Nonzero Degree Maps Between Three Dimensional Manifolds

by

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Abstract

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The main result of this dissertation shows that every orientable closed 3-manifold admits a nonzero degree map onto at most finitely many homeomorphically distinct non-geometric prime 3-manifolds. Furthermore, for any integer $d > 0$, every orientable closed 3-manifold admits a map of degree d onto only finitely many homeomorphically distinct 3-manifolds. This answers a question of Yongwu Rong. The finiteness of JSJ piece of the targets under nonzero degree maps was known earlier by the results of Soma and Boileau–Rubinstein–Wang, and a new proof is provided is this dissertation. We also prove analogous results for dominations relative to boundary. As an application, we describe the degree set of dominations onto integral homology 3-spheres.

To my homeland

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Chapter 1

Introduction

In the present dissertation, we study finiteness associated with nonzero degree maps between 3-manifolds, from the viewpoint of geometrization. For convenience, we often stay in the piecewise linear category of 3-manifolds for topological discussions, and throughout this dissertation, a 3-manifold is always assumed to be connected, unless explicitly stated otherwise. In this chapter, we provide an overview of known results and the main result of the present dissertation.

1.1 Background

Let M, N be two orientable closed 3-manifolds. For an integer $d > 0$, we say that M *d-dominates* N if there is a map $f : M \to N$ of degree d up to sign. We say M *dominates* N if M d-dominates N for some integer $d > 0$. The notion of domination can certainly be extended to orientable compact 3-manifolds. However, for most of the topics discussed below, the general case can be derived easily from the essential case of closed 3-manifolds, so we shall not consider dominations relative to boundary until Chapter 6.

Dominations of degree one naturally induces a partial ordering on the set of homeomorphism classes of orientable closed 3-manifolds, sometimes attributed to Mikhail Gromov in literature. According to [CT89], Gromov suggested studying the degree set of dominations between closed orientable manifolds of general dimensions in a lecture given in 1978. In dimension three, the following two problems are the basic aspects of our interest:

Problem 1.1.1. For every pair of orientable closed 3-manifolds M and N, describe the set of mapping degrees that are realizable by maps between M and N.

Problem 1.1.2. For every orientable closed 3-manifold M, describe the set of 3-manifolds up to homeomorphism that are dominated by M.

If the word 'describe' here was taken in the strong sense as to decide, as far as we are concerned, both of these problems are still widely open except for a few very special cases. However, if the word was taken in a weaker sense as to determine the finiteness, much has been known since the 1990s, when the study of nonzero degree maps started to become active. In this course, William Thurston's revolutionary program of geometrization played an influential role, not only because it provided deep insight into the topology of 3-manifolds, but also because it naturally brought maps between 3-manifolds as the next stage of exploration.

When the target manifold N is the same as M, one of the pioneer results in this area, due to Shicheng Wang [Wan93], implies that the set $D(M,N)$ of mapping degrees in Problem 1.1.1 is infinite if and only if either M is prime supporting one of the geometries $\mathbb{H}^2 \times \mathbb{E}^1$, $\mathbb{E}^3$, Nil, or Sol, or that every prime factor of M supports one of the geometries $\mathbb{S}^3$ or $\mathbb{S}^2 \times \mathbb{E}^1$. Later work of various people [Du09, SWW10, SWWZ12] fully characterized the set of self-mapping degree. More recently, Pierre Derbez, Hongbin Sun, and Shicheng Wang [DSW11] showed that for a given N, there exists an M so that $D(M,N)$ is infinite if and only if $D(P,P)$ is infinite for every prime factor P of N.

Problem 1.1.2 has been answered restricted to geometric targets by the work of various people, cf. [Som00, BBW08, BRW]. As a summary of their conclusions, every closed orientable 3-manifold M dominates at most finitely many geometric 3-manifolds that support none of the geometries $\mathbb{S}^3$, $\widetilde{\mathrm{SL}}_2$ or Nil. Note that any 3-manifold supporting one of the excluded three geometries above dominates infinitely many homeomorphically distinct 3-manifolds of the same geometry. It is remarkable that in [BRW], Michel Boileau, Hyam Rubinstein, and Shicheng Wang actually proved the finiteness of possible homeomorphism types of JSJ pieces in the target N. They also wondered if every closed orientable 3-manifold M dominates finitely many irreducible 3-manifolds supporting none of the geometries $\mathbb{S}^3$, $\widetilde{\mathrm{SL}}_2$ or Nil.

1.2 Results

In the present dissertation, we shall show that every orientable closed 3-manifold dominates at most finitely many homeomorphically distinct non-geometric prime 3-manifolds, (Theorem 5.4.2). This answers affirmatively the question of Boileau, Rubinstein, and Wang in [BRW]. Our proof also provides an alternative approach to the finiteness of JSJ pieces previously obtained by [Som00, BRW]. Furthermore, we shall also show that for any integer $d > 0$, every orientable closed 3-manifold d-dominates only finitely many homeomorphically distinct 3-manifolds, (Theorem 6.1.1). In particular, this answers an earlier question of Yongwu Rong [Kir97, Problem 3.100], which was concerned about 1-dominations. This was known before only under the assumption of geometric targets, cf. [Som00, HLWZ02, WZ02, BBW08]. Analogous results also hold for dominations relative to boundary (Theorems 6.2.1, 6.2.2). As an application, we provide a description of the degree set of dominations onto integral homology spheres, partially resolving Problem 1.1.1. We show that for any oriented closed 3-manifold M, there are only finitely many integral homology 3-spheres N dominated by M, as previously obtained by [BRW, Theorem 1.2]; moreover, we show that the (signed) degree set of dominations of M onto N is either finite or a translationally periodic subset of $\mathbf{Z}$ with zero removed, (Theorem 6.3.1). This provides some description, beyond the finite-versus-infinite dichotomy, about the degree set of dominations onto integral homology 3-spheres.

A traditional approach to Problems 1.1.1 and 1.1.2 is via volume estimation. For example, when M is given, the simplicial volume of M imposes an upper bound on the simplicial volume of the target N under the domination assumption. Such a bound provides certain restrictions to the topology of hyperbolic pieces of N. As a variation of this idea, the Seifert volume of a 3-manifold was introduced by Robert Brooks and William Goldman [BG84]. It is analogous to the hyperbolic volume in the representation sense, and there has been interesting applications of this notion to dominations onto graph manifolds. For example, Derbez and Wang [DW09a, DW09b] showed that nontrivial graph manifolds have virtually positive Seifert volume, so the mapping degree set $D(M,N)$ is finite if N is a nontrivial graph manifold. While it has been successful dealing with Problem 1.1.1 in many situations, the volume estimation approach has its weakness in solving Problem 1.1.2, mainly because there are usually infinitely many manifolds with uniformly bounded volume of either version.

Our main technique is a new type of estimation as was developed in [AL12], inspired by the idea from an unpublished paper of Matthew White [Whi]. Heuristically speaking, whenever there is a map $f : M \to N$, one may geometrize the map in a certain manner with respect to the geometrization of N. If N has either a deep Margulis tube in a hyperbolic piece, or a sharp cone point in a Seifert fibered piece, or heavy distortion along a cut torus, then the map would have to fail to be surjective homologically localized to these significant elementary parts. In particular, it would not be a domination. To be more precise, one may regard the geometric features of N above as a certain form of complexity, then in fact, we shall show that under the assumption of domination, such complexity can be bounded in terms of the triangulation number $\tau(M)$ of M, namely, the minimal number of triangles in any triangulation of M. In [AL12], the role of triangulation number was played by the presentation length of $\pi_1(M)$.

1.3 Organization

In Chapter 2, we provide a brief review on topology of 3-manifold, especially the geometric decomposition. In Chapter 3, we discuss a general process known as straightening a map $f : M \to N$ between 3-manifolds. Heuristically, this homotopes f to a position of minimal area with respect to a metric of N close to the geometric metric in each piece. In formulation, we shall adopt ruled surfaces instead of minimal surfaces to avoid unnecessary technicalities. In Chapter 4, we provide an alternative proof of the finiteness of geometric pieces using the techniques from Chapter 3. In Chapter 5, we prove the finiteness of gluings under dominations. This will complete the proof of the main theorem (Theorem 5.4.2). In Chapter 6, we consider the case of bounded-degree dominations, and deduce Corollary 6.1.1 from Theorem 5.4.2. We also provide generalizations of Theorems 5.4.2 and 6.1.1 to the boundary-relative case. Finally, we shall describe the degree set of dominations from any closed oriented 3-manifold to integral homology 3-spheres.

Chapter 2

Preliminaries

In this chapter, we review topology of 3-manifold from the perspective of geometrization, cf. [Thu80, MF10]. We also refer to [Jac80] for standard terminology and facts of 3-manifold topology.

2.1 Geometric decomposition

Suppose N is an orientable compact 3-manifold, possibly with boundary. We say that N is *geometric*, if it supports one of Thurston's Eight Geometries: $\mathbb{E}^3$, $\mathbb{S}^3$, $\mathbb{H}^3$, $\mathbb{S}^2 \times \mathbb{E}^1$, $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2$, Nil, or Sol, in the interior of finite volume. Under this circumstance, it is necessary that N is prime; and that every component of ∂N, if any, is an incompressible torus; and that N is not homeomorphic to an *orientable thickened-torus*, or an *orientable thickened-Klein-bottle*, i.e. the trivial interval-bundle over a torus, or the twisted interval-bundle over a Klein bottle, respectively.

In general, if N is an orientable compact 3-manifold satisfying the necessary conditions above, the Thurston–Perelman Geometrization Theorem implies that there exists a canonical *geometric decomposition* of N, namely, a minimal finite collection of essential tori or Klein-bottles, unique up to isotopy, cutting N into geometric pieces. Recall that by the Kneser–Milnor Theorem, every orientable compact 3-manifold is homeomorphic to the connected sum of a finite collection of prime 3-manifolds, unique up to homeomorphism. It follows that every orientable closed prime 3-manifold admits a canonical geometric decomposition.

We shall only speak of the geometric decomposition for orientable closed prime 3-manifolds. Such a manifold, N, is either geometric or non-geometric. When N is itself geometric, it is either atoroidal, supporting the $\mathbb{H}^3$-geometry, or Seifert-fibered, supporting one of the six geometries $\mathbb{H}^2 \times \mathbb{E}^1$, $\widetilde{\mathrm{SL}}_2$, $\mathbb{E}^3$, Nil, $\mathbb{S}^2 \times \mathbb{E}^1$ or $\mathbb{S}^3$, or otherwise, supporting the Sol-geometry. The geometry of the Seifert-fibered case can be determined according to the sign of the Euler characteristic $\chi \in \mathbf{Q}$ of the base orbifold and to whether the Euler number $e \in \mathbf{Q}$ of the fiberation vanishes. When N is not geometric, there are only two types of its geometric pieces, namely, $\mathbb{H}^3$ or $\mathbb{H}^2 \times \mathbb{E}^1$. In other words, every geometric piece is either homeomorphic to a cusped hyperbolic 3-manifold of finite volume, or homeomorphic to an orientable Seifert-fibered space with boundary over a cusped hyperbolic 2-orbifold of finite area.

The geometric decomposition splits N as a *graph-of-spaces*, where each vertex is decorated by a geometric piece, and each edge is decorated a cutting torus or Klein-bottle, joining vertices decorated by the adjacent pieces. Since the regular neighborhood of a cutting Klein-bottle in N has only one boundary component, the cutting Klein-bottle decorates an edge with only one end. This suggests that such an edge should be regarded as a 'semi-edge'. Similarly, it will be convenient to regard a Seifert-fibered piece (over a non-orientable hyperbolic base orbifold) containing an essential Klein-bottle as a 'semi-vertex'. For this reason, we shall think of the underlying graph of the geometric decomposition as a graph with semi-objects. See Definition 2.2.1 for a rigorous formulation.

2.2 Gluing geometrics

Gluing geometrics is the opposite procedure of the geometric decomposition. The purpose of this section is to lay down some notations for the rest of our discussion.

Definition 2.2.1. A *graph with semi-objects*, or simply a *graph*, is a finite CW 1-complex Λ with a (possibly empty) subset of loop-edges marked as *semi-edges*, and with a (possibly empty) subset of vertices marked as *semi-vertices*. We shall refer to other vertices and edges as *entire-vertices* and *entire-edges*, respectively. A entire-edge has two *ends*, but a semi-edge has only one. The *valence* of a vertex v is the number of distinct ends adjacent to v. For a graph Λ, we denote its set of vertices as $\mathrm{Ver}(\Lambda)$, and its set of edges as $\mathrm{Edg}(\Lambda)$. The set of ends-of-edges $\widetilde{\mathrm{Edg}}(\Lambda)$ is a branched two-covering of $\mathrm{Edg}(\Lambda)$ singular over all the semi-edges. The covering transformation takes every end δ to its *opposite end* $\bar{\delta}$, of the same edge that δ belongs to.

Definition 2.2.2. A *preglue graph-of-geometrics* is a finite graph Λ, together with an assignment of each vertex $v \in \mathrm{Ver}(\Lambda)$ to an oriented, compact, geometric 3-manifold J_v whose boundary consists of exactly n_v incompressible tori components, where n_v is the valence of v, and with an assignment of each end-of-edge

$\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ adjacent to v to a distinct component T_δ of ∂J_v with the induced orientation. We require a semi-vertex be assigned to a J_v containing an embedded orientable thickened-Klein-bottle, and an entire vertex be assign to a J_v not as above. Let $\mathcal{J}$ be the disjoint union of all J_v's. We often ambiguously denote the preglue graph-of-geometrics as $(\Lambda, \mathcal{J})$.

Definition 2.2.3. Two preglue graphs-of-geometrics $(\Lambda, \mathcal{J})$ and $(\Lambda', \mathcal{J}')$ are said to be *isomorphic* if there is a homeomophism $\mathcal{J} \to \mathcal{J}'$, which compatibly (in an obvious sense) induces a graph isomorphism $\Lambda \to \Lambda'$.

Definition 2.2.4. A *gluing* of a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$ is an assignment of each end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ to an orientation-reversing homeomorphism $\phi_\delta : T_\delta \to T_{\bar{\delta}}$ between the tori assigned to δ and its the opposite end $\bar{\delta}$, up to isotopy, such that $\phi_{\bar{\delta}} = \phi_\delta^{-1}$ for any end-of-edge δ. Let:

$$\phi : \partial\mathcal{J} \to \partial\mathcal{J},$$

be the orientation-reversing involution defined by all ϕ_δ's. We often denote the gluing as ϕ, and denote the set of all gluings of $(\Lambda, \mathcal{J})$ as $\Phi(\Lambda, \mathcal{J})$.

A gluing ϕ is said to be *nondegenerate* if it does not match up ordinary-fibers in any pair of (possibly the same or via semi-edges) adjacent Seifert-fibered pieces.

For any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$, there is a naturally associated oriented closed 3-manifold N_ϕ from $\mathcal{J}$ obtained by identifying points in $\partial\mathcal{J}$ with their images under ϕ. It is clear that N_ϕ has the same geometric decomposition as prescribed by $(\Lambda, \mathcal{J})$ and ϕ if and only if ϕ is nondegenerate, and in this case, N_ϕ is by definition non-geometric.

Let $\mathrm{Mod}(\partial\mathcal{J})$ be the special mapping class group of $\partial\mathcal{J}$, consisting of isotopy classes of component-preserving, orientation-preserving self-homeomorphisms of $\partial\mathcal{J}$. There is a natural (right) action of $\mathrm{Mod}(\partial\mathcal{J})$ on $\Phi(\Lambda, \mathcal{J})$. In fact, abusing the notations of isotopy classes and their representatives, for any $\tau \in \mathrm{Mod}(\partial\mathcal{J})$, and $\phi \in \Phi(\Lambda, \mathcal{J})$, one may define $\phi^\tau \in \Phi(\Lambda, \mathcal{J})$ to be:

$$\phi^\tau = \tau^{-1} \circ \phi \circ \tau,$$

namely, $(\phi^\tau)_\delta = \tau_{\bar{\delta}}^{-1} \circ \phi_\delta \circ \tau_\delta$ for each end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, where $\tau_\delta \in \mathrm{Mod}(T_\delta)$ is the restriction of τ on the torus T_δ. It is straightforward to check that this is a well-defined, transitive action.

Definition 2.2.5. Two gluings $\phi, \phi' \in \Phi(\Lambda, \mathcal{J})$ are said to be *equivalent* if $\phi' = \phi^\tau$ for some $\tau \in \mathrm{Mod}(\partial\mathcal{J})$ that extends over $\mathcal{J}$ as a self-homeomorphism. Hence equivalent gluings yield homeomorphic 3-manifolds.

We close this section with a discussion of a special type of elements in $\mathrm{Mod}(\partial\mathcal{J})$, called fiber-shearings. Recall that for an oriented torus T and a slope $\gamma \subset T$, the (right-hand) *Dehn-twist* along γ is self-homeomorphism $D_\gamma \in \mathrm{Mod}(T)$ so that $D_\gamma(\zeta) = \zeta + \langle\zeta,\gamma\rangle\,\gamma$ for any slope ζ, where $\langle\cdot,\cdot\rangle$ denotes the intersection form. Note this does not depend on the direction of γ. For any integer k, a *k-times Dehn-twist* along γ is known as the k-times iteration D_γ^k.

Definition 2.2.6. Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. We say an automorphism $\tau \in \mathrm{Mod}(\partial\mathcal{J})$ is a *fiber-shearing* with respect to $(\Lambda, \mathcal{J})$, if for each end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$ adjacent to a vertex v, $\tau_\delta \in \mathrm{Mod}(T_\delta)$ is either the identity, if J_v is atoroidal, or a k_δ-times Dehn-twist along the ordinary-fiber, where k_δ is an integer, if J_v is Seifert-fibered. The *index* of τ at a Seifert-fibered vertex v is the integer:

$$k_v(\tau) = \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} k_\delta,$$

where $\widetilde{\mathrm{Edg}}(v)$ denotes the set of ends adjacent to v. For any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$, the *fiber-shearing* of ϕ under τ is the gluing $\phi^\tau \in \Phi(\Lambda, \mathcal{J})$.

Note that the index is additive for products of fiber-shearings.

Lemma 2.2.7. *Fiber-shearings of the same index at all Seifert-fibered vertices yield equivalent gluings.*

Proof. It suffices to show that a fiber-shearing with zero index at all Seifert-fibered vertices does not change the equivalence class of a gluing. This follows immediately from the fact that for any pair of boundary tori T, T' in a Seifert-fibered piece J, there is a properly embedded annulus A bounding a pair of ordinary-fibers, one on each component. As the annulus A is two-sided when J is oriented, there is a well-defined Dehn-twist on J along this annulus, restricting to a right-hand Dehn-twist on T and a left-hand Dehn-twist (i.e. the inverse of a right-hand Dehn-twist) on T'. ◻

Chapter 3

Maps and geometrization

In this chapter, we introduce a general process that homotopes a map between 3-manifolds to a position respecting the geometrization of the target, known as *straightening*. Note that this process is nontrivial only if the target is either non-geometric, or supports one of the geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$ or $\widetilde{\mathrm{SL}}_2$. In these cases, straightening will allow us to study the local behaviour of the maps from a hyperbolic geometric point of view. Our treatment here is an extension of the techniques developed in [AL12]. In an unpublished paper of Matthew White [Whi] about diameters of closed 3-manifolds, essentially equivalent estimations was considered, although in slightly different formulations. His idea inspired [AL12] and the generalizations that we shall discuss in this chapter.

3.1 Straightening a map

Let M be an orientable closed 3-manifold, and N be an orientable closed prime 3-manifold. Suppose $f : M \to N$ is a map from M to N. We triangulate M, and geometrize N, and homotope f to a nice position with respect to these structures, as follows.

Take a minimal triangulation of M, namely, a finite 3-dimensional simplicial complex structure on M with the fewest possible 3-simplices. We often denote $M^{(i)} \subset M$ for the i-skeleton of M, where $0 \leq i \leq 3$. The number of tetrahedra:

$$\tau(M),$$

in this triangulation will be called the *triangulation number* of M. Hence $M^{(2)}$ contains exactly $2\tau(M)$ triangles.

Suppose N is an orientable closed prime 3-manifold. We often denote the underlying graph of the geometric decomposition of N as $\Lambda = \Lambda(N)$. Let:

$$\mathcal{T} = \bigsqcup_{e \in \mathrm{Edg}(\Lambda)} T_e \subset N,$$

be the union of cutting tori or Klein bottles of N in its geometric decomposition, and let:

$$\mathcal{U} = \bigsqcup_{e \in \mathrm{Edg}(\Lambda)} \mathcal{U}_e \subset N,$$

be a compact regular neighborhood of $\mathcal{T}$. Note $\partial\mathcal{U}$ can be naturally identified as the disjoint union of tori T_δ's, where $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, and the complement in N of the interior of $\mathcal{U}$ can be naturally identified with the disjoint union of the geometric pieces $\mathcal{J}$. Making this identification, we have:

$$N = \mathcal{J} \cup_{\partial\mathcal{U}} \mathcal{U}.$$

Let $\epsilon_3 > 0$ be the Margulis constant of $\mathbb{H}^3$, so every $0 < \epsilon < \epsilon_3$ is a proper Margulis number of $\mathbb{H}^3$ (hence also of $\mathbb{H}^2$). For any $0 < \epsilon < \epsilon_3$, we may endow N with a Riemannian metric ρ_ϵ that approximates its geometrization, namely, the complete Riemannian 3-manifold:

$$(N, \rho_\epsilon),$$

satisfies the following requirements. For every $\mathbb{H}^3$-geometric piece J_v of N, (J_v, ρ_ϵ) is isometric to the corresponding complete hyperbolic 3-manifold J_v^{geo} with open ϵ-thin horocusps removed; or if J_v is Seifert-fibered, (J_v, ρ_ϕ) is isometric to a corresponding complete $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\widetilde{\mathrm{SL}}_2$-geometric 3-manifold J_v^{geo} with open horizontal-ϵ-thin horocusps removed. Here by *horizontal* we mean with respect to the pseudo-metric pulled back from the metric on the hyperbolic base orbifold, so for instance, a horizontal-ϵ-thin horocusp means the preimage in J_v^{geo} of a ϵ-thin horocusp in O^{geo}, the base orbifold with the naturally induced Riemannian metric. We do not impose further conditions for ρ_ϵ on the rest of N.

Note that with the Riemanian metric ρ_ϵ on N, one may speak of the *area* for any piecewise-linearly immersed CW 2-complex $f : K \to N$, or for any integral (cellular) 2-chain of K. Specifically, note that for each hyperbolic J_v, there is an area measure on $f^{-1}(J_v) \cap K$ pulling back the hyperbolic area measure on J_v, and for each $\mathbb{H}^2 \times \mathbb{E}^1$-geometric J_v, there is an area measure on $f^{-1}(J_v) \cap K$ puling back the horizontal-area measure on J_v, (namely, the area pulled back from the base orbifold). Thus, the *area* of K with respect to f is known as the sum of the area measures of $K \cap f^{-1}(J_v)$ for all $v \in \mathrm{Ver}(\Lambda)$, denoted as $\mathrm{Area}(f(K))$; and the area of an integral 2-chain of K is the sum of the areas of its simplices weighted by the absolute values of their coefficients.

We first homotope f to be piecewise linear so that $f^{-1}(\mathcal{T}) \subset M$ becomes a normal surface in minimal position with respect to the triangulation of M, and that $f^{-1}(\mathcal{U}) \subset M$ is an interval bundle over $f^{-1}(\mathcal{T})$. Here minimal position means that the cardinality of $f^{-1}(\mathcal{T}) \cap M^{(1)}$ is minimized. Furthermore, we pull straight $f|_{M^{(2)}}$ within each J_v relative to ∂J_v, with respect to the Riemannian metric ρ_ϵ, namely:

Lemma 3.1.1. *If $\epsilon > 0$ is sufficiently small, then the map $f : M \to N$ can be homotoped relative to $f^{-1}(\mathcal{U})$, so that $f(M^{(2)}) \cap \mathcal{J}$ is ruled on each component of the image of the* 2*-simplices of M, and that the area of $M^{(2)}$ is at most $2\pi\tau(M)$, where $\tau(M)$ is the number of tetrahedra in the triangulation of M.*

Proof. To sketch the proof, pick a subdivision of the components of $M^{(2)} \setminus (f^{-1}(\mathcal{U}) \cup M^{(1)})$ into the fewest possible triangles. First homotope f relative to $f^{-1}(\mathcal{U})$, so that the image of the sides of these triangles becomes geodesic in their corresponding pieces. Then relatively homotope f further, so that the image of these triangles becomes ruled in their corresponding pieces. If we fix $f|_{M^{(0)}}$, as $\epsilon \to 0$, the image of these triangles converges to geodesic (possibly degenerate) triangles in hyperbolic pieces, and to horizonally-geodesic (possibly degenerate) triangles in $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\widetilde{\mathrm{SL}}_2$-geometric pieces (in the sense of being geodesic after projecting onto the base orbifold). Moreover, for each 2-simplex of $M^{(2)}$, all except at most one triangle above contained in this 2-simplex becomes degenerate in the above sense, while the exceptional one has area at most π. Thus, for sufficiently small $\epsilon > 0$, the area of $M^{(2)}$ can be bounded by $2\tau(M)\pi$ where $2\tau(M)$ is the number of 2-simplices of $M^{(2)}$ with our notations. □

We shall say that $f : M \to N$ is *straightened* if it has been homotoped to a position satisfying the conclusion of Lemma 3.1.1. Note this process depends on the choice of the minimal triangulation of M, and the Riemannian metric ρ_ϵ of N for a sufficiently small $\epsilon > 0$, but for the sake of simplicity, we shall not mention such a choice explicitly as long as it causes no confusion.

3.2 Local geometry of straightened maps

Let M be an orientable closed 3-manifold, and N be an orientable closed prime 3-manifold. Suppose $f : M \to N$ is a straightened map from M to N.

By a *local region* $\mathcal{W} \subset N$, we mean a connected compact 3-submanifold of N whose boundary lies entirely in $N \setminus \mathcal{U}$, such that any component of the preimage of W is a convex submanifold of the Riemmanian universal cover of N. Suppose

$\mathcal{W} \subset N$ is a local region, generic in the sense that $f^{-1}(\partial\mathcal{W})$ intersects $M^{(2)}$ in general position, i.e. that any 2-simplex of $M^{(2)}$ is transversal to $\partial\mathcal{W}$ under f. We write $M^{(2)}_{\mathcal{W}}$ for $f^{-1}(\mathcal{W}) \cap M^{(2)}$, and $M^{(2)}_{\partial\mathcal{W}}$ for $f^{-1}(\partial\mathcal{W}) \cap M^{(2)}$. Then $M^{(2)}_{\mathcal{W}}$ has a natural CW 2-complex structure, induced from the triangulation of M, and $M^{(2)}_{\partial\mathcal{W}}$ is a 1-subcomplex.

We are interested in two aspects of the local behavior of f over $\mathcal{W}$. On one hand, the local second relative $\mathbf{R}$-homology of $(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}})$ has a bounded generating set of relative $\mathbf{Z}$-cycles, (Lemma 3.2.1); and on the other hand, the domination property can be inherited locally, yielding a surjection on the second local relative $\mathbf{R}$-homology, (Lemma 3.2.2)

Lemma 3.2.1. *If $f : M \to N$ is a straightened map, and $\mathcal{W} \subset N$ is a local region, then there is an $\mathbf{R}$-spanning set of $H_2(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}}; \mathbf{R})$, in which the elements are represented by relative $\mathbf{Z}$-cycles each with area bounded by $A(2\tau(M))$. Here $A(n) = 27^n(9n^2 + 4n)\pi$, and $\tau(M)$ is the triagulation number of M.*

Proof. Let $\mathcal{N}$ be an open regular neighborhood of $M^{(0)}$ in $M^{(2)}$. Let $K_{\mathcal{W}} = M^{(2)}_{\mathcal{W}} \setminus \mathcal{N}$, and $K_{\partial\mathcal{W}} = (M^{(2)}_{\partial\mathcal{W}} \cup \bar{\mathcal{N}}) \setminus \mathcal{N}$. As $H_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{R}) \cong H_2(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}}; \mathbf{R})$ via an obvious quotient map $K_{\mathcal{W}} \to M^{(2)}_{\mathcal{W}}$, it suffices to find an $\mathbf{R}$-spanning set of $H_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{R})$ represented by relative $\mathbf{Z}$-cycles of area at most $A(2\tau(M))$.

Because $\mathcal{W}$ is local and f is straightened, $K_{\mathcal{W}}$ is a finite union of 0-*handles* (half-disks), 1-*handles* (bands), *monkey-handles* (hexagons), and possibly a few *isolated disks* (disks whose boundary do not meet the 1-skeleton of $M^{(2)}$). It is clear that the number of monkey-handles is at most the number of 2-simplices $2\tau(M)$, and the union of 1-handles in $K_{\mathcal{W}}$ is an interval-bundle over a (possible disconnected) graph. By fixing an orientation for each of them, the handles and the isolated disks give a CW-complex structure on $K_{\mathcal{W}}$ in an obvious fashion. Let $C_*(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$, $\mathcal{Z}_*(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$, $\mathcal{B}_*(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$ denote the free $\mathbf{Z}$-modules of cellular relative chains, cycles and boundaries, respectively. Note that $C_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$ has a natural basis consisting of the handles and the isolated disks.

To prove the lemma, it suffices to find a generating set for $\mathcal{Z}_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{Q})$ whose elements are in $\mathcal{Z}_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}) \leq C_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$ with bounded coefficients over the natural basis. Decompose $K_{\mathcal{W}}$ as:

$$K_{\mathcal{W}} = S_{\mathcal{W}} \sqcup E_{\mathcal{W}} \sqcup K'_{\mathcal{W}},$$

where $S_{\mathcal{W}}$ is the union of the isolated disk components, $E_{\mathcal{W}}$ is the union of the components that contain no monkey-handles, and $K'_{\mathcal{W}}$ is the union of the components that contain at least one monkey-handle. Let $S_{\partial\mathcal{W}}$, $E_{\partial\mathcal{W}}$, $K'_{\partial\mathcal{W}}$ be the intersection of $S_{\mathcal{W}}$, $E_{\mathcal{W}}$, $K'_{\mathcal{W}}$ with $K_{\partial\mathcal{W}}$, respectively.

$$\mathcal{Z}_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{Q}) = \mathcal{Z}_2(S_{\mathcal{W}}, S_{\partial\mathcal{W}}; \mathbf{Q}) \oplus \mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}}; \mathbf{Q}) \oplus \mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q}).$$

It suffices to find bounded generating relative $\mathbf{Z}$-cycles for the direct-summands separately.

First, consider $\mathcal{Z}_2(S_{\mathcal{W}}, S_{\partial\mathcal{W}}; \mathbf{Q})$. Clearly, it has a generating set whose elements are the isolated disks. Hence absolute value of the coefficients over the natural basis are bounded ≤ 1 for every element of the generating set.

Secondly, consider $\mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}}; \mathbf{Q})$. We show that it has a generating set whose elements have coefficients bounded ≤ 2 in absolute value over the natural basis. To see this, note that $E = E_I \cup D_1 \cup \cdots \cup D_s$ is a union of an I-bundle E_I over a (possibly disconnected) graph Γ_I together with 0-handles D_j, $1 \leq i \leq s$. Note also that $K_{\partial\mathcal{W}} \cap E_I$ is an embedded ∂I-bundle $E_{\partial I}$. Now $\mathcal{Z}_2(E_I, E_{\partial I}; \mathbf{Q})$ can be generated by all the relative $\mathbf{Z}$-cycles, in fact finitely many, of the following forms: (i) $A_I \in \mathcal{Z}_2(E_I, E_{\partial I})$, where $(A_I, A_{\partial I}) \subset (E_I, E_{\partial I})$ is a sub-I-bundle which is an embedded annulus; or (ii) $R_I + 2B_I + R'_I \in \mathcal{Z}_2(E_I, E_{\partial I})$, where $(R_I, R_{\partial I}), (R'_I, R'_{\partial I}) \subset (E_I, E_{\partial I})$ are sub-I-bundles which are embedded Möbius strips, and $(B_I, B_{\partial I}) \subset (E_I, E_{\partial I})$ is a sub-I-bundle which is an embedded band joining R_I and R'_I. Moreover, $\mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}}; \mathbf{Q}) \,/\, \mathcal{Z}_2(E_I, E_{\partial I}; \mathbf{Q})$ can be generated by the residual classes represented by all the relative $\mathbf{Z}$-cycles, in fact finitely many, of the following forms: (i) $D_j + B_I \pm D_{j'} \in \mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}})$, where D_j, $D_{j'}$ are distinct 0-handles, and $(B_I, B_{\partial I}) \subset (E_I, E_{\partial I})$ is a sub-I-bundle which is an embedded band joining D_j and $D_{j'}$; or (ii) $2D_j + 2B_I + R_I \in \mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}})$, where D_j is a 0-handle, and $(R_I, R_{\partial I}) \subset (E_I, E_{\partial I})$ is a sub-I-bundle which is an embedded Möbius strip, and $(B_I, B_{\partial I}) \subset (E_I, E_{\partial I})$ is a sub-I-bundle which is an embedded band joining D_j and R_I. All these relative $\mathbf{Z}$-cycles together generate $\mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}}; \mathbf{Q})$, and each of them has coefficients bounded ≤ 2 in absolute value over the natural basis.

Finally, consider $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$. We show that it has a generating set whose elements have coefficients bounded $\leq 27^t(9t + 4)$ in absolute value over the natural basis, where $t \leq 2\tau(M)$ is the number of monkey-handles. We write the 1-handles of $K'_{\mathcal{W}}$ as $B_1, \cdots, B_r$, and the 0-handles as $D_1, \cdots, D_s$ in $K'_{\mathcal{W}}$, and the monkey-handles as $F_1, \cdots, F_t$. Pick a maximal union of 1-handles K'_I so that K'_I is homeomorphic to a trivial I-bundle over a (possibly disconnected) graph. We write its components as $K'_{I,1}, \cdots, K'_{I,p}$, where $p \leq 3t$.

Let $\bar{\partial} : \mathcal{C}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}) \to \mathcal{C}_1(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}})$ be the relative boundary operator. Then $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$ is by definition the solution space of:

$$\bar{\partial} U = 0,$$

for $U \in \mathcal{C}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$. We shall first solve the residual equation $\bar{\partial} U \equiv 0$ modulo $\mathcal{B}_1(K'_I, K'_{\partial I})$, then lift a set of fundamental solutions to solutions of $\bar{\partial} U = 0$ by adding chains from $C_2(K'_I, K'_{\partial I})$. This set of solutions together with a generating set of $\mathcal{Z}_2(K'_I, K'_{\partial I}; \mathbf{Q})$ will be a generating set of $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$.

To solve $\partial U = 0$ modulo $\mathcal{B}_1(K'_I, K'_{\partial I})$, we write:

$$U = \sum_{i=1}^{r} x_i\, B_i + \sum_{j=1}^{s} y_j\, D_j + \sum_{k=1}^{t} z_k\, F_k.$$

The topological interpretation of $\bar{\partial} U$ modulo $\mathcal{B}_1(K'_I, K'_{\partial I})$ is the total 'contribution' of the base elements B_i, D_j, F_k's to the fiber of each component of K'_I.

To make sense of this, on each component $K'_{I,l}$ of K'_I, we pick an oriented fiber φ_l, $1 \leq l \leq p$. Note that $C_1(K'_I, K'_{\partial I}) = C_1(K'_{I,1}, K'_{\partial I,1}) \oplus \cdots \oplus C_1(K'_{I,p}, K'_{\partial I,p})$, and that:

$$C_1(K'_I, K'_{\partial I}) \,/\, \mathcal{B}_1(K'_I, K'_{\partial I}) \cong \mathbf{Z}^{\oplus p},$$

generated by $\varphi_1, \cdots, \varphi_p$ mod $\mathcal{B}_1(K'_I, K'_{\partial I})$. The contribution of B_i, D_j, F_k on φ_l is formally the value of $\bar{\partial} B_i$, $\bar{\partial} D_j$, $\bar{\partial} F_k$ modulo $\mathcal{B}_1(K'_I, K'_{\partial I})$ on the l-th direct-summands. In other words, we count algebraically how many components of $\bar{\partial} B_i$ is parallel to φ_l in $K'_{I,l}$, and similarly for $\bar{\partial} D_j$, $\bar{\partial} F_k$. In this sense, on any φ_l, each B_i contributes 0 or ± 2, each D_j contributes 0 or ± 1, and each F_k contributes 0, ± 1, ± 2 or ± 3. Let $\vec{u}$ be the column vector of coordinates $(x_1, \cdots, x_r, y_1, \cdots, y_s, z_1, \cdots, z_t)^T$, and $q = r+s+t$. Let a_{lm} be the contribution of the m-th basis vector (corresponding to some B_i, D_j or F_k) on φ_l. Thus, a_{lm} are integers satifying $|a_{lm}| \leq 3$, for $1 \leq l \leq p$, $1 \leq m \leq q$, and $\sum_{l=1}^{p} |a_{lm}| \leq 3$, for $1 \leq m \leq q$. The residual equation $\bar{\partial} U = 0$ mod $\mathcal{B}_1(K'_I, K'_{\partial I})$ becomes a linear system of equations:

$$A\vec{u} = \vec{0},$$

where $A = (a_{lm})$ is a $p \times q$ integral matrix. Every column of A has at most 3 nonzero entries, and the sum of their absolute values is at most 3. Our aim is to find a set of fundamental solutions over $\mathbf{Q}$ with bounded integral entries.

Picking out a maximal independent collection of equations if necessary, we may assume p equals the rank of A over $\mathbf{Q}$. We may also re-order the coordinates and assume the first p columns of A are linearly independent over $\mathbf{Q}$. Let $A = (P, Q)$ where P consists of the first p columns and Q of the rest $q - p$ columns. Let $\vec{u} = \begin{pmatrix} \vec{v} \\ \vec{w} \end{pmatrix}$ be the corresponding decomposition of coordinates. Then the linear system becomes $P\vec{v} + Q\vec{w} = \vec{0}$. Basic linear algebra shows that a set of fundamental solutions is $\vec{v}_n = -P^{-1}Q\,\vec{e}_n$, $\vec{w}_n = \vec{e}_n$, where $1 \leq n \leq q - p$ and $(\vec{e}_1, \cdots, \vec{e}_{q-p})$ is the natural basis of $\mathbf{R}^{q-p}$. We clear the denominator by letting $\vec{v}^*_n = -P^*Q\,\vec{e}_n$, $\vec{w}^*_n = \det(P)\,\vec{e}_n$, where P^* is the adjugate matrix of P. The corresponding $\vec{u}^*_1, \cdots, \vec{u}^*_{q-p}$ is a set of fundamental solutions over $\mathbf{Q}$ of the linear system $A\vec{u} = \vec{0}$ with integral entries.

For each $1 \leq n \leq q-p$, $\vec{u}_n^*$ has at most $p+1$ non-zero entries, and the absolute value of the entries are all bounded 3^p. Indeed, $\vec{u}^*$ has at most $p+1$ non-zero entries by the way we picked $\vec{v}_n^*$ and $\vec{w}_n^*$. To bound the absolute value of entries, note each column of P has at most 3 nonzero entries whose absolute value sum ≤ 3. It is easy to see $|\det(P)| \leq 3^p$ by an induction on p using column expansions. Similarly, the absolute value of each entry of P^* is at most 3^{p-1}, and each column of Q has at most 3 nonzero entries whose absolute value sum ≤ 3, so the absolute value of any entry of $-P^*Q$ is also $\leq 3^p$.

Let $U_1^*, \cdots, U_{q-p}^* \in C_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}})$ be the relative 2-chains corresponding to the fundamental solutions $\vec{u}_1^*, \cdots, \vec{u}_{q-p}^*$ respectively as obtained above. Then the U_n^*'s form a set of fundamental solutions to $\partial U = 0 \bmod \mathcal{B}_1(K'_I, K'_{\partial I})$. To lift U_n^* to a solution of $\partial U = 0$, note ∂U_n^* is the $\mathbf{Z}$-algebraic sum of 1-simplices each parallel to a fiber φ_l. For a 1-simplex σ parallel to φ_l coming from ∂U_n^*, we pick a sub-I-bundle of $K'_{I,l}$ which is an embedded band joining σ and φ_l, and let $L_n \in C_2(K'_I, K'_{\partial I})$ be the relative $\mathbf{Z}$-chain which is the algebraic sum of all such sub-I-bundles. Since each sub-I-bundle as a relative $\mathbf{Z}$-chain has coefficient bounded by 1 in absolute value over the natural basis, the absolute values of coefficients of L_n are bounded $\leq 3 \cdot 3^p(p+1) = 3^{p+1}(p+1)$. Let $\hat{U}_n = U_n^* - L_n$, $1 \leq n \leq q-p$, then $\partial\hat{U}_n = 0$, with coefficients bounded $\leq 3^{p+1}(p+1) + 3^p = 3^p(3p+4)$ in absolute value.

In other words, $\hat{U}_n \in \mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}})$, $1 \leq n \leq q-p$. Moreover, $\hat{U}_n$'s together with a generating set of $\mathcal{Z}_2(K'_I, K'_{\partial I}; \mathbf{Q})$ generate $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$. Since K'_I has no monkey-handle, the no-monkey-handle case implies that $\mathcal{Z}_2(K'_I, K'_{\partial I}; \mathbf{Q})$ has a generating set of relative $\mathbf{Z}$-cycles with coefficients bounded by 2 in absolute value. Therefore, $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$ has a generating set of relative $\mathbf{Z}$-cycles, consisting of $\hat{U}_n$'s and the generating set of $\mathcal{Z}_2(K'_I, K'_{\partial I}; \mathbf{Q})$ as above, with coefficients bounded by $3^p(3p+4)$ in absolute value. Remember $p \leq 3t$, the absolute values of coefficients are bounded $\leq 3^{3t}(3 \cdot 3t + 4) = 27^t(9t+4)$.

Now a generating set of $\mathcal{Z}_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{Q})$ is obtained by putting together the generating sets of its direct summands $\mathcal{Z}_2(S_{\mathcal{W}}, S_{\partial\mathcal{W}}; \mathbf{Q})$, $\mathcal{Z}_2(E_{\mathcal{W}}, E_{\partial\mathcal{W}}; \mathbf{Q})$, and $\mathcal{Z}_2(K'_{\mathcal{W}}, K'_{\partial\mathcal{W}}; \mathbf{Q})$ as constructed above. It consists of relative $\mathbf{Z}$-cycles with coefficients bounded by $27^t(9t+4)$ over the natural basis. In particular, they represent homology classes that generate $H_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}}; \mathbf{Q})$. Remember that the natural basis of $C_2(K_{\mathcal{W}}, K_{\partial\mathcal{W}})$ consists of handles and isolated disks, whose total area is bounded by $2\pi\tau(M)$, (Lemma 3.1.1). Therefore, the generating set consists of relative $\mathbf{Z}$-cycles with area bounded $\leq 27^t(9t+4) \cdot \mathrm{Area}(K_{\mathcal{W}}) \leq A(2\tau(M))$, where $A(n) = 27^n(9n^2+4n)\pi$. □

Lemma 3.2.2. *If $f : M \to N$ is a straightened domination, and $\mathcal{W} \subset N$ is a local region, then the induced homomorphism:*

$$f|_* : H_2(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}}; \mathbf{R}) \to H_2(\mathcal{W}, \partial\mathcal{W}; \mathbf{R}),$$

is surjective.

Proof. We first decompose the homomorphism $f|_*$ as:

$$\begin{array}{ccccc} H_2(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}}; \mathbf{R}) & & & & \\ \cong\downarrow & & & & \\ H_2(M^{(2)}, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) & \xrightarrow{i_*} & H_2(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) & \xrightarrow{\bar{f}_*} & H_2(N, N\setminus\mathring{\mathcal{W}}; \mathbf{R}) \\ & & & & \cong\downarrow \\ & & & & H_2(\mathcal{W}, \partial\mathcal{W}; \mathbf{R}), \end{array}$$

where the vertical isomorphisms are homology excisions. The homomorphism i_* induced by the inclusion is surjective by the long exact sequence of relative homology:

$$\cdots \longrightarrow H_2(M^{(2)}, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) \xrightarrow{i_*} H_2(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) \longrightarrow H_2(M, M^{(2)}; \mathbf{R}) \longrightarrow \cdots,$$

where $H_2(M, M^{(2)}; \mathbf{R}) \cong 0$. It suffices to show $\bar{f}_*$ is surjective.

Because $f : M \to N$ has nonzero degree, the commutative diagram:

$$\begin{array}{ccc} H^3(N, N\setminus\mathring{\mathcal{W}}; \mathbf{R}) & \xrightarrow{\bar{f}^*} & H^3(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) \\ \downarrow & & \downarrow \\ H^3(N; \mathbf{R}) & \xrightarrow{f^*} & H^3(M; \mathbf{R}), \end{array}$$

implies that $\bar{f}^*$ is injective on the third $\mathbf{R}$-coefficient relative cohomology.

Thus,

$$\bar{f}^* : H^*(N, N\setminus\mathring{\mathcal{W}}; \mathbf{R}) \to H^*(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}),$$

is injective on all dimensions, following from the commutative diagram:

$$\begin{array}{ccccc} H^i(\mathring{\mathcal{W}}; \mathbf{R}) & \times H^{3-i}(N, N\setminus\mathring{\mathcal{W}}; \mathbf{R}) & \xrightarrow{\smile} & H^3(N, N\setminus\mathring{\mathcal{W}}; \mathbf{R}) \\ \bar{f}^*\downarrow & \bar{f}^*\downarrow & & \bar{f}^*\downarrow \\ H^i(M - M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) & \times H^{3-i}(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}) & \xrightarrow{\smile} & H^3(M, M^{(2)}_{N\setminus\mathring{\mathcal{W}}}; \mathbf{R}), \end{array}$$

where the cup-product pairings are nonsingular and the rightmost vertical homomorphism is injective.

Therefore, $\overline{f_*} : H_*(M, M^{(2)}_{N \setminus \mathring{\mathcal{W}}}; \mathbf{R}) \to H_*(N, N \setminus \mathring{\mathcal{W}}; \mathbf{R})$ is indeed surjective on all dimensions, and in particular, on dimension two as desired. $\square$

Chapter 4

Preglue finiteness

In this chapter, we study finiteness of preglue graph-of-geometrics under domination. In particular, we show that given an orientable closed 3-manifold M, then there are at most finitely many possible homeomorphism types of JSJ pieces that could appear in a non-geometric prime 3-manifold dominated by M. This result was known due to Soma [Som00] and Boileau–Rubinstein–Wang [BRW], and our treatment provides an alternative approach.

4.1 Short hyperbolic geodesics

In this section, we give a lower bound estimation for the length of geodesics in an $\mathbb{H}^3$-geometric piece of the target, under the assumption of domination. Note that if N is an orientable closed prime 3-manifold, and if J is a hyperbolic piece in N, then it follows from the Mostow-Prasad Rigidity Theorem that the interior of J has a unique complete hyperbolic metric of finite volume, up to isometry. In this sense, we may speak of geodesics in this piece.

Proposition 4.1.1. *Suppose M is an orientable closed 3-manifold, then there exists a constant $\delta > 0$, depending only on M, satisfying the following. If $f : M \to N$ is a domination, and J is an $\mathbb{H}^3$-geometric piece of N, then the length of every closed geodesic in J is at least δ.*

Proof. Without loss of generality, we may assume f has been straightened, with respect to a minimal triangulation of M and a Riemannian metric ρ_ϵ of N approximating its geometrization for some sufficiently small Margulis number $\epsilon > 0$, (Lemma 3.1.1). Suppose γ is a closed geodesic in J. A theorem of Chun Cao, Frederick Gehring and Gavin Martin says that if γ has length $l < \frac{\sqrt{3}}{2\pi}(\sqrt{2}-1)$, then there is an embedded tube $V \subset M$ of radius r with the core geodesic γ, such that:

$$\sinh^2(r) = \frac{\sqrt{1 - (4\pi l/\sqrt{3})}}{4\pi l/\sqrt{3}} - \frac{1}{2},$$

[CGM97]. This means if γ is very short, it lies in a very deep tube V. In particular, any meridian disk of V will have very large area.

Up to a small adjustment of the radius of V, we may assume the image of two skeleton $f(M^{(2)})$ intersects ∂V in general position. Picking V as the local region $\mathcal{W}$, we apply Lemma 3.2.1 and see that the homomorphism $f|_* : H_2(M^{(2)}_{\mathcal{W}}, M^{(2)}_{\partial\mathcal{W}}; \mathbf{R}) \to H_2(\mathcal{W}, \partial\mathcal{W}; \mathbf{R})$ would vanish if the area of the meridian disk of the tube V was larger than $A(2\tau(M))$. This would violate the assumption that f is a domination by Lemma 3.2.2. Therefore, the radius r of V must satisfy:

$$\pi \sinh^2(r) > A(2\tau(M)),$$

which implies a lower bound of the length of γ depending only on the triangulation number $\tau(M)$ of M. □

4.2 Sharp cone angles

In this section, we give a lower bound estimation for the cone angle of the base orbifold in an $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\widetilde{\mathrm{SL}}_2$-geometric piece of the target, under the assumption of domination. Note that if N is an orientable closed prime 3-manifold, and if J is a $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\widetilde{\mathrm{SL}}_2$-geometric piece in N, then J is a Seifert fibered space over a hyperbolic base orbifold, and the cone angle at any cone point equals 2π divided by its order. In this sense, we may speak of angle of cone points on the base orbifold of this piece.

Proposition 4.2.1. *Suppose M is an orientable closed 3-manifold, then there exists a constant $\delta > 0$, depending only on M, satisfying the following. If $f : M \to N$ is a domination, and J is an $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\widetilde{\mathrm{SL}}_2$-geometric piece of N, then the angle of every cone point on the base orbifold of J is at least δ.*

Proof. Without loss of generality, we may assume f has been straightened, with respect to a minimal triangulation of M and a Riemannian metric ρ_ϵ of N approximating its geometrization for some sufficiently small Margulis number $\epsilon > 0$, (Lemma 3.1.1). Suppose γ is the (geodesic) fiber over a cone point of the base orbifold of J. A result of Gavin Martin implies that for any complete hyperbolic 2-orbifold O

with a cone point of angle $\frac{2\pi}{q}$, there is an embedded cone centered at the point with radius r satisfying:

$$\cosh(r) = \frac{1}{2\sin\frac{\pi}{q}},$$

which is optimal in $S^2(2,3,q)$, (cf. [Mar96, Theorem 2.2]). Applying to the cone point we are concerned about, the preimage of the embedded cone in J is a tube V, which will have very large radius if the cone is very sharp. The rest of the proof is a verbatim repeat of the second paragraph in the proof of Proposition 4.1.1. □

4.3 Finiteness of preglue graph-of-geometrics

In this section, we bound number of allowable preglue graph-of-geometrics under domination.

Theorem 4.3.1. *Suppose M is an orientable closed 3-manifold. Then there are at most finitely many isomorphically distinct preglue graph-of-geometrics $(\Lambda, \mathcal{J})$, which admit a nondegenerate gluing $\phi \in \Phi(\Lambda, \mathcal{J})$ yielding a 3-manifold dominated by M.*

Remark 4.3.2. Theorem 4.3.1 implies that if M dominates a non-geometric prime 3-manifold N, then there are at most finitely many homeomorphism types of JSJ pieces that can appear in N. This reproves [BRW, Theorem 1.1] modulo some easy geometric cases. Moreover, the part of the argument about hyperbolic pieces does not appeal to the fact that N is non-geometric, so it works for the geometric case as well, reproving [Som00, Theorem 1].

Proof. Suppose $(\Lambda, \mathcal{J})$ is a preglue graph-of-geometrics with a nondegenerate gluing $\phi \in \Phi(\Lambda, \mathcal{J})$ that yields a 3-manifold N_ϕ dominated by M. As the Kneser–Haken number $h(M)$, namely i.e. the maximal possible number of components of essential subsurfaces of M, bounds that of N_ϕ (cf. [Wan91, Propositio 4]), the number of edges of Λ is at most $h(M)$, and the number of vertices of Λ is at most $h(M) + 1$. Thus there are at most finitely many allowable isomorphism types of Λ. Note also that N_ϕ is by definition non-geometric. It suffices to bound the number of homeomorphism types of geometric pieces that can appear as components of $\mathcal{J}$, which are either $\mathbb{H}^3$-geometric or $\mathbb{H}^2 \times \mathbb{E}^1$-geometric.

Suppose J is an $\mathbb{H}^3$-geometric piece of N_ϕ. As N_ϕ is dominated by M, the simplicial volume of N_ϕ is bounded by that of M. It is a well-known result due to Teruhiko Soma [Som81, Theorem 1] that the simplicial volume of a closed 3-manifold equals the sum of the hyperbolic volumes of its hyperbolic pieces. Thus the volume of J is bounded above by the simplicial volume of M. Moreover, Proposition 4.1.1 implies that the length of the shortest geodesic in J is bounded below in terms of the triangulation number of M. It follows from the Jørgensen–Thurston Theorem (cf. [Thu80, Theorem 5.12.1]) that there are at most finitely many allowable homeomorphism types of J.

Suppose J is an $\mathbb{H}^2 \times \mathbb{E}^1$-geometric piece of N_ϕ. It suffices to bound the number of allowable isomorphism types of the base orbifold $\mathcal{O}$. Applying the Kneser–Haken finiteness again, we see the genus and the number of cone points of $\mathcal{O}$ are both bounded above in terms of $h(M)$. Moreover, Proposition 4.2.1 implies that the cone angles of $\mathcal{O}$ are bounded below in terms of the triangulation number of M. Thus there are at most finitely many allowable isomorphism types of $\mathcal{O}$, and hence finitely many allowable homeomorphism types of J. This completes the proof. $\square$

Chapter 5

Gluing finiteness

In this chapter, we study finiteness of gluings under domination for any given preglue graph-of-geometrics. This will lead to the main result of the present dissertation, namely, that every orientable closed 3-manifold dominates at most finitely many non-geometric prime 3-manifolds up to homeomorphism, (Theorem 5.4.2).

5.1 Distortions in a gluing

In this section, we introduce the notion of distortion measuring the complexity of a nondegenerate gluing. There are distortions at vertices of Λ, or along edges of Λ, which, roughly speaking, measure the local obstruction to extending the geometry beyond the corresponding pieces, or across the corresponding cutting tori or Klein bottles, respectively.

Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics, and $\phi \in \Phi(\Lambda, \mathcal{J})$ be a nondegenerate gluing. Remember that in this situation, the associated 3-manifold N_ϕ is non-geometric, and every geometric piece J of N_ϕ is a component of $\mathcal{J}$ with nonempty tori boundary, and supports either the geometry $\mathbb{H}^3$ or the geometry $\mathbb{H}^2 \times \mathbb{E}^1$.

We first introduce a natural positive-semidefinite quadratic form:

$$\mathfrak{q}_J : H_1(\partial J; \mathbf{R}) \to \mathbf{R},$$

on $H_1(\partial J; \mathbf{R})$, for each geometric piece $J \subset \mathcal{J}$, as follows.

If J is $\mathbb{H}^3$-geometric, the interior of J has a unique complete hyperbolic metric of finite volume, so we denote the cusped hyperbolic 3-manifold as J^{geo}. Then the induced conformal structures on the cusps endow $H_1(\partial J; \mathbf{R})$ with a canonical norm. Specifically, let $\epsilon > 0$ be a sufficiently small Margulis number of $\mathbb{H}^3$, so that the compact ϵ-thick part of J^{geo} removes only horocusps of J^{geo}. Then the boundary of ϵ-thick part is a disjoint union of tori $T^1 \sqcup \cdots \sqcup T^q$ with induced Euclidean metrics, canonical up to rescaling for sufficiently small ϵ. We rescale the Euclidean metric on each T^j so that the shortest simple closed geodesic on T^j has length 1. This rescaled metric induces a Euclidean metric on the universal covering of T^j, and hence defines a canonical positive-definite quadratic form $\mathfrak{q}_{T^j}$ via the naturally induced inner product on $H_1(T^j;\mathbf{R})$. We define the positive-definite quadratic form $\mathfrak{q}_J$ on $H_1(\partial J;\mathbf{R}) \cong H_1(T^1;\mathbf{R}) \oplus \cdots \oplus H_1(T^q;\mathbf{R})$ to be the direct sum of the quadratic forms on its components.

If J is $\mathbb{H}^2 \times \mathbb{E}^1$-geometric, it is a Seifert fibered space over a hyperbolic base orbifold $\mathcal{O}$. Let $p : \pi_1(J) \to \pi_1(\mathcal{O})$ be the naturally induced homomorphism, where $\pi_1(\mathcal{O})$ is the fundamental group in the orbifold sense. For any component $T \subset \partial J$, we regard $\pi_1(T)$ as a subgroup of $\pi_1(J)$, so we may first define for any $\zeta \in H_1(T;\mathbf{Z}) \cong \pi_1(T)$ that $\mathfrak{q}_T(\zeta)$ equals the square of the divisibility of $p(\zeta) \in \pi_1(\mathcal{O})$ if $p(\zeta)$ is nontrivial, and equals zero if $p(\zeta)$ is trivial. This extends to a unique positive-semidefinite quadratic form $\mathfrak{q}_T$ on $H_1(T;\mathbf{R})$ which vanishes on the ordinary-fiber dimension. We define the positive-semidefinite quadratic $\mathfrak{q}_J$ on $H_1(\partial J;\mathbf{R})$ by summing up the quadratic forms on its components.

These local quadratic forms allows us to define a positive-definite quadratic form $\mathfrak{q}_\phi$ associated to a nondegenerate gluing $\phi \in \Phi(\Lambda,\mathcal{J})$:

Definition 5.1.1. Suppose $(\Lambda,\mathcal{J})$ is a preglue graph-of-geometrics, and $\phi \in \Phi(\Lambda,\mathcal{J})$ is a nondegenerate gluing. For any end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, let v, v' be the vertices adjacent to δ and its opposite $\bar{\delta}$, respectively. For any $\zeta \in H_1(T_\delta;\mathbf{R})$, we define:

$$\mathfrak{q}_\phi(\zeta) = \mathfrak{q}_{J_v}(\zeta) + \mathfrak{q}_{J_{v'}}(\phi_\delta(\zeta)).$$

Note that this is also well-defined when $\delta = \bar{\delta}$, and that this is positive-definite as ϕ is nondegenerate. We define the positive-definite quadratic from $\mathfrak{q}_\phi$ on:

$$H_1(\partial\mathcal{J};\mathbf{R}) = \bigoplus_{\delta\in\widetilde{\mathrm{Edg}}(\Lambda)} H_1(T_\delta;\mathbf{R}),$$

to be the direct sum of the quadratic forms on its components. Note that $\mathfrak{q}_\phi$ depends only on the equivalent class of ϕ.

We are now ready to introduce the notion of distortions of a nondegenerate gluing. Recall that for any free $\mathbf{Z}$-module V of finite rank $n \geq 0$, and a quadratic form $\mathfrak{q}$ on $V_{\mathbf{R}} = V \otimes_{\mathbf{Z}} \mathbf{R}$ over $\mathbf{R}$, the discriminant:

$$\Delta(V,\mathfrak{q}) \in \mathbf{R},$$

is the determinant of the associated bilinear form of $\mathfrak{q}$ over a (hence any) basis of V. When $\mathfrak{q}$ is positive-definite, it equals the square of the volume of the n-dimensional flat torus $V_{\mathbf{R}} \,/\, V$ with the Euclidean structure of $V_{\mathbf{R}}$ induced from $\mathfrak{q}$.

Definition 5.1.2. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a gluing, and let $e \in \mathrm{Edg}(\Lambda)$ be an (entire or semi) edge. We define the *average distortion* (or simply, the *distortion*) of ϕ along e as:

$$\mathscr{D}_e(\phi) = \Delta\left(H_1(T_\delta; \mathbf{Z}),\ \mathrm{q}_\phi\right)^{\frac{1}{4}},$$

where δ is an end of e. Note the definition does not depend on the choice of the end.

Definition 5.1.3. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a gluing, and let $v \in \mathrm{Ver}(\Lambda)$ be a vertex of valence n_v. Suppose $n_v > 0$. If v is an entire-vertex, we define the *average distortion* (or simply, the *distortion*) of ϕ at v as:

$$\mathscr{D}_v(\phi) = \Delta\left(\partial_* H_2(J_v, \partial J_v; \mathbf{Z}),\ \mathrm{q}_\phi\right)^{\frac{1}{2n_v}},$$

where $\partial_* H_2(J_v, \partial J_v; \mathbf{Z})$ denotes the image of $H_2(J_v, \partial J_v; \mathbf{Z})$ in $H_1(\partial \mathcal{J}; \mathbf{R})$ under the natural boundary homomorphism. If v is a semi-vertex, J_v is Seifert-fibered with a non-orientable base orbifold. Let $\tilde{J}_v$ be the double covering of J_v corresponding to the centralizer of its ordinary-fiber, and let $\tilde{\mathrm{q}}_\phi$ on $H_1(\partial \tilde{J}_v; \mathbf{R})$ be the direct sum of the quadratic forms on each component $H_1(\tilde{T}; \mathbf{R})$ pulled back from q_ϕ, where $\tilde{T} \subset \partial \tilde{J}_v$. We define:

$$\mathscr{D}_v(\phi) = \Delta\left(\partial_* H_2(\tilde{J}_v, \partial \tilde{J}_v; \mathbf{Z}),\ \tilde{\mathrm{q}}_\phi\right)^{\frac{1}{4n_v}}.$$

We also define $\mathscr{D}_v(\phi) = 0$ if $n_v = 0$.

Remark 5.1.4. Note the definition of average distortion along entire edges can be restated in a similar fashion if one takes a compact regular neighborhood $\mathcal{U}_e$ of T_e in place of the role of J_v above, because $\partial_* H_2(\mathcal{U}_e, \partial \mathcal{U}_e; \mathbf{Z}) \cong H_1(T_\delta)$ is a canonical isomorphism. One can also restate the definition of average distortion along semi-edges.

5.2 Gluing with bounded distortions

In this section, we show a finiteness result that there are only finitely many homeomorphically distinct orientable closed irreducible 3-manifolds obtained from nondegenerate gluings of a preglue graph-of-geometrics with bounded distortions. This is an immediate consequence of the following:

Proposition 5.2.1. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. For any $C > 0$, there are at most finitely many distinct nondegenerate gluings $\phi \in \Phi(\Lambda, \mathcal{J})$ up to equivalence, such that $\mathscr{D}_v(\phi) < C$ for every vertex $v \in \mathrm{Ver}(\Lambda)$, and that $\mathscr{D}_e(\phi) < C$ for every edge $e \in \mathrm{Edg}(\Lambda)$.*

We prove Proposition 5.2.1 in the rest of this section. Our strategy is as follows: using distortion along edges, we bound the allowable gluings up to fiber-shearings (Definition 2.2.6); then using the distortion at Seifert-fibered vertices, we shall bound the allowable indices of fiber-shearings, and hence the allowable gluings up to equivalence. This will prove Proposition 5.2.1.

Firstly, we show that distortion along edges bounds nondegenerate gluings up to fiber-shearings. This follows from a general fact about twisted sum of positive semi-definite quadratic forms. Although we shall only apply the rank two case of Lemma 5.2.3 in our estimations, it might be worth pursuing a little more generality for certain independent interest.

The following an easy fact in linear algebra will be used later.

Lemma 5.2.2. *Let V be a free $\mathbf{Z}$-module of finite rank $n > 0$, and $\mathfrak{q}$ be a positive-definite quadratic form on $V_{\mathbf{R}} = V \otimes_{\mathbf{Z}} \mathbf{R}$. For any $C > 0$, and any integer $0 \leq k \leq n$, there are at most finitely many rank-k submodules W of V with the discriminant $\Delta(W, \mathfrak{q}) < C$.*

Proof. Fix a basis $e_1, \cdots, e_n$ of V. It suffices to prove for the Euclidean form $\mathfrak{q}_0$ induced by the fixed basis as an orthonormal basis, since the nondegeneracy ensures $\Delta(W, \mathfrak{q}_0) < \lambda \cdot \Delta(W, \mathfrak{q})$ for some $\lambda > 0$ depending only $\mathfrak{q}$. Note that rank-k submodules of V are in bijection with rank-1 submodules of $\wedge^k V$, represented by primitive elements $w \in \wedge^k V$ up to sign. As $\wedge^k V$ has a natural inner product with a standard orthonormal basis $\{e_{i_1} \wedge \cdots \wedge e_{i_k} \,|\, 1 \leq i_1 < \cdots < i_k \leq n\}$, for any $\pm w \in \wedge^k V$ representing W, the well-known Cauchy–Binet formula implies:

$$\Delta(W, \mathfrak{q}_0) = \|w\|^2,$$

where $\|\cdot\|$ is the norm induced from the inner product structure. As w is an integral linear combination of the basis vectors, there are at most finitely many primitive w's satisfying $\|w\| < C$. $\square$

Let V be a free $\mathbf{Z}$-module of finite rank $n \geq 0$. The special linear group $\Gamma = \mathrm{SL}(V)$ acts naturally (from the right) on the space of quadratic forms on $V_{\mathbf{R}}$, namely, any $\tau \in \Gamma$ transforms a quadratic form $\mathfrak{q}$ into the composition $\mathfrak{q}\tau$. We write the stabilizer of $\mathfrak{q}$ in Γ as $\Gamma_{\mathfrak{q}}$. We say a quadratic form $\mathfrak{q}$ has *rational kernel* with respect to the lattice $V \subset V_{\mathbf{R}}$, if the kernel $U_{\mathbf{R}}$ of (the associated bilinear form of) $\mathfrak{q}$ in $V_{\mathbf{R}}$ intersects V in a lattice (i.e. a discrete cocompact subgroup) $U \subset U_{\mathbf{R}}$.

Lemma 5.2.3. *With notations above, let $\mathfrak{q}, \mathfrak{q}'$ be two positive-semidefinite quadratic forms on $V_{\mathbf{R}}$ over $\mathbf{R}$ with rational kernels with respect to V. Note that the value of*

$\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')$ *depends only on the double-coset* $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$. *Then for any* $C > 0$, *there are at most finitely many distinct double-cosets* $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$ *of* Γ, *such that the discriminant:*

$$0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C.$$

Proof. We denote the unit-balls of $\mathfrak{q}$ and $\mathfrak{q}'$ as B and B', respectively. The unit-ball B_σ of $\mathfrak{q}\sigma + \mathfrak{q}'$ is clearly contained in $\sigma^{-1}(B) \cap B'$. When $\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') > 0$, B_σ is compact, but B or B' may be noncompact if $\mathfrak{q}$ or $\mathfrak{q}'$ are degenerate.

We claim that for any $C > 0$, there exists some compact subset:

$$K \subset V_{\mathbf{R}},$$

such that for any $\sigma \in \Gamma$ with $0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C$, there is some $\tau' \in \Gamma_{\mathfrak{q}'}$, such that the unit-ball $B_{\sigma\tau'}$ of $\mathfrak{q}(\sigma\tau') + \mathfrak{q}'$ is contained in K.

To prove this claim, we need to understand the action of $\Gamma_{\mathfrak{q}'}$. Let $U'_{\mathbf{R}}$ be the kernel of $\mathfrak{q}'$, of dimension k', and let $U' = U'_{\mathbf{R}} \cap V$ be the sublattice intersecting V. As $\mathfrak{q}'$ has rational kernel, U' also has rank k', and V splits as $U' \oplus L'$ for some sublattice L' of rank $n - k'$. Pick a basis $\xi'_1, \cdots, \xi'_{k'}$ of U' and a basis $\xi'_{k'+1}, \cdots, \xi'_n$ of L'. Hence they form a basis of V. Now $\Gamma_{\mathfrak{q}'}$ has a free abelian subgroup Π' generated by the 'elementary shearings' $\tau'_{ij} \in \Gamma$, defined for any $1 \le i \le k'$, and $k' + 1 \le j \le n$, by the identity on all the basis vectors except for:

$$\tau'_{ij}(\xi'_j) = \xi'_i + \xi'_j.$$

In particular, Π fixes the subspace $U'_{\mathbf{R}}$. Moreover, $\Gamma_{\mathfrak{q}'}$ has a natural subgroup isomorphic to $\mathrm{SL}(U')$, acting on the $U'_{\mathbf{R}}$ factor while fixing the $L'_{\mathbf{R}}$ factor. In fact, these two subgrops generate a finite-index normal subgroup of $\Gamma_{\mathfrak{q}'}$, which is a semidirect product $\Pi' \rtimes \mathrm{SL}(U')$.

We fix a reference Euclidean metric on $V_{\mathbf{R}}$ with the orthonormal basis $\xi'_1, \cdots, \xi'_n$, and denote the induced m-dimensional volume measure on any m-dimensional subspace as μ_m. It will be also conventient to make the convention that the zero-dimensional volume of the origin is one. The volume of B_σ is proportional to the reciprocal of the square root of $\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')$, indeed:

$$\mu_n(B_\sigma) = \frac{\omega_n}{\Delta(V, \mathfrak{q}\sigma + \mathfrak{q}')^{\frac{1}{2}}},$$

where $\omega_n = \frac{\pi^{n/2}}{\Gamma(1+n/2)}$ is the volume of an n-dimensional Euclidean unit-ball. Thus the assumption $0 < \Delta(V, \mathfrak{q}\sigma + \mathfrak{q}') < C$ is equivalent to:

$$\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) < \infty.$$

Up to a composition by some τ' in $\mathrm{SL}(U') \leq \Gamma_{\mathfrak{q}'}$, we may first assume $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ is bounded within a uniform distance $D_1 > 0$ from the origin. In fact, for any $\sigma \in \Gamma$, we have:

$$\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) \leq \mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}}) \cdot \mu_{n-k'}(B' \cap (U'_{\mathbf{R}})^{\perp}),$$

so $\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}})$ is bounded below in terms of C. On the other hand,

$$\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}}) = \frac{\omega_{k'}}{\Delta(U', \mathfrak{q}\sigma)^{\frac{1}{2}}} = \frac{\omega_{k'}}{\Delta(\sigma(U'), \mathfrak{q})^{\frac{1}{2}}},$$

and $\Delta(\sigma(U'), \mathfrak{q})$ further equals the discriminant of the embedded image $\overline{\sigma(U')}$ of $\sigma(U')$ in the quotient $V\,/\,U$, with respect to the induced nondegenerate quadratic form $\overline{\mathfrak{q}}$. Thus, the uniform lower bound of $\mu_{k'}(\sigma^{-1}(B) \cap U'_{\mathbf{R}})$ yields a uniform upper bound of $\Delta(\overline{\sigma(U')}, \overline{\mathfrak{q}})$. By Lemma 5.2.2, at most finitely many rank-k' submodules of $V\,/\,U$ are allowed to be the image $\overline{\sigma(U')}$. Furthermore, if two images $\overline{\sigma_0(U')}$ and $\overline{\sigma_1(U')}$ coincide, the identification pulls back to be an isomorphism τ' in $\mathrm{SL}(U')$, so that $\sigma_1 = \sigma_0\tau'$ restricted to U'. In other words, there are at most finitely many $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ up to compositions by elements of $\mathrm{SL}(U') \leq \Gamma_{\mathfrak{q}'}$. Hence they can be bounded uniformly within a uniform distance $D_1 > 0$ from the origin.

Now we also pick a splitting $V = U \oplus L$, and correspondingly pick a basis $\xi_1, \cdots, \xi_k$ of U and a basis $\xi_{k+1}, \cdots, \xi_n$ of L, in a similar fashion as before. For any $\sigma \in \Gamma$ with $\frac{\omega_n}{\sqrt{C}} < \mu_n(B_\sigma) < \infty$, we can find $k + 1 \leq j_1 < j_2 < \cdots < j_h \leq n$, where $h = n - k - k'$, such that $U'_{\mathbf{R}}$ is transversal to the subspace:

$$\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}),$$

where $H_{\mathbf{R}}$ is spanned by $\xi_{j_1}, \cdots, \xi_{j_h}$.

Let σ be as above. Up to a composition by some τ' in $\Pi \leq \Gamma_{\mathfrak{q}'}$, we may further assume $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ bounded within a uniform distance $D_2 > 0$ from the origin. In fact, we may find vectors:

$$\eta'_j = y_{1j}\,\xi'_1 + \cdots + y_{k'j}\,\xi'_{k'} + \xi'_j,$$

for each $k' + 1 \leq j \leq n$, such that the η'_j together span $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}})$. Note τ_{ij} fixes η'_t when $t \neq j$, and changes only the i-th coordinate of η'_j by $+1$. Thus, using $\sigma\tau'$ instead of σ for some $\tau' \in \Pi$, we may assume that $0 \leq y_{ij} < 1$ for all the y_{ij}'s above. Let $R > 0$ be sufficiently large, so that every point in $L'_{\mathbf{R}} \cap B'$ is bounded within in the radius R ball centered at the origin. Then every point in $\sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ is bounded within the radius $\sqrt{k' + 1}R$ ball centered at the origin, so we take this radius as the uniform $D_2 > 0$. Note that for a different $\sigma \in \Gamma$ one may need to pick a different coordinate subspace $H_{\mathbf{R}} \leq L_{\mathbf{R}}$, (indeed, there could be up to $\binom{n-k}{h}$ choices), but the constant $D_2 > 0$ depends only on $\mathfrak{q}'$. Note also that this does not affect $\sigma^{-1}(B) \cap U'_{\mathbf{R}}$ which we have already taken care of.

Under the adjustment assumptions above, every vector $v \in B_\sigma$ can be written as $u' + w$, where $u' \in \sigma^{-1}(B) \cap U'_{\mathbf{R}}$, and $w \in \sigma^{-1}(U_{\mathbf{R}} \oplus H_{\mathbf{R}}) \cap B'$ for some appropriate $H_{\mathbf{R}}$. Thus, for any $\sigma \in \Gamma$ with $0 < \Delta(\mathfrak{q}\sigma + \mathfrak{q}') < C$, we have shown that there is some $\tau' \in \Gamma_{\mathfrak{q}'}$ so that $B_{\sigma\tau'}$ is bounded within the uniform radius $D_1 + D_2$ ball centered at the origin. Taking this uniform large ball as K, we have proved the claim.

To complete the proof of Lemma 5.2.3, observe that there is a uniform positive lower bound of the length of the short-axis of the ellipsoid B_σ, provided that B_σ is bounded within K. This is clear because the volume $\mu_n(B_\sigma)$ is at least $\frac{\omega_n}{\sqrt{C}}$. It follows that for all such σ's, $(\mathfrak{q}\sigma + \mathfrak{q}')(\xi'_t)$ is bounded by some uniform constant, for every $1 \leq t \leq n$. Suppose:

$$\sigma(\xi'_t) = x_{t1}\,\xi_1 + \cdots + x_{tn}\,\xi_n,$$

where $x_{t1}, \cdots, x_{tn}$ are integers. Because $\mathfrak{q}$ vanishes restricted to $U_{\mathbf{R}}$ and is nondegenerate restricted to $L_{\mathbf{R}}$, we obtain a uniform upper bound for every x_{jt}, where $k + 1 \leq j \leq n$, and $1 \leq t \leq n$. Hence at most finitely many integers are allowed to be the the coefficients of the $\xi_{k+1}, \cdots, \xi_n$ components. Moreover, whenever two σ_0, σ_1 coincide on these coefficients, they differ only by a post-composition of some $\tau \in \Gamma$, which preserves U and induces the trivial action on V / U. Such a τ belongs to $\Gamma_{\mathfrak{q}}$, so $\Gamma_{\mathfrak{q}}\sigma_0 = \Gamma_{\mathfrak{q}}\sigma_1$.

To sum up, we have shown that for every $\sigma \in \Gamma$ with $0 < \Delta(\mathfrak{q}\sigma + \mathfrak{q}') < C$, every left-coset $\sigma\Gamma_{\mathfrak{q}'}$ contains a representative so that B_σ is bounded in some uniform compact set K, and that these representatives belong to at most finitely many distinct right-cosets $\Gamma_{\mathfrak{q}}\sigma$. This means there are at most finitely many distinct double-cosets $\Gamma_{\mathfrak{q}}\sigma\Gamma_{\mathfrak{q}'}$. □

Lemma 5.2.4. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics. For any $C > 0$, there are at most finitely many nondegenerate distinct gluings $\phi \in \Phi(\Lambda, \mathcal{J})$ up to fiber-shearings, such that $\mathscr{D}_e(\phi) < C$ for every edge $e \in \mathrm{Edg}(\Lambda)$.*

Proof. Let $\phi \in \Phi(\Lambda, \mathcal{J})$ be a nondegenerate gluing satisfying the conclusion. For any end-of-edge $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, $\phi_\delta : T_\delta \to T_{\bar{\delta}}$ induces the quadratic form $\mathfrak{q}_\phi| = \mathfrak{q}_{J'}\phi_\delta + \mathfrak{q}_J$ on $H_1(T_\delta; \mathbf{R})$, where J, J' are the pieces containing T_δ, $T_{\bar{\delta}}$, respectively. Pick a reference gluing $\psi_\delta : T_\delta \to T_{\bar{\delta}}$, then $\phi_\delta = \psi_\delta\sigma$ for some $\sigma \in \mathrm{Mod}(T_\delta)$. Write $\mathfrak{q} = \mathfrak{q}_J$, and $\mathfrak{q}' = \mathfrak{q}_{J'}\psi_\delta$, and $\Gamma = \mathrm{Mod}(T_\delta)$, then $\mathfrak{q}_\phi$ on $H_1(T_\delta; \mathbf{R})$ equals $\mathfrak{q}\sigma + \mathfrak{q}'$ for some $\sigma \in \Gamma$. Clearly the stabilizer $\Gamma_{\mathfrak{q}}$ of $\mathfrak{q}$ in Γ is nontrivial only if J is Seifert-fibered, in which case $\Gamma_{\mathfrak{q}}$ is generated by a Dehn-twist along an ordinary-fiber on T_δ; and the stabilizer $\Gamma_{\mathfrak{q}'}$ is nontrivial only if J' is Seifert-fibered, in which case $\Gamma_{\mathfrak{q}'}$ is generated by a Dehn-twist along an ordinary-fiber on $T_{\bar{\delta}}$ pulled back on T_δ

via ψ_δ. By the assumption and the definition of edge distortion, $\Delta(H_1(T_\delta;\mathbf{Z}),\mathfrak{q}\sigma+\mathfrak{q}') < C$. Moreover, $\Delta(H_1(T_\delta;\mathbf{Z}),\mathfrak{q}\sigma+\mathfrak{q}') > 0$ because ϕ is nondegenerate. Thus Lemma 5.2.3 implies that there are at most finitely many allowable types of ϕ_δ up to fiber-shearings. As $\phi : \partial\mathcal{J} \to \partial\mathcal{J}$ is defined by all the ϕ_δ's where $\delta \in \widetilde{\mathrm{Edg}}(\Lambda)$, we conclude there are at most finitely many nondegenerate gluings ϕ up to fiber-shearings, which have edge distortions all bounded by C. □

Next, we show that distortion at Seifert-fibered vertices bounds nondegenerate fiber-shearings of a given gluing up to equivalence.

Lemma 5.2.5. *Let $(\Lambda,\mathcal{J})$ be a preglue graph-of-geometrics, and $\phi \in \Phi(\Lambda,\mathcal{J})$ be a nondegenerate gluing. Suppose $v \in \mathrm{Ver}(\Lambda)$ is a Seifert-fibered vertex. Then for any $C > 0$, there exists some $K > 0$, depending on C and ϕ, such that whenever ϕ^τ is a fiber-shearing of ϕ with $\mathscr{D}_v(\phi^\tau) < C$, the fiber-shearing index $k_v(\tau)$ satisfies $|k_v(\tau)| < K$.*

Proof. There are two cases according to v being entire or semi.

Case 1. v is an entire-vertex, i.e. J_v has an orientable base orbifold.

In this case, we pick consistent directions for all the fibers of J_v, and for any end-of-edge δ adjacent to v, let λ_δ be the directed slope on $T_\delta \subset \partial J_v$. Suppose the valence of v is $n_v > 0$. It is not hard to see that $\partial_* H_2(J_v,\partial J_v;\mathbf{Z}) < H_1(\partial J_v;\mathbf{R})$ has a rank-(n_v-1) submodule:

$$L_v = \left\{ \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} l_\delta\,[\lambda_\delta] \,\Big|\, \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} l_\delta = 0,\ \text{where } l_\delta \in \mathbf{Z} \right\},$$

and that there is an element:

$$[\mu_v] = \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} [\mu_\delta] \in \partial_* H_2(J_v,\partial J_v;\mathbf{Z}),$$

such that for each $\delta \in \widetilde{\mathrm{Edg}}(v)$, $[\mu_\delta] \in H_1(T_\delta,\mathbf{Z})$ and the intersection number $\langle \mu_\delta,\lambda_\delta\rangle = m_v$ where $m_v > 0$ is the least common multiple of the orders of cone-points on the base orbifold. Moreover,

$$\partial_* H_2(J_v,\partial J_v;\mathbf{Z}) = L_v \oplus \mathbf{Z}\cdot[\mu_v].$$

For simplicity, we write $\mathfrak{q},\mathfrak{q}^\tau$ for $\mathfrak{q}_\phi,\mathfrak{q}_{\phi^\tau}$. Note that $\mathfrak{q}^\tau = \mathfrak{q}$ restricted to L_v.

We estimate the value of $\mathfrak{q}^\tau$ over the coset $[\mu_v] + L_v\otimes\mathbf{R}$ of $\partial_* H_2(J_v,J_{\partial v};\mathbf{R})$. For any $[\xi] = \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} l_\delta\,[\lambda_\delta] \in L_v\otimes\mathbf{R}$,

$$\mathfrak{q}^\tau([\mu_v]+[\xi]) = \sum_{\delta\in\widetilde{\mathrm{Edg}}(v)} \mathfrak{q}\left([\mu_\delta] + (l_\delta + m_v k_\delta)\,[\lambda_\delta]\right),$$

where $m_v > 0$ is as above, and k_δ is the Dehn-twist number on T_δ as in the definition of fiber-shearings. We have:

$$\sum_{\delta \in \widetilde{\mathrm{Edg}}(v)} (l_\delta + mk_\delta) = m_v k_v(\tau),$$

so if $|k_v(\tau)| \geq K$, there must be one end $\delta^* \in \widetilde{\mathrm{Edg}}(v)$, such that $|l_{\delta^*} + m_v k_{\delta^*}| \geq K/n_v$. Thus:

$$\begin{aligned} \mathfrak{q}^\tau([\mu_v] + [\xi]) &\geq \mathfrak{q}([\mu_{\delta^*}] + (l_{\delta^*} + m_v k_{\delta^*})[\lambda_{\delta^*}]) \\ &\geq \frac{1}{2}\mathfrak{q}((l_{\delta^*} + m_v k_{\delta^*})[\lambda_{\delta^*}]) - \mathfrak{q}([\mu_{\delta^*}]) \\ &\geq \frac{K^2 r_v}{2n_v} - R_v, \end{aligned}$$

where $r_v = \min_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathfrak{q}([\lambda_\delta])$ and $R_v = \max_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathfrak{q}([\mu_\delta])$ are constants depending only on J_v and ϕ. Note $r_v > 0$ because ϕ is nondegenerate.

Now we have:

$$\mathscr{D}_v(\phi^\tau) = \left(\Delta(L_v, \mathfrak{q}) \cdot \inf_{[\xi] \in L_v \otimes \mathbf{R}} \{\mathfrak{q}([\mu_v] + [\xi])\}\right)^{\frac{1}{2n_v}} \geq \left(\Delta_{L_v} \cdot (\frac{K^2 r_v}{2n_v} - R_v)\right)^{\frac{1}{2n_v}},$$

where $\Delta_{L_v} = \Delta(L_v, \mathfrak{q}) > 0$ because ϕ is nondegenerate. In other words, if $\mathscr{D}_v(\phi^\tau) < C$, we obtain an upperbound $K > 0$ so that the absolute value of the fiber-shearing index $k_v(\tau)$ is bounded by K.

Case 2. v is a semi-vertex, i.e. J_v has a non-orientable base orbifold.

In this case, let $\tilde{J}_v$ be the double covering of J_v corresponding to the centralizer of ordinary-fiber as in the definition of the vertex distortion. Then $\partial\tilde{J}_v$ is a trivial double covering of ∂J_v, and every fiber-shearing $\tau \in \mathrm{Mod}(\partial J_v)$ at v of index $k_v(\tau)$ lifts to a unique $\tilde{\tau} \in \mathrm{Mod}(\partial\tilde{J}_v)$ of index $2k_v(\tau)$. As now $\tilde{J}_v$ is Seifert-fibered over an orientable base orbifold, we reduce to the previous case, bounding the absolute value of $2k_v(\tau)$ by some K depending on C and ϕ. □

Now we are ready to prove Proposition 5.2.1.

Proof of Proposition 5.2.1. By Lemma 5.2.4, there are at most finitely many allowable types of gluings up to fiber-shearings. By Lemma 5.2.5, for each allowable fiber-shearing family $\{\phi^\tau\}$ as $\tau \in \mathrm{Mod}(\partial\mathcal{J})$ runs over all fiber-shearings where ϕ is a reference nondegenerate gluing, there are at most finitely many allowable indices of τ at any Seifert-fibered vertex. Hence by Lemma 2.2.7, there are at most finitely many distinct nondegenerate gluings up to equivalence with bounded distortions at all vertices and along all edges. □

The reader may have noticed that distortion at atoroidal (i.e. $\mathbb{H}^3$-geometric) vertices are not used in the proof of Proposition 5.2.1. We close this section with the following lemma, which provides some reason behind.

Lemma 5.2.6. *Let $(\Lambda, \mathcal{J})$ be a preglue graph-of-geometrics, and $v \in \mathrm{Ver}(\Lambda)$ be a vertex of valence n_v, corresponding to an atoroidal piece $J_v \subset \mathcal{J}$. Then for any gluing $\phi \in \Phi(\Lambda, \mathcal{J})$,*

$$\mathscr{D}_v(\phi) \leq C \cdot \left(\prod_{\delta \in \widetilde{\mathrm{Edg}}(v)} \mathscr{D}_{e(\delta)}(\phi) \right)^{\frac{2}{n_v}},$$

where $\widetilde{\mathrm{Edg}}(v)$ denotes the ends-of-edges adjacent to v, and $e(\delta)$ denotes the edge containing the end-of-edge δ, and $C > 0$ is some constant depending only on the topology of J_v.

Proof. We simply rewrite J_v as J, and n_v as n. Write the submodule $\partial_* H_2(J, \partial J; \mathbf{Z})$ of $H_1(\partial J; \mathbf{Z})$ as W, and the subspace $\partial_* H_2(J, \partial J; \mathbf{R})$ of $H_1(\partial J; \mathbf{R})$ as $W_{\mathbf{R}}$. From the definition, we have $\mathfrak{q}_\phi \geq \mathfrak{q}_J$, both positive-definite on $H_1(\partial J; \mathbf{R})$, so the unit-ball B_ϕ of $\mathfrak{q}_\phi$ is contained the (compact) unit-ball B_J of $\mathfrak{q}_J$. It suffices to show for some $C_0 > 0$ independent of ϕ,

$$\Delta(W, \mathfrak{q}_\phi) \leq C_0 \cdot \Delta(H_1(\partial J; \mathbf{Z}), \mathfrak{q}_\phi).$$

Picking a basis of $H_1(\partial J; \mathbf{Z})$ as an orthonormal basis, we fix a reference inner product of $H_1(\partial J; \mathbf{R})$. Denote the induced $2n$-dimensional volume measure as μ_{2n}, and denote the induced n-dimensional volume measure on $W_{\mathbf{R}}$ and on $W_{\mathbf{R}}^\perp$ as μ_n. It suffices to show for some $C_1 > 0$ independent of ϕ,

$$\mu_{2n}(B_\phi) \leq C_1 \cdot \mu_n(W_{\mathbf{R}} \cap B_\phi).$$

Note that:

$$\mu_{2n}(B_\phi) = \frac{\omega_{2n}}{\omega_n^2} \cdot \mu_n(W_{\mathbf{R}} \cap B_\phi) \cdot \mu_n(\bar{B}_\phi),$$

where ω_m denotes the volume of an m-dimensional Euclidean unit-ball, and $\bar{B}_\phi$ is the image of the orthogonal projection of B_ϕ to $W^\perp$. Therefore, the last inequality follows immediately because:

$$\mu_n(\bar{B}_\phi) \leq \mu_n(\bar{B}_J),$$

where $\bar{B}_J$ is the image of the orthogonal projection of B_J to $W^\perp$. The right-hand side is finite, independent of ϕ. □

5.3 Bounding distortions

In this section, we bound the distortions of a nondegenerate gluing under the assumption of domination, namely:

Proposition 5.3.1. *Suppose M is an orientable closed 3-manifold, and N_ϕ is an orientable closed irreducible 3-manifold obtained from a nondegenerate gluing $\phi \in \Phi(\Lambda, \mathcal{J})$ of a preglue graph-of-geometrics $(\Lambda, \mathcal{J})$. Then there exists some $C > 0$, such that if M dominates N_ϕ, then $\mathscr{D}_v(\phi) < C$ for every vertex $v \in \mathrm{Ver}(\Lambda)$, and $\mathscr{D}_e(\phi) < C$ for every edge $e \in \mathrm{Edg}(\Lambda)$.*

The rest of this section is devoted to the proof of Proposition 5.3.1. The idea is similar to the proofs of Propositions 4.1.1 and 4.2.1.

To start with, we reduce the proof to the case when the underlying graph Λ is *loopless* and *entire*, namely, such that it contains no loop edges, and that there is no semi-edges or semi-vertices:

Lemma 5.3.2. *If Proposition 5.3.1 holds under the assumption that Λ is loopless and entire, it holds in general as well.*

Proof. The idea is that Λ, as an 'orbi-graph', has a finite cover $\kappa : \tilde{\Lambda} \to \Lambda$ of index at most four which is loopless and entire. To be precise, suppose $f : M \to N_\phi$ is a nonzero degree map. We rewrite N_ϕ as N for simplicity. Take two copies X_0, X_1 of the compact 3-manifold obtained by cutting N along a maximal disjoint union of incompressible Klein-bottles, and glue each component of ∂X_0 to a unique component of ∂X_1 according to the gluing pattern of N. Then we obtain a double cover $\tilde{N}'$ of N, whose graph $\tilde{\Lambda}'$ is entire, (possibly disconnected if Λ is itself entire). Now cut $\tilde{N}'$ along the tori corresponding to the loop edges of $\tilde{\Lambda}'$, and glue two copies of the resulting compact 3-manifold up according to the gluing pattern of $\tilde{N}'$. Then we obtain a double cover $\tilde{N}''$ of $\tilde{N}'$, whose graph $\tilde{\Lambda}''$ is loopless and entire, (possibly disconnected if $\tilde{\Lambda}'$ is already loopless). Pick a connected component of $\tilde{N}''$, and rewrite as $\tilde{N}$. Thus $\tilde{N}$ covers N of index at most four, and has a loopless entire graph $\tilde{\Lambda}$. Indeed, $\tilde{N}$ may be regarded as the associated 3-manifold $N_{\tilde{\phi}}$ for a nondegenerate gluing $\tilde{\phi} \in \Phi(\tilde{\Lambda}, \tilde{\mathcal{J}})$. Moreover, it is clear from the definition that distortions are preserving passing to covers induced by the graph, namely, $\mathscr{D}_{\tilde{v}}(\tilde{\phi}) = \mathscr{D}_{\kappa(\tilde{v})}(\phi)$, and $\mathscr{D}_{\tilde{e}}(\tilde{\phi}) = \mathscr{D}_{\kappa(\tilde{e})}(\phi)$. However, since $\tilde{N}$ is dominated by a (connected) cover $\tilde{M}$ of M with index at most four, the distortions of $\tilde{\phi}$ are bounded by $c(\tilde{M})$, where $c(\tilde{M}) > 0$ is a constant guaranteed by the assumption. Note there are only finitely many such $\tilde{M}$'s, since $\pi_1(M)$ is finitely generated. Let $C > 0$ be the maximum among all possible $c(\tilde{M})$, as $\tilde{M}$ runs over all the coverings of M with index at most four. Thus the distortions of ϕ are bounded by C as well. □

Without loss of generality, we assume Λ is loopless and entire in the rest of this section. To simplify the notations, we rewrite N_ϕ as N in the rest of this subsection. Let:

$$f : M \to N,$$

be a domination as assumed. Let $\epsilon_3 > 0$ denote the Margulis constant of $\mathbb{H}^3$.

For any sufficiently small Margulis number ϵ with $0 < \epsilon < \epsilon_3$, by Lemma 3.1.1, we may straighten the map f via homotopy, with respect to a minimal triangulation of M and a Riemannian metric ρ_ϵ of N approximating its geometrization. We still write the straightened map as f. Remember that N has the decomposition:

$$N = \mathcal{J} \cup_{\partial \mathcal{U}} \mathcal{U},$$

where $\mathcal{U}$ are regular neighborhood of the cutting tori $\mathcal{T}$, and components of $\mathcal{J}$ are ϵ-thick or horizontally-ϵ-thick, dependint on whether they are $\mathbb{H}^3$-geometric or $\mathbb{H}^2 \times \mathbb{E}^1$-geometric, respectively.

For each edge $e \in \mathrm{Edg}(\Lambda)$, let:

$$\mathcal{W}_e \subset N,$$

be the union of $\mathcal{U}_e$ together with the compact ϵ_3-thin (or horizontal-ϵ_3-thin) horocusp neighborhoods of its adjacent pieces. Possibly after an arbitrarily small shrinking of $\mathcal{W}_e$, we may assume the union of $\mathcal{W}_e$'s is still a compact regular neighborhood of $\mathcal{T}$, properly containing $\mathcal{U}$ whenever $\epsilon < \epsilon_3$; and we may also assume that $f^{-1}(\partial\mathcal{W}_e)$ intersects $M^{(2)}$ in general positions. As Λ is a loopless graph, each $\mathcal{W}_e$ is local, and deformation-retracts to T_e. Thus there is a quadratic form on the subspace $\partial_* H_2(\mathcal{W}_e, \partial\mathcal{W}_e; \mathbf{R})$ of $H_1(\partial\mathcal{W}_e; \mathbf{R})$, naturally induced from $\mathfrak{q}_\phi$ on $H_1(T_\delta; \mathbf{R}) \oplus H_1(T_{\bar{\delta}}; \mathbf{R})$, where $\delta, \bar{\delta}$ are the two ends of e. Furthermore, for each vertex $v \in \mathrm{Ver}(\Lambda)$, let:

$$\mathcal{W}_v \subset N,$$

be the union of J_v together with all the $\mathcal{W}_e$'s where e runs over edges adjacent to v. As Λ is a loopless graph, each $\mathcal{W}_v$ is local, and deformation-retracts to J_v. Thus there is a quadratic form on the subspace $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ of $H_1(\partial\mathcal{W}_v; \mathbf{R})$, naturally induced from $\mathfrak{q}_\phi$ on $H_1(\partial J_v; \mathbf{R})$.

These $\mathcal{W}_e$'s and $\mathcal{W}_v$'s are natural geometric objects associated with the geometric decomposition of N. The following comparison plays the role of the meridianal area estimation in the proofs of Propositions 4.1.1 and 4.2.1:

Lemma 5.3.3. *For any vertex $v \in \mathrm{Ver}(\Lambda)$, if $j : (S, \partial S) \to (\mathcal{W}_v, \partial\mathcal{W}_v)$ is a properly piecewise-linearly immersed oriented compact surface, then:*

$$\mathrm{Area}(j(S)) \geq 4 \left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])},$$

where $j_*[\partial S] \in \partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$*. The same holds for any edge* $e \in \mathrm{Edg}(\Lambda)$ *in place of v above.*

Remark 5.3.4. An easy computation in hyperbolic geometry yields that $4\sinh(\frac{\epsilon}{2})$ is the Euclidean length of the shortest geodesic on the boundary of a hyperbolic horocusp whose injectivity radius is at most ϵ (realized at points on the boundary). Moreover, the right-hand side of the inequality may be replaced by $\left(1 - 4\sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])}$, if one takes mutually disjoint maximal horocusps instead of the Margulis horocusps in the definition of $\mathcal{W}_e$. This follows because the length of shortest geodesic on each component of ∂W_e in this case is at least 1, (cf. [Ada92]).

Proof. We only prove the vertex case, and the edge case is similar.

Let $v \in \mathrm{Ver}(\Lambda)$ be a vertex. Write $\mathrm{Edg}(v)$ for the edges adjacent to v, and $\widetilde{\mathrm{Edg}}(e)$ for the two ends of an edge e. As Λ is loopless, $e \in \mathrm{Edg}(v)$ has two ends $\delta, \bar{\delta}$, corresponding to the two components of $\mathcal{W}_e \setminus \mathring{\mathcal{U}}_e$ which we write as $\mathcal{W}_\delta, \mathcal{W}_{\bar{\delta}}$ respectively. Suppose $j_*[\partial S] = \sum_{e\in\mathrm{Edg}(v)} \alpha_e$, corresponding to the direct-sum decomposition:

$$H_1(\partial\mathcal{W}_v; \mathbf{R}) \cong \bigoplus_{e\in\mathrm{Edg}(v)} H_1(T_e; \mathbf{R}).$$

It follows from an easy calibration argument that the area (or the horizontal-area) of $j(S) \cap \mathcal{W}_\delta$ is at least $4\,(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})) \cdot \sqrt{\mathfrak{q}_{J_\delta}(\alpha_e)}$, for any $\delta \in \widetilde{\mathrm{Edg}}(e)$ and any $e \in \mathrm{Edg}(v)$, where $J_\delta \subset \mathcal{J}$ denotes the piece corresponding to the vertex that e is adjacent to on the end δ, for the definition of $\mathfrak{q}_{J_\delta}$). We have:

$$\begin{aligned}
\mathrm{Area}(j(S)) &\geq \sum_{e\in\mathrm{Edg}(v)} \sum_{\delta\in\widetilde{\mathrm{Edg}}(e)} 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\mathfrak{q}_{J_\delta}(\alpha_e)} \\
&\geq 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\sum_{e\in\mathrm{Edg}(v)} \sum_{\delta\in\widetilde{\mathrm{Edg}}(e)} \mathfrak{q}_{J_\delta}(\alpha_e)} \\
&= 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\sum_{e\in\mathrm{Edg}(v)} \mathfrak{q}_\phi(\alpha_e)} \\
&= 4\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right) \cdot \sqrt{\mathfrak{q}_\phi(j_*[\partial S])},
\end{aligned}$$

as desired. □

Lemma 5.3.5. *For any vertex* $v \in \mathrm{Ver}(\Lambda)$*, if* $\alpha_1, \cdots, \alpha_m$ *is a collection of elements in* $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$ *spanning* $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ *over* $\mathbf{R}$*, then for at least one* $1 \leq k \leq m$*,*

$$\sqrt{\mathfrak{q}_\phi(\alpha_k)} \geq \mathscr{D}_v(\phi).$$

The same holds for any edge $e \in \mathrm{Edg}(\Lambda)$ *in place of v above.*

Proof. This is a Minkowski-type estimation for lattices. Without loss of generality, we may assume that m is minimal, and hence equal to the valence of v. Consider the volume of the parallelogram spanned by the α_i's with respect to the inner product induced by $\mathfrak{q}_\phi$ on $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$, then clearly:

$$\prod_{i=1}^{n_v} \sqrt{\mathfrak{q}_\phi(\alpha_i)} \geq |\det(\alpha_1, \cdots, \alpha_{n_v})| \cdot \sqrt{\Delta(\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z}), \mathfrak{q}_\phi)},$$

where $\det(\alpha_1, \cdots, \alpha_{n_v})$ is the determinant regarding α_i's as column coordinate vectors over a basis of $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$, which is a nonzero integer and hence at least one in absolute value. Thus,

$$\prod_{i=1}^{n_v} \sqrt{\mathfrak{q}_\phi(\alpha_i)} \geq \sqrt{\Delta(\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z}), \mathfrak{q}_\phi)} = \mathscr{D}_v(\phi)^{n_v},$$

by the definition of vertex distortion. The lemma follows immediately from this estimation, and the edge case is similar. □

We are now ready to prove Proposition 5.3.1.

Proof of Proposition 5.3.1. By Lemma 5.3.2, we may assume Λ is loopless and entire. Rewrite N_ϕ as N Let:

$$f : M \to N,$$

be a domination as assumed. Let $\epsilon_3 > 0$ denote the Margulis constant of $\mathbb{H}^3$. For a sufficiently small Margulis number ϵ with $0 < \epsilon < \epsilon_3$, straighten the map f via homotopy with respect to $\epsilon > 0$. We only prove the vertex case, and the edge case is similar.

Let $v \in \mathrm{Ver}(\Lambda)$ be a vertex. Taking $\mathcal{W}_v$ as $\mathcal{W}$, by Lemma 3.2.1, there is an $\mathbf{R}$-spanning set $[S_1], \cdots, [S_m]$ of $H_2(M^{(2)}_{\mathcal{W}_v}, M^{(2)}_{\partial\mathcal{W}_v}; \mathbf{R})$ represented by relative $\mathbf{Z}$-cycles each with area bounded by $A(2\tau(M))$, where $A(n) = 27^n(9n^2+4n)\pi$, and where $\tau(M)$ is the triangulation number of M. From the construction, these relative $\mathbf{Z}$-cycles can be regarded as proper immersions of compact oriented surfaces: $j_i : (S_i, \partial S_i) \to (\mathcal{W}_v, \partial\mathcal{W}_v)$, where $1 \leq i \leq m$. By Lemma 5.3.3,

$$\begin{aligned} \sqrt{\mathfrak{q}_\phi(j_{i*}[\partial S_i])} &\leq \frac{1}{4}\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)^{-1} \cdot \mathrm{Area}(j_i(S_i)) \\ &\leq \frac{1}{4}\left(\sinh(\frac{\epsilon_3}{2}) - \sinh(\frac{\epsilon}{2})\right)^{-1} \cdot A(2\tau(M)), \end{aligned}$$

for all $1 \leq i \leq m$. Note that $j_{i*}[\partial S_i] = \partial_* j_{i*}[S_i] \in \partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{Z})$ for all $1 \leq i \leq m$. On the other hand, Lemma 3.2.2 implies that all the $j_{i*}[\partial S_i]$'s together span $\partial_* H_2(\mathcal{W}_v, \partial\mathcal{W}_v; \mathbf{R})$ over $\mathbf{R}$, as f is a domination. Thus, by Lemma 5.3.5,

$$\mathscr{D}_v(\phi) \leq \max_{1\leq i\leq m} \sqrt{\mathfrak{q}_\phi(j_{i*}[\partial S_i])} \leq \frac{1}{4}\left(\sinh\left(\frac{\epsilon_3}{2}\right) - \sinh\left(\frac{\epsilon}{2}\right)\right)^{-1} \cdot A(2\tau(M)).$$

As $\epsilon > 0$ can be arbitrarily small, we obtain:

$$\mathscr{D}_v(\phi) \leq \frac{A(2\tau(M))}{4\sinh\left(\frac{\epsilon_3}{2}\right)},$$

where the right-hand side depends only on M. In fact, one can show $\mathscr{D}_v(\phi) \leq A(2\tau(M))$ with the stronger estimation as mentioned in Remark 5.3.4. ◻

5.4 Finiteness of gluings

To summarize our discussions so far, we have the following finiteness of nondegenerate gluings as an immediate consequence of Propositions 5.2.1 and 5.3.1:

Theorem 5.4.1. *Suppose M is an orientable closed 3-manifold, and $(\Lambda, \mathcal{J})$ is a preglue graph-of-geometrics. Then there are at most finitely many equivalently distinct nondegenerate gluings $\phi \in \Phi(\Lambda, \mathcal{J})$ yielding a 3-manifold dominated by M.*

Finally, we obtain the following theorem, which is the main result of the present dissertation as mentioned in the introduction:

Theorem 5.4.2. *Every orientable closed 3-manifold dominates at most finitely many homeomorphically distinct non-geometric prime 3-manifolds.*

Proof. Note that every orientable closed non-geometric prime 3-manifold is obtained from a nondegenerate gluing induced from its geometric decomposition. Because equivalent gluings yield homeomorphic 3-manifolds, the theorem is an immediate consequence of Theorems 4.3.1 and 5.4.1. ◻

Chapter 6

Further results

In this chapter, we discuss consequences of Theorem 5.4.2.

6.1 Domination of bounded degree

In this section, we study finiteness for dominations of bounded degree.

Theorem 6.1.1. *For any integer $d > 0$, every orientable closed* 3*-manifold d-dominates only finitely many homeomorphically distinct* 3*-manifolds.*

The rest of this section is devoted to the proof of Theorem 6.1.1. We shall focus on the case when the target is Seifert fibered, and reduce to that case to prove Theorem 6.1.1.

We start by the following estimation of the size of torsion under dominations of bounded degree, directly generalizing a lemma previously obtained for the 1-dominations, cf. [HLWZ02, Lemma 3], [WZ02, Lemma 3 (1)].

Lemma 6.1.2. *For any integer $d > 0$, if M is an orientable closed* 3*-manifold d-dominating an orientable closed* 3*-manifold N, then:*

$$|\mathrm{Tor}\, H_1(N;\mathbf{Z})| \leq d \cdot |H_1(M;\mathbf{Z}_d)| \cdot |\mathrm{Tor}\, H_1(M;\mathbf{Z})|,$$

where Tor *denotes the submodule of torsion elements, and where $|\cdot|$ denotes the cardinality.*

Proof. This follows from an easy algebraic topology argument. Suppose $f : M \to N$ is a map of degree d (after approriately orientating M and N), then the umkehr homomorphism:

$$f_! : H_*(N;\mathbf{Z}) \to H_*(M;\mathbf{Z}),$$

is known as $f_!(\alpha) = [M] \frown f^*(\check{\alpha})$ for $\alpha \in H_*(N;\mathbf{Z})$, where $\check{\alpha} \in H^{3-*}(N;\mathbf{Z})$ denotes the Poincaré dual of α. It is straightforward to check $f_* \circ f_! : H_*(N;\mathbf{Z}) \to H_*(N;\mathbf{Z})$ is the scalar multiplication by d. In particular, $d \cdot \mathrm{Tor}\, H_1(N;\mathbf{Z})$ is surjected by $f_!(\mathrm{Tor}\, H_1(N;\mathbf{Z})) \leq \mathrm{Tor}\, H_1(M;\mathbf{Z})$. On the other hand, from the long exact sequence:

$$\cdots \longrightarrow H_1(N;\mathbf{Z}) \xrightarrow{d} H_1(N;\mathbf{Z}) \longrightarrow H_1(N;\mathbf{Z}_d) \longrightarrow 0,$$

we have $\mathrm{Tor}\, H_1(N;\mathbf{Z}) \,/\, d \cdot \mathrm{Tor}\, H_1(N;\mathbf{Z}) \leq H_1(N;\mathbf{Z}_d)$. Note as $f : M \to N$ has degree d, the image of $H_1(M;\mathbf{Z}_d)$ in $H_1(N;\mathbf{Z}_d)$ has index at most d. This gives our inequality as desired. □

The Seifert-fibered case was previously obtained when d equals one, due to Claude Hayat-Legrand, Shicheng Wang, Heiner Zieschang [HLWZ02] for the $\mathbb{S}^3$-geometric case, and later due to Shicheng Wang, Qing Zhou [WZ02] for the general cases. Their techniques actually work in general when d is greater than one, using the updated version of torsion size estimation above:

Lemma 6.1.3. *For any integer $d > 0$, any orientable closed 3-manifold d-dominates at most finitely many Seifert fibered spaces.*

Proof. We give a brief outline of the proof, cf. [HLWZ02, WZ02].

Suppose M is an orientable closed 3-manifold d-dominating a Seifert fibered space N. We may focus on the case when the Euler class of N is nonvanishing. In fact, this happens exactly when N supports one of the geometries $\mathbb{S}^3$, Nil or $\widetilde{\mathrm{SL}}_2$, according to the sign of χ. Other cases have already been covered by the finiteness result of [BRW, Theorem 1.1] concerning JSJ pieces using a Kneser–Haken finiteness argument. Furthermore, we may assume without loss of generality that N has an orientable base orbifold, because every Seifert fibered space has a finite cover of index at most two with this property, and because M has only finitely many homeomorphically distinct index two covers.

We shall denote an orientable closed Seifert fibered 3-manifold as:

$$N = \Sigma(g; b_0, \frac{b_1}{a_1}, \cdots, \frac{b_s}{a_s}),$$

normalized so that $s, g \geq 0$ and b_0 are integers, and that $0 < b_i < a_i$ are coprime integers for $1 \leq i \leq s$. When the base orbifold is orientable, it can be denoted as $F_g(a_1, \cdots, a_s)$, which means the orientable closed surface of genus g with cone-points of order a_i's. The base orbifold has the Euler characteristic:

$$\chi = 2 - 2g - \sum_{i=1}^{s}(1 - \frac{1}{a_i}),$$

and the Seifert fibration has the Euler class (as a rational number):

$$e = -b_0 - \sum_{i=1}^{s} \frac{b_i}{a_i}.$$

When e is not vanishing, the torsion size in its first homology is:

$$|\mathrm{Tor}\, H_1(N;\, \mathbf{Z})| = |e| \cdot \prod_{i=1}^{s} a_i.$$

As we have assumed that N fibers over an orientable base orbifold with nonvanishing Euler class, the rest of the proof falls into three cases according to the sign of χ.

When $\chi > 0$, N supports the $\mathbb{S}^3$-geometry. It is an easy exercise to check that such a manifold is covered by a lens space of index at most 60. In fact, we have $g = 0$ and $s \leq 3$. For $0 \leq s \leq 2$, N is a lens space (possibly the 3-sphere). For $s = 3$, N is N is either a prism 3-manifold $\Sigma(0; b_0, \frac{1}{2}, \frac{1}{2}, \frac{b_3}{a_3})$, or of one of the types $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{3})$, $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{4})$, or $\Sigma(g; b_0, \frac{1}{2}, \frac{b_2}{3}, \frac{b_3}{5})$. In each of these cases, there is a cover of the base orbifold of order at most 60 with at most two cone points. Thus N is covered by a lens space of index at most 60. Applying the covering trick as above, we may assume N is indeed a lens space, without loss of generality. As a lens space has cyclic fundamental group, its order can be bounded by Lemma 6.1.2, so there are only finitely many allowable N's up to homeomorphism.

When $\chi = 0$, N supports the Nil-geometry, and there are only finitely many allowable values of s, g and a_i's by the formula of χ. For each possibility, there are only finitely allowable values of b_i's because $0 < b_i < a_i$ for $1 \leq i \leq s$, and because b_0 can be bounded by the torsion-size comparison. Thus there are at most finitely many homeomorphically distinct N's with $\chi = 0$.

When $\chi < 0$, N supports the $\widetilde{\mathrm{SL}}_2$-geometry. By Proposition 4.2.1 and the Kneser–Haken finiteness, there are only a finite number of allowable isomorphism types of the base orbifold of N, which does not depend on d. Thus by the covering trick, we may assume N fibers over a closed orientable surface. In this case, it is straightforward to check that the Euler number of the fiber is an integer whose absolute value is bounded by the torsion-size of $H_1(N)$, and hence is bounded in terms of M and d. This yields the finiteness of allowable homeomorphism types of N, so we have completed the proof of Lemma 6.1.3. □

Proof of Theorem 6.1.1. One may first reduce to the case when the target is irreducible, because any orientable closed 3-manifold 1-dominates any of its connected-sum components in the Kneser–Milnor decomposition, and because the number of connect-sum components in the target is bounded in terms of the Kneser–Haken number of the source. Moreover, Theorem 5.4.2 reduces the statement to the case of geometric targets. The $\mathbb{H}^3$-geometric target case was proved by Teruhiko Soma [Som00, Theorem 1], (cf. Remark 4.3.2). The Sol-geometric target case was proved by Michel Boileau, Steve Boyer, and Shicheng Wang [BBW08] using torsion polynomials. We also remark that provided the upper bound of mapping degree d, there is an easy and direct argument bounding the number of allowable monodromies, using the torsion-size estimation (Lemma 6.1.2). In the cases of the rest six geometries, the target is a Seifert fibered space, so we may apply Lemma 6.1.3. This completes the proof of Theorem 6.1.1 □

6.2 Domination relative to boundary

In this section, we consider extension of Theorems 5.4.2 and 6.1.1 to orientable compact 3-manifolds with boundary. For two orientable compact 3-manifolds M and N, and an integer $d > 0$, N is said to be *d-dominated* by M relative to boundary if there exists a proper map $f : (M, \partial M) \to (N, \partial N)$ of degree d up to sign, namely, that the induced homomorphism $f_* : H_1(M, \partial M; \mathbf{Z}) \to H_1(N, \partial N; \mathbf{Z})$ can be identified as the scalar multiplication by $d : \mathbf{Z} \to \mathbf{Z}$. We say *$M$ dominates N* relative to boundary if M d-dominates N for some integer $d > 0$. Note that under domination, N has nonempty boundary if and only if M has nonempty boundary.

Theorem 6.2.1. *Every orientable compact* 3*-manifold with nonempty boundary dominates only finitely many irreducible, and ∂-irreducible* 3*-manifolds relative to boundary, up to homeomorphism.*

Theorem 6.2.2. *For any integer $d > 0$, every orientable compact* 3*-manifold d-dominates only finitely many* 3*-manifolds relative to boundary, up to homeomorphism.*

The rest of this section is devoted to the proof of Theorems 6.2.1 and 6.2.2. The idea is to perform a 'doubling trick' to reduce to the closed case, cf. [BRW, Remark 4.7]. Recall that for an oriented compact 3-manifold M with nonempty boundary, the *double* of M along boundary, denoted as $\mathrm{Dbl}(M)$, is the oriented closed 3-manifold $M \cup_{\partial M} (-M)$, obtained from gluing M to its orientation-reversal via the natural identification on the boundary. The double of an orientable compact 3-manifold is the double of the 3-manifold picking an orientation, so it is canonical up to homeomorphism.

Lemma 6.2.3. *If Q is an irreducible, ∂-irreducible, orientable compact* 3*-manifold with nonempty boundary, then* $\mathrm{Dbl}(Q)$ *is either non-geometric, or supports one of the geometries $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{E}^1$, or $\mathbb{E}^3$, unless Q is homeomorphic to a ball.*

Proof. As Q is irreducible and ∂-irreducible, $\mathrm{Dbl}(Q)$ is irreducible as well. Suppose $\mathrm{Dbl}(Q)$ is geometric. If $\mathrm{Dbl}(Q)$ is $\mathbb{S}^3$-geometric, ∂Q is necessarily a sphere bounding a ball in $\mathrm{Dbl}(Q)$, so Q has to be a ball and $\mathrm{Dbl}(Q)$ is a sphere. The irreducibility of $\mathrm{Dbl}(Q)$ excludes the possibility of the geometry $\mathbb{S}^2 \times \mathbb{E}^1$. For the geometries Nil, Sol, or $\widetilde{\mathrm{SL}}_2$, the double $\mathrm{Dbl}(Q)$ could not support them since two-sided incompressible subsurfaces in these cases would be tori. In fact, cutting along tori in these cases results in Seifert fibered spaces with boundary, but if those were Q, the double $\mathrm{Dbl}(Q)$ would be either $\mathbb{H}^2 \times \mathbb{E}^1$-geometric or $\mathbb{E}^3$-geometric. ◻

In the statement of the following lemma, a *reflection* of a 3-manifold is known as an orientation-reversing self-homeomorphism whose square equals the identity.

Lemma 6.2.4. *If Q is an irreducible, ∂-irreducible, orientable compact* 3*-manifold with nonempty boundary, then there are at most finitely many (possibly disconnected) incompressible subsurfaces of* $\mathrm{Dbl}(Q)$ *up to homeomorphism of* $\mathrm{Dbl}(Q)$*, which could be the fixed point set of a reflection of* $\mathrm{Dbl}(Q)$.

Proof. If Q is not a ball, $\mathrm{Dbl}(Q)$ is Haken by the assumption. The conclusion follows from the fact that $\mathrm{Out}(\pi_1(\mathrm{Dbl}(Q)))$ has only finitely many finite subgroups up to conjugacy [Zim86, Theorem 4.1], and uniqueness of the homeomorphism realization for involutions up to homeomorphisms, [Tol81, Corollary 1]. ◻

Proof of Theorem 6.2.1. Let M be an orientable compact 3-manifold with nonempty boundary. Suppose N is an irreducible, ∂-irreducible, orientable compact 3-manifold dominated by M relative to boundary. Then there is a naturally induced domination of the same degree between the double of M and the double of N. Applying Theorem 5.4.2 for the non-geometric case and [BRW, Theorem 1.1] for the geometric cases, it follows from Lemma 6.2.3 that there are only finitely many allowable homeomorphism types of $\mathrm{Dbl}(N)$, under the assumption of domination. Furthermore, Lemma 6.2.4 implies that there are hence only finitely many allowable homeomorphism types of N. ◻

Proof of Theorem 6.2.2. Let M be an orientable compact 3-manifolds. Suppose $f : M \to N$ is a d-domination relative to boundary of M onto an orientable compact 3-manifold N. Cut N along a maximal union $\mathcal{S}$ of mutually non-parallel disjoint essential spheres and ∂-essential disks into a disjoint union of compact components $\mathcal{Q}$. The number of components of $\mathcal{S}$ can be bounded in terms of M by a Kneser–Haken type argument, so there is an upper bound on the number of components of $\mathcal{Q}$.

For each component $Q \subset \mathcal{Q}$, let $\hat{Q}$ be the manifold obtained by filling up sphere boundary-components with balls. Note that $\hat{Q}$ is irreducible and ∂-irreducible. Thus exactly one of the following three possibilities holds: either that $\hat{Q}$ is homeomorphic to S^3, so Q is a 3-sphere with finitely many disjoint balls removed; or that $\hat{Q}$ is closed with nontrivial fundamental group, so two copies of $\hat{Q}$ are connected-sum components of the Kneser–Milnor decomposition of $\mathrm{Dbl}(N)$; or that $\hat{Q}$ has nonempty boundary, so $\mathrm{Dbl}(\hat{Q})$ is a connected-sum component of the Kneser–Milnor decomposition of $\mathrm{Dbl}(N)$. Moreover, $\mathrm{Dbl}(N)$ is homeomorphic to the connected sum of the $\hat{Q}$ or $\mathrm{Dbl}(\hat{Q})$ in the latter two cases, respectively, together with possibly a finite number of $S^1 \times S^2$.

As $\mathrm{Dbl}(N)$ 1-dominates each of its connected-sum components via pinching, $\mathrm{Dbl}(M)$ d-dominates each $\hat{Q}$ or $\mathrm{Dbl}(\hat{Q})$ as above. By Theorem 6.1.1, there are at most finitely many allowable homeomorphism types of $\hat{Q}$ or $\mathrm{Dbl}(\hat{Q})$. Furthermore, if $\hat{Q}$ has nonempty boundary, the finiteness of allowable homeomorphism types of $\mathrm{Dbl}(\hat{Q})$ and Lemma 6.2.4 implies the finiteness of allowable homeomorphism types of $\hat{Q}$ as well. Thus the homeomorphism types of $\hat{Q}$, and hence of Q, are always bounded in terms of M and d.

Since the number of components of Q and the number of allowable homeomoprhism types of components of Q are both bounded in terms of M and d, there are only finitely many allowable homeomorphism types of N that are d-dominated by M relative to boundary, as Q was obtained by cutting N along spheres and disks. □

6.3 Integral homology sphere and mapping degree

It is in general difficult to characterize the the degree set of maps between 3-manifolds. Nevertheless, we attempt to offer a more comprehensive answer to Problem 1.1.1 for dominations onto integral homology 3-spheres in this section. It will be natural to work in the oriented category, where the degree of a map becomes a signed integer.

Suppose M is an oriented closed 3-manifold. For any oriented closed 3-manifold N, denote the *degree set* of maps between M and N as:

$$\mathrm{D}_M(N) = \{\deg(f) \mid f : M \to N,\ \deg(f) \neq 0\}.$$

By an integral homology 3-sphere we mean a closed 3-manifold whose homology coincides with that of S^3. In particular, it is orientable. We denote the orientation-preserving homeomoprhism classes of oriented integral homology 3-spheres as $\mathbf{Z}HS^3$, often abusing the notation of elements of $\mathbf{Z}HS^3$ and that of their representatives.

By a *periodic* subset of $\mathbf{Z}$, we mean a subset invariant under the translation by some positive integer.

Theorem 6.3.1. *Every oriented closed* 3*-manifold M dominates only finitely many $N \in \mathbf{Z}\mathrm{H}S^3$, and the mapping degree set $\mathrm{D}_M(N)$ is either finite, or an infinite, periodic subset of $\mathbf{Z}$ with zero removed. The latter case happens if and only if $N \cong \Pi^{\#m} \# \overline{\Pi}^{\#n}$, where $m, n \geq 0$ are integers, and where Π, $\overline{\Pi}$ denote the oriented Poincaré dodecahedral space and its orientation-reversal, respectively up to a choice of orientation.*

We prove Theorem 6.3.1 in the rest of this section. The finiteness of $\mathbf{Z}\mathrm{H}S^3$ targets was previously obtained by [BRW, Theorem 1.2], but with Theorem 5.4.2 we have a quick argument:

Lemma 6.3.2. *If M is an oriented closed* 3*-manifold, then M dominates only finitely many $N \in \mathbf{Z}\mathrm{H}S^3$.*

Proof. This follows from the same argument as that of Theorem 6.1.1, once we make the following two modifications in the proof of 6.1.3. Firstly, every Seifert fibered integral homology 3-sphere either supports the $\widetilde{\mathrm{SL}}_2$ geometry, fibering over an orientable base orbifold, or supports the $\mathbb{S}^3$ geometry, homeomorphic to either S^3 or the Poincaré dodecahedral space. Thus we only need the $\widetilde{\mathrm{SL}}_2$ fibered case in the argument of Lemma 6.1.3, and we do not need to pass to a finite cover to ensure the orientable base. Secondly, we replace the torsion size estimation by the trivial fact that $\mathrm{Tor}\, H_1(N) = 0$, instead of invoking Lemma 6.1.2. □

By a π_1-*surjective* map, we mean a map $f : M \to N$ which induces an epimorphism between the fundamental groups. The following lemma was implicitly proved in [SWWZ12, Section 3] for π_1-isomorphic self-maps of orientable closed 3-manifolds with only spherical prime factors, and earlier in [HLKWZ01] for the prime indecomposable case. We give a different argument in slightly more general context.

Lemma 6.3.3. *Let L be an oriented closed* 3*-manifold with $\pi_1(L)$ a free product of finite groups. Then there exists a positive integer l such that the following holds. For any π_1-surjective map $f : M \to L$ from an oriented closed* 3*-manifold onto L of degree $d \in \mathbf{Z}$, there exists a map $f_n : M \to L$ of degree $(d + nl)$, for every integer n. Moreover, f_n induces the same epimorphism between the fundamental groups as that of f.*

Proof. Because $\pi_1(L)$ is a free product of finite groups, it is virtually torsion free. Let $\tilde{L}$ be a regular finite cover of L corresponding to a torsion-free subgroup of $\pi_1(L)$ of finite index l.

We first claim that any π_1-surjective map $f' : M \to L$ of degree d can be modified to be a map $f' : M \to L$ of degree $d - l$, inducing the same epimorphism between the fundamental groups. Suppose $f : M \to L$ is such a map. Let $\kappa : \tilde{L} \to L$ be the regular covering above. Using the classification of 3-manifold groups ensured by the Geometrization, $\tilde{L}$ is the connected sum of finitely many copies of $S^2 \times S^1$. In other words, it is homeomorphically the double of a handlebody, namely, $\tilde{L} \cong Y \cup Y'$ where Y and Y' are two oppositely oriented copies of a compact handlebody, glued up along the boundary $\partial Y = \partial Y'$. We denote the natural orientation-reversing reflection swapping Y and Y' as $\sigma : \tilde{L} \to \tilde{L}$. Possibly after an isotopy of $\tilde{L}$, we may assume $\kappa(Y)$ to be embedded in L. As $f : M \to L$ is π_1-surjective, possibly after an isotopy of M, we may find an embedded handlebody $X \subset M$ which maps homeomorphically onto $\kappa(Y)$. Moreover, restricted on X there is a lift $\tilde{f} : X \to \tilde{L}$ such that $\kappa \circ \tilde{f} = f$. In particular, X is embedded as Y via $\tilde{f}$. Now modify the map $\tilde{f}$ to be the reflected embedding $\tilde{f}' : X \to \tilde{L}$, namely, such that $\tilde{f}' = \sigma \circ \tilde{f}$. This induces a modified map $f' : M \to L$, such that $f'|_{M \setminus X} = f$, and that $f'|_X = \kappa \circ \tilde{f}'$. It is clear from the construction that f' satisfies our claim.

Similarly, we may obtain another π_1-surjective map $f'' : M \to L$ of degree $d+l$, inducing the same epimorphism between the fundamental groups as that of f. With the notations above, suppose $\tilde{f}''| : \partial X \times [0,1] \to Y$ is a map of zero degree relative to boundary homotoping $\tilde{f}|_{\partial X}$ to an orientation reversing homeomorphism. For example, if $c_1, \cdots, c_m$ is a collection of disjoint essential simple closed curves on ∂X, bounding disjoint embedded disks in Y under $\tilde{f}$, we may take an orientation reversing self-homeomorphism $\tau : \partial X \to \partial X$ that sends each c_i onto itself reversing orientation. Then $\tilde{f}''|$ may be picked so that $\tilde{f}''|_{\partial X \times \{0\}} = \tilde{f}$ while $\tilde{f}''|_{\partial X \times \{1\}} = \tilde{f} \circ \tau$. Regarding $\partial X \times [0,1]$ as a collar neighborhood of ∂X in X, with $\partial X \times \{0\}$ as ∂X, we may extend $\tilde{f}''|$ as a map $\tilde{f}'' : X \to \tilde{L}$ which sends $\partial X \times [0,1]$ onto Y via $\tilde{f}''|$ and which sends the rest homeomorphically onto Y'. Now modify f to be $f'' : M \to L$ so that $f''|_{M \setminus X} = f$, and that $f''|_X = \kappa \circ \tilde{f}''$. We see the degree of f'' equals $d + l$.

Iterating these two constructions we see that for each $n \in \mathbf{Z}$, there is a map $f_n : M \to L$ of degree $(d+nl)$, inducing the same epimorphism between the fundamental groups as that of f. □

A π_1-surjective map of nonzero degree is sometimes called an *essential domination*. Every domination factors canonically as an essential domination followed by a finite covering up to homotopy. We denote the subset of $\mathrm{D}_M(N)$ consisting of degrees of essential dominations as $\mathrm{ED}_M(N)$, and denote the subset of $\mathrm{D}_M(N)$ of degrees covering maps as $\mathrm{CD}_M(N)$. Both of them could be empty. For a triple of oriented closed 3-manifolds M, L, N, it is clear that the elementwise product:

$$\mathrm{CD}_L(N) \cdot \mathrm{ED}_M(L) \subset \mathrm{D}_M(N),$$

consists of degrees of dominations factoring via L.

Proof of Theorem 6.3.1. By Lemma 6.3.1, M dominates only finitely many integral homology 3-spheres. Suppose $N \in \mathbf{Z}\mathrm{H}S^3$ is dominated by M. The argument of Lemma 6.3.1 made clear that every Seifert fibered prime factor of N is either $\widetilde{\mathrm{SL}}_2$-geometric, or the Poincaré dodecahedral space, unless N is itself S^3. Note also that an orientable Sol-geometric 3-manifold has nonvanishing $\mathbf{Z}_2$-homology, and hence cannot be a prime factor of $N \in \mathbf{Z}\mathrm{H}S^3$.

If a prime factor Q of N contains at least one $\mathbb{H}^3$-geometric piece, the simplicial volume of Q is nonvanishing by the result of Soma [Som81]; if Q is a nontrivial graph manifold or is $\widetilde{\mathrm{SL}}_2$-geometric, the Seifert volume is nonvanishing for some finite cover of Q by Brooks–Goldman [BG84] and Derbez–Wang [DW09b]. In both the cases above, $D_M(N)$ is a finite set.

We are left with the case when every prime factor of N is $\mathbb{S}^3$-geometric, hence homeomorphic to the Poincaré dodecahedral space unless N is itself S^3. In this case, there are only finitely many $L_1, \cdots, L_s$ covering N, up to orientation-preserving homeomorphism, which are essentially dominated by M. In fact, if N is itself prime, it has finitely many distinct covers up to orientation-preserving homeomorphism; if N is not prime, every essential sphere lifts homeomorphically to be essential spheres in the cover, so the index of the cover, and hence the number of covers essentially dominated by M, can be bounded by the Kneser–Haken number of M. Therefore,

$$\mathrm{D}_M(N) = \bigcup_{i=1}^{s} \mathrm{CD}_{L_i}(N) \cdot \mathrm{ED}_M(L_i).$$

The size of each $\mathrm{CD}_{L_i}(N)$ is bounded by either the size of $\pi_1(N)$ or the Kneser–Haken number of L_i, so it is finite. By Lemma 6.3.3, $\mathrm{ED}_M(L_i)$ is an infinite, periodic subset of $\mathbf{Z}$ with zero removed. Because any union of finitely many periodic sets is still periodic, we see in this case that $\mathrm{D}_M(N)$ is an infinite, periodic subset of $\mathbf{Z}$ with zero removed. Indeed, the argument of Lemma 6.3.3 implies that the period can be taken as 120, the order of the fundamental group of the Poincaré dodecahedral space. ◻

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