasiconvex. It is unknown whether there exists a non-quasiconvex subgroup of a hyperbolic group that has finite height.

Quasiconvex subgroups can also be used to build new hyperbolic groups from old ones. A subgroup H of a group Γ is *malnormal* if $gHg^{-1} \cap H$ is trivial for all $g \in \Gamma \backslash H$ (see also Subsection 3.3.1). This property is stronger than having finite height. The following is a particular case of Bestvina-Feighn combination theorem.

Theorem 2.4.19 (Bestvina–Feighn [BF92])**.** *Let Γ_1, Γ_2 be hyperbolic groups and let H be a group that embeds as a quasiconvex and malnormal subgroup of both Γ_1 and Γ_2. Then $\Gamma_1 *_H \Gamma_2$ is hyperbolic.*

2.5 Relatively hyperbolic groups

Roughly speaking, relatively hyperbolic groups are those in which non-hyperbolic behavior can only be found inside some isolated subgroups, which form at most finitely many conjugacy classes. Depending on the perspective, there are several (but equivalent) definitions for relative hyperbolicity [Hru10], some of them valid for arbitrary countable groups. For convenience, we will restrict to the finitely generated case and introduce relative hyperbolicity in terms of *cusped spaces*. See [GM08] or [Ago13, Appendix A] for more details about cusped spaces associated to relatively hyperbolic groups.

Let Γ be a group generated by a finite, symmetric set S, and let $\mathcal{P} = \{P_1, \dots, P_n\}$ be a finite collection of subgroups of Γ such that $S \cap P_i$ generates P_i for every i.

For each $P \in \mathcal{P}$, let $S_0 = S \cap P \backslash \{o\}$, and for $n > 0$ let $S_n := S_{n-1} \cup \{s_1s_2 \neq o \colon s_1, s_2 \in S_{n-1}\}$. Given a left coset gP with $g \in \Gamma$, define the 1-complex $\mathcal{H}(gP)$ as the vertex set $\mathcal{H}(gP)^{(0)} = gP \times \mathbb{Z}_{\geq 0}$ and edges given by:

1. (vertical) If $(v, n) \in \mathcal{H}(gP)^{(0)}$, then an edge joins (v, n) and $(v, n+1)$.
2. (horizontal) If $(v, n) \in \mathcal{H}(gP)^{(0)}$ and $s \in S_n$, there is an edge from (v, n) to (vs, n) (so that if for instance, s has order 2, then there are two different edges between (v, n) and (vs, n)).

Note that there is a natural way to glue $\mathcal{H}(gP)$ and the Caley graph $\mathrm{Cay}(\Gamma, S)$ along $gP = gP \times \{0\}$.

Definition 2.5.1. The *cusped space* $X(\Gamma, \mathcal{P}, S)$ is obtained from $\mathrm{Cay}(\Gamma, S)$ by gluing all the complexes $\mathcal{H}(gP)$ for $g \in \Gamma$ and $P \in \mathcal{P}$ in the previously mentioned way. The group Γ is *hyperbolic relative to* $\mathcal{P}$ if the cusped space $X(\Gamma, \mathcal{P}, S)$ is hyperbolic when endowed with the graph metric [Ago13, Rmk. A.13]. In that case, we say that $\mathcal{P}$ is a *peripheral structure* on Γ.

Remark 2.5.2. With the definition of cusped space given above, the natural isometric action of Γ on $X(\Gamma, \mathcal{P}, S)$ is *free*, and the distance function on vertices coincides with the one constructed in [GM08]. Therefore, the coarse geometry is unchanged, and the classical results about cusped spaces also hold for this slightly different construction (see [Ago13, Rmk. A.13]).

If $(\Gamma, \mathcal{P})$ is relatively hyperbolic, then a *parabolic subgroup* will be any subgroup of Γ that can be conjugated into a member of $\mathcal{P}$, and a maximal parabolic subgroup will be called *peripheral subgroup*. An element of Γ is *loxodromic* if it is infinite order and the group it generates is non-parabolic. This classification is consistent with the one given in Section 2.3.3. A *horoball* of $X(\Gamma, \mathcal{P}, S)$ is a full subgraph on the vertices $gP \times \mathbb{Z}_{\geq R}$ for some $R \geq 0$ and some left coset gP with $P \in \mathcal{P}$. Note that peripheral subgroups correspond to stabilizers in Γ of horoballs. An infinite subgroup of Γ is non-parabolic if and only if it contains a loxodromic element. Namely, if $H < \Gamma$ is infinite and with no loxodromics, then any H-orbit of a point in $X(\Gamma, \mathcal{P}, S)$ is unbounded and H must fix a point in the Gromov boundary $\partial X(\Gamma, \mathcal{P}, S)$ (see e.g. [DSU17, Thm. 6.2.3]). The action of Γ on $\partial X(\Gamma, \mathcal{P}, S)$ is *geometrically finite* [Hru10, Sec. 3], so this fixed point corresponds to a horoball by [Tuk98, Thm. 3A (a)].

Any finitely generated group is hyperbolic relative to the peripheral structure corresponding to the whole group. Also, any hyperbolic group is hyperbolic relative to any (possibly empty) finite set of finite subgroups. Also, if $(\Gamma, \mathcal{P})$ is relatively hyperbolic and each $P \in \mathcal{P}$ is hyperbolic, then Γ is hyperbolic [Far98, Thm. 3.8].

Example 2.5.3 (Cusped hyperbolic manifolds)**.** Let M be finite-volume hyperbolic 3-manifold with cusps. Then $\Gamma = \pi_1(M)$ is hyperbolic relative to the cusps subgroups, which are free abelian of rank 2 [Gro87]. For example, the fundamental group of the figure-8 complement is relatively hyperbolic. More generally, if M is a pinched negatively curved Riemannian manifold of finite volume, then $\Gamma = \pi_1(M)$ is hyperbolic relative to the cusp subgroups, which are virtually nilpotent.

Example 2.5.4 (Small cancellation free products)**.** Generalizing Example 2.4.5, let $G = G_1 * \cdots * G_n$ be the free product of finitely many groups $G_1, \ldots, G_n$, which are called the *free factors* of G. Every non-trivial element of G can be represented in a unique way as a product $w = h_1 \cdots h_m$, called the *normal form*, where h_i is a non-trivial element in some G_j and no two consecutive h_i, h_{i+1} belong to the same free factor. The *free product length* of w is given by $|w| := m$. The normal form of w is *weakly cyclically reduced* if $|w| \leq 1$ or $h_1 \neq h_{m-1}$. If $u, v \in G$ are such that $u = h_1 \cdots h_m, v = k_1 \cdots k_l$, and $h_m = k_1^{-1}$, then h_m and k_1 *cancel* in the product uv. Otherwise, we say that the product uv is *weakly reduced*. Now, let $R \subset G$ be a subset whose elements are represented by weakly cyclically reduced normal forms, and assume that R is stable under taking weakly cyclically reduced conjugates and inverses. Let $\Gamma := G/\langle\langle R \rangle\rangle_G$ be the quotient of G by the normal closure of R in G. An element $p \in G$ is a *piece* if there are distinct relators $r_1, r_2 \in R$ such that the products $r_1 = pu_1$ and $r_2 = pu_2$ are weakly reduced.

Given $0 < \lambda < 1$, the set R satisfies the $C'(\lambda)$ *small cancellation condition* (over G) if for every piece p and every relator $r \in R$ such that the product $r = pu$ is weakly reduced, we have that $|p| < \lambda |r|$. To avoid pathological cases, let us in addition assume that for all $r \in R$ we have that $|r| > 1/\lambda$. If these conditions are satisfied, we say that Γ is a $C'(\lambda)$*–group* (over G). As in the hyperbolic case, we have that if Γ is a $C'(1/6)$ small cancellation quotient of the free product $G_1 * \cdots * G_n$, then the free factors embed as subgroups of Γ, and Γ is hyperbolic relative to these factors [Osi06, p. 2 Ex. (II)].

Example 2.5.5 (Limit groups)**.** A group Γ is *fully residually* G (G is a group) if for any finite set $A \subset \Gamma$ there exists a homomorphism $\Gamma \to G$ that is injective on A. A group is *fully residually free* (also called a *limit group*) if it is fully residually F_∞, where $F_\infty = F(\mathbb{N})$. Every finitely generated limit group is hyperbolic relative to its maximal abelian subgroups [Dah03, Thm. 0.3].

2.5.1 Relatively quasiconvexity

Let $H < \Gamma$ be a finitely generated subgroup of a relatively hyperbolic group $(\Gamma, \mathcal{P})$ with cusped space $X = X(\Gamma, \mathcal{P}, S)$, and suppose there exists a finite set $\mathcal{D}$ of representatives for H-conjugacy classes of the infinite groups of the form $H \cap P^g$ with $g \in \Gamma$ and $P \in \mathcal{P}$ (we use the notation $A^x = xAx^{-1}$). Given $D \in \mathcal{D}$, there is a a unique $P_D \in \mathcal{P}$ and some $c_D \in \Gamma$ of shortest word length (with respect to S) so that $D = H \cap P_D^{c_D}$. Also, assume that each group in $\mathcal{D}$ is finitely generated, and let X_H be a combinatorial cusped space for the pair $(H, \mathcal{D})$ with respect to some compatible finite generating set S' of H, in the sense that S' is symmetric and $S' \cap D$ generates D for each $D \in \mathcal{D}$. We extend the inclusion $\iota : H \hookrightarrow G$ to an H-equivariant Lipschitz map $\check{\iota} : X_H^{(0)} \to X$ as follows: a vertex in a horoball of X_H is a tuple (sD, h, n) with $s \in H$, $D \in \mathcal{D}$, $h \in sD$ and $n \in \mathbb{Z}_{\geq 0}$, so define its image by $\check{\iota}(sD, h, n) := (sc_D P_D, hc_D, n)$.

Definition 2.5.6. The pair $(H, \mathcal{D})$ is *relatively quasiconvex* in $(\Gamma, \mathcal{P})$ if the image $\check{\iota}(X_H^{(0)}) \subset X$ is λ-quasiconvex for some λ, which will be called a *quasiconvexity constant* for $(H, \mathcal{D})$ in $(\Gamma, \mathcal{P})$. Sometimes we will omit the peripheral structures and simply say that H is relatively quasiconvex in Γ.

As noted in [GM21, Def. 2.9], this definition is equivalent to other notions of relative quasiconvexity existing in literature, at least in the finitely generated case [Hru10]. In particular, if $H < \Gamma$ is relatively quasiconvex, then the collection $\mathcal{D}$ defined above makes $(H, \mathcal{D})$ into a relatively hyperbolic group [Hru10, Thm. 9.1], and we call $\mathcal{D}$ an *induced relatively hyperbolic structure* (or *peripheral structure*) on H. The characterization of relative quasiconvexity given above will be particularly helpful in Section 3.3, but for most of our purposes, the following criterion due to Hruska will suffice.

Theorem 2.5.7 (Hruska [Hru10, Thm. 1.5])**.** *If Γ is finitely generated and $H < \Gamma$ is an undistorted subgroup, then H is relatively quasiconvex in $(\Gamma, \mathcal{P})$ with respect to any possible peripheral structure $\mathcal{P}$ on Γ.*

An important subclass of relatively quasiconvex subgroups is given by those which are *full*, and in a sense, this is the correct generalization to quasiconvex subgroups of hyperbolic groups.

Definition 2.5.8. A relatively quasiconvex subgroup $H < \Gamma$ of a relatively hyperbolic group $(\Gamma, \mathcal{P})$ is *fully relatively quasiconvex* if for any peripheral subgroup P of Γ, the group $H \cap P$ is either finite or finite index in P.

2.5.2 Dehn filling

Dehn filling was introduced independently by Groves and Manning [GM08] and Osin [Osi07] as a group theoretical generalization of the corresponding concept to cusped hyperbolic 3-manifolds, due to Thurston [Thu79].

Definition 2.5.9. Let $(\Gamma, \mathcal{P} = \{P_1, \dots, P_n\})$ be a relatively hyperbolic group and consider normal subgroups $N_i \trianglelefteq P_i$. The *group theoretical Dehn filling* of Γ (or simply the *filling*) is the quotient map

$$\phi : \Gamma \to \Gamma(N_1, \dots, N_n) = \Gamma(\mathcal{N}) := \Gamma / \langle\langle \bigcup N_i \rangle\rangle_\Gamma,$$

where $\mathcal{N} = \{N_1, \dots, N_n\}$ is the collection of *filling kernels* of ϕ. Let $\overline{\mathcal{P}}$ denote the set of images of the subgroups P_i under ϕ.

Example 2.5.10 (Dehn filling for 3-manifolds)**.** Let M be a finite-volume hyperbolic 3-manifold with n cusps, so that it is homeomorphic to the interior of a compact 3-manifold $\hat{M}$ with boundary a disjoint union of n tori $T_1, \dots, T_n$. Let $\Gamma = \pi_1(M) = \pi_1(\hat{M})$, and for each i let $P_i = \pi_1(T_i)$ be the fundamental group of T_i seen as a subgroup of Γ. If we glue n solid tori $S_1, \dots, S_n$ to $\hat{M}$ along $T_1, \dots, T_n$, we obtain a closed 3-manifold $\overline{M}$ with fundamental group $\overline{\Gamma}$. We identify each S_i with $S^1 \times D^2$ and let $\phi_i : \partial S_i = S^1 \times S^1 \to T_i$ denote the homeomorphisms in the construction of $\overline{M}$. If $N_i \trianglelefteq P_i$ is the subgroup generated by the image of the loop $\phi_i(\{pt\} \times S^1)$ in T_i, then Γ is hyperbolic relative to $\mathcal{P} = \{P_1, \dots, P_n\}$ and $\overline{\Gamma}$ is isomorphic to the Dehn filling $\Gamma(N_1, \dots, N_n)$. A remarkable theorem of Thurston [Thu79] states that for all but finitely many choices of homeomorphisms $\phi_1, \dots, \phi_n$ (up to isotopy) the closed manifold $\overline{M}$ admits a hyperbolic Riemannian metric, so in particular $\overline{\Gamma}$ is hyperbolic.

If $H < (\Gamma, \mathcal{P})$ is a relatively quasiconvex subgroup, we need further conditions on a filling to guarantee good properties for the image of H. In [GM21], Groves and Manning introduced *H-wide fillings* as a generalization of H-fillings, but with enough flexibility to behave nicely even when H is not necessarily full. Let $\mathcal{D}$ be an induced peripheral structure on H, such that every $D \in \mathcal{D}$ is of the form $D = H \cap P_{i_D}^{c_D}$ for some $P_{i_D} \in \mathcal{P}$ and some $c_D \in \Gamma$ of shortest word length (with respect to a fixed compatible generating set of Γ).

Definition 2.5.11. If $S \subset (\bigcup \mathcal{P})\backslash\{o\}$ is a finite set, then a filling $\Gamma \to \Gamma(N_1, \ldots, N_n)$ is (H, S)-*wide* if for any $D \in \mathcal{D}$, and for any $d \in D$ and $w \in S \cap P_{i_D}$, we have $c_D w c_D^{-1} \in D$ whenever $d c_D w c_D^{-1} \in N_{i_D}^{c_D}$.

More generally, if $\mathcal{H} = \{(H_1, \mathcal{D}_1), \ldots, (H_k, \mathcal{D}_k)\}$ is a collection of relatively quasiconvex subgroups of Γ, then a filling is $(\mathcal{H}, S)$-*wide* if it is (H_j, S)-wide for each $1 \leq j \leq k$.

Given a collection $\{N_i \trianglelefteq P_i\}_i$ of filling kernels of $(\Gamma, \mathcal{P})$, the *induced filling kernels* of $(H, \mathcal{D})$ are the groups of the collection $\mathcal{N}_H = \{K_D := D \cap N_{i_D}^{c_D}\}_{D \in \mathcal{D}}$. These groups define the *induced filling*

$$\phi_H : (H, \mathcal{D}) \to (H(\mathcal{N}_H), \overline{\mathcal{D}}),$$

with $\overline{\mathcal{D}}$ being the set of images of the elements of $\mathcal{D}$ in $H(\mathcal{N}_H) := H/\langle\langle \bigcup_D K_D \rangle\rangle_H$. Note that there is a natural map from $H(\mathcal{N}_H)$ into $\overline{\Gamma} = \Gamma(N_1, \ldots, N_n)$. If $\Gamma \to \overline{\Gamma}$ is a Dehn filling of $(\Gamma, \mathcal{P})$ with kernel K, and X is a cusped space for $(\Gamma, \mathcal{P})$ as defined above, then a combinatorial cusped space for $(\overline{\Gamma}, \overline{\mathcal{P}})$ is obtained from $\overline{X} = K\backslash X$ by removing self-loops. This removing process does not affect the metric on the 0-skeleton, so we will ignore this ambiguity and simply set $\overline{X} = K\backslash X$ [GM21, p. 4].

Definition 2.5.12. A property P holds for *all sufficiently long and H-wide fillings* if there is a finite set $S \subset (\bigcup \mathcal{P})\backslash\{o\}$ so that P holds for any (H, S)-wide filling $\Gamma \to \Gamma(N_1, \ldots, N_n)$ with $S \cap (\bigcup_i N_i) = \emptyset$. In general, if $\mathcal{H}$ is a collection of relatively quasiconvex subgroups of Γ, then P holds for *all sufficiently long and $\mathcal{H}$-wide fillings* if there is a finite set S so that P holds for any $(\mathcal{H}, S)$-wide filling with $S \cap (\bigcup_i N_i) = \emptyset$.

The next result summarizes the main required properties about wide Dehn fillings of relatively hyperbolic groups [AGM16, Sec. 7], [GM21, Sec. 3 & Sec. 4].

Theorem 2.5.13. *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic group and let X be a cusped space for $(\Gamma, \mathcal{P})$, which we assume is δ-hyperbolic. Let $H < \Gamma$ be a relatively quasiconvex subgroup with quasiconvexity constant λ with respect to X, and let $A \subset \Gamma$ be a finite set. Then there exist positive numbers $\delta' = \delta'(\delta)$ and $\lambda' = \lambda'(\lambda, \delta)$ such that for all sufficiently long and H-wide fillings $\phi : \Gamma \to \overline{\Gamma} := \Gamma(N_1, \ldots, N_n)$:*

1. *$(\overline{\Gamma}, \overline{\mathcal{P}})$ is relatively hyperbolic.*

2. *$\phi(A) \cap \phi(H) = \phi(A \cap H)$.*

3. $\overline{H} := \phi(H)$ *is relatively quasiconvex in* $(\overline{\Gamma}, \overline{\mathcal{P}})$, *and* λ' *is a quasiconvexity constant for* $\overline{H}$ *with respect to* $\overline{X}$. *In addition, if* H *is fully relatively quasiconvex, then* $\overline{H}$ *is fully relatively quasiconvex.*

4. $\overline{H}$ *is canonically isomorphic to the induced filling* $H(\mathcal{N}_H)$.

To guarantee the existence of wide fillings, we require the notion of *separability* for subsets of groups, which will be introduced in Section 2.6. The next definitions are motivated by [SW15].

Definition 2.5.14. Let $(H, \mathcal{D})$ be a relatively quasiconvex subgroup of the relatively hyperbolic group $(\Gamma, \mathcal{P})$. We say that H is *peripherally separable* in $(\Gamma, \mathcal{P})$ if D is separable in $P_{i_D}^{c_D}$ for every $D \in \mathcal{D}$. We say that H is *strongly peripherally separable* if D' is separable in $P_{i_D}^{c_D}$ for any finite index subgroup $D' < D$ and any $D \in \mathcal{D}$.

The existence of wide fillings is then guaranteed by the following lemma.

Lemma 2.5.15 ([GM21, Lem. 5.2])**.** *Let* $(\Gamma, \mathcal{P})$ *be relatively hyperbolic and consider a finite collection* $\mathcal{H} = \{(H_1, \mathcal{D}_1), \ldots, (H_k, \mathcal{D}_k)\}$ *of relatively quasiconvex and peripherally separable subgroups of* Γ. *Then for any finite set* $S \subset (\bigcup \mathcal{P}) \backslash \{o\}$ *there exist finite index subgroups* $\dot{N}_i \trianglelefteq P_i$ *such that any filling* $\Gamma \to \Gamma(N_1, \ldots, N_n)$ *with* $N_i < \dot{N}_i$ *is* $(\mathcal{H}, S)$*-wide.*

2.6 Separability properties on groups

One of the main properties of groups and their subsets that we will consider is the notion of *separability*, which guarantees the existence of plenty of finite index subgroups satisfying desirable properties. Although it is not explicit in the statements, this concept is key in the proof of the main results of Chapter 3.

Definition 2.6.1. Let Γ be any group. A subset S of Γ is *separable* in Γ (or simply *separable* if the ambient group Γ is understood) if for any $a \in \Gamma \backslash S$ there exists a finite quotient of Γ in which the image of a does not lie in the image of S. In particular, a subgroup $H < \Gamma$ is separable if it is the intersection of finite index subgroups of Γ. The group Γ is *residually finite* if the trivial subgroup $\{o\}$ is separable.

For separability of subgroups, a topological interpretation is given by Scott's criterion, in terms of the existence of finite-sheeted covers.

Proposition 2.6.2 (Scott's Criterion [Sco78])**.** *Let* X *be a connected complex with fundamental group* Γ *and let* $H < \Gamma$ *be a subgroup with corresponding cover* $\pi : X^H \to X$ *Then* H *is separable in* Γ *if and only if for every compact subcomplex* $Y \subset X^H$, *there exists an intermediate finite-sheeted cover* $\pi_1 : X^H \to \hat{X}, \pi_2 : \hat{X} \to X$ *such that* $\pi_1 : Y \to \hat{X}$ *is an embedding.*

Example 2.6.3 (Linear groups)**.** Since every non-zero integer is relatively prime to some prime number, we see that $\mathbb{Z}$ is residually finite. In a similar way, by considering the (kernels of the) quotients $\mathrm{GL}_m(\mathbb{Z}) \to \mathrm{GL}_m(\mathbb{Z}/p\mathbb{Z})$ for p prime, the groups $\mathrm{GL}_m(\mathbb{Z})$ are residually finite for all $m \geq 1$. This argument was pushed further by Mal'cev, who proved that every finitely generated linear group is residually finite (see [DK18, Ch. 26]).

Example 2.6.4 (Free and surface groups)**.** By Mal'cev's theorem in the previous example, we deduce that finitely generated free groups are residually finite, as well as surface groups. More generally, for these groups every finitely generated subgroup is separable, which can be proved by using Scott's Criterion 2.6.2.

Non-Example 2.6.5 (Baumslag-Solitar groups)**.** Given m, n non-zero integers, the *Baumslag-Solitar group* $B(m, n)$ is defined according to the presentation

$$B(m, n) = \langle a, b | ba^m b^{-1} = a^n \rangle.$$

The group $B(m, n)$ is residually finite if and only if $|m| = 1$ or $|n| = 1$ or $|m| = |n|$ [Mes72].

Non-Example 2.6.6 (Simple groups)**.** Note that infinite, finitely generated simple groups are never simple. This is because, for finitely generated groups, every finite index subgroup contains a finite index subgroup that is normal in the ambient group.

Example 2.6.7 (Hopfian groups)**.** A group Γ is *Hopfian* if any surjective endomorphism of Γ is an isomorphism. Examples of Hopfian groups include the rational numbers, torsion-free hyperbolic groups, and finitely generated residually finite groups. On the other hand, the Baumslag-Solitar group $B(2, 3)$ is not Hopfian. Every finitely generated residually finite group is Hopfian [Mal40].

Example 2.6.8 (Non-linear, residually finite groups)**.** There are examples of residually finite groups that are not linear. Among 1-relator groups, the first example was the group with presentation $\langle a, t | t^2 a t^{-2} = a^2 \rangle$, given by Druţu and Sapir [DS05a]. Recently, Tholozan and Tsouvalas constructed the first examples of residually finite, non-linear hyperbolic groups [TT22].

Example 2.6.9 (Maximal subgroups satisfying a law)**.** A group H satisfies a *law* if there is a word $w(x_1, \ldots, x_n)$ in the variables $x_1, \ldots, x_n$ and its inverses such that $w(h_1, \ldots, h_n) = o$ for all $h_1, \ldots, h_n \in H$. If Γ is a residually finite group, then every maximal subgroup satisfying a given law is separable. In particular, maximal abelian subgroups of residually finite groups are separable.

Example 2.6.10 (Retracts)**.** A subgroup $H < \Gamma$ is a *retract* if there exists an endomorphism $\phi : \Gamma \to \Gamma$ such that $\phi(\Gamma) = H$ and ϕ is the identity when restricted to H. Such ϕ is called a *retraction homomorphism*. In this setting, we can also deduce separability of double cosets. The next result is [HW08, Lem. 9.3], and it will be useful in the proof of Lemma 3.2.3.

Proposition 2.6.11. *Let Γ be a residually finite group, and let $\phi : \Gamma \to \Gamma$ be a retraction homomorphism with $H = \phi(\Gamma)$. Then $H < \Gamma$ is a separable subgroup, and if $K < \Gamma$ is separable with $\phi(K) \subset K$, then the double coset HK is separable in Γ.*

2.7 CAT(0) cube complexes

In this section we introduce non-positively curved cube complexes, CAT(0) cube complexes, and cubulated groups. The latter class of groups is particularly well-behaved among CAT(0) groups, as we will see in Section 2.8 and Chapter 3. For more references about the geometry of CAT(0) cube complexes and groups acting on them, see [BH99; Sag14].

2.7.1 Cubulated groups

Before discussing cube complexes, we introduce the class of CAT(0) metric spaces.

Definition 2.7.1. Let (X, d) be a geodesic metric space. Given points $x, y, z \in X$ and a geodesic triangle $\Delta_{x,y,z} \subset X$ with vertices x, y, z, consider a *comparison triangle* $\overline{\Delta}_{\overline{x},\overline{y},\overline{x}} \subset \mathbb{R}^2$, consisting of three points $\overline{x}, \overline{y}, \overline{z} \in \mathbb{R}^2$ such that

$$d(x, y) = d_{\mathbb{R}^2}(\overline{x}, \overline{y}), \;\; d(y, z) = d_{\mathbb{R}^2}(\overline{y}, \overline{z}), \quad \text{and} \quad d(z, x) = d_{\mathbb{R}^2}(\overline{z}, \overline{x}),$$

where $d_{\mathbb{R}^2}$ is the Euclidean distance. From this data, there is a natural comparison map $\Phi : \Delta_{x,y,z} \to \overline{\Delta}_{\overline{x},\overline{y},\overline{z}}$ such that x, y, z are sent to $\overline{x}, \overline{y}, \overline{z}$ respectively and each side of $\Delta_{x,y,z}$ is mapped isometrically. The metric space (X, d) is CAT(0) is for any geodesic triangle $\Delta_{x,y,z} \subset X$ and any comparison map $\Phi : \Delta_{x,y,z} \to \overline{\Delta}_{\overline{x},\overline{y},\overline{z}}$ we have

$$d(p, q) \leq d_{\mathbb{H}^2}(\Phi(p), \Phi(q)) \text{ for all } p, q \in \Delta_{x,y,z}.$$

Remark 2.7.2. If in the definition above we replace $\mathbb{R}^2$ with $\mathbb{H}^2$ we obtain the class of CAT(-1) *spaces.*

CAT(0) spaces are always contractible and uniquely geodesic. A group is CAT(0) if it acts properly and cocompactly by isometries on a CAT(0) space. For more properties about CAT(0) spaces and groups, see [BH99].

Example 2.7.3 (Non-positively curved manifolds)**.** A Riemannian manifold is CAT(0) with respect to its length metric if and only if it is simply connected and its sectional curvatures are non-positive [BH99, Thm. II.1.1A.6]. In consequence, if Γ is the fundamental group of a closed Riemannian manifold with non-positive sectional curvatures, then Γ is CAT(0).

Most of the examples of CAT(0) spaces and groups that we will consider come by looking at cube complexes.

Definition 2.7.4. A *cube complex* is a metric polyhedral complex in which all polyhedra are unit-length Euclidean cubes. Such a complex is *non-positively curved* (NPC) if its universal cover is a CAT(0) metric space when endowed with the induced length metric [BH99].

There is a combinatorial description of this property, due to Gromov. Recall that the *link* of a vertex v in the complex X is the complex $Lk_X(v)$ with vertices the edges of X incident to v, and in which n such edges are the 1-skeleton of an $(n-1)$-face in $Lk_X(v)$ if and only if they are incident to a common n-cell at v. Also, a *flag simplicial complex* is a complex determined by its 1-skeleton: for every complete subgraph of the 1-skeleton there is a simplex with 1-skeleton equal to that subgraph.

Proposition 2.7.5 (Gromov, see e.g. [BH99, 7, Thm. II.5.2])**.** *A cube complex X is NPC if and only if the link of each vertex is a flag complex.*

An n-cube $C = [0,1]^n \subset \mathbb{R}^n$ has n-midcubes obtained by setting one coordinate equal to $1/2$. Since the face of a midcube of C is the midcube of a face of C, the set of midcubes of a cube complex X has a cube complex structure, called the *wall complex*. A *wall* of X is a connected component of the wall complex. When X is a CAT(0) cube complex any wall W is 2-sided and embeds as a convex subspace (with respect to the CAT(0) metric), so we will not make any distinction between a wall and its embedded image. Also, the complement of a wall in X has exactly two connected components, whose closures are called *half-spaces* of X. Since walls are convex subcomplexes of the cubical barycentric subdivision of X, they are CAT(0) cube complexes as well. For an edge e in the cube complex X, let $W(e)$ denote the unique wall it intersects, and say that e is *dual* to $W(e)$.

As we mentioned, we are interested in group actions on CAT(0) cube complexes.

Definition 2.7.6. Suppose Γ is a group acting properly and cocompactly by cubical isometries on the CAT(0) cube complex X with CAT(0) metric d_X. In that case we say that Γ is a *cubulable group* and that (Γ, X) is a *cubulated group* with X being a *cubulation* of Γ.

Remark 2.7.7. Although in this thesis we will restrict to proper and cocompact actions on CAT(0) cube complexes, these assumptions can be relaxed and we still can deduce interesting consequences. For example, when Γ is relatively hyperbolic we can consider cubulations that are non-necessarily cocompact, but *cosparse* [HW14; SW15]. Also, there has been some progress in understanding non-proper actions of hyperbolic groups [GM18]. Recently, Einstein and Groves introduced *relatively geometric cubulations* of relatively hyperbolic groups [EG20].

If (Γ, X) is a cubulated group, then X is finite-dimensional, locally finite, and Γ is finitely generated and quasi-isometric to X [She21, Prop. 4.2]. Cocompactness and properness also imply the following lemma (cf. [Ago13, p. 1052]).

Lemma 2.7.8. *If (Γ, X) is a cubulated group, then:*

1. *There are only finitely many conjugacy classes of torsion elements in Γ.*

2. *If $Y \subset X$ is a non-empty subset and $H < \Gamma$ preserves Y and acts cocompactly on it, then for any $R > 0$ the set of double cosets*

$$A_{Y,H,R} := \{HgH \colon g \in \Gamma, d_X(Y, gY) \leq R\}$$

is finite.

Proof. Part (1) holds for arbitrary proper and cocompact actions by isometries on CAT(0) spaces [BH99, Cor. II.2.8 (2)].
For part (2), let $g \in \Gamma$ be such that $d_X(gY, Y) \leq R$ and let γ be a geodesic segment of length $\leq R{+}1$ joining Y and gY. Consider $D \subset Y$ a compact subset with $H{\cdot}D = Y$ and $w_1, w_2 \in H$ such that γ intersects $w_1 D$ in Y and $gw_2 D$ in gY. Then $d_X(w_1^{-1} g w_2 D, D) \leq R + 1$, and so by local finiteness and properness of the action, there is a finite set $F \subset G$ such that $w_1^{-1} g w_2 \in F$. Therefore $g \in w_1 F w_2^{-1} \subset HFH$. □

We will also require the following result about cubulated relatively hyperbolic groups, which is key in the proof of Proposition 3.8.12. We use the notation $[p, q]$ for the geodesic segment joining the points p and q, and recall that $N_R(A)$ denotes the R-neighborhood of the set A.

Proposition 2.7.9. *If $(\Gamma, \mathcal{P})$ is relatively hyperbolic and cubulated by X, then there exists $\delta \geq 0$ such that if $h \in \Gamma$ is loxodromic and preserves two axes $\gamma_1, \gamma_2 \subset X$, then $d_X(\gamma_1, \gamma_2) \leq \delta$.*

Proof. Let $x \in X$ be a base-point. The map $\Gamma \xrightarrow{\mu} X$, $g \mapsto gx$ is a quasi-isometry for X considered with the CAT(0) metric, so there exists some $\delta' \geq 0$ such that for any geodesic triangle $\Delta \subset X$ with vertices a, b, c there is some peripheral left coset $Q_\Delta = g_\Delta P_\Delta$ with $g_\Delta \in \Gamma$ and $P_\Delta \in P$, such that for any point $p \in \Delta$, either:

(i) p lies in the δ'-neighborhood of the union of the sides of Δ not containing p; or,

(ii) $p \in N_{\delta'}(Q_\Delta \cdot x)$

(see e.g. [SW15, Thm. 4.1 & Prop. 4.2] or [DS05b, Sec. 8.1.3]). In addition, for such a δ' there exists some $\lambda \geq 0$ so that if gP and $g'P'$ are distinct peripheral left cosets, then

$$\operatorname{diam}(N_{\delta'}(gP \cdot x) \cap N_{\delta'}(g'P' \cdot x)) < \lambda, \tag{2.6}$$

see [Ein19, 10, Cor. 2.3].

Let $\delta := 4\delta'$, and suppose by contradiction that $h \in \Gamma$ is loxodromic and preserves two axes $\gamma_1, \gamma_2 \subset X$ with $r := d_X(\gamma_1, \gamma_2) > \delta$. We claim that there is some peripheral left coset gP such that $\gamma_1 \subset N_{\delta'}(gP \cdot x)$.

By [BH99, Thm. II.2.13], γ_1 and γ_2 are asymptotic and bound a flat strip isometric to $\mathbb{R} \times [0, r]$. Let $a \in \gamma_1$, and let b be its closest point projection into γ_2. Choose an isometry $\alpha : \mathbb{R} \to \gamma_1$ sending 0 to a, and for $\eta > r$ consider the geodesic triangle Δ_η with vertices a, b, and $\alpha(\eta)$. After using some Euclidean trigonometry we can prove that the segment

$[\alpha(\sqrt{2}\delta'), \alpha(\eta/2)]$ lies outside the δ'-neighborhood of $[a,b] \cup [\alpha(\eta), b]$, so by item (ii) above there exists some peripheral left coset $Q_\eta = g_\eta P_\eta$ with $[\alpha(\sqrt{2}\delta'), \alpha(\eta/2)] \subset N_{\delta'}(Q_\eta \cdot x)$. Also, by (2.6) we have $Q_\eta = Q_{\eta'}$ for any $\eta \geq \eta' := \max(2\lambda + 2\sqrt{2}\delta', r)$, implying $\alpha([\sqrt{2}\delta', +\infty)) \subset N_{\delta'}(Q_{\eta'} \cdot x)$. In fact, by a completely analogous argument, we can prove $\gamma_1 \subset N_{\delta'}(Q_{\eta'} \cdot x)$, and so the claim follows with $gP := Q_{\eta'}$.

Now, let $L = d_X(hx, \gamma_1)$ which equals $d_X(h^n x, \gamma_1)$ for any $n \in \mathbb{Z}$. By our previous claim we have $d_X(h^n x, gP \cdot x) \leq L + \delta'$ for all n, so by means of the quasi-isometry μ we can find a constant $C \geq 0$ such that

$$d_S(h^n, gP) \leq C$$

for any $n \in \mathbb{Z}$, for d_S the word metric with respect to some finite, symmetric generating set $S \subset \Gamma$. This is our desired contradiction since in that case the infinite cyclic group generated by h would be bounded in Γ for the word metric $d_{(S \cup \bigcup \mathcal{P})}$, contradicting that h is loxodromic [HW09, Lem. 8.3 (1)]. □

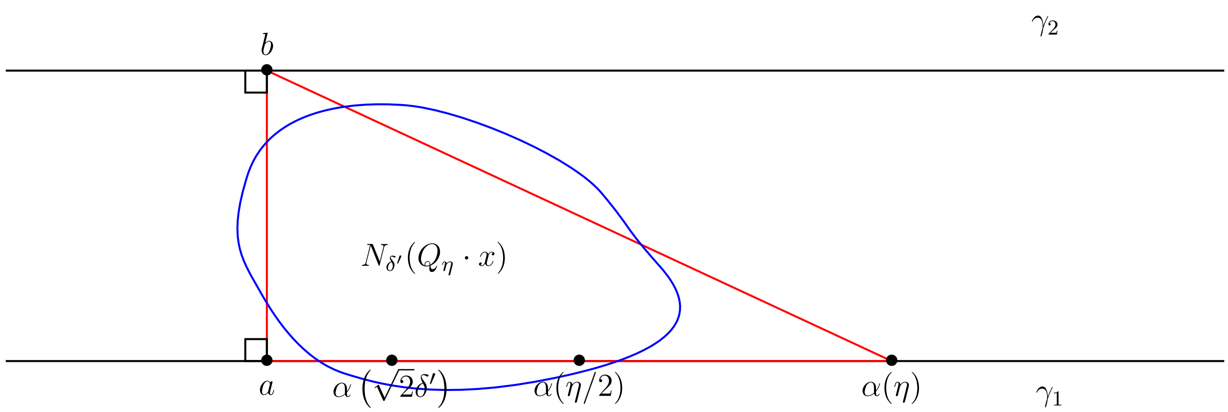

Figure 2.1: Proof of Proposition 2.7.9.

2.7.2 Convex subgroups

Let (Γ, X) be a cubulated group, and for W a wall of X let $\Gamma_W < \Gamma$ denote its setwise stabilizer. The group Γ_W acts properly and cocompactly on W [She21, Rmk. 2.3], and since W is convex in X for the CAT(0) metric, (Γ_W, W) is a cubulated group with Γ_W is undistorted in Γ. This is the prototypical example of a convex subgroup.

Definition 2.7.10. If (Γ, X) is a cubulated group, we say that a subgroup $H < \Gamma$ is *convex* in (Γ, X) if there is an H-invariant convex subcomplex $Z \subset X$ such that the action of H on Z is cocompact. Such a subcomplex Z will be called a *convex core* for H. Similarly to wall stabilizers, convex subgroups are undistorted.

Definition 2.7.11. If Y is a NPC cube complex and $S \subset Y$ is any subset, then the *cubical neighborhood* of S is the subcomplex $\mathcal{N}(S) \subset Y$ consisting of the cubes of Y intersecting S.

If (Γ, X) is a cubulated group and W is a wall of X, then $\mathcal{N}(W)$ is a convex subcomplex of X [HW08, Lem. 13.4], and hence it is a convex core for Γ_W. If in addition Γ is relatively hyperbolic, and since Γ quasi-isometric to X, by Theorem 2.5.7 every convex subgroup of Γ is relatively quasiconvex. As a partial converse, Sageev and Wise proved the following result.

Theorem 2.7.12 (Sageev–Wise [SW15, Thm. 1.1])**.** *If (Γ, X) is a cubulated group with Γ relatively hyperbolic and $H < \Gamma$ is fully relatively quasiconvex, then for any compact subset $B \subset X$ there exists a convex core $Z \subset X$ for H that contains B. In particular, any peripheral subgroup of Γ is convex.*

Remark 2.7.13. When (Γ, X) is cubulated with Γ relatively hyperbolic and $P < \Gamma$ is a peripheral subgroup with convex core $Z \subset X$, then for any wall $W \subset X$ with $W \cap Z \neq \emptyset$ we have $P_{W \cap Z} = P \cap G_W$. The inclusion $P \cap \Gamma_W \subset P_{W \cap Z}$ is evident, and if e is an edge of Z dual to W, then e is also dual to the wall $W \cap Z$ of Z. Therefore, for any $g \in P$ with $g(W \cap Z) = W \cap Z$, the edge ge is dual to $W \cap Z \subset W$, and hence $gW = gW(e) = W(ge) = W$, implying $P_{W \cap Z} = P \cap \Gamma_W$.

When dealing with subcomplexes, sometimes it is useful to work with the *combinatorial metric* instead. That is, if X is a CAT(0) cube complex, then we consider the path metric on its 1-skeleton $X^{(1)}$, where each edge has length 1. In this way, a subcomplex of X is convex for the CAT(0) metric if and only if it is full and its 1-skeleton is convex in $X^{(1)}$ for the combinatorial metric [HW08, Prop. 13.7]. By abuse of notation, by combinatorial metric we might also mean the restriction of the combinatorial metric to $X^{(0)}$. The combinatorial metric can be used, for example, to show that the intersection of convex subgroups is convex.

Lemma 2.7.14. *If (Γ, X) is a cubulated group and $H_1, H_2 < \Gamma$ are convex subgroups with convex cores $Y_1, Y_2 \subset X$ respectively and such that $Y_1 \cap Y_2 \neq \emptyset$, then $H_1 \cap H_2$ is a convex subgroup with convex core $Y_1 \cap Y_2$.*

Proof. Let x_0 be a vertex in the convex subcomplex $Y_1 \cap Y_2$, and let $R \geq 0$ be such that if $D \subset X^{(0)}$ is the combinatorial R-ball around x_0, then $Y_1 \subset H_1 \cdot D$ and $Y_2 \subset H_2 \cdot D$. Since $X^{(0)}$ is a proper metric space with the combinatorial distance induced by $X^{(1)}$ and Γ acts properly on $X^{(0)}$, it can be proven that for any $K \geq 0$ there exists some $L = L(K)$ such that

$$N_K(H_1 \cdot x_0) \cap N_K(H_2 \cdot x_0) \subset N_L((H_1 \cap H_2) \cdot x_0),$$

with $N_r(S)$ denoting the combinatorial r-neighborhood of $S \subset X^{(0)}$ (cf. [Mar09, Lem. 4.2]). In particular we have

$$Y_1 \cap Y_2 \subset H_1 \cdot D \cap H_2 \cdot D \subset N_R(H_1 \cdot x_0) \cap N_R(H_2 \cdot x_0) \subset N_{L(R)}((H_1 \cap H_2) \cdot x_0),$$

implying that $H_1 \cap H_2$ acts cocompactly on $Y_1 \cap Y_2$ by the local finiteness of X. □

The lemma above requires two convex cores to intersect, which can be always assumed after enlarging the convex cores.

Lemma 2.7.15. *If (Γ, X) is a cubulated group and $H < (\Gamma, X)$ is a convex subgroup, then for any compact set $B \subset X$ there is a convex core $Y \subset X$ for H such that $B \subset Y$. In addition, for any two convex cores $Y_1, Y_2 \subset X$ for the subgroup H, there exists a third convex core $Y_3 \subset X$ containing both Y_1 and Y_2.*

Proof. Let Y' be any convex core for H. Since X is finite-dimensional, by [HW08, Lem. 13.15] the iterated cubical neighborhoods $\mathcal{N}^k(Y') = \mathcal{N}(\mathcal{N}^{k-1}(Y'))$ are convex cores for H for all $k \geq 1$, and we can choose k such that $Y := \mathcal{N}^k(Y')$ contains B. The second statement follows by considering a compact set $B_i \subset Y_i$ such that $H \cdot B_i \supset Y_i$ for each $i = 1, 2$, and then finding a convex core Y_3 containing $B_1 \cup B_2$. □

2.7.3 Examples of cubulable groups

We end this section by presenting examples of cubulable groups, some of them for which a cubulation can be hard to describe explicitly.

Example 2.7.16 (RAAGS)**.** If G is a simplicial graph with vertex set V and set of edges $E \subset V \times V$, then the *right-angled Artin group* associated to Γ is the group with presentation

$$A_G := \langle V \mid [v, w] \text{ if and only if } (v, w) \in E \rangle.$$

That is, A_G is generated by the vertices of G, and two of them commute if and only they are joined by an edge of G. In this way, if G has no edges then A_G is a free group and if G is a full graph then A_G is free abelian.

If G is a simplicial graph we can associate it to a cube complex S_G in the following way. We start with a single vertex and we add a loop for each vertex of G. This is the 1-skeleton of S_G. Now, for every maximal n-clique H in G we consider an n-cube whose sides are labeled by the vertices in H, so that each of its vertices is used as a label and opposite sides are labeled by the same vertex. to the complex. We see this n-torus as the quotient of a cube by identifying opposite sides, and the edges of this cube descend to n-loops that can be identified with the corresponding loops in S_G. Finally, we identify any pair of tori whose edges match in the 1-skeleton of S_G. This complex is the *Salvetti complex* associated to G, and it can be checked directly that it satisfies Gromov's Proposition 2.7.5. Moreover, the fundamental group of S_G is exactly A_G, so that the universal cover $\widetilde{S}_G$ of S_G is a cubulation for A_G when G is finite. Note that if $G' \subset G$ is a full subgraph of G, then $S_{G'}$ is a locally convex subcomplex of S_G, and hence $A_{G'}$ is a convex subgroup of $(A_G, \widetilde{S}_G)$.

Example 2.7.17 (Cubulating surface groups)**.** There are plenty of 2-dimensional NPC cube complexes homeomorphic to the closed orientable surface of genus 2, and by considering finite-sheeted covers we can cubulate every surface group.

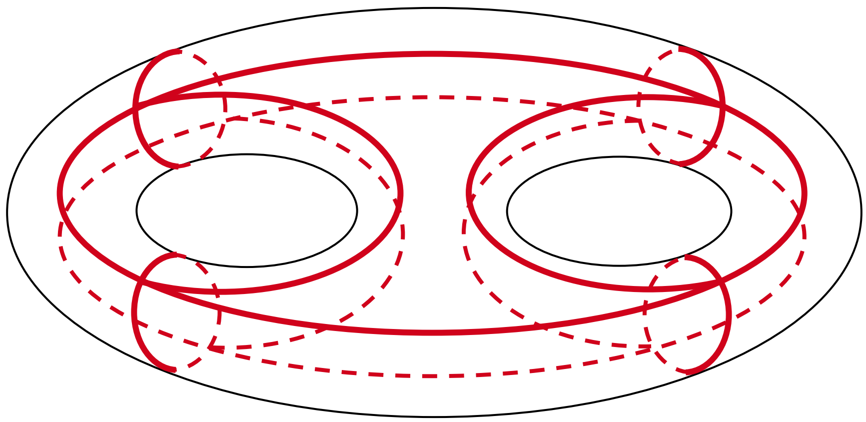

Figure 2.2: Cubulation of a genus 2 surface.

Example 2.7.18 (Cubulable simple groups)**.** Not all cubulable groups are residually finite. Indeed, Burger and Mozes constructed infinite simple groups acting simplicially, freely and cocompactly on products of two locally finite trees [BM00]. From this, we also deduce that residual finiteness is not a quasi-isometric invariant. For more examples of non-residually finite cubulable groups, see e.g. [Wis07].

If Γ is a finitely generated group, a subgroup $H < \Gamma$ is *codimension-1* if the coset graph $H\backslash \mathrm{Cay}(\Gamma, S)$ has more than one end for some (any) finite generating subset $S \subset \Gamma$. For example, if Γ is cubulated and $H < \Gamma$ is a wall stabilizer for the wall W such that each half-space determined by W contains points arbitrarily far from W, then H is codimension-1.

As noted by Sageev [Sag95], codimension-1 subgroups can be used to construct non-trivial actions on CAT(0) cube complexes. Among hyperbolic groups, Bergeron and Wise used Sageev's construction to prove the following criterion for cubulability.

Theorem 2.7.19 (Bergeron–Wise [BW12, Thm. 1.4])**.** *Let Γ be a hyperbolic group and assume that for any pair of distinct points in $\partial\Gamma$ there exists a codimension-1 quasiconvex subgroup $H < \Gamma$ such that the two points lie in different components of $\partial\Gamma \setminus \partial H$. Then Γ is cubulable.*

Example 2.7.20 (Free-by-cyclic groups)**.** A group Γ is *free-by-cyclic* if there exists a free group F and an automorphism $\phi : F \to F$ such that Γ is isomorphic to the semidirect product

$$(F * \langle t \rangle)/\langle\langle \phi(x) = txt^{-1} \forall x \in F \rangle\rangle.$$

When F is finitely generated and of rank at least 2, then Γ is hyperbolic if and only if ϕ is *atoroidal*, meaning that no positive power of ϕ fixes the conjugacy class of a non-trivial element in F [BF92; Bri00]. In that case, Hagen and Wise produced plenty of quasiconvex codimension-1 subgroups, and by Theorem 2.7.19 they deduced that Γ is cubulable [HW15].

Example 2.7.21 (hyperbolic 3-manifolds)**.** If M is a closed hyperbolic 3-manifold, then $\Gamma = \pi_1(M)$ contains many quasiconvex surface subgroups, as mentioned in Example 2.4.17. All these subgroups are codimension-1, and hence Theorem 2.7.19 implies that Γ is cubulable [BW12, Thm. 1.5].

Example 2.7.22 (Arithmetic manifolds of simplest type)**.** Let k be a totally real number field equipped with an *identity embedding* $k \subseteq \mathbb{R}$, and a non-degenerate quadratic form $q\colon k^{n+1} \to k$. We say that (q, k^{n+1}) is *of simplest type* if the signature of q is $(n, 1)$ under the identity embedding $k \subseteq \mathbb{R}$ and $(n+1, 0)$ for every non-trivial Galois automorphism of $k/\mathbb{Q}$.

In this case, the group $\Gamma = \mathrm{SO}(q, k^{n+1})$ is a lattice in $\prod_{\sigma \in \mathrm{Gal}(k/\mathbb{Q})} \mathrm{SO}(q_\sigma, k^{n+1})$. But since the map onto the factor corresponding to the trivial element of $\mathrm{Gal}(k/\mathbb{Q})$ is an open map, the image of this lattice is a lattice in $\mathrm{SO}(q, \mathbb{R}^{n+1})$, hence Γ is a lattice in $\mathbb{H}^n$. We call Γ an *arithmetic lattice of simplest type*. By applying Sageev's construction, Bergeron, Haglund and Wise proved that cocompact arithmetic lattices of simplest type are cubulable [BHW11].

In some cases, Sageev's construction also allows us to cubulate relatively hyperbolic groups.

Example 2.7.23 (Small cancellation free products)**.** If Γ is a $C'(1/6)$-small cancellation quotient of the free product $G_1 * \cdots * G_n$, and each G_i is cubulated, then Martin and Steenbock [MS17] showed that Γ is cubulated (Theorem 1.1.4). Indeed, from their proof, it can be deduced that for each i, the cubulation X_i of G_i embeds as a convex subcomplex of the cubulation of Γ, so that it is a convex core for G_i.

We will see more examples of cubulable groups in the next section, once we have introduced virtual specialness.

2.8 Virtually special groups

As we saw in Example 2.7.18, the existence of non-trivial finite index subgroups in a finitely generated group is not immediate from cubulability. However, Haglund and Wise [HW08] introduced the class of virtually special groups, which is extremely well-behaved among cubulable groups, particularly in terms of separability.

2.8.1 Characterizations of virtual specialness

The original definition of special cube complexes is in terms of forbidding some pathological behavior of the walls [HW08, Def. 3.2]. However, to obtain applications in a more explicit way it is convenient to introduce them in terms of convex embeddings [HW08, Thm. 4.2].

Definition 2.8.1. A connected NPC cube complex X is *special* if there exists a finite simplicial graph G and a locally convex, immersion of X into the Salvetti complex S_G. Equivalently, X is special if $\Gamma = \pi_1(X)$ is a subgroup of some RAAG A_G, and $\widetilde{X}$ isometrically embeds as a convex subcomplex of $\widetilde{S}_G$ that is invariant under the group Γ seen as a subgroup A_G.

Note that in the definition above we do not require the complex X to be finite. In that case, if X is special then Γ is a convex subgroup of $(A_G, \widetilde{S}_G)$ with convex core $\widetilde{X}$.

Definition 2.8.2. We say that a cubulated group (Γ, X) is *special* if Γ acts freely on X and the (compact) quotient $\Gamma\backslash X$ is a special cube complex, and that (Γ, X) is *virtually special* if there is a finite index subgroup $\Gamma' < \Gamma$ such that (Γ, X) is special.

By abuse of notation, sometimes we will simply say that Γ is (virtually) special without mentioning the cubulation X, and it will be implicit that the quotient $\Gamma\backslash X$ is compact. Since RAAGS are residually finite, the same holds for virtually special groups. Similarly, all the groups in this class are linear over $\mathbb{Z}$.

In practice, we will not work directly with the definition of virtual specialness given above. Instead, we present some of their main properties and some criteria for virtual specialness in the case of cubulated hyperbolic groups. We will extend some of these results in Sections 3.2 and 3.4.

One of the main properties of virtually special groups is the following result by Haglund and Wise.

Theorem 2.8.3 (Haglund–Wise [HW08, Thm. 1.3, Cor. 7.9 & Prop. 13.7])**.** *If Γ is hyperbolic and virtually special then every quasiconvex subgroup of Γ is separable. In general, if (Γ, X) is a virtually special group then every convex subgroup of (Γ, X) is separable.*

As a partial converse, they also proved the following characterization of virtual specialness, known as the double coset criterion.

Theorem 2.8.4 (Haglund–Wise [HW08, Thm. 9.19])**.** *The cubulated group (Γ, X) is virtually special if and only if:*

(i) Γ_W is separable for every wall $W \subset X$; and,

(ii) the double coset $\Gamma_{W_1}\Gamma_{W_2}$ is separable for any pair W_1, W_2 of intersecting walls of X.

The second characterization of virtual specialness is the groundbreaking result of Wise, in terms of the *quasiconvex virtual hierarchy* $\mathcal{QVH}$ (see also [AGM16, Thm. 10.2]).

Definition 2.8.5. Let $\mathcal{QVH}$ be the smallest class of hyperbolic groups closed under the operations:

1. $\{o\} \in \mathcal{QVH}$.
2. If $\Gamma = A *_B C$ with $A, C \in \mathcal{QVH}$ and such that B is quasiconvex in Γ, then $\Gamma \in \mathcal{QVH}$.
3. If $\Gamma = A *_B$ with $A \in \mathcal{QVH}$ and such that B is quasiconvex in Γ, then $\Gamma \in \mathcal{QVH}$.

4. If $H < \Gamma$ with $|\Gamma : H| < \infty$ and $H \in \mathcal{QVH}$, then $\Gamma \in \mathcal{QVH}$.

Theorem 2.8.6 (Wise [Wis21, Thm 13.3])**.** *A hyperbolic group is virtually special if and only if it belongs to $\mathcal{QVH}$.*

Example 2.8.7 (1-relator groups with torsion)**.** In [Wis21, Cor. 19.2], Wise used Theorem 2.8.6 to prove that 1-relator groups with torsion are virtually special, and hence residually finite, solving a conjecture of Baumslag. More precisely, Wise proved that if $\Gamma = \langle S | r^n \rangle$ is a 1-relator group with torsion, then its *Magnus-Moldavanskii* hierarchy is quasiconvex. Roughly speaking, this hierarchy allows us to describe the free product $\Gamma * \mathbb{Z}$ as an HNN extension $K*_M$, where M is a free group and $K = \langle \overline{S} | \overline{r}^n \rangle$ is a 1-relator group with torsion whose *complexity* is strictly less than that of Γ. Since 1-relator groups with torsion and complexity 0 are of the form $F * \mathbb{Z}/n\mathbb{Z}$ for some free group F and some positive integer n, the proof follows by induction on the complexity. For more details about the proof, see [Wis21, Ch. 19].

Wise also gave a relative version of Theorem 2.8.6 for groups that are hyperbolic relative to virtually abelian groups [Wis21, Thm. 15.1]. He used this result to cubulate (and deduce virtual specialness) for some interesting classes of relatively hyperbolic groups.

Example 2.8.8 (Cusped hyperbolic 3-manifolds)**.** In [Wis21, Thm. 17.4], Wise proved that the fundamental group of a compact and atoroidal 3-manifold with non-empty boundary is virtually special. In particular, non-compact, hyperbolic 3-manifolds with finite volume are virtually special. In this case, cubulability can also be deduced from the existence of plenty of relatively quasiconvex codimension-1 subgroups, see [CF19].

Example 2.8.9 (Limit groups)**.** Wise also proved that finitely generated limit groups are virtually special. Indeed, by [KM98], each such group is a subgroup of a group Γ_k constructed as follows:

1. $\Gamma_0 = \{o\}$.
2. For each $k \geq 0$ we have Γ_{k+1} is isomorphic to $\Gamma_k *_{C_k} A_k$, where C_k is a malnormal abelian subgroup of Γ_k and $A_k = C_k \times B_k$ is a free-abelian group of finite rank.

By [Wis21, Lem. 18.2], each of the groups Γ_k is virtually special, and from this Wise deduced that finitely generated limit groups are virtually special.

2.8.2 Canonical completion and retraction

We continue this section by recalling the *canonical completion and retraction*, introduced by Haglund and Wise [HW08, Sec. 6]. We will make use of this construction in the proofs of Theorems 2.8.3 and 2.8.4.

Definition 2.8.10. Let $f : A \to B$ be a local isometry of NPC special cube complexes, and assume that B is *fully clean* in the sense of [HW08, Def. 6.1] and has simplicial 1-skeleton. Then there exists a covering map $p : \mathsf{C}(A, B) \to B$ (of finite degree if A is finite), an injection $j : A \to \mathsf{C}(A, B)$ and a cellular map $r : \mathsf{C}(A, B) \to A$ such that $rj = \mathrm{Id}_A$ and $pj = f$. The covering is defined as follows:

The 0-skeleton of $\mathsf{C}(A, B)$ is $A^{(0)} \times B^{(0)}$ with $j(a) = (a, f(a))$ for $a \in A^{(0)}$, and $r : A^{(0)} \times B^{(0)} \to A^{(0)}$ and $p : A^{(0)} \times B^{(0)} \to B^{(0)}$ being the projections to the first and second coordinate, respectively.

Since the 1-skeletons of A and B are simplicial, edges in A and B are determined by their endpoints. The edges of $\mathsf{C}(A, B)$ are of two types:

- *Horizontal*: pairs of the form $\{(a, b), (a, b')\}$ with $\{b, b'\}$ an edge of B and such that there is no edge e of A incident to a with $f(e)$ and $\{b, b'\}$ dual to the same wall.
- *Diagonal*: pairs of the form $\{(a, b), (a', b')\}$ with $\{b, b'\}$ an edge of B and $e = \{a, a'\}$ an edge of A with $f(e)$ and $\{b, b'\}$ dual to the same wall (note that $\{(a', b), (a, b')\}$ is also a diagonal edge).

It follows from this definition that $j(\{a, a'\}) = \{(a, f(a)), (a', f(a'))\}$ is a diagonal edge of $\mathsf{C}(A, B)$ for any edge $\{a, a'\}$ of A. Also, for an edge $e = \{(a, b), (a, b')\}$ as above, we define $p(e) = \{b, b'\}$ and $r(e) = a$ if $a = a'$ and e is horizontal, and $r(e) = \{a, a'\}$ if e is diagonal.

Clearly we have $rj = \mathrm{Id}_A$ and $pj = f$ at the level of 1-skeletons, and full cleanliness implies that p is a covering map from $\mathsf{C}(A, B)^{(1)}$ onto $B^{(1)}$. It can be proven that any lifting of the 1-skeleton of a square of B is a closed 4-circuit, so the 2-skeleton of $\mathsf{C}(A, B)$ is constructed in such a way that the boundaries of squares coincide with liftings of boundaries of squares of B.

The maps p and j then naturally extend to the 2-skeleton, and since any wall of the 2-skeleton of $\mathsf{C}(A, B)$ only consists of either horizontal or diagonal edges, we can extend r to this 2-skeleton by mapping a square $Q \subset \mathsf{C}(A, B)$ either to a square (if the walls dual to Q are both diagonal), to an edge (if one wall dual to Q is diagonal and the other one is horizontal, then collapse the horizontal edge to a point), or to a point (if both walls dual to Q are horizontal).

There is a unique way to complete this 2-skeleton to produce a NPC cube complex $\mathsf{C}(A, B)$ [HW08, Lem. 3.13], for which we can naturally extend j, r and p to maps satisfying the desired commuting properties (see [HW08, Cor. 6.7]). We call $\mathsf{C}(A, B)$ the *canonical completion* of $f : A \to B$ with (*canonical*) *inclusion* j and (*canonical*) *retraction* r.

Remark 2.8.11. In general, $\mathsf{C}(A, B)$ may be a disconnected covering of B. Also, from the construction above, it follows that if e is an edge of A and e' is an edge of $\mathsf{C}(A, B)$ dual to the wall $W(j(e))$, then e' is diagonal and $r(e')$ is dual to the wall $W(e) \subset A$. It is not hard to see that j maps distinct walls of A to distinct walls of $\mathsf{C}(A, B)$.

There are two relevant instances of this construction.

1. Suppose that $x \in \Gamma := \pi_1(B)$ is non-trivial and represented by the loop $\gamma \subset B$ and A is a convex, compact subcomplex of the universal cover of B containing a lift of γ. Then the component C of $\mathsf{C}(A, B)$ containing $j(A)$ is a finite cover for which γ lifts to a non-closed curve, so that x is not contained in the finite index subgroup $\hat{\Gamma} := \pi(\mathsf{C}) < \Gamma$. Applying this to all the non-trivial elements $x \in \Gamma$, we deduce that Γ is residually finite, and this gives a direct proof that virtually special groups are residually finite.

2. Suppose now that B is a (fully clean) compact special cube complex with universal cover X, so that X is a cubulation of $\Gamma = \pi_1(B)$. If $H < \Gamma$ is a convex subcomplex with convex core $Y \subset X$, then $A = H \backslash Y$ is a compact NPC cube complex and the inclusion $Y \subset X$ induces a local isometry $f : A \to B$. It follows that A is also special. The retraction $r : \mathsf{C}(A, B) \to A$ induces a retraction homomorphism from $\hat{\Gamma} = \pi_1(\mathsf{C})$ onto $H = \pi_1(A) < \hat{\Gamma}$ for the component C of $\mathsf{C}(A, B)$ containing $j(A)$. Since $\hat{\Gamma}$ is residually finite, by Proposition 2.6.11 we get that H is separable in $\hat{\Gamma}$, and hence in Γ. As being fully clean can be ensured by passing to an appropriate finite-sheeted cover, this proves Theorem 2.8.3.

In section 3.2 we will extend these results and show that double cosets of convex subgroups of virtually special groups are separable.

2.8.3 Agol's Theorem and the Virtual Haken and Fibering Theorems

The strong separability properties of virtually special groups have relevant applications in low-dimensional topology. As mentioned in Example 2.7.21, fundamental groups of hyperbolic 3-manifolds are cubulable. Therefore, virtual specialness of these groups follows from the remarkable theorem of Agol (Theorem 1.1.1 in the Introduction), which we restate now.

Theorem 2.8.12 (Agol [Ago13, Thm. 1.1])**.** *Let (Γ, X) be a cubulated group. If Γ is hyperbolic, then (Γ, X) is virtually special.*

By Theorem 2.8.3, an implication of this is that the quasiconvex surface subgroups in the fundamental group of a closed hyperbolic 3-manifold constructed by Kahn and Markovic are separable (Example 2.4.17). Since all these subgroups correspond to surface immersions, we can use Scott's Criterion 2.6.2 to promote these immersions to embeddings in finite-sheeted covers of the 3-manifold. This gives a positive solution to the Virtual Haken Conjecture for closed hyperbolic 3-manifolds.

Theorem 2.8.13 (Agol, Virtual Haken Theorem [Ago13, Thm. 9.1])**.** *Let M be a closed hyperbolic 3-manifold. Then it has a finite-sheeted cover $\hat{M} \to M$ such that $\hat{M}$ contains a π_1-injective, embedded closed surface of negative Euler characteristic.*

Theorem 2.8.12 also has consequences for virtual fibering. We say that a 3-manifold *fibers* if it is homeomorphic to the mapping torus of the homeomorphism of a surface, and that it *virtually fibers* if it has a finite-sheeted cover that fibers. In [Ago08], Agol gave a criterion of virtual fibering for 3-manifolds in terms of their fundamental groups. We say that a group Γ is *residually finite rationally solvable* (or *RFRS*) if there is a sequence of finite index subgroups $\Gamma = \Gamma_0 > \Gamma_1 > \Gamma_2 > \dots$ such that $\Gamma_k \trianglelefteq \Gamma$ for each k, $\bigcap_k \Gamma_k = \{o\}$, and $\ker\{\Gamma_k \to H_1(\Gamma_k; \mathbb{Q})\} \leq \Gamma_{k+1}$ for each k.

Theorem 2.8.14 (Agol, Fibering Criterion [Ago08, Thm. 5.1])**.** *Let M be a compact irreducible orientable 3-manifold with zero Euler characteristic. If $\pi_1(M)$ is RFRS, then M virtually fibers.*

In that same paper, Agol proved that subgroups of RAAGs are virtually RFRS [Ago08, Cor. 2.3]. As virtually special groups contain finite index subgroups that are contained in RAAGs, they are also virtually RFRS. Also, since fundamental groups of cusped hyperbolic 3-manifolds are virtually special by the work of Wise (Example 2.8.8), these manifolds are virtually fibered. Theorem 2.8.12 implies the same result in the closed case.

Theorem 2.8.15 (Agol, Wise, Virtual Fibering Theorem [Ago13, Thm. 9.2])**.** *Let M be a finite-volume hyperbolic 3-manifold. Then M virtually fibers.*

We end this chapter mentioning an application of Theorem 2.8.12 to *Cannon's Conjecture*, which asserts that if Γ is a hyperbolic group with Gromov boundary homeomorphic to a 2-sphere, then there exists a finite index subgroup $\Gamma_0 < \Gamma$ such that Γ_0 acts geometrically on $\mathbb{H}^3$. In [Mar13], Markovic used the separability of quasiconvex subgroups given by Theorem 2.8.3 to prove Cannon's conjecture under the assumption that Γ is cubulable (see also [Haï15, Cor. 1.11]).

Theorem 2.8.16 (Markovic [Mar13, Thm. 1.1])**.** *Let Γ be a hyperbolic cubulable group whose Gromov boundary is homeomorphic to a 2-sphere. Then there is a finite index subgroup $\Gamma_0 < \Gamma$ such that Γ_0 acts geometrically on $\mathbb{H}^3$.*

For more consequences of Theorem 2.8.12 to 3-manifold topology, see [Ago14]. In the next chapter we will extend Agol's Theorem 2.8.12 to the setting of relatively hyperbolic groups.

Chapter 3

Virtual specialness of relatively hyperbolic groups

Both hyperbolic and cubulated groups are very well-behaved classes of groups, even among the CAT(0) groups. These two properties combined have stronger implications, such as the celebrated Agol's Theorem 2.8.12. In this chapter we relax the hyperbolicity requirements and analyze groups acting geometrically on CAT(0) cube complexes which are only assumed to be relatively hyperbolic. Our goal is to prove Theorem 1.1.2 (now Theorem 3.1.2), which is an analog of Agol's theorem in the relative setting. Our main result relies on the notion of compatibility of virtually special peripheral subgroups, which is discussed in Section 3.1.

Then in Section 3.2 we prove Theorem 3.2.1, which says that double cosets of convex subgroups of virtually compact special groups are separable. This is done by studying the functoriality of the canonical completion, and implies Proposition 3.1.3 that is used in subsequent sections. We discuss Dehn fillings in Section 3.3 where we extend some properties of H-wide fillings, our main results being Theorems 3.3.2 and 3.3.14. In this section we also introduce Einstein's relative malnormal special quotient theorem. These results will be used in Section 3.4 to obtain hyperbolic virtually special fillings for relatively hyperbolic groups in $\mathcal{CMVH}$. This will imply Theorem 1.1.7 (now Theorem 3.4.5), a relative analog of Wises's quasiconvex hierarchy theorem.

The rest of the chapter consists in proving Theorem 1.1.8 (now Theorem 3.5.1), which asserts that a cubulated relatively hyperbolic group (Γ, X) with compatible virtually special peripheral subgroups belongs to $\mathcal{CMVH}$. We do this by adapting Agol's proof that cubulated hyperbolic groups belong to $\mathcal{QVH}$. In section 3.5 we construct a quotient cube complex $\mathcal{X}$ for $\dot{X}$ with finite embedded walls, which will be used to model the desired hierarchy. We color the walls of $\mathcal{X}$ in Section 3.6, and in Section 3.7 we use this coloring to start the construction of the *cubical polyhedra*, an inductively defined collection of (disconnected) cube complexes with *boundary walls* that encode the description of $(\Gamma, \dot{X})$ as a cubulated group in $\mathcal{CMVH}$. We study these cubical polyhedra in more detail in Section 3.8, and conclude the proof of Theorem 3.5.1 in Section 3.9, where we show how to perform the inductive construction.

3.1 Compatible virtually special peripheral subgroups

As we mentioned in the Introduction, we will work with cubulated relatively hyperbolic groups whose peripheral subgroups are virtually special and compatible, in the sense of the following definition.

Definition 3.1.1. A cubulated and relatively hyperbolic group (Γ, X) is *hyperbolic relative to compatible virtually special subgroups* if for any peripheral subgroup $P < \Gamma$ there exists a convex core $Z \subset X$ for P such that the cubulated group (P, Z) is virtually special.

From Theorems 2.7.12 and 2.8.3, it follows that if (Γ, X) is relatively hyperbolic and virtually special, then (Γ, X) is hyperbolic relative to compatible virtually special subgroups. We recall our main result (Theorem 1.1.2), which tells us that this compatibility condition is enough to guarantee virtual specialness.

Theorem 3.1.2. *Let (Γ, X) be a cubulated group that is hyperbolic relative to compatible virtually special subgroups. Then (Γ, X) is virtually special.*

Since any subgroup or double coset of a finitely generated virtually abelian group is separable, Theorem 2.8.4 implies that any cubulated group that is hyperbolic relative to virtually abelian subgroups satisfies Definition 3.1.1. By Theorem 1.1.2, this proves Corollary 1.1.3. In a similar way we can deduce Corollary 1.1.5 from Theorem 1.1.4.

Before starting the proof of Theorem 3.1.2, we need some facts about cubulations with compatible virtually special peripheral subgroups. First, we observe that the existence of a virtually special convex core for a convex subgroup implies that any other convex core gives a virtually special cubulation. This follows from the next proposition, which will be proven in Section 3.2 as a consequence of Theorem 3.2.1.

Proposition 3.1.3. *Let (Γ, X) be a cubulated group and let $H < \Gamma$ be a convex subgroup with convex core $Y \subset X$. If (H, Y) is virtually special, then (H, Y') is virtually special for any other convex core $Y' \subset X$ for H.*

We also note that the property of having compatible virtually special peripheral subgroups is preserved under considering convex subgroups. In particular, all the relevant subgroups mentioned in points (2) and (3) of Definition 3.4.4 in Section 3.4 (Definition 1.1.6) are hyperbolic relative to compatible virtually special subgroups.

Lemma 3.1.4. *If (Γ, X) is a cubulated group such that $(\Gamma, \mathcal{P})$ is hyperbolic relative to compatible virtually special subgroups, and $H < \Gamma$ is a convex subgroup with convex core $Y \subset X$, then (H, Y) is also hyperbolic relative to compatible virtually special subgroups when endowed with its induced peripheral structure.*

Proof. By Theorem 2.5.7, H is relatively quasiconvex in Γ. Let $P < \Gamma$ be a peripheral subgroup such that $H \cap P$ is infinite, and let $U \subset Y$ be a convex core for $H \cap P$. We claim that $(H \cap P, U)$ is virtually special. If $Z \subset X$ is any convex core for $P < G$, by Proposition

3.1.3 the cubulation (P, Z) is virtually special, and so by the characterization of special cube complexes given in [HW12, Def. 2.4] it is enough to show that there is a convex core Z with $U \subset Z$. But $(H \cap P)\backslash U$ is compact, so there is a compact subset $B \subset Z$ such that $(H \cap P) \cdot B = U$. Therefore by Theorem 2.7.12 we can choose Z containing B, and hence containing U. □

We end this section noticing that having compatible virtually special subgroups implies strong peripheral separability of convex subgroups (Definition 2.5.14).

Lemma 3.1.5. *Let (Γ, X) be a cubulated group that is hyperbolic relative to compatible virtually special subgroups. Then any convex subgroup of (Γ, X) is strongly peripherally separable.*

Proof. Let $H < \Gamma$ be a convex subgroup with convex core $Y \subset X$, and let $P < \Gamma$ be a peripheral subgroup with $H \cap P$ infinite. We claim that if $D' < H \cap P$ is a finite index subgroup, then it is separable in P. To prove the claim, let $Z \subset X$ be a convex core for P, for which we can assume to have non-trivial intersection with Y by Theorem 2.7.12. In this case, Lemma 2.7.14 implies that D' is a convex subgroup of (P, Z). But the cubulated group (P, Z) is virtually special by Proposition 3.1.3, and so the claim follows from Theorem 2.8.3. □

3.2 Functoriality of the canonical completion

This section is devoted to proving the following theorem, which generalizes one of the implications of Theorem 2.8.4. This result will be needed to prove Proposition 3.1.3.

Theorem 3.2.1. *Let (Γ, X) be a cubulated virtually special group. Then for any pair $H, K < \Gamma$ of convex subgroups the double coset HK is separable in Γ.*

Recall that $\dot{X}$ denotes the cubical barycentric subdivision of the CAT(0) cube complex X.

Corollary 3.2.2. *Let (Γ, X) be a cubulated group. Then (Γ, X) is virtually special if and only if $(\Gamma, \dot{X})$ is virtually special.*

Proof. If W' is a wall of $\dot{X}$, then $\Gamma_{W'}$ is a finite index subgroup of Γ_W for some wall W of X, hence a convex subgroup of (Γ, X). Therefore, by Theorems 2.8.3, 3.2.1, and 2.8.4, if (Γ, X) is virtually special then $(\Gamma, \dot{X})$ is virtually special. The converse follows by a similar argument. □

Our first lemma is a generalization of [HW08, Prop. 9.7]. We follow the exact same argument, relying on the canonical completion discussed in Section 2.8.

Lemma 3.2.3. *If (Γ, X) is a virtually special group, $H < (\Gamma, X)$ is a convex subgroup and $W \subset X$ is a wall, then $H\Gamma_W \subset \Gamma$ is separable.*

Proof. By using Lemma 2.7.15 we can find a convex core $Y \subset X$ for H such that $Y \cap W$ is non-empty, and let a be an edge of Y dual to W and incident to the vertex $y \in Y$. Since Γ is residually finite, by Lemma 2.7.8 there is a finite index subgroup $\hat{\Gamma} < \Gamma$ acting freely on X such that $\overline{X} := \hat{G}\backslash X$ is fully clean and special with simplicial 1-skeleton, and such that the projection $X \to \overline{X}$ maps squares to squares [HW08, Rmk. 6.8]. In that case, if $\hat{H} = H \cap \hat{\Gamma}$, then $\overline{Y} = \hat{H}\backslash Y$ is compact and the composition $f : \overline{Y} \to \hat{H}\backslash X \to \overline{X}$ is a local isometry. We can also assume that $\hat{H}_W\backslash(W \cap Y) \to \overline{Y}$ and $\hat{\Gamma}_W\backslash W \to \hat{\Gamma}\backslash X$ embed as walls, that we denote respectively by $W_{\overline{a}}$ and $W_{f(\overline{a})}$, with $\overline{a}$ being the image of a under the projection $Y \to \overline{Y}$. Let $p : \mathsf{C}(\overline{Y}, \overline{X}) \to \overline{X}$ be the canonical completion induced by f, with retraction r and inclusion map j.

Let $\mathcal{N}_{f(\overline{a})} \subset \overline{X}$ denote the cubical neighborhood of $W_{f(\overline{a})}$, which is lifted by p to the cubical neighborhood $\mathcal{N}_{j(\overline{a})}$ of the wall $W_{j(\overline{a})} \subset \mathsf{C}(\overline{Y}, \overline{X})$ dual to $j(\overline{a})$. By Remark 2.8.11, r maps any edge of $\mathsf{C}(\overline{Y}, \overline{X})$ dual to $W_{j(\overline{a})}$ to an edge dual to $W_{\overline{a}} \subset \overline{Y}$. In particular, since r maps cubes to cubes (possibly of lower dimension), we have $r(\mathcal{N}_{j(\overline{a})}) \subset \mathcal{N}_{\overline{a}}$, where $\mathcal{N}_{\overline{a}}$ is the cubical neighborhood of $W_{\overline{a}}$. On the other hand, since $X \to \overline{X}$ maps squares to squares, j and f also map squares to squares, so in fact we have $r(\mathcal{N}_{j(\overline{a})}) = \mathcal{N}_{\overline{a}}$. Therefore, if $\overline{y} \in \overline{Y}$ denotes the projection of y, there is a commutative diagram

$$\begin{array}{ccc} (\mathcal{N}_{j(\overline{a})}, j(\overline{y})) & \xrightarrow{\subset} & (\mathsf{C}(\overline{Y}, \overline{X}), j(\overline{y})) \\ \downarrow r & & \downarrow r \\ (\mathcal{N}_{\overline{a}}, \overline{y}) & \xrightarrow{\subset} & (\overline{Y}, \overline{y}) \end{array}$$

At the level of fundamental groups, and after considering the corresponding isomorphisms induced by $\hat{\Gamma} \cong \pi_1(\overline{X}, f(\overline{y}))$, we obtain a retraction homomorphism $r_* : \Gamma' \to \hat{H}$, where $\Gamma' < \hat{\Gamma}$ is the subgroup corresponding to $\pi_1(\mathsf{C}(\overline{Y}, \overline{X}), j(\overline{y}))$, and such that $r_*(\Gamma'_W) \subset \hat{H}_W$. It is not hard to see that the group $\widetilde{\Gamma}_W := \Gamma'_W \cap r_*^{-1}(\Gamma'_W) < \Gamma'$ satisfies $r_*(\widetilde{\Gamma}_W) \subset \widetilde{\Gamma}_W$, so by Proposition 2.6.11 the double coset $\hat{H}\widetilde{\Gamma}_W$ is separable in Γ', and hence in Γ since $\Gamma' < \Gamma$ is of finite index. But $\Gamma'_W = \Gamma_W \cap \Gamma' < \Gamma_W$ and $\hat{H} < H$ are of finite index, and hence $\widetilde{\Gamma}_W = \Gamma'_W \cap r_*^{-1}(\Gamma'_W) = \Gamma'_W \cap r_*^{-1}(\Gamma'_W \cap \hat{H}) < \Gamma'_W \cap r_*^{-1}(\Gamma_W \cap \hat{H}) = \Gamma'_W$ is also of finite index. We conclude that $H\Gamma_W$ is a finite union of translates of $\hat{H}\widetilde{\Gamma}_W$, so it is also separable in Γ. □

Corollary 3.2.4. *Let (Γ, X) be a virtually special group, and let $H < \Gamma$ be a convex subgroup with convex core $Y \subset X$. Then there exists a finite index subgroup $\Gamma' < \Gamma$ such that for any wall $W \subset X$ intersecting Y:*

1. *If $g \in \Gamma'$ satisfies $g\mathcal{N}(W) \cap Y \neq \emptyset$, then in fact $gW \cap Y \neq \emptyset$.*

2. *If W' is another wall of X intersecting Y and $g \in \Gamma'$ is such that $W' = gW$, then in fact $W' \cap Y = h'(W' \cap Y)$ for some $h' \in \Gamma' \cap H$.*

Remark 3.2.5. When we project down to the corresponding quotients, conclusion (1) of the corollary above may be thought of as a version of *no inter-osculation* for a wall and a locally convex subcomplex of a compact special cube complex [HW08].

Proof. The proof closely follows the idea of [HW08, Lem. 9.14]. Let $W_1, \dots, W_n$ be a complete set of representatives of H-orbits of walls intersecting Y, and for each i define the sets

$$I(\Gamma, Y, i) = \{g \in \Gamma \colon gW_i \cap Y \neq \emptyset\}, \ \ J(\Gamma, Y, i) = \{g \in \Gamma \colon g\mathcal{N}(W_i) \cap Y \neq \emptyset\}.$$

These sets are clearly (H, Γ_{W_i})-invariant, and also $I(\Gamma, Y, i) \subset J(\Gamma, Y, i)$, so there are subsets $\mathcal{I}_i \subset \mathcal{J}_i \subset \Gamma$ such that $I(\Gamma, Y, i) = \bigsqcup_{g \in \mathcal{I}_i} Hg\Gamma_{W_i}$ and $J(\Gamma, Y, i) = \bigsqcup_{g \in \mathcal{J}_i} Hg\Gamma_{W_i}$.

Let us prove first that each of the sets $\mathcal{J}_i$ is finite. Fix $1 \leq i \leq n$, and consider finite sets $D_i \subset \mathcal{N}(W_i)^{(0)}$ and $E \subset Y^{(0)}$ such that $\mathcal{N}(W_i)^{(0)} = \Gamma_{W_i} \cdot D$ and $Y^{(0)} = H \cdot E$. Given $g \in \Gamma$ such that $g\mathcal{N}(W_i) \cap Y$ is non-empty, there is a vertex v of $\mathcal{N}(W_i)$ with $gv \in Y$, and so there are group elements $w \in \Gamma_{W_i}$ and $h \in H$ satisfying $wv \in D_i$ and $hgv \in E$. In particular, since the action of Γ on X is proper, the composition hgw^{-1} lies in the finite set $\mathcal{F}_i$ of group elements $g' \in \Gamma$ such that $g'D_i \cap E \neq \emptyset$, and hence we can choose $\mathcal{J}_i \subset \mathcal{F}_i$.

Next, note that since by assumption $W_i \cap Y \neq \emptyset$, we have $H\Gamma_{W_i} \subset I(\Gamma, Y, i)$, so we may assume $o \in \mathcal{I}_i$. The finite set $\mathcal{J}_i \backslash \{o\}$ is then disjoint from $H\Gamma_{W_i}$ which is separable in Γ by Lemma 3.2.3. Therefore, there exists a finite index normal subgroup $\hat{\Gamma}_i \trianglelefteq \Gamma$ such that $(\bigcup_{g \in \mathcal{J}_i \backslash \{o\}} g\hat{\Gamma}_i) \cap H\Gamma_{W_i} = \emptyset$. We claim that (any finite index subgroup of) $\hat{\Gamma} := \bigcap_i \hat{\Gamma}_i$ satisfies conclusion (1).

Indeed, let $W \subset X$ be a wall intersecting Y, and let $1 \leq i \leq n$ and $h \in H$ such that $W = hW_i$. Assume by contradiction that there is some $g \in \hat{\Gamma}$ such that $g\mathcal{N}(W) \cap Y \neq \emptyset$ but $gW \cap Y = \emptyset$. Since $\hat{\Gamma}$ is normal, this implies $h^{-1}gh \in \hat{\Gamma} \cap J(\Gamma, Y, i) \backslash I(\Gamma, Y, i) \subset \hat{\Gamma}_i \cap J(\Gamma, Y, i) \backslash I(\Gamma, Y, i)$, and hence $h^{-1}gh = vg_iw$, for $v \in H$, $g_i \in \mathcal{J}_i \backslash \mathcal{I}_i$ and $w \in \Gamma_{W_i}$. This is a contradiction, because otherwise we would have $g_i((wh^{-1})g^{-1}(wh^{-1})^{-1}) = v^{-1}w^{-1} \in g_i\hat{\Gamma}_i \cap H\Gamma_{W_i}$, and so conclusion (1) follows.

For conclusion (2), since (Γ, X) is virtually special we can assume that $\hat{\Gamma} \backslash X$ is special, fully clean, and with simplicial 1-skeleton, so that the composition $f : (H \cap \hat{G}) \backslash Y \to (H \cap \hat{\Gamma}) \backslash X \to \hat{\Gamma} \backslash X$ is a local isometry. But (H, Y) is also virtually special, so by using Lemma 2.7.8 (2) and the separability of wall stabilizers in H we can find a finite index subgroup $H' < H$ such that for any further finite index subgroup $H'' < H'$ and for any wall $W \subset X$ intersecting Y, the map $(H'' \cap G_W) \backslash (W \cap Y) \to H'' \backslash Y$ is an embedding and the image is a wall of $H'' \backslash Y$.

The group $H' < \Gamma$ is convex, hence separable by Theorem 2.8.3, so by a separability argument we may assume that $H' \cap \hat{\Gamma} = H \cap \hat{\Gamma}$. With this in mind, let C be the connected component of $\mathsf{C}((H \cap \hat{\Gamma}) \backslash Y, \hat{\Gamma} \backslash X)$ including $(H \cap \hat{\Gamma}) \backslash Y$, and let $\Gamma' < \hat{\Gamma}$ correspond to its fundamental group, which is finite index in Γ since $(H \cap \hat{\Gamma}) \backslash Y$ is compact. Also, we have $H \cap \hat{\Gamma} < \Gamma'$, and so $H \cap \Gamma' = H \cap \hat{\Gamma}$. Since the inclusion of $(H \cap \Gamma') \backslash Y$ into C maps distinct walls to distinct walls (see Remark 2.8.11), our assumptions about H' imply that the group Γ' satisfies conclusion (2). ◻

The key ingredient in the proof of Theorem 3.2.1 is the following proposition, which says that under some mild assumptions, the canonical completion is functorial.

Proposition 3.2.6. *Let $\overline{V}, \overline{X}, \overline{Y}, \overline{Z}$ be special cube complexes such that the following is a commutative diagram of local isometries.*

$$\begin{array}{ccc} \overline{V} & \xrightarrow{f} & \overline{Y} \\ \downarrow s & & \downarrow t \\ \overline{Z} & \xrightarrow{g} & \overline{X} \end{array} \tag{3.1}$$

In addition, assume that

(i) $\overline{X}$ and $\overline{Y}$ are fully clean and have simplicial 1-skeleton.

(ii) t maps distinct walls to distinct walls.

(iii) If e is an edge of $\overline{X}$ incident to a vertex $t(y)$ of $t(\overline{Y})$ with e dual to a wall intersecting $t(\overline{Y})$, then $e = t(e')$ for some edge e' of $\overline{Y}$ incident to y.

(iv) For every vertex $v \in \overline{V}$, if there exist edges e of $\overline{Y}$ and e' of $\overline{Z}$ incident to $f(v)$ and $s(v)$ respectively and such that $t(e) = g(e')$, then there is an edge e'' of $\overline{V}$ incident to v and such that $e = f(e'')$ and $e' = s(e'')$.

Then there is a local isometry $\hat{t} : \mathsf{C}(\overline{V}, \overline{Y}) \to \mathsf{C}(\overline{Z}, \overline{X})$ of the canonical completions commuting with the corresponding inclusions and projections in the sense that the following diagrams commute.

$$\begin{array}{ccccc} \overline{V} & \xrightarrow{j} & \mathsf{C}(\overline{V}, \overline{Y}) & \xrightarrow{r} & \overline{V} \\ \downarrow s & & \downarrow \hat{t} & & \downarrow s \\ \overline{Z} & \xrightarrow{j'} & \mathsf{C}(\overline{Z}, \overline{X}) & \xrightarrow{r'} & \overline{Z} \end{array} \qquad \begin{array}{ccc} \mathsf{C}(\overline{V}, \overline{Y}) & \xrightarrow{\hat{t}} & \mathsf{C}(\overline{Z}, \overline{X}) \\ \downarrow p & & \downarrow p' \\ \overline{Y} & \xrightarrow{t} & \overline{X} \end{array} \tag{3.2}$$

Remark 3.2.7. As we will see below, items (i) (ii) and (iii) in the previous statement can be obtained for a general commutative diagram of local isometries between compact special cube complexes after passing to finite coverings. Item (iv) may be achieved if, for instance, the universal cover of $\overline{V}$ coincides with the intersection of the universal covers of $\overline{Y}$ and $\overline{Z}$, when we see these complexes naturally embedded in the universal cover of $\overline{X}$.

Proof. We first construct the map $\hat{t}$ for lower dimensional cubes of $\mathsf{C}(\overline{V}, \overline{Y})$, starting with the 0-skeleton where we define $\hat{t} : \overline{V}^{(0)} \times \overline{Y}^{(0)} \to \overline{Z}^{(0)} \times \overline{X}^{(0)}$ by $(v, y) \mapsto (s(v), t(y))$. In this way, $\hat{t}$ clearly satisfies (3.2). For the 1-skeleton, we will check that the image under $\hat{t}$ of a pair of vertices of $\mathsf{C}(\overline{V}, \overline{Y})$ representing a horizontal (resp. diagonal) edge is a pair of vertices representing a horizontal (resp. diagonal) edge of $\mathsf{C}(\overline{Z}, \overline{X})$. Let $e = \{(v, y), (v, y')\}$

be a horizontal edge of $\mathsf{C}(\overline{V},\overline{Y})$, for which we claim that the pair $\{(s(v),t(y)),(s(v),t(y'))\}$ represents a horizontal edge. Assume by contradiction that there exists an edge b of $\overline{Z}$ incident to $s(v)$, with $g(b)$ dual to the wall $W(\{t(y),t(y')\}) \subset \overline{X}$ (note that $\{t(y),t(y')\}$ is an edge since $\overline{X}$ has simplicial 1-skeleton and t is local isometry). The edge $g(b)$ is incident to $g(s(v)) = t(f(v))$, so by item (iii), $g(b)$ equals $t(b')$ for an edge b' incident to $f(v)$. Item (ii) then implies that b' is dual to the wall $W(\{y,y'\}) \subset \overline{Y}$, and item (iv) gives us an edge b'' of $\overline{V}$ incident to v with $f(b'') = b'$. But this would imply that e is not horizontal, and this contradiction proves the claim. The case of $e = \{(v,y),(v',y')\}$ diagonal is easier since t maps walls to walls, and hence $g(\{s(v),s(v')\})$ is dual to $W(\{t(y),t(y')\})$. Therefore, the image of a horizontal/diagonal edge of $\mathsf{C}(\overline{V},\overline{Y})$ is defined as the expected horizontal/diagonal edge of $\mathsf{C}(\overline{Z},\overline{X})$, and since the image of an edge under r or r' only depends on whether the edge is horizontal or vertical, $\hat{t}$ also satisfies (3.2) at the level of 1-skeleton.

Now, let Q be a square of $\mathsf{C}(\overline{V},\overline{Y})$, say with 1-skeleton determined by the vertices $\{(v_i,y_i)\}_{i=1}^4$. By definition, this means that $p(Q)$ is also a square with 0-skeleton $\{y_i\}_{i=1}^4$, and by item (i) the vertices $\{t(y_i)\}_{i=1}^4$ are the 0-skeleton of the square $t(p(Q))$ of $\overline{X}$. Since $\mathsf{C}(\overline{Z},\overline{X})$ is a covering, these vertices lift under p' to the set $\{(s(v_i),t(y_i))\}_{i=1}^4$ that is the 0-skeleton a square Q', that we define as the image of Q under $\hat{t}$. Again, since the image of a square under a retraction only depends on whether its 1-skeleton consists of horizontal/diagonal edges, the diagrams (3.2) still commute.

Finally, by [HW08, Lem. 2.5] there is a unique way to extend $\hat{t}$ to a combinatorial map $\mathsf{C}(\overline{V},\overline{Y}) \to \mathsf{C}(\overline{Z},\overline{X})$, which is clearly a local isometry. Also, since the maps r, r', p and p' restricted to a higher dimensional cube depend only on its 2-skeleton, by the uniqueness of $\hat{t}$, it must satisfy (3.2). □

Remark 3.2.8. In the proof of Theorem 3.2.1 below, we will be interested in the commutative diagrams of fundamental groups induced by (3.2), so we will only require these diagrams to commute at the level of 2-skeletons.

Proof of Theorem 3.2.1. Let us use Lemma 2.7.15 to find convex cores Y and Z for H and K respectively, such that $V = Y \cap Z$ is non-empty. By Lemma 2.7.14 this implies that V is a convex core for $H \cap K$. We will prove first that there exists a finite index subgroup $\hat{\Gamma} < \Gamma$ such that if $\hat{H} = H \cap \hat{\Gamma}$ and $\hat{K} = K \cap \hat{\Gamma}$, then after defining $\overline{V} = (\hat{H} \cap \hat{K})\backslash V$, $\overline{X} = \hat{\Gamma}\backslash X$, $\overline{Y} = \hat{H}\backslash Y$ and $\overline{Z} = \hat{K}\backslash Z$ the induced diagram (3.1) is of local isometries and satisfies the items (i)-(iv) of Proposition 3.2.6.

By [HW08, Cor. 8.9] we can find $\hat{\Gamma} < \Gamma$ of finite index such that item (i) holds, and by possibly replacing $\hat{\Gamma}$ by a further finite index subgroup satisfying Corollary 3.2.4, we can ensure that $\hat{\Gamma}$ also satisfies (ii).

To prove item (iii), let e be an edge of $\overline{X}$ incident to $t(y)$ for a vertex $y \in \overline{Y}$, and let $\widetilde{e} \subset X$ be a lifting of e incident to the lifting $\widetilde{y} \in Y$ of y. If e is dual to the wall $W(t(b)) \subset \overline{X}$ for some edge $b \subset \overline{Y}$, then there exists a lifting $\widetilde{b} \subset Y$ of b and some $g \in \hat{\Gamma}$ such that $W(\widetilde{e}) = gW(\widetilde{b})$. Since $\widetilde{y} \in Y$, we have $W(\widetilde{b}) \cap Y \neq \emptyset$ and $g\mathcal{N}(W(\widetilde{b})) \cap Y \neq \emptyset$, so by conclusion (1) of Corollary 3.2.4 we have $W(\widetilde{e}) \cap Y \neq \emptyset$ and $\widetilde{e} \subset Y$, implying item (iii).

Finally, let $v \in \overline{V}$ be a vertex lifting to $\widetilde{v} \in V$, and let $\widetilde{e}$ and $\widetilde{e}'$ be edges of Y and Z respectively, incident to $\widetilde{v}$ and such that there exists some $g \in \hat{\Gamma}$ with $g\widetilde{e} = \widetilde{e}'$. Since the action of $\hat{\Gamma}$ is free, we have $g = 1$ and $\widetilde{e} = \widetilde{e}' \in V$. Projecting to the corresponding quotients we deduce (iv).

Therefore, we are in the assumptions of Proposition 3.2.6, and there is a local isometry $\hat{t} : \mathsf{C}(\overline{V}, \overline{Y}) \to \mathsf{C}(\overline{Z}, \overline{X})$ such that the diagrams (3.2) commute. The proof now goes as in Lemma 3.2.3. If $H' < \hat{H}$ and $\Gamma' < \hat{\Gamma}$ are the finite index subgroups representing the fundamental groups of the (appropriate connected components of the) canonical completions $\mathsf{C}(\overline{V}, \overline{Y})$ and $\mathsf{C}(\overline{Z}, \overline{X})$ respectively, then $H' < \Gamma'$ and there is a retraction homomorphism $r_* : \Gamma' \to \Gamma'$ with image $\hat{K}$ and such that $r_*(H') = \hat{H} \cap \hat{K}$. Again, we can check that $\widetilde{H} := H' \cap r_*^{-1}(H')$ satisfies $r_*(\widetilde{H}) \subset \widetilde{H}$, and so Proposition 2.6.11 implies that $\widetilde{H}\hat{K}$ is separable in Γ'. Since $\Gamma' < \Gamma$, $\widetilde{H} < H$ and $\hat{K} < K$ are all of finite index, we conclude that HK is separable in Γ, completing the proof. □

We now see how Theorem 3.2.1 implies Proposition 3.1.3. In fact, by Lemma 2.7.15, Proposition 3.1.3 follows from the next proposition.

Proposition 3.2.9. *Let (Γ, X) be a cubulated group, and let $Y \subset X$ be a Γ-invariant convex subcomplex. If the cubulated group (Γ, Y) is virtually special, then (Γ, X) is also virtually special.*

Before proving this result, we first recall the definition of gate map projection [BHS17, Sec. 2].

Definition 3.2.10. Let X be a CAT(0) cube complex, and consider a convex subcomplex $Y \subset X$. The *gate map* is defined as the unique cubical map $\mathfrak{g} : X \to Y$ characterized by the following property: for any point $x \in X$, the wall $W \subset X$ separates x from $\mathfrak{g}(x)$ if and only if it separates x from Y.

The next lemma is well-known by experts and is implicit, for instance, in [BHS17], so we provide proof in the absence of a precise reference.

Lemma 3.2.11. *Let (Γ, X) be a cubulated group and let $Y \subset X$ be a Γ-invariant convex subcomplex. Then for any convex subcomplex $K \subset X$, its image $\mathfrak{g}(K) \subset Y$ is also a convex subcomplex. Moreover, if $H < \Gamma$ preserves K and acts cocompactly on it, then it also acts cocompactly on $\mathfrak{g}(K)$.*

Proof. Since CAT(0) convexity coincides with combinatorial convexity for subcomplexes, for the first assertion it is enough to prove that if $x, y \in X$ are vertices and β is a combinatorial geodesic segment in $X^{(1)}$ joining $\mathfrak{g}(x)$ and $\mathfrak{g}(y)$, then $\beta = \mathfrak{g}(\alpha)$, for some geodesic α joining x and y. We will prove this by induction on the sum of combinatorial distances $d = d(x, \mathfrak{g}(x)) + d(y, \mathfrak{g}(y))$, where the case $d = 0$ holds since Y is convex. So, assume that the claim follows for $d \geq 0$, and let $x, y \in X$ be vertices with $d(x, \mathfrak{g}(x)) + d(y, \mathfrak{g}(y)) = d + 1$, for which we can assume $d(x, \mathfrak{g}(x)) > 0$. Thus, let γ be a combinatorial geodesic path in $X^{(1)}$ joining x and

$\mathfrak{g}(x)$, and u be the vertex on this geodesic at distance 1 to x. Except for the wall dual to the edge e determined by x and u, any other wall dual to an edge of γ separates u and $\mathfrak{g}(x)$, so $\mathfrak{g}(u) = \mathfrak{g}(x)$, and by our inductive assumption there is a geodesic α' joining u and y, such that $\mathfrak{g}(\alpha') = \beta$.

If e separates x from y, the concatenation of e and α' defines a geodesic α projecting to β. Otherwise, there is an edge e' of α' dual to $W(e)$, say determined by the vertices p and q with p between u and q. In this case, the segment α'' of α' between u and p lies in one of the sides of $\mathcal{N}(W(e))$ which we know is a convex subcomplex, so every vertex of α'' lies in an edge dual to $W(e)$. If we follow the extreme points of these edges lying on the other side of $\mathcal{N}(W(e))$, we will obtain a geodesic path joining x and q. By concatenating this path with the segment of α between q and y, we will obtain a geodesic path α (there is no repetition in the walls dual to α), and it is easy to see that $\mathfrak{g}(\alpha) = \beta$.

The second assertion follows easily since Y is Γ-invariant, and since $H\backslash\mathfrak{g}(K)$ is the image of the compact set $H\backslash K$ under the induced projection $\mathfrak{g} : \Gamma\backslash X \to H\backslash Y$. □

Proof of Proposition 3.2.9. Let $W_1, W_2 \subset X$ be walls with stabilizers Γ_1 and Γ_2, respectively. By Theorem 2.8.4, to prove the proposition it is enough to show that Γ_1 and $\Gamma_1\Gamma_2$ are separable in Γ. Consider then the gate map $\mathfrak{g} : X \to Y$ and the projections $\mathfrak{g}(\mathcal{N}(W_1))$ and $\mathfrak{g}(\mathcal{N}(W_2))$, which by Lemma 3.2.11 are convex subcomplexes of Y. This same lemma also implies that each subgroup Γ_i acts cocompactly on $\mathfrak{g}(\mathcal{N}(W_i))$, and so Γ_1 and Γ_2 are convex subgroups of (Γ, Y), which by assumption is virtually special. The conclusion then follows by Theorems 2.8.3 and 3.2.1. □

3.3 Relative height and weak separability of double cosets

In this section we prove some results about Dehn fillings of relatively hyperbolic groups. We discuss the notions of (relative) height and malnormality, as well as weak separability of double cosets of relatively quasiconvex subgroups under some assumptions. We also review Einstein's malnormal special quotient theorem in the setting of relatively hyperbolic groups. The main results of the section are Theorems 3.3.2, 3.3.14, and Proposition 3.3.19.

3.3.1 Relative Height, and R-Hulls

The relative height of a relatively quasiconvex subgroup of a relatively hyperbolic group was introduced by Hruska and Wise [HW09, Def. 8.1] as the appropriate analog of the height for quasiconvex subgroups of hyperbolic groups [Git+98]. In particular, they proved that the relative height is always finite. In this subsection we introduce the corresponding version for a finite collection of relatively quasiconvex subgroups, we show that it is finite, and prove that it is non-increasing under sufficiently long Dehn filling, extending the results in [GM21, Sec. 7] (c.f. [Ago13, A.3]).

Definition 3.3.1. Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic group and consider a set $\mathcal{H} = \{H_1, \ldots, H_k\}$ of *distinct* relatively quasiconvex subgroups of Γ. The *relative height* of $\mathcal{H}$ in Γ is the maximum number $n \geq 0$ so that there are distinct left cosets $\{g_1 H_{\sigma(1)}, \ldots, g_n H_{\sigma(n)}\}$ with $\bigcap_{i=1}^n H_{\sigma(i)}^{g_i}$ containing a loxodromic element of Γ.

The following is the main result of the subsection.

Theorem 3.3.2. *For all sufficiently long and $\mathcal{H}$-wide fillings, the groups in $\overline{\mathcal{H}} = \{\overline{H}_1 \ldots, \overline{H}_k\}$ are all distinct and the relative height of $\overline{\mathcal{H}}$ in $(\overline{\Gamma}, \overline{\mathcal{P}})$ is at most the relative height of $\mathcal{H}$ in $(\Gamma, \mathcal{P})$.*

In the case where $\mathcal{H}$ consists of a single subgroup H, Theorem 3.3.2 reduces to [AGM16, Thm. 7.12] for H-fillings and to [GM21, Thm. 7.15] for general H-wide fillings.

Definition 3.3.3. A collection $\{H_1, \ldots, H_k\}$ of distinct subgroups of $(\Gamma, \mathcal{P})$ is *relatively malnormal* if whenever $H_i \cap H_j^g$ contains a loxodromic element, then $i = j$ and $g \in H_i$. The collection is *almost malnormal* if $H_i \cap H_j^g$ is finite unless $i = j$ and $g \in H_i$.

Note that relative malnormality is equivalent to relative height at most 1 and that relative malnormality coincides with almost malnormality when $\mathcal{P}$ is empty or consists of finite groups (in which case Γ is hyperbolic). These observations together with Theorem 3.3.2 imply the following corollary. Recall that a filling $\Gamma \to \Gamma(N_1, \ldots, N_n)$ is *peripherally finite* if $N_i \trianglelefteq P_i$ is finite index for each $1 \leq i \leq n$.

Corollary 3.3.4. *Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a relatively malnormal collection of relatively quasiconvex subgroups of $(\Gamma, \mathcal{P})$. Then for all sufficiently long and $\mathcal{H}$-wide peripherally finite fillings, the collection $\{\overline{H}_1, \ldots, \overline{H}_k\}$ is almost malnormal in $\overline{\Gamma}$.*

The proof of Theorem 3.3.2 will take us the rest of the subsection, in which we refine the techniques from [AGM16] and [GM21] about R-hulls over cusped spaces and relative multiplicity.

Let $X = X(\Gamma, \mathcal{P}, S)$ be a cusped space for $(\Gamma, \mathcal{P})$, and let $\delta \geq 0$ be given by Theorem 2.5.13 (1), so that X and $\overline{X}$ are δ-hyperbolic and have δ-slim geodesic triangles for all sufficiently long fillings (see Subsection 2.3.1). The *depth* of a vertex of X is its distance to the Caley graph $\mathrm{Cay}(\Gamma, S) \subset X$. Note that the action of Γ on X is depth-preserving.

Definition 3.3.5 (c.f. [GM21, Def. 7.3])**.** Let $\widetilde{*} = o \in \Gamma \subset X$, and consider $(H, \mathcal{D}) < (\Gamma, \mathcal{P})$ a relatively quasiconvex subgroup and $R \geq 0$. An *R-hull for H acting on X* is a connected H-invariant full subgraph $\widetilde{Z_H} \subset X$ satisfying:

1. $\widetilde{*} \in \widetilde{Z_H}$.
2. If $\gamma \subset X$ is a geodesic with endpoints in the *limit set* $\Lambda(H)$, then $N_R(\gamma) \cap N_R(\Gamma) \subset \widetilde{Z_H}$.
3. If A is a horoball in X containing a vertex a at depth greater than 0 in the image $\breve{\imath}(X_H)$, then $\widetilde{Z_H} \cap A^{(0)}$ contains every vertex of the maximal vertical ray in A containing a.

4. The action of $(H, \mathcal{D})$ on $\widetilde{Z_H}$ is *weakly geometrically finite* in the sense of [Ago13, A.27].

By [GM21, Lem. 7.6], R-hulls for the action of H on X exist for any $R \geq 0$, and moreover, we can construct them in such a way that they have only finitely many H-orbits of vertices at depth 0. Therefore, anytime we consider an R-hull, implicitly we will assume it satisfies this property. The relevance of R-hulls is that they allow us to extract some algebraic information about their corresponding relatively quasiconvex subgroups, as we will see in Proposition 3.3.9 below.

Let $Y = \Gamma \backslash X$, and consider an R-hull $\widetilde{Z_H}$ for the action of H on X with quotient $Z_H := H \backslash \widetilde{Z_H}$. The natural map $\mathscr{I}_H : Z_H \to Y$ induces the inclusion $H \to \Gamma$ in the following way. If $*_H \in Z_H$ and $* \in Y$ are the respective projections of $\widetilde{*}$, we obtain canonical surjections $s : \pi_1(Z_H, *_H) \to H$ and $s : \pi_1(Y, *) \to \Gamma$ such that the following diagram commutes

$$\begin{array}{ccc} \pi_1(Z_H, *_H) & \xrightarrow{(\mathscr{I}_H)_*} & \pi_1(Y, *) \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ H & \longrightarrow & \Gamma \end{array} \tag{3.3}$$

with the bottom map being the inclusion.

Let $\mathcal{H} = \{H_1, \ldots, H_k\}$ be a collection of relatively quasiconvex subgroups, and assume that $H_i \neq H_j$ whenever $i \neq j$. By Theorem 2.5.13 (3), let λ be a quasiconvexity constant for each H_j and $\overline{H}_j$ for all sufficiently long and $\mathcal{H}$-wide fillings. Consider R-hulls $\widetilde{Z_j} = \widetilde{Z_{H_j}}$ for each $1 \leq j \leq k$, with quotients $Z_j = H_j \backslash \widetilde{Z_j}$ and maps $\mathscr{I}_j = \mathscr{I}_{H_j} : Z_j \to Y$ as above.

Definition 3.3.6 (cf. [Ago13, Def. A.15])**.** Given $m > 0$ and an m-tuple $\sigma = (\sigma(1), \ldots, \sigma(m)) \in \{1, \ldots, k\}^m$, we write $|\sigma| = m$ and define

$$S_\sigma := \{(z_1, \ldots, z_m) \in Z_{\sigma(1)} \times \cdots \times Z_{\sigma(m)} \colon \mathscr{I}_{\sigma(1)}(z_1) = \cdots = \mathscr{I}_{\sigma(m)}(z_m)\} \backslash \Delta_\sigma$$

where $\Delta_\sigma = \{(z_1, \ldots, z_m) \colon \exists i \neq j \text{ s.t. } \sigma(i) = \sigma(j) \text{ and } z_i = z_j\}$ should be understood as the fat diagonal.

Points in S_σ have a well-defined depth which is the depth of their corresponding images in Y. Let C be a component of S_σ containing a base-point $p = (p_1, \ldots, p_m)$ at depth 0, and fix maximal trees T_j in Z_j inducing preferred paths β_v between $*_j = *_{H_j}$ and the vertex $v \in Z_j$. The paths $\beta_i = \beta_{p_i}$ give isomorphisms $\pi_1(Z_{\sigma(i)}, p_i) \to \pi_1(Z_{\sigma(i)}, *_{\sigma(i)})$, and composing them with the push-forwards of the projections $\omega_i : (C, p) \to (Z_{\sigma(i)}, p_i)$ we obtain homomorphisms $(\omega_i)_* : \pi_1(C, p) \to \pi_1(Z_{\sigma(i)}, *_{\sigma(i)})$. Define then $\tau_{C,i} : s \circ (\omega_i)_* : \pi_1(C, p) \to H_{\sigma(i)}$, where $s : \pi_1(Z_{\sigma(i)}, *_{\sigma(i)}) \to H_{\sigma(i)}$ is as in (3.3).

Definition 3.3.7 (cf. [GM21, Def. 7.7])**.** The *relative multiplicity* of $\{\mathscr{I}_j : Z_j \to Y\}_{1 \leq j \leq k}$ is the largest m so that there is some σ with $|\sigma| = m$ and S_σ containing a component C such that the group $\tau_{C,i}(\pi_1(C, p))$ is infinite non-parabolic for all $1 \leq i \leq m$.

Remark 3.3.8. As we mentioned after Definition 3.3.5, we are considering R-hulls with only finitely many orbits of vertices at depth 0. This implies that for any j, the number r_j of vertices in Z_j at depth 0 is finite. In particular, if $m > r_1 + \cdots + r_k$ and $|\sigma| = m$, then any tuple in $Z_{\sigma(1)} \times \cdots \times Z_{\sigma(m)}$ at depth 0 lies in Δ_σ, and consequently the relative multiplicity of $\{\mathscr{J}_j : Z_j \to Y\}_{1\leq j\leq k}$ is bounded by $r_1 + \cdots + r_k$.

The main property of relative multiplicity is given by the following proposition.

Proposition 3.3.9 (cf. [GM21, Thm. 7.8])**.** *For sufficiently large R, depending only on δ and λ, if each $\widetilde{Z_j}$ is an R-hull for the action of H_j on X, then the relative height of $\{H_1, \dots, H_k\}$ in Γ is equal to the relative multiplicity of $\{\mathscr{J}_j : Z_j \to Y\}_{1\leq j\leq k}$.*

For its proof, we need a preliminary lemma.

Lemma 3.3.10 (cf. [Ago13, Lem. A.37])**.** *Let C be a fixed component of S_σ based at the point $p = (p_1, \dots, p_m)$ at depth 0. Then for any $1 \leq i, j \leq m$ there is some $g_{i,j} \in \Gamma$ such that $g_{i,j}\tau_{C,j}([\alpha]){g_{i,j}}^{-1} = \tau_{C,1}([\alpha])$ for any homotopy class $[\alpha] \in \pi_1(C, p)$.*

Proof. Recall the paths β_i, β_j in the construction of the maps $\tau_{C,i}, \tau_{C,j}$. In virtue of the commutative diagram (3.3), the element $\tau_{C,i}([\alpha]) \in \Gamma$ coincides with $s([(\mathscr{I}_i \circ \beta_i) \cdot (\mathscr{I}_i \circ \omega_i \circ \alpha) \cdot (\mathscr{I}_i \circ \overline{\beta}_i)])$. But α is a loop in C, so $(\mathscr{I}_i \circ \omega_i \circ \alpha) = (\mathscr{I}_j \circ \omega_j \circ \alpha)$, and in particular $(\mathscr{I}_i \circ \beta_i) \cdot (\mathscr{I}_j \circ \overline{\beta}_j)$ is a loop in Y based at $*$. This loop defines the element $g_{i,j} \in \Gamma$, for which it is easy to check that the requirements are satisfied. □

Proof of Proposition 3.3.9. We proceed in the same way as in the proof of [GM21, Thm. 7.8]. First we prove that the relative height is at most the relative multiplicity, so suppose that the relative height is at least m, and let $\sigma = (\sigma(1), \dots, \sigma(m))$ and $g_1 H_{\sigma(1)}, \dots, g_m H_{\sigma(m)}$ be distinct left cosets such that $\bigcap_{i=1}^m H_{\sigma(i)}^{g_i}$ contains a loxodromic element, say h. In [GM21] it was proven that by choosing an appropriate R, and by replacing h by a power we may suppose that h has a quasi-geodesic axis $\widetilde{\gamma}$ contained in $\bigcap_{i=1}^m g_i \widetilde{Z_{\sigma(i)}}$. In this case the path $\widetilde{\gamma}$ induces a loop $\gamma : \mathbb{R}/\mathbb{Z} \to Z_{\sigma(1)} \times \cdots \times Z_{\sigma(m)}$ defined by

$$\gamma(t) = (\pi_{\sigma(1)}(g_1^{-1}\widetilde{\gamma}(t)), \dots, \pi_{\sigma(m)}(g_m^{-1}\widetilde{\gamma}(t))).$$

Since the cosets $g_i H_{\sigma(i)}$ are distinct and Γ acts freely on X, γ misses the fat diagonal Δ_σ, and hence defines a loop in a connected component C of S_σ containing a vertex p at depth 0. It is not hard to see that $\tau_{C,i}(\pi_1(C, p))$ contains a conjugate of h for each i, so the relative multiplicity is at least m.

For the converse, suppose that the relative multiplicity of $\{\mathscr{J}_j : Z_j \to Y\}_j$ is m, and let $\sigma = (\sigma(1), \dots, \sigma(m))$ and C be a component of S_σ containing a point p at depth 0, such that for each $1 \leq i \leq m$ the group $A_i = \tau_{C,i}(\pi_1(C, p)) < H_{\sigma(i)} < G$ contains a loxodromic element. Consider the elements $g_i = g_{1,i} \in \Gamma$ given by Lemma 3.3.10 so that $A_i^{g_i} = A_1$ for each i, which implies $A_1 \subset \bigcup_{i=1}^m H_{\sigma(i)}^{g_i}$.

It is only left to show that the cosets $g_i H_{\sigma(i)}$ are all different. Indeed, let $p = (p_1, \dots, p_m)$ and suppose there exist $i \neq j$ with $g_i H_{\sigma(i)} = g_j H_{\sigma(j)}$, implying $\sigma(i) = \sigma(j)$ and $g_i = g_j h$

for some $h \in H_{\sigma(i)}$. Let $\widetilde{\beta}_i, \widetilde{\beta}_j$ be the unique liftings of β_i and β_j to $\widetilde{Z_{\sigma(i)}} \subset X$ starting at $\widetilde{*}$, respectively, and note that $g_j^{-1} g_i$ is represented by the loop $(\mathscr{I}_{\sigma(j)} \circ \beta_j) \cdot (\mathscr{I}_{\sigma(i)} \circ \overline{\beta}_i)$ in $\pi_1(Y, *)$. This implies $\widetilde{p}_j = h\widetilde{p}_i$, where $\widetilde{p}_i$ and $\widetilde{p}_j$ are the corresponding endpoints of $\widetilde{\beta}_i$ and $\widetilde{\beta}_j$, and by projecting back into $Z_{\sigma(i)}$ we get $p_i = p_j$, contradicting $p \notin \Delta_\sigma$. □

The previous proposition together with Remark 3.3.8 implies the following corollary, generalizing [HW09, Thm. 1.4].

Corollary 3.3.11. *If Γ is a relatively hyperbolic group and $\mathcal{H}$ is a finite collection of distinct relatively quasiconvex subgroups of Γ, then the relative height of $\mathcal{H}$ in Γ is finite.*

Before starting the proof of Theorem 3.3.2 we state the following lemma.

Lemma 3.3.12 ([GM21, Lem. 7.12 & Lem. 7.14])**.** *Suppose $(\Gamma, \mathcal{P})$ is relatively hyperbolic and $\mathcal{H} = \{H_1, \ldots, H_k\}$ is a collection of relatively quasiconvex subgroups of Γ. Then for all R there is some R' satisfying the following: for all sufficiently long and $\mathcal{H}$-wide fillings $\phi : \Gamma \to \Gamma/K$, if $\widetilde{Z_j}$ is an R'-hull for the action of H_j on X for each $1 \leq j \leq k$, then its image $\overline{\widetilde{Z_j}} \subset \overline{X}$ under ϕ is an R-hull for the action of $\overline{H}_j$ on $\overline{X}$ and is the embedded image of $(H_j \cap K)\backslash\widetilde{Z_j}$.*

Proof of Theorem 3.3.2. Let δ be a constant such that the cusped spaces X and $\overline{X}$ are both δ-hyperbolic and have δ-slim geodesic triangles for all sufficiently long fillings, and let λ be a common quasiconvexity constant for the groups H_j, which we may also assume is a quasiconvexity constant for each $\overline{H}_j$ in any sufficiently long and $\mathcal{H}$-wide filling $\phi : \Gamma \to \Gamma/K$. For these fillings, by Theorem 2.5.13 (2),(4) we may assume $\overline{H}_i \neq \overline{H}_j$ for all $i \neq j$ (just take a set A containing elements in $H_i \backslash H_j \cup H_j \backslash H_i$ for any pair $i \neq j$), and also that each $\overline{H}_j$ is naturally isomorphic to the image of the induced filling of H_j. Let R be sufficiently large so that Proposition 3.3.9 applies for δ and λ, and let $R' = R'(R)$ be given by Lemma 3.3.12.

For each $1 \leq j \leq k$ consider the following commutative diagram

$$\begin{array}{ccc} H_j \curvearrowright \widetilde{Z_j} & \longrightarrow & X \curvearrowleft \Gamma \\ \downarrow & & \downarrow \\ H_j/K_j \curvearrowright \overline{\widetilde{Z_j}} & \longrightarrow & \overline{X} \curvearrowleft \Gamma/K \end{array}$$

where $\widetilde{Z_j}$ is an R'-hull for the action of H_j on X, and $\overline{X} = K\backslash X$, $K_j = K \cap H_j$, and $\overline{\widetilde{Z_j}} = K_j\backslash\widetilde{Z_j}$. From Lemma 3.3.12, $\overline{\widetilde{Z_j}}$ embeds in $\overline{X}$ and is an R-hull for the action of H_j/K_j on $\overline{X}$.

Taking quotients under the respective groups we obtain the diagram

$$\begin{array}{ccc} Z_j & \xrightarrow{\mathscr{I}_j} & Y \\ \downarrow & & \downarrow \\ \overline{Z}_j & \xrightarrow{\overline{\mathscr{I}}_j} & \overline{Y} \end{array} \tag{3.4}$$

the vertical maps being homeomorphisms. By our choice of R, Proposition 3.3.9 implies that the relative heights of $\mathcal{H}$ in Γ and of $\overline{\mathcal{H}}$ in $\overline{\Gamma}$ coincide with the relative multiplicities of $\{\mathscr{I}_j : Z_j \to Y\}_j$ and $\{\overline{\mathscr{I}}_j : \overline{Z}_j \to \overline{Y}\}_j$, respectively.

The vertical maps of the diagram (3.4) induce depth-preserving homeomorphisms between

$$S_\sigma = \{(z_1, \dots, z_m) \in Z_{\sigma(1)} \times \dots Z_{\sigma(m)} \colon \mathscr{I}_{\sigma(1)}(z_1) = \dots = \mathscr{I}_{\sigma(m)}(z_m)\} \backslash \Delta_\sigma$$

and

$$\overline{S}_\sigma = \{(z_1, \dots, z_m) \in \overline{Z}_{\sigma(1)} \times \dots \overline{Z}_{\sigma(m)} \colon \overline{\mathscr{I}}_{\sigma(1)}(z_1) = \dots = \overline{\mathscr{I}}_{\sigma(m)}(z_m)\} \backslash \overline{\Delta}_\sigma$$

for any $\sigma = (\sigma(1), \dots, \sigma(m))$ with $|\sigma| = m > 0$, where Δ_σ and $\overline{\Delta}_\sigma$ are the corresponding fat diagonals.

If $\pi_\sigma : S_\sigma \to \overline{S}_\sigma$ is this homeomorphism, since $\overline{H}_j \cong H_j/K_j$ for all j, then for any component C of S_σ with base-point p at depth 0 and for any $1 \le i \le m$, the following diagram commutes

$$\begin{array}{ccc} \pi_1(C, p) & \xrightarrow{\tau_{C,i}} & H_{\sigma(i)} \\ \downarrow{\scriptstyle (\pi_\sigma)_*} & & \downarrow{\scriptstyle \phi_{H_{\sigma(i)}}} \\ \pi_1(\overline{C}, \overline{p}) & \xrightarrow{\tau_{\overline{C},i}} & \overline{H}_{\sigma(i)} \end{array}$$

where $\phi : \Gamma \to \overline{\Gamma}$ is the filling map. In particular, if $\tau_{\overline{C},i}(\pi_1(\overline{C}, \overline{p})) < \overline{\Gamma}$ contains a loxodromic element of $(\overline{\Gamma}, \overline{\mathcal{P}})$, then $\tau_{C,i}(\pi_1(C, p))$ contains a loxodromic element of $(\Gamma, \mathcal{P})$ as well, and therefore the relative height of $\mathcal{H}$ in Γ is at least the relative height of $\overline{\mathcal{H}}$ in $\overline{\Gamma}$. □

3.3.2 Weak separability of double cosets

In this subsection we study the behavior of double cosets of relatively quasiconvex subgroups under Dehn filling. Similarly to the peripheral separability we require for a single relatively quasiconvex subgroup, we need some assumptions on the double cosets.

Definition 3.3.13. Let H, L be relatively quasiconvex subgroups of the relatively hyperbolic group $(\Gamma, \mathcal{P})$. The pair H, L is said to be *doubly peripherally separable* if the double coset $(H^{g_1} \cap P)(L^{g_2} \cap P)$ is separable in P for any $g_1, g_2 \in \Gamma$ and any peripheral subgroup $P < \Gamma$ such that $H^{g_1} \cap P$ and $L^{g_2} \cap P$ are infinite.

Our next theorem establishes weak separability for double cosets of doubly peripherally separable pairs of relatively quasiconvex subgroups under Dehn fillings. This extends [GM21, Prop. 6.2], where the peripheral subgroups are assumed to be abelian, and hence peripheral separability and double peripheral separability trivially hold. It also generalizes [GM18, Thm. 6.4], where it is assumed that the relatively quasiconvex subgroups are full (though they proved a result for several subgroups, and also allowing some finite subsets removed).

Theorem 3.3.14. *Let $(\Gamma, \mathcal{P} = \{P_1, \ldots, P_n\})$ be relatively hyperbolic and let $H, L < \Gamma$ be relatively quasiconvex and peripherally separable subgroups of $(\Gamma, \mathcal{P})$ such that the pair H, L is doubly peripherally separable. Also, let $S \subset (\bigcup \mathcal{P}) \setminus \{o\}$ be a finite set and consider $a \in \Gamma \setminus HL$.*

Then there exist finite index subgroups $K_i \trianglelefteq P_i$ such that for any subgroups $N_i < K_i$ the filling $\Gamma \to \Gamma(N_1, \ldots, N_n) = \Gamma/K$ is $(\{H, L\}, S)$-wide, and moreover $a \notin KHL$.

The lemmas below are [MM10, Prop. A.6] and [GM21, Lem. 4.1 & Lem. 4.4], respectively, where for [GM21, Lem. 4.1] we cite the statement for the case $L_1 = 10\delta$.

Proposition 3.3.15. *Let H be a relatively quasiconvex subgroup of a relatively hyperbolic group $(\Gamma, \mathcal{P})$. For a cusped space X for $(\Gamma, \mathcal{P})$, there exists a positive constant R such that for any horoball A in X, if there is a geodesic in X with endpoints in H and intersecting A in a point at depth R, then $H \cap \operatorname{Stab}(A)$ is infinite.*

Lemma 3.3.16. *Let $(\Gamma, \mathcal{P})$ be with cusped space X, and let δ be a hyperbolicity constant for X and for the induced cusped space of any sufficiently long filling. We also assume that geodesic triangles in all these spaces are δ-slim. Then for any $L_2 \geq 10\delta$, and for all sufficiently long fillings $\phi : \Gamma \to \Gamma/K$ the following holds: if γ is a geodesic in X such that $\phi(\gamma)$ is a loop in $\overline{X}$, then:*

- *there is a horoball A in X so that γ intersects A in a segment $[x, y]$ with $x, y \in \Gamma$ and containing a point at depth L_2, and*
- *there is some $k \in K \cap \operatorname{Stab}(A)$ so that $d_X(x, ky) \leq 20\delta + 3$.*

Proof of Theorem 3.3.14. Almost all the work has been done in [GM21, Prop. 6.2], so we will only check the details where the doubly peripheral separability assumption is required.

By Lemma 2.5.15 there exist finite index subgroups $\dot{N}_i \trianglelefteq P_i$ such that any filling $\Gamma \to \Gamma(N_1, \ldots, N_n)$ with $N_i < \dot{N}_i$ is $(\{H, L\}, S)$-wide. The rest of the proof is devoted to finding finite index subgroups $\dot{K}_i \trianglelefteq P_i$ such that a is separated from HL in any filling $\Gamma \to \Gamma(N_1, \ldots, N_n)$ with $N_i < \dot{K}_i$. In that case, the theorem follows by defining $K_i := \dot{N}_i \cap \dot{K}_i$ for each i.

Consider an induced peripheral structure $\mathcal{D}$ on H, with each $D \in \mathcal{D}$ of the form $D = H \cap P_{j_D}^{c_D}$ for some $1 \leq j_D \leq n$ and some shortest $c_D \in \Gamma$. Similarly, let $\mathcal{E}$ be an induced peripheral structure on L with each $E \in \mathcal{E}$ of the form $E = L \cap P_{k_E}^{d_E}$ for some $1 \leq k_E \leq n$ and some shortest $d_E \in \Gamma$. Also, let X, X_H, X_L be cusped spaces for $(\Gamma, \mathcal{P})$, $(H, \mathcal{D})$ and $(L, \mathcal{E})$ respectively, with induced maps $\breve{\iota}_H : X_H^{(0)} \to X$ and $\breve{\iota}_L : X_L^{(0)} \to X$. Let $\delta \geq 1$ be a hyperbolicity constant for X and for the induced cusped space of any sufficiently long filling of Γ, and let $\lambda \geq 2\delta + 1$ be such that the images $\breve{\imath}_H(X_H^{(0)})$ and $\breve{\imath}_L(X_L^{(0)})$ are λ-quasiconvex in X. We assume that δ and λ are integer numbers and geodesic triangles are δ-slim in X or any cusped space for a sufficiently long filling.

For H and L, consider the constants R_H, R_L given by Proposition 3.3.15, and define $M = d_X(o,a)$ and $L_2 = \max\{20\delta + M + \lambda + 4, R_H, R_L\}$. Given $1 \leq i \leq n$, let $S_i \subset P_i$ be a finite set containing

$$\{p \in P_i \colon d_X(o,p) \leq 48\delta + 8M + 16\lambda + 5\}.$$

Also, consider the collection $\mathcal{C}_i$ of pairs of subgroups of P_i of the form $(B_1, B_2) = (D^{\alpha c_D^{-1}}, E^{\beta d_E^{-1}})$ with $D \in \mathcal{D}$ and $E \in \mathcal{E}$ such that $j_D = k_E = i$, and $\alpha, \beta \in S_i$. Note that there are finitely many such pairs, and that by assumption for each of them the double coset B_1B_2 is separable in P_i. This implies the existence of finite index subgroups $\dot{K}_{B_1,B_2} \trianglelefteq P_i$ such that

$$S_i \cap \dot{K}_{B_1,B_2} B_1 B_2 \subset B_1 B_2. \tag{3.5}$$

Set $\dot{K}_i := \bigcap_{(B_1,B_2)\in\mathcal{C}_i} \dot{K}_{B_1,B_2}$, for which we expect the separability conditions to hold.

Let $N_i < \dot{K}_i$ be filling kernels inducing the filling $\phi : \Gamma \to \Gamma(N_1, \ldots, N_n) = \Gamma/K$, for which we claim that $a \notin KHL$. Suppose by contradiction that there is some $g \in K$ of the form $g = hla^{-1}$ for some $h \in H$ and $l \in L$, and assume $d_X(o,g)$ is minimal among the elements of $K \cap HLa^{-1}$.

Consider a geodesic quadrilateral in X with vertices o, h, hl, g, and let ξ_1, ξ_2, η and ρ be the geodesics from o to h, from h to hl, from hl to g and from o to g, respectively. By assumption $g \neq o$.

The map $X \to \overline{X}$ induced by ϕ sends ρ to a loop, so by Lemma 3.3.16 there is a horoball A in X intersecting ρ in a geodesic segment $[x,y]$ (with x between o and y), such that $[x,y]$ contains points at depth L_2, and there is some $k \in K \cap \mathrm{Stab}(A)$ with $d_X(x,ky) \leq 20\delta + 3$. This implies $d_X(o,kg) < d_X(o,g)$, since $d_X(x,ky) \leq 20\delta + 3$ and $d_X(x,y) \geq 2L_2 > 20\delta + 4$. Our contradiction will be obtained by showing that $kg \in HLa^{-1}$, contrary to our minimality assumption on g.

By [GM08, Lem. 3.10] we can assume that the segment $[x,y]$ is a geodesic through A consisting of a vertical segment down from x, a horizontal segment of length at most 3, and then a vertical segment with endpoint y. We will make a similar assumption in case A intersects ξ_1 or ξ_2. Let x', y' be the points on $[x,y]$ directly below x and y, respectively, at depth $3\delta + M + \lambda$. The δ-slim condition applied to the quadrilateral $\xi_1 \cup \xi_2 \cup \eta \cup \rho$ implies that x' and y' lie in the 2δ-neighborhood of $\xi_1 \cup \xi_2 \cup \eta$, but since η is of length M and has extreme points at depth 0, in fact x' and y' are within 2δ of $\xi_1 \cup \xi_2$.

The geodesics ξ_1 and ξ_2 join points in H and hL, respectively, and so λ-quasiconvexity gives us points $u_0, v_0 \in \breve{\imath}_H(X_H^{(0)}) \cup h\breve{\imath}_L(X_L^{(0)})$ with $d_X(u_0,x') \leq 2\delta + \lambda$ and $d_X(v_0,y') \leq 2\delta + \lambda$. The points u_0 and v_0 lie in A since x' and y' have depth greater than $2\delta + \lambda$. Let $u, v \in A$ be the points at depth 0 directly above u_0 and v_0 respectively. We have $d_X(u,x) \leq d_X(u,u_0) + d_X(u_0,x') + d_X(x',x) \leq 2(d_X(x,x') + d_X(u_0,x')) \leq 10\delta + 2M + 4\lambda$, and similarly $d_X(v,y) \leq 10\delta + 2M + 4\lambda$.

This gives us four cases depending on whether each of u_0, v_0 are contained in $\breve{\iota}_H(X_H^{(0)})$ or $h\breve{\iota}_L(X_L^{(0)})$. The cases where u_0 and v_0 are both contained in $\breve{\iota}_H(X_H^{(0)})$ or $h\breve{\iota}_L(X_L^{(0)})$ are completely analogous, and the proof given in [GM21, Prop. 6.2] still works here since it only requires peripheral separability. As noted at the end of that proof, the case where u_0 is contained in $h\breve{\iota}_L(X_L^{(0)})$ and v_0 is contained in $\breve{\iota}_H(X_H^{(0)})$ essentially becomes the case where u_0, v_0 are contained in $\breve{\iota}_H(X_H^{(0)})$. Therefore, we are only left to deal with the case where u_0 is contained in $\breve{\iota}_H(X_H^{(0)})$ and v_0 is contained in $h\breve{\iota}_L(X_L^{(0)})$, which we now check.

We can write $u_0 = (sc_D P_{j_D}, h_u c_D, n)$ for some $s \in H, h_u \in sD$ and $n \in \mathbb{N}$, where $D \in \mathcal{D}$, and similarly $v_0 = (htd_E P_{k_E}, hl_v d_E, m)$ for $t \in L, l_v \in tE, m \in \mathbb{N}$ and $E \in \mathcal{E}$. This implies $u = h_u c_D$ and $v = hl_v d_E$. We write $c = c_D, d = d_E$, and $j_D = k_E = i$, where A is the horoball based on the coset $scP_i = htdP_i$.

The geodesic ξ_1 intersects A in a segment with extreme point h_1 closer to h. Since we may assume that x' lies within 2δ of ξ_1, there exists a point h_1' in $\xi_1 \cap A$ directly below h_1 and at depth $\delta + M + \lambda$. By λ-quasiconvexity of $\breve{\iota}_H(X_H)$ we can find some $w' \in \breve{\iota}_H(X_H)$ with $d_X(w', h_1') \leq \lambda$, which indeed lies in A, so that the group element $w \in A$ directly above w' is of the form $w = h_w c \in sDc \subset scP_i$ with $h_w \in H$ and $d_X(w, h_1) \leq d_X(w, w') + d_X(w', h_1') + d_X(h_1', h_1) \leq 2\delta + 2M + 4\lambda$. In the same way, if g_2 is the extreme point of the segment $\xi_2 \cap A$ closer to h, then it is directly above the vertex $g_2' \in A$ at depth $\delta + M + \lambda$ and there is some $z = hl_z d \in htEd \subset htdP_i$ with $d_X(g_2, z) \leq 2\delta + 2M + 4\lambda$.

Let $[h, h_1'] \cup [h, g_2'] \cup [h_1', g_2']$ be a geodesic triangle in X with $[h, h_1'] \subset \xi_1$ and $[h, g_2'] \subset \xi_2$. Since h_1' and g_2' are at depth $\delta + M + \lambda \geq 3\delta + 1$ (recall that $\lambda \geq 2\delta + 1$), it follows from [GM08, Lemm. 3.26] that every point of $[h_1', g_2']$ is at depth at least $3\delta + 1$. By the δ-slim assumption, this implies that the vertex in $[h, h_1']$ directly below h_1 at depth $\delta + 1$ lies within δ of a point in $[h, g_2']$, which must lie directly below g_2 and at depth at most $2\delta + 1$. In particular, $d_X(h_1, g_2) \leq 4\delta + 2$ and we have $d_X(w, z) \leq 8\delta + 4M + 8\lambda + 2$.

Let $p = (u^{-1}kv)(z^{-1}w) = (u^{-1}ku)(u^{-1}w)(z^{-1}v)^{w^{-1}z}$. Since $k \in K \cap \mathrm{Stab}(A) = K \cap P_i^{c_D}$ and K is normal, we have $u^{-1}ku \in K \cap P_i = N_i \trianglelefteq P_i$. Also, note that $u^{-1}w \in D^{c^{-1}}, z^{-1}v \in E^{d^{-1}}$ and $w^{-1}z \in P_i$, implying $p \in N_i D^{c^{-1}} E^{(w^{-1}z)d^{-1}}$. In addition, $d_X(u, kv) \leq d_X(u, x) + d_X(x, ky) + d(y, v) \leq 40\delta + 4M + 8\lambda + 3$, so $d_X(o, p) \leq d_X(w, z) + 40\delta + 4M + 8\lambda + 3 \leq 48\delta + 8M + 16\lambda + 5$, and hence $p \in S_i$. The pair $(D^{c^{-1}}, E^{(w^{-1}z)d^{-1}})$ is then in $\mathcal{C}_i$, so in virtue of (3.5) there exist $\hat{d} \in D^{c^{-1}}$ and $\hat{e} \in E^{d^{-1}}$ such that $p = \hat{d}(w^{-1}z)\hat{e}(z^{-1}w)$. In consequence, khl can be written as

$$\begin{aligned} khl &= u\left(u^{-1}kvz^{-1}w\right)\left(w^{-1}z\right)v^{-1}hl \\ &= up\left(w^{-1}z\right)v^{-1}hl \\ &= u\left(\hat{d}\left(w^{-1}z\right)\hat{e}\left(z^{-1}w\right)\right)\left(w^{-1}z\right)v^{-1}hl \\ &= (h_u c)\,\hat{d}\left(c^{-1}h_w^{-1}hl_z d\right)\hat{e}\left(d^{-1}l_v^{-1}h^{-1}\right)hl \\ &= \left(h_u \hat{d}^c h_w^{-1}h\right)\left(l_z \hat{e}^d l_v^{-1} l\right), \end{aligned}$$

and the last expression lies in HL since $\hat{d}^c \in D$ and $\hat{e}^d \in E$. Therefore, $kg = khla^{-1} \in$

HLa^{-1}, implying the desired contradiction and concluding the proof of this case and the theorem. □

The next corollary roughly says that under some mild assumptions, double cosets of doubly peripherally separable pairs of relatively quasiconvex subgroups are "almost" the intersection of double cosets of fully relatively quasiconvex subgroups.

Corollary 3.3.17. *Let $(\Gamma, \mathcal{P})$ be relatively hyperbolic and let $H, L < \Gamma$ be relatively quasiconvex subgroups such that H, L is a doubly peripherally separable pair. Also, suppose that $\dot{H}, \dot{L}$ and $\dot{P}$ are separable in Γ for any finite index subgroups $\dot{H} < H, \dot{L} < L$ or $\dot{P} < P$, with P being a peripheral subgroup of Γ.*
Then for any $a \in \Gamma \setminus HL$ there exist fully relatively quasiconvex subgroups $\hat{H}, \hat{L}$ with $H \cap \hat{H} < H$ and $L \cap \hat{L} < L$ of finite index, such that $a \notin H\hat{H}\hat{L}L$.

Proof. For a, H and L as in the statement, let $K_1, \ldots K_n$ be the filling kernels given by Theorem 3.3.14, and let $K = K(a, H, L) \trianglelefteq \Gamma$ be the kernel of the filling $\Gamma \to \Gamma(K_1, \ldots, K_n)$. Since by assumption each K_i is finite index in P_i and separable in Γ, the subgroups $K_i \cap H$ are separable in H, and there exists a finite index subgroup $\dot{H} \trianglelefteq H$ such that $\dot{H} \cap P_i < K_i$ for all i, implying $\dot{H} \cap P < K$ for any peripheral subgroup P. Similarly, we can find a finite index subgroup $\dot{L} \trianglelefteq L$ so that $\dot{L} \cap P < K$ for any peripheral subgroup P.

We claim the existence of parabolic subgroups $Q_1, \ldots, Q_s, R_1, \ldots, R_t < \Gamma$, each of them a finite index subgroup of a conjugate of some K_i, and such that $\hat{H} := \langle \dot{H}, Q_1, \ldots, Q_s \rangle$ and $\hat{L} := \langle \dot{L}, R_1, \ldots, R_t \rangle$ are fully relatively quasiconvex subgroups of Γ. By assuming this claim, the corollary follows since $a \notin KHL$ and

$$H\hat{H}\hat{L}L = H\langle \dot{H}, Q_1, \ldots, Q_s \rangle \langle \dot{L}, R_1, \ldots, R_t \rangle L \subset H\langle \dot{H}, K \rangle \langle \dot{L}, K \rangle L = H\dot{H}K\dot{L}L = KHL.$$

To prove the claim, we will only construct the groups $Q_1, \ldots, Q_s$, since the groups $R_1, \ldots, R_t$ can be found in exactly the same way. Let us choose $\mathcal{D}_0 = \{D_1, \ldots, D_s\}$ an induced peripheral structure on $\dot{H}_0 := \dot{H}$ with $D_i = \dot{H} \cap P_{j_i}^{c_i}$ for some $1 \leq j_i \leq n$ and $c_i \in \Gamma$, and inductively construct $Q_1, \ldots, Q_s$ with each $Q_i < K_{j_i}^{c_i}$ of finite index, such that $\mathcal{D}_i := \{Q_1, \ldots, Q_i, D_{i+1}, \ldots, D_s\}$ is an induced peripheral structure on $\dot{H}_i := \langle \dot{H}, Q_1, \ldots, Q_i \rangle = \langle \dot{H}_{i-1}, Q_i \rangle$.

Assume we have found $\dot{H}_{i-1}$ and $Q_1, \ldots, Q_{i-1}$, and note that every parabolic subgroup of $\dot{H}_{i-1}$ is a subgroup of K. If $D_i < K_{j_i}^{c_i}$ is finite index, define $Q_i = D_i$. Otherwise, by [Mar09, Thm. 1.1] there exists a finite set $F \subset P_{j_i}^{c_i} \setminus (\dot{H}_{i-1} \cap P_{j_i}^{c_i})$ such that for any $P' < P_{j_i}^{c_i}$ containing $\dot{H}_{i-1} \cap P_{j_i}^{c_i}$ and disjoint from F, the group $\langle \dot{H}_{i-1}, P' \rangle$ is relatively quasiconvex and has $\{Q_1, \ldots, Q_{i-1}, P'\}$ as induced peripheral structure. It is then enough to find a subgroup $P' < K_{j_i}^{c_i}$ of finite index, containing $\dot{H}_{i-1} \cap P_{j_i}^{c_i}$ and disjoint from F, so in that case we can define $Q_i = P'$.

To find such P', note that each infinite parabolic subgroup of $\dot{H}_{i-1}$ is $\dot{H}_{i-1}$-conjugate into some group in $\mathcal{D}_{i-1}$, implying $\dot{H}_{i-1} \cap P_{j_i}^{c_i} = D_i$. Also, since $\dot{H}$ is separable in Γ, we have that D_i is separable in $P_{j_i}^{c_i}$. But, by construction D_i is contained in K, so in fact D_i

is separable in $K_{j_i}^{c_i}$, and there exists a finite index subgroup $P' < K_{j_i}^{c_i}$, disjoint from F and such that $P' \supset \dot{D}_i$. This solves the claim, and since each subgroup in $\mathcal{D}_s$ is finite index in some peripheral subgroup, the group $\hat{H} := \dot{H}_s$ is fully relatively quasiconvex. □

3.3.3 The malnormal special quotient theorem

In this subsection we present a result that will be needed in the proofs of Theorem 3.5.4 and Proposition 3.8.14. It depends on the relative version of Wise's malnormal special quotient theorem, due to Einstein.

Theorem 3.3.18 (Einstein [Ein19, Thm. 2])**.** *If $(\Gamma, \{P_1, \dots, P_n\})$ is a relatively hyperbolic group with Γ cubulated and virtually special, then there exist finite index subgroups $\dot{P}_i \trianglelefteq P_i$ such that if $\overline{\Gamma} = \Gamma(N_1, \dots, N_n)$ is any filling with each $N_i < \dot{P}_i$ of finite index, then $\overline{\Gamma}$ is hyperbolic and virtually special.*

By combining Theorem 2.5.13, Lemma 2.5.15, Theorem 3.3.2 and Einstein's Theorem 3.3.18, we deduce the following.

Proposition 3.3.19. *Let $(\Gamma, \mathcal{P})$ be a relatively hyperbolic group with each $P \in \mathcal{P}$ being residually finite, and let $\mathcal{H} = \{H_0, H_1, H_2, \dots, H_k\}$ be a collection of relatively quasiconvex subgroups of Γ. Assume that the groups $H_1, \dots, H_k$ are all distinct and contained in H_0, that H_l is peripherally separable for $0 \le l \le k$, and H_0 is strongly peripherally separable. Consider also a finite set $A \subset \Gamma$.*

Then there exist finite index subgroups $\dot{P}_i \trianglelefteq P_i$ such that for any further finite index subgroups $N_j < \dot{P}_j$ with $N_j \trianglelefteq P_j$, the filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(\mathcal{N}) = \Gamma(N_1, \dots, N_n)$ satisfies:

1. *$\overline{\Gamma}$ is hyperbolic.*
2. *$\overline{H}_l = \phi(H_l)$ is quasiconvex in $\overline{\Gamma}$ and naturally isomorphic to the induced filling $H_l(\mathcal{N}_{H_l})$ for each $0 \le l \le k$.*
3. *$\phi|_A : A \to \overline{\Gamma}$ is injective and $\phi(A \cap H_l) = \phi(A) \cap \overline{H}_l$ for all $0 \le l \le k$.*
4. *For each l, if H_l is virtually special and strongly peripherally separable, then $\overline{H}_l$ is also virtually special.*
5. *For $1 \le l \le k$, $\overline{H}_l$ is isomorphic to the filling induced by $H_0 \to \overline{H}_0$.*
6. *The height of $\{\overline{H}_1, \dots, \overline{H}_k\}$ in $\overline{H}_0$ is at most the relative height of $\{H_1, \dots, H_k\}$ in H_0.*

Proof. First, consider a peripheral structure $\mathcal{D}_0 = \{D_{0,1}, \dots, D_{0,s_0}\}$ on H_0 induced by Γ so that $D_{0,i} = H_0 \cap P_{\alpha_{0,i}}^{c_{0,i}}$ for some $1 \le \alpha_{0,i} \le n$ and some shortest $c_{0,i} \in \Gamma$. Also, for any $1 \le l \le k$ consider a peripheral structure $\mathcal{D}_l = \{D_{l,1}, \dots, D_{l,s_l}\}$ on H_l induced by $(H_0, \mathcal{D}_0)$, so that $D_{l,i} = H_l \cap D_{0,\beta_{l,i}}^{d_{l,i}}$ for some $1 \le \beta_{l,i} \le s_0$ and some shortest $d_{l,i} \in H_0$. Since $\mathcal{D}_l$ is also a peripheral structure induced by Γ, we have $D_{l,i} = H_l \cap P_{\alpha_{l,i}}^{c_{l,i}}$ for $1 \leq \alpha_{l,i} \leq n$ and $c_{l,i} \in \Gamma$. The equation

$$D_{l,i} = H_l \cap P_{\alpha_{l,i}}^{c_{l,i}} = H_l \cap D_{0,\beta_{l,i}}^{d_{l,i}} = H_l \cap P_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}$$

then implies

$$\alpha_{l,i} = \alpha_{0,\beta_{l,i}}, \quad \text{and} \quad (c_{l,i})^{-1} d_{l,i} c_{0,\beta_{l,i}} \in P_{\alpha_{l,i}}. \tag{3.6}$$

The relevance of this equation is that if $\phi : \Gamma \to \Gamma(N_1, \dots, N_n)$ induces the filling $\phi_0 : H_0 \to H_0(K_{0,1}, \dots, K_{0,s_0})$, then for any $1 \leq l \leq k$ the filling $\phi_l : H_l \to H_l(K_{l,1}, \dots, K_{l,s_l})$ induced by ϕ is the same as the one induced by ϕ_0. Indeed, a filling kernel of ϕ_l induced by ϕ is of the form $K_{l,i} = D_{l,i} \cap N_{\alpha_{l,i}}^{c_{l,i}}$, while the one induced by ϕ_0 takes the form $K_{l,i}^{(0)} = D_{l,i} \cap K_{0,\beta_{l,i}}^{d_{l,i}} = D_{l,i} \cap N_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}$. The identity $K_{l,i} = K_{l,i}^{(0)}$ then follows from $N_{\alpha_{l,i}}^{c_{l,i}} = N_{\alpha_{0,\beta_{l,i}}}^{d_{l,i}c_{0,\beta_{l,i}}}$, which is consequence of (3.6) and the fact that $N_{\alpha_{l,i}}$ is normal in $P_{\alpha_{l,i}}$.

With that in mind, by Theorem 2.5.13 we can find a finite set $S \subset (\bigcup \mathcal{P}) \setminus \{o\}$ such that all $(\mathcal{H}, S)$-wide and peripherally finite fillings $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \dots, N_n)$ with $S \cap (\bigcup N_i) = \emptyset$ satisfy items (1)-(3) (for the injectivity of $\phi|_A$, include the trivial group into $\mathcal{H}$ and consider the finite set $\{ab^{-1} \neq 1 : a, b \in A\} \subset \Gamma$). For such S, Lemma 2.5.15 gives us finite index subgroups $\dot{N}_j \trianglelefteq P_j$ such that any filling $\Gamma \to \Gamma(N_1, \dots, N_n)$ with $N_j < \dot{N}_j$ is $(\mathcal{H}, S)$-wide.

Now, let I be the set of $0 \leq l \leq k$ such that H_l is virtually special, and apply Theorem 3.3.18 to each pair $(H_l, \mathcal{D}_l)$ with $l \in I$ to obtain finite index subgroups $\dot{D}_{l,i} \trianglelefteq D_{l,i}$ such that for any further finite index subgroups $\widetilde{D}_{l,i} < \dot{D}_{l,i}$ the filling $H_l(\widetilde{D}_{l,1}, \dots, \widetilde{D}_{l,s_l})$ is virtually special.

Given $l \in I$ and $1 \leq i \leq s_l$, there is a chain of inclusions

$$\dot{D}_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} \trianglelefteq D_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} < \dot{N}_{\alpha_{l,i}}^{c_{l,i}},$$

the first one being of finite index. Since H_l is strongly peripherally separable, the group $\dot{D}_{l,i}$ is separable in $P_{\alpha_{l,i}}^{c_{l,i}}$, and hence the inclusion $\dot{D}_{l,i} \cap \dot{N}_{\alpha_{l,i}}^{c_{l,i}} < \dot{N}_{\alpha_{l,i}}^{c_{l,i}}$ is also separable. Therefore we can find a finite index subgroup $\widetilde{N}_{\gamma_{l,i}} \trianglelefteq P_{\alpha_{l,i}}$ contained in $\dot{N}_{\alpha_{l,i}}$, such that $\dot{D}_{l,i} \cap \widetilde{N}_{\gamma_{l,i}}^{c_{l,i}} = D_{l,i} \cap \widetilde{N}_{\gamma_{l,i}}^{c_{l,i}}$. We conclude that $K_{l,i} := D_{l,i} \cap \widetilde{N}_{\gamma_{l,i}}^{c_{l,i}}$ is contained in $\dot{D}_{l,i}$ as a finite index subgroup.

For $1 \leq j \leq n$, let I_j be the set of pairs (l,i) with $l \in I$ and $1 \leq i \leq s_l$, such that $j = \alpha_{l,i}$. Let $\widetilde{P}_j := \bigcap_{(l,i) \in I_j} \widetilde{N}_{\gamma_{l,i}}$ if $I_j \neq \emptyset$, and $\widetilde{P}_j = \dot{N}_j$ otherwise. Since each $\dot{P}_j$ is of finite index in P_j and by assumption, the groups P_j are residually finite, we can find finite index subgroups $\dot{P}_j^{(1)} \trianglelefteq P_j$ contained in $\widetilde{P}_j$ and disjoint from S. By construction, any filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \dots, N_n)$ with $N_j < \dot{P}_j^{(1)}$ of finite index satisfies the conclusions (1)-(4).

To deal with (5)-(6), note that the groups $H_1, \dots, H_k$ are peripherally separable in $(H_0, \mathcal{D}_0)$, and that each group in $\mathcal{D}_0$ is residually finite. Therefore, by applying Theorem 2.5.13, Lemma 2.5.15, and Theorem 3.3.2, and in the same way we constructed the groups $\dot{P}_j^{(1)}$ above, we obtain finite index subgroups $\dot{K}_{0,i} \trianglelefteq D_{0,i}$ such that for any filling $\phi_0 : H_0 \to H_0(K_{0,1}, \dots, K_{0,s_0})$ with $K_{0,i} < \dot{K}_{0,i}$, we have:

(*i*) for any $1 \leq l \leq k$ the induced filling ϕ_l of H_l satisfies $\ker \phi_l = \ker \phi_0 \cap H_l$; and,

(*ii*) the height of $\{\phi_0(H_1), \ldots, \phi_0(H_k)\}$ in $\phi_0(H_0)$ is at most the relative height of $\{H_1, \ldots, H_k\}$ in H_0.

By our strong peripheral separability assumption, and by the same separability argument used to find the groups $\widetilde{N}_{\gamma_{l,i}}$, there exist finite index subgroups $\dot{P}_j \trianglelefteq P_j$ contained in $\dot{P}_j^{(1)}$, and such that $D_{0,i} \cap \dot{P}_{\alpha_{0,i}}^{c_{0,i}} < \dot{K}_{0,i}$ for any $1 \leq i \leq s_0$.

By construction, if $\phi : \Gamma \to \Gamma(N_1, \ldots, N_n)$ is a peripherally finite filling with $N_i < \dot{P}_j$, then it satisfies conclusions (1)-(4). In addition, by (*i*) we have $\ker \phi \cap H_j = (\ker \phi_0 \cap H_0) \cap H_j = \ker \phi_j$, so it also satisfies (5). Finally, note that by the discussion after equation (3.6), the embedding $\phi_0(H_0) \hookrightarrow \phi(\Gamma)$ induces isomorphisms $\phi_0(H_j) \xrightarrow{\sim} \phi(H_j)$ for each j, and so the height of $\{\phi_0(H_1), \ldots, \phi_0(H_k)\}$ in $\phi_0(H_0)$ coincides with the height of $\{\phi(H_1), \ldots, \phi(H_k)\}$ in $\phi(H_0)$. This fact together with (*ii*) proves (6). □

3.4 A relative quasiconvex hierarchy theorem

One of the main tools in Agol's proof of Theorem 2.8.12 is Wise's quasiconvex hierarchy Theorem 2.8.6. In this section recall the class $\mathcal{CMVH}$ for relatively hyperbolic groups defined in the Introduction. The main result of this section is Theorem 3.4.5 (Theorem 1.1.7), a relative quasiconvex hierarchy theorem for groups in $\mathcal{CMVH}$.

First, we recall the notion of graphs of groups, referring to the work of Bass [Bas93].

Definition 3.4.1. A *graph of groups* is a pair $(G, \mathcal{G})$ consisting of:

1. a connected, non-empty graph G with vertex set $V = V(G)$, and an oriented edge set $E = E(G)$ with an involution $e \mapsto \overline{e}$ that switches the orientation of each edge;
2. an assignment $\mathcal{G} : V \sqcup E \to \mathbf{Grp}$ of a group $x \mapsto \Gamma_x = \mathcal{G}(x)$ for any vertex or edge x, such that $\Gamma_e = \Gamma_{\overline{e}}$ for any edge e; and,
3. a set of attachment monomorphisms $\psi_e : \Gamma_e \to \Gamma_{t(e)}$ where $t(e)$ is the terminal vertex of the edge e.

Given a graph of groups $(G, \mathcal{G})$ we consider the group

$$F(G, \mathcal{G}) = (*_{v \in V} \Gamma_v) * F(E))/N,$$

where $F(E)$ is the free group generated by the set E and N is the normal subgroup generated by the relations $e^{-1} = \overline{e}$ and $e\psi_e(g)e^{-1} = \psi_{\overline{e}}(g)$ for any edge $e \in E$ and any $g \in \Gamma_e$.

Definition 3.4.2. The *fundamental group* of $(G, \mathcal{G})$ based at $v_0 \in V$ is the subgroup

$$\pi_1(G, \mathcal{G}, v_0) < F(G, \mathcal{G})$$

consisting of the elements of the form

$$g = g_0 e_1 g_1 e_2 \cdots e_n g_n,$$

where $e_1, e_2, \ldots, e_n$ form a *circuit* in G based at v_0 (i.e. $t(e_i) = t(\overline{e}_{i+1})$ for $1 \leq i \leq n-1$ and $t(e_n) = t(\overline{e}_1) = v_0$), and $g_i \in \Gamma_{t(\overline{e}_{i+1})}$ for any $0 \leq i \leq n$, with the convention $t(\overline{e}_{n+1}) = v_0$.

The isomorphism class of $\pi_1(G, \mathcal{G}, v_0)$ is independent of the base-point, and the canonical maps from vertex groups into $F(G, \mathcal{G})$ are injective, so we can consider any vertex group as a subgroup of $\pi_1(G, \mathcal{G}, v_0)$ by means of choosing a maximal tree of G containing v_0.

Definition 3.4.3. We say that a group Γ *splits* as a graph of groups $(G, \mathcal{G})$ if Γ is isomorphic to $\pi_1(G, \mathcal{G}, v_0)$ for some (any) vertex v_0 of G.

As we mentioned, we will work with the class $\mathcal{CMVH}$ (Definition 1.1.6), whose definition we recall.

Definition 3.4.4. $\mathcal{CMVH}$ is the smallest class of cubulated and relatively hyperbolic groups (Γ, X) (here Γ acts properly and cocompactly on the cubulation X) relative to compatible virtually special subgroups, that is closed under the following operations:

1. $(\{o\}, X) \in \mathcal{CMVH}$ for any finite CAT(0) cube complex X.
2. If Γ splits as a finite graph of groups $(G, \mathcal{G})$ satisfying:
   - each edge/vertex group is convex in (Γ, X);
   - if v is a vertex of G then the collection $\mathcal{A}_v := \{\Gamma_e \colon e \text{ an edge attached to } v\}$ is relatively malnormal in Γ_v; and,
   - if Γ_v is a vertex group, then it has a convex core $X_v \subset X$ with $(\Gamma_v, X_v) \in \mathcal{CMVH}$;

   then $(\Gamma, X) \in \mathcal{CMVH}$.
3. If $H < \Gamma$ with $|\Gamma : H| < \infty$ and $(H, X) \in \mathcal{CMVH}$, then $(\Gamma, X) \in \mathcal{CMVH}$.

The main result of the section is that groups in $\mathcal{CMVH}$ are virtually special (see Theorem 1.1.7).

Theorem 3.4.5. *If $(\Gamma, X) \in \mathcal{CMVH}$, then (Γ, X) is virtually special.*

A key ingredient in the proof of this theorem is the proposition below, which guarantees the existence of hyperbolic and virtually special fillings for groups in $\mathcal{CMVH}$.

Proposition 3.4.6. *Let $(\Gamma, \mathcal{P} = \{P_1, \ldots, P_n\})$ be a relatively hyperbolic group with each P_i being residually finite. Suppose Γ splits as a finite graph of groups $(G, \mathcal{G})$ satisfying:*

- *each edge group is relatively quasiconvex in Γ and peripherally separable;*

- *if v is a vertex of G, then the collection $\mathcal{A}_v := \{\Gamma_e : e$ attached to $v\}$ is relatively malnormal in Γ_v; and,*
- *each vertex group is virtually special and strongly peripherally separable.*

Then:

1. *Every relatively quasiconvex and peripherally separable subgroup of Γ is separable.*
2. *Every double coset of a doubly peripherally separable pair of relatively quasiconvex and strongly peripherally separable subgroups of Γ is separable.*

Proposition 3.4.7. *Let $(\Gamma, \mathcal{P})$ and $(G, \mathcal{G})$ be as in the statement of Proposition 3.4.6. Then there exist finite index subgroups $\dot{P}_j \trianglelefteq P_j$ such that for any further finite index subgroups $N_j < \dot{P}_j$ there are subgroups $M_j < N_j$ (normal in P_j) with $\overline{\mathbb{G}} = \Gamma(M_1, \ldots, M_n)$ hyperbolic and virtually special.*

Remark 3.4.8. Since we are restricting to finitely generated groups, relative quasiconvexity of edge groups implies relative quasiconvexity of vertex groups [BW13, Lem. 4.9].

Before proving Proposition 3.4.7, we need a lemma. Item (2) below will be used in the proof of Proposition 3.8.14.

Lemma 3.4.9. *Let $\overline{\mathbb{G}}$ be a group splitting as a finite graph of groups $(G, \overline{\mathcal{G}})$ with hyperbolic vertex groups, and edge groups quasiconvex in their corresponding vertex groups. Suppose that either:*

1. *for each vertex $v \in V = V(G)$ the collection $\overline{\mathcal{P}}_v := \{\overline{\Gamma}_e : e$ is an edge attached to $v\}$ is almost malnormal in $\overline{\Gamma}_v$; or,*
2. *G is bipartite with $V(G) = V_1 \sqcup V_2$ and each edge joining vertices of V_1 and V_2, and such that for each $v \in V_1$ the collection $\overline{\mathcal{P}}_v := \{\overline{\Gamma}_e : e$ is an edge attached to $v\}$ is almost malnormal in $\overline{\Gamma}_v$.*

Then $\overline{\mathbb{G}}$ is hyperbolic and the edge groups are quasiconvex in $\overline{\mathbb{G}}$.

Proof. Since edge groups are quasiconvex in the vertex groups, by almost malnormality the pair $(\overline{\Gamma}_v, \overline{\mathcal{P}}_v)$ is relatively hyperbolic for each $v \in V$ (resp. $v \in V_1$) [Osi06]. Also, for $v \in V_2$ consider the trivial peripheral structure $(\overline{\Gamma}_v, \{\overline{\Gamma}_v\})$. With these conventions, each edge group is maximal parabolic in the vertex groups of V (resp. V_1), and no two of them are conjugate into a common vertex group $\overline{\Gamma}_v$ with $v \in V$ (resp. $v \in V_1$) unless they are finite. Therefore, we are in the assumptions of [BW13, Cor. 4.6] (resp. [BW13, Cor. 1.5]), and hence $\overline{\mathbb{G}}$ is hyperbolic relative to $\bigcup_{v \in V} \overline{\mathcal{P}}_v - \{\text{repeats}\}$ (resp. $\{\overline{G}_v \colon v \in V_2\}$), and the edge groups are quasiconvex in some peripheral subgroup of $\overline{\mathbb{G}}$. In both cases $\overline{\mathbb{G}}$ will be hyperbolic relative to hyperbolic groups, hence hyperbolic as well (see e.g. [Far98, Thm. 3.8]), and edge groups will be quasiconvex since the peripheral subgroups are quasiconvex in $\overline{\mathbb{G}}$. □

Proof of Proposition 3.4.7. Let us fix a vertex $v_0 \in V = V(G)$, an isomorphism $\Gamma \cong \pi_1(G, \mathcal{G}, v_0)$, and a maximal tree T of G containing v_0 that induces embeddings of the vertex/edge groups of $(G, \mathcal{G})$ into Γ. For each vertex of G, apply Proposition 3.3.19 to the collection $\mathcal{A}_v$ and the group Γ_v, to find finite index subgroups $\dot{P}_j(v) \trianglelefteq P_j$ such that for any choice of peripherally finite filling kernels $\mathcal{N} = \{N_1, \ldots, N_n\}$ with $N_i < \dot{P}_i(v)$, the filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \ldots, N_n)$ satisfies:

- $\overline{\Gamma}$ is hyperbolic.
- $\overline{\Gamma}_v := \phi(\Gamma_v)$ is virtually special, quasiconvex (hence hyperbolic) in $\overline{\Gamma}$, and isomorphic to the image of the induced filling $\phi_v : \Gamma_v \to \Gamma_v(\mathcal{N}_v)$.
- The collection $\overline{\mathcal{A}}_v$ of images under ϕ of groups in $\mathcal{A}_v$ is almost malnormal in $\overline{\Gamma}_v$.
- Each $\overline{\Gamma}_e := \phi(\Gamma_e)$ in $\overline{\mathcal{A}}_v$ is naturally isomorphic to the image of the filling $\phi_e : \Gamma_e \to \Gamma_e(\mathcal{N}_e)$ induced by both ϕ and ϕ_v (that is, $\ker \phi_e = \ker \phi_v \cap \Gamma_e = \ker \phi \cap \Gamma_e$).

For $1 \leq j \leq n$ define $\dot{P}_j := \bigcap_{v \in V} \dot{P}_j(v)$, which is a finite index normal subgroup of P_j, and consider finite index subgroups $N_j < \dot{P}_j$ inducing the filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \ldots, N_n)$.

To construct $\overline{\mathbb{G}}$, consider a new graph of groups $(G, \overline{\mathcal{G}})$ with the same underlying graph G, with $\overline{\mathcal{G}}$ assigning the group $\overline{\Gamma}_x$ to each $x \in V \sqcup E$, and with attaching maps being the inclusions $\overline{\psi}_e : \overline{\Gamma}_e \hookrightarrow \overline{\Gamma}_{t(e)}$ induced by ϕ. Define $\overline{\mathbb{G}}$ as $\pi_1(G, \overline{\mathcal{G}}, v_0)$, and choose embeddings of the vertex groups according to the same maximal tree T. This choice of embeddings induces commuting diagrams

$$\begin{array}{ccc} \Gamma_e & \xrightarrow{\phi_e} & \overline{\Gamma}_e \\ \big\downarrow{\scriptstyle \varphi_e} & & \big\downarrow{\scriptstyle \overline{\varphi}_e} \\ \Gamma_{t(e)} & \xrightarrow{\phi_{t(e)}} & \overline{\Gamma}_{t(e)} \end{array}$$

all of them together inducing a homomorphism of graphs of groups (see e.g. Bass [Bas93])

$$\Phi : \Gamma \to \overline{\mathbb{G}}$$

such that $\Phi(x) = \phi_v(x)$ for any vertex v and any $x \in \Gamma_v$.

By our choice of ϕ, the splitting $(G, \overline{\mathcal{G}})$ of $\overline{\mathbb{G}}$ satisfies the assumptions of Lemma 3.4.9 (1), and so $\overline{\mathbb{G}}$ is hyperbolic and the edge groups $\Phi(\Gamma_e) = \overline{\Gamma}_e$ are quasiconvex in $\overline{\mathbb{G}}$. In addition, by assumption, the vertex groups of $(G, \overline{\mathcal{G}})$ are hyperbolic and virtually special, and so Theorem 2.8.6 implies that $\overline{\mathbb{G}}$ is virtually special.

Finally, we need to show that $\ker \Phi = \langle\langle \bigcup_j M_j \rangle\rangle_\Gamma$ for some $M_j \trianglelefteq P_j$ contained in N_j. To do this, first, note that $\ker \Phi = \langle\langle \bigcup_{v \in V} \ker \phi_v \rangle\rangle_\Gamma$, and that for each v we have the identity $\ker \phi_v = \langle\langle \bigcup_{D \in \mathcal{D}_j} K_D \rangle\rangle_{\Gamma_v}$, with $\mathcal{D}_v$ being a peripheral structure on Γ_v induced by $(\Gamma, \mathcal{P})$ and $\{K_D\}_{D \in \mathcal{D}_v}$ being the filling kernels induced by ϕ.

Suppose each $D \in \mathcal{D}_v$ is of the form $D = \Gamma_v \cap P_{i_D}^{c_D}$ with $1 \leq i_D \leq n$ and $c_D \in \Gamma$, so that $K_D = H \cap D \cap N_{i_D}^{c_D}$. For $1 \leq j \leq n$, let $\mathcal{D}_j$ denote the set of all $D \in \bigcup_v \mathcal{D}_v$ such that $i_D = j$, and define $M_j := \langle\langle \bigcup_{D\in\mathcal{D}_j} K_D^{c_D^{-1}} \rangle\rangle_{P_j}$ if $\mathcal{D}_j$ is non-empty, and $M_j := \{o\}$ otherwise. Note that $M_j < N_j$ for each j. We claim that $\langle\langle \bigcup_j M_j \rangle\rangle_\Gamma = \langle\langle \bigcup_{v\in V} \ker\phi_v \rangle\rangle_\Gamma$ for these choices of M_j.

For the inclusion "$\subset$", it is enough to show $M_j \subset \langle\langle \bigcup_{v\in V} \ker\phi_v \rangle\rangle_\Gamma$ for any j, which holds because when $\mathcal{D}_j$ is non-empty, we have

$$M_j = \langle\langle \bigcup_{D\in\mathcal{D}_j} K_D^{c_D^{-1}} \rangle\rangle_{P_j} \subset \langle\langle \bigcup_{D\in\mathcal{D}_j} K_D^{c_D^{-1}} \rangle\rangle_\Gamma = \langle\langle \bigcup_{D\in\mathcal{D}_j} K_D \rangle\rangle_\Gamma \subset \langle\langle \bigcup_{v\in V} \bigcup_{D\in\mathcal{D}_v} K_D \rangle\rangle_\Gamma \subset \langle\langle \bigcup_{v\in V} \ker\phi_v \rangle\rangle_\Gamma.$$

On the other hand, for any $v \in V$ we obtain

$$\ker\phi_v = \langle\langle \bigcup_{D\in\mathcal{D}_v} K_D \rangle\rangle_{\Gamma_v} \subset \langle\langle \bigcup_{D\in\mathcal{D}_v} K_D \rangle\rangle_\Gamma \subset \langle\langle \bigcup_{D\in\mathcal{D}_v} K_D^{c_D^{-1}} \rangle\rangle_\Gamma \subset \langle\langle \bigcup_j \bigcup_{D\in\mathcal{D}_j} K_D^{c_D^{-1}} \rangle\rangle_\Gamma = \langle\langle \bigcup_j M_j \rangle\rangle_\Gamma,$$

which proves "$\supset$". □

Proof of Proposition 3.4.6. To prove conclusion (1), let H be a relatively quasiconvex and peripherally separable subgroup of Γ and consider $a \in \Gamma\backslash H$, which we want to separate from H in a finite quotient of Γ. Instead of using Dehn filling directly on H, we will use [SW15, Cor. 6.3] and the peripheral separability of H to find a fully relatively quasiconvex subgroup $\hat{H} < \Gamma$ containing H and such that $a \notin \hat{H}$, so now it is enough to separate a from $\hat{H}$. By Theorem 2.5.13 (1)-(3) there exists a finite set $S \subset (\bigcup\mathcal{P})\backslash\{o\}$ such that for any $(\hat{H}, S)$-wide filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \dots, N_n)$ with $S \cap (\bigcup_j N_j) = \emptyset$, the pair $(\overline{\Gamma}, \overline{\mathcal{P}})$ is relatively hyperbolic with $\overline{\hat{H}} = \phi(\hat{H})$ fully relatively quasiconvex in $(\overline{\Gamma}, \overline{\mathcal{P}})$ and $\phi(a) \notin \overline{\hat{H}}$. Since the groups P_j are residually finite, there exist finite index subgroups $Q_j \trianglelefteq P_j$ with each Q_j being disjoint from S. Also, since $\hat{H}$ is fully relatively quasiconvex, it is peripherally separable and by Lemma 2.5.15 there exist finite index subgroups $\dot{N}_j \trianglelefteq P_j$ such that any filling $\Gamma \to \Gamma(N_1, \dots, N_n)$ with $N_j < \dot{N}_j$ for each j is $(\hat{H}, S)$-wide.

Let $\dot{P}_j \trianglelefteq P_j$ be given by Proposition 3.4.7, and set $N_j := \dot{N}_j \cap \dot{P}_j \cap Q_j \trianglelefteq P_j$. By construction, $\phi : \Gamma \to \overline{\Gamma} = \Gamma(N_1, \dots, N_n)$ is an $(\hat{H}, S)$-wide filling factoring through the $(\hat{H}, S)$-wide filling $\Phi : (\Gamma, \mathcal{P}) \to (\overline{\mathbb{G}}, \overline{\mathbb{P}})$, with $(\overline{\mathbb{G}}, \overline{\mathbb{P}})$ relatively hyperbolic, $\Phi(\hat{H})$ fully relatively quasiconvex in $(\overline{\mathbb{G}}, \overline{\mathbb{P}})$ and $\Phi(a) \notin \Phi(\hat{H})$ (here $\overline{\mathbb{P}}$ denotes the set of images of groups in $\mathcal{P}$ under Φ). Also, $\overline{\mathbb{G}}$ is hyperbolic and virtually special, and therefore, to solve the problem it is enough to show that $\Phi(\hat{H})$ is separable in $\overline{\mathbb{G}}$.

By Theorem 2.8.3 it suffices to show that $\Phi(\hat{H})$ is quasiconvex in $\overline{\mathbb{G}}$, i.e. that $\Phi(\hat{H})$ is relatively quasiconvex in $(\overline{\mathbb{G}}, \emptyset)$. But $(\overline{\mathbb{G}}, \overline{\mathbb{P}})$ is an extended peripheral structure for $(\overline{\mathbb{G}}, \emptyset)$, and so by [Yan14, Thm. 1.3] it is enough to show that $\Phi(\hat{H}) \cap \Phi(P)$ is quasiconvex in $\overline{\mathbb{G}}$ for any peripheral subgroup $P \subset \Gamma$. This last statement follows from the quasiconvexity of $\Phi(P)$ in $\overline{\mathbb{G}}$ [DS05b, Lem. 4.15], and the full relative quasiconvexity of $\Phi(\hat{H})$ in $\overline{\mathbb{G}}$, since quasiconvexity is stable under finite index inclusions.

Now, let $H, L < \Gamma$ be relatively quasiconvex and strongly peripherally separable subgroups such that H, L is doubly peripherally separable, and consider $a \in \Gamma\backslash HL$. Conclusion (1) implies that the peripheral subgroups of Γ are separable, as well as any of their finite index subgroups. In addition, by our strong peripheral separability assumption, any finite index subgroup of H or L will be peripherally separable, hence also separable. Therefore, H and L satisfy the assumptions of Corollary 3.3.17, a so there are fully relatively quasiconvex subgroups $\hat{H}$ and $\hat{L}$ with $H \cap \hat{H} < H$ and $L \cap \hat{L} < L$ of finite index, and such that $a \notin H\hat{H}\hat{L}L$.

Thus, to prove conclusion (2) we only need to verify that $H\hat{H}\hat{L}L$ is separable, and since this set is a finite union of translates of $\hat{H}\hat{L}$, it is enough to show that double cosets of fully relatively quasiconvex subgroups are separable. This can be done in the same way as in the proof of conclusion (1), by using Proposition 3.4.7 together with Theorem 3.3.14 and the separability of double cosets of quasiconvex subgroups of hyperbolic virtually special groups [Min06, Thm. 1.1]. Details are left to the reader. □

Remark 3.4.10. The last part in the proof of conclusion (2) can also be deduced directly from [McC19, Cor. 4.4].

Proof of Theorem 3.4.5. Let (Γ, X) be a cubulated group in $\mathcal{CMVH}$. We will prove that (Γ, X) is virtually special by induction on the minimal number of operations $(1) - (3)$ used in a description of (Γ, X) (see Definition 3.4.4), where clearly $(\{o\}, X)$ is virtually special if X is a finite CAT(0) cube complex.

First, suppose that $H < \Gamma$ is of finite index with $(H, X) \in \mathcal{CMVH}$. Our inductive assumption implies that (H, X) is virtually special, so clearly (Γ, X) is also virtually special.

Now, let Γ be splitting as a finite graph of groups $(G, \mathcal{G})$ such that:

- if $u \in V(G) \sqcup E(G)$ then Γ_u has convex core $X_u \subset X$;
- if $v \in V(G)$ then $(\Gamma_v, X_v) \in \mathcal{CMVH}$ (and hence (Γ_v, X_v) is virtually special by our inductive assumption and Lemma 3.1.4); and,
- $\mathcal{A}_v$ is relatively malnormal in Γ_v for any $v \in V(G)$.

To show that (Γ, X) is virtually special we will use Theorem 2.8.4, but first, we need to show that Γ satisfies the hypotheses of Proposition 3.4.6. This follows because by our compatibility assumption, the peripheral subgroups are residually finite and because by Lemma 3.1.5, all convex subgroups of (Γ, X) are strongly peripherally separable.

Therefore, (Γ, X) satisfies Proposition 3.4.6, so any relatively quasiconvex and peripherally separable subgroup of Γ is separable, as well as any double coset of a doubly peripherally separable pair of relatively quasiconvex and strongly peripherally separable subgroups. In particular, the wall stabilizers of (Γ, X) are separable, so by Theorem 2.8.4 it is enough to show that the pair $\Gamma_{W_1}, \Gamma_{W_2}$ is doubly peripherally separable for any pair of walls $W_1, W_2 \subset X$. To prove this, let $g_1, g_2 \in \Gamma$ and let $P < \Gamma$ be a peripheral subgroup so that $\Gamma_{g_1{}^{-1}W_1} \cap P = (\Gamma_{W_1})^{g_1} \cap P$ and $\Gamma_{g_2{}^{-1}W_2} \cap P = (\Gamma_{W_2})^{g_2} \cap P$ are both infinite. By Theorem 2.7.12 choose a convex core $Z \subset X$ for P such that $g_1{}^{-1}W_1 \cap Z$ and $g_2{}^{-1}W_2 \cap Z$ are both non-empty, so that (P, Z) is virtually special by Proposition 3.1.3. The subgroups $\Gamma_{g_1{}^{-1}W_1} \cap P$ and $\Gamma_{g_2{}^{-1}W_2} \cap P$ are then convex in (P, Z) by Remark 2.7.13, and the conclusion follows from Theorem 3.2.1. □

3.5 Construction of the complex with finite walls

The goal of this and the next sections is to prove the following theorem (Theorem 1.1.8), which combined with Theorem 3.4.5 allows us to deduce Theorem 3.1.2.

Theorem 3.5.1. *If (Γ, X) is cubulated and hyperbolic relative to compatible virtually special subgroups, then $(\Gamma, \dot{X}) \in \mathcal{CMVH}$, where $\dot{X}$ is the cubical barycentric subdivision of X.*

The proof proceeds by induction on the dimension of X (the 0-dimensional case is evident). Henceforth, throughout the rest of this chapter, we will work under the following assumptions.

Assumption 3.5.2. (1) (Γ, X) is a cubulated relatively hyperbolic group with compatible virtually special peripheral subgroups.

(2) $(H, \dot{Y}) \in \mathcal{CMVH}$ for every cubulated and relatively hyperbolic group (H, Y) with compatible virtually special peripheral subgroups and such that $\dim Y < \dim \dot{X} = \dim X$.

(3) δ is the constant for $(\Gamma, \mathcal{P})$ and $\dot{X}$ given by Proposition 2.7.9.

(4) R is a number satisfying $R \geq \delta + 2\sqrt{\dim X}$.

Remark 3.5.3. From Corollary 3.2.2, $(\Gamma, \dot{X})$ is also hyperbolic relative to compatible virtually special subgroups.

Let $W_1, \ldots, W_m$ be a complete set of representatives of Γ-orbits of walls in $\dot{X}$. The following is the main result of the section.

Theorem 3.5.4. *There exists a torsion-free normal subgroup $K \trianglelefteq \Gamma$ such that the quotient cube complex $\mathcal{X} := K \backslash \dot{X}$ satisfies:*

1. *Γ acts cocompactly on $\mathcal{X}$.*
2. *All walls of $\mathcal{X}$ are finite.*
3. *If W is a wall of $\dot{X}$ then the R-neighborhood $N_R(W)$ quotiented by $K \cap \Gamma_W$ embeds in $\mathcal{X}$. In particular, all walls of $\mathcal{X}$ are embedded, and distinct walls in $\dot{X}$ which are less than R apart map to distinct walls in $\mathcal{X}$.*

We will need the following weak separability result for virtually special quasiconvex subgroups of hyperbolic groups, which is the main result in the Appendix of [Ago13].

Theorem 3.5.5 (Agol–Groves–Manning [Ago13, Thm. A.1])**.** *Let Γ be a hyperbolic group and $H < \Gamma$ be a quasiconvex virtually special subgroup. Then for any $g \in \Gamma \backslash H$ there is a hyperbolic group $\mathcal{G}$ and a surjective homomorphism $\psi : \Gamma \to \mathcal{G}$ such that $\psi(g) \notin \psi(H)$ and $\psi(H)$ is finite.*

Proof of Theorem 3.5.4. For every $1 \leq i \leq m$, consider a complete set $\mathcal{A}_i$ of representatives of double cosets $\Gamma_{W_i} g \Gamma_{W_i}$ with $g \notin \Gamma_{W_i}$ and $d_{\dot{X}}(W_i, gW_i) \leq 2R$. Also, choose a complete set $\mathcal{T} \subset \Gamma$ of representatives of conjugacy classes of non-trivial torsion elements of Γ. These sets are finite by Lemma 2.7.8. Set $\mathcal{H} = \{\Gamma_{W_1}, \dots, \Gamma_{W_m}\}$.

As a first step, since each subgroup Γ_{W_i} is convex in $(\Gamma, \dot{X})$, Lemma 3.1.5 and Proposition 3.3.19 imply that we can find a peripherally finite Dehn filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(\dot{P}_1, \dots, \dot{P}_n)$ so that $\overline{\Gamma}$ is hyperbolic, each $\overline{\Gamma}_i := \phi(\Gamma_{W_i})$ is quasiconvex, virtually special and disjoint from $\phi(\mathcal{A}_i)$, and $o \notin \phi(\mathcal{T})$.

Our second step is to find a quasiconvex and virtually special subgroup $\overline{H} < \overline{\Gamma}$ such that $\overline{H} \cap \overline{\Gamma}_i < \overline{\Gamma}_i$ is finite index for $1 \leq i \leq m$. We will do this by inducting on k, and finding for each $1 \leq j \leq m$ a quasiconvex and virtually special subgroup $\overline{H}_j < \overline{\Gamma}$ such that $\overline{H}_j \cap \overline{\Gamma}_i < \overline{\Gamma}_i$ is finite index for $1 \leq i \leq j$. For the base case we choose $\overline{H}_1 = \overline{\Gamma}_1$.

Suppose we have found $\overline{H}_j$, and consider the intersection $L = \overline{H}_j \cap \overline{\Gamma}_{j+1}$. By Gitik's ping-pong theorem [Git99, Thm. 1], there exists a finite set $F \subset (\overline{H}_j \cup \overline{\Gamma}_{j+1}) \backslash L$ such that if $\widetilde{H}_j < \overline{H}_j$ and $\widetilde{\Gamma}_{j+1} < \Gamma_{j+1}$ are finite index subgroups with $\widetilde{H}_j \cap \widetilde{\Gamma}_{j+1} = L$, and $\widetilde{H}_j$, $\widetilde{\Gamma}_{j+1}$ disjoint from F, then $\langle \widetilde{H}_j \cup \widetilde{\Gamma}_{j+1} \rangle$ is quasiconvex in $\overline{\Gamma}$ and isomorphic to $\widetilde{H}_j *_L \widetilde{\Gamma}_{j+1}$. The existence of finite index subgroups $\widetilde{H}_j < \overline{H}_j$ and $\widetilde{\Gamma}_{j+1} < \overline{\Gamma}_{j+1}$ disjoint from F is guaranteed by subgroup separability since by assumption $\overline{H}_j$, $\overline{\Gamma}_{j+1}$ are virtually special and L is quasiconvex in both groups, so we can apply Theorem 2.8.3.

Define $\overline{H}_{j+1} := \langle \widetilde{H}_j \cup \widetilde{\Gamma}_{j+1} \rangle$. By construction $\overline{H}_{j+1} \cap \overline{\Gamma}_{W_i}$ is finite index in $\overline{\Gamma}_{W_i}$ for $1 \leq i \leq j+1$. The virtual specialness of $\overline{H}_{j+1}$ follows from Theorem 2.8.6 and the isomorphism $\overline{H}_{j+1} \cong \widetilde{H}_j *_L \widetilde{\Gamma}_{j+1}$, where $\widetilde{H}_j$ and $\widetilde{\Gamma}_{j+1}$ are virtually special (finite index subgroups of virtually special groups), and L is quasiconvex in $\overline{H}_{j+1}$ since both $\widetilde{H}_j$ and $\widetilde{\Gamma}_{j+1}$ are quasiconvex subgroups of $\overline{\Gamma}$. The induction is then complete, so we define $\overline{H} := \overline{H}_m$.

Before ending the proof, and after possibly replacing $\overline{H}$ by a finite index subgroup, we can assume that $\overline{H}$ does not intersect $\phi(\mathcal{T})$. This is because $\overline{H}$ is virtually special, hence residually finite, and $\phi(\mathcal{T})$ is finite. This modification does not affect the expected properties for $\overline{H}$.

For $1 \leq i \leq m$, let $\overline{N}_i = \overline{H} \cap \overline{\Gamma}_i$, and define $N_i := \phi^{-1}(\overline{N}_i) \cap \Gamma_{W_i} < \Gamma_{W_i}$. Note that N_i is finite index in Γ_{W_i} for each i, so N_i acts cocompactly on W_i, and by Lemma 2.7.8 (2) there is a finite set $\widetilde{\mathcal{A}}_i$ of representatives of double cosets $N_i g N_i$ with $g \notin N_i$ and $d_{\dot{X}}(gW_i, W_i) \leq 2R$. We claim that $\phi(\widetilde{\mathcal{A}}_i) \cap \overline{N}_i = \emptyset$ for each i. Indeed, let $a \in \widetilde{\mathcal{A}}_i$, and suppose first that $a \notin \Gamma_{W_i}$. In that case, $a = g_1 b g_2$ for some $b \in \mathcal{A}_i$ and $g_1, g_2 \in \Gamma_{W_i}$, implying $\phi(a) = \phi(g_1)\phi(b)\phi(g_2) \notin \overline{\Gamma}_i \supset \overline{N}_i$ since by construction $\phi(b) \notin \overline{\Gamma}_i$. In the case $a \in \Gamma_{W_i}$, the conclusion follows from the definition of N_i. Proven our claim, the groups $\overline{N}_i$ are quasiconvex in $\overline{H}$, hence separable by Theorem 2.8.3, and we can find finite index subgroups $\hat{H}_i < \overline{H}$ such that $\hat{H}_i \cap \overline{\Gamma}_i = \overline{N}_i$ and $\hat{H}_i \cap \phi(\widetilde{\mathcal{A}}_i) = \emptyset$ for all i. Note that the groups $\hat{H}_i$ are quasiconvex in $\overline{\Gamma}$ and virtually special.

Now we construct K. For each $1 \leq i \leq m$ and each $g \in \phi(\widetilde{\mathcal{A}}_i)$ we apply Theorem 3.5.5 to find a quotient homomorphism φ_g of $\overline{\Gamma}$ with $\varphi_g(\hat{H}_i)$ finite and $\varphi_g(g) \notin \varphi_g(\hat{H}_i)$. Similarly, for each $g \in \phi(\mathcal{T})$ we construct a quotient homomorphism τ_g of $\overline{\Gamma}$ with $\tau_g(g) \notin \tau_g(\overline{H})$ and

$\tau_g(\overline{H})$ finite.

Define $K := \phi^{-1}\left(\left(\bigcap_{g\in\bigcup_i \widetilde{\mathcal{A}}_i} \ker\varphi_g\right) \cap \left(\bigcap_{g\in\mathcal{T}} \ker\tau_g\right)\right) \trianglelefteq \Gamma$ and $\mathcal{X} := K\backslash\dot{X}$. Note that K is disjoint from $\mathcal{T}$, hence torsion-free, so $\mathcal{X}$ is a cube complex. Clearly Γ acts cocompactly on $\mathcal{X}$, so (1) holds, and by construction $K\cap\Gamma_W$ is finite index in Γ_W for each wall W, implying (2). Finally, if $x,y\in N_R(W_i)$ and $k\in K$ are such that $x=ky$, then $d_{\dot{X}}(kW_i,W_i)\le 2R$. So if $k\notin K_i$, there is some $g\in\widetilde{\mathcal{A}}_i$ with $k=h_1gh_2$ for some $h_1,h_2\in N_i$. But this is impossible since it would imply

$$\varphi_{\phi(g)}(\phi(g)) = (\varphi_{\phi(g)}\circ\phi)(h_1^{-1}kh_2^{-1}) = (\varphi_{\phi(g)}\circ\phi)(h_1^{-1}h_2^{-1}) \in \varphi_{\phi(g)}(\overline{N}_i) \subset \varphi_{\phi(g)}(\hat{H}_i).$$

Therefore, $k\in N_i\subset\Gamma_{W_i}$ and the map $(K\cap\Gamma_{W_i})\backslash N_R(W_i)\to\mathcal{X}$ is an embedding for all $1\le i\le m$. Property (3) then follows from the normality of K and by considering translates of the W_i. □

The point of working with $\dot{X}$ instead of X, is that for every wall W of $\dot{X}$, the subgroup Γ_W does not exchange the sides of W. This allows us to find a Γ-equivariant *co-orientation* on the walls of $\dot{X}$, that is, a labelling W^+ and W^- for the half-spaces of each wall W of $\dot{X}$, such that $(gW)^\pm = g(W^\pm)$ for any $g\in\Gamma$ and for any wall $W\subset\dot{X}$.

Let $q:\dot{X}\to\mathcal{X}$ denote the quotient map from Theorem 3.5.4. This map will send a vertex x (resp. an edge e and wall W) of $\dot{X}$ to a vertex $\overline{x}$ (resp. an edge $\overline{e}$ and wall $\overline{W}$ of $\mathcal{X}$).

3.6 Coloring walls in $\mathcal{X}$

We keep working with the notation of the previous section. Now we proceed to coloring the walls of $\mathcal{X}$, in the same way as in [Ago13, Sec. 5].

Definition 3.6.1. Let $G(\mathcal{X})$ be the simplicial graph with vertices the walls of $\mathcal{X}$ and with an edge joining the walls $\overline{W}_1$ and $\overline{W}_2$ if and only $d_{\mathcal{X}}(\overline{W}_1,\overline{W}_2)\le R$, with $d_{\mathcal{X}}$ being the induced (locally CAT(0)) distance on $\mathcal{X}$ and R as in Theorem 3.5.4.

There is a natural action of Γ on $G(\mathcal{X})$, and since $\mathcal{X}$ is locally finite and with finite walls, there are only finitely many Γ-orbits of vertices in $G(\mathcal{X})$. This implies that there exists some k such that the degree of any vertex of $G(\mathcal{X})$ is bounded above by k, and also that Γ acts cocompactly on $G(\mathcal{X})$.

Definition 3.6.2. A *coloring* of $G(\mathcal{X})$ is a map $c: V(G(\mathcal{X}))\to\{1,\dots,k+1\}$ such that if $\overline{W}_1,\overline{W}_2\in V(G(\mathcal{X}))$ are adjacent walls, then $c(\overline{W}_1)\neq c(\overline{W}_2)$. Let $C_{k+1}(G(\mathcal{X}))$ denote the set of colorings, which is non-empty since vertices of $G(\mathcal{X})$ have degree $\le k$.

The action of Γ on $G(\mathcal{X})$ induces an action on $C_{k+1}(G(\mathcal{X}))$ via pullback $g: c\mapsto gc := c\circ g^{-1}$ for each $g\in\Gamma$. We use this action to define several equivalence classes related to $C_{k+1}(G(\mathcal{X}))$, following the notation of [She21, Def. 6.2].

1. If W is a wall of $\dot{X}$, define the equivalence class $[c]_W$ of $c \in C_{k+1}(G(\mathcal{X}))$ as

$$[c]_W := \{c' \in C_{k+1}(G(\mathcal{X}))\colon c = c' \text{ on the ball of radius } c(\overline{W}) \text{ in } G(\mathcal{X}) \text{ centered at } \overline{W}\}.$$

2. If e is an edge of $\dot{X}$ dual to the wall W, then $[c]_e := [c]_W$ for any $c \in C_{k+1}(G(\mathcal{X}))$.

3. If x is a vertex of $\dot{X}$ we define $[c]_x := \bigcap \{[c]_e \colon e \text{ incident to } x\}$.

4. We also define equivalence classes on $V(\dot{X}) \times C_{k+1}(G(\mathcal{X}))$ and $E(\dot{X}) \times C_{k+1}(G(\mathcal{X}))$ according to $[e, c] := \{e\} \times [c]_e$ and $[x, c] := \{x\} \times [c]_x$.

There are natural actions of Γ on these sets of equivalence classes, given by $g[e, c] := [ge, gc]$ and $g[x, c] := [gx, gc]$. For each edge e, the class $[-]_e$ depends only on the colors of vertices in some $(k+1)$-ball of $G(\mathcal{X})$, so there are only finitely many equivalence classes $[-]_e$, and similarly for $[-]_x$. Since there are finitely many Γ-orbits of edges and vertices in $\dot{X}$, there are only finitely many Γ-orbits of equivalence classes on $E(\dot{X}) \times C_{k+1}(G(\mathcal{X}))$ and $V(\dot{X}) \times C_{k+1}(G(\mathcal{X}))$.

3.7 Cubical polyhedra and the gluing construction

We continue with the notation from the previous two sections. In this section we introduce the main construction used to prove Theorem 3.5.1. As in [She21, Sec. 7 & Sec. 8], we will mainly work with the universal covers instead of the "cubical polyhedra" used in [Ago13], so our notation will be similar to that from [She21]. Inductively, we will construct non-empty sets $\mathcal{V}_{k+1}, \ldots, \mathcal{V}_0$, where each $\mathcal{V}_j$ is a finite collection of triplets $(Z, H, (c_x))$ satisfying:

- $Z \subset \dot{X}$ is a non-empty intersection of half-spaces (thus closed and convex);
- for each vertex $x \in Z$ we have a coloring $c_x \in C_{k+1}(G(\mathcal{X}))$; and,
- $H < \Gamma$ acts freely and cocompactly on Z, and $c_{hx} = hc_x$ for each $h \in H$ and vertex $x \in Z$.

The subset $Z \subset \dot{X}$ is not a subcomplex of $\dot{X}$, but H also acts cocompactly on its cubical neighborhood $\mathcal{N}(Z)$. Since Z is the intersection of half-spaces, the vertices inside Z span a convex subcomplex. The cubical neighborhood of this subcomplex is, therefore, convex, and also equals $\mathcal{N}(Z)$, so we have proven:

Lemma 3.7.1. *$\mathcal{N}(Z)$ is a convex subcomplex of $(\Gamma, \dot{X})$, hence a convex core for H.*

We permit $\mathcal{V}_j$ to contain duplicates of some triplets, and sometimes we will write $Z \in \mathcal{V}_j$ instead of $(Z, H, (c_x)) \in \mathcal{V}_j$, and also $(Z, H, (c_x); \alpha) \in \mathcal{V}_j$ to make explicit that there are exactly $\alpha \in \mathbb{N}$ duplicates of $(Z, H, (c_x))$ in $\mathcal{V}_j$.

Definition 3.7.2. We require each triplet $(Z, H, (c_x)) \in \mathcal{V}_j$ to satisfy four conditions:

1. If $e \in E(\dot{X})$ joins vertices $x, y \in Z \in \mathcal{V}_j$, then $[e, c_x] = [e, c_y]$.

2. If $e \in E(\dot{X})$ joins the vertices $x \in Z \in \mathcal{V}_j$ and $y \in \dot{X}$, then $y \in Z$ if and only if $c_x(\overline{W(e)}) > j$.

3. If $(Z, H, (c_x)) \in \mathcal{V}_j$, then $(H, \mathcal{N}(Z)) \in \mathcal{CMVH}$, when H is endowed with the peripheral structure induced from $(\Gamma, \mathcal{P})$ (see Lemma 3.1.4).

4. For $e \in E(\dot{X})$ with endpoints $x_+ \in W(e)^+$ and $x_- \in W(e)^-$, and $c \in C_{k+1}(G(\mathcal{X}))$, define

$$\mathcal{V}_j^{\pm}(e, c) := \{(H \cdot x, Z) \colon x \in Z \in \mathcal{V}_j, \text{ and } \exists g \in \Gamma \text{ s.t. } gx = x_{\pm}, [e, gc_x] = [e, c]\},$$

where duplicates of $Z \in \mathcal{V}_j$ are counted separately. The collection $\mathcal{V}_j$ must satisfy the *Gluing Equations*

$$|\mathcal{V}_j^+(e, c)| = |\mathcal{V}_j^-(e, c)|$$

for any $e \in E(\dot{X})$ and $c \in C_{k+1}(G(\mathcal{X}))$.

Remark 3.7.3. By Property (1), for an edge e intersecting Z we can consider a coloring $c_e \in C_{k+1}(G(\mathcal{X}))$ (in fact an equivalence class), such that if e is incident to $x \in Z$ then $[e, c_e] = [e, c_x]$.

Let us see how the existence of $\mathcal{V}_0$ implies Theorem 3.5.1. Consider an arbitrary triplet $(Z, H, (c_x))$ in $\mathcal{V}_0$. By conditions (1)-(2) of $\mathcal{V}_0$, any vertex of $\dot{X}$ is contained in Z, and since Z is intersection of half-spaces, we must have $\mathcal{N}(Z) = Z = \dot{X}$. But then $H < \Gamma$ acts cocompactly on $\dot{X}$, implying that H is of finite index in Γ. Condition (3) implies that $(H, \dot{X}) \in \mathcal{CMVH}$, so $(\Gamma, \dot{X})$ also belongs to $\mathcal{CMVH}$.

The rest of the chapter concerns the inductive construction of the sequence $\mathcal{V}_{k+1}, \mathcal{V}_k, \ldots, \mathcal{V}_0$. In the hyperbolic case, the existence of $\mathcal{V}_{k+1}$ was given by Agol by means of an ingenious argument regarding invariant measures on $G(\mathcal{X})$ [Ago13, Sec. 7] (see also [She21, Lem. 7.1]). Definition 3.7.2 (3) differs from that in [She21] since it uses $\mathcal{CMVH}$ rather than $\mathcal{QVH}$, but the proof of [She21, Lem. 7.1] still applies to $\mathcal{CMVH}$ since the groups H used there are finite. Therefore we obtain:

Proposition 3.7.4. *There exists $\mathcal{V}_{k+1}$ satisfying all the conditions (1)-(4) of Definition 3.7.2.*

In the next sections, we will need to modify our collections $\mathcal{V}_j$, and for that, we will use the following lemma.

Lemma 3.7.5. *Let $\mathcal{V}_j$ consist of the weighted triplets $(Z, H, (c_x)); \alpha_Z)$ and for each Z consider a finite index normal subgroup $H_0 \trianglelefteq H$ of index i_Z. Then after replacing each $(Z, H, (c_x); \alpha_Z)$ by $(Z, H_0, (c_x); (\prod_{Z' \neq Z} i_{Z'})\alpha_Z)$, the collection $\mathcal{V}_j$ still satisfies properties (1)-(4).*

Proof. Properties (1) and (2) are immediate. To verify (4), let e be an edge of $\dot{X}$, $c \in C_{k+1}(G(\mathcal{X}))$ be a coloring, and let $\widetilde{\mathcal{V}}_j^{\pm}(e,c)$ be the set of pairs $(H_0 \cdot x, Z)$ such that $(H \cdot x, Z)$ is in $\mathcal{V}_j^{\pm}(e,c)$. The contribution of a triplet $(Z, H_0, (c_x))$ to $\widetilde{\mathcal{V}}_j^{\pm}(e,c)$ is i_Z times the contribution of a triplet $(Z, H, (c_x))$ to $\mathcal{V}_j^{\pm}(e,c)$. Let $\widetilde{C}_Z^{\pm}$ and $C_Z^{\pm}$ denote these contributions, respectively. Then if we choose $\widetilde{\alpha}_Z := \prod_{Z' \neq Z} i_{Z'} \alpha_Z$, we have

$$\left|\widetilde{\mathcal{V}}_j^{+}(e,c)\right| = \sum_Z \widetilde{\alpha}_Z \widetilde{C}_Z^{+} = \left(\prod_Z i_Z\right) \sum_Z \alpha_Z C_Z^{+} = \left(\prod_Z i_Z\right) \sum_Z \alpha_Z C_Z^{-} = \sum_Z \widetilde{\alpha}_Z \widetilde{C}_Z^{-} = \left|\widetilde{V}_j^{-}(e,c)\right|,$$

so the gluing equations are also satisfied by the modified $\mathcal{V}_j$. Finally, property (3) follows from the lemma below that will also be used in Section 3.9. □

Lemma 3.7.6. *If $(H, Y) \in \mathcal{CMVH}$ and $H_0 \trianglelefteq H$ is a finite index normal subgroup, then $(H_0, Y) \in \mathcal{CMVH}$.*

Proof. The lemma follows by induction on the minimal number of operations $(1)-(3)$ used in a description of (H, Y) as a group in $\mathcal{CMVH}$, after noting that finite index subgroups of convex subgroups are convex, and that if (H, Y) splits as a graph of groups satisfying the properties of condition (3) in Definition 3.4.4, then the induced splitting of (H_0, Y) also satisfies (3) when H_0 is considered with the induced peripheral structure, see e.g. [AGM16, Prop. 3.18]. □

Definition 3.7.7. Any change of $\mathcal{V}_j$ by first considering finite index subgroups $H_0 \trianglelefteq H$ for each $(Z, H, (c_x))$ and then duplicating the triplets $(Z, H_0, (c_x))$ as in the previous lemma will be called a *virtual modification* of $\mathcal{V}_j$.

3.8 Boundary walls and graphs of groups

We keep the notation from the previous sections. We assume the existence of a collection $\mathcal{V}_j$ as in Definition 3.7.2, and in this section we consider $(Z, H, (c_x)) \in \mathcal{V}_j$. We will introduce the main definitions that will be used in Section 3.9 to construct $\mathcal{V}_{j-1}$ from $\mathcal{V}_j$.

Definition 3.8.1. A *boundary wall* of Z is a wall W dual to an edge e crossing out of Z. By property (2) of Definition 3.7.2 and Remark 3.7.3, W is a boundary wall if and only if $W = W(e)$ for an edge e intersecting Z and $c_e(\overline{W(e)}) \leq j$.

The next lemma is implicit in [Ago13, p. 1062] and is stated as the *Zipping Lemma* in [She21, Lem. 8.4].

Lemma 3.8.2. *If $W = W(e_1) = W(e_2)$ is a boundary wall of Z with e_1, e_2 edges crossing out of Z, then $[c_{e_1}]_W = [c_{e_2}]_W$.*

Remark 3.8.3. By the lemma above, the color $c_e(\overline{W})$ is independent of the choice of e, and should be thought of as the *color* of the boundary wall W.

Definition 3.8.4. Let W be a boundary wall of Z with color j in the sense of the previous remark. We say that W is a *j-boundary wall* of Z, and that $P(W) := W \cap Z$ is the *portal of W leading to Z*. If an edge e dual to W crosses out of Z, then we say that e is *dual* to $P(W)$.

The next lemmas are from [She21].

Lemma 3.8.5 ([She21, Lem. 8.6])**.** *A vertex in $Z \in \mathcal{V}_j$ cannot be incident to distinct edges dual to j-boundary walls.*

Definition 3.8.6. For a wall $\overline{W}$ in $\mathcal{X}$ and $c \in C_{k+1}(G(\mathcal{X}))$, let $B(\overline{W}, c) := \overline{W} \cap c^{-1}([1, j])$ be the intersection of $\overline{W}$ with other walls in $\mathcal{X}$ colored $\leq j$ by c (j is fixed in this section). We define $\overline{W}$ *split along* c by $\overline{W} - c := \overline{W} - B(\overline{W}, c)$, and for a vertex $\overline{x}$ in $\overline{W}$, we let $(\overline{W} - c)(\overline{x})$ denote the component of $\overline{W} - c$ containing $\overline{x}$.

Lemma 3.8.7 ([She21, Lem. 8.10])**.** *Let W be a j-boundary wall with portal $P = Z \cap W$ and let e be an edge dual to P with midpoint x_0. Let $\mathring{P}$ denote the interior of P as a subspace of W. Then the following holds.*

1. *The quotient map $q : \dot{X} \to \mathcal{X}$ restricts to a universal covering map*
$$q|_{\mathring{P}} : \mathring{P} \to (\overline{W} - c_e)(\overline{x}_0).$$
2. *$(\overline{W} - c_e)(\overline{x}_0) = (\overline{W} - c)(\overline{x})$ for any other vertex $x \in P$ of W and any $c \in [c_e]_W$.*
3. *The group of deck transformations of $q|_{\mathring{P}}$ is $K_P := \{g \in K : gx_0 \in P\} = \mathrm{Stab}_K(P)$ (where K is from Theorem 3.5.4), hence K_P acts cocompactly on P.*

If P is a portal leading to $(Z, H, (c_x))$, let H_P denote its setwise stabilizer in H. Suppose P lies in the wall W and let $x \in P$ and $h \in H$ be such that $hx \in P$. If x' is the vertex closest to x in P, then x' is the midpoint of the edge e dual to P with he also dual to $P \subset W$, implying $hx' \in P$, $hW = W$, and $hP = h(Z \cap W) = Z \cap W = P$. We conclude that $h \in H_P$, so the map $H_P \backslash P \to H \backslash Z$ is an embedding, and also that H_P acts properly and cocompactly on P because $H \backslash Z$ is compact.

This last observation and properness of the action of Γ on $\dot{X}$, together with the previous two lemmas, imply that $H_P \cap K$ is finite index in H_P for any portal P leading to $(Z, H, (c_x))$. Thus we can use Lemma 3.7.5 to modify our set $\mathcal{V}_j$.

Corollary 3.8.8. *We can virtually modify $\mathcal{V}_j$ (in the sense of Lemma 3.7.5) so that $H_P < K$ for any portal P leading to Z.*

Proof. Let $P_1, \dots, P_k$ be a set of representatives of H-orbits for portals of Z, and note that by Lemma 3.7.5 it is enough to replace H by a finite index normal subgroup $H_0 \trianglelefteq H$ such that $H_0 \cap H_{P_i} \cap K = H_0 \cap H_{P_i}$ for each i. But each $H_{P_i} \cap K$ is finite index in H_{P_i}, so we just need each subgroup $H_{P_i} \cap K$ to be separable in H, which is true by Theorems 2.8.3 and 3.4.5 since $(H, \mathcal{N}(Z)) \in \mathcal{CMVH}$ and all the subgroups $H_{P_i} \cap K$ are convex in $(H, \mathcal{N}(Z))$ (they preserve the convex subcomplexes $\mathcal{N}(P_i) \cap \mathcal{N}(Z)$ respectively). $\square$

Definition 3.8.9. We say that two portals P and P' leading to $(Z, H, (c_x)), (Z', H', (c'_x)) \in \mathcal{V}_j$ respectively are *compatible* if there are edges e and e' dual to P and P' respectively such that $[e, c_e] \in \Gamma \cdot [e', c'_{e'}]$.

Let P and P' be compatible portals as above, say lying in walls W and W'. Take $g \in \Gamma$ and edges e and e' dual to P and P' such that $[e, c_e] = g[e', c'_{e'}]$, and let x_0 and x'_0 be the midpoints of e and e'. We have $e = ge'$, $W = gW'$ and $x_0 = gx'_0$. At the level of $\mathcal{X}$ the action of g translates to

$$g(\overline{W}' - c'_{e'})(\overline{x}'_0) = (\overline{W} - gc'_{e'})(\overline{x}_0) = (\overline{W} - c_e)(\overline{x}_0)$$

where we used Lemma 3.8.7 (2) and the fact that $[c_e]_W = [gc'_{e'}]_W$.

Since q restricts to coverings for $\mathring{P}$ and $\mathring{P}'$, we deduce that g restricts to a cube isomorphism $P' \to P$ which is equivariant with respect to the group isomorphism $K_{P'} \to K_P; k \mapsto gkg^{-1}$. This induces an isomorphism $K_{P'} \backslash P' \xrightarrow{\sim} K_P \backslash P$.

Compatibility of portals can be described in terms of the following lemma.

Lemma 3.8.10 ([She21, Lem. 9.2])**.** *Portals P and P' leading to $(Z, H, (c_x)), (Z', H', (c'_x)) \in \mathcal{V}_j$ are compatible if and only if there exists $g \in \Gamma$ such that*

$$\{[e, c_e] \colon e \text{ is dual to } P\} = g\{[e', c'_{e'}] \colon e' \text{ is dual to } P'\}.$$

In particular, compatibility of portals is an equivalence relation.

Following the notation from [She21], in the case of P, P' and g as above we say that P is a *g-teleport* of P'.

3.8.1 The graph of groups $(\mathbb{A}, \mathcal{A})$

As we saw previously, compatible portals P and P' leading to $(Z, H, (c_x)), (Z', H', (c'_x)) \in \mathcal{V}_j$ are isomorphic and induce an isomorphism $K_{P'} \backslash P' \to K_P \backslash P$. However, we would like to glue $H' \backslash Z'$ and $H \backslash Z$ along $H'_{P'} \backslash P'$ and $H_P \backslash P$, which are only isomorphic up to a finite-sheeted cover by Corollary 3.8.8. If we want isomorphisms $g : H'_{P'} \backslash P' \to H_P \backslash P$ we need to virtually modify $\mathcal{V}_j$ again, and for that, we will construct a graph of groups.

Definition 3.8.11. Let $(\mathbb{A}, \mathcal{A})$ be the finite bipartite (and possibly disconnected) graph of groups defined as follows.

- *Type I* vertices of $\mathbb{A}$ are triplets $(Z, H, (c_x)) \in \mathcal{V}_j$ with corresponding vertex group H. Here repeated triplets are counted separately.
- *Type II* vertices of $\mathbb{A}$ are portals $\{P_i\}$ forming a complete set of representatives for the compatibility classes of portals, with corresponding vertex groups K_{P_i}.

- Edges attached to the Type I vertex $(Z, H, (c_x))$ will be portals P leading to $(Z, H, (c_x)) \in \mathcal{V}_j$, such that we choose just one P from each H-orbit of portals (repeated triplets will have the same conjugacy representatives). The edge P will be attached to the Type II vertex in its compatibility class of portals, and its edge group will be H_P.

For a portal P leading to $(Z, H, (c_x))$, the injection of the edge group H_P into its type I vertex group is just the inclusion $H_P \hookrightarrow H$, while the map into a type II vertex group is the composition $H_P \hookrightarrow K_P \xrightarrow{g(-)g^{-1}} K_{P_i}$, where $g \in \Gamma$ is so that P_i is a g-teleport of P (for the case $g : P_i \to P_i$ we take $g = 1$, and same portals corresponding to repeated triplets will have the same choice of g).

The next proposition may be thought of as a relative version of the acylindricity of the graph of groups $\mathbb{A}$ proven in the absolute case (cf. [Ago13, p. 1063] and [She21, Lem. 8.8]). The proof is practically the same, the only difference is that we require Proposition 2.7.9.

Proposition 3.8.12. *If $(Z, H, (c_x))$ is a type I vertex group of $(\mathbb{A}, \mathcal{A})$, then the collection*

$$\{H_P \colon P \text{ is an edge attached to } Z\}$$

is relatively malnormal in H. That is, if P_1 and P_2 are edges attached to Z and $h \in H$ is so that $H_{P_1} \cap H_{P_2}^h$ contains a loxodromic element, then $P_1 = P_2$ and $h \in H_{P_1}$.

Proof. Assume $\lambda \in H_{P_1} \cap H_{P_2}$ is a loxodromic element, in which case we claim that $P_1 = P_2$. If W_1 and W_2 are the walls containing P_1 and P_2 respectively, then it is enough to prove that $W_1 = W_2$, since that implies $P_1 = Z \cap W_1 = Z \cap W_2 = P_2$.

The element λ acts freely on P_1 and P_2, so it acts loxodromically on them, and there exist axes $\gamma_i \subset P_i \subset W_i$ in which λ acts by non-trivial translation. These axes are asymptotic in Z, thus γ_1 and γ_2 bound a flat strip in Z of width $r \geq 0$. Since λ is loxodromic, by Proposition 2.7.9 we have $r \leq \delta$. Let us assume $W_1 \neq W_2$, so that $\overline{W}_1 \neq \overline{W}_2$ and $d_{\mathcal{X}}(\overline{W}_1, \overline{W}_2) \leq \delta \leq R$ (by Assumption 3.5.2), and get a contradiction by showing that $\overline{W}_1$ and $\overline{W}_2$ are colored equal by some coloring.

If p is any point in P_1, then it is contained in a cube C of $\dot{X}$ and we can find a vertex $x \in Z$ incident to an edge dual to P_1, with $d_{\dot{X}}(p, x) \leq \frac{1}{2}\sqrt{\dim C} \leq \frac{1}{2}\sqrt{\dim X}$. The same is true for P_2, so there are vertices $x_1, x_2 \in Z$ with each x_i being incident to an edge dual to P_i so that the geodesic segment α joining x_1 and x_2 has length at most $\delta + \sqrt{\dim X}$. By considering the sequence of cubes that α travels through, we can find an edge path β in Z from x_1 to x_2 with $\beta \subset N_{\sqrt{\dim X}}(\alpha)$. Let $e_1, \ldots, e_s$ be the edges of β, and $x_1 = y_1, y_2, \ldots, y_{s+1} = x_2$ be their vertices, with e_i joining y_i and y_{i+1}. Since $R \geq \delta + 2\sqrt{\dim X}$ (by Assumption 3.5.2) we have $d_{\mathcal{X}}(\overline{W(e_i)}, \overline{W}_1) \leq d_{\dot{X}}(W(e_i), W_1) \leq R$ for all $1 \leq i \leq s$, and so $\overline{W(e_i)}$ and $\overline{W}_1$ are adjacent vertices in $G(\mathcal{X})$. For each $1 \leq i \leq s$, from property (1) of $\mathcal{V}_j$ we deduce $[c_{y_i}]_{e_i} = [c_{y_{i+1}}]_{e_i}$, so $c_{y_i}(\overline{W}_1) = c_{y_{i+1}}(\overline{W}_1)$ and hence $c_{x_1}(\overline{W}_1) = c_{x_2}(\overline{W}_1) = j$, because W_1 and W_2 are j-boundary walls. Thus c_{x_2} is a coloring with $c_{x_2}(\overline{W}_1) = c_{x_2}(\overline{W}_2)$, contradicting $W_1 \neq W_2$.

To finish the proposition, let P_1 and P_2 be edges attached to $(Z, H, (c_x))$ and $h \in H$ be such that $H_{P_2} \cap H_{P_2}^h = H_{P_1} \cap H_{P_2}^h$ contains a loxodromic. By our previous claim, we obtain

$P_1 = hP_2$, and since different edges correspond to distinct representatives of H-orbits of portals we must have $P_1 = P_2$, implying $h \in H_{P_1}$. □

Definition 3.8.13. Let $\mathbb{G} = \mathbb{G}_c$ be the fundamental group of a connected component $(\mathbb{A}_c, \mathcal{A}_c)$ of $(\mathbb{A}, \mathcal{A})$ with respect to some vertex w_0 of $\mathbb{A}_c$, and fix a maximal subtree T of $\mathbb{A}_c$ containing w_0 that fixes inclusions of the edge/vertex groups of $(\mathbb{A}_c, \mathcal{A}_c)$ into $\mathbb{G}$.

Proposition 3.8.14. *Let Γ_{e_0} be an edge group of $(\mathbb{A}_c, \mathcal{A}_c)$ attached to the type II vertex group Γ_{v_0}, and let $a \in \Gamma_{v_0} \setminus \Gamma_{e_0} \subset \mathbb{G}$. Then there exists a finite index subgroup $\dot{\mathbb{G}}_{e_0,a} < \mathbb{G}$ containing Γ_{e_0} with $a \notin \dot{\mathbb{G}}_{e_0,a}$.*

Proof. The proof is almost the same as the one given for Proposition 3.4.7, so we just give a sketch of it. Recall that by Lemma 3.1.5, each vertex/edge group is convex and strongly peripherally separable in Γ, so by several applications of Proposition 3.3.19 we can find finite index subgroups $\dot{P}_j < P_j$ such that the filling $\phi : \Gamma \to \overline{\Gamma} = \Gamma(\mathcal{N} = \{\dot{P}_1, \ldots, \dot{P}_n\})$ satisfies:

- $\overline{\Gamma}_v := \phi(\Gamma_v)$ is hyperbolic and virtually special, and isomorphic to the image of the induced filling $\phi_v : \Gamma_v \to \Gamma_v(\mathcal{N}_v)$ for any vertex v of $\mathbb{A}_c$ (images of type II vertex groups will be virtually special because every type II vertex is a finite index extension of an edge group, which is virtually special by Theorem 3.4.5, Corollary 3.2.2 and our inductive assumption, see the beginning of Section 3.5).
- The image $\overline{\Gamma}_e := \phi(\Gamma_e)$ of the edge group Γ_e of $(\mathbb{A}_c, \mathcal{A}_c)$ with terminal vertex of type I is naturally isomorphic to the image of the filling $\phi_e : \Gamma_e \to \Gamma_e(\mathcal{N}_e)$ induced by both ϕ and $\phi_{t(e)}$ (that is, $\ker \phi_e = \ker \phi_{t(e)} \cap \Gamma_e = \ker \phi \cap \Gamma_e$).
- The collection of images under ϕ of groups in $\{\overline{\Gamma}_e \colon e \text{ attached to } v\}$ is almost malnormal in $\overline{\Gamma}_v$ for any type I vertex v of $\mathbb{A}_c$.
- $\phi(a) \notin \overline{\Gamma}_{e_0}$.

We then consider the graph of groups $(\mathbb{A}_c, \overline{\mathcal{A}}_c)$ with the same underlying graph $\mathbb{A}_c$, and $\overline{\mathcal{A}}_c$ assigning the group $\overline{\Gamma}_x$ to each vertex/edge x of $\mathbb{A}_c$, and with attaching maps induced by ϕ and the attaching maps of $(\mathbb{A}_c, \mathcal{A}_c)$ (every attaching map of $(\mathbb{A}_c, \mathcal{A}_c)$ is composition of inclusions and conjugations in Γ). Let $\overline{\mathbb{G}} = \pi_1(\mathbb{A}_c, \overline{\mathcal{A}}_c, w_0)$ and choose embeddings of edge/vertex groups according to the same maximal subtree T of $\mathbb{A}_c$.

The homomorphism ϕ restricted to each edge/vertex group induces a homomorphism $\Phi : \mathbb{G} \to \overline{\mathbb{G}}$ such that $\Phi(x) = \phi_v(x)$ for any vertex v of $\mathbb{A}_c$ and for any $x \in \Gamma_v$, and since the splitting $(\mathbb{A}_c, \overline{\mathcal{A}}_c)$ of $\overline{\mathbb{G}}$ satisfies the assumptions of Lemma 3.4.9 (2), $\overline{\mathbb{G}}$ is hyperbolic and $\overline{\Gamma}_{e_0}$ is quasiconvex in $\overline{\mathbb{G}}$. Then Theorem 2.8.6 implies that $\overline{\mathbb{G}}$ is virtually special, and since $\Phi(a) = \phi(a) \notin \overline{\Gamma}_{e_0} = \Phi(\Gamma_{e_0})$, Theorem 2.8.3 gives us the separability of $\Phi(\Gamma_{e_0})$ in $\overline{\mathbb{G}}$, and hence the existence of $\dot{\mathbb{G}}_{a,e_0}$. □

Corollary 3.8.15. *There exists a finite index subgroup $\mathbb{N}_c \trianglelefteq \mathbb{G}_c$ such that if $\Gamma_e < \mathbb{G}_c$ is an edge group attached to (and hence contained in) the type II vertex group $\Gamma_v < \mathbb{G}_c$, then*

$$\Gamma_e \cap \mathbb{N}_c = \Gamma_v \cap \mathbb{N}_c. \tag{3.7}$$

Proof. Recall that an edge group Γ_e is finite index in the type II vertex group Γ_v it is attached to. So, for each edge e let $S_e \subset \Gamma_v \backslash \Gamma_e$ be any finite set of representatives of non-trivial left cosets of Γ_e in Γ_v. By Proposition 3.8.14, for each $a \in S_e$ there is a finite index subgroup $\dot{\mathbb{G}}_{a,e_0} < \mathbb{G}_c$ separating Γ_e from a, and so the intersection of the finitely many conjugates of $\dot{\mathbb{G}}_{a,e_0}$ in $\mathbb{G}_c$ is a finite index normal subgroup, that we denote $\mathbb{N}_{e,a}$. The group $\mathbb{N} := \bigcap_{e \in E(\mathbb{A}_c)} \bigcap_{a \in S_v} \mathbb{N}_{e,a}$ then satisfies the required identities from (3.7). ☐

3.9 Constructing $\mathcal{V}_{j-1}$ from $\mathcal{V}_j$

In this section we continue with the notation from the previous sections. Our goal is the construction of $\mathcal{V}_{j-1}$ from $\mathcal{V}_j$, which implies Theorem 3.5.1.

Let $(\mathbb{A}_c, \mathcal{A}_c)$ be a component of the graph of groups $(\mathbb{A}, \mathcal{A})$ with fundamental group $\mathbb{G}_c$, as in Definition 3.8.11, and let $\mathbb{N}_c \trianglelefteq \mathbb{G}_c$ be given by Corollary 3.8.15. For each type I vertex group H of $(\mathbb{A}_c, \mathcal{A}_c)$, define $\hat{H} := \mathbb{N}_c \cap H \trianglelefteq H$ and see these groups as subgroups of Γ. After doing this for each connected component $(\mathbb{A}_c, \mathcal{A}_c)$ of $(\mathbb{A}, \mathcal{A})$ we obtain a finite index subgroup of H for each $(Z, H, (c_x)) \in \mathcal{V}_j$ and so Lemma 3.7.5 gives us a virtual modification $\hat{\mathcal{V}}_j$ of $\mathcal{V}_j$ with triplets $(Z, \hat{H}, (c_x))$.

We will use $\hat{\mathcal{V}}_j$ to construct a (possibly disconnected) graph of spaces $(\mathbb{S}, \mathcal{S})$. As before, let $\{P_i\}$ be the set of type II vertices of $\mathbb{A}$, and for each $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$ choose a set $\mathbb{B}_Z$ of representatives of portals leading to Z, with exactly one portal P for each $\hat{H}$-orbit of translates of portals (with same representatives for repeated triplets). Set $\mathbb{B} := \bigsqcup_{Z \in \hat{\mathcal{V}}_j} \mathbb{B}_Z$, and choose the representatives of portals in such a way that each edge of $\mathbb{A}$ lies in $\mathbb{B}$.

As in [She21, Def. 9.11], the *size* of an edge P of $\mathbb{A}$ is defined by

$$\mathrm{sz}(P) := |\{H_P \cdot e \colon e \text{ is an edge dual to } P\}|,$$

where P leads to $(Z, H, (c_x)) \in \mathcal{V}_j$. Similarly, for $P \in \mathbb{B}$ leading to $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, define the *size* of P as

$$\hat{\mathrm{sz}}(P) := |\{\hat{H}_P \cdot e \colon e \text{ is an edge dual to } P\}|,$$

where $\hat{H}_P := \hat{H} \cap H_P$ is the stabilizer of P in $\hat{H}$. Note that $\hat{\mathrm{sz}}(P) = |H_P : \hat{H}_P| \cdot \mathrm{sz}(P)$ for any edge P of $\mathbb{A}$ attached to $(Z, H, (c_x))$. Also, by equation (3.7), $\hat{\mathrm{sz}}(P) = \hat{\mathrm{sz}}(P')$ whenever $P, P' \in \mathbb{B}$ are compatible portals.

If P is a portal leading to Z and contained in the wall W, with Z in either $\mathcal{V}_j$ or $\hat{\mathcal{V}}_j$, and P_i is a g-teleport of P contained in the wall W_i that is a type II vertex group of $\mathbb{A}$, we say that P is a P_i^+*-portal* if $gZ \cap W_i^+ \neq \emptyset$, and a P_i^-*-portal* if $gZ \cap W_i^- \neq \emptyset$. For $Z \in \mathcal{V}_j$ define $\mathbb{A}_Z(P_i, \pm)$ as the set of $P_i^{\pm}$-portals P leading to Z which are an edge of $\mathbb{A}$, and let $\mathbb{A}(P_i, \pm) := \bigsqcup_{Z \in \mathcal{V}_j} \mathbb{A}_Z(P_i, \pm)$. The sets $\mathbb{B}_Z(P_i, \pm)$ for $Z \in \hat{\mathcal{V}}_j$ and $\mathbb{B}(P_i, \pm)$ are defined similarly.

Before defining $(\mathbb{S}, \mathcal{S})$ we need some combinatorial results.

Lemma 3.9.1 ([She21, Lem. 9.13])**.** *If P_i is a type II vertex of $\mathbb{A}$ then*

$$\sum_{P \in \mathbb{A}(P_i,+)} \mathrm{sz}(P) = \sum_{P \in \mathbb{A}(P_i,-)} \mathrm{sz}(P).$$

Corollary 3.9.2. *For each type II vertex P_i of $\mathbb{A}$, the number of P_i^+-portals P in $\mathbb{B}$ equals the number of P_i^--portals P in $\mathbb{B}$.*

Proof. Let $s := \hat{\mathrm{sz}}(P)$ be the size of any portal P in $\mathbb{B}$ compatible with P_i. From the proof of Lemma 3.7.5, it follows that there exists a positive integer d such that for each triplet $(Z, H, (c_x)) \in \mathcal{V}_j$ there are $d/|H : \hat{H}|$ triplets of $(Z, \hat{H}, (c_x))$ in $\hat{\mathcal{V}}_j$. Also, if P is any portal leading to $(Z, H, (c_x))$, then the set of H-orbits of P is the disjoint union of $|H : \hat{H}|/|H_P : \hat{H}_P|$ $\hat{H}$-orbits of translates of portals. With this in mind we have

$$\begin{aligned}
\sum_{P \in \mathbb{B}(P_i,+)} 1 = \sum_{Z \in \hat{\mathcal{V}}_j} \sum_{P \in \mathbb{B}_Z(P_i,+)} 1 &= \sum_{(Z,H,(c_x)) \in \mathcal{V}_j} \frac{d}{|H : \hat{H}|} \left(\sum_{P \in \mathbb{B}_Z(P_i,+)} 1 \right) \\
&= \sum_{(Z,H,(c_x)) \in \mathcal{V}_j} \frac{d}{|H : \hat{H}|} \left(\sum_{P \in \mathbb{A}_Z(P_i,+)} \frac{|H : \hat{H}|}{|H_P : \hat{H}_P|} \right) \\
&= \frac{d}{s} \sum_{(Z,H,(c_x)) \in \mathcal{V}_j} \left(\sum_{P \in \mathbb{A}_Z(P_i,+)} \frac{\hat{\mathrm{sz}}(P)}{|H_P : \hat{H}_P|} \right) \\
&= \frac{d}{s} \sum_{(Z,H,(c_x)) \in \mathcal{V}_j} \left(\sum_{P \in \mathbb{A}_Z(P_i,+)} \mathrm{sz}(P) \right) \\
&= \frac{d}{s} \sum_{P \in \mathbb{A}(P_i,+)} \mathrm{sz}(P).
\end{aligned}$$

The same is true for the P_i^--portals, and so the conclusion follows by Lemma 3.9.1. $\square$

Definition 3.9.3. Let $(\mathbb{S}, \mathcal{S})$ be the graph of spaces defined as follows.

- The vertices of $\mathbb{S}$ will be the triplets $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, with corresponding vertex spaces $\hat{H} \backslash Z$.

- If P_i is a type II vertex of $\mathbb{A}$, by Corollary 3.9.2 there exists a perfect matching between $\mathbb{B}(P_i, +)$ and $\mathbb{B}(P_i, -)$. If $p := (P, P')$ is an oriented pair given by this matching with

P and P' leading to $(Z, \hat{H}, (c_x))$ and $(Z', \hat{H}', (c'_x))$ respectively, then P is a g_p-teleport of P' for some fixed $g_p \in \Gamma$, and there are embeddings

$$\hat{H}'_P \backslash P' \hookrightarrow \hat{H}' \backslash Z', \quad \text{and} \quad \hat{H}'_{P'} \backslash P' \xrightarrow{g_p} \hat{H}_P \backslash P \hookrightarrow \hat{H} \backslash Z, \tag{3.8}$$

where $\hat{H}'_{P'} \backslash P' \xrightarrow{g_p} \hat{H}_P \backslash P$ is an isomorphism for g_P chosen appropriately, due to Corollary 3.8.15.

- The edges of $\mathbb{S}$ are oriented pairings $p := (\hat{H}_P, \hat{H}'_{P'})$ attached to Z and Z' as above, with edge spaces $\hat{H}'_{P'} \backslash P'$ and attaching maps given by (3.8). For the reverse pairing $\overline{p} = (P', P)$, the attaching maps are constructed in the same way with $g_{\overline{p}} = g_p^{-1}$.

Consider a component $(\mathbb{S}_c, \mathcal{S}_c)$ of $(\mathbb{S}, \mathcal{S})$, with underlying space $\mathcal{T}_c$ obtained by gluing vertex spaces along images of attaching maps. We want to construct an embedding $\widetilde{\mathcal{T}_c} \hookrightarrow \dot{X}$ of the universal cover of $\mathcal{T}_c$ into $\dot{X}$.

First of all, fix a base-point $x \in Z$ for each $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$. If $\hat{H} \backslash Z$ and $\hat{H}' \backslash Z'$ are vertex spaces joined by the oriented edge $p = (P, P')$ then, up to homotopy, there exists a unique path α_p in $Z \cup g_p Z'$ from x to $g_p x$ (this is because $Z \cup g_p Z'$ is simply-connected). Let β_p be the projection of α_p into $\mathcal{T}_c$ via $Z \to \hat{H} \backslash Z$ and $Z' \to \hat{H}' \backslash Z'$. We can choose our paths so that $\beta_{\overline{p}}$ is the reverse path of β_p. Also, for each $(Z, \hat{H}, (c_x))$ and $h \in \hat{H}$ let γ_h be a loop in $\hat{H} \backslash Z$ lifting to a path from x to hx.

Fix a base vertex $(Z_0, \hat{H}_0, (c_x)_0)$ of $\mathbb{S}_c$, and for edges $p_1, \ldots, p_n$ in $\mathbb{S}_c$ forming a path through vertex spaces

$$\hat{H}_0 \backslash Z_0 \xrightarrow{p_1} \hat{H}_1 \backslash Z_1 \xrightarrow{p_2} \ldots \xrightarrow{p_n} \hat{H}_n \backslash Z_n,$$

and $h_i \in \hat{H}_i$ for $i = 0, 1, \ldots, n$, consider the concatenation

$$\gamma = \gamma_{h_0} \cdot \beta_{p_1} \cdot \gamma_{h_{n-1}} \cdots \beta_{p_n} \cdot \gamma_{h_n}, \tag{3.9}$$

which gives a path in $\mathcal{T}_c$. Note that every path in $\mathcal{T}_c$ between (projections of) base-points of vertex spaces and starting in $\hat{H}_0 \backslash Z_0$ is homotopic to a path of this form. For each such γ, we define

$$g(\gamma) = h_0 g_{p_1} h_1 \cdots g_{p_n} h_n,$$

and we take $\widetilde{\mathcal{T}_c} \subset \dot{X}$ to be the union of all the possible Γ-translates $g(\gamma) Z_n$. The covering map $\mu : \widetilde{\mathcal{T}_c} \to \mathcal{T}_c$ restricts to $g(\gamma) Z_n$ by

$$\mu : g(\gamma) Z_n \xrightarrow{g(\gamma)^{-1}} Z_n \to \hat{H}_n \backslash Z_n \to \mathcal{T}_c.$$

Lemma 3.9.4 ([She21, Lem. 9.17 & Lem. 9.20])**.** *$\mu : \widetilde{\mathcal{T}_c} \to \mathcal{T}_c$ is a universal covering map and $\widetilde{\mathcal{T}_c} \subset \dot{X}$ is a non-empty intersection of half-spaces.*

For a loop γ as in (3.9) and $g(\beta)Z$ in $\widetilde{\mathcal{T}}_c$, we have that $g(\gamma)g(\beta) = g(\gamma \cdot \beta)$, hence $g(\gamma)g(\beta)Z = g(\gamma \cdot \beta)Z \subset \widetilde{\mathcal{T}}_c$. This holds for all translates $g(\beta)Z$ in $\widetilde{\mathcal{T}}_c$, thus $g(\gamma)\widetilde{\mathcal{T}}_c = \widetilde{\mathcal{T}}_c$ and $\mu \circ g(\gamma) = \mu$.

Define

$$H(\mathcal{T}_c) := \{g(\gamma) \colon \gamma \text{ is a loop of form (3.9)}\} < \Gamma.$$

Since $g(\gamma)g(\beta) = g(\gamma \cdot \beta)$ for any loop γ and path β, $H(\mathcal{T}_c)$ is a subgroup of Γ preserving $\widetilde{\mathcal{T}}_c$. Moreover, $\mu \circ g(\gamma) = \mu$ for every $g(\gamma) \in H(\mathcal{T}_c)$, so $H(\mathcal{T}_c)$ is a subgroup of the group of Deck transformations of μ. In addition, by construction the orbit of the base-point x_0 of Z_0 is $H(\mathcal{T}_c) \cdot x_0 = \mu^{-1}(\mu(x_0))$, implying that $H(\mathcal{T}_c) \cong \pi_1(\mathcal{T}_c)$ is the full group of Deck transformations of μ, and hence it acts freely and cocompactly on $\widetilde{\mathcal{T}}_c$.

Finally, if $x \in Z$ is any vertex with $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$ and γ is as in (3.9), then we endow $g(\gamma)x$ with the coloring $c^{\mathcal{T}_c}_{g(\gamma)x} := g(\gamma)c_x$. It is evident that $H(\mathcal{T}_c)$ preserves these colorings.

Definition 3.9.5. Let $\mathcal{V}_{j-1}$ consist of the set of triplets $(\widetilde{\mathcal{T}}_c, H(\mathcal{T}_c), (c^{\mathcal{T}_c}_x))$, with one triplet for each underlying space $\mathcal{T}_c$ for a component $(\mathbb{S}_c, \mathcal{S}_c)$ of $(\mathbb{S}, \mathcal{S})$.

The next proposition is our last step in the proof of Theorem 3.5.1.

Proposition 3.9.6. *$\mathcal{V}_{j-1}$ satisfies all the desires properties (1)-(4) of Definition 3.7.2.*

Proof. For properties (1) and (2), the proof is the same as in [She21, p. 34] so it will be omitted.

To show property (3), we first note that by Lemma 3.1.4 each subgroup $H(\mathcal{T}_c)$ is convex in Γ with convex core $\mathcal{N}(\widetilde{\mathcal{T}}_c)$, so it is also hyperbolic relative to compatible virtually special subgroups by Lemma 3.7.1. In addition, since any edge space embedding into a vertex space of $(\mathbb{S}_c, \mathcal{S}_c)$ is π_1-injective, by Van Kampen's theorem there is an induced splitting $(\mathbb{S}_c, \mathcal{U}_c)$ of $H(\mathcal{T}_c) \cong \pi_1(\mathcal{T}_c)$ with vertex groups $\hat{H}$ for $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$, and with edge groups of the form $\hat{H}_P \cong \hat{H}'_{P'}$ for each edge $p = (P, P')$ with P, P' leading to $(Z, \hat{H}, (c_x))$ and $(Z', \hat{H}', (c'_x))$ respectively. By Assumption 3.5.2 and Lemma 3.7.6, each cubulated vertex group $(\hat{H}, \mathcal{N}(Z))$ of $(\mathbb{S}_c, \mathcal{U}_c)$ is in $\mathcal{CMVH}$, and by construction $\mathcal{N}(Z)$ is the Γ-translate of a convex subcomplex of $\mathcal{N}(\widetilde{\mathcal{T}}_c)$. The same is true for the embedding of an edge group into $H(\mathcal{T}_c)$ since it acts cocompactly on a translate of the cubical neighborhood $\mathcal{N}(P)$ of a portal P, which is a convex subcomplex of $\mathcal{N}(\widetilde{\mathcal{T}}_c)$. This implies that each vertex/edge group is convex in $(H(\mathcal{T}_c), \mathcal{N}(\widetilde{\mathcal{T}}_c))$. Finally, note that the conclusion of Proposition 3.8.12 holds also for vertex groups in $(\mathbb{S}_c, \mathcal{U}_c)$, and hence the collection of embeddings of edge groups into a vertex group of $(\mathbb{S}_c, \mathcal{U}_c)$ is relatively malnormal. Therefore $(H(\mathcal{T}_c), \mathcal{N}(\widetilde{T}_c)) \in \mathcal{CMVH}$.

For property (4), note that since each $\mathcal{T}_c$ is obtained by gluing quotients $\hat{H}\backslash Z$ for $(Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$ with each triplet being used in exactly one component $\mathcal{T}_c$ (repeated triplets are counted separately), there is a canonical bijection Λ from the set of vertices of $\bigsqcup_{(Z,\hat{H},(c_x)) \in \hat{\mathcal{V}}_j} \hat{H}\backslash Z$ onto the set of vertices of $\bigcup_c \mathcal{T}_c$, where c runs among the components of $(\mathbb{S}, \mathcal{S})$ (here, by vertex we mean an image of a vertex of $\dot{X}$ contained in some Z). Also, any vertex of some $\widetilde{\mathcal{T}}_c$ takes the form $\widetilde{x} = g(\gamma)x$ for some path γ as in (3.9) and some vertex $x \in (Z, \hat{H}, (c_x)) \in \hat{\mathcal{V}}_j$. Since by definition $c_{\widetilde{x}}^{\mathcal{T}_c} = g(\gamma)c_x$, Λ restricts to a bijection from $\mathcal{V}_j^{\pm}(e, c)$ onto $\mathcal{V}_{j-1}^{\pm}(e, c)$ for any equivalence class $[e, c]$ with e an edge and c a coloring. Since $\mathcal{V}_j$ satisfies the gluing equations, $\mathcal{V}_{j-1}$ also does. $\square$

Chapter 4

The space of metric structures on hyperbolic groups

Let Γ be a non-elementary hyperbolic group. If Γ is either a surface group or a free group, there are contractible spaces parametrizing nice geometric structures on Γ, namely the Teichmüller space $\mathscr{T}_\Gamma$ and the Culler-Vogtmann outer space $\mathscr{C}\mathscr{V}_\Gamma$, respectively. These spaces have been fundamental tools for understanding surface and free groups, as well as their outer automorphisms groups. However, for a general hyperbolic group, there are no analogs of these constructions, unless the group satisfies very strong topological/geometric/algebraic assumptions. In general, these assumptions do not hold, so one might wonder to what extent significant deformation spaces exist for arbitrary hyperbolic groups.

The goal of this chapter is to study a deformation space that is valid for any non-elementary hyperbolic group Γ. We study the space of *metric structures* $\mathscr{D}_\Gamma$, which parameterizes the geometric actions of Γ on Gromov hyperbolic spaces. This space is equipped with a natural metric, so that $\mathscr{D}_\Gamma$ contains embedded copies of Teichmüller space for Γ a surface group (indeed, copies of quasi-Fuchsian space and all Hitchin components) and of outer space when Γ is a free group.

In Section 4.1 we study the class $\mathcal{D}_\Gamma$ of pseudo metrics on Γ and introduce Manhattan curves, which are our main tool to analyze these pseudo metrics. We push this analysis further in Section 4.2, where we introduce the larger class $\mathcal{D}_\Gamma^{hf}$ of hyperbolic distance-like functions. Our main result there is Theorem 4.2.14, which relates optimal quasi-isometry constants for hyperbolic distance-like functions and their marked length spectra. The space of metric structures is defined in Section 4.3, as well as its metric. There we also explain various ways to describe metric structures and discuss many constructions inducing subspaces of $\mathscr{D}_\Gamma$. In Section 4.4 we prove some topological and metric properties about $\mathscr{D}_\Gamma$. In particular, we prove Theorem 1.2.4 (now Propositions 4.4.1, 4.4.3 and 4.4.13), Theorem 1.2.6 (now Theorem 4.4.22), Theorem 1.2.7 (now Theorems 4.4.29 and 4.4.30), and Theorem 1.2.9 (now Theorems 4.4.4 and 4.4.16). Continuity properties of functions associated to $\mathscr{D}_\Gamma$ are studied in Section 4.5, where we prove Theorem 1.2.8 (now Theorem 4.5.1). In Section 4.6 we discuss the Manhattan boundary by introducing the class $\overline{\mathcal{D}}_\Gamma$, also consisting of pseudo metrics on Γ, and interpreting it as limits at infinity of Manhattan geodesics. Many examples of points in this boundary are presented in Section 4.7, where we prove Theorem 1.2.12 (now Corollaries 4.7.7 and 4.7.13, and Propositions 4.7.17 and 4.7.18). Finally, Section 4.8 studies the extension of the marked length spectrum of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$ to the space of geodesic currents, and we use this to prove Theorem 1.2.13 (now Theorem 4.8.4). Most of the results of Section 4.3, Subsections 4.4.1, 4.4.4 and 4.4.5, and Section 4.5 are part of the paper [Ore23], whereas Section 4.2, Subsections 4.4.2 and 4.4.3, and Sections 4.6, 4.7 and 4.8 are included in the preprint [CR22], joint work with Stephen Cantrell.

Unless otherwise explicit, throughout this chapter Γ will denote a non-elementary hyperbolic group.

4.1 Hyperbolic pseudo metrics on Γ

In this section we discuss the space $\mathcal{D}_\Gamma$ consisting of well-behaved pseudo metrics on Γ. To understand this space from a global perspective, one of our main tools will be the Manhattan curves, which we also introduce.

4.1.1 The class $\mathcal{D}_\Gamma$

We start this subsection by recalling the definition of the set $\mathcal{D}_\Gamma$ given in the Introduction.

Definition 4.1.1. $\mathcal{D}_\Gamma$ is the set of all the left-invariant pseudo metrics on Γ that are hyperbolic and quasi-isometric to a word metric for a finite, symmetric generating subset of Γ.

Remark 4.1.2. In the definition above, all the assumptions for pseudo metrics in $\mathcal{D}_\Gamma$ are necessary. Indeed, as long as Γ is infinite, for any $d \in \mathcal{D}_\Gamma$ the new pseudo metric

$$d'(x,y) := d(x,y) + \log(1 + d(x,y))$$

is left-invariant and quasi-isometric to d, but it is not hyperbolic [BHM11, Sec. A.4].

Remark 4.1.3. For any pseudo metric d in $\mathcal{D}_\Gamma$ and $x, y \in \Gamma$, we have $d(x,y) = d(y^{-1}x, o) \geq \ell_d[y^{-1}x]$. Since $\ell_d[y^{-1}x] > 0$ when $y^{-1}x$ is non-torsion, we deduce that the subgroup $\{x \in \Gamma : d(o,x) = 0\}$ is torsion [BH99, Cor. III.Γ.3.10], hence it is finite. In particular, if Γ is torsion-free, then any pseudo metric in $\mathcal{D}_\Gamma$ is a genuine metric.

Natural examples of pseudo metrics in $\mathcal{D}_\Gamma$ are word metrics with respect to finite and symmetric generating sets. More generally, if Γ acts geometrically on the geodesic metric space (X, d), then for any $w \in X$ the orbit pseudo metric d_X^w belongs to $\mathcal{D}_\Gamma$. This follows from the Milnor-Schwarz Lemma 2.2.2 and Corollary 2.3.9. Less obvious examples can be constructed from random walks on Γ.

Example 4.1.4 (Green metrics)**.** Let λ be a probability measure on Γ, and assume that the support of λ is finite and generates Γ, and that $\lambda(x) = \lambda(x^{-1})$ for all $x \in \Gamma$ (under these assumptions we say that λ is *admissible*). Then we consider the random walk $(Z_n)_{n\geq 0}$ on Γ with transition probabilities given by λ. That is, $\mathbb{P}(Z_{n+1} = Z_n x) = \lambda(x)$ for all $x \in \Gamma$. From this data, the *Green metric* on Γ is defined according to

$$d_\lambda(x, y) := -\log \mathbb{P}(\exists n \text{ s.t. } xZ_n = y)$$

for $x, y \in \Gamma$. If λ is admissible, Blachère, Haïssinsky and Mathieu proved that $d_\lambda \in \mathcal{D}_\Gamma$ [BHM11, Coro. 1.2].

We will see more examples of pseudo metrics belonging to $\mathcal{D}_\Gamma$ in Subsection 4.2.1. Since all the pseudo metrics belonging to $\mathcal{D}_\Gamma$ are quasi-isometric to each other, their marked length spectra can be uniformly compared among infinite order elements (recall Definition 2.2.3). Therefore, the next definition makes sense, following [CT21].

Definition 4.1.5. Given two pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$, we define its *(positive) dilation* by the formula

$$\mathrm{Dil}(d, d_*) := \sup_{[x]\in \mathbf{conj}'} \frac{\ell_d[x]}{\ell_{d_*}[x]} \in (0, \infty), \tag{4.1}$$

Recall that $\mathbf{conj}' = \mathbf{conj}'_\Gamma$ is the set of all the conjugacy classes of infinite order elements of Γ. The dilation satisfies

$$\mathrm{Dil}(d, d_{**}) \leq \mathrm{Dil}(d, d_*)\,\mathrm{Dil}(d_*, d_{**}) \quad \text{for all } d, d_*, d_{**} \in \mathcal{D}_\Gamma.$$

4.1.2 Manhattan curves

In this subsection we discuss the Manhattan curves, which are primordial tools for our study of pseudo metrics in $\mathcal{D}_\Gamma$. These curves were first introduced by Burger for the displacement functions associated to isometric actions on rank-1 symmetric spaces [Bur93]. In our setting, the main references are the works of Cantrell and Tanaka [CT22; CT21].

Definition 4.1.6. Consider two pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$. The *Manhattan curve* associated to the pair d, d_* is the boundary of the convex set

$$\mathcal{C}^M_{d_*/d} = \left\{ (a, b) \in \mathbb{R}^2 : \sum_{x\in\Gamma} e^{-ad_*(o,x)-bd(o,x)} < \infty \right\}.$$

Convexity of $\mathcal{C}^M_{d_*/d}$ follows from Hölder's inequality. The Manhattan curve for d, d_* can be parameterized using a function $\theta_{d_*/d} : \mathbb{R} \to \mathbb{R}$ which is defined in the following way. For each $t \in \mathbb{R}$ let $\theta_{d_*/d}(t)$ be the critical exponent of

$$s \mapsto \sum_{x\in\Gamma} e^{-td_*(o,x)-sd(o,x)}. \tag{4.2}$$

By abuse of notation, we will also call $\theta_{d_*/d}$ the *Manhattan curve* for d, d_*.

The next result summarizes the main properties of the Manhattan curves for pseudo metrics in $\mathcal{D}_\Gamma$.

Theorem 4.1.7 (Cantrell–Tanaka [CT21])**.** *For $d, d_* \in \mathcal{D}_\Gamma$ we have the following:*

1. *$\theta_{d_*/d}$ is convex, decreasing, and continuously differentiable.*
2. *$\theta_{d_*/d}$ goes through the points $(0, v_d)$ and $(v_{d_*}, 0)$ and is a straight line between these points if and only if d, d_* are roughly similar.*
3. *We have*
$$-\theta'_{d_*/d}(v_{d_*}) \le \frac{v_d}{v_{d_*}} \le -\theta'_{d_*/d}(0),$$
and both equalities occur if and only if d and d_* are roughly similar.
4. *We have*
$$\lim_{t\to-\infty} \frac{\theta(t)}{t} = -\operatorname{Dil}(d_*, d) \quad \text{and} \quad \lim_{t\to\infty} \frac{\theta(t)}{t} = -\operatorname{Dil}(d, d_*)^{-1} \ .$$
5. *d, d_* are roughly similar if and only if they have proportional marked length spectra.*

Remark 4.1.8. In fact, for certain pairs of pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$, the associated Manhattan curve is known to be analytic [CT22]. This is the case for pairs of word metrics or Green metrics.

Manhattan curves can be used to define invariants for pairs of pseudo metrics in $\mathcal{D}_\Gamma$, such as in the next definition.

Definition 4.1.9. Given two pseudo metrics d, d_* in $\mathcal{D}_\Gamma$, the *mean distortion* of d_* over d is the quantity
$$\tau(d_*/d) = \lim_{r\to\infty} \frac{1}{\#\{x \in \Gamma\colon d(o,x) \le r\}} \sum_{d(o,x)\le r} \frac{d_*(o,x)}{r}. \tag{4.3}$$

This limit is well-defined [CT21, Thm. 1.2], finite, and positive. The mean distortion has been considered in the case that d and d_* are word metrics [CF10] and appears in the study of automorphisms of hyperbolic groups as the generic stretching factor [KKS07]. The relation between the mean distortion and the Manhattan curve if given by the identity
$$\tau(d_*/d) = -\theta'(0) \tag{4.4}$$
for all $d, d_* \in \mathcal{D}_\Gamma$. In particular, from item (3) of the theorem above we have the inequality
$$\frac{v_d}{v_{d_*}} \le \tau(d_*/d), \tag{4.5}$$
where the equality occurs if and only if d and d_* are roughly similar. Also, it follows from the definition of the Manhattan curve that $\theta^{-1}_{d_*/d} = \theta_{d/d_*}$ and so by the inverse function theorem we have that
$$-\theta'_{d_*/d}(v_{d_*}) = (-\theta'_{d/d_*}(0))^{-1} = \tau(d/d_*)^{-1}.$$

Example 4.1.10. Given a probability measure λ on Γ, its *entropy* is given by

$$h_\lambda = \lim_{n\to\infty} -\frac{1}{n}\sum_{x\in\Gamma} \lambda^{*n}(x)\cdot\log(\lambda^{*n}(x)) \in [0,\infty],$$

where λ^{*n} denotes the nth convolution of λ and the entropy is well-defined by subadditivity. Similarly, given a left-invariant pseudo metric d on Γ, the *drift* of d with respect to λ is the number

$$l_\lambda = l_\lambda(d) = \lim_{n\to\infty} \frac{1}{n}\sum_{x\in\Gamma} \lambda^{*n}(x)\cdot d(o,x) \in [0,\infty].$$

If $d \in \mathcal{D}_\Gamma$ and λ is an admissible probability measure on Γ with corresponding Green metric d_λ, then both h_λ and l_λ are positive and we have

$$\tau(d/d_\lambda) = l_\lambda/h_\lambda.$$

Since Green metrics for admissible probability measures have exponential growth rate 1 [BHM08, Rmk. 3.2], in this case the inequality (4.5) reduces to the *fundamental inequality* [BHM08, Prop. 3.4]

$$h_\lambda \le l_\lambda(d)v_d.$$

4.2 Hyperbolic distance-like functions

In this section we introduce hyperbolic distance-like functions, a class of functions on $\Gamma\times\Gamma$ that extends the class $\mathcal{D}_\Gamma$. As we will see, these functions appear naturally in a variety of contexts (see also Section 4.7), and most of the results about pseudo metrics belonging to $\mathcal{D}_\Gamma$ can be proven for functions in this class. The main result of this section is Theorem 4.2.14, which recovers the dilation of hyperbolic distance-like functions as an optimal quasi-isometry constant.

4.2.1 The class $\mathcal{D}_\Gamma^{hf}$

We start with the definition of a hyperbolic distance-like function.

Definition 4.2.1. A *hyperbolic distance-like function* on Γ is a function $\psi : \Gamma\times\Gamma \to \mathbb{R}$ satisfying the following:

1. Positivity: $\psi(x,y) \ge 0$ and $\psi(x,x) = 0$ for all $x,y\in\Gamma$.
2. Triangle inequality: $\psi(x,z) \le \psi(x,y)+\psi(y,z)$ for all $x,y,z\in\Gamma$.
3. Γ-invariance: $\psi(gx,gy) = \psi(x,y)$ for all $g,x,y\in\Gamma$.

4. For any $d_0 \in \mathcal{D}_\Gamma$ and $C \geq 0$ there exists $D \geq 0$ such that the following holds: if $x, y, w \in \Gamma$ are such that $(x|y)_{w,d_0} \leq C$, then the *Gromov product* for ψ satisfies

$$(x|y)_{w,\psi} := \frac{(\psi(x,w) + \psi(w,y) - \psi(x,y))}{2} \leq D.$$

Let $\mathcal{D}_\Gamma^{hf}$ denote the set of all the hyperbolic distance-like functions on Γ.

The following lemma is immediate from the definition of hyperbolic distance-like function.

Lemma 4.2.2. *If $\psi, \psi_* \in \mathcal{D}_\Gamma^{hf}$, then $\psi + \psi_* \in \mathcal{D}_\Gamma^{hf}$.*

By Proposition 2.3.11 we have $\mathcal{D}_\Gamma \subset \mathcal{D}_\Gamma^{hf}$. Also, if $S \subset \Gamma$ is a (non-necessarily symmetric) set that is finite and generates Γ as a semigroup, then its (right) word metric

$$d_S(x,y) := |x^{-1}y|_S$$

belongs to $\mathcal{D}_\Gamma^{hf}$ and is quasi-isometric to any pseudo metric in $\mathcal{D}_\Gamma$. More examples of hyperbolic distance-like functions come from Anosov representations, which we briefly recall. Suppose Γ is a (non-necessarily hyperbolic) finitely generated group equipped with a finite generating set S.

Definition 4.2.3. A representation $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is said to be *j-dominated* for $j \in \{1, \ldots, m-1\}$ if there exist constants $C, \mu > 0$ such that

$$\frac{\sigma_j(\rho(x))}{\sigma_{j+1}(\rho(x))} \geq Ce^{\mu|x|_S} \quad \text{for all } x \in \Gamma. \tag{4.6}$$

Here, for $A \in \mathrm{PSL}_m(\mathbb{R})$, $\sigma_1(A) \geq \sigma_2(A) \geq \ldots \geq \sigma_m(A)$ represent the singular values of A. This condition was studied by Bochi, Potrie and Sambarino in [BPS19], where they showed that being 1-dominated is equivalent to being *projective Anosov*, as defined for surface groups by Labourie in [Lab06] and extended to all groups in [GW12]. It is known that for a group to admit a 1-dominated representation, it must be hyperbolic [BPS19, Thm. 3.2]. As we continue we will stop using the term 1-dominated and will instead use projective Anosov. The following result is due to Cantrell and Tanaka.

Lemma 4.2.4 (Cantrell–Tanaka [CT22, Lem. 7.1])**.** *If $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is a projective Anosov representation and $\|\cdot\|$ is a norm on $\mathbb{R}^m$, then the function*

$$\psi_\rho(x,y) := \log \|\rho(x^{-1}y)\|$$

belongs to $\mathcal{D}_\Gamma^{hf}$ and is quasi-isometric to any pseudo metric belonging to $\mathcal{D}_\Gamma$.

Note that a representation $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is j-dominated if and only if its jth exterior power representation $\Lambda^j \rho : \Gamma \to \mathrm{PSL}_{\binom{m}{j}}(\mathbb{R})$ is 1-dominated. Also, by considering the inequality (4.6) for x^{-1} instead of x, we see that ρ is j-dominated if and only if it is $(m-j)$-dominated. Combining these observations with the fact that $\|\Lambda^{m-1} A\| = \|A^{-1}\|$ for all $A \in \mathrm{PSL}_m(\mathbb{R})$ and $\|\cdot\|$ any Euclidean norm on $\mathbb{R}^m$, by Lemma 4.2.4 we deduce the following corollary.

Corollary 4.2.5. *Let $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ be a projective Anosov representation and $\|\cdot\|$ be a norm on $\mathbb{R}^m$. Then*

$$d_\rho(x,y) := \psi_\rho(x,y) + \psi_{\Lambda^{m-1}\rho}(x,y) = \log \|\rho(y^{-1}x)\| + \log \|\rho(x^{-1}y)\|$$

defines a pseudo metric on Γ that belongs to $\mathcal{D}_\Gamma$.

Remark 4.2.6. The hyperbolicity of d_ρ was also proven by Dey and Kapovich [DK22, Cor. 4.8].

Example 4.2.7 (Hitchin representations)**.** Let Γ be a surface group, and fix $m \geq 2$. A representation of Γ into $\mathrm{PSL}_m(\mathbb{R})$ is *m-Fuchsian* if it is the composition of a Fuchsian representation of Γ into $\mathrm{PSL}_2(\mathbb{R})$ and an irreducible representation of $\mathrm{PSL}_2(\mathbb{R})$ into $\mathrm{PSL}_m(\mathbb{R})$. A representation $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is *Hitchin* if it can be continuously deformed to an m-Fuchsian representation. Labourie showed in [Lab06] that Hitchin representations are *Borel Anosov*, meaning that they are j-dominated for all $1 \leq j \leq m-1$.

We will see more examples of hyperbolic distance-like functions in Section 4.7 once we have defined the class $\overline{\mathcal{D}}_\Gamma$, while for the rest of this subsection we discuss some properties of functions in $\mathcal{D}_\Gamma^{hf}$. A key fact is that these functions are roughly geodesic, in the sense that they satisfy condition (2.1) for some $\alpha \geq 0$.

Proposition 4.2.8. *If $\psi \in \mathcal{D}_\Gamma^{hf}$ then ψ is roughly geodesic. Moreover, for every $d_0 \in \mathcal{D}_\Gamma$ and $\alpha_0 \geq 0$ there is some $\alpha \geq 0$ such that if $x = z_0, \dots, z_m = y$ is an (α_0, d_0)-rough geodesic, then we can find a non-decreasing subsequence $0 = i(0) \leq i(1) \leq \cdots \leq i(n) = m$ such that $z_{i(0)}, \dots, z_{i(n)}$ is an (α, ψ)-rough geodesic.*

In particular, we deduce that pseudo metrics in $\mathcal{D}_\Gamma$ are roughly geodesic. Conversely, if d is a roughly geodesic and left-invariant pseudo metric on Γ that is quasi-isometric to pseudo metrics in $\mathcal{D}_\Gamma$, then it is hyperbolic by Corollary 2.3.9. These observations imply the following corollaries.

Corollary 4.2.9. *Every pseudo metric in $\mathcal{D}_\Gamma$ is roughly geodesic. Indeed, if a left-invariant pseudo metric on Γ is quasi-isometric to pseudo metric in $\mathcal{D}_\Gamma$, then it belongs to $\mathcal{D}_\Gamma$ if and only if it is roughly geodesic.*

Corollary 4.2.10. *If $d, d_* \in \mathcal{D}_\Gamma$, then $d + d_* \in \mathcal{D}_\Gamma$.*

The proof of Proposition 4.2.8 requires the following lemma.

Lemma 4.2.11. *Let $a_0, \dots, a_m$ be a sequence of real numbers and $L \geq 0$ be such that $a_0 \leq a_m$ and $|a_i - a_{i+1}| \leq L$ for $0 \leq i < m$. Assume that $[a_0, a_m] \cap \mathbb{Z} = \{k, k+1, \dots, k+n\}$. Then there exists a non-decreasing subsequence $0 = i(0) \leq i(1) \leq \cdots \leq i(n) = m$ such that $\left|(k+j) - a_{i(j)}\right| \leq (L+2)/2$ for all $0 \leq j \leq n$.*

Proof. First, we observe that for $a_0, \dots, a_m$ and L as in the statement, for any $t \in [a_0 - L/2, a_m + L/2]$ there exists $0 \leq j \leq m$ such that $|t - a_j| \leq L/2$. With this in mind, we define $i(0) = 0$, and suppose we have found $i(0) \leq \cdots \leq i(j)$ such that $|(k+s) - a_{i(s)}| \leq (L+1)/2$ for $0 \leq s \leq j$. Assume also that $k+j+1 \leq a_m$. If $a_{i(j)} > (k+j+1) - (L+1)/2$, then we can choose $i(j+1) = i_j$. If $a_{i(j)} \leq (k+j+1) - (L+1)/2$, then by our observation there exists some $i(j) \leq l \leq m$ such that $a_l \in [a_{i(j)} + 1, a_{i(j)} + L + 1]$, and we choose $i(j+1) = l$. Finally, we can assume $i(n) = m$. □

Proof of Proposition 4.2.8. Let $d_0 \in \mathcal{D}_\Gamma$, and let $x = z_0, \dots, z_m = y$ be an (α_0, d_0)-rough geodesic. Let $L := \max\{\psi(o,u) \colon d_0(o,u) \leq 1 + \alpha_0\}$, which is finite because Γ is finitely generated and d_0 is proper. Since ψ is Γ-invariant and satisfies the triangle inequality, the sequence $a_i := \psi(z_0, z_i)$ satisfies the assumptions of Lemma 4.2.11. Therefore, for each integer j between 0 and $\psi(o,y)$ there exists some $x_j := z_{i(j)}$ with $|\psi(x, x_j) - j| \leq (L+2)/2$ and such that $i(j) \leq i(j+1)$ for all j. Let us suppose this sequence is $x = x_0, \dots, x_n = y$, which lies in an (α_0, d)-rough geodesic. From this we obtain $(x|x_j)_{x_i, d_0} \leq 3\alpha_0/2$ for $0 \leq i \leq j \leq n$. But ψ is a hyperbolic distance-like function, implying that $(x|x_j)_{x_i, \psi} \leq D$ for some D independent of the sequence. We conclude $\psi(x_i, x_j) \leq 2D + \psi(x, x_j) - \psi(x, x_i) \leq j - i + 2D + L + 2$ and $\psi(x_i, x_j) \geq \psi(x, x_j) - \psi(x, x_i) \geq j - i - (L+2)$, and $x_0, \dots, x_n$ is an (α, ψ)-rough geodesic with $\alpha := 2D + L + 2$. □

We will also need the fact that any hyperbolic distance-like function can be "approximated" by word metrics. The next lemma is a variation of [Bes+21, Lem. 4.6].

Lemma 4.2.12. *Let $\psi \in \mathcal{D}_\Gamma^{hf}$ be a hyperbolic distance-like function that is α-roughly geodesic. For $n > \alpha + 1$, let $S_n := \{x \in \Gamma : \psi(o,x) \leq n\}$. Then S_n generates Γ as a semigroup and for all $x, y \in \Gamma$ we have*

$$(n - \alpha - 1) d_{S_n}(x,y) - (n-1) \leqslant \psi(x,y) \leq n d_{S_n}(x,y).$$

Remark 4.2.13. We do not require ψ to be quasi-isometric to a pseudo metric in $\mathcal{D}_\Gamma$, so the sets S_n in the lemma above can be infinite. Also, note that if ψ is symmetric, then all the sets S_n are symmetric. Therefore, if $d \in \mathcal{D}_\Gamma$ then all the sets S_n are finite and symmetric.

Proof. Let $n > \alpha + 1$ be such that S_n generates Γ. If $x, y \in \Gamma$ are such that $d_{S_n}(x,y) = k > 0$, then $x^{-1}y = x_1 \cdots x_k$ with $x_i \in S_n$, and hence $\psi(x,y) \leq \psi(o, x_1) + \cdots \psi(o, x_k) \leq nk = n d_{S_n}(x,y)$. This proves the second inequality.

For the first inequality, let $x, y \in \Gamma$ and consider an (α, ψ)-rough geodesic sequence $x = x_0, \dots, x_l = y$, so that $j - i - \alpha \leq \psi(x_i, x_j) \leq j - i + \alpha$ for all $0 \leq i \leq j \leq l$. Let m be the greatest integer with $m + \alpha \leq n$ (note that m is positive), and say $l = mk + r$ with k, r

integers such that $0 \le r < m$. If we define $y_j = x_{m(j-1)}^{-1} x_{mj}$ for $1 \le j \le k$ and $y_{k+1} = x_{mk}^{-1} y$, then $x^{-1}y = y_1 \cdots y_{k+1}$, and each y_j lies in S_n, so that $d_{S_n}(x,y) \le k+1$. This implies

$$\begin{aligned}\psi(x,y) \ge l - \alpha \ge mk - \alpha &\ge (n-1-\alpha)(d_{S_n}(x,y)-1) - \alpha \\ &= (n-1-\alpha)d_{S_n}(x,y) - (n-1).\end{aligned}$$

□

4.2.2 Optimal quasi-isometry constants

In this subsection we introduce the stable translation length of hyperbolic distance-like functions, as well as their dilations, extending Definition 4.1.5. Then we prove Theorem 4.2.14, whose proof relies on the existence of strongly Markov structures on non-elementary hyperbolic groups.

Similarly as in the case of pseudo metrics in $\mathcal{D}_\Gamma$, for hyperbolic distance-like functions $\psi, \psi_* \in \mathcal{D}_\Gamma^{hf}$ we define the *stable translation length* function

$$\ell_\psi[x] := \lim_{n\to\infty} \frac{1}{n}\psi(o, x^n) \text{ for } x \in \Gamma,$$

and the *dilation* of ψ and ψ_*:

$$\mathrm{Dil}(\psi,\psi_*) := \inf\{\lambda > 0 \colon \ell_\psi[x] \le \lambda \ell_{\psi_*}[x] \text{ for all } [x] \in \mathbf{conj}\} \in [0,\infty],$$

where we define $\mathrm{Dil}(\psi,\psi_*) = \infty$ if no such λ exists. Our main result states that for hyperbolic distance-like functions, the dilations are the optimal multiplicative errors for quasi-isometries between them.

Theorem 4.2.14. *Let $\psi, \psi_* \in \mathcal{D}_\Gamma^{hf}$ be such that $\mathrm{Dil}(\psi_*,\psi) < \infty$. Then there exists $C \ge 0$ such that for any $x, y \in \Gamma$*

$$\psi_*(x,y) \le \mathrm{Dil}(\psi_*,\psi)\psi(x,y) + C.$$

As an immediate consequence, we extend the (weak) marked length spectrum rigidity, known for pseudo metrics in $\mathcal{D}_\Gamma$, generalizing item 5 of Theorem 4.1.7.

Corollary 4.2.15. *For $\psi, \psi_* \in \mathcal{D}_\Gamma^{hf}$ the following are equivalent:*

(i) There exists $C \ge 0$ such that $|\psi(o,x) - \psi_*(o,x)| \le C$ for all $x \in \Gamma$.

*(ii) $\ell_\psi[x] = \ell_{\psi_*}[x]$ for all $[x] \in$ **conj**.*

From Theorem 4.2.14 we also obtain corollaries to Anosov representations, when we combine it with Lemma 4.2.4.

Corollary 4.2.16. *Suppose that ρ and ρ_* are two projective Anosov representations (not necessarily of the same dimension). Then there exists $C \ge 0$ such that for every $x \in \Gamma$ we have*

$$\mathrm{Dil}(\psi_\rho,\psi_{\rho_*})^{-1} \log\|\rho(x)\| - C \le \log\|\rho_*(x)\| \le \mathrm{Dil}(\psi_{\rho_*},\psi_\rho)\log\|\rho(x)\| + C.$$

Corollary 4.2.17. *Suppose that $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is projective Anosov and $S \subset \Gamma$ is any (not necessarily symmetric) finite generating set. Then there exists a constant $C \geq 0$ such that for all $x \in \Gamma$ we have*

$$\mathrm{Dil}(d_S, \psi_\rho)^{-1}|x|_S - C \leq \log \|\rho(x)\| \leq \mathrm{Dil}(\psi_\rho, d_S)|x|_S + C.$$

Remark 4.2.18. (1) It may be possible to prove these results using ideas involving the semi-simplification representation discussed by Tsouvalas in [Tso20].
(2) The corollaries above complement the spectral rigidity results of Bridgeman, Canary, Labourie and Sambarino [Bri+15], and Cantrell and Tanaka [CT22].

The rest of the subsection is devoted to the proof of Theorem 4.2.14. To this end, we fix a finite, symmetric generating set $S \subset \Gamma$ with word metric d_S, and let $\ell_S = \ell_{d_S}$ denote the stable translation length function for this metric. Similarly, $(\cdot|\cdot)_S$ denotes the Gromov product for d_S based at the identity o, and $|\cdot|_S$ denotes the word length with respect to S. We start with a lemma.

Lemma 4.2.19. *There exist constants $C', R' \geq 0$ such that for any $x \in \Gamma$ there is some $\gamma_x \in \Gamma$ such that $d_S(x, \gamma_x) \leq R'$ and*

$$(\gamma_x^{-m}|\gamma_x^n)_S \leq C' \quad \text{for all } m, n \geq 0.$$

Proof. Let $\mathcal{A} = (\mathcal{G}, \pi, S)$ be a strongly Markov automatic structure on Γ. If $\mathcal{C}$ is a maximal recurrent component of the transition matrix for $\mathcal{G}$, then by Proposition 2.4.12 there is a finite set $B \subset \Gamma$ such that $B\Gamma_{\mathcal{C}}B = \Gamma$. Since $\mathcal{C}$ is finite and recurrent, there is some N such that every two vertices in $\mathcal{C}$ can be joined by a directed path in $\mathcal{C}$ of length at most N.

Let $x \in \Gamma$, and write $x = s_1 r s_2$ for $s_1, s_2 \in B$ and $r \in \Gamma_{\mathcal{C}}$. Suppose that $r = \pi_*(\omega)$, for ω a path in $\mathcal{C}$ with initial vertex v and final vertex v', and let ω' be a directed path in $\mathcal{C}$ of length at most N from v' to v. Then $\omega\omega'$ is a loop in $\mathcal{C}$, and consider $w = \pi_*(\omega')$.

Define $\gamma_x = s_1 r w {s_1}^{-1}$. Then

$$d_S(x, \gamma_x) = d_S(s_2, w{s_1}^{-1}) \leq N + 2\max_{t \in B} |t|_S =: R',$$

and since $(\omega\omega')^m$ is also a loop in $\mathcal{C}$ for each $m \geq 1$, the word $(rw)^m$ is geodesic (Γ, S) and for $m, n \geq 1$ we get

$$(\gamma_x^{-m}|\gamma_x^n)_S = \frac{|s_1\gamma_x^m s_1^{-1}|_S + |s_1\gamma_x^n s_1^{-1}|_S - |s_1\gamma_x^{m+n}s_1^{-1}|_S}{2} \leq 3|s_1|_S \leq 3\max_{t \in B}|t|_S =: C'. \quad \square$$

Lemma 4.2.20. *Let $\gamma : \Gamma \to \Gamma$, $x \mapsto \gamma_x$ be the assignment from Lemma 4.2.19. Then for any $\psi \in \mathcal{D}_\Gamma^{hf}$ there exist $C_0, R_0 \geq 0$ such that for any $x \in \Gamma$ we have $\max(\psi(x, \gamma_x), \psi(\gamma_x, x)) \leq R_0$ and*

$$(\gamma_x^{-m}|\gamma_x^n)_{o,\psi} \leq C_0$$

for all $m, n \geq 0$. In particular we have $\psi(o, \gamma_x) \leq \ell_\psi[\gamma_x] + 2C_0$ for all $x \in \Gamma$.

Proof. Let C', R' be the constants from Lemma 4.2.19. Then for any $x \in \Gamma$ we have $d_S(x, \gamma_x) \leq R'$ and $(\gamma_x^{-m}|\gamma_x^n)_S \leq C'$ for all $m, n \geq 0$. If ψ is a hyperbolic distance-like function, let $C_0 \geq 0$ (resp. $R_0 \geq 0$) be such that $(x|y)_S \leq C'$ (resp. $(x|y)_S \leq R'$) implies $(x|y)_{o,\psi} \leq C_0$ (resp. $(x|y)_{o,\psi} \leq R_0/2$) for all $p, q \in \Gamma$. Therefore, for all $x \in \Gamma$ we have $\max(\psi(x, \gamma_x), \psi(\gamma_x, x)) \leq 2(x^{-1}\gamma_x|x^{-1}\gamma_x)_{o,\psi} \leq R_0$ and

$$\psi(o, \gamma_x^m) + \psi(o, \gamma_x^n) \leq \psi(o, \gamma_x^{m+n}) + 2C_0 \tag{4.7}$$

for all $m, n \geq 0$, which proves the first assertion of the lemma. For the second assertion, we apply (4.7) to $m = 1$ and obtain $\psi(o, \gamma_x) \leq \psi(o, \gamma_x^{n+1}) - \psi(o, \gamma_x^n) + 2C_0$ for all n. By adding these inequalities for $0 \leq n \leq k$ we get

$$(k+1)\psi(o, \gamma_x) \leq \psi(o, \gamma_x^{k+1}) + 2(k+1)C_0.$$

The proof concludes after dividing by $(k+1)$ and letting k tend to infinity. □

Corollary 4.2.21. *There exists a finite set $B \subset \Gamma$ such that the following holds. Given $\psi \in \mathcal{D}_\Gamma^{hf}$ there exists $C_1 \geq 0$ such that for any $x, y \in \Gamma$ we have*

$$\psi(x, y) \leq \max_{u \in B} \ell_\psi[x^{-1}yu] + C_1.$$

Proof. Let $R' \geq 0$ and $x \mapsto \gamma_x$ be as in Lemma 4.2.19. Suppose they are induced by the generating set $S \subset \Gamma$, and set $B := \{u \in \Gamma \colon d_S(o, u) \leq R'\}$. If $\psi \in \mathcal{D}_\Gamma^{hf}$ and C_0, R_0 are the constants found in Lemma 4.2.20, we set $C_1 := 2C_0 + R_0$. Since for all $x, y \in \Gamma$ we have $yx^{-1}\gamma_{x^{-1}y} \in B$, we deduce

$$\psi(x, y) = \psi(o, x^{-1}y) \leq \psi(o, \gamma_{x^{-1}y}) + R_0 \leq \ell_\psi[\gamma_{x^{-1}y}] + 2C_0 + R_0 \leq \max_{u \in B} \ell_\psi[x^{-1}yu] + C_1. \quad \square$$

Proof of Theorem 4.2.14. Let $B \subset \Gamma$ be the finite set given by Corollary 4.2.21, and let C_1 be the corresponding constant for ψ_*. Then for any $x, y \in \Gamma$ we get

$$\begin{aligned} \psi_*(x, y) &\leq \max_{u \in B} \ell_{\psi_*}[x^{-1}yu] + C_1 \\ &\leq \mathrm{Dil}(\psi_*, \psi) \cdot \max_{u \in B} \ell_\psi[x^{-1}yu] + C_1 \\ &\leq \mathrm{Dil}(\psi_*, \psi) \cdot \psi(x, y) + \left(C_1 + \mathrm{Dil}(\psi_*, \psi) \cdot \max_{u \in B} \psi(o, u) \right), \end{aligned}$$

This concludes the proof since the last term on the right-hand side is finite and independent of x, y. □

4.3 The space of metric structures

In this section we introduce the space of metric structures on Γ, which is the main construction of the chapter. It was previously defined by Furman in [Fur02], and it can be understood as the deformation space for geometric actions of Γ on geodesic spaces. We will present several characterizations and examples of metric structures, and will introduce a natural metric on this space.

4.3.1 The space $\mathscr{D}_\Gamma$

The space $\mathcal{D}_\Gamma$ is already rich enough to parameterize geometric actions of Γ, but in principle, there are infinitely many pseudo metrics in $\mathcal{D}_\Gamma$ with the same asymptotic geometry. If we only want to keep track of the large-scale behavior of these pseudo metrics, it is more convenient to consider equivalence classes under rough similarity. We recall the definition of $\mathscr{D}_\Gamma$ from the Introduction.

Definition 4.3.1. The *space of metric structures* on Γ is the set $\mathscr{D}_\Gamma$ of rough similarity equivalence classes of pseudo metrics in $\mathcal{D}_\Gamma$. The class of $d \in \mathcal{D}_\Gamma$ will be denoted by $[d]$, and points in $\mathscr{D}_\Gamma$ will be called *metric structures*.

Each geometric action of Γ on a geodesic hyperbolic space induces a metric structure. Indeed, if we consider such an action of Γ on (X, d), then all the orbit pseudo metrics d^w_X with $w \in X$ are roughly isometric and belong to $\mathcal{D}_\Gamma$, and hence $\rho_X := [d^w_X] \in \mathscr{D}_\Gamma$ is independent of the choice of w and invariant under rescaling of the metric d on X. In fact, every metric structure can be recovered in this way, as the next lemma shows.

Lemma 4.3.2. *The assignment $(X, d) \mapsto \rho_X = [d^w_X]$ induces a bijection from the set of equivalence classes of geometric actions on Γ under Γ-equivariant rough similarity onto $\mathscr{D}_\Gamma$.*

Proof. We just need to define an inverse to the assignment in the statement. Indeed, if $d \in \mathcal{D}_\Gamma$ we let $i : (\Gamma, d) \to (X_d, \hat{d})$ be the injective hull of (Γ, d) given by Proposition 2.3.7. Since d is hyperbolic and roughly geodesic, we have that the induced isometric action of Γ on $(X_d, \hat{d})$ is geometric, and hence $d = d^o_{X_d}$ for $o = i(o) \in X_d$. By an axiom of choice-like argument, if d and d_* are roughly similar, we can construct a Γ-equivariant map from $(X_d, \hat{d})$ to $(X_{d_*}, \hat{d}_*)$, which must be a rough similarity. □

From the lemma above we can give an alternative definition of $\mathscr{D}_\Gamma$ as the quotient space of the class of all the geometric actions of Γ on geodesic metric spaces, under the equivalence relation of Γ-equivariant rough similarity. However, in practice it is more convenient to see metric structures as classes of pseudo metrics on Γ. By item 5 of Theorem 4.1.7, we can also characterize metric structures in terms of their marked length spectra: two metric structures $\rho = [d], \rho_* = [d_*]$ coincide if and only if ℓ_d and ℓ_{d_*} are proportional. This was first proven by Krat [Kra01, Thm. 1.1] in the case of pseudo metrics induced by geometric actions on proper geodesic spaces. We will recover this result as a consequence of Proposition 4.3.21.

The space of metric structures encodes many interesting deformation spaces for actions of hyperbolic groups, as we will see in the following examples.

Example 4.3.3 (Teichmüller space)**.** Suppose S is a closed orientable surface of negative Euler characteristic and Γ is isomorphic to the fundamental group of S. A *marked negatively curved Riemannian (resp. hyperbolic) metric* is a pair $(\mathfrak{g}, \phi)$, where $\mathfrak{g}$ is a Riemannian metric on S with negative sectional curvatures (resp. constant negative curvature equal to -1) and $\phi : \Gamma \to \pi_1(S)$ is an isomorphism. Two marked negatively curved metrics $(\mathfrak{g}, \phi), (\mathfrak{g}_*, \phi_*)$ are *equivalent* if there exists a homothety $f : (S, \mathfrak{g}) \to (S, \mathfrak{g}_*)$ that is isotopic to the identity and such that $f_* \circ \phi = \phi_*$.

Definition 4.3.4. The *Teichmüller space* of S is the space $\mathscr{T}_S$ of equivalence classes or marked hyperbolic metrics on S.

A marked hyperbolic metric $(\mathfrak{g}, \phi)$ induces a geometric action of Γ on the hyperbolic plane $\mathbb{H}^2$ by orientation-preserving isometries, via Deck transformations on the universal cover of $(S, \mathfrak{g})$ and composing with ϕ. If two marked hyperbolic metrics $(\mathfrak{g}, \phi), (\mathfrak{g}_*, \phi_*)$ are equivalent (related by the isometry f), then the induced geometric actions $\rho, \rho_* : \Gamma \to \mathrm{Isom}(\mathbb{H}^2)$ are Γ-*equivariant*: the lift $\tilde{f}$ of f to the universal cover of S induces an isometry $\overline{f} : \mathbb{H}^2 \to \mathbb{H}^2$ such that $\overline{f}(\rho(x)w) = \rho_*(x)\overline{f}(w)$ for all $w \in \mathbb{H}^2$ and $x \in \Gamma$. Conversely, if such Γ-equivariant isometric $\overline{f}$ exists, then projecting down to S we obtain an isometry $f : (S, \mathfrak{g}) \to (S, \mathfrak{g}_*)$ that induces the identity on $\pi_1(S)$, and hence is isotopic to the identity.

In conclusion, we get a description of $\mathscr{T}_\Gamma$ as the space of equivalence classes of orientation-preserving geometric actions of Γ on $\mathbb{H}^2$, where we mod out by Γ-equivariant isometry. In particular, $\mathscr{T}_S$ only depends on Γ, and hence we will use the notation $\mathscr{T}_\Gamma$ to denote the Teichmüller space of S. Since $\mathbb{H}^2$ is geodesic, from this description we also get a natural map $\mathscr{T}_\Gamma \to \mathscr{D}_\Gamma$. In fact, by the solution of the marked length spectrum rigidity conjecture for negatively curved Riemannian metrics [Cro90; Ota90], two marked negatively curved Riemannian metrics on S are equivalent if and only if their corresponding marked length spectra are homothetic. This implies that $\mathscr{D}_\Gamma$ contains a copy of the space of all marked negatively curved Riemannian metrics on S up to equivalence, and in particular, the map $\mathscr{T}_\Gamma \to \mathscr{D}_\Gamma$ is injective.

Example 4.3.5 (Negatively curved metrics)**.** Generalizing the example above, if Γ is isomorphic to the fundamental group of a closed negatively curved manifold M, we can talk of marked negatively curved Riemannian metrics on M. If $\mathscr{T}_M^{<0}$ denotes the space of the equivalence classes of such metrics, then we get a natural map from $\mathscr{T}_M$ into $\mathscr{D}_\Gamma$. Whether this map is injective is equivalent to a positive solution to the mark length rigidity conjecture, which is known to be true only in dimension 2 by Otal and Croke [Cro90; Ota90].

Example 4.3.6 (Outer space)**.** Let Γ be a free group. As in the case of closed surfaces, we can define a *marked metric graph* as a pair (G, ϕ), where G is a metric graph with each vertex of valence at least 3 and $\phi : \Gamma \to \pi_1(G)$ is an isomorphism. As above, two marked graphs $(G, \phi), (G_*, \phi_*)$ are *equivalent* if there exists a homothety $f : G \to G_*$ such that $f_* \circ \phi = \phi_*$.

Definition 4.3.7 (Culler–Vogtmann [CV86])**.** The *outer space* of Γ is the space $\mathscr{CV}_\Gamma$ of equivalence classes of marked metric graphs.

By a lifting procedure as in the case of surfaces, we can also describe $\mathscr{CV}_\Gamma$ as the space of equivalence classes of geometric and minimal actions of Γ on metric trees, up to Γ-equivariant homothety. In this case we also get a natural injective map from $\mathscr{DV}_\Gamma$ into $\mathscr{D}_\Gamma$ [FM11].

From the examples above, we can see $\mathscr{D}_\Gamma$ as a "thickened" version of Teichmüller and outer spaces for general non-elementary hyperbolic groups. This will be also justified once we define a metric on $\mathscr{D}_\Gamma$ (see Remark 4.3.23).

Example 4.3.8 (Quasi-Fuchsian space)**.** We can also generalize Teichmüller space from the point of view of representation theory. For Γ a surface group, a representation $\rho : \Gamma \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{Isom}^+(\mathbb{H}^3)$ is *quasi-Fuchsian* if for some (any) $w \in \mathbb{H}^3$ the map $x \mapsto \rho(x)w$ defines a quasi-isometric embedding from Γ into $\mathbb{H}^3$. If ρ is quasi-Fuchsian, the induced action of Γ on $\mathbb{H}^3$ is not cobounded, but it has a *convex core*. That is, there exists a Γ-invariant convex subset $X \subset \mathbb{H}^3$ such that the action of Γ on X is geometric. In this way, the representation ρ induces a metric structure on $\mathscr{D}_\Gamma$.

The *quasi-Fuchsian space* of Γ is the space $\mathscr{QF}_\Gamma$ of quasi-Fuchsian representations of Γ, quotiented by conjugation in $\mathrm{PSL}_2(\mathbb{C})$. It is clear that in this way we obtain a natural map from $\mathscr{QF}_\Gamma$ into $\mathscr{D}_\Gamma$ that extends the inclusion of $\mathscr{T}_\Gamma$. By the work of Burger [Bur93], this map is also injective. If Γ is isomorphic to the fundamental group of the closed surface S, Fricker and Furman showed that the intersection in $\mathscr{D}_\Gamma$ of $\mathscr{QF}_\Gamma$ and $\mathscr{T}_\Gamma^{<0}$ is precisely $\mathscr{T}_\Gamma$ [FF22, Thm. A].

Example 4.3.9 (Hitchin components)**.** For Γ a surface group and $m \geq 2$, let $\mathcal{H}_\Gamma^m$ denote the space of Hitchin representations of Γ into $\mathrm{PSL}_m(\mathbb{R})$, up to conjugation in $\mathrm{PSL}_m(\mathbb{R})$. By the Hitchin rigidity of Bridgeman, Canary, Labourie and Sambarino [Bri+15, Cor. 1.5], the assignment $\rho \mapsto [d_\rho]$ given by Corollary 4.2.5 induces an injection from $\mathcal{H}_\Gamma^m$ into $\mathscr{D}_\Gamma$.

In general, it is hard to decide whether two given metric structures are the same. A useful invariant is the notion of arithmeticity.

Definition 4.3.10. A pseudo metric $d \in \mathcal{D}_\Gamma$ has *arithmetic* marked length spectrum if the set $\{\ell_d[x] \colon [x] \in \mathbf{conj}\}$ is contained in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that the spectrum of d is *non-arithmetic*. Similarly, we can talk of a metric structure on $\mathscr{D}_\Gamma$ having arithmetic or non-arithmetic marked length spectrum.

For example, the marked length spectrum of word metrics for finite, symmetric generating sets is always arithmetic [BH99, Thm. III.Γ.3.17], as well as metric structures associated to geometric actions on CAT(0) cube complexes with the combinatorial metric [Hag21]. On the other, Gouëzel, Mathéus and Maucourant proved that if Γ is not virtually free and d is a Green metric for an admissible probability measure on Γ, then d has non-arithmetic marked length spectrum. Their proof shows something stronger: if $d \in \mathcal{D}_\Gamma$ is *strongly hyperbolic* (see e.g. [NŠ16]) and Γ is not virtually free, then d has non-arithmetic marked length spectrum. This is the case for all the orbit pseudo metrics associated to geometric actions on CAT(-1) spaces, so in particular points in $\mathscr{T}_M^{<0}$ and $\mathscr{QF}_\Gamma$ are non-arithmetic.

Producing invariants that detect different metric structures with non-arithmetic marked length spectra is much more difficult. For example, for a surface group Γ it is unknown whether there exists a metric structure induced by a Green metric associated to an admissible probability measure on Γ that belongs to $\mathscr{T}_\Gamma$. Recently, there has been some progress in this question, see [KT22].

4.3.2 Characterizing metric structures from measures

In this subsection we introduce quasiconformal measures and geodesic currents, which can be understood as the (measure-theoretical) traces at infinity of pseudo metrics in $\mathcal{D}_\Gamma$. Indeed, under the expected equivalence relations, these objects completely characterize metric structures. Recall that since Γ is non-elementary, its Gromov boundary $\partial\Gamma$ is an infinite compact metrizable space with a natural topological action of Γ.

Definition 4.3.11. Let $d \in \mathcal{D}_\Gamma$. A Borel probability measure ν on $\partial\Gamma$ is *quasiconformal* for d if $x\nu \ll \nu$ for all $x \in \Gamma$ and there exists a constant $C \geq 1$ such that for every $x \in \Gamma$ and ν-almost every $p \in \partial\Gamma$ we have

$$C^{-1}e^{-v_d\beta(x,o;p)} \leq \frac{dx\nu}{d\nu}(p) \leq Ce^{-v_d\beta(x,o;p)}, \tag{4.8}$$

where $\beta = \beta_d$ is the Busemann function for d (see Subsection 2.3.2).

It is clear that if $d, d_* \in \mathcal{D}_\Gamma$ are roughly similar, then ν is quasiconformal for d if and only if it is quasiconformal for d_*. In [Coo93], Coornaert proved the following for arbitrary $d \in \mathcal{D}_\Gamma$:

- d admits a quasiconformal measure.
- If ν_1, ν_2 are quasiconformal measures for d, then they are *equivalent*, in the sense that there is some $C \geq 1$ such that $C^{-1}\nu_2(A) \leq \nu_1(A) \leq C\nu_2(A)$ for any $A \subset \partial\Gamma$ Borel.
- Every quasiconformal measure ν for d is *ergodic*: if $A \subset \partial\Gamma$ is a Γ-invariant Borel subset, then either $\nu(A) = 0$ or $\nu(A) = 1$.

Moreover, the multiplicative constant in (4.8) can be chosen uniformly in the following sense: for all $\delta \geq 0$ there is some $D_\delta \geq 1$ such that any δ-hyperbolic pseudo metric $d \in \mathcal{D}_\Gamma$ with $v_d = 1$ admits a quasiconformal measure ν such that for all $x \in \Gamma$ and ν-almost every $p \in \partial\Gamma$ we have

$$D_\delta^{-1}e^{-\beta_d(x,o;p)} \leq \frac{dx\nu}{d\nu}(p) \leq D_\delta e^{-\beta_d(x,o;p)}. \tag{4.9}$$

This uniformity is crucial in our proof of Theorem 4.5.1 (see Proposition 4.5.3).

Quasiconformal measures indeed characterize metric structures. That is, two pseudo metrics belonging to $\mathcal{D}_\Gamma$ determine the same metric structure if and only if they have equivalent quasiconformal measures. To show this, let ν be any Borel probability measure on $\partial\Gamma$ that is *quasi-invariant* in the sense that for all $x \in \Gamma$ there is some $C \geq 1$ such that $C^{-1}\nu \leq x\nu \leq C\nu$. Then we can define the pseudo metric d_ν on Γ according to

$$d_\nu(x,y) = \max\left(\log\left|\left|\frac{dx\nu}{dy\nu}\right|\right|_\infty, \log\left|\left|\frac{dy\nu}{dx\nu}\right|\right|_\infty\right).$$

It is immediate that d_ν is left-invariant and that the rough similarity class of d_ν only depends on the measure class of ν. The key property is that if ν is quasiconformal for $d \in \mathcal{D}_\Gamma$, then d_ν is roughly isometric to $v_d d$ [BHM11, Lem. B.6].

Example 4.3.12 (Lebesgue measure)**.** Let Γ be a group acting geometrically on the n-dimensional real hyperbolic space $\mathbb{H}^n$ for $n \geq 2$. This action identifies (up to conjugacy) the boundary $\partial\Gamma$ with the $(n-1)$-dimensional round sphere $\mathbb{S}^{n-1}$, and let $\nu \in \text{Prob}(\partial\Gamma)$ be the pullback of Lebesgue measure under this identification. Then ν is quasiconformal for any orbit pseudo metric for this action.

Example 4.3.13 (Cylinder measures)**.** Let $\Gamma = F(a_1, \ldots, a_k)$ be a free group with free basis $a_1, \ldots, a_k$ and let $S = \{a_1^\pm, \ldots, a_k^\pm\}$ induce the word metric $d = d_S \in \mathcal{D}_\Gamma$. Then $\partial\Gamma$ can be identified with the subset of $S^\mathbb{N}$ consisting of the infinite sequences $(x_1, x_2, x_3, \ldots)$ such that $x_{i+1} \neq x_i^{-1}$ for all i. Given $y_1, \ldots, y_n \in S$ such that $y_{i+1} \neq y_i^{-1}$ for $1 \leq i \leq n-1$, the *cylinder* $C[y_1, \ldots, y_n]$ is the subset of all the sequences $(x_1, x_2, \ldots)$ in $\partial\Gamma$ such that $x_i = y_i$ for $1 \leq i \leq n$. The *cylinder measure* on $\partial\Gamma$ is the unique Borel probability measure ν such that

$$\nu(C[y_1, \ldots, y_n]) = \frac{1}{2k} \cdot \frac{1}{(2k-1)^{n-1}}$$

for all $n \geq 1$ and all the cylinders $C(y_1, \ldots, y_n)$ as above. This measure is quasiconformal for d_S.

Example 4.3.14 (Harmonic measures)**.** Let λ be an admissible probability measure on Γ with corresponding Green metric d_λ. If $(Z_n)_{n\geq 1}$ is the random walk on Γ with transition probabilities λ, then for almost every trajectory the limit $Z_\infty = \lim_n Z_n \in \partial\Gamma$ exists. The *harmonic* measure ν_λ is the hitting measure of Z_∞. That is,

$$\nu_\lambda(A) = \mathbb{P}(Z_\infty \in A)$$

for every Borel set $A \subset \partial\Gamma$. The harmonic measure ν_λ is quasiconformal for the Green metric d_λ [BHM11, Sec. 3.4].

There is another class of measures that are related to pseudo metrics in $\mathcal{D}_\Gamma$. They are defined on the *double boundary* of Γ, which is the set $\partial^2\Gamma$ of ordered pairs of distinct points in $\partial\Gamma$, endowed with the expected topology and the diagonal action of Γ.

Definition 4.3.15. A *geodesic current* on Γ is a locally finite, flip-invariant, and Γ-invariant measure η on $\partial^2\Gamma$, meaning that $\eta(K)$ is finite for any compact subset $K \subset \partial^2\Gamma$. flip-invariance means that η is invariant under the involution $(p, q) \leftrightarrow (p, q)$ on $\partial^2\Gamma$. We let $\mathcal{C}urr(\Gamma)$ denote the space of geodesic currents equipped with the weak-$*$ topology.

That is, a sequence $(\eta_n)_n$ of geodesic currents converges to the geodesic current η if and only if $\int f d\eta_n \to \int f d\eta$ for every compactly supported continuous function $f \in C_c(\partial^2\Gamma)$. We will also consider the quotient space $\mathbb{P}\mathcal{C}urr(\Gamma) := (\mathcal{C}urr(\Gamma)\backslash\{0\})/\mathbb{R}^+$ of *projective geodesic* *currents* endowed with the quotient topology, which makes it into a compact metrizable space [Bon91, Prop. 6].

Geodesic currents were introduced by Bonahon, first for surface groups [Bon88], and later for general hyperbolic groups [Bon91], and were used to give a new interpretation of Thurston's compactification for Teichmüller spaces (see also Example 4.7.10). In the case of negatively curved manifolds, geodesic currents can be understood in terms of invariant measures on the geodesic flow [Kai90, Thm. 2.2]. Let us examine this case in more detail.

Example 4.3.16 (Currents from geodesic flows)**.** Let M be a closed manifold endowed with the Riemannian metric $\mathfrak{g}$. The *geodesic flow* associated to $\mathfrak{g}$ is the continuous flow $\varphi_t : T^1M \to T^1M$ on the unit tangent bundle of M defined as follows: for a unit tangent vector $v \in T^1M$, consider the oriented Riemannian geodesic in $\gamma : \mathbb{R} \to (M, \mathfrak{g})$ such that tangent vector of γ at $\gamma(0)$ is v. Then $\varphi_t(v)$ is defined as the unit vector tangent to γ at $\gamma(t)$.

Let μ be a Borel $(\varphi_t)_t$-invariant probability measure on T^1M. If Γ is isomorphic to $\pi_1(M)$, then given a marking ϕ for $\mathfrak{g}$ we obtain an isometric action of Γ on the universal cover $(\widetilde{M}, \widetilde{\mathfrak{g}})$ of $(M, \mathfrak{g})$, and hence on the unit tangent bundle $T^1\widetilde{M}$. We lift μ to a Γ-equivariant, locally finite measure $\widetilde{\mu}$ on $T^1\widetilde{M}$. The marking also induces a Γ-equivariant homeomorphism between $\partial^2\Gamma$ and the space $\mathcal{G}(\widetilde{M})$ of oriented geodesics on $\widetilde{M}$, since each of these geodesics is uniquely determined by it endpoints in $\partial\widetilde{M}$. In this way we obtain a continuous Γ-equivariant map $\pi : T^1\widetilde{M} \to \partial^2\Gamma$, where we send a unit tangent vector v to the pair (v^-, v^+) determined by the geodesic v is tangent to, with v pointing towards v^+.

It turns out that there exists a unique geodesic current $\eta_\mu \in \mathcal{C}urr(\Gamma)$ such that for every Borel subset $A \subset T^1\widetilde{M}$ we have

$$\widetilde{\mu} = \int_{\pi(A)} \mathrm{Leb}_{(p,q)}(\pi^{-1}((p,q)))d\eta_\mu(p,q),$$

where $\mathrm{Leb}_{(p,q)}$ is the Lebesgue measure along the geodesic in $(\widetilde{M}, \widetilde{\mathfrak{g}})$ determined by (p,q).

The assignment $\mu \mapsto \eta_\mu$ induces a bijection from the space of Borel probability measures on T^1M that are invariant under the geodesic flow and $\mathbb{P}\mathcal{C}urr(\Gamma)$.

The space of geodesic currents can also be thought of as a completion of the space of conjugacy classes of Γ in the following sense: if $[x] \in \mathbf{conj'}$ is the conjugacy class of the non-torsion element $x \in \Gamma$, then $x = y^n$ for some $n \neq 0$ and $y \in \Gamma$ a primitive element. If $y_\infty, y_{-\infty}$ denote the two points in $\partial\Gamma$ that are fixed by y, then the set $\mathcal{A}_{[y]} = \{(gy_{\mp\infty}, gy_{\pm\infty}) \colon g \in \Gamma\}$ is a discrete, Γ-invariant subset of $\partial^2\Gamma$. In this way, the *rational* current associated to $[y]$ is given by

$$\eta_{[y]} = \sum_{(p,q) \in \mathcal{A}_{[y]}} \delta_{(p,q)},$$

(here δ denotes the Dirac measure) and similarly we define $\eta_{[x]} = |n|\eta_{[y]}$. The set $\{\lambda\eta_{[x]} \colon \lambda > 0, [x] \in \mathbf{conj'}\}$ turns out to be dense in $\mathcal{C}urr(\Gamma)$ [Bon91, Thm. 7].

Given $\rho \in \mathscr{D}_\Gamma$, we can construct a projective geodesic current $BM(\rho) \in \mathbb{P}\mathcal{C}urr(\Gamma)$ as follows. If $\rho = [d]$ and ν is a quasiconformal measure for d, there exists a geodesic current η in the same measure class of $\nu \otimes \nu$, and any other geodesic current in this measure class is a positive multiple of η.

Indeed, if d is δ-hyperbolic and has exponential growth rate 1 and ν satisfies (4.9) with respect to d, then there is a constant $R = R_\delta$ depending only on δ and D_δ and a geodesic current η satisfying

$$R^{-1} \int_A e^{2(p|q)_{o,d}} d\nu(p) d\nu(q) \leq \eta(A) \leq R \int_A e^{2(p|q)_{o,d}} d\nu(p) d\nu(q) \tag{4.10}$$

for any Borel subset $A \subset \partial^2\Gamma$, see e.g. [Fur02, Sec. 3]. This induces a projective geodesic current, which turns out to be independent of the choices of d and ν (see [Fur02, Prop. 3.1]).

Definition 4.3.17. Given $\rho = [d] \in \mathscr{D}_\Gamma$ with $v_d = 1$, the *Bowen-Margulis current* is the unique projective geodesic current $BM(\rho) \in \mathbb{P}\mathcal{C}urr(\Gamma)$ represented by a geodesic current η satisfying condition (4.10) above for some $R \geq 0$ and every Borel subset $A \subset \partial^2\Gamma$.

Example 4.3.18 (Measures maximizing the entropy)**.** Let $(\phi, \mathfrak{g})$ be a marked negatively curved Riemannian metric on the closed manifold M with fundamental group isomorphic to Γ. The geodesic flow on T^1M is *Anosov* [Ano69], and hence it has a unique flow-invariant probability measure maximizing the entropy, that we call μ_{BM}. Under the correspondence explained Example 4.3.16, the measure μ_{BM} is assigned to the Bowen-Margulis current of the metric structure induced by the marking $(\mathfrak{g}, \phi)$ [Kai90, Sec. 3.5].

In [Fur02, Thm. 4.1], Furman proved that the Bowen-Margulis map $BM : \mathscr{D}_\Gamma \to \mathbb{P}\mathcal{C}urr(\Gamma)$ is injective for any non-elementary hyperbolic group Γ. This can be seen as a characterization of metric structures in terms of geodesic currents.

4.3.3 The symmetrized Lipschitz metric

In this subsection we give a topology to the space $\mathscr{D}_\Gamma$ by recalling the (symmetrized) Lipschitz metric Δ on $\mathscr{D}_\Gamma$ that we defined in the Introduction. This metric quantifies how far are two quasi-isometric pseudo metrics from being roughly similar.

Definition 4.3.19. Given $\rho = [d], \rho_* = [d_*] \in \mathscr{D}_\Gamma$, we define

$$\Lambda(\rho, \rho_*) := \inf\{\lambda_1\lambda_2 \colon \exists A \geq 0 \text{ s.t. } \frac{1}{\lambda_1} d - A \leq d_* \leq \lambda_2 d + A\}, \tag{4.11}$$

and

$$\Delta(\rho, \rho_*) := \log \Lambda(\rho, \rho_*).$$

The quantities Λ and Δ are well-defined, and if $\Delta(\rho, \rho_*) = 0$ then d and d_* have proportional marked length spectra, and hence $\rho = \rho_*$ by item 5 of Theorem 4.1.7. We can easily verify that Δ satisfies all the other axioms of a metric.

Corollary 4.3.20. *Δ defines a metric on $\mathscr{D}_\Gamma$.*

A key property of the metric Δ is that the optimal quasi-isometry constants λ_1, λ_2 in (4.11) are given by the dilations of d and d_*. This is the content of the next proposition, which is a direct consequence of Theorem 4.2.14.

Proposition 4.3.21. *For any $d, d_* \in \mathcal{D}_\Gamma$ there exists $C \geq 0$ such that*

$$\operatorname{Dil}(d, d_*)^{-1}(x|y)_{o,d} - C \leq (x|y)_{o,d_*} \leq \operatorname{Dil}(d_*, d)(x|y)_{o,d} + C \tag{4.12}$$

for all $x, y \in \Gamma$. In particular, for all $x, y \in \Gamma$ we have

$$\operatorname{Dil}(d, d_*)^{-1}d(x, y) - C \leq d_*(x, y) \leq \operatorname{Dil}(d_*, d)d(x, y) + C. \tag{4.13}$$

Proof. Let $d, d_* \in \mathcal{D}_\Gamma \subset \mathcal{D}_\Gamma^{hf}$, so that they are hyperbolic distance-like functions satisfying $\operatorname{Dil}(d, d_*), \operatorname{Dil}(d_*, d) < \infty$. Then by Theorem 4.2.14 there is $A \geq 0$ such that

$$\operatorname{Dil}(d, d_*)^{-1}d(x, y) - A \leq d_*(x, y) \leq \operatorname{Dil}(d_*, d)d(x, y) + A$$

for all $x, y \in \Gamma$. This proves (4.13), and then (4.12) follows from Proposition 2.3.11. □

Corollary 4.3.22. *For any $d, d_* \in \mathcal{D}_\Gamma$ we have*

$$\Lambda([d], [d_*]) = \operatorname{Dil}(d, d_*) \operatorname{Dil}(d_*, d).$$

Proof. By Proposition 4.3.21 we have $\Lambda([d], [d_*]) \leq \operatorname{Dil}(d, d_*) \operatorname{Dil}(d_*, d)$, and the converse follows since $d_* \leq \lambda d + A$ for $d, d_* \in \mathcal{D}_\Gamma$ and $\lambda, A > 0$ implies $\operatorname{Dil}(d_*, d) \leq \lambda$. □

Remark 4.3.23. From the corollary above we deduce that for Γ a surface group, the restriction of Δ to $\mathscr{T}_\Gamma$ coincides with the (symmetrized) Thurston's metric [Thu98], so that the inclusion $\mathscr{T}_\Gamma \to \mathscr{D}_\Gamma$ is a continuous embedding. By the work of Francaviglia and Martino [FM11], the analogous continuity result holds for a free group Γ, so that the injection $\mathscr{CV}_\Gamma \to \mathscr{D}_\Gamma$ is also continuous. In the setting of arbitrary hyperbolic groups, this metric Δ also appeared in [FF22].

In some cases we will deal with pseudo metrics with exponential growth rate 1. For these pseudo metrics, the logarithm of the dilation is always non-negative.

Lemma 4.3.24. *For all $d, d_* \in \mathcal{D}_\Gamma$, if $v_d = v_{d_*} = 1$ then $\operatorname{Dil}(d, d_*) \geq 1$.*

Proof. By Proposition 4.3.21 we can find $C \geq 0$ such that $d(o, x) \leq \operatorname{Dil}(d, d_*)d_*(o, x) + C$ for any $x \in \Gamma$. In particular we have

$$\{x \in \Gamma : d_*(o, x) \leq n\} \subset \{x \in \Gamma : d(o, x) \leq \operatorname{Dil}(d, d_*)n + C\}$$

for all $n \geq 0$, implying that

$$\begin{aligned} 1 = v_{d_*} &= \lim_{n\to\infty} \frac{\log \#\{x \in \Gamma : d_*(o,x) \leq n\}}{n} \\ &\leq \lim_{n\to\infty} \frac{\log \#\{x \in \Gamma : d(o,x) \leq \mathrm{Dil}(d,d_*)n + C\}}{n} \\ &\leq \mathrm{Dil}(d,d_*) \lim_{n\to\infty} \frac{\log \#\{x \in \Gamma : d(o,x) \leq n\}}{n} = \mathrm{Dil}(d,d_*). \end{aligned}$$ □

The previous lemma motivates the following definition.

Definition 4.3.25. For $\rho \in \mathscr{D}_\Gamma$, we set $\hat{\rho} := \{d \in \rho : v_d = 1\}$.

Remark 4.3.26. Given $\rho, \rho_* \in \mathscr{D}_\Gamma$ one can define

$$\Delta^+(\rho, \rho_*) = \log \mathrm{Dil}(d, d_*),$$

where $d \in \hat{\rho}$ and $d_* \in \hat{\rho}_*$. Lemma 4.3.24 then tells us that Δ^+ is non-negative, and by Theorem 4.1.7 we get that it is an asymmetric distance on $\mathscr{D}_\Gamma$ (i.e. $\Delta^+(\rho, \rho_*) = 0$ implies $\rho = \rho_*$). In the case Γ is a surface group and $\rho, \rho_* \in \mathscr{T}_\Gamma \subset \mathscr{D}_\Gamma$, since cocompact lattices in $\mathbb{H}^2$ have exponential growth rate 1, from (4.1) we get that $\Delta^+(\rho, \rho_*)$ is *equal* to the asymmetric Thurston's distance of ρ and ρ_* [Thu98]. The content of Lemma 4.3.22 is then that Δ is twice the symmetrization of Δ^+.

4.4 Geometry and topology of $\mathscr{D}_\Gamma$

In this section we study metric and topological properties of the metric space $(\mathscr{D}_\Gamma, \Delta)$, in analogy with Teichmüller and outer spaces. We prove that $(\mathscr{D}_\Gamma, \Delta)$ is unbounded and separable in Subsection 4.4.1, geodesic and contractible in Subsection 4.4.2, and has a natural geodesic bicombing in Subsection 4.4.3. We also discuss the subspaces $\mathscr{D}_\Gamma^\delta$ and $\mathscr{D}_\Gamma^{\delta,\alpha}$ in Subsection 4.4.4, and study the isometric action of $\mathrm{Out}(\Gamma)$ on $\mathscr{D}_\Gamma$ in Subsection 4.4.5. Recall that Γ is always a non-elementary hyperbolic group.

4.4.1 $\mathscr{D}_\Gamma$ is unbounded and separable

We start by proving that $(\mathscr{D}_\Gamma, \Delta)$ is an unbounded metric space. In particular, $\mathscr{D}_\Gamma$ is always infinite.

Proposition 4.4.1. *The metric space $(\mathscr{D}_\Gamma, \Delta)$ is unbounded.*

Proof. Given $x \in \Gamma$ a non-torsion element, it is enough to construct a sequence $(\rho_n)_n \subset \mathscr{D}_\Gamma$ and $d_n \in \hat{\rho}_n$ such that $\ell_{d_n}[x] \to 0$ as $n \to \infty$. To do this, fix S a finite, symmetric generating set for Γ, and for $x \in \Gamma$ as above consider the sequence of sets $S_n = S \cup \{x^n, x^{-n}\}$ and define $\rho_n = [d_{S_n}]$. If $d_n = v_{S_n} d_{S_n}$ for all n, we have $v_{d_n} = 1$ and

$$\ell_{d_n}[x] = v_{S_n} \ell_{d_{S_n}}[x] \leq \frac{v_{S_n}|x^n|_{S_n}}{n} \leq \log(\#S+1)/n,$$

and hence $\ell_{d_n}[x]$ tends to 0 and $\Delta(\rho_n, \rho_1)$ tends to infinity as n tends to infinity. □

Remark 4.4.2. Unboundedness of $\mathscr{D}_\Gamma$ can also be deduced from Theorem 4.4.4.

Also, by Lemma 4.2.12 we deduce that any point in $\mathscr{D}_\Gamma$ can be approximated by a sequence of metric structures induced by word metrics.

Proposition 4.4.3. *The set $\{[d_S] \in \mathscr{D}_\Gamma : S \subset \Gamma$ is finite, symmetric, generating$\}$ is dense in $(\mathscr{D}_\Gamma, \Delta)$. In particular, $(\mathscr{D}_\Gamma, \Delta)$ is separable.*

Proof. Let $d \in \mathcal{D}_\Gamma$ be α-roughly geodesic, and for $n > \alpha+1$ let $S_n = \{x \in \Gamma : d(o,x) \leq n\}$. Lemma 4.2.12 implies that $\mathrm{Dil}(d, d_{S_n}) \leq n$ and $\mathrm{Dil}(d_{S_n}, d) \leq \frac{1}{n-\alpha-1}$, so that $\Delta([d],[d_{S_n}]) \leq \log\left(\frac{n}{n-\alpha-1}\right)$, which tends to 0 as n tends to infinity. □

4.4.2 $\mathscr{D}_\Gamma$ is geodesic

In this subsection we apply Proposition 4.3.21 to prove that $(\mathscr{D}_\Gamma, \Delta)$ is a geodesic metric space. Indeed, every pair of distinct metric structures lie in a *bi-infinite* geodesic. The following is the main result of the subsection.

Theorem 4.4.4. *For any pair $d, d_* \in \mathcal{D}_\Gamma$ such that $[d] \neq [d_*] \in \mathscr{D}_\Gamma$, there exists a continuous, injective map $\rho_\bullet = \rho_\bullet^{d_*/d} : \mathbb{R} \to \mathscr{D}_\Gamma$ satisfying:*

(i) $\rho_0 = [d]$ *and* $\rho_{v_{d_*}} = [d_*]$ *where v_{d_*} is the exponential growth rate of d_*;*

(ii) $\Delta(\rho_r, \rho_t) = \Delta(\rho_r, \rho_s) + \Delta(\rho_s, \rho_t)$ *for all $r < s < t$; and,*

(iii) $\lim_{t\to\infty} \Delta(\rho_t, \rho_{v_{d_*}}) = \lim_{t\to-\infty} \Delta(\rho_t, \rho_0) = \infty$.

In particular, we get another proof that $\mathscr{D}_\Gamma$ is unbounded. The result above seems surprising when we contrast it with the case of outer space, which is not geodesic for the symmetrized Lipschitz metric [FM11, Sec. 6].

Throughout this subsection we fix two pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$, which we assume are *not* roughly similar, and let $\theta = \theta_{d_*/d} : \mathbb{R} \to \mathbb{R}$ be the Manhattan curve for d, d_*. The core of the theorem above is the following proposition.

Proposition 4.4.5. *For any $t \in \mathbb{R}$ there exists a pseudo metric $d_t = d_t^\theta \in \mathcal{D}_\Gamma$ and some $C_t \geq 0$ such that*

$$|d_t - (td_* + \theta(t)d)| \leq C_t.$$

The constants C_t can be chosen so that

$$C_t = \begin{cases} 0 & \text{if} \quad 0 \leq t \leq v_{d_*}, \\ -\theta(t)C & \text{if} \quad t > v_{d_*}, \\ -tC & \text{if} \quad t < 0, \end{cases}$$

where C is a constant depending only on d and d_*.

Remark 4.4.6. (1) By the definition of θ we see that $v_{d_t} = 1$ for all $t \in \mathbb{R}$.
(2) For any $t \in \mathbb{R}$, the quasiconformal measures $\mu_{t,\theta(t)}$ from [CT21, Cor. 2.10] are actually quasiconformal for d_t in the sense of Definition 4.3.11.

For the proof of Proposition 4.4.5 we will use the notation $D_{d_*,d} = \mathrm{Dil}(d_*, d), D_{d,d_*} = \mathrm{Dil}\,(d, d_*)$, and for $t \in \mathbb{R}$ we define $\widehat{d_t} := td_* + \theta(t)d$.

Lemma 4.4.7. *If $t > v_{d_*}$, then $\theta(t)D_{d,d_*} + t > 0$.*

Proof. Consider the function $g(r) = -\theta(r)/r$ on $(0, \infty)$. Since $g(r)$ tends to D_{d,d_*}^{-1} as $r \to \infty$ by Theorem 4.1.7 (4), it is enough to show that g is strictly increasing. As θ is differentiable, we can compute

$$g'(r) = \frac{-\theta'(r) \cdot r + \theta(r)}{r^2},$$

which gives $g'(r) > 0$ for all $0 < r \leq v_{d_*}$ since $\theta'(r) < 0 \leq \theta(r)$. Also, θ is strictly convex and decreasing, so for $0 < r < s$ we have

$$\theta'(r)(s - r) < \theta(s) - \theta(r) < \theta'(s)(s - r),$$

implying that

$$-s^2 g'(s) = \theta'(s)s - \theta(s) > \theta'(s)r - \theta(r) > \theta'(r)r - \theta(r) = -r^2 g'(r). \tag{4.14}$$

As we already checked $g'(r) > 0$ for $0 < r \leq v_{d_*}$, by (4.14) we deduce that g is increasing, concluding the proof of the lemma. □

Proof of Proposition 4.4.5. There are three cases to consider.

Case 1) If $0 \leq t \leq v_{d_*}$, then $\widehat{d_t} \in \mathcal{D}_\Gamma$ by Corollary 4.2.10, so we take $d_t = \widehat{d_t}$ and $C_t = 0$.

Case 2) If $t > v_{d_*}$. We use the notation $(\cdot|\cdot) = (\cdot|\cdot)_{o,d}$, $(\cdot|\cdot)_* = (\cdot|\cdot)_{o,d_*}$, and $\widehat{(\cdot|\cdot)}_t = t(\cdot|\cdot)_* + \theta(t)(\cdot|\cdot)$. By Proposition 4.3.21 there is a constant $C \geq 0$ such that

$$(x|y) \leq D_{d,d_*}(x|y)_* + C$$

for all $x, y \in \Gamma$. Therefore, for all $x, y \in \Gamma$ we have

$$\begin{aligned} \widehat{(x|y)}_t &= t(x|y)_* + \theta(t)(x|y) \\ &= [t + D_{d,d_*}\theta(t)]\,(x|y)_* - \theta(t)\,[D_{d,d_*}(x|y)_* - (x|y)] \\ &\geq [t + D_{d,d_*}\theta(t)]\,(x|y)_* + \theta(t)C. \end{aligned} \tag{4.15}$$

Since $\widehat{d_t}$ is Γ-invariant, by (4.15) and Lemma 4.4.7, the function

$$d_t(x,y) := \begin{cases} \widehat{d_t}(x,y) - 2\theta(t)C & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

is a Γ-invariant pseudo metric on Γ. By (4.15) and Lemma 4.4.7 we also get that d_t is quasi-isometric to d_*, and hence to any word metric.

Finally, if $(\cdot|\cdot)_t$ denotes the Gromov product for d_t based at the identity element, then for all $x, y \in \Gamma$ we have

$$(x|y)_t \leq \widehat{(x|y)}_t + |\theta(t)|C \leq t(x|y)_* + |\theta(t)|C,$$

and hence d_t is roughly geodesic and hyperbolic by Proposition 4.2.8 and Corollary 2.3.9, concluding that $d_t \in \mathcal{D}_\Gamma$. From the definition of d_t, we see that $C_t = -2\theta(t)C$ works.

Case 3) If $t < 0$, let ψ be the Manhattan curve for d_*, d. Then $\psi = \theta^{-1}, t = \psi(s)$ for some $s > v_d$, and $\widehat{d_t} = \widehat{d}_t^\theta = \widehat{d}_s^\psi$, so the conclusion follows from Case 2. □

Given $t \in \mathbb{R}$, let $\rho_t = \rho_t^\theta = [d_t] \in \mathscr{D}_\Gamma$ be the metric structure induced by the pseudo metric d_t from Proposition 4.4.5. Let $\rho = [d] = \rho_0$ and $\rho_* = [d_*] = \rho_{v_{d_*}}$.

For the rest of the subsection we will use the notation $D_{s,t} = \mathrm{Dil}\,(d_s, d_t)$ and $\Delta_{s,t} = \Delta\,(\rho_s, \rho_t)$ for $s, t \in \mathbb{R}$, so that $\Delta_{s,t} = \log\,(D_{s,t}D_{t,s})$. We will also use the notation $v = v_d$ and $v_* = v_{d_*}$, so that $d_0 = vd$ and $d_{v_*} = v_* d_*$. Theorem 4.4.4 then follows from the following estimates.

Proposition 4.4.8. *For any $t \in \mathbb{R}$ we have*

$$D_{0,t} = \begin{cases} v\left(tD_{d_*,d} + \theta(t)\right)^{-1} & \text{if } t < 0 \\ v\left(tD_{d,d_*}^{-1} + \theta(t)\right)^{-1} & \text{if } t > 0 \end{cases}$$

$$D_{t,0} = \begin{cases} v^{-1}(tD_{d,d_*}^{-1} + \theta(t)) & \text{if } t < 0 \\ v^{-1}(tD_{d_*,d} + \theta(t)) & \text{if } t > 0 \end{cases}$$

$$D_{v_*,t} = \begin{cases} v_*\left(\theta(t)D_{d_*,d}^{-1} + t\right)^{-1} & \text{if } t < 0 \\ v_*\left(\theta(t)D_{d,d_*} + t\right)^{-1} & \text{if } t > 0 \end{cases}$$

$$D_{t,v_*} = \begin{cases} {v_*}^{-1}(\theta(t)D_{d,d_*} + t) & \text{if } t < 0 \\ {v_*}^{-1}(\theta(t)D_{d_*,d}^{-1} + t) & \text{if } t > 0 \end{cases}$$

and hence

$$e^{\Delta_{t,0}} = \begin{cases} \left(\dfrac{tD_{d,d_*}^{-1} + \theta(t)}{tD_{d_*,d} + \theta(t)}\right) & \text{if } \ t < 0 \\ \left(\dfrac{tD_{d_*,d} + \theta(t)}{tD_{d,d_*}^{-1} + \theta(t)}\right) & \text{if } \ t > 0 \end{cases} \qquad e^{\Delta_{t,v_*}} = \begin{cases} \left(\dfrac{\theta(t)D_{d,d_*} + t}{\theta(t)D_{d_*,d}^{-1} + t}\right) & \text{if } \ t < v_* \\ \left(\dfrac{\theta(t)D_{d_*,d}^{-1} + t}{\theta(t)D_{d,d_*} + t}\right) & \text{if } \ t > v_*. \end{cases}$$

We begin the proof of Proposition 4.4.8 with some lemmas. For two functions f, g on a set X, the notation $f \lesssim g$ means that there is some $C \geq 0$ such that $f(x) \leq g(x) + C$ for all $x \in X$. We also write $f \approx g$ if $f \lesssim g$ and $g \lesssim f$.

Lemma 4.4.9. *If $t > 0$, then*

(i) $D_{0,t} = v\left(tD_{d,d_*}^{-1} + \theta(t)\right)^{-1}$, *and*

(ii) $D_{t,0} = v^{-1}(tD_{d_*,d} + \theta(t))$.

If $t < v_*$, then

(iii) $D_{t,v_*} = {v_*}^{-1}(\theta(t)D_{d,d_*} + t)$, *and*

(iv) $D_{v_*,t} = v_*\left(\theta(t)D_{d_*,d}^{-1} + t\right)^{-1}$.

Proof. We have $d \lesssim D_{d,d_*}d_*$, and hence

$$d_t \approx td_* + \theta(t)d \gtrsim \left(tD_{d,d_*}^{-1} + \theta(t)\right)d = v^{-1}\left(tD_{d,d_*}^{-1} + \theta(t)\right)d_0.$$

By Lemma 4.4.7, $tD_{d,d_*}^{-1} + \theta(t) \geq 0$, and so $D_{0,t} \leq v\left(tD_{d,d_*}^{-1} + \theta(t)\right)^{-1}$. The reverse inequality of (i) is similar. From $d_0 \leq D_{0,t}d_t$ we get

$$d = v^{-1}d_0 \lesssim v^{-1}[D_{0,t}td_* + D_{0,t}\theta(t)d]$$

and hence

$$\left(1 - v^{-1}D_{0,t}\theta(t)\right)d \lesssim v^{-1}D_{0,t}td_*. \tag{4.16}$$

The left hand side of (4.16) is positive for $t \geq v_*$, and for $0 < t < v_*$ we have

$$v^{-1}\left(1 + \frac{t}{\theta(t)}D_{d,d_*}^{-1}\right)d_0 = \left(1 + \frac{t}{\theta(t)}D_{d,d_*}^{-1}\right)d \lesssim d + \frac{t}{\theta(t)}d_* \approx \theta(t)^{-1}d_t.$$

We deduce

$$D_{0,t} \leq v\theta(t)^{-1}\left(1 + \frac{t}{\theta(t)}D_{d,d_*}^{-1}\right)^{-1} < v\theta(t)^{-1}$$

and the left hand side of (4.16) is positive for any $t > 0$. This gives

$$D_{d,d_*} \leq v^{-1}D_{0,t}t\left(1 - v^{-1}D_{0,t}\theta(t)\right)^{-1}$$

or equivalently $D_{0,t} \geq v\left(tD_{d,d_*}^{-1} + \theta(t)\right)^{-1}$.

We can prove (ii) in the same way, and identities (iii) and (iv) follow from (i) and (ii) applied to $\psi = \theta^{-1}$, and noting that $D_{v_*,t} = D^{\theta}_{v_*,t} = D^{\psi}_{0,s}$ and $D_{t,v_*} = D^{\psi}_{s,0}$, for $s = \theta(t)$. □

Lemma 4.4.10. *If $r < s < t$, then*

$$D_{r,t} = D_{r,s} \cdot D_{s,t} \text{ and } D_{t,r} = D_{t,s} \cdot D_{s,r}.$$

Proof. For the case when $r = 0$ and $t = v_*$, the conclusion follows easily from Lemma 4.4.9. For the general case, let ψ be the Manhattan curve for d_r, d_t, such that

$$d_a^\psi \eqsim ad_t + \psi(a)d_r \eqsim (at + \psi(a)r)\, d_* + (a\theta(t) + \psi(a)\theta(r))d$$

for every $a \in \mathbb{R}$. In particular, ψ satisfies

$$\theta(at + \psi(a)r) = a\theta(t) + \psi(a)\theta(r)$$

for every $a \in \mathbb{R}$. Note that $d_0^\psi \eqsim d_r$ and $d_1^\psi \eqsim d_t$, and that $\lambda(a) := at + \psi(a)r$ is an increasing bijection on $\mathbb{R}$ such that $d_a^\psi \eqsim d_{\lambda(a)}^\theta$ for all a. Since $v_{d_t} = 1$, the general case then follows from the first case applied to ψ and the value $0 < \tilde{s} < 1$ satisfying $\lambda(\tilde{s}) = s$. □

Lemma 4.4.11. *If $t < 0$, then*

(i) $D_{0,t} = v\,(tD_{d_*,d} + \theta(t))^{-1}$, *and*

(ii) $D_{t,0} = v^{-1}(tD_{d,d_*}^{-1} + \theta(t))$.

Also, if $t > v_*$, then

(iii) $D_{t,v_*} = v_*^{-1}(\theta(t)D_{d_*,d}^{-1} + t)$, *and*

(iv) $D_{v_*,t} = v_*\,(\theta(t)D_{d,d_*} + t)^{-1}$.

Proof. From Lemmas 4.4.9 and 4.4.10, for $t < 0$ we have

$$D_{0,t} = D_{v_*,t}/D_{v_*,0} = v\,(tD_{d_*,d} + \theta(t))^{-1} \quad \text{and} \quad D_{t,0} = D_{t,v_*}/D_{0,v_*} = v^{-1}(tD_{d,d_*}^{-1} + \theta(t)).$$

Identities (iii) and (iv) are deduced in an analogous way. □

Proof of Proposition 4.4.8. Lemmas 4.4.9 and 4.4.11 imply the result, since from them we can already verify the formulas for $\Delta_{t,0}$ and Δ_{t,v_*}. □

Proof of Theorem 4.4.4. For each $t \in \mathbb{R}$, let $\rho_t = [d_t]$ as above, for which statement (i) holds by definition and statement (ii) follows from Lemma 4.4.10. For statement (iii) we compute

$$\lim_{t\to\infty} e^{\Delta_{t,v_*}} = \lim_{t\to\infty}\left(\frac{\theta(t)D_{d_*,d}^{-1} + t}{\theta(t)D_{d,d_*} + t}\right) = \lim_{t\to\infty}\left(\frac{1 + \frac{\theta(t)}{t}D_{d_*,d}^{-1}}{1 + \frac{\theta(t)}{t}D_{d,d_*}}\right) = \lim_{t\to\infty}\left(\frac{1 - D_{d,d_*}^{-1}\cdot D_{d_*,d}^{-1}}{1 - D_{d,d_*}^{-1}\cdot D_{d,d_*}}\right) = \infty,$$

where we used $\lim_{t\to\infty}\frac{\theta(t)}{t} = -D_{d,d_*}^{-1}$ by Theorem 4.1.7 (4). Similarly $\lim_{t\to-\infty}\Delta_{t,0} = \infty$.

Finally, note that $\Delta_{0,t}$ and $\Delta_{v_*,t}$ are continuous functions on t, so that $\lim_{s\to t}\Delta_{s,t} = 0$ for any t. Since $[d] \neq [d_*]$, we have $D_{d,d_*}D_{d_*,d} > 1$ and from Proposition 4.4.8 we deduce that $\Delta_{s,t} > 0$ for $s \neq t$, and hence $\rho_\bullet^\theta$ is continuous and injective. □

Remark 4.4.12. From Proposition 4.4.8 we deduce that

$$0 < tD_{d_*,d} + \theta(t) \leq v \quad \text{for } t < 0, \qquad \text{and} \qquad 0 < tD_{d,d_*}^{-1} + \theta(t) \leq v_* D_{d,d_*}^{-1} \quad \text{for } t > v_*.$$

Therefore,

$$\theta(t) = -tD_{d_*,d} + O(1) \quad \text{and} \quad \theta(-t) = tD_{d,d_*}^{-1} + O(1) \quad \text{as } t \to -\infty,$$

which generalizes [CT21, Prop. 4.22] to arbitrary pairs of pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$.

From the proof of Theorem 4.4.4 we can deduce that $\mathscr{D}_\Gamma$ is contractible. This fact combined with Propositions 4.4.1 and 4.4.3 imply Theorem 1.2.4 from the Introduction.

Proposition 4.4.13. *$(\mathscr{D}_\Gamma, \Delta)$ is contractible.*

Proof. Fix $\rho_0 \in \mathscr{D}_\Gamma$ and $d_0 \in \hat{\rho}_0$. We will construct a map $H : \mathscr{D}_\Gamma \times [0,1] \to \mathscr{D}_\Gamma$ which is constant at $t = 0$ and the identity at $t = 1$ as follows. For each $\rho \in \mathscr{D}_\Gamma$ consider some $d \in \hat{\rho}$, and given $t \in [0,1]$, let $\rho_t := [td + (1-t)d_0]$. By Corollary 4.2.10, $d_t \in \mathcal{D}_\Gamma$ and its rough isometry class is independent of the choice of d_0 and d, so that ρ_t is well-defined. It follows that $H(\rho, 0) = \rho_0$ and $H(\rho, 1) = \rho$ for all ρ, and in the same way as we deduced Proposition 4.4.8, we can verify that H is continuous. □

Remark 4.4.14. In general, $(\mathscr{D}_\Gamma, \Delta)$ is not uniquely geodesic. For instance, let $\Gamma = F_3$ be the rank-3 free group with free basis a_1, a_2, a_3 and let $S = \{a_1^{\pm 1}, a_2^{\pm 1}, a_3^{\pm 1}\}$. Given a triplet of positive numbers $\overline{\lambda} = (\lambda_1, \lambda_2, \lambda_3)$, let $|\cdot|_{\overline{\lambda}} : \Gamma \to \mathbb{R}$ be given by

$$|x|_{\overline{\lambda}} := \inf \left\{ \sum_{j=1}^{n} \lambda_{i_j} : x = a_{i_1}^{\varepsilon_1} \cdots a_{i_n}^{\varepsilon_n}, \varepsilon_j \in \{\pm 1\}, 1 \leq i_j \leq 3 \right\}$$

if $x \neq o$ and $|o|_{\overline{\lambda}} := 0$. Then $d_{\overline{\lambda}}(x,y) := |x^{-1}y|_{\overline{\lambda}}$ defines a pseudo metric on Γ belonging to $\mathcal{D}_\Gamma$. If $\overline{\lambda}' = (\lambda_1', \lambda_2', \lambda_3')$ is another triplet of positive numbers, we can verify that

$$\mathrm{Dil}(d_{\overline{\lambda}}, d_{\overline{\lambda}'}) = \max \left(\frac{\lambda_1}{\lambda_1'}, \frac{\lambda_2}{\lambda_2'}, \frac{\lambda_3}{\lambda_3'} \right),$$

so in particular we have

$$\Delta([d_{(1,1,1)}], [d_{(1,2,3)}]) = \Delta([d_{(1,1,1)}], [d_{(1,a,2)}]) + \Delta([d_{(1,a,2)}], [d_{(1,2,3)}])$$

for all $4/3 \leq a \leq 2$. The only value a for which $[d_{(1,a,2)}]$ lies on the geodesic for $[d_{(1,1,1)}], [d_{(1,2,3)}]$ given by Theorem 4.4.4 is $a = 3/2$, and for any other such value a we can construct a geodesic from $[d_{(1,1,1)}]$ to $[d_{(1,2,3)}]$ passing through $[d_{(1,a,2)}]$ (for example, we can concatenate the geodesic segments from $[d_{(1,1,1)}]$ to $[d_{(1,a,2)}]$ and from $[d_{(1,a,2)}]$ to $[d_{(1,2,3)}]$ given by Theorem 4.4.4).

4.4.3 The geodesic bicombing

By appropriately reparametrizing the curves $\rho_\bullet^{d_*/d}$ given by Theorem 4.4.4 we can produce a geodesic bicombing on $\mathscr{D}_\Gamma$ consisting of bi-infinite geodesics.

Definition 4.4.15. For two distinct metric structures $\rho = [d], \rho_* = [d_*]$ in $\mathscr{D}_\Gamma$, the *Manhattan geodesic* of the pair ρ, ρ_* is the map $\sigma_\bullet^{\rho_*/\rho} : \mathbb{R} \to \mathscr{D}_\Gamma$ given by the arc-length reparametrization of the map $\rho_\bullet^{d_*/d}$ such that $\sigma_0^{\rho_*/\rho} = \rho$ and $\sigma_{\Delta(\rho,\rho_*)}^{\rho_*/\rho} = \rho_*$.

More precisely, if $\rho = [d]$ and $\rho_* = [d_*]$, then $\sigma_t^{\rho_*/\rho}$ equals $\rho_{\gamma(t)}^{d_*/d}$, where $\gamma(t)$ is the unique number such that

$$\Delta(\rho, \rho_{\gamma(t)}) = t \quad \text{and} \quad t \cdot \gamma(t) \geq 0. \tag{4.17}$$

Manhattan geodesics are well-defined, since for ρ and ρ_* as in the preceding definition, the image $\rho^{d_*/d}(\mathbb{R}) \subset \mathscr{D}_\Gamma$ and the orientation of the curve $\rho_\bullet^{d_*/d}$ do not depend on the representatives d and d_*. The next theorem summarizes some properties of the bicombing given by the Manhattan geodesics.

Theorem 4.4.16. *The geodesic bicombing $(\rho, \rho_*) \mapsto \sigma_\bullet^{\rho_*/\rho}$ satisfies the following.*

- Continuity*: if $\rho^n \to \rho$, $\rho_*^n \to \rho_*$ and $\rho \neq \rho_*$ in $\mathscr{D}_\Gamma$, then $\sigma_\bullet^{\rho_*^n/\rho^n}$ converges to $\sigma_\bullet^{\rho_*/\rho}$ uniformly on compact subsets of $\mathbb{R}$.*
- Consistency*: if $\rho \neq \rho_*$ and $\tau = \sigma_s^{\rho_*/\rho}, \tau_* = \sigma_{s_*}^{\rho_*/\rho}$ for $s \neq s_*$, then*

$$\sigma_t^{\tau_*/\tau} = \sigma_{T(s,s_*,t)}^{\rho_*/\rho} \quad \textit{where} \quad T(s, s_*, t) = t\left(\frac{s_* - s}{\Delta(\tau, \tau_*)}\right) + s \quad \textit{for each } t \in \mathbb{R}.$$

Remark 4.4.17. The geodesic bicombing above also satisfies Out(Γ)-*invariance*, see Remark 4.4.28. Therefore, Theorems 4.4.4 and 4.4.16 imply Theorem 1.2.9 from the Introduction.

Proof. To prove continuity, consider sequences $\rho^n = [d^n]$ and $\rho_*^n = [d_*^n]$ in $\mathscr{D}_\Gamma$ converging to $\rho = [d]$ and $\rho_* = [d_*]$ as n tends to infinity, respectively. We can assume that d, d_*, d^n, and d_*^n have exponential growth rates equal to 1 for all n. Under this assumption, if we let $\theta_n = \theta_{d_*^n/d^n}$ and $\theta = \theta_{d_*/d}$ then θ_n converges to θ uniformly on compact subsets of $\mathbb{R}$ (see also the proof of Theorem 4.5.6. From this we deduce that if $\rho_\bullet^n = \rho_\bullet^{d_*^n/d^n}$, then $\rho_\bullet^n$ converges to $\rho_\bullet$ uniformly on compact subsets of $\mathbb{R}$. Continuity follows from this property and (4.17).

Finally, consistency follows from the fact that if $\rho = [d] \neq \rho_* = [d_*] \in \mathscr{D}_\Gamma$ and $\tau = \tau_* \in \sigma^{\rho_*/\rho}(\mathbb{R})$, then $\sigma^{\tau_*/\tau}(\mathbb{R}) = \sigma^{\rho_*/\rho}(\mathbb{R})$. To prove this fact, say $\tau = [\overline{d}] = \rho_s^{d_*/d}$ and $\tau_* = [\overline{d}_*] = \rho_{s_*}^{d_*/d}$ for some $s \neq s_*$. Then

$$h(\overline{d})\overline{d} \eqsim sd_* + \theta(s)d \quad \text{and} \quad h(\overline{d}_*)\overline{d}_* \eqsim s_*d_* + \theta(s_*)d,$$

and if $\overline{\theta} = \theta_{\overline{d}_*/\overline{d}}$, then for all $t \in \mathbb{R}$ we get

$$\begin{aligned} t\overline{d}_* + \overline{\theta}(t)\overline{d} &\eqsim th(\overline{d}_*)^{-1}(sd_* + \theta(s)d) + \theta(t)h(\overline{d})^{-1}(s_*d_* + \theta(s_*)d) \\ &= [th(\overline{d}_*)^{-1}s_* + \overline{\theta}(t)h(\overline{d})^{-1}s]d_* + [th(\overline{d}_*)^{-1}\theta(s_*) + \overline{\theta}(t)h(\overline{d})^{-1}\theta(s)]d. \end{aligned}$$

This implies the identity $\overline{\theta}(\alpha(t)) = \beta(t)$, for $\alpha(t) = th(\overline{d}_*)^{-1}s_* + \overline{\theta}(t)h(\overline{d})^{-1}s$ and $\beta(t) = th(\overline{d}_*)^{-1}\theta(s_*) + \overline{\theta}(t)h(\overline{d})^{-1}\theta(s)$. Since $s \neq s_*$, α and β are bijections on $\mathbb{R}$, such that $\rho_t^{\overline{d}_*/\overline{d}} = \rho_{\alpha(t)}^{d_*/d}$ for all t. This concludes the proof of the fact, and hence the theorem. □

4.4.4 The subspaces $\mathscr{D}_\Gamma^\delta$ and $\mathscr{D}_\Gamma^{\delta,\alpha}$

In this subsection we study the (family of) subspaces $(\mathscr{D}_\Gamma^\delta)_\delta$ and $(\mathscr{D}_\Gamma^{\delta,\alpha})_{\delta,\alpha}$ of $\mathscr{D}_\Gamma$. The main result of the section is Theorem 4.4.22, which implies Theorem 1.2.6 from the Introduction. We first recall the definition of these subspaces.

Definition 4.4.18.
- Given $\delta \geq 0$, let $\mathscr{D}_\Gamma^\delta$ be the space of all metric structures $\rho = [d]$ such that d is a δ-hyperbolic pseudo metric with exponential growth rate 1.
- Given $\delta, \alpha \geq 0$, let $\mathscr{D}_\Gamma^{\delta,\alpha}$ be the space of all metric structures $\rho = [d]$ such that d is a δ-hyperbolic and α-roughly geodesic pseudo metric with exponential growth rate 1.

Example 4.4.19. If Γ is a free group, then $\mathscr{D}_\Gamma^0$ is exactly the outer space $\mathscr{CV}_\Gamma$. Indeed, if $d \in \mathcal{D}_\Gamma$ is 0-hyperbolic, then the injective hull $(\Gamma, d) \overset{i}{\hookrightarrow} (X, \hat{d})$ is a complete $\mathbb{R}$-tree. If $T \subset X$ is the union of the axes of all the non-trivial elements of Γ, then T is also an $\mathbb{R}$-tree, with each vertex of finite valence since d is proper. The action of Γ on T is then geometric and minimal, and hence $[d] = \rho_T \in \mathscr{CV}_\Gamma$. Conversely, we have that $\mathscr{D}_\Gamma^0$ is non-empty if and only if Γ is virtually free.

Example 4.4.20. If Γ is a surface group, then the exponential growth rate of any geometric action of Γ on $\mathbb{H}^2$ is 1. Since $\mathbb{H}^2$ is $\log 2$-hyperbolic [NŠ16, Cor. 5.4] we deduce that $\mathscr{T}_\Gamma$ is contained in $\mathscr{D}_\Gamma^{\log 2}$. Similarly, the exponential growth rate of any quasi-Fuchsian representation of Γ equals the Hausdorff dimension of its limit set in the 2-sphere with its standard metric (see e.g. [Sul79, Thm. 7]). As $\mathbb{H}^3$ is also $\log 2$-hyperbolic, we deduce that $\mathscr{QF}_\Gamma \subset \mathscr{D}_\Gamma^{\log 4}$.

Example 4.4.21 (Bounded subsets of negatively curved metric structures)**.** Let M be a closed manifold with fundamental group isomorphic to Γ and let $\mathcal{B}$ be a bounded set of negatively curved Riemannian metrics on M in the following sense: there exists $C \geq 1$ and $\mathfrak{g}_0 \in \mathcal{B}$ such that

$$C^{-1}\mathfrak{g}_0(v,v) \leq \mathfrak{g}(v,v) \leq \mathfrak{g}_0(v,v)C$$

for every tangent vector $v \in TM$ and $\mathfrak{g} \in \mathcal{B}$. If $\widetilde{\mathfrak{g}}$ is the lifting of $\mathfrak{g}$ to the universal cover $\widetilde{M}$ of M, then we have

$$C^{-1}d_{\widetilde{\mathfrak{g}}_0} \leq d_{\widetilde{\mathfrak{g}}} \leq Cd_{\widetilde{\mathfrak{g}}_0} \tag{4.18}$$

for every $\mathfrak{g} \in \mathcal{B}$, where $d_{\tilde{\mathfrak{g}}}$ denotes the induced length distance on $\widetilde{M}$. In particular, the diameter of $(M, \mathfrak{g})$ is uniformly bounded as a function of $\mathfrak{g} \in \mathcal{B}$. In addition, if $d_{\tilde{\mathfrak{g}}_0}$ is δ_0-hyperbolic, then Corollary 2.3.9 implies that $d_{\tilde{\mathfrak{g}}}$ is $\hat{\delta}$-hyperbolic for all $\mathfrak{g} \in \mathcal{B}$, for $\hat{\delta}$ depending only on δ_0 and C.

Now, fix any marking ϕ of Γ, which induces an isometric action on $\widetilde{M}$. By (4.18) we deduce that the exponential growth rate of the action of Γ on $(\widetilde{M}, d_{\tilde{\mathfrak{g}}})$ is uniformly bounded for $\mathfrak{g} \in \mathcal{B}$. We conclude that there exist $\delta, \alpha \geq 0$ such that the set of metric structures on Γ induced by the marked metrics $(\mathfrak{g}, \phi)$ with $\mathfrak{g} \in \mathcal{B}$ is contained in $\mathscr{D}_\Gamma^{\delta,\alpha}$.

The following is the main result of the subsection, which by Examples 4.4.19 and 4.4.20 generalizes the properness of Teichmüller, quasi-Fuchsian, and outer spaces. Recall that a metric space is *proper* if its closed balls are compact.

Theorem 4.4.22. *For any $\delta, \alpha \geq 0$ we have that:*

(i) $\mathscr{D}_\Gamma^{\delta}$ is either empty or a proper subspace of $(\mathscr{D}_\Gamma, \Delta)$, and;

(ii) $\mathscr{D}_\Gamma^{\delta,\alpha}$ is either empty or proper.

In particular, $\mathscr{D}_\Gamma^{\delta}$ and $\mathscr{D}_\Gamma^{\delta,\alpha}$ are always complete subspaces of $\mathscr{D}_\Gamma$. The rest of the subsection is devoted to the proof of this theorem, where a key ingredient is Theorem 2.3.15. Our first lemma will be used to find good representatives for metric structures.

Lemma 4.4.23. *Let $S \subset \Gamma$ be a finite, symmetric generating set, and $B \subset \mathscr{D}_\Gamma^{\delta,\alpha}$ a bounded subset. Then there exists some $C \geq 1$ depending only on B and S, such that for every $\rho \in B$ there exists a δ-hyperbolic and α-roughly geodesic pseudo metric $d_\rho \in \hat{\rho}$ satisfying*

$$C^{-1} d_S(x,y) - C \leq d_\rho(x,y) \leq C d_S(x,y) \tag{4.19}$$

for all $x, y \in \Gamma$.

Proof. Since B is bounded, there exists some $\Lambda \geq 1$ such that for all $\rho \in B$, $d \in \hat{\rho}$ and $[x] \in \mathbf{conj}$, we have

$$\Lambda^{-1} \ell_S[x] \leq \ell_d[x] \leq \Lambda \ell_S[x], \tag{4.20}$$

where $\ell_S = \ell_{d_S}$. Now, for each $\rho \in B$, let $d'_\rho \in \hat{\rho}$ be a δ-hyperbolic and α-roughly geodesic pseudo metric. We want to modify these pseudo metrics so that they satisfy (4.19) for some C independent of ρ. To do this, for any $\rho \in B$ we apply Theorem 2.3.15 to the finite set S and the pseudo metric d'_ρ to find a point $z_\rho \in \Gamma$ such that

$$\max_{s \in S} d'_\rho(sz_\rho, z_\rho) \leq \frac{1}{2} \max_{s_1, s_2 \in S} \ell_{d'_\rho}[s_1 s_2] + K\delta + 4\alpha + 1 \leq \frac{\Lambda}{2} \max_{s_1, s_2 \in S} \ell_S[s_1 s_2] + K\delta + 4\alpha + 1, \tag{4.21}$$

where in the last inequality we used (4.20). We define C_1 as the last term in (4.21), which is independent of $\rho \in B$. In this way, the pseudo metric $d_\rho(x,y) := d'_\rho(xz_\rho, yz_\rho)$ is also left-invariant, δ-hyperbolic and α-roughly geodesic, and for each $x, y \in \Gamma$ we have

$$d_\rho(x,y) \leq C_1 d_S(x,y). \tag{4.22}$$

For the other inequality in (4.19), we apply Corollary 4.2.21 to $\psi = d_S$ to find a finite set $U \subset \Gamma$ and a constant $C_2 \geq 0$ such that

$$d_S(x,y) \leq \max_{u \in U} \ell_S[x^{-1}yu] + C_2$$

for all $x, y \in \Gamma$. This last inequality together with (4.20) and (4.22) imply that for $\rho \in B$ and all $x, y \in \Gamma$

$$\begin{aligned} d_\rho(x,y) &\geq \max_{u\in U} d_\rho(x^{-1}yu, o) - \max_{u \in U} d_\rho(u,o) \\ &\geq \max_{u\in U} \ell_{d_\rho}[x^{-1}yu] - C_1 \max_{u\in U} d_S(u,o) \\ &= \max_{u\in U} \ell_{d'_\rho}[x^{-1}yu] - C_1 \max_{u\in U} d_S(u,o) \\ &\geq \Lambda^{-1} \max_{u\in U} \ell_S[x^{-1}yu] - C_1 \max_{u\in U} d_S(u,o) \\ &\geq \Lambda^{-1}(d_S(x,y) - C_2) - C_1 \max_{u\in U} d_S(u,o) \\ &= \Lambda^{-1} d_S(x,y) - \left(\Lambda^{-1}C_2 + C_1 \max_{u\in U} d_S(u,o)\right), \end{aligned}$$

and inequality (4.19) holds for all ρ in B, with $C = \max(C_1, \Lambda, \Lambda^{-1}C_2 + C_1 \max_{u\in U} d_S(u,o))$. □

Now we begin the proof of Theorem 4.4.22. Our first step is to prove completeness of $\mathscr{D}_\Gamma^{\delta,\alpha}$.

Proposition 4.4.24. *For all $\delta, \alpha \geq 0$ the set $\mathscr{D}_\Gamma^{\delta,\alpha}$ is either empty, or a complete subspace of $\mathscr{D}_\Gamma$.*

Proof. Assume that $\mathscr{D}_\Gamma^{\delta,\alpha}$ is non-empty and let $(\rho_n)_n \subset \mathscr{D}_\Gamma^{\delta,\alpha}$ be a Cauchy sequence of metric structures, for which we expect it to converge to some $\rho_\infty \in \mathscr{D}_\Gamma^{\delta,\alpha}$.

Let $S \subset \Gamma$ be a finite, symmetric generating set. The sequence ρ_n is bounded, and hence by Lemma 4.4.23 there exists a constant $C \geq 1$ independent of n, and δ-hyperbolic and α-roughly geodesic pseudo metrics $d_n \in \hat{\rho}_n$ such that for each $x, y \in \Gamma$ we have

$$C^{-1} d_S(x,y) - C \leq d_n(x,y) \leq C d_S(x,y). \tag{4.23}$$

This last inequality allows us to find a subsequence $(n_k)_k$ such that for each $x, y \in \Gamma$, the sequence $d_{n_k}(x,y)$ has a limit, so that for $x, y \in \Gamma$ we define $d_\infty(x,y) := \lim_k d_{n_k}(x,y)$. To conclude the result, there are some claims to verify:

Claim 1: $d_\infty \in \mathcal{D}_\Gamma$.

Clearly, d_∞ is a left-invariant pseudo metric on Γ. Since each d_n is δ-hyperbolic, d_∞ is also δ-hyperbolic, and it is quasi-isometric to d_S as a consequence of (4.23), so that $d_\infty \in \mathcal{D}_\Gamma$.

Claim 2: The sequence ρ_n converges to $\rho_\infty := [d_\infty]$.

We first prove that $\ell_{d_{n_k}}[x]$ converges to $\ell_{d_\infty}[x]$ for all $[x] \in \mathbf{conj}$. Let $x \in \Gamma$, for which we have

$$\ell_{d_\infty}[x] = \lim_{m\to\infty} \frac{d_\infty(o,x^m)}{m} = \lim_{m\to\infty} \frac{\lim_k d_{n_k}(o,x^m)}{m} \geq \limsup_k \ell_{d_{n_k}}[x],$$

where in the last inequality we used that $\ell_d[x] = \inf_{m\geq 1} \frac{d(o,x^m)}{m}$ for all $d \in \mathcal{D}_\Gamma$ and $x \in \Gamma$. For the inequality $\ell_{d_\infty}[x] \leq \liminf_k \ell_{d_{n_k}[x]}$ we use [Ore18, Thm. 1.1], which implies that

$$\ell_d[x] = \sup_{m\geq 1}\left(\frac{d(o,x^{2m}) - d(o,x^m) - 2\delta}{m}\right)$$

for each δ-hyperbolic pseudo metric $d \in \mathcal{D}_\Gamma$. From this we get

$$\begin{aligned}\ell_{d_\infty}[x] &= \lim_{m\to\infty}\left(\frac{d_\infty(o,x^{2m}) - d_\infty(o,x^m) - 2\delta}{m}\right)\\ &= \lim_{m\to\infty}\left(\frac{\lim_k (d_{n_k}(o,x^{2m}) - d_{n_k}(o,x^m) - 2\delta)}{m}\right) \leq \liminf_k \ell_{d_{n_k}}[x].\end{aligned}$$

This convergence implies that for all n and $[x] \in \mathbf{conj}$

$$(\limsup_{k\to\infty} \mathrm{Dil}(d_n, d_{n_k}))^{-1}\ell_{d_n}[x] \leq \ell_{d_\infty}[x] \leq \limsup_{k\to\infty} \mathrm{Dil}(d_{n_k}, d_n)\ell_{d_n}[x]. \tag{4.24}$$

As each d_n has exponential growth rate 1, by Proposition 4.3.22 and Lemma 4.3.24 we have that

$$\max(\mathrm{Dil}(d_n, d_m), \mathrm{Dil}(d_m, d_n)) \leq \Lambda(\rho_m, \rho_n) \tag{4.25}$$

for all m, n. Also, since $(\rho_n)_n$ is Cauchy, the sequence $\Lambda_n := \limsup_{k\to\infty} \Lambda(\rho_{n_k}, \rho_n)$ tends to 1, and hence ρ_n converges to ρ_∞.

Claim 3: $\rho_\infty \in \mathscr{D}_\Gamma^{\delta,\alpha}$.

We already proved that d_∞ is δ-hyperbolic, so we now show that $v_{d_\infty} = 1$. Indeed, for each $d \in \mathcal{D}_\Gamma$, the quantity v_d is also the critical exponent of

$$b \mapsto \sum_{[x]\in\mathbf{conj}'} e^{-b\ell_d[x]},$$

(see Equation (4.2), the case $t = 0$). Applying this to d_∞ and each d_n, and using (4.24) and (4.25), we get that

$$\Lambda_n^{-1} v_{d_n} \leq v_{d_\infty} \leq \Lambda_n v_{d_n}$$

for all n. Since $v_{d_n} = 1$ for each n and Λ_n tends to 1, we deduce $v_{d_\infty} = 1$.

We are only left to show that d_∞ is α-roughly geodesic. To do this, fix $x \in \Gamma$, and for each k, let $o = x_{0,k}, x_{1,k}, \ldots, x_{r_k,k} = x$ be an (α, d_{n_k})-rough geodesic joining o and x. By (4.23) we have $r_k \leq \alpha + d_{n_k}(o,x) \leq \alpha + C|x|_S$, so that the sequence $(r_k)_k$ is bounded, and after replacing n_k by a further subsequence and reindexing, we can assume that $r_k = r$ for all k. From this, it is enough to prove that for all $0 \leq j \leq r$, the sequence $(x_{j,k})_k$ is bounded in d_∞, since in that case, and after extracting a subsequence, the constant sequence of points $x_j = x_{j,k}$ will define an (α, d_∞)-rough geodesic joining o and x.

To prove this, note that for $0 \leq j \leq r$ and any k we have $d_{n_k}(o, x_{j,k}) \leq j + \alpha$, so it is enough to show that for all $y \in \Gamma$ and k we have

$$d_\infty(o, y) \leq A' d_{n_k}(o, y) + B',$$

for some constants A', B' independent of y and k. For this we apply Corollary 4.2.21 to $\psi = d_\infty$ to get a finite set $V \subset \Gamma$ and a constant $E \geq 0$ such that $d_\infty(o, y) \leq \max_{v \in V} \ell_{d_\infty}[vy] + E$ for all y. Therefore, by this and (4.23) we obtain

$$\begin{aligned} d_\infty(o, y) &\leq \max_{v \in V} \ell_{d_\infty}[yv] + E \\ &\leq \Lambda(\rho_\infty, \rho_{n_k}) \max_{v \in V} \ell_{d_{n_k}}[yv] + E \\ &\leq \Lambda(\rho_\infty, \rho_{n_k}) d_{n_k}(o, y) + E + \Lambda(\rho_\infty, \rho_{n_k}) \max_{v \in V} d_{n_k}(o, v) \\ &\leq \Lambda(\rho_\infty, \rho_{n_k}) d_{n_k}(o, y) + E + C\Lambda(\rho_\infty, \rho_{n_k}) \max_{v \in V} |v|_S, \end{aligned}$$

and the conclusion follows since the sequence $k \mapsto \Lambda(\rho_\infty, \rho_{n_k})$ is bounded. This completes the proof of Claim 3 and the proposition. □

Proposition 4.4.25. *For each $\delta, \alpha \geq 0$, the set $\mathscr{D}_\Gamma^{\delta,\alpha}$ is either empty or a proper subspace of $\mathscr{D}_\Gamma$.*

Proof. Assume $\mathscr{D}_\Gamma^{\delta,\alpha}$ is non-empty. Then it is complete by Proposition 4.4.24, so it is enough to show that for any bounded set $B \subset \mathscr{D}_\Gamma^{\delta,\alpha}$ and any $\varepsilon > 0$, B can be covered by finitely many subsets of diameter at most ε.

Let $S \subset \Gamma$ be a finite, symmetric generating set. As B is bounded and contained in $\mathscr{D}_\Gamma^{\delta,\alpha}$, by Lemma 4.4.23 we can find some $C \geq 1$ such that for every $\rho \in B$ there is some δ-hyperbolic and α-roughly geodesic pseudo metric $d_\rho \in \hat{\rho}$ satisfying

$$C^{-1} d_S(x, y) - C \leq d_\rho(x, y) \leq C d_S(x, y) \tag{4.26}$$

for all $x, y \in \Gamma$.

Now, for every $\rho \in B$ and $n \geq 0$, let $S_{n,\rho} := \{x \in \Gamma : d_\rho(o, x) \leq n\}$. By (4.26), for every $\rho \in B$ we have $S \subset S_{C,\rho}$, so that $S_{n,\rho}$ generates Γ for all $n \geq C$. Therefore, by Lemma 4.2.12 we have that for all $\rho \in B$, $n \geq \max(C, \alpha + 2)$ and $x \in \Gamma$:

$$(n - 1 - \alpha) d_{S_{n,\rho}}(o, x) - (n - 1) \leq d_\rho(o, x) \leq n d_{S_{n_\rho}}(o, x).$$

Given $\varepsilon > 0$, let $n_0 \geq \max(C, \alpha + 2)$ be such that $\frac{n_0}{n_0 - 1 - \alpha} < e^{\varepsilon/2}$, and let $\mathcal{A} \subset \mathscr{D}_\Gamma$ be the set of all the metric structures $[d_T]$, with T a finite generating set contained in $\{x \in \Gamma : |x|_S \leq Cn_0 + C^2\}$. Note that $\mathcal{A}$ is finite.

By (4.26) we obtain that $[d_{S_{n_0,\rho}}] \in \mathcal{A}$ for every $\rho \in B$, and that $\Delta(\rho, [d_{S_{n_0,\rho}}]) \leq \log(\frac{n_0}{n_0 - 1 - \alpha}) < \varepsilon/2$, so that B is covered by the finite collection of balls $\{\tau \in \mathscr{D}_\Gamma : \Delta(\sigma, \tau) \leq \varepsilon/2\}$ with $\sigma \in \mathcal{A}$. □

To finish the proof of Theorem 4.4.22, we are left to show that $\mathscr{D}_\Gamma^\delta$ is either empty or proper for any $\delta \geq 0$. In virtue of Proposition 4.4.25, it is enough to prove that bounded subsets of $\mathscr{D}_\Gamma^\delta$ are contained in $\mathscr{D}_\Gamma^{\delta,\alpha}$ for some $\alpha \geq 0$. The following lemma is an adaptation of Lemma 4.4.23 for bounded subsets of $\mathscr{D}_\Gamma^\delta$.

Lemma 4.4.26. *Let $S \subset \Gamma$ be a finite, symmetric generating set, and $B \subset \mathscr{D}_\Gamma^\delta$ a bounded subset. Then there exists some $C \geq 1$ depending only on B and S, such that for every $\rho \in B$ there exists a δ-hyperbolic pseudo metric $d_\rho \in \hat{\rho}$ satisfying*

$$C^{-1} d_S(x,y) - C \leq d_\rho(x,y) \leq C d_S(x,y)$$

for all $x, y \in \Gamma$.

Sketch of proof. For each $\rho \in B$, let $d'_\rho \in \hat{\rho}$ be a δ-hyperbolic pseudo metric, and by Proposition 2.3.7 consider its injective hull $i_\rho : (\Gamma, d'_\rho) \hookrightarrow (X_\rho, \hat{d}_\rho)$ which is δ-hyperbolic and geodesic. By this same proposition, Γ also acts isometrically on $(X_\rho, \hat{d}_\rho)$, and the isometric map i_ρ is Γ-equivariant.

As in the proof of Lemma 4.4.23, we can use that B is bounded and apply Theorem 2.3.15 to the set S and each $(X_\rho, \hat{d}_\rho)$, to find a constant $C \geq 1$ such that for any $\rho \in B$ there is a point $z_\rho \in X_\rho$ satisfying

$$C^{-1} d_S(x,y) - C \leq \hat{d}_\rho(x z_\rho, y z_\rho) \leq C d_S(x,y)$$

for all $x, y \in \Gamma$. The proof is completed by taking $d_\rho(x,y) := \hat{d}_\rho(x z_\rho, y z_\rho)$ for $x, y \in \Gamma$ and $\rho \in B$, details are left to the reader. □

Proposition 4.4.27. *For any $\delta \geq 0$, if $B \subset \mathscr{D}_\Gamma^\delta$ is bounded, then $B \subset \mathscr{D}_\Gamma^{\delta,\alpha}$ for some α.*

Proof. Let $S \subset \Gamma$ be a finite, symmetric generating set. We apply Lemma 4.4.26 to S and B to find a constant $C \geq 1$ such that for any $\rho \in B$ there is a δ-hyperbolic pseudo metric $d_\rho \in \hat{\rho}$ satisfying

$$C^{-1} d_S(x,y) - C \leq d_\rho(x,y) \leq C d_S(x,y)$$

for all $x, y \in \Gamma$. In particular, for each $\rho \in B$, any pair of points in (Γ, d_ρ) can be joined by a (C, C, C)-quasigeodesic. Therefore, by Lemma 2.3.10 applied to each (Γ, d_ρ) we deduce that there is some $\alpha \geq 0$ such that for every $\rho \in B$, the pseudo metric d_ρ is δ-hyperbolic and α-roughly geodesic, which concludes the proof of the proposition. □

4.4.5 The action of $\mathrm{Out}(\Gamma)$

One of the main properties of Teichmüller space and outer space is that they admit proper isometric actions of $\mathrm{Out}(\Gamma)$, where Γ is a surface or a free group respectively. In our more general setting, there is a natural isometric action of $\mathrm{Out}(\Gamma)$ on $(\mathscr{D}_\Gamma, \Delta)$ via pullback: if $\phi \in \mathrm{Aut}(\Gamma)$ and $d \in \mathcal{D}_\Gamma$, then $\phi([d]) = [\phi(d)]$, where $\phi(d)(x,y) = d(\phi^{-1}x, \phi^{-1}y)$ for all $x, y \in \Gamma$. This action is clearly isometric for the metric Δ, and if $\phi \in \mathrm{Inn}(\Gamma)$ then ϕ acts trivially on $\mathscr{D}_\Gamma$ so that the action descends to an isometric action of $\mathrm{Out}(\Gamma)$.

Remark 4.4.28. It is easy that the action of $\mathrm{Out}(\Gamma)$ preserves the Manhattan geodesic bicombing $\sigma_\bullet$ introduced in Section 4.4.3, in the sense that $\phi \circ \sigma_\bullet^{\rho_*/\rho} = \sigma_\bullet^{\phi(\rho_*)/\phi(\rho)}$ for any $\phi \in \mathrm{Out}(\Gamma)$ and $\rho \neq \rho_*$ in $\mathscr{D}_\Gamma$. This follows since for any $d, d_* \in \mathcal{D}_\Gamma$ and $\phi \in \mathrm{Aut}(\Gamma)$ we have $\rho_\bullet^{\phi(d_*)/\phi(d)} = \phi \circ \rho_\bullet^{d_*/d}$.

Extending the cases of Teichmüller and outer spaces, the action of $\mathrm{Out}(\Gamma)$ on $\mathscr{D}_\Gamma$ is proper.

Theorem 4.4.29. *The action of* $\mathrm{Out}(\Gamma)$ *on* $\mathscr{D}_\Gamma$ *is isometric and metrically proper. That is, for any* $\rho \in \mathscr{D}_\Gamma$ *and* $R \geq 0$*, there exist at most finitely many* $\phi \in \mathrm{Out}(\Gamma)$ *such that* $\Delta(\rho, \phi(\rho)) \leq R$*.*

Proof. To prove the assertion, suppose by contradiction that there exists some $\rho \in \mathscr{D}_\Gamma$, some $R > 0$ and infinitely many $\phi \in \mathrm{Out}(\Gamma)$ such that $\Delta(\rho, \phi(\rho)) \leq R$. Consider $d \in \hat{\rho}$ and let $\mathcal{A} \subset \mathrm{Aut}(\Gamma)$ be a complete set of representatives of the elements $\phi \in \mathrm{Out}(\Gamma)$ such that $\ell_{\phi(d)} \leq e^R \ell_d$, which is infinite by assumption.

Let $S \subset \Gamma$ be a finite, symmetric generating set, and for $\phi \in \mathcal{A}$ consider the quantity $\lambda_\phi = \inf_{x \in \Gamma} \max_{s \in S} d(x, \phi^{-1}(s)x)$. As $\mathcal{A}$ is infinite, the set $\{\lambda_\phi\}_{\phi \in \mathcal{A}}$ is unbounded, see e.g. [Pau91, p. 338]. On the other hand, since each pseudo metric $\phi(d)$ is δ-hyperbolic and α-roughly geodesic for some uniform δ and α, by Theorem 2.3.15 there exists a constant $L > 0$ such that for all $\phi \in \mathcal{A}$ we have

$$\lambda_\phi \leq L + \frac{1}{2} \max_{s_1, s_2 \in S} \ell_{\phi(d)}[s_1 s_2] \leq L + \frac{e^R}{2} \max_{s_1, s_2 \in S} \ell_d[s_1 s_2].$$

The last term does not depend on ϕ, implying the desired contradiction. ☐

The situation is also interesting when we restrict to the $\mathrm{Out}(\Gamma)$-invariant subspaces $\mathscr{D}_\Gamma^\delta$ and $\mathscr{D}_\Gamma^{\delta,\alpha}$ for $\delta, \alpha \geq 0$, since by Theorem 4.4.22, metric properness of the action implies proper discontinuity. Indeed, when Γ is torsion-free, we can prove cocompactness for the action on $\mathscr{D}_\Gamma^{\delta,\alpha}$. The next theorem combined with Theorem 4.4.29 implies Theorem 1.2.7 from the Introduction.

Theorem 4.4.30. *Assume* Γ *is torsion-free, and let* $\alpha, \delta \geq 0$ *be such that* $\mathscr{D}_\Gamma^{\delta,\alpha}$ *is non-empty. Then the action of* $\mathrm{Out}(\Gamma)$ *on* $\mathscr{D}_\Gamma^{\delta,\alpha}$ *is cocompact. In particular, if* $\mathrm{Out}(\Gamma)$ *is finite then* $\mathscr{D}_\Gamma^{\delta,\alpha}$ *is compact.*

Example 4.4.31 (Thick part for Teichmüller space)**.** Let S be a closed orientable surface with fundamental group isomorphic to Γ, and let $\varepsilon > 0$. A *thick part* for $\mathscr{T}_\Gamma$ is the set $\mathscr{T}_{\Gamma,\varepsilon}$ of points in $\mathscr{T}_\Gamma$ represented by a marked hyperbolic metric whose closed geodesics have lengths at least ε. Since the mapping class group of S is a finite index subgroup of $\mathrm{Out}(\Gamma)$, it follows from Mumford's compactness theorem [Mum71] that the action of $\mathrm{Out}(\Gamma)$ on $\mathscr{T}_{\Gamma,\varepsilon}$ is cocompact. In consequence, by Proposition 4.4.27 and Example 4.4.20 we can find $\alpha_\varepsilon \geq 0$ such that $\mathscr{T}_{\Gamma,\varepsilon} \subset \mathscr{D}_\Gamma^{\log 2, \alpha_\varepsilon}$. Since Γ is torsion-free, Theorem 4.4.30 applies, and indeed we have that for all $\delta, \alpha \geq 0$ there exists $C = C(\delta, \alpha, \varepsilon) \geq 0$ such that $\mathscr{D}_\Gamma^{\delta,\alpha}$ is contained in the C-neighborhood of $\mathscr{T}_{\Gamma,\varepsilon}$ in $(\mathscr{D}_\Gamma, \Delta)$.

Example 4.4.32 (Thick part for outer space)**.** Let Γ be a free group, and let $\varepsilon > 0$. As in the case of surface groups, we can define a *thick part* for $\mathscr{CV}_\Gamma$ as the set $\mathscr{CV}_{\Gamma,\varepsilon}$ of the points in $\mathscr{CV}_\Gamma$ represented by a marked graph (G, ϕ) such the minimal length of a loop in G is at least ε times the sum of the lengths of edges of G. In this case the action of $\mathrm{Out}(\Gamma)$ on $\mathscr{CV}_{\Gamma,\varepsilon}$ is also cocompact, and as in the example above we deduce that for all $\delta, \alpha \geq 0$ there exists $C = C(\delta, \alpha, \varepsilon) \geq 0$ such that $\mathscr{D}_\Gamma^{\delta,\alpha}$ is contained in the C-neighborhood of $\mathscr{CV}_{\Gamma,\varepsilon}$.

For the proof of Theorem 4.4.30 we need some notation. A *marked group* is a pair (Γ, S) where Γ is a group and $S \subset \Gamma$ is a finite, symmetric generating subset. If Γ is hyperbolic, we say that (Γ, S) is δ-hyperbolic if the word metric d_S is δ-hyperbolic. Similarly, the exponential growth rate of (Γ, S) is $v(\Gamma, S) = v_S$. An *isomorphism of marked groups* $\phi : (\Gamma, S) \to (\Gamma', S')$ is an isomorphism $\phi : \Gamma \to \Gamma'$ such that $\phi(S) = S'$ and the restriction $\phi|_S$ is injective, and two marked groups are *isometrically isomorphic* if there exists an isomorphism of marked groups between them. Note that the isomorphism of marked groups induces an isometry $(\Gamma, d_S) \to (\Gamma', d_{S'})$, and when $\Gamma = \Gamma'$, the corresponding element in $\mathrm{Out}(\Gamma)$ induced by ϕ maps $[d_S]$ to $[d_{S'}]$.

The key ingredient in the proof is the following finiteness result, which is a particular case of a theorem of Besson, Courtois, Gallot and Sambusetti.

Theorem 4.4.33 (Besson-Courtois-Gallot-Sambusetti [Bes+21, Thm. 1.4])**.** *Let Γ be a non-elementary, torsion-free hyperbolic group. For any $\delta, H \geq 0$ there are only finitely many marked groups (Γ, S) (up to isometric isomorphism) such that (Γ, S) is δ-hyperbolic and $v(\Gamma, S) \leq H$.*

Proof of Theorem 4.4.30. Assume $\mathscr{D}_\Gamma^{\delta,\alpha}$ is non-empty, and let $\rho \in \mathscr{D}_\Gamma^{\delta,\alpha}$ and $d_\rho \in \hat{\rho}$ be a δ-hyperbolic and α-roughly geodesic pseudo metric representative. For this pseudo metric we have that $S_\rho := \{x \in \Gamma : d_\rho(o, x) \leq 2 + \lfloor\alpha\rfloor\}$ generates Γ. Lemma 4.2.12 then implies that

$$(1 + \lfloor\alpha\rfloor - \alpha)d_{S_\rho}(x, y) - (n - 1) \leq d_\rho(x, y) \leq (2 + \lfloor\alpha\rfloor)d_{S_\rho}(x, y)$$

for all $x, y \in \Gamma$. Thus, we get that $\Delta(\rho, [d_{S_\rho}]) \leq R := \log(\frac{2+\lfloor\alpha\rfloor}{1+\lfloor\alpha\rfloor-\alpha})$. In addition, we also get $v(G, S_\rho) \leq (2 + \lfloor\alpha\rfloor)v_{d_\rho} = 2 + \lfloor\alpha\rfloor$, and by applying Corollary 2.3.9, we can deduce that (Γ, S_ρ) is $\widetilde{\delta}$-hyperbolic, for $\widetilde{\delta}$ depending only on δ and α. From this, we can apply Theorem 4.4.33 and conclude that there is a finite set $(\Gamma, S_1), \ldots, (\Gamma, S_k)$ of marked groups such that each marked group (Γ, S_ρ) with $\rho \in \mathscr{D}_\Gamma^{\delta,\alpha}$ is isometrically isomorphic to some (Γ, S_i). Define K as the intersection of $\mathscr{D}_\Gamma^{\delta,\alpha}$ with the closure of the union of the closed balls in $\mathscr{D}_\Gamma$ of radius R and with centers $[d_{S_1}], \ldots, [d_{S_k}]$, which is compact by Proposition 4.4.25. As marked isomorphisms are induced by automorphisms of Γ exchanging the corresponding word metrics, and since $\mathrm{Out}(\Gamma)$ acts isometrically, we get $\mathscr{D}_\Gamma^{\delta,\alpha} = \mathrm{Out}(\Gamma) \cdot K$ and conclude the proof of the theorem. □

4.5 Continuity results

In this section we prove continuity results for some relevant functions on $\mathscr{D}_\Gamma$. These are the Bowen-Margulis map (Theorem 4.5.1), the mean distortion (Theorem 4.5.6), the hyperbolicity constant functional (Proposition 4.5.8), and the visual dimension functional (Proposition 4.5.10). Theorem 4.5.1 confirms Theorem 1.2.8 from the Introduction.

4.5.1 Continuity of the Bowen-Margulis map

In Definition 4.3.17, we introduced the (projective) Bowen-Margulis current $BM(\rho) \in \mathbb{P}\mathcal{C}urr(\Gamma)$ associated to the metric structure $\rho \in \mathscr{D}_\Gamma$. If Γ is a surface group and $\rho \in \mathscr{T}_\Gamma$, the Bowen-Margulis current $BM(\rho)$ has a canonical normalization in $\mathcal{C}urr(\Gamma)$, which coincides with the Liouville current of ρ (see Example 4.7.9). The work of Bonahon [Bon88] then implies that the Bowen-Margulis map $BM : \mathscr{T}_\Gamma \to \mathbb{P}\mathcal{C}urr(\Gamma)$ is continuous. Similar continuity results for the Bowen-Margulis current still hold if we consider marked (variable) negatively curved metrics on surfaces [Kat+89] (indeed, sufficiently regular perturbations of the metric will give higher regularity of the Bowen-Margulis current). In the case of a free group Γ, the continuity of $BM : \mathscr{C}\mathscr{V}_\Gamma \to \mathbb{P}\mathcal{C}urr(\Gamma)$ is due to Kapovich and Nagnibeda [KN07, Thm. A]. Our next theorem generalizes all these previous continuity results.

Theorem 4.5.1. *For any $\delta \geq 0$ such that $\mathscr{D}_\Gamma^\delta$ is non-empty, the Bowen-Margulis map $BM : \mathscr{D}_\Gamma^\delta \to \mathbb{P}\mathcal{C}urr(\Gamma)$ is continuous.*

For the proof of this result we fix a sequence $(\rho_n)_n \subset \mathscr{D}_\Gamma^\delta$ converging to ρ_∞. Our goal is to prove that $BM(\rho_n)$ converges to $BM(\rho_\infty)$ in $\mathbb{P}\mathcal{C}urr(\Gamma)$. As $\mathbb{P}\mathcal{C}urr(\Gamma)$ is metrizable, our strategy will be to prove that each subsequence of $(BM(\rho_n))_n$ has a further subsequence converging to $BM(\rho_\infty)$.

Fix a finite, symmetric generating set $S \subset \Gamma$. By Proposition 4.4.27, there exists some $\alpha \geq 0$ such that $\rho_n \in \mathscr{D}_\Gamma^{\delta,\alpha}$ for all n, and by Lemma 4.4.23 we can find a constant $C \geq 1$ and δ-hyperbolic, α-roughly geodesic pseudo metrics $d_n \in \hat{\rho}_n$ such that

$$C^{-1} d_S(x,y) - C \leq d_n(x,y) \leq C d_S(x,y) \tag{4.27}$$

for all n and all $x, y \in \Gamma$.

We consider a subsequence of $(\rho_n)_n$, that for simplicity we still denote by ρ_n. Due to our assumptions, and after extracting a further subsequence, we can assume that d_n converges pointwise to a pseudo metric $d_\infty \in \hat{\rho}_\infty$, so that d_∞ is also δ-hyperbolic and α-roughly geodesic and satisfies (4.27) with d_∞ in the place of d_n. The next lemma will guarantee a good large-scale convergence of the pseudo metrics d_n, and might be considered as a characterization of convergence of metric structures in terms of pointwise convergence of convenient pseudo metric representatives.

Lemma 4.5.2. *Let $\delta, \alpha \geq 0$ and $(d_n)_n \subset \mathcal{D}_\Gamma$ be a sequence of δ-hyperbolic, α-roughly geodesic pseudo metrics on Γ. Suppose that d_n converges pointwise to a pseudo metric d_∞, also δ-hyperbolic and α-roughly geodesic. Then for any $\varepsilon > 0$ there is some n_0 such that if $n \geq n_0$ and $x \in \Gamma$ then*

$$d_n(o,x) \leq (1+\varepsilon)d_\infty(o,x) + 3\alpha + 1.$$

In addition, suppose that there exist $A, B \geq 1$ such that $d_\infty \leq Ad_n + B$ for all n. Then for any $\varepsilon > 0$ there is some n_1 such that if $n \geq n_1$ and $x \in \Gamma$ then

$$d_\infty(o,x) \leq (1+\varepsilon)d_n(o,x) + 3\alpha + 1.$$

Proof. Without loss of generality we can assume that $\varepsilon \in (0,1)$, and for $x \in \Gamma$ consider an (α, d_∞)-rough geodesic $o = x_0, \ldots, x_m = x$ joining o and x. Given $L > \frac{2\alpha+\varepsilon}{2\varepsilon}$ an integer, let n_0 be such that $|d_n(p,q) - d_\infty(p,q)| < \varepsilon/2$ for all $n \geq n_0$ and all p, q such that $d_\infty(p,q) \leq L + \alpha$. If $m = sL + r$ with s, r non-negative integers and $0 \leq r < L$, then

$$\begin{aligned} d_n(o,x) &\leq \sum_{i=0}^{s-1} d_n(x_{iL}, x_{(i+1)L}) + d_n(x_{sL}, x_m) \\ &\leq \sum_{i=0}^{s-1} \big(d_\infty(x_{iL}, x_{(i+1)L}) + \varepsilon/2\big) + d_\infty(x_{sL}, x_m) + \varepsilon/2 \\ &\leq s(L + \alpha + \varepsilon/2) + (r + \alpha + \varepsilon/2) \\ &= m + (s+1)(\alpha + \varepsilon/2) \\ &\leq d_\infty(o,x) + \alpha + (s+1)(\alpha + \varepsilon/2). \end{aligned}$$

Since $s \leq d_\infty(o,x)/L + \alpha/L$ we deduce

$$d_n(o,x) \leq (1 + \alpha/L + \varepsilon/(2L))d_\infty(o,x) + \alpha + (\alpha/L + 1)(\alpha + \varepsilon/2) \leq (1+\varepsilon)d_\infty(o,x) + 3\alpha + 1,$$

where we used $\alpha/L + \varepsilon/(2L) < \varepsilon < 1$. This proves the first statement.

The second statement is proven similarly, where we choose n_1 such that $|d_n(p,q) - d_\infty(p,q)| < \varepsilon/2$ for all $n \geq n_1$ and all p, q such that $d_\infty(p,q) \leq A(L+\alpha) + B$, where $L > \frac{2\alpha+\varepsilon}{2\varepsilon}$ is an integer, and $x_0, \ldots, x_m$ is now an (α, d_n)-rough geodesic with $n \geq n_1$. ☐

After extracting a subsequence, by Lemma 4.5.2 we can assume that there is a *decreasing* sequence $\Lambda_n \to 1$ such that

$$\Lambda_n^{-1} d_\infty - 3\alpha - 2 \leq d_n \leq \Lambda_n d_\infty + 3\alpha + 1 \tag{4.28}$$

for all n. By Proposition 2.3.11 this implies that

$$\Lambda_n^{-1}(\cdot|\cdot)_{\cdot,d_\infty} - Q \leq (\cdot|\cdot)_{\cdot,d_n} \leq \Lambda_n(\cdot|\cdot)_{\cdot,d_\infty} + Q \tag{4.29}$$

for all n, where Q is a constant independent of n.

Given n, let ν_n be a quasiconformal measure for d_n with uniform multiplicative constant $D = D_\delta$ as in (4.9). As $\partial\Gamma$ is compact metrizable, and after extracting a subsequence, we can assume that ν_n weak-$*$ converges to ν_∞. We claim that ν_∞ is quasiconformal for d_∞.

Proposition 4.5.3. *There is a constant $M \geq 1$ such that for all $x \in \Gamma$ and all $f \in C(\partial\Gamma)$ with $f \geq 0$:*

$$M^{-1} \int f(p) e^{-\beta_{d_\infty}(x,o;p)} d\nu_\infty(p) \leq \int f(p) dx\nu_\infty(p) \leq M \int f(p) e^{-\beta_{d_\infty}(x,o;p)} d\nu_\infty(p). \tag{4.30}$$

In consequence, ν_∞ is quasiconformal for d_∞.

We will need the next lemma, which is an immediate consequence of (2.3).

Lemma 4.5.4. *Let $d \in \mathcal{D}_\Gamma$ be δ-hyperbolic. Then for all $x \in \Gamma$, the functions $\overline{h}_x, \underline{h}_x : \partial\Gamma \to \mathbb{R}$ defined by*

$$\overline{h}_x(p) := \limsup_{q \to p} e^{(x|q)_{o,d}}, \quad \underline{h}_x(p) := \liminf_{q \to p} e^{(x|q)_{o,d}}$$

are upper and lower semi-continuous respectively, bounded above by $e^{d(x,o)}$, and satisfy the inequalities

$$\underline{h}_x(p) \leq \overline{h}_x(p), \quad e^{-\delta}\overline{h}_x(p) \leq e^{(x|p)_{o,d}} \leq e^{\delta}\underline{h}_x(p)$$

for all $p \in \partial\Gamma$.

Proof of Proposition 4.5.3. quasiconformality follows immediately from (4.30) since ν_∞ is regular. We will only prove the second inequality in (4.30), as the first one is proven in the same way. Let $f \in C(\partial\Gamma)$ with $f \geq 0$ and $x \in \Gamma$. By inequalities (4.9) and (2.4) we get

$$\begin{aligned} \int f dx\nu_\infty &= \lim_{n\to\infty} \int f dx\nu_n = \lim_{n\to\infty} \int f \frac{dx\nu_n}{d\nu_n} d\nu_n \\ &\leq D \limsup_{n\to\infty} \int f(p) e^{-\beta_{d_n}(x,o;p)} d\nu_n(p) \\ &= D \limsup_{n\to\infty} \left(e^{-d_n(x,o)} \int f(p) e^{2(x|p)_{o,d_n}} d\nu_n(p) \right). \end{aligned}$$

Since the sequence $(\Lambda_n)_n$ is monotone, by (4.28) and (4.29) we obtain

$$\begin{aligned} \int f dx\nu_\infty &\leq D e^{3\alpha+2Q+2} \limsup_{n\to\infty} \left(e^{-\Lambda_n^{-1} d_\infty(x,o)} \int f(p) e^{2\Lambda_n (x|p)_{o,d_\infty}} d\nu_n(p) \right) \\ &\leq D e^{3\alpha+2Q+2} e^{-d_\infty(x,o)} \liminf_{k\to\infty} \limsup_{n\to\infty} \left(\int f(p) (e^{2(x|p)_{o,d_\infty}})^{\Lambda_k} d\nu_n(p) \right). \end{aligned}$$

At this point, we would like to use the convergence $\nu_n \xrightarrow{*} \nu_\infty$ to deduce

$$\limsup_{n\to\infty} \left(\int f(p) (e^{2(x|p)_{o,d_\infty}})^{\Lambda_k} d\nu_n(p) \right) = \int f(p) (e^{2(x|p)_{o,d_\infty}})^{\Lambda_k} d\nu_\infty(p).$$

However, the function $p \to (e^{2(x|p)_{o,d_\infty}})^{\Lambda_k}$ is not necessarily continuous, so this last equation might not hold. Instead, we consider the function $\overline{h} = \overline{h}_x : \partial\Gamma \to \mathbb{R}$ from Lemma 4.5.4

with respect to $d = d_\infty$. Since $\partial\Gamma$ is metrizable and $\overline{h}$ is upper semi-continuous, there exists a decreasing sequence $\overline{h}_m$ of continuous functions on $\partial\Gamma$ that converges pointwise to $\overline{h}$. Moreover, since $\overline{h}$ is bounded, we can assume that the sequence $\overline{h}_m$ is uniformly bounded. Therefore, the convergence $\nu_n \xrightarrow{*} \nu_\infty$ and the dominated convergence theorem yields

$$\begin{aligned}\limsup_{n\to\infty}\left(\int f(p)(e^{2(x|p)_{o,d_\infty}})^{\Lambda_k}d\nu_n(p)\right) &\leq e^{2\Lambda_k\delta}\limsup_{n\to\infty}\left(\int f\cdot(\overline{h})^{2\Lambda_k}d\nu_n\right)\\ &\leq e^{2\Lambda_k\delta}\liminf_{m\to\infty}\limsup_{n\to\infty}\left(\int f\cdot(\overline{h}_m)^{2\Lambda_k}d\nu_n\right)\\ &= e^{2\Lambda_k\delta}\liminf_{m\to\infty}\left(\int f\cdot(\overline{h}_m)^{2\Lambda_k}d\nu_\infty\right)\\ &= e^{2\Lambda_k\delta}\int f\cdot(\overline{h})^{2\Lambda_k}d\nu_\infty\\ &\leq e^{4\Lambda_k\delta}\int f(p)(e^{2(x|p)_{o,d_\infty}})^{\Lambda_k}d\nu_\infty(p).\end{aligned}$$

Combining these two inequalities, and applying the dominated convergence theorem to the sequence $k \mapsto (p \mapsto f(p)(e^{2(x|p)_{o,d_\infty}})^{\Lambda_k})$ we deduce

$$\begin{aligned}\int f dx\nu_\infty &\leq De^{4\delta+3\alpha+2Q+2}\int f(p)e^{-d_\infty(x,o)+2(x|p)_{o,d_\infty}}d\nu_\infty(p)\\ &= De^{4\delta+3\alpha+2Q+2}\int f(p)e^{-\beta_{d_\infty}(x,o;p)}d\nu_\infty(p),\end{aligned}$$

where in the last inequality we used (2.4). Since $M = De^{4\delta+3\alpha+2Q+2}$ is independent of f and x, the conclusion follows. □

As the quasiconformal measures ν_n satisfy (4.8) and each d_n is δ-hyperbolic, by (4.10) there exists $R > 0$ such that for every n there is a geodesic current η_n representing $BM(\rho_n)$ and satisfying

$$R^{-1}\int_A e^{2(p|q)_{o,d_n}}d\nu_n(p)d\nu_n(q) \leq \eta_n(A) \leq R\int_A e^{2(p|q)_{o,d_n}}d\nu_n(p)d\nu_n(q) \tag{4.31}$$

for any Borel subset $A \subset \partial^2\Gamma$.

Lemma 4.5.5. *After taking a subsequence, η_n weak-$*$ converges to a positive Radon measure ω.*

Proof. Since $\partial^2\Gamma$ is locally compact and metrizable, it is σ-compact, so it is enough to show that for any compact set $C \subset \partial^2\Gamma$ with non-empty interior, the sequence $(\eta_n(C))_n$ is bounded by above and below by positive constants. For the upper bound, we apply (4.31), (4.29) and the monotonicity of $(\Lambda_n)_n$ to obtain

$$\begin{aligned}\eta_n(C) &\leq R\int_C e^{2(p|q)_{o,d_n}}d\nu_n(p)d\nu_n(q)\\ &\leq R\int_C e^{2\Lambda_1(p|q)_{o,d_\infty}+2Q}d\nu_n(p)d\nu_n(q)\\ &\leq R\sup_{(p,q)\in C}\left(e^{2\Lambda_1(p|q)_{o,d_\infty}+2Q}\right)(\nu_n\otimes\nu_n)(C)\leq R\sup_{(p,q)\in C}\left(e^{2\Lambda_1(p|q)_{o,d_\infty}+2Q}\right).\end{aligned}$$

The last quantity is finite by compactness of C and is independent of n.

For the lower bound, we may assume that C contains a non-empty open set $U=U_1\times U_2$ with each $U_i\subset\partial\Gamma$, so that the convergence $\nu_n\overset{*}{\rightharpoonup}\nu_\infty$ gives

$$\liminf_n \eta_n(C)\geq\liminf_n \eta_n(U)\geq R^{-1}\liminf_n \nu_n(U_1)\nu_n(U_1)\geq R^{-1}\nu_\infty(U_1)\nu_\infty(U_2).$$

The last term is positive since the measure ν_∞ has full support, and hence the sequence $(\eta_n(C))_n$ is bounded below by a positive number. □

Now we finish the proof of Theorem 4.5.1, for which we are left to show that ω represents $BM(\rho_\infty)$. Indeed, given $f\in C_c(\partial^2\Gamma)$ with $f\geq 0$, by (4.31) and (4.29) we have

$$\begin{aligned}\int fd\omega &= \lim_{n\to\infty}\int fd\eta_n\\ &\leq R\limsup_{n\to\infty}\int f(p,q)e^{2(p|q)_{o,d_n}}d\nu_n(p)d\nu_n(q)\\ &\leq Re^{2Q}\liminf_{k\to\infty}\limsup_{n\to\infty}\left(\int f(p,q)e^{2\Lambda_k(p|q)_{o,d_\infty}}d\nu_n(p)d\nu_n(q)\right).\end{aligned}$$

Let $\varepsilon>0$ be such $\varepsilon\delta<\log 2$, and let ϱ_ε be a visual metric on $\partial\Gamma$ satisfying (2.5) with respect to d_∞. As f has compact support in $\partial^2\Gamma$, the functions $(p,q)\mapsto f(p,q)(\varrho_\varepsilon(p,q)^{-1/\varepsilon})^{2\Lambda_k}$ are continuous on $(\partial\Gamma)^2$ and uniformly bounded. Also, since $\nu_n\overset{*}{\rightharpoonup}\nu_\infty$ and $\partial\Gamma$ is separable, $\nu_n\otimes\nu_n$ weak-$*$ converges to $\nu_\infty\otimes\nu_\infty$ [Bil99, Thm. 2.8]. Therefore, by the dominated convergence theorem we obtain

$$\begin{aligned}\int fd\omega &\leq Re^{2Q}\liminf_{k\to\infty}\limsup_{n\to\infty}\left(\int f(p,q)(\varrho_\varepsilon(p,q)^{-1/\varepsilon})^{2\Lambda_k}d\nu_n(p)d\nu_n(q)\right)\\ &\leq Re^{2Q}\liminf_{k\to\infty}\left(\int f(p,q)(\varrho_\varepsilon(p,q)^{-1/\varepsilon})^{2\Lambda_k}d\nu_\infty(p)d\nu_\infty(q)\right)\\ &\leq Re^{2Q}\int f(p,q)\varrho_\varepsilon(p,q)^{-2/\varepsilon}d\nu_\infty(p)d\nu_\infty(q)\\ &\leq Re^{2Q}(2\varepsilon\delta)^{2/\varepsilon}\int f(p,q)e^{2(p|q)_{o,d_\infty}}d\nu_\infty(p)d\nu_\infty(q).\end{aligned}$$

As ν_∞ is quasiconformal for d_∞, we can find a measure η_∞ representing $BM(\rho_\infty)$ in the same class of $e^{2(p|q)_{o,d_\infty}} d\nu_\infty(p)d\nu_\infty(q)$ and with essentially uniformly bounded Radon-Nikodym derivatives, and hence there is a constant $L \geq 1$ independent of f such that

$$\int f d\omega \leq L \int f dm_\infty.$$

In the same way, we can prove that there is a constant L' such that $\int f d\eta_\infty \leq L' \int f d\omega$, implying that ω and η_∞ are absolutely continuous with respect to each other and that the Radon-Nikodym derivative $\frac{d\omega}{d\eta_\infty}$ is essentially bounded by above, and by below by a positive number. To conclude the proof of Theorem 4.5.1, note that ω is Γ-invariant (resp. flip-invariant), being the limit of Γ-invariant (resp. flip-invariant) measures, and that η_∞ is Γ-ergodic [BF17, Thm. 1.4]. In consequence, $\frac{d\omega}{d\eta_\infty}$ is constant almost everywhere, and there is some $\lambda > 0$ such that $\omega = \lambda\eta_\infty$, so that $BM(\rho_n) \to BM(\rho_\infty)$. □

4.5.2 Continuity of the mean distortion

Given metric structures $\rho, \rho_* \in \mathscr{D}_\Gamma$, we define the *mean distortion* of ρ_* over ρ by

$$\tau(\rho_*/\rho) = \tau(d_*/d),$$

where $d \in \rho$ and $d_* \in \rho$ have exponential growth rate 1 and $\tau(d_*/d)$ is the mean distortion of d_* over d introduced in Subsection 4.1.2. It follows from (4.5) that $\tau(\rho_*/\rho) \geq 1$, with equality if and only if $\rho = \rho_*$. It is also clear from (4.3) that $\tau(\rho_*/\rho)$ is continuous in the variable ρ_*, and the next theorem states that it is actually continuous in both variables.

Theorem 4.5.6. *The mean distortion $\tau : \mathscr{D}_\Gamma \times \mathscr{D}_\Gamma \to \mathbb{R}$ is continuous.*

Proof. Given $d, d_* \in \mathcal{D}_\Gamma$, the recall that the Manhattan curve $\theta = \theta_{d_*/d}$ for d, d_* is the function $\theta : \mathbb{R} \to \mathbb{R}$ which maps a to the critical exponent of

$$b \mapsto \mathcal{P}_{d,d*}(a,b) := \sum_{x\in\Gamma} e^{-ad_*(x,o)-bd(x,o)}.$$

Now, consider $\rho, \rho_* \in \mathscr{D}_\Gamma$ and sequences $(\rho^n)_n, (\rho^n_*)_n$ in $\mathscr{D}_\Gamma$ converging to ρ and ρ_* respectively, for which we claim that $\tau(\rho^n_*/\rho^n)$ tends to $\tau(\rho_*/\rho)$. For each $n \geq 1$ choose representatives $d^n \in \hat{\rho}^n$ and $d^n_* \in \hat{\rho}^n_*$, and let θ_n be the Manhattan curve for the pair d^n, d^n_*. Similarly, choose $d \in \hat{\rho}$ and $d_* \in \hat{\rho}_*$ and let θ be the Manhattan curve for d, d_*. From (4.4) we obtain $\tau(\rho^n_*/\rho^n) = -\theta'_n(0)$ and $\tau(\rho_*/\rho) = -\theta'(0)$, so it is enough to show that θ'_n converges to θ' pointwise.

Let $(\Lambda_n)_n$ and $(C_n)_n$ be sequences such that $\Lambda_n \to 1$ and satisfying

$$\Lambda_n^{-1}d - C_n \leq d_n \leq \Lambda_n d + C_n \quad \text{and} \quad \Lambda_n^{-1}d' - C_n \leq d'_n \leq \Lambda_n d' + C_n \tag{4.32}$$

for all n. For any $a, b \in \mathbb{R}$, from (4.32) we get

$$e^{-(|a|+|b|)C_n}\mathcal{P}_{d,d_*}(\max(\Lambda_n a, \Lambda_n^{-1}a), \max(\Lambda_n b, \Lambda_n^{-1}b)) \leq \mathcal{P}_{d^n,d_*^n}(a,b),$$

so that

$$\theta(\max(\Lambda_n a, \Lambda_n^{-1}a)) \leq \max(\Lambda_n\theta_n(a), \Lambda_n^{-1}\theta_n(a)),$$

and by the same argument, we also get

$$\theta(\min(\Lambda_n a, \Lambda_n^{-1}a)) \geq \min(\Lambda_n\theta_n(a), \Lambda_n^{-1}\theta_n(a)).$$

Continuity of θ then implies that θ_n converges pointwise to θ, and since θ is differentiable and all the curves θ_n are convex and differentiable, we conclude that θ_n' converges pointwise to θ', as desired. □

4.5.3 The hyperbolicity constant and visual dimension functionals

It is natural to define functions on $\mathscr{D}_\Gamma$ by minimizing over quantities that we can associate to metric structure representatives. An illustration of this is considering the optimal hyperbolicity constant (up to normalization).

Definition 4.5.7. Let $\delta : \mathscr{D}_\Gamma \to \mathbb{R}$ be the function that sends $\rho \in \mathscr{D}_\Gamma$ to the infimal δ such there exists a δ-hyperbolic pseudo metric $d \in \hat{\rho}$. This is the *hyperbolicity constant function.*

Note that $\delta(\rho)$ is actually a minimum. Indeed, if $d_n \in \hat{\rho}$ is a sequence with each d_n being δ_n-hyperbolic and δ_n converging to $\delta(\rho)$, then an argument similar as in the proof of Proposition 4.4.24 allows us to assume that, after conjugating the pseudo metrics d_n and extracting a subsequence, the sequence d_n pointwise converges to $d_\infty \in \hat{\rho}$, and hence d_∞ is $\delta(\rho)$-hyperbolic. The same argument shows the following.

Proposition 4.5.8. *The hyperbolicity constant functional $\delta : \mathscr{D}_\Gamma \to \mathbb{R}$ is lower semi-continuous.*

For example, if Γ is a free group then $\delta(\rho) = 0$ if and only if $\rho \in \mathscr{CV}_\Gamma$ (see Example 4.4.19). This allows us to recover the outer in a *purely* coarse geometric way. Similarly, for Γ a surface group we have $\delta(\rho) \leq \log 2$ for any $\rho \in \mathscr{T}_\Gamma$ (see Example 4.4.20).

We can define another functional on $\mathscr{D}_\Gamma$ in terms of invariants associated to metrics on $\partial\Gamma$. For $\rho = [d] \in \mathscr{D}_\Gamma$, we say that the metric ϱ on $\partial\Gamma$ is *visual* for ρ if there exists some $\varepsilon > 0$ and $C \geq 1$ such that

$$C^{-1}e^{-\varepsilon(p|q)_{o,d}} \leq \varrho(p,q) \leq Ce^{-\varepsilon(p|q)_{o,d}} \tag{4.33}$$

for all $p, q \in \partial\Gamma$ (see Subsection 2.3.2). Note that this definition is independent of the representative d.

Definition 4.5.9. Let $\dim : \mathscr{D}_\Gamma \to \mathbb{R}$ be the function that sends $\rho \in \mathscr{D}_\Gamma$ to the infimal Hausdorff dimension of a visual metric for ρ on $\partial\Gamma$. This is the *visual dimension functional.*

If the metric ϱ on $\partial\Gamma$ satisfies (4.33) and ν is a quasiconformal metric for d, then the measure metric space $(\partial\Gamma, \varrho, \nu)$ is v_d/ε-*Ahlfors regular*, meaning that there exists some $D \geq 1$ such that for all $p \in \partial\Gamma$ and every $0 < r \leq \operatorname{diam}(\partial\Gamma, \varrho)$ we have

$$D^{-1} r^{v_d/\varepsilon} \leq \nu(\{q \in \partial\Gamma\colon \varrho(p,q) \leq r\}) \leq D r^{v_d/\varepsilon},$$

see e.g. [BHM11, Thm. 2.3]. In consequence, the Hausdorff dimension of ϱ is v_d/ε and hence $\dim(\rho)$ can be recovered as the infimal ε^{-1} such that there exists a metric on $\partial\Gamma$ satisfying (4.33) for some $d \in \hat{\rho}$. From this perspective, we have the identity

$$\dim(\rho) = -K_u(\Gamma, d)^{-1/2},$$

where $d \in \hat{\rho}$ is arbitrary and $K_u(\Gamma, d)$ is the *asymptotic upper curvature* of (Γ, d) introduced by Bonk and Foertsch [BF06, Def. 1.2 & Thm. 1.5]. Indeed, from this identity and [BF06, Prop. 3.4] we deduce that

$$e^{-\Delta(\rho,\rho_*)} \dim(\rho) \leq \dim(\rho_*) \leq e^{\Delta(\rho,\rho_*)} \dim(\rho) \tag{4.34}$$

for all $\rho, \rho_* \in \mathscr{D}_\Gamma$. This last inequality implies the continuity of the visual dimension functional.

Proposition 4.5.10. *For any Γ the visual dimension functional $\dim : \mathscr{D}_\Gamma \to \mathbb{R}$ is continuous. Indeed, either:*

1. *Γ is virtually free and $\dim(\rho) = 0$ for every ρ; or,*
2. *Γ is not virtually free, $\dim(\rho) \geq 1$ for every ρ and*

$$|\log \dim(\rho) - \log \dim(\rho_*)| \leq \Delta(\rho, \rho_*)$$

for all ρ, ρ_*.

Proof. If Γ is virtually free, then $\mathscr{D}_\Gamma^0$ is non-empty, and by (2.5), for points in $\mathscr{D}_\Gamma^0$ we can construct visual metrics of arbitrarily small Hausdorff dimension. Then dim vanishes on $\mathscr{D}_\Gamma^0$, so it is constant on $\mathscr{D}_\Gamma$ by (4.34).

On the other hand, if Γ is not virtually free, then the topological dimension of $\partial\Gamma$ is at least 1 (see e.g. [KB02, Thm. 6.5 & Thm. 8.1]). Since the topological dimension is a lower bound for the Hausdorff dimension of any Ahlfors regular metric compatible with the topology, we have that $\dim(\rho) \geq 1$ for all $\rho \in \mathscr{D}_\Gamma$. The second assertion then follows from (4.34). ∎

The hyperbolicity and visual dimension functionals are related by the inequality

$$\dim(\rho) \leq \frac{1}{\log 2}\delta(\rho) \tag{4.35}$$

valid for any metric structure ρ, which follows from (2.5). By looking at virtually free groups, from Proposition 4.5.10 we see that this inequality is not achieved in general. This inequality also allows us to compute the visual dimension for uniform lattices in real hyperbolic spaces.

Corollary 4.5.11. *Let Γ act geometrically on the real hyperbolic space of dimension n, so that this action induces a metric structure $\rho_{\mathbb{H}^n} \in \mathscr{D}_\Gamma$. Then*

$$\inf_{\rho \in \mathscr{D}_\Gamma} \dim(\rho) = \dim(\rho_{\mathbb{H}^n}) = n-1.$$

Proof. The action of Γ on $\mathbb{H}^n$ has exponential growth rate $n-1$, and since $\mathbb{H}^n$ is $\log 2$-hyperbolic we have $\rho_{\mathbb{H}^n} \in \mathscr{D}_\Gamma^{(n-1)\log 2}$ and $\dim(\rho_{\mathbb{H}^n}) \leq n-1$ by (4.35). The reverse inequality follows since the topological dimension of $\partial\Gamma = \partial\mathbb{H}^n$ is $n-1$. □

4.6 The Manhattan boundary

In this section we study the Manhattan boundary of $\mathscr{D}_\Gamma$, which encodes the limits at infinity of Manhattan geodesics. This is done by extending the space $\mathcal{D}_\Gamma$ and allowing pseudo metrics not necessarily quasi-isometric to a word metric. We start by recalling the definition of $\overline{\mathcal{D}}_\Gamma$ and $\partial_M\mathcal{D}_\Gamma$ from the Introduction.

Definition 4.6.1. $\overline{\mathcal{D}}_\Gamma$ is the set of all the left-invariant pseudo metrics d on Γ such that its stable translation length function is non-constant and there are $\lambda > 0$ and $d_0 \in \mathcal{D}_\Gamma$ such that

$$(x|y)_{o,d} \leq \lambda(x|y)_{o,d_0} + \lambda \tag{4.36}$$

for all $x, y \in \Gamma$. We also have $\partial_M\mathcal{D}_\Gamma := \overline{\mathcal{D}}_\Gamma \backslash \mathcal{D}_\Gamma$.

Note that $\overline{\mathcal{D}}_\Gamma \subset \mathcal{D}_\Gamma^{hf}$, so pseudo metrics in $\overline{\mathcal{D}}_\Gamma$ are roughly geodesic by Proposition 4.2.8. Also, since hyperbolicity is preserved under quasi-isometry among roughly geodesic metric spaces, we have that $\mathcal{D}_\Gamma \subset \overline{\mathcal{D}}_\Gamma$. Moreover, a pseudo metric in $\overline{\mathcal{D}}_\Gamma$ belongs to $\mathcal{D}_\Gamma$ if and only if it is quasi-isometric to a word metric.

Definition 4.6.2. The *Manhattan boundary* of $\mathscr{D}_\Gamma$ is the set $\partial_M\mathscr{D}_\Gamma$ of rough similarity equivalence classes of pseudo metrics in $\partial_M\mathcal{D}_\Gamma$, and we call its elements *boundary metric structures*. The *(Manhattan) closure* of $\mathscr{D}_\Gamma$ is $\overline{\mathscr{D}}_\Gamma := \mathscr{D}_\Gamma \cup \partial_M\mathscr{D}_\Gamma$.

As in the case of $\mathscr{D}_\Gamma$, we will use the notation $[d]$ for the rough similarity class of $d \in \overline{\mathcal{D}}_\Gamma$. The goal of this section is to describe the points in $\partial_M\mathscr{D}_\Gamma$ as limits of Manhattan geodesics in $\mathscr{D}_\Gamma$. First, we apply Proposition 4.3.21 to deduce that for any two pseudo metrics $d, d_* \in \mathcal{D}_\Gamma$

that are not roughly similar, there exist pseudo metrics $d_\infty, d_{-\infty} \in \partial_M \mathcal{D}_\Gamma$ which are roughly isometric to $\mathrm{Dil}(d, d_*)d_* - d$ and $\mathrm{Dil}(d_*, d)d - d_*$, respectively.

Let $d, d_* \in \mathcal{D}_\Gamma$ be a pair of non-roughly similar pseudo metrics, let θ be its Manhattan curve, and $t \mapsto \rho_t = [d_t]$ be the reparametrization of the Manhattan geodesic for $\rho = [d], \rho_* = [d_*]$ defined in terms of θ, according to Proposition 4.4.5. We will use the notation $\eqsim$ from Subsection 4.4.2.

Proposition 4.6.3. *There are left-invariant pseudo metrics $d_{-\infty} = d^\theta_{-\infty}$ and $d_\infty = d^\theta_\infty$ on Γ and a constant $C \geq 0$ such that*

$$|d_\infty - (\mathrm{Dil}(d, d_*)d_* - d)| \leq C \quad \text{and} \quad |d_{-\infty} - (\mathrm{Dil}(d_*, d)d - d_*)| \leq C. \tag{4.37}$$

The pseudo metrics d_∞ and $d_{-\infty}$ satisfy:

1. $\ell_\infty := \ell_{d_\infty} = \lim_{t\to\infty} \frac{1}{-\theta(t)} \ell_{d_t}$ *and* $\ell_{-\infty} := \ell_{d_{-\infty}} = \lim_{t\to-\infty} \frac{1}{-t} \ell_{d_t}$; *and,*
2. *they both belong to $\partial_M \mathcal{D}_\Gamma$.*

Proof. By Proposition 4.3.21, there is a constant $C' \geq 0$ such that

$$\mathrm{Dil}(d, d_*)^{-1}(x|y)_{o,d} - \mathrm{Dil}(d, d_*)^{-1}C' \leq (x|y)_{o,d_*} \leq \mathrm{Dil}(d_*, d)(x|y)_{o,d} + C'$$

for all $x, y \in \Gamma$. Therefore, as in the proof of Proposition 4.4.5, the functions

$$d_\infty(x, y) := \begin{cases} \mathrm{Dil}(d, d_*)d_*(x, y) - d(x, y) + 2C' & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

and

$$d_{-\infty}(x, y) := \begin{cases} \mathrm{Dil}(d_*, d)d(x, y) - d_*(x, y) + 2C' & \text{if } x \neq y \\ 0 & \text{otherwise} \end{cases}$$

define left-invariant pseudo metrics on Γ verifying (4.37) with $C = 2C'$.

Now we check the desired properties for $d_{-\infty}$ and d_∞.

First, we compute

$$\lim_{t\to\infty} \frac{1}{-\theta(t)} \ell_{d_t} = \lim_{t\to\infty} \frac{(t\ell_{d_*} + \theta(t)\ell_d)}{-\theta(t)} = \mathrm{Dil}(d, d_*)\ell_{d_*} - \ell_d,$$

where we use $\lim_{t\to\infty} \frac{t}{-\theta(t)} = \mathrm{Dil}(d, d_*)$. Similarly, the identity $\lim_{t\to-\infty} \frac{\theta(t)}{-t} = \mathrm{Dil}(d_*, d)$ gives the analogous result for $\ell_{-\infty}$. The functions ℓ_∞ and $\ell_{-\infty}$ are non-constant since d and d_* are not roughly isometric, and d_∞ and $d_{-\infty}$ satisfy (1).

In addition, we have

$$(x|y)_{o,d_\infty} \leq \mathrm{Dil}(d, d_*)(x|y)_{d_*} + C/2 \quad \text{and} \quad (x|y)_{o,d_{-\infty}} \leq \mathrm{Dil}(d_*, d)(x|y)_{o,d} + C/2,$$

so d_∞ and $d_{-\infty}$ belong to $\overline{\mathcal{D}}_\Gamma$.

Finally, by the definition of $\mathrm{Dil}(d, d_*)$ there is a sequence $[x_n] \in \textbf{conj}$ such that

$$\ell_d[x_n] \geq \left(\mathrm{Dil}(d, d_*) - \frac{1}{n}\right) \ell_{d_*}[x_n] > 0$$

for all n, and hence

$$\ell_\infty[x_n] = \mathrm{Dil}(d_*, d)\ell_{d_*}[x_n] - \ell_d[x_n] \leq \frac{1}{n}\ell_{d_*}[x_n].$$

This implies that d_∞ is not quasi-isometric to d_1. Similarly, $d_{-\infty}$ is not quasi-isometric to d_0, which proves that $d_{-\infty}, d_\infty \in \partial_M \mathcal{D}_\Gamma$, and hence (2). □

The proposition above motivates the following definition.

Definition 4.6.4. If $\sigma = \sigma_\bullet^{\rho_*/\rho}$ is the Manhattan geodesic for the pair $\rho = [d], \rho_* = [d_*]$ with $\rho \neq \rho_*$, the *limit at infinity* of σ is the unique boundary metric structure $\sigma_\infty^{\rho_*/\rho} \in \partial_M \mathscr{D}_\Gamma$ such that every pseudo metric representing $\sigma_\infty^{\rho_*/\rho}$ is roughly similar to $\mathrm{Dil}(d, d_*)d_* - d$. Analogously, the *limit at negative infinity* of σ is the unique boundary metric structure $\sigma_{-\infty}^{\rho_*/\rho}$ whose pseudo metric representatives are roughly similar to $\mathrm{Dil}(d_*, d)d - d_*$.

As we will see now, every boundary metric structure is the limit at infinity of some Manhattan geodesic. Indeed, we can choose this geodesic to contain any given metric structure in $\mathscr{D}_\Gamma$.

Theorem 4.6.5. *For any $\rho \in \mathscr{D}_\Gamma$ and $\rho_\infty \in \partial_M \mathscr{D}_\Gamma$, there exists some $\rho_* \in \mathscr{D}_\Gamma$ such that $\rho_\infty = \sigma_\infty^{\rho_*/\rho}$. Moreover, if $\rho'_* \in \mathscr{D}_\Gamma$ satisfies $\rho_\infty = \sigma_\infty^{\rho'_*/\rho}$ then $\rho'_* \in \sigma^{\rho_*/\rho}(0, \infty)$.*

We need a preliminary lemma.

Lemma 4.6.6. *If $d \in \mathcal{D}_\Gamma$ and $d_\infty \in \partial_M \mathcal{D}_\Gamma$, then $d + d_\infty \in \mathcal{D}_\Gamma$.*

Proof. Clearly, $d + d_\infty$ is a left-invariant pseudo metric on Γ. It also satisfies (4.36) for some $\lambda > 0$ and $d_0 \in \mathcal{D}_\Gamma$, since d and d_∞ do. Therefore, $d + d_\infty$ is roughly geodesic by Proposition 4.2.8 and quasi-isometric to a word metric, so it is hyperbolic by Corollary 2.3.9. This concludes the proof of the lemma. □

Proof of Theorem 4.6.5. Let $\rho_\infty = [d_\infty]$ and $\rho = [d]$. Define $d_* := d + d_\infty$, which is a pseudo metric in $\mathcal{D}_\Gamma$ by Lemma 4.6.6. Since $d_\infty \in \partial_M \mathcal{D}_\Gamma$, we have that d and d_* are not roughly similar.

We claim that $\mathrm{Dil}(d, d_*) = 1$. Indeed, since $d = d_* - d_\infty \leq d_*$, we get $\mathrm{Dil}(d, d_*) \leq 1$. In addition, by Proposition 4.3.21 there is some $C \geq 0$ such that

$$(1 - \mathrm{Dil}(d, d_*))d_* \leq d_* - d + C = d_\infty + C,$$

and since d_∞ is not quasi-isometric to a word metric, we get $\mathrm{Dil}(d, d_*) \geq 1$.

Therefore, by our claim we deduce $d_\infty = \mathrm{Dil}(d, d_*)d_* - d$, and $\rho_\infty = \sigma_\infty^{\rho_*/\rho}$ for $\rho_* = [d_*]$.

Finally, suppose that $\rho_\infty = \sigma_\infty^{\tilde{\rho}_*/\rho}$ for some $\tilde{\rho}_* = [\tilde{d}_*]$. Then there exists $\lambda > 0$ such that

$$\lambda(d_* - d) = \lambda d_\infty \eqsim \mathrm{Dil}(d, \tilde{d}_*)\tilde{d}_* - d.$$

We get

$$v_{\tilde{d}_*}\tilde{d}_* \eqsim v_{\tilde{d}_*}\lambda\,\mathrm{Dil}(d, \tilde{d}_*)^{-1}d_* + v_{\tilde{d}_*}(1-\lambda)\,\mathrm{Dil}(d, \tilde{d}_*)^{-1}d,$$

and we conclude $\tilde{\rho}_* = \rho_t^{d_*/d}$ for $t = v_{\tilde{d}_*}\lambda\,\mathrm{Dil}(d, \tilde{d}_*)^{-1} > 0$, so that $\tilde{\rho}_* \in \sigma^{\rho_*/\rho}(0, \infty)$. □

From the proof of Theorem 4.6.5 we deduce:

Corollary 4.6.7. *For any $d_\infty \in \partial_M \mathcal{D}$ and $d \in \mathcal{D}_\Gamma$ there exists $d_* \in \mathcal{D}_\Gamma$ such that*

$$d_\infty = \mathrm{Dil}(d, d_*)d_* - d.$$

We end this section by characterizing when two boundary metric structures are the positive and negative limits of a Manhattan geodesic.

Definition 4.6.8. Two boundary metric structures $\rho, \rho_* \in \partial_M \mathscr{D}_\Gamma$ are *transverse* if for some (any) $d \in \rho$ and $d_* \in \rho_*$ we have $d + d_* \in \mathcal{D}_\Gamma$.

Proposition 4.6.9. *The boundary metric structures $\rho, \rho_* \in \partial_M \mathscr{D}_\Gamma$ are transverse if and only if there is a Manhattan geodesic $\sigma_\bullet$ such that $\rho = \sigma_{-\infty}$ and $\rho_* = \sigma_\infty$.*

Proof. Suppose first that $\rho = \sigma_{-\infty}$ and $\rho_* = \sigma_\infty$ for $\sigma_\bullet = \sigma_\bullet^{\tau_*/\tau}$ the Manhattan geodesic for $\tau = [d], \tau_* = [d_*]$, so that d and d_* are not roughly similar. We consider $d_{-\infty} \in \rho$ and $d_\infty \in \rho_*$, which up to rescaling, we can assume satisfy

$$d_\infty \eqsim \mathrm{Dil}(d, d_*)d_* - d \quad \text{and} \quad d_{-\infty} \eqsim \mathrm{Dil}(d_*, d)d - d_*,$$

where $\eqsim$ is the notation introduced right before Lemma 4.4.9. In particular, we have

$$d_\infty + d_{-\infty} \eqsim (\mathrm{Dil}(d_*, d) - 1)d + (\mathrm{Dil}(d, d_*) - 1)d_*,$$

and this last pseudo metric is in $\mathcal{D}_\Gamma$ since $\mathrm{Dil}(d, d_*) \cdot \mathrm{Dil}(d_*, d) > 1$. This implies that ρ and ρ_* are transverse.

For the reverse implication, suppose that $\rho = [d_{-\infty}]$ and $\rho_* = [d_\infty]$ in $\partial_M \mathscr{D}_\Gamma$. By assumption, $ad_\infty + bd_{-\infty} \in \mathcal{D}_\Gamma$ for any $a, b > 0$, and in particular the pseudo metrics

$$d := d_\infty + 2d_{-\infty} \quad \text{and} \quad d_* := 2d_\infty + d_{-\infty}, \tag{4.38}$$

belong to $\mathcal{D}_\Gamma$. We have that d and d_* are not roughly isometric, since otherwise we would get $d \eqsim \lambda d_*$ for some $\lambda > 0$ and hence $(1 - 2\lambda)d_\infty \eqsim (\lambda - 2)d_{-\infty}$, contradicting $d_\infty + d_{-\infty} \in \mathcal{D}_\Gamma$.

From (4.38) we get

$$d = d_\infty + 2d_{-\infty} = d_\infty + 2(d_* - d_\infty) = 2d_* - 3d_\infty,$$

and $2d_* - d = 3d_\infty \geq 0$. This gives $d \leq 2d_*$ and hence $\mathrm{Dil}(d, d_*) \leq 2$. But, if $2 = \mathrm{Dil}(d, d_*) + \alpha$ for some $\alpha > 0$, then we would have

$$3d_\infty = 2d_* - d = (\mathrm{Dil}(d, d_*) + \alpha)d_* - d = \alpha d_* + (\mathrm{Dil}(d, d_*)d_* - d) \gtrsim \alpha d_*,$$

contradicting that $d_\infty \in \partial_M \mathcal{D}_\Gamma$. We obtain $\mathrm{Dil}(d, d_*) = 2$, and by the same argument we deduce $\mathrm{Dil}(d_*, d) = 2$.

To conclude the result, if $\sigma_\bullet = \sigma_\bullet^{\tau_*/\tau}$ is the Manhattan geodesic for $\tau = [d]$ and $\tau_* = [d_*]$, then by Proposition 4.6.3 there are pseudo metrics $d^\sigma_{\pm\infty} \in \partial_M \mathcal{D}_\Gamma$ representing $\sigma_{\pm\infty}$ and satisfying

$$d^\sigma_\infty \approx \mathrm{Dil}(d, d_*)d_* - d = 2d_* - d = 3d_\infty,$$
$$d^\sigma_{-\infty} \approx \mathrm{Dil}(d_*, d)d - d_* = 2d - d_* = 3d_{-\infty}.$$

We get that d^σ_∞ is roughly similar to d_∞ and $d^\sigma_{-\infty}$ is roughly similar to $d_{-\infty}$, so that $\rho = \sigma_{-\infty}$ and $\rho_* = \sigma_\infty$, as desired. □

Corollary 4.6.10. *If $[d_\infty], [d_{-\infty}] \in \partial_M \mathscr{D}_\Gamma$ are transverse, then there exist $d, d_* \in \mathcal{D}_\Gamma$ such that*

$$d_\infty \approx \mathrm{Dil}(d, d_*)d_* - d \quad \textit{and} \quad d_{-\infty} \approx \mathrm{Dil}(d_*, d)d - d_*.$$

4.7 Examples of boundary metric structures

Many interesting and widely studied isometric actions on hyperbolic spaces induce pseudo metrics in $\mathcal{D}_\Gamma$, and the same holds for $\partial_M \mathcal{D}_\Gamma$. In this section we provide concrete examples of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$ and $\partial_M \mathcal{D}_\Gamma$. From this we will deduce that when Γ is a surface (resp. free group), the Manhattan boundary is an extension of the Thurston (resp. Culler-Vogtmann) boundary for Teichmüller (resp. outer) space, see Corollaries 4.7.7 and 4.7.13. Theorem 1.2.12 from the Introduction will be a consequence of Corollaries 4.7.7 and 4.7.13 and Propositions 4.7.17 and 4.7.18.

4.7.1 Useful criteria

In general, verifying condition (4.36) in Definition 4.6.1 is not at all direct. Instead, we will rely on the following criterion, which will be used in subsequent subsections.

Lemma 4.7.1. *A left-invariant pseudo metric d on Γ belongs to $\overline{\mathcal{D}}_\Gamma$ if and only if ℓ_d is non-identically zero and additionally:*

(i) d is α-roughly geodesic for some $\alpha \geq 0$; and,

(ii) if $d_0 \in \mathcal{D}_\Gamma$ is α_0-roughly geodesic, then there exists some $C \geq 0$ such that if $\gamma \subset \Gamma$ is an (α_0, d_0)-rough geodesic with endpoints x, y, then γ is C-Hausdorff close in d to an (α, d)-rough geodesic with endpoints x, y.

This lemma follows immediately from the next statement.

Lemma 4.7.2. *Let d_0 and d be left-invariant pseudo metrics on Γ, and assume $d_0 \in \mathcal{D}_\Gamma$ is δ_0-hyperbolic and α_0-roughly geodesic. Then the following conditions are equivalent:*

1. *There exists $\lambda > 0$ such that*
$$(x|y)_{w,d} \leq \lambda (x|y)_{w,d_0} + \lambda \quad \text{for all } x, y, w \in \Gamma.$$
2. *d is α-roughly geodesic for some α, and there is $C > 0$ such that for all $x, y \in \Gamma$ the following holds: if γ is an (α_0, d_0)-rough geodesic with endpoints x, y, and β is an (α, d)-rough geodesic with endpoints x, y, then β and γ are C-Hausdorff close in the pseudo metric d.*

Remark 4.7.3. By applying similar computations as in the lemma above, it can be proven that indeed *every* pseudo metric in $\overline{\mathcal{D}}_\Gamma$ is hyperbolic. Details are left to the reader.

Proof. Suppose first that d satisfies (1), so that it is α-roughly geodesic by Proposition 4.2.8. To prove (2), let γ and β be (α_0, d_0) and (α, d)-rough geodesics respectively, with endpoints $x, y \in \Gamma$. We claim that these geodesics are C-Hausdorff close in d for some C independent of β and γ.

To this end, let $u \in \gamma$, so that $(x|y)_{u,d_0} \leq 3\alpha_0/2$ and by (1) we get $(x|y)_{u,d} \leq 3\lambda\alpha_0/2 + \lambda$. As in the proof of Lemma 4.2.11 we can find some $v \in \beta$ such that

$$|d(x,u) - d(x,v)| \leq 3\lambda\alpha_2 + 2\lambda + \alpha + 1, \tag{4.39}$$

and hence

$$|d(u,y) - d(v,y)| \leq 6\lambda\alpha_2 + 4\lambda + 4\alpha + 1. \tag{4.40}$$

Also, by δ_0 hyperbolicity of d_0 we have

$$\min\{(x|v)_{u,d}, (v|y)_{u,d}\} \leq \lambda(x|y)_{u,d_0} + \lambda\delta_0 + \lambda \leq 3\lambda\alpha_0/2 + \lambda\delta_0 + \lambda. \tag{4.41}$$

Independently on which Gromov product achieves the minimum on the left-hand side of (4.41), by applying inequalities (4.39) or (4.40) we end up concluding that $d(u,v) \leq \lambda(9\alpha_0 + 2\delta_0 + 6) + 4\delta + 1 =: C_1$. This implies that γ is contained in the C_1-neighborhood of β with respect to d.

Now take $v \in \beta$, so that $(x|y)_{v,d} \leq 3\alpha/2$. By Proposition 4.2.8, we can deduce that there exists $D \geq 0$ independent of β and γ and some $u \in \gamma$ such that

$$\max(|d(x,u) - d(x,v)|, |d(u,y) - d(v,y)|) \leq D.$$

As before, we can conclude that $d(u,v) \leq C_2$ for some uniform C_2, and hence α and β are C-Hausdorff close with respect to d, for $C = \max(C_1, C_2)$. This proves our claim and the implication (1) $\Rightarrow$ (2).

Conversely, suppose d satisfies (2) so that it is α-roughly geodesic for some $\alpha \geq 0$. Let $x, y, w \in \Gamma$, and let p be a (κ_0, d_0)-quasi-center for x, y, w. We claim that p is a $(\tilde{\kappa}, d)$-quasi-center for x, y, w, with $\tilde{\kappa}$ independent of x, y, w.

Let $\gamma_1, \gamma_2, \gamma_3$ be (α_0, d_0)-rough geodesics joining x and y, y and w, and w and x, respectively. Then there is some D_0 depending only on δ_0, α_0 and κ_0 such that $d_0(p, \gamma_i) \leq D_0$ for $i \in \{1,2,3\}$. If $\beta_1, \beta_2, \beta_3$ are (α, d)-rough geodesics joining x and y, y and w, and w and x, respectively, then by (2), there is some $C \geq 0$ depending only on d_0 and d such that β_i and γ_i are C-Hausdorff close in (Γ, d).

Also, since d_0 is quasi-isometric to a word metric and Γ is finitely generated, we can find some $\lambda_0 > 0$ such that $d \leq \lambda_0 d_0 + \lambda_0$. In particular we get

$$d(p, \beta_i) \leq \lambda_0 D_0 + \lambda_0 + C$$

for all $i \in \{1,2,3\}$, implying

$$\max\{(x|y)_{p,d}, (y|w)_{p,d}, (w|x)_{p,d}\} \leq 3\alpha/2 + \lambda_0 D_0 + \lambda_0 + C.$$

Therefore, p is a $(\tilde{\kappa}, d)$-quasi-center, with $\tilde{\kappa} = 3\alpha/2 + \lambda_0 D_0 + \lambda_0 + C$, which proves the claim.

From this, we deduce

$$\begin{aligned}(x|y)_{w,d} &\leq \tilde{\kappa}/2 + d(p,w) \leq \tilde{\kappa}/2 + \lambda_0 d_0(p,w) + \lambda_0 \\ &\leq \tilde{\kappa}/2 + \lambda_0 [\kappa_0 + (x|y)_{w,d_0}] + \lambda_0 \\ &= \lambda_0 (x|y)_{w,d_0} + \lambda_0 + \lambda_0\kappa_0 + \tilde{\kappa}/2,\end{aligned}$$

and d satisfies 1) with $\lambda = \lambda_0 + \lambda_0\kappa_0 + \tilde{\kappa}/2$. □

We also need a criterion that guarantees non-triviality of the stable translation length.

Lemma 4.7.4. *Let d be a left-invariant, δ-hyperbolic, and α-roughly geodesic pseudo metric on the (non-necessarily hyperbolic) group Γ. Then ℓ_d is non-identically zero if and only if (Γ, d) is unbounded.*

Proof. It is enough to prove that if $\operatorname{diam}(\Gamma, d) \geq L := 9\alpha + 12\delta + 2$, then there is some $x \in \Gamma$ with $\ell_d[x] > 0$. To this end, let $x \in \Gamma$ be such that $d(x, o) \geq L$. By our α-rough geodesic assumption there is some $u \in \Gamma$ such that if we set $v := u^{-1}x$, then

$$|d(v,o) - d(x,o)/2| \leq (3\alpha+1)/2 \quad \text{and} \quad d(u,o) + d(v,o) \leq d(x,o) + 3\alpha.$$

Also, by [Ore18, Thm. 1.2] applied to $f = u, g = v$ and with base-point the identity element $o \in \Gamma$, we get

$$d(x,o) \leq \max\left\{d(u,o) + \ell_d[v], d(v,o) + \ell_d[u], \frac{d(u,o) + d(v,o) + \ell_d[x]}{2}\right\} + 6\delta.$$

Therefore, either some of the elements x, u, v have positive stable translation lengths, or

$$L \leq d(x,o) \leq \max\left\{d(u,o), d(v,o), \frac{d(u,o) + d(v,o)}{2}\right\} + 6\delta \leq d(x,o)/2 + (9\alpha+1)/2 + 6\delta,$$

which is a contradiction since $L > 9\alpha + 1 + 12\delta$. □

We also require the following lemma, which asserts that $\overline{\mathcal{D}}_\Gamma$ is closed under equivariant quasi-isometry among roughly geodesic pseudo metrics. It is an immediate consequence of Corollary 2.3.9 and Proposition 2.3.11.

Lemma 4.7.5. *Let $d \in \overline{\mathcal{D}}_\Gamma$, and let $\tilde{d}$ be a roughly geodesic, left-invariant pseudo metric on Γ such that the identity map $id : (\Gamma, d) \to (\Gamma, \tilde{d})$ is a quasi-isometry. Then $\tilde{d} \in \overline{\mathcal{D}}_\Gamma$, and $d \in \partial_M \mathcal{D}_\Gamma$ if and only if $\tilde{d} \in \partial_M \mathcal{D}_\Gamma$.*

4.7.2 Bounded backtracking and actions on $\mathbb{R}$-trees

Some hyperbolic groups act naturally and non-trivially on $\mathbb{R}$-trees, see for instance [BF95], [Pau91]. Extending the definition given in [Gab+98], we say that the isometric action of the hyperbolic group Γ on the $\mathbb{R}$-tree (T, d_T) has *bounded backtracking* if the following holds: for some (any) finite, symmetric, generating subset $S \subset \Gamma$ and some (any) $p \in T$, there exists $C \geq 0$ such that if $\gamma \subset \Gamma$ is a geodesic in d_S joining o and x, then $\gamma \cdot p$ is C-Hausdorff close to the geodesic in T joining p and xp. The next proposition relates bounded backtracking and pseudo metrics belonging to $\overline{\mathcal{D}}_\Gamma$.

Proposition 4.7.6. *Suppose Γ acts isometrically on the $\mathbb{R}$-tree T, so that the action has no global fixed point. Then the orbit pseudo metrics for the action of Γ on T belong to $\overline{\mathcal{D}}_\Gamma$ if and only if the action has bounded backtracking. In particular, when Γ is not virtually free, isometric actions with bounded backtracking induce pseudo metrics belonging to $\partial_M \mathcal{D}_\Gamma$.*

Recall that an action of a hyperbolic group on an $\mathbb{R}$-tree is *small* if the pointwise stabilizer of any set of two points in the $\mathbb{R}$-tree is virtually cyclic. When Γ is a finitely generated non-abelian free group, Guirardel showed that every small, minimal, and isometric action of Γ on an $\mathbb{R}$-tree has bounded backtracking [Gui98, Cor. 2]. In addition, the Culler-Morgan compactification $\overline{\mathscr{CV}}_\Gamma$ of the outer space coincides with the space of (rough similarity) classes of orbit pseudo metrics induced by very small isometric actions of Γ on $\mathbb{R}$-trees [BF93, Thm. 2.2]. Therefore, from Proposition 4.7.6 we deduce that $\overline{\mathscr{CV}}_\Gamma$ naturally injects into $\overline{\mathscr{D}}_\Gamma$.

Corollary 4.7.7. *Let Γ be a finitely generated non-abelian free group acting isometrically on the $\mathbb{R}$-tree T so that the action is small. Then the orbit pseudo metrics induced by this action belong to $\overline{\mathcal{D}}_\Gamma$. In particular, there exists a natural injective map $\overline{\mathscr{CV}}_\Gamma \hookrightarrow \overline{\mathscr{D}}_\Gamma$ that sends the Culler-Vogtmann boundary $\partial\mathscr{CV}_\Gamma$ into $\partial_M \mathscr{D}_\Gamma$.*

For the proof of Proposition 4.7.6, we need a preliminary lemma.

Lemma 4.7.8. *Let Γ be a (not necessarily hyperbolic) finitely generated group acting isometrically on the $\mathbb{R}$-tree (T, d_T). Then for any $p \in T$, the pseudo metric $d_T^p(x, y) = d_T(xp, yp)$ on Γ is hyperbolic and roughly geodesic.*

Proof. Clearly, d is hyperbolic. To show it is roughly geodesic, let $S \subset \Gamma$ be a finite, symmetric generating set, and let $\phi : \text{Cay}(\Gamma, S) \to T$ be the unique Γ-equivariant map such that $\phi(o) = p$, and each edge from o to $s \in S$ in $\text{Cay}(\Gamma, S)$ is linearly mapped to the geodesic in T joining p and sp. Then ϕ is L-Lipschitz, with $L = \max_{s \in S} d_T(p, sp)$.

Now, let $x, y \in \Gamma$, and let $[x, y]_T$ denote the unique geodesic segment in T joining xp and yp. Since ϕ is continuous, for any geodesic segment $\gamma \subset \text{Cay}(\Gamma, S)$ joining x and y, the image $\phi(\gamma)$ contains $[x, y]_T$. Therefore, if $xp = p_0, p_1, \ldots, p_n = yp$ is a 1-rough geodesic in T, then for any i there is some $q_i \in \gamma$ such that $d_T(p_i, \phi(q_i)) \leq 3/2$. Also, for each q_i there is some vertex $x_i \in \Gamma$ such that $d_S(q_i, x_i) \leq 1$, and hence $d_T(p_i, x_i p) = d_T(p_i, \phi(x_i)) \leq d_T(p_i, \phi(q_i)) + d_T(\phi(q_i), \phi(x_i)) \leq 3/2 + L/2$. If we choose $x_0 = x$ and $x_n = y$, we conclude that the sequence $x = x_0, x_1, \ldots, x_n = y$ is a $(4 + L, d_T^p)$-rough geodesic joining $x, y \in \Gamma$, which completes the proof. □

Proof of Proposition 4.7.6. Let Γ act on the $\mathbb{R}$-tree (T, d_T) as in the statement, and for $p \in T$, consider the pseudo metric d_T^p. This pseudo metric has non-constant stable translation length function since the action has no global fixed point. As a consequence of Lemma 4.7.8, d_T^p also satisfies property (i) of Lemma 4.7.1. Therefore, the theorem follows by Lemma 4.7.1, since d_T^p satisfying property (ii) of that lemma is equivalent to the action having bounded backtracking. □

4.7.3 Liouville embedding of the space of projective geodesic currents

Throughout this subsection let Γ be a surface group, and fix a geometric action of Γ on the hyperbolic plane $(\mathbb{H}^2, d_{\mathbb{H}^2})$ with quotient surface S (i.e., a marking ϕ). In this case the space of geodesic currents $\mathcal{C}urr(\Gamma)$ can be equipped with an *intersection number*

$$i : \mathcal{C}urr(\Gamma) \times \mathcal{C}urr(\Gamma) \to \mathbb{R}.$$

This function was introduced by Bonahon [Bon88] and can be characterized by the following properties:

- it is continuous and symmetric;
- $i(\lambda_1 \eta_1, \lambda_2 \eta_2) = \lambda_1 \lambda_2 i(\eta_1, \eta_2)$ for all for all $\eta_1, \eta_2 \in \mathcal{C}urr(\Gamma)$ and $\lambda_1, \lambda_2 \geq 0$; and,
- if $[x], [y] \in \mathbf{conj}'$ are conjugacy classes represented by closed geodesics $\gamma_{[x]}, \gamma_{[y]}$ (with respect to any hyperbolic metric on S), then $i(\eta_{[x]}, \eta_{[y]})$ is the geometric intersection number of $\gamma_{[x]}$ and $\gamma_{[y]}$ on S (here $\eta_{[x]}, \eta_{[y]}$ are the corresponding rational currents).

The action of Γ on $\mathbb{H}^2$ induces a Γ-equivariant bijection between the set $\mathcal{G}$ of geodesics in $\mathbb{H}^2$ and $\partial^2\Gamma$. In this way, we consider geodesic currents as measures on $\mathcal{G}$ that are Γ-invariant, flip-invariant, and locally finite.

Following [Bur+21, Sec. 4], to each $\mu \in \mathcal{C}urr(\Gamma)$ we construct the pseudo metric d_μ on $\mathbb{H}^2$ as follows: for $p, q \in \mathbb{H}^2$, let $[p, q]$ denote the closed geodesic interval in $\mathbb{H}^2$ joining p and q, and we also define $(p, q] = [p, q]\backslash\{p\}$ and $[p, q) = [p, q]\backslash\{q\}$. If $I \subset \mathbb{H}^2$ is any subset, we let $\mathcal{G}_I^\perp$ denote the set of geodesics in $\mathbb{H}^2$ intersecting I exactly once. In this way, the pseudo metric d_μ is given by

$$d_\mu(p, q) = \frac{1}{2}(\mu(\mathcal{G}^\perp_{[p,q)}) + \mu(\mathcal{G}^\perp_{(p,q]})).$$

In [Bur+21, Prop. 4.1] it was proven that d_μ is indeed a *straight* pseudo metric, meaning that for $p, r \in \mathbb{H}^2$ and $q \in [p, r]$ it holds that

$$d_\mu(p, r) = d_\mu(p, q) + d_\mu(q, r).$$

This fact together with [Bur+21, Lem. 4.7] implies that

$$i(\mu, \eta_{[x]}) = \ell_{d_\mu}[x] \tag{4.42}$$

for all $\mu \in \mathcal{C}urr(\Gamma)$ and $[x] \in \mathbf{conj}$.

This construction can be seen as an inverse map to some geodesic currents encoding different geometric structures associated to S and Γ. Now we review some examples of such currents, for which a more detailed discussion appears in [DM22].

Example 4.7.9 (Liouville currents)**.** Let $\mathfrak{g}$ be a negatively curved Riemannian metric on S. The *Liouville measure* $\nu_\mathfrak{g}$ on T^1S is given locally by the product of the Riemannian volume on $(S, \mathfrak{g})$ and the usual Lebesgue measure on the unit sphere, and normalized so that it is a probability measure. This measure is invariant under the geodesic flow on T^1S, so as in Example 4.3.16, the action of Γ on $\widetilde{S}$ induces the *Liouville current* $L_\mathfrak{g} = L_{(\mathfrak{g},\phi)} \in \mathcal{C}urr(\Gamma)$. This current has full support, has no atoms, and satisfies

$$i(L_\mathfrak{g}, \eta_{[x]}) = \ell_{(\widetilde{S},\widetilde{\mathfrak{g}})}[x] \quad \text{for all } [x] \in \mathbf{conj}.$$

The Liouville current induces an injective map from $\mathscr{T}_S^{<0}$ into $\mathbb{P}\mathcal{C}urr(\Gamma)$, which is an embedding when restricted to $\mathscr{T}_S = \mathscr{T}_\Gamma$ [Bon88]. For a description of $L_\mathfrak{g}$ in terms of cross-ratios, see [DM22, Sec. 2.3.3].

Example 4.7.10 (Measured laminations)**.** A non-zero geodesic current $\alpha \in \mathcal{C}urr(\Gamma)$ is a *measured lamination* if $i(\alpha, \alpha) = 0$. Equivalently, α is a geodesic lamination if any two pairs of distinct geodesics $\gamma_1, \gamma_2 \in \mathcal{G}$ in the support of α are disjoint. If α is a geodesic lamination, then the complement in $\mathbb{H}^2$ to the set of geodesics in the support of α consists of simply connected open subsets, and from this data, we can produce an isometric action of Γ on an $\mathbb{R}$-tree T_α. This is the tree *dual* to the lamination α. It can be checked that the metric identification of $(\mathbb{H}^2, d_\alpha)$ is Γ-equivariantly isometric to T_α (see e.g. [DM22, Appendix A]). In particular, we have

$$i(\alpha, \eta_{[x]}) = \ell_{T_\alpha}[x] \quad \text{for all } [x] \in \mathbf{conj}.$$

Bonahon proved in [Bon88] that when projected to $\mathbb{P}\mathcal{C}urr(\Gamma)$, the set of projective measured laminations is the boundary of the image of $\mathscr{T}_\Gamma$ in $\mathbb{P}\mathcal{C}urr(\Gamma)$ discussed in Example 4.7.9. This gives a description of the *Thurston boundary* of $\mathscr{T}_\Gamma$.

Example 4.7.11 (Currents for Hitchin components)**.** If $\rho : \Gamma \to \mathrm{PSL}_m(\mathbb{R})$ is a Hitching representation of Γ in the sense of Example 4.2.7, Martone and Zhang associated in [MZ19] a geodesic current $\omega_\rho \in \mathcal{C}urr(\Gamma)$ satisfying

$$i(\eta_{[x]}, \omega_\rho) = \ell_{d_\rho}[x]$$

for all $[x] \in \mathbf{conj}$, where d_ψ is the pseudo metric from Corollary 4.2.5 [MZ19, Thm. 1.1 & Thm. 3.4]. They also constructed geodesic currents dual other types of Anosov representations [MZ19, Sec. 3].

From equation (4.42) and the work of Otal [Ota90] and Croke [Cro90], we see that two pseudo metrics d_μ and $d_{\mu'}$ on $\mathbb{H}^2$ are roughly isometric if and only if $\mu = \mu'$. For non-zero geodesic currents, these pseudo metrics induce metric structures in $\overline{\mathscr{D}}_\Gamma$. Recall that a geodesic current $\eta \in \mathcal{C}urr(\Gamma)$ is *filling* if $i(\eta, \mu) > 0$ for any non-zero current $\mu \in \mathcal{C}urr(\Gamma)$.

Proposition 4.7.12. *Let Γ be a surface group acting geometrically on $\mathbb{H}^2$. Then for any non-zero geodesic current $\mu \in \mathcal{C}urr(\Gamma)$, the orbit pseudo metrics induced by the action of Γ on $(\mathbb{H}^2, d_\mu)$ belong to $\overline{\mathcal{D}}_\Gamma$, and they belong to $\mathcal{D}_\Gamma$ if and only if μ is filling.*

Since measured laminations are never filling, Example 4.7.10 implies the next corollary.

Corollary 4.7.13. *The assignment $\mu \mapsto \rho_{(\mathbb{H}^2, d_\mu)}$ induces an injective map from the space $\mathbb{P}\mathcal{C}urr(\Gamma)$ of projective geodesic currents into $\overline{\mathscr{D}}_\Gamma$. This map sends the Thurston boundary $\partial\mathscr{T}_\Gamma$ into $\partial_M \mathscr{D}_\Gamma$.*

Remark 4.7.14. All the actions of Γ on $\mathbb{R}$-trees dual to geodesic laminations are small. Conversely, by Skora's theorem [Sko96] every such small action arises in this way, and hence the Thurston boundary of Teichmüller space $\partial\mathscr{T}_\Gamma$ can be described as the space of (rough similarity) classes of orbit pseudo metrics induced by small isometric actions of Γ on $\mathbb{R}$-trees. Therefore, Proposition 4.7.6 and Corollary 4.7.13 imply that if Γ is a surface group, then every small action of Γ on an $\mathbb{R}$-tree induces pseudo metrics in $\overline{\mathcal{D}}_\Gamma$.

We begin the proof of Proposition 4.7.12, for which we fix a non-zero current $\mu \in \mathcal{C}urr(\Gamma)$.

Lemma 4.7.15. *There exists $\lambda_0 > 0$ such that*

$$d_\mu(p,q) \leq \lambda_0 d_{\mathbb{H}^2}(p,q) + \lambda_0 \quad \textit{for all } p, q \in \mathbb{H}^2.$$

Proof. Given $A \geq 0$, we claim that there exists $B_A \geq 0$ such that $d_{\mathbb{H}^2}(p,q) \leq A$ implies $d_\mu(p,q) \leq B_A$. Indeed, since the action of Γ on $\mathbb{H}^2$ is cocompact, there exists a compact set $K \subset \mathbb{H}^2$ such that if $d_{\mathbb{H}^2}(p,q) \leq A$, then $xp, xq \in K$ for some $x \in \Gamma$. The set $\mathcal{G}_K \subset \mathcal{G}$ of geodesics intersecting K is compact, so that $B_A := \mu(\mathcal{G}_K)$ is finite. Therefore, if $p, q \in \mathbb{H}^2$ satisfy $d_{\mathbb{H}^2}(p,q) \leq A$ and x is as above, we deduce that $d_\mu(p,q) = d_\mu(xp, xq) \leq \mu(\mathcal{G}_K) = B_A$, which proves the claim.

Let $\lambda_0 := B_1$. If $p, q \in \mathbb{H}^2$ and $n = \lfloor d_{\mathbb{H}^2}(p,q) \rfloor$, let $p = p_0, p_1, \dots, p_n \in [p,q]$ be such that $d_{\mathbb{H}^2}(p, p_i) = i$ for all $0 \leq i \leq n$. We get

$$d_\mu(p,q) \leq d_\mu(p_0,p_1) + \cdots + d_\mu(p_{n-1},p_n) + d_\mu(p_n,q) \leq (n+1)\lambda_0 \leq \lambda_0 d_{\mathbb{H}^2}(p,q) + \lambda_0,$$

as desired. □

Proof of Proposition 4.7.12. Let $\mu \in \mathcal{C}urr(\Gamma)$ be non-zero, and let λ_0 be the constant from Lemma 4.7.15. We claim that there exists $\lambda_1 > 0$ such that

$$(p|q)_{w,d_\mu} \leq \lambda_1 (p|q)_{w,d_{\mathbb{H}^2}} + \lambda_1 \tag{4.43}$$

for all $p, q, w \in \mathbb{H}^2$. To this end, let m be a $(\kappa, d_{\mathbb{H}^2})$-quasi-center for p, q, w, with κ independent of this triple. If u is the point in $[p,q]$ closest to m, then $d_{\mathbb{H}^2}(m,u) \leq (p|q)_{m,\mathbb{H}^2} \leq \kappa$, so that $d_\mu(m,u) \leq \lambda_0\kappa + \lambda_0$. Since d_μ is straight, we also have

$$d_\mu(p,m) + d_\mu(m,q) \leq 2\lambda_0\kappa + 2\lambda_0 + d_\mu(p,u) + d_\mu(u,q) = 2\lambda_0\kappa + 2\lambda_0 + d_\mu(p,q),$$

and hence $(p|q)_{m,d_\mu} \leq \tilde{\kappa} := \lambda_0\kappa + \lambda_0$. Similarly, we obtain $(p|w)_{m,d_\mu}, (w|q)_{m,d_\mu} \leq \tilde{\kappa}$, so that m is a $(\tilde{\kappa}, d_\mu)$-quasi-center for p, q and w. In particular, we deduce $(p|q)_{w,d_\mu} \leq d_\mu(w,m) + \tilde{\kappa}$ $\leq \lambda_0(p|q)_{w,d_{\mathbb{H}^2}} + \lambda_0 + \tilde{\kappa} + 2\lambda_0\kappa$, which proves the claim with $\lambda_1 = \lambda_0 + \tilde{\kappa} + 2\lambda_0\kappa$.

Now take $w \in \mathbb{H}^2$ and let $d^w_\mu, d^w_{\mathbb{H}^2}$ be the corresponding orbit pseudo metrics induced by the action of Γ on $\mathbb{H}^2$. These pseudo metrics also satisfy a version of (4.43), and since $d^w_{\mathbb{H}^2} \in \mathcal{D}_\Gamma$ we see that d^w_μ satisfies the inequality (4.36) in Definition 4.6.1. Also, since μ is non-zero, there exists $x \in \Gamma$ such that $i\left(\mu, \eta_{[x]}\right) = \ell_{d_\mu}[x] > 0$, implying $d^w_\mu \in \overline{\mathcal{D}}_\Gamma$.

Finally, if $\eta \in \mathcal{C}urr(\Gamma)$ is any filling current (for example, a Liouville current for a marked negatively curved Riemannian metric), then $i(\eta,\beta) > 0$ for all $\beta \in \mathcal{C}urr(\Gamma)\backslash\{0\}$, and hence the function $\Phi : \mathbb{P}\mathcal{C}urr(\Gamma) \to \mathbb{R}_{\geq 0}$ given by

$$[\beta] \mapsto \frac{i(\mu,\beta)}{i(\eta,\beta)}$$

is well-defined, continuous, and positive. Since $\mathbb{P}\mathcal{C}urr(\Gamma)$ is compact and $i(\eta, \eta_{[x]}) = \ell_{d_{\mathbb{H}^2}}[x]$ for all $[x] \in \mathbf{conj}$, we deduce that μ is filling if and only if there exists $A > 0$ such that $\ell_{d_\mu}[x] \geq A\ell_{d_{\mathbb{H}^2}}[x]$ for all $[x] \in \mathbf{conj}$, which happens if and only if $d^w_\mu \in \mathcal{D}_\Gamma$. □

4.7.4 Combinatorial examples

In this subsection we provide examples of (boundary) metric structures of combinatorial nature. We start with a connected graph X with graph metric d_X, and let $\mathbb{K} = \{X_j\}_{j\in J}$ be a family of subgraphs of X. From this data, we construct the connected graph $X_{\mathbb{K}}$ obtained by adding to X the new edges $e_{x,y,j}$ with endpoints x, y whenever $j \in J$ and x, y are vertices of X_j (thus X is a subgraph of $X_{\mathbb{K}}$). Let $d_{X,\mathbb{K}}$ be the simplicial metric on $X_{\mathbb{K}}$. The following result is due to Kapovich and Rafi, and will be used along with Lemma 4.7.1 to find examples of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$.

Proposition 4.7.16 (Kapovich–Rafi [KR14, Prop. 2.6])**.** *Let X be a connected graph such that the simplicial metric d_X is hyperbolic, and let $\mathbb{K}$ be a family of uniformly quasiconvex subgraphs of X. Then $(X_{\mathbb{K}}, d_{X,\mathbb{K}})$ is also hyperbolic, and there is a constant $C \geq 0$ such that whenever $x, y \in X^{(0)}$, $[x, y]_X$ is a d_X-geodesic from x to y in X and $[x, y]_{X,\mathbb{K}}$ is a $d_{X,\mathbb{K}}$-geodesic from x to y in $X_{\mathbb{K}}$, then $[x, y]_X$ and $[x, y]_{X,\mathbb{K}}$ are C-Hausdorff close in $(X_{\mathbb{K}}, d_{X,\mathbb{K}})$.*

Now, let $S \subset \Gamma$ be a finite, symmetric generating set with Cayley graph $(\mathrm{Cay}(\Gamma, S), d_S)$. If $\mathcal{H}$ is a set of subgroups of Γ, the *coned-off Cayley graph* $(\mathrm{Cay}(\Gamma, S, \mathcal{H}), d_{S,\mathcal{H}})$ is defined as follows. For each left coset xH with $x \in \Gamma$ and $H \in \mathcal{H}$, add a new vertex $v(xH)$ to $\mathrm{Cay}(\Gamma, S)$, and add an edge of length $1/2$ from this new vertex to each element of xH.

When we cone-off finitely many quasiconvex subgroups, the orbit pseudo metrics induced by the action of Γ on the corresponding coned-off Cayley graphs will belong to $\overline{\mathcal{D}}_\Gamma$.

Proposition 4.7.17. *Let $\mathcal{H}$ be a finite set of quasiconvex subgroups of Γ, and for a finite, symmetric generating set $S \subset \Gamma$, consider the coned-off Cayley graph $\mathrm{Cay}(\Gamma, S, \mathcal{H})$. If all the subgroups in $\mathcal{H}$ are infinite index in Γ, then the orbit pseudo metrics induced by the action of Γ on $\mathrm{Cay}(\Gamma, S, \mathcal{H})$ belong to $\overline{\mathcal{D}}_\Gamma$. In addition, these pseudo metrics belong to $\partial_M \mathcal{D}_\Gamma$ if and only if some subgroup in $\mathcal{H}$ is infinite.*

Proof. By Lemma 4.7.5, the conclusion of the proposition is independent of the chosen finite generating set S, so without loss of generality we can assume S contains finite generating sets for each $H \in \mathcal{H}$. In this way, for each $H \in \mathcal{H}$, we can consider its Cayley graph $\mathrm{Cay}(H, S \cap H)$ as a subgraph of Cay (Γ, S). Therefore, we can apply Proposition 4.7.16 to $X = \mathrm{Cay}(\Gamma, S)$ and $\mathbb{K} = \{x \,\mathrm{Cay}(H, S \cap H) \colon x \in \Gamma, H \in \mathcal{H}\}$, so that the inclusion $\mathrm{Cay}(\Gamma, S) \to X_{\mathbb{K}}$ maps geodesics in $\mathrm{Cay}(\Gamma, S)$ uniformly close to geodesics in $X_{\mathbb{K}}$. Since $X_{\mathbb{K}}$ is geodesic by construction, any orbit pseudo metric from the isometric action of Γ on $(X_X, d_{X,\mathbb{K}})$ will be roughly geodesic, and hence will satisfy properties (i) and (ii) of Lemma 4.7.1. As all the subgroups $H \in \mathcal{H}$ are infinite index in Γ, $(X_{\mathbb{K}}, d_{X,\mathbb{K}})$ is unbounded, so by Lemmas 4.7.1 and 4.7.4 the orbit pseudo metrics induced by the action of Γ on $(X_{\mathbb{K}}, d_{X,\mathbb{K}})$ belong to $\overline{\mathcal{D}}_\Gamma$.

Finally, since the coned-off Cayley graph $\mathrm{Cay}(\Gamma, S, \mathcal{H})$ is both geodesic and Γ-equivariantly quasi-isometric to $X_{\mathbb{K}}$, by Lemma 4.7.5 we conclude that orbit pseudo metrics induced by the action of Γ on $\mathrm{Cay}(\Gamma, S, \mathcal{H})$ belong to $\overline{\mathcal{D}}_\Gamma$ as well. It is clear that these pseudo metrics belong to $\mathcal{D}_\Gamma$ if and only if all the subgroups in $\mathcal{H}$ are finite. □

We can apply the proposition above to show that cocompact actions on CAT(0) cube complexes induce pseudo metrics in $\overline{\mathcal{D}}_\Gamma$, as long as the wall stabilizers are quasiconvex.

Proposition 4.7.18. *Let X be* CAT(0) *cube complex with combinatorial metric $d_{X^{(1)}}$ on $X^{(1)}$, and assume Γ acts cocompactly on X by simplicial isometries. Also, suppose that:*

1. *wall stabilizers are quasiconvex; and,*

2. *the action has no global fixed point.*

Then the orbit pseudo metrics for the action of Γ *on* $(X^{(1)}, d_{X^{(1)}})$ *belong to* $\overline{\mathcal{D}}_\Gamma$*. In addition, they belong to* $\partial_M \mathcal{D}_\Gamma$ *if and only if some vertex stabilizer is infinite.*

Specializing the proposition above to 1-dimensional CAT(0) cube complexes, we obtain that Bass-Serre tree actions with quasiconvex edge groups induce pseudo metrics in $\overline{\mathcal{D}}_\Gamma$.

Corollary 4.7.19. *Let* T *be a Bass-Serre tree for a finite graph of groups decomposition of* Γ*. Suppose this action satisfies:*

1. *the edge subgroups are quasiconvex in* Γ*; and*
2. *the vertex subgroups are infinite index in* Γ*.*

Then the orbit pseudo metrics for the action of Γ *on* T *belong to* $\overline{\mathcal{D}}_\Gamma$*. In addition, they belong to* $\partial_M \mathcal{D}_\Gamma$ *if and only if some vertex stabilizer is infinite.*

Proof of Proposition 4.7.18. Let $\mathcal{H}$ be a complete set of representatives of the conjugacy classes of vertex stabilizers for the action of Γ on X. This set is finite, and since wall stabilizers are quasiconvex, by [GM18, Thm. A] all the subgroups in $\mathcal{H}$ are quasiconvex. Also, since the action of Γ on X is cocompact and has no global fixed point, it has unbounded orbits, so all the subgroups in $\mathcal{H}$ are infinite index in Γ. Therefore, Proposition 4.7.17 applies to $\mathcal{H}$, and hence the orbit pseudo metrics induced by the action of Γ on the coned-off Cayley graph $\mathrm{Cay}(\Gamma, S, \mathcal{H})$ belong to $\overline{\mathcal{D}}_\Gamma$.

To conclude the result, by [CC07, Thm. 5.1], $(X^{(1)}, d_{X^{(1)}})$ is Γ-equivariantly quasi-isometric to $\mathrm{Cay}(\Gamma, S, \mathcal{H})$, and the first conclusion follows from Lemma 4.7.5. The second conclusion follows from the cocompactness of the action since in this case, properness is equivalent to the finiteness of all the vertex stabilizers. □

4.7.5 An exotic example

So far, most of the examples of pseudo metrics d in $\partial_M \mathcal{D}_\Gamma$ that we have exhibited satisfy $\ell_d[x] = 0$ for some non-torsion conjugacy $[x] \in \mathbf{conj}'$. We end this section by showing an example of a boundary pseudo metric for which this property does not hold. In [Kap16], Kapovich constructed an example of a hyperbolic graph (Y, d_Y) and an isometric action of the rank-2 free group $\Gamma = F(a, b)$ on Y satisfying the following properties:

$(1)_Y$ The action is *acylindrical*.

$(2)_Y$ The action is *purely loxodromic*. That is, every non-trivial element of $F(a, b)$ acts loxodromically on Y.

$(3)_Y$ If $x \in F(a, b)$ is non-trivial, then $\ell_Y[x] \geq 1/7$.

$(4)_Y$ For any $p \in Y$, the orbit $F_2 \cdot p \subset Y$ is quasiconvex in Y.

$(5)_Y$ For any $p \in Y$, there exists $C \geq 1$ such that for any $x, y \in F(a,b)$, if γ is the vertex set of a geodesic joining x and y in $\mathrm{Cay}(F(a,b), \{a^{\pm}, b^{\pm}\})$, and if β is a geodesic joining xp and yp in Y , then $\gamma \cdot p$ and β are C-Hausdorff close in Y.

$(6)_Y$ For any $p \in Y$, the orbit map $F(a,b) \to Y$, $x \to xp$, is not a quasi-isometric embedding.

Given $p \in Y$ we consider the orbit pseudo metric $d := d_Y^p$ on Γ. Since Y is hyperbolic, d is also hyperbolic, and property $(4)_Y$ implies that d is roughly geodesic, so that d satisfies (i) in Lemma 4.7.1. Similarly, $(5)_Y$ implies (ii) and $(2)_Y$ implies that ℓ_d is non-trivial. Therefore, by Lemma 4.7.1 we get that $d \in \overline{\mathcal{D}}_\Gamma$. In addition, by $(6)_Y$, d is not quasi-isometric to a word metrics, so we indeed obtain:

Proposition 4.7.20. *The orbit pseudo metrics induced by the action of $F(a,b)$ on Y described above belong to $\partial_M \mathcal{D}_{F(a,b)}$.*

4.8 Marked length spectrum in $\overline{\mathcal{D}}_\Gamma$

In this section we deduce some results about the stable translation length function for pseudo metrics in $\overline{\mathcal{D}}_\Gamma$. We apply these results to some of the examples found in Section 4.7 to disprove a conjecture of Bonahon about the marked length spectra of small actions on $\mathbb{R}$-trees. This proves Theorem 1.2.13 in the Introduction.

4.8.1 Continuous extension of the stable translation length function

Given a function $F : \mathbf{conj} \to \mathbb{R}_{\geq 0}$ satisfying $F[x^n] = |n|F[x]$ for all $[x] \in \mathbf{conj}$ and $n \in \mathbb{Z}$, we can ask whether it extends continuously to a function on $\mathcal{C}urr(\Gamma)$. That is, whether there exists a continuous function $\overline{F} : \mathcal{C}urr(\Gamma) \to \mathbb{R}$ such that $\overline{F}(\lambda_n \eta_{[x_n]})$ converges to $\overline{F}(\eta)$ whenever $\lambda_n > 0$ and $[x_n] \in \mathbf{conj}$ are sequences such that $\lambda_n \eta_{[x_n]}$ converges to the geodesic current η.

When Γ is torsion-free and d is the orbit pseudo metric induced by geometric action on a geodesic metric space, Erlandsson, Parlier and Souto proved that ℓ_d extends continuously to $\mathcal{C}urr(\Gamma)$ [EPS20, Thm. 1.5]. This result can be extended to any pseudo metric in $\overline{\mathcal{D}}_\Gamma$.

Proposition 4.8.1. *If Γ is virtually torsion-free, then for any $d \in \overline{\mathcal{D}}_\Gamma$, its stable translation length ℓ_d extends uniquely to a continuous function $\ell_d : \mathcal{C}urr(\Gamma) \to \mathbb{R}$.*

Proof. Suppose first that Γ is torsion-free. By [EPS20, Thm. 1.5], the result follows if $d \in \mathcal{D}_\Gamma$ is a word metric. For arbitrary $d \in \mathcal{D}_\Gamma$, we apply Proposition 4.4.3 to find a sequence of word metrics $d_n \in \mathcal{D}_\Gamma$ such that $\Lambda_n := \Delta([d], [d_n])$ tends to 1 as n tends to ∞. By choosing $\lambda_n > 0$ such that $v_{\lambda_n d_n} = v_d$ for each n, by Lemma 4.3.24 we have ${\Lambda_n}^{-1} \ell_{\lambda_n d_n} \leq \ell_d \leq \Lambda_n \ell_{\lambda_n d_n}$ for all n, and hence the functions $\log \ell_{\lambda_n d_n} : \mathcal{C}urr(\Gamma) \to \mathbb{R}$ converge uniformly to some function f.

Continuity of each ℓ_{d_n} then guarantees the continuity of e^f, which is the desired extension of ℓ_d to $\mathcal{C}urr(\Gamma)$.

Suppose now that $d \in \partial_M \mathcal{D}_\Gamma$, and given $d_0 \in \mathcal{D}_\Gamma$, let $d_1 \in \mathcal{D}_\Gamma$ be such that

$$\ell_d = \mathrm{Dil}(d_0, d_1)\ell_{d_1} - \ell_{d_0},$$

which can be found by Corollary 4.6.7. By our first case, the stable translation lengths ℓ_{d_0} and ℓ_{d_1} can be extended continuously to $\mathcal{C}urr(\Gamma)$, so the same holds for ℓ_d.

In the general case that Γ contains the torsion-free group Γ_0 as a finite index subgroup, the conclusion follows since every geodesic current on Γ is a geodesic current on Γ_0. In all these cases, uniqueness is deduced from the density of rational currents [Bon91, Thm. 7]. □

Remark 4.8.2. In a forthcoming preprint of Kapovich and Martínez-Granado, the conclusion of the proposition above is obtained without the virtual torsion-free assumption [KM].

The corollary above implies that when Γ is virtually torsion-free, there exists a left-invariant pseudo metric d_0 on Γ, quasi-isometric to a word metric, for which ℓ_{d_0} does not extend continuously to $\mathcal{C}urr(\Gamma)$ (in particular, d_0 is not hyperbolic, cf. [BHM11, Prop. A.11]). Indeed, let $\phi : \Gamma \to \overline{\Gamma}$ be a non-elementary hyperbolic quotient of Γ with infinite kernel, which can be constructed from group theoretical Dehn filling. Given any pseudo metric $d \in \mathcal{D}_{\overline{\Gamma}}$, consider the pseudo metric $\overline{d}(x,y) = d(\phi(x), \phi(y))$ on Γ. The same argument as in the proof of [Bon91, Prop. 11] implies that the marked length spectrum $\ell_{\overline{d}}$ does not extend continuously to $\mathcal{C}urr(\Gamma)$. Therefore, for $d_1 \in \mathcal{D}_\Gamma$, the metric $d_0 = \overline{d} + d_1$ is left-invariant and quasi-isometric to the word metric, and by Corollary 4.8.1 its marked length spectrum $\ell_{d_0} = \ell_{\overline{d}} + \ell_{d_1}$ does not extend continuously to $\mathcal{C}urr(\Gamma)$.

At this point, we may wonder if continuity is the main obstruction for a stable translation length function to represent a point in $\mathscr{D}_\Gamma$. That is, if d is a left-invariant pseudo metric on Γ quasi-isometric to pseudo metrics in $\mathcal{D}_\Gamma$ and ℓ_d extends continuously to $\mathcal{C}urr(\Gamma)$, does there exist $\hat{d} \in \mathcal{D}_\Gamma$ such that $\ell_d = \ell_{\hat{d}}$? The following example shows that it is not the case.

Example 4.8.3 (A bad pseudo metric)**.** Let $\Gamma = F(a,b)$ be a rank-2 free group freely generated by a, b, and consider generating sets $S = \{a^\pm, b^\pm\}$ and $T = \{a^\pm, (ab)^\pm\}$. Let d be the metric $d(x,y) = \max\{d_S(x,y), d_T(x,y)\}$ on Γ, which is left-invariant and quasi-isometric to pseudo metrics on $\mathcal{D}_\Gamma$. Note that $\ell_d[x] = \max\{\ell_S[x], \ell_T[x]\}$ for all $[x] \in \mathbf{conj}$, so that ℓ_d extends continuously to $\mathcal{C}urr(\Gamma)$. We claim that there is no pseudo metric in $\mathcal{D}_\Gamma$ with stable translation length equal to ℓ_d.

First we prove that $d \notin \mathcal{D}_\Gamma$. To do this, consider the sequences $x_n, y_n \in \Gamma$, $n \geq 1$ given by $x_n = b^{-n}$ and $y_n = (ab)^n$. It can be checked that $d(x_n, o) = |x_n|_T = 2n$, $d(o, y_n) = |y_n|_S = 2n$ and $d(x_n, y_n) = |b^n(ab)^n|_S = |b^n(ab)^n|_T = 3n$ for all n, and hence $(x_n|y_n)_{o,d} = n/2$ tends to infinity ad $n \to \infty$, whereas $(x_n|y_n)_S = 0$ for all $n \geq 1$. This shows that d does not satisfy Definition 4.2.1 of a hyperbolic distance-like function, and in particular d does not belong to $\mathcal{D}_\Gamma$.

Now, assume for the sake of contradiction that $\ell_d = \ell_{\hat{d}}$ for some $\hat{d} \in \mathcal{D}_\Gamma$. By Corollary 4.2.21 there is a finite set $B \subset \Gamma$ and a constant $C > 0$ such that for all $d_* \in \{d_S, d_T, \hat{d}\}$ and

$x, y \in \Gamma$ we have

$$d_*(x,y) \leq \max_{u \in B} \ell_{d_*}[x^{-1}yu] + C.$$

We can verify that the inequality above is also verified for $d_* = d$, and hence for all $x, y \in \Gamma$ we get

$$d(x,y) \leq \max_{u \in B} \ell_d[x^{-1}yu] + C = \max_{u \in B} \ell_{\hat{d}}[x^{-1}yu] + C \leq \max_{u \in B} \hat{d}(x, yu) + C \leq \hat{d}(x,y) + D$$

for $D = C + \max_{u \in B} \hat{d}(o, v)$. Similarly, we can find D' such that $\hat{d}(x,y) \leq d(x,y) + D_1$ for all x, y, implying that d and $\hat{d}$ are roughly isometric. This is the desired contradiction since this would imply that $d \in \mathcal{D}_\Gamma$.

By the same argument, we can prove that the stable translation length function of $\tilde{d} = \sqrt{d_S^2 + d_T^2}$ is continuous, but not the stable length function of any pseudo metric in $\mathcal{D}_\Gamma$.

4.8.2 Application: Counterexamples to a conjecture of Bonahon

As we saw in Corollaries 4.7.7 and 4.7.13 (see also Remark 4.7.14), many small actions of hyperbolic groups on $\mathbb{R}$-trees induce pseudo metrics in $\overline{\mathcal{D}}_\Gamma$, and hence their translation length functions extend continuously to $\mathcal{C}urr(\Gamma)$ by Proposition 4.8.1. In [Bon91, p. 164], Bonahon conjectured that the converse of this should hold. That is, the only isometric actions of a hyperbolic group Γ on $\mathbb{R}$-trees whose stable translation length continuously extends to $\mathcal{C}urr(\Gamma)$ are those that are small. However, according to Corollary 4.7.19 and Proposition 4.8.1, such a continuous extension exists for every Bass-Serre tree action with quasiconvex edge subgroups. As we can construct examples of splittings over quasiconvex, non-virtually cyclic subgroups, we settle Bonahon's conjecture in the negative and confirm Theorem 1.2.13.

Theorem 4.8.4. *There exist hyperbolic groups Γ for which there is a minimal, isometric action of Γ on an $\mathbb{R}$-tree (T, d_T) such that:*

1. *the action is not small; and,*
2. *the stable translation length ℓ_T extends continuously to $\mathcal{C}urr(\Gamma)$.*

Example 4.8.5. If M_0 is any closed hyperbolic 3-manifold, there exists a finite cover M of M_0 and an embedded, incompressible connected, closed surface $S \subset M_0$ such that $H = \pi_1(S)$ is quasiconvex in $\Gamma = \pi_1(M)$. Cutting M along S gives us a splitting of Γ over H, and the stable translation length of the Bass-Serre tree corresponding to this splitting extends continuously to $\mathcal{C}urr(\Gamma)$ by quasiconvexity of H. The action of Γ on this tree is not small.

Example 4.8.6. Generalizing the example above, let (Γ, X) be any cubulated hyperbolic group having a non-virtually cyclic wall stabilizer $H < \Gamma$ of infinite index, and assume it stabilizes the wall $W \subset X$. By Agol's Theorem 2.8.12 there exists a finite index subgroup $\Gamma_0 < \Gamma$ such that if $H_0 := H \cap \Gamma_0$, then $\Gamma_0 \backslash X$ is a special cube complex and $H_0 \backslash W$ is an embedded, two-sided wall of $\Gamma_0\backslash X$ that does not self-osculate. This implies that Γ_0 splits over H_0, and since H_0 is non-virtually cyclic, the action of Γ_0 on the corresponding Bass-Serre tree is also a counterexample to Bonahon's conjecture.

The theorem above suggests that for actions of hyperbolic groups on $\mathbb{R}$-trees, instead of asking for the more restrictive assumption of having virtually cyclic interval stabilizers, we should only impose quasiconvex interval stabilizers. It seems reasonable to expect that, when non-trivial, all these actions induce (boundary) metric structures. Also, this would allow us to work with hyperbolic groups admitting no non-trivial small actions on $\mathbb{R}$-trees (see e.g. [Gui00]).

Chapter 5

Questions and future directions

In this chapter we mention some questions that arise from the projects discussed above, and that the author considers worth investigating. Some of them are works in progress, in collaboration with Nic Brody, Stephen Cantrell, Dídac Martínez-Granado, and Gabriel Pallier.

Redundancy of the compatibility assumption in Theorem 1.1.2. The criterion of virtual specialness for cubulated relatively hyperbolic groups given by Theorem 1.1.2 depends on the compatibility of virtually special peripheral subgroups (Definition 3.1.1). It may happen that this condition is always satisfied if we assume virtually special peripheral subgroups. This is equivalent to a positive answer to the following question.

Question 5.1. *Let X, Y be compact NPC cube complexes that are homotopy equivalent, and assume X is special. Does Y have a special finite-sheeted cover?*

By Theorem 1.1.2, the question above has a positive answer when $\Gamma \cong \pi_1(X) \cong \pi_1(Y)$ is hyperbolic relative to virtually abelian subgroups. Also, by Haglund-Wise's Criterion 2.8.4 and Theorem 3.2.1, the conclusion holds if the cubulations $(\Gamma, \widetilde{X})$ and $(\Gamma, \widetilde{Y})$ have the same sets of convex subgroups, where $\widetilde{X}, \widetilde{Y}$ are the corresponding universal covers. By the recent work of Fioravanti, Levcovitz and Sageev [FLS22], this last condition holds when Γ is a *twistless* right-angled Artin group (for example, if $\mathrm{Out}(\Gamma)$ is finite) [FLS22, Thm. A] or Γ is a right-angled Coxeter group *without loose squares* [FLS22, Cor. C].

Relaxing the cocompactness assumption in Theorem 1.1.2. In general, Sageev's construction does not give cocompact cubulations for relatively hyperbolic groups, but only *relatively cocompact* ones [HW14, Sec. 7]. It would be desirable to have a version of Theorem 1.1.2 in this setting, and in the case of virtually abelian peripheral subgroups, the notion to consider is that of cosparse actions [Wis21, Sec. 7.e].

A *quasiflat* F is a locally finite CAT(0) cube complex with a proper and cubical action by a finitely generated virtually abelian group P with only finitely many P-orbits of walls. If

$(\Gamma, \mathcal{P} = \{P_1, \ldots, P_n\})$ is relatively hyperbolic with each P_i being virtually abelian, then the action of Γ on the CAT(0) cube complex X is *cosparse* if there exists a compact subspace $K \subset X$ and quasiflats $F_1, \ldots, F_n \subset X$ satisfying:

- For each i the group P_i is the setwise stabilizer of F_i in Γ.
- $X = \Gamma \cdot (K \cup \bigcup_i F_i)$.
- For each i there is a compact set $K_i \subset X$ such that $(F_i \cap \Gamma \cdot K) \subset P_i \cdot K_i$.
- For all i, j and $g \in \Gamma$, either $(F_i \cap gF_j) \subset \Gamma \cdot K$ or else $i = j$ and $F_i = gF_j$.

Question 5.2. *Let Γ be a relatively hyperbolic group with virtually abelian peripheral subgroups, and let it act cosparsely on the* CAT(0) *cube complex X. Does there exist a finite index subgroup $\Gamma' < \Gamma$ acting freely on X such that the quotient $\Gamma' \backslash X$ is special?*

More generally, we can ask the following (cf. [Ago14, Sec. 11, Question 9]).

Question 5.3. *Let Γ be a group acting properly and by cubical isometries on the* CAT(0) *cube complex X. Assume that Γ is hyperbolic (relative to virtually abelian subgroups) and that its action on X has only finitely many Γ-orbits of walls. Does there exist a finite index subgroup $\Gamma' < \Gamma$ acting freely on X such that the quotient $\Gamma' \backslash X$ is special?*

Producing cubulations from the relative quasiconvex hierarchy theorem. In Theorems 1.1.7 and 1.1.8 we work with the class $\mathcal{CMVH}$ of relatively hyperbolic groups (Definition 1.1.6). We impose these groups to be cubulated, but a priori this condition is not required for the hyperbolic groups in the class $\mathcal{QVH}$. We wonder if the cubulation assumption in the definition of $\mathcal{CMVH}$ can be dropped somehow. Inspired by Proposition 3.4.6, a tempting conjecture is the following.

Conjecture 5.4. *Let $(\Gamma, \mathcal{P} = \{P_1, \ldots, P_n\})$ be a relatively hyperbolic group with each P_i being residually finite, and suppose Γ splits as a finite graph of groups $(G, \mathcal{G})$ satisfying:*

- *each edge/vertex group is relatively quasiconvex and strongly peripherally separable (Definition 2.5.14), and each pair of edge/vertex groups is doubly peripherally separable (Definition 3.3.13);*
- *each vertex group Γ_v has a cubulation X_v such that (Γ_v, X_v) is virtually special;*
- *if e is an edge attached to the vertex v of G, then Γ_e is convex in (Γ_v, X_v);*
- *if $P < \Gamma$ is a peripheral subgroup and Γ_v is a vertex group, then $P \cap \Gamma_v$ is convex in (Γ_v, X_v); and,*
- *if v is a vertex of G then the collection $\mathcal{A}_v := \{\Gamma_e \colon e$ an edge attached to $v\}$ is relatively malnormal in Γ_v.*

Then Γ has a cubulation X such that each vertex group Γ_v is convex in (Γ, X) with a convex core Γ_v-equivariantly isometric to X_v.

Of course, it might happen that more assumptions are necessary, or that the cubulation in the conclusion is not necessarily cocompact. See e.g. [Wis21, Thm. 15.1 & Conj. 15.5].

Dense subsets of $\mathscr{D}_\Gamma$. For Γ a non-elementary hyperbolic group, we saw in Proposition 4.4.3 that metric structures induced by word metrics of finite symmetric generating sets form a dense subset of $\mathscr{D}_\Gamma$. We can ask if there are other classes of pseudo metrics inducing dense subsets of metric structures.

A natural candidate is the set of metric structures induced by Green metrics. Suppose $\rho = [d] \in \mathscr{D}_\Gamma$ is such that $v_d = 1$, and let μ_n be the uniform probability measure on the set $S_n = \{x \in \Gamma \colon d(o,x) \leq n\}$ with Green metric $d_n = d_{\mu_n}$. The work of Gouëzel, Mathéus and Maucourant [GMM18, Thm. 1.4] implies that $\tau(d/d_n) = l_{\mu_n}(d)/h_{\mu_n}$ tends to 1 as n tends to ∞. Therefore, the density of metric structures induced by Green metrics would follow from a positive answer to the following question.

Question 5.5. *Let $\rho \in \mathcal{D}_\Gamma$ and let $(\rho_n)_n \subset \mathscr{D}_\Gamma$ be a sequence satisfying $\tau(\rho/\rho_n) \to 1$ as $n \to \infty$. Does it hold that ρ_n converges to ρ in $(\mathscr{D}_\Gamma, \Delta)$ as $n \to \infty$?*

When Γ is cubulable, another candidate is the set of metric structures induced by geometric actions of Γ on CAT(0) cube complexes with the combinatorial metric.

Question 5.6 (Futer–Wise [FW21, Question. 7.8])**.** *Let Γ be a hyperbolic group, and let $\mathscr{D}_\Gamma^{cub} \subset \mathscr{D}_\Gamma$ be the subset of metric structures induced by geometric actions on* CAT(0) *cube complexes. If $\mathscr{D}_\Gamma^{cub} \subset \mathscr{D}_\Gamma$ is non-empty, is it dense in $\mathscr{D}_\Gamma$?*

In a forthcoming work with Nic Brody, we approximate by cubulations uniform lattices in the hyperbolic space $\mathbb{H}^n$ for $n = 2, 3$, as well as uniform arithmetic lattices of simplest type in arbitrary dimensions, as conjectured by Futer and Wise [FW21, Conj. 7.7]. By applying a theorem of Brooks [Bro86, Thm. 1], we can also approximate by cubulations points in the quasi-Fuchsian spaces of surface groups. More precisely, we claim the following.

Claim 5.7 (Brody–Reyes)**.** *For $\rho \in \mathscr{D}_\Gamma$, suppose that either:*

- *Γ is a surface group and $\rho \in \mathbb{P}\mathcal{C}urr_f(\Gamma) \cup \mathscr{Q}\mathscr{F}_\Gamma$;*
- *ρ represents a geometric action of Γ on $\mathbb{H}^3$; or,*
- *ρ represents a geometric action of Γ on $\mathbb{H}^n$ and Γ is of simplest type.*

Then ρ belongs to the closure of $\mathscr{D}_\Gamma^{cub}$ in $(\mathscr{D}_\Gamma, \Delta)$.

More examples of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$. As we saw in Sections 4.3 and 4.7, many non-trivial actions of Γ on $\mathbb{R}$-trees represent points in $\overline{\mathscr{D}}_\Gamma$. In addition, many of the examples of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$ are induced by acylindrical actions. Recall that an action of Γ on the hyperbolic metric space (X,d) is *acylindrical* if for all $R \geq 0$ there exist $L \geq 1$ and $M \geq 1$ such that whenever $p,q \in X$ satisfy $d(x,y) \geq L$ then

$$\#\{x \in \Gamma \colon d(p,xp) \leq R \text{ and } d(q,xq) \leq R\} \leq M.$$

We suspect that all these types of actions induce (boundary) metric structures.

Conjecture 5.8. *Let Γ act isometrically and coboundedly on the hyperbolic and geodesic metric space (X,d). Suppose that the action has at least one loxodromic element, and that either:*

- *(X,d) is an $\mathbb{R}$-tree and the action is small; or,*
- *the action is acylindrical.*

Then the orbit pseudo metrics induced by the action of Γ on (X,d) belong to $\overline{\mathcal{D}}_\Gamma$.

Topological properties of $\overline{\mathscr{D}}_\Gamma$. There are still some desirable properties for the space of metric structures that need to be studied. The most basic one is related to the completeness of $\mathscr{D}_\Gamma$.

Question 5.9. *If $(\mathscr{D}_\Gamma,\Delta)$ complete?*

Another aspect not covered in this thesis is about defining topologies on $\partial_M\mathscr{D}_\Gamma$ and $\overline{\mathscr{D}}_\Gamma$. In a future project with Stephen Cantrell, we define a "sphere" topology on $\partial_M\mathscr{D}_\Gamma$, making it homeomorphic to *any* sphere of positive radius in $\mathscr{D}_\Gamma$. In the same way, we can endow $\overline{\mathscr{D}}_\Gamma$ with a topology so that it is homeomorphic to any closed ball of positive radius in $\mathscr{D}_\Gamma$. We prove that $\partial_M\mathscr{D}_\Gamma$ (and hence $\overline{\mathscr{D}}_\Gamma$) is not compact for the sphere topology, which implies that $\mathscr{D}_\Gamma$ is not proper. This also has strong consequences for the topological homogeneity of $\mathscr{D}_\Gamma$.

Claim 5.10 (Cantrell–Reyes)**.** *The space $(\mathscr{D}_\Gamma,\Delta)$ satisfies the following properties:*

- *It is topologically homogeneous.*
- *It is not proper.*
- *All the closed balls of positive radius in $(\mathscr{D}_\Gamma,\Delta)$ are homeomorphic to each other.*
- *All the spheres of positive radius in $(\mathscr{D}_\Gamma,\Delta)$ are homeomorphic to each other.*

For these topologies, it is natural to ask how relevant subspaces of $\mathscr{D}_\Gamma$ sit as topological subspaces of $\overline{\mathscr{D}}_\Gamma$. For example, it can be proven that $\overline{\mathscr{T}}_\Gamma = \mathscr{T}_\Gamma \cup \partial \mathscr{T}_\Gamma$ and $\overline{\mathscr{C}\mathscr{V}}_\Gamma = \mathscr{C}\mathscr{V}_\Gamma \cup \partial \mathscr{C}\mathscr{V}_\Gamma$ embed as closed (indeed, compact) subspaces of $\overline{\mathscr{D}}_\Gamma$ for Γ a surface and a free group, respectively. However, the existence of quasi-Fuchsian representations converging to non-Fuchsian ones in the representation variety implies that for Γ a surface group, the closure of $\mathscr{D}_\Gamma^\delta$ is not compact for any $\delta \geq \log 4$. We expect this behavior to be particular to 2-dimensional manifolds.

Conjecture 5.11. *Let Γ be the fundamental group of a closed negatively curved Riemannian manifold of dimension at least 3 (or more generally, a hyperbolic PD_n group for $n \geq 3$). Then $\mathscr{D}_\Gamma^\delta$ is compact for every $\delta \geq 0$.*

Question 5.12. *What non-elementary hyperbolic groups Γ satisfy that there exists some δ_0 such that the closure of $\mathscr{D}_\Gamma^\delta$ in $\overline{\mathscr{D}}_\Gamma$ is non-compact for any $\delta \geq \delta_0$?*

We also expect a better behavior for the subspaces $\mathscr{D}_\Gamma^{\delta,\alpha}$.

Conjecture 5.13. *The closure of $\mathscr{D}_\Gamma^{\delta,\alpha}$ in $\overline{\mathscr{D}}_\Gamma$ is compact for every $\delta, \alpha \geq 0$ such that $\mathscr{D}_\Gamma^{\delta,\alpha}$ is non-empty.*

By Theorems 1.2.6 and 1.2.7, this question is only interesting if Γ has torsion or $\mathrm{Out}(\Gamma)$ is infinite. Also, by adapting the argument in [KL15], a positive answer to the question above would imply the following, generalizing [KL15, Thm. 1.6].

Conjecture 5.14. *For any non-elementary hyperbolic group Γ and $d \in \mathcal{D}_\Gamma$ there exists $\lambda_d > 0$ such that for any $\phi \in \mathrm{Aut}(\Gamma)$ we have*

$$\lambda_d \leq \frac{\tau(\phi(d)/d)}{\mathrm{Dil}(\phi(d), d)} \leq 1,$$

where $\phi(d)(x,y) = d(\phi^{-1}x, \phi^{-1}y)$.

Some of the questions above can be addressed by studying the relationship between the sphere topology and other topologies, such as the length topology or the equivariant Gromov-Hausdorff topology (see also [DM22, Conj. 10.6]).

Minimizing functionals on $\mathscr{D}_\Gamma$. As we saw in Subsection 4.5.3, we have a visual dimension functional $\dim : \mathscr{D}_\Gamma \to \mathbb{R}$. Since $\dim(\rho)$ is an upper bound for the Hausdorff dimension for any metric on $\partial\Gamma$ that is visual for $\rho \in \mathscr{D}_\Gamma$, we have the inequality

$$\inf_{\rho \in \mathscr{D}_\Gamma} \dim(\rho) \geq AR(\partial\Gamma), \tag{5.1}$$

where $AR(\partial\Gamma)$ is the *Ahlfors regular conformal dimension* of $\partial\Gamma$ endowed with its canonical quasisymmetric gauge (see e.g. [BK02]).

Question 5.15. *For what hyperbolic groups is the inequality in* (5.1) *an equality?*

By Proposition 4.5.10 and Corollary 4.5.11, the equality holds when Γ is a free group or admits a geometric action on $\mathbb{H}^n$ for some n. Similarly, we can ask for the set of metric structures minimizing the hyperbolicity constant functional $\delta : \mathscr{D}_\Gamma \to \mathbb{R}$. Let δ_Γ be the infimum of $\delta(\rho)$ among all $\rho \in \mathscr{D}_\Gamma$, and set $\mathscr{D}_\Gamma^{min} := \mathscr{D}_\Gamma^{\delta_\Gamma}$.

Conjecture 5.16. *Let Γ be a hyperbolic group.*

- *If Γ is a surface group then $\mathscr{D}_\Gamma^{min} = \mathscr{T}_\Gamma$.*
- *If Γ acts geometrically on $\mathbb{H}^n$ and $n \geq 3$, then $\mathscr{D}_\Gamma^{min}$ is the finite set of metric structures induced by the geometric actions of Γ on $\mathbb{H}^n$.*

Question 5.17. *Let Γ be such that $\mathscr{D}_\Gamma^{min}$ is non-empty. Is $\mathscr{D}_\Gamma^{min}$ finite-dimensional? Does $\mathscr{D}_\Gamma^{min}$ have only finitely many connected components?*

Hyperplane geodesic currents. For a surface group, many metric structures can be represented in terms of geodesic currents, via the pairing given by the intersection number (see Subsection 4.7.3). In an ongoing project with Dídac Martínez-Granado, we attempt to extend this construction, and understand cubulations on arbitrary hyperbolic groups via their dual *"hyperplane geodesic currents"*.

We also want to construct a *continuous intersection number* by pairing hyperplane geodesic currents and geodesic currents. To define these objects canonically, it is convenient to understand walls of cubulations in terms of their limit sets in the Gromov boundary, which are closed subsets whose complements in the boundary are split into two open sets (cf. [BW12]). If this construction succeeds, we might get some ideas to answer Question 5.6.

Deformation spaces for non-hyperbolic groups. For a hyperbolic group Γ, a pseudo metric on Γ belongs to $\mathcal{D}_\Gamma$ if and only if it is left-invariant, quasi-isometric to a word metric in $\mathcal{D}_\Gamma$, and roughly geodesic (see Corollary 2.3.9). Therefore, it is natural to consider this as a potential definition of $\mathcal{D}_\Gamma$ for finitely generated groups that are non-necessarily hyperbolic. That is, if Γ is a finitely generated group, we let $\mathscr{D}_\Gamma$ be the space of rough similarity classes of left-invariant pseudo metrics on Γ that are roughly geodesic and quasi-isometric to word metrics for finite symmetric generating subsets of Γ.

We expect this definition to be appropriate for relatively hyperbolic groups, and potentially for acylindrically hyperbolic groups. In a forthcoming work with Gabriel Pallier, we plan to study the space $\mathscr{D}_\Gamma$ (as well as the pseudo metric Δ) when Γ is virtually abelian, the Heisenberg group (and some other nilpotent groups), or a uniform lattice in a higher rank semi-simple Lie group.

The space of hyperbolic structures on Γ. By definition, all the pseudo metrics in $\mathcal{D}_\Gamma$ are quasi-isometric to each other, but that is not the case for pseudo metrics in $\overline{\mathcal{D}}_\Gamma$. For a non-elementary hyperbolic group Γ, we let $\mathscr{H}_\Gamma$ be the set of *quasi-isometry classes* of pseudo metrics in $\overline{\mathcal{D}}_\Gamma$. That is, two pseudo metrics $d, d_* \in \overline{\mathcal{D}}_\Gamma$ determine the same point in $\mathscr{H}_\Gamma$ if the identity map $(\Gamma, d) \to (\Gamma, d_*)$ is a quasi-isometry. This is the space of *hyperbolic metric structures* on Γ. Note that this space is naturally a subset of the space $\mathcal{H}(\Gamma)$ of *hyperbolic structures* on Γ introduced by Abbot, Balasubramanya and Osin [ABO19]. In particular, $\mathscr{H}_\Gamma$ has a natural partial order $\preceq$: if $\mathfrak{h}, \mathfrak{h}_* \in \mathscr{H}_\Gamma$ are represented by pseudo metrics $d, d_* \in \overline{\mathcal{D}}_\Gamma$ respectively, then $\mathfrak{h} \preceq \mathfrak{h}_*$ if and only if there exists $\lambda > 0$ such that $d \leq \lambda d_* + \lambda$.

If S is a closed orientable surface, we can see every conjugacy class in $\Gamma = \pi_1(S)$ representing a simple closed curve in S as a point in $\mathscr{H}_\Gamma$. Indeed, for such conjugacy class $[x]$, the group Γ splits over the cyclic group generated by x, and we can define $\mathfrak{h}_{[x]}$ as the point in $\mathscr{H}_\Gamma$ represented by any Bass-Serre tree corresponding to such splitting. The assignment $[x] \mapsto \mathfrak{h}_{[x]}$ is well-defined and induces an embedding of the vertex set of the curve graph $\mathcal{C}_S$ into $\mathscr{H}_\Gamma$. Since the automorphism group of the graph $\mathcal{C}_S$ is exactly $\mathrm{Out}(\Gamma)$ [Iva97], we ask whether the same holds to the poset $(\mathscr{H}_\Gamma, \preceq)$.

Question 5.18. *Is* $\mathrm{Aut}(\mathscr{H}_\Gamma, \preceq)$ *equal to* $\mathrm{Out}(\Gamma)$*?*

The natural isometric action of $\mathrm{Out}(\Gamma)$ on the curve graph $\mathcal{C}_S$ motivates the following question.

Question 5.19. *For arbitrary* Γ*, does there exist a reasonably interesting* $\mathrm{Out}(\Gamma)$*-invariant metric on* $\mathscr{H}_\Gamma$*?*

Given a point $\mathfrak{h} \in \mathscr{H}_\Gamma$ represented by the pseudo metric d, we define

$$K_\mathfrak{h} := \{\eta \in \mathbb{P}\mathcal{C}urr(\Gamma) \colon \ell_d(\eta') = 0 \text{ for some (any) } \eta' \text{ representing } \eta\},$$

which is a non-empty compact subset of $\mathbb{P}\mathcal{C}urr(\Gamma)$ that is independent of the representative d of $\mathfrak{h}$. By [ABO19, Thm. 2.14], we have that $\mathfrak{h} \preceq \mathfrak{h}_*$ if and only if $K_\mathfrak{h} \subset K_{\mathfrak{h}_*}$.

Question 5.20. *Can we give a reasonable description of the set* $\{K_\mathfrak{h} \colon \mathfrak{h} \in \mathscr{H}_\Gamma\}$*?*

Question 5.21. *Can the set* $K_\mathfrak{h}$ *help us determine whether* $\mathfrak{h} \in \mathscr{H}_\Gamma$ *is induced by an action of* Γ *on an* $\mathbb{R}$*-tree?*

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