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On the Compatibility of Two Conjectures Concerning p-adic Gross-Stark Units

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ON THE COMPATIBILITY OF TWO CONJECTURES CONCERNING P-ADIC GROSS-STARK UNITS

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Shawn Tsosie

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Abstract

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by

Shawn Tsosie

We give a proof of the consistency of Dasgupta's conjectural p-adic formula for Gross-Stark units with Dasgupta and Spiess's alternative conjectural formula for these units. We give details of the proof when F is a totally real number field of degree 2, which had been previously proven by Dasgupta and Spiess. We present work towards proving the case for a general totally real number field. Finally, we give a proof when F is a totally real number field of degree 3.

Always to Claire and the light within.

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Introduction

In a series of four papers, [11], [12], [13], and [14] written from 1971 to 1980, Stark outlined a series of conjectures about the value of the derivative of abelian L-functions at $s = 0$. One such conjecture, the Brumer-Stark Conjecture, was stated in the following manner by Tate:

Conjecture ([7], Conjecture 7.4)**.** Let F be a totally real number field and let H be an abelian extension of F. Let S be a non-empty finite set of places that contains the Archimedean places and a place $\mathfrak{p}$ which splits completely in H. Let T be an auxiliary finite set of places which is disjoint from S.

There is a unique $u \in H^*$ that satisfies the following conditions:

1. u is a $\mathfrak{p}$-unit and $|u|_v = 1$ for each Archimedean place v of H;

2. fix a prime $\mathfrak{P}$ of H which divides $\mathfrak{p}$, for all $\sigma \in \mathrm{Gal}(H/F)$, $\zeta'_T(\sigma, 0) = \log|u^\sigma|_{\mathfrak{P}}$;

3. $u \equiv 1 \pmod{T}$, i.e. if $\mathfrak{q} \in T$, then $|u - 1|_{\mathfrak{q}} < 1$.

Further, in 1988, Benedict H. Gross gave a p-adic refinement of the Brumer-Stark conjecture.

Conjecture ([7], Conjecture 7.6)**.** We denote the Artin reciprocity map from local class field theory by

$$\mathrm{rec}_{\mathfrak{p}} : F_{\mathfrak{p}}^{\times} \to \mathbb{A}_F^{\times} \to \mathrm{Gal}(H/F).$$

If $u \in H^{\times} \subset H_{\mathfrak{P}}^{*} \subset F_{\mathfrak{p}}^{*}$ is the unit given in the previous conjecture, then we have

$$\mathrm{rec}_{\mathfrak{p}}(u^{\sigma}) = \prod_{\substack{\tau \in \mathrm{Gal}(H/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(H/F,\sigma^{-1},0)} \quad \text{for all } \sigma \in \mathrm{Gal}(H/F).$$

Using Gross's refinement of Stark's Conjectures, Samit Dasgupta was led to an explicit conjectural formula for u in $F_{\mathfrak{p}}^{\times}$. This formula is given in a definition:

Definition ([4], Definition 3.18)**.** Given a conductor $\mathfrak{f}$, let e be the order of $\mathfrak{p}$ in the narrow ray class group $G_{\mathfrak{f}}$. So, we have $\mathfrak{p}^e = (\pi)$, where π is totally positive and $\pi \equiv 1 \pmod{\mathfrak{f}}$. If $\mathcal{D}$ is a Shintani domain, $R = S - \{\mathfrak{p}\}$, and T satisfies a technical condition, which will be given later, then we define

$$u_T(\mathfrak{b}, \mathcal{D}) := \epsilon \cdot \pi^{\zeta_{R,T}(H_{\mathfrak{f}}/F,\mathfrak{b},0)} \cdot \times\!\!\!\!\!\!\int_{\mathcal{O}_{\mathfrak{p}} - \pi\mathcal{O}_{\mathfrak{p}}} x \, d\nu(\mathfrak{b}, \mathcal{D}, x) \in F_{\mathfrak{p}}^{\times}$$

where ϵ is a specific unit of F that will be defined later, $\nu(\mathfrak{b}, \mathcal{D}, x)$ is a measure that will also be defined later, and $\times\!\!\!\!\!\!\int$ denotes the multiplicative integral as defined in Section 3 of [4].

Remark. We note that $u_T(\mathfrak{b}, \mathcal{D})$ is the notation used in [4]. We will specialize to the case where $T = \{\lambda\}$ and so use the notation $\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})$.

In [5], Michael Spiess and Samit Dasgupta gave a conjectural formula for the minors of the Gross regulator matrix. Suppose that F is a totally real number field of degree n and suppose that $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}$ is a totally odd character. We take a fixed prime number p and we fix two embeddings $\overline{\mathbb{Q}} \subset \mathbb{C}$ and $\overline{\mathbb{Q}} \subset \mathbb{C}_p$. Let H be the field fixed by $\ker \chi$ and let R be the primes of F above p, which split in H.

For all $\mathfrak{p}$ above p, we denote the group of $\mathfrak{p}$-units of H by

$$U_{\mathfrak{p}} := \{u \in H^{\times} : \mathrm{ord}_{\mathfrak{P}}\, u = 0 \text{ for all } \mathfrak{P} \nmid \mathfrak{p}\}$$

Further, consider the subsets of $U_{\mathfrak{p}} \otimes \overline{\mathbb{Q}}$:

$$U_{\mathfrak{p},\chi} := \{u \in U_{\mathfrak{p}} \otimes \overline{\mathbb{Q}} : \sigma(u) = u \otimes \chi^{-1}(\sigma) \text{ for all } \sigma \in G\}.$$

Then

$$\dim_{\overline{Q}} U_{\mathfrak{p},\chi} = \begin{cases} 1 & \mathfrak{p} \in R \\ 0 & \text{otherwise.} \end{cases}$$

So, if $\mathfrak{p} \in R$, $U_{\mathfrak{p},\chi}$ is generated by any non-zero element. We fix a generator and denote it by $u_{\mathfrak{p},\chi}$.

Now, consider the continuous homomorphisms

$$o_{\mathfrak{p}} := \mathrm{ord}_{\mathfrak{p}} : \quad F_{\mathfrak{p}}^{\times} \to \mathbb{Z}$$

$$\ell_{\mathfrak{p}} := \log_p \circ \mathrm{Norm}_{F_{\mathfrak{p}}/\mathbb{Q}_p} : \quad F_{\mathfrak{p}}^{\times} \to \mathbb{Z}_p.$$

As $\mathfrak{p}$ splits completely we can evaluate elements U_p with $o_{\mathfrak{p}}$ and $\ell_{\mathfrak{p}}$, as $U_{\mathfrak{p}} \subset H \subset H_{\mathfrak{P}} \cong F_{\mathfrak{p}}$.

In their paper [5], they also introduced a conjectural cohomological formula for the Gross-Stark units. Suppose that $E_R^{\times}$ is the group of totally positive units of F, let $F_R = \prod_{\mathfrak{p} \in R} F_{\mathfrak{p}}$, and let K be a finite extension of $\mathbb{Q}_p$, which contains the image of

χ. There is an action of $F_R^\times$ on the space of compactly supported continuous functions from F_R to K, denoted by $C_c(F_R, K)$, given by

$$(g \cdot f)(x) := f(g^{-1}x)$$

where $g \in F_R^\times$, $x \in F_R$, and $f \in C_c(F_R, K)$. Further, by restriction this action induces an action of $E_R^\times$ on $C_c(F_R, K)$.

We are also concerned with the p-adic measures $\text{Meas}(F_R, K)$, which are the p-adically bounded linear forms on $C_c(F_R, K)$. The action of $E_R^\times$ on $C_c(F_R, K)$ induces an action on $\text{Meas}(F_R, K)$. Further, we have a cocycle, called an Eisenstein cocycle,

$$\kappa_\chi \in H^{n-1}(E_R^\times, \text{Meas}(F_R, K)),$$

which will be given a precise definition later.

Given a prime $\mathfrak{p} \in R$, there are cocycles $c_\ell, c_o \in H^r(E_R^\times, C_c(F_R, K))$ that are defined in terms of the homomorphisms $o_\mathfrak{p}$ and $\ell_\mathfrak{p}$. Finally, we let

$$\vartheta \in H_{n+r-1}(E_R^\times, \mathbb{Z}) \cong \mathbb{Z}.$$

This ϑ is a generator of the $n + r - 1$-st homology group.

The cap product sends $\kappa_\chi \cap \vartheta$ to $H_r(E_R^\times, \text{Meas}(F_R, K))$. We take the cap product of this with c_ℓ and c_o to obtain the following element of K:

$$(-1)^{\#J} \frac{c_\ell \cap (\kappa_\chi \cap \vartheta)}{c_o \cap (\kappa_\chi \cap \vartheta)} \in K.$$

Further, in their paper, they proved the following theorem:

Theorem ([5], Theorem 1.5)**.** If $[F : \mathbb{Q}] = 2$ and $\mathfrak{p} \in R$, then [5], Conjecture 1.2 is consistent with [4], Conjecture 3.21.

Further, this theorem gives the following lemma:

Lemma. If [5], Conjecture 1.2 is true, then we have the Gross-Stark unit is given by the following formula:

$$c_{id} \cap (\kappa_\chi \cap \vartheta) = \mathcal{U}_{\mathfrak{p},\chi}.$$

where $id : F_{\mathfrak{p}}^\times \to F_{\mathfrak{p}}^\times$ is the identity homomorphism and

$$\mathcal{U}_{\mathfrak{p},\chi} = \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}) \otimes \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell}.$$

In this work, we will prove the $n = 3$ and $\mathfrak{p} \in R$ case and we will give work towards the general n case. We begin by providing background information, including some background on the cap product and calculating the cap product. We provide some work towards proving the general case. Afterwards, we then explicitly prove the $n = 2$ case as motivation towards the proof of the $n = 3$ case. Finally, we prove the $n = 3$ case.

Chapter 1

Background

In this section, we will flesh out everything that was mentioned in the introduction. We will give no proofs, but we will include references that contain the relevant proofs.

1.1 Dasgupta's Conjectural p-adic Formula for Gross-Stark Units

Suppose that we have a totally real number field F, whose degree is $[F : \mathbb{Q}] = n$. As F is totally real, there are n embeddings $\iota : F \hookrightarrow \mathbb{R}$. We will fix an ordering for these embeddings and denote this ordered set of embeddings by I. With this ordering, we can embed F into $\mathbb{R}^n$, via $\alpha \mapsto (\alpha^\iota)_{\iota \in I}$.

There is an action of $F^\times$ on $\mathbb{R}^n$ via componentwise multiplication. Explicitly, if $\alpha \mapsto (\alpha^\iota)_{\iota \in I} \in \mathbb{R}^n$ and $(x_\iota)_{\iota \in I}$, then we have

$$\alpha \cdot (x_\iota)_{\iota \in I} = (\alpha^\iota x_\iota)_{\iota \in I}.$$

If we restrict to the totally positive units of F, which we denote by E_+, then we get an action of E^+ on the positive orthant of $\mathbb{R}^n$, which we denote by $\mathbb{R}^n_{>0}$.

In 1976, [9], Takuro Shintani showed that there exists a fundamental domain for the action of E_+ on $\mathbb{R}^n_{>0}$. An explicit formula for this fundamental domain was proven by Francisco Diaz y Diaz and Eduardo Friedman using topological degree theory, in a 2014 paper, [6]. In a 2015 paper, [2], this formula was also independently proven by Pierre Charollois, Samit Dasgupta, and Matthew Greenberg, using a cohomological argument to reduce it to a result of Pierre Colmez. We will recount the construction of this domain and how it relates to zeta functions and L-functions.

We begin by defining a Shintani Cone:

Definition 1.1.1. *Let $v_1, \ldots, v_r \in \mathbb{R}^n$ be a collection of linearly independent vectors in $\mathbb{R}^n_{>0}$. If $v_1, \ldots, v_r \in F \cap \mathbb{R}^n_{>0}$ and they generate the simplicial cone:*

$$C(v_1, \ldots, v_r) = \left\{ \sum_{i=1}^{r} \alpha_i v_i : \alpha_i > 0 \right\},$$

then we say that $C(v_1, \ldots, v_r)$ is a **Shintani cone***.*

Further, if $\mathcal{D} \subset \mathbb{R}^n_{>0}$ is the finite disjoint union of Shintani cones, then we call $\mathcal{D}$ a **Shintani set***.*

Note that $C(v_1, \ldots, v_r)$ may also be referred to as an F-rational simplicial cone.

We can now state Shintani's Unit Theorem and we use the version from Jürgen Neukirch's *Algebraic Number Theory*, [8]:

Theorem 1.1.2 ([9], Proposition 4)**.** *Let U be a finite index subgroup of E_+, then there exists a Shintani set $\mathcal{D}_U$ such that*

$$\mathbb{R}^n_{>0} = \bigcup_{\epsilon \in U} \epsilon \mathcal{D}_U \quad \textit{(disjoint union).}$$

Explicitly, for all $(x_\iota)_{\iota \in I} \in \mathbb{R}^n_{>0}$ there exists a unique $(y_\iota)_{\iota \in I} \in \mathcal{D}_U$ and a unique $\epsilon \in U$ such that

$$(x_\iota)_{\iota \in I} = (\epsilon^\iota y_\iota)_{\iota \in I}.$$

This theorem leads us to make the following definition:

Definition 1.1.3. *If a disjoint set of Shinani cones, denoted by $\mathcal{D}$, satisfies Theorem 1.1.2 for U, then we say that $\mathcal{D}$ is a* **Shintani domain** *for U.*

We introduce the concept of Colmez perturbation. This allows us to write the Shintani domain in a relatively simple and clean manner. We do this by defining it through the characteristic functions. With this method, we also define a generalization of the fundamental domain: a signed fundamental domain.

Following the exposition given in [2], consider a linear independent collection of vectors $v_1, \dots, v_r \in \mathbb{R}^n$. Further, let $Q \in \mathbb{R}^n$ be a vector such that $\mathrm{Span}(Q) \oplus \mathrm{Span}_{i \neq j}(v_i) = \mathrm{Span}_{1 \leq i \leq r}(v_i)$ for all $1 \leq j \leq r$. We define the set $C_Q(v_1, \dots, v_r)$, which is the disjoint union of $C(v_1, \dots, v_n)$ and some boundaries. If $c_Q(v_1, \dots, v_r)$ is the characteristic function of $C_Q(v_1, \dots, v_r)$, then it satisfies:

$$c_Q(v_1, \dots, v_r)(w) = \begin{cases} \lim_{\epsilon \to 0^+} \mathbf{1}_{C(v_1, \dots, v_r)}(w + \epsilon Q) & \text{the } v_i \text{ are linearly independent,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.1}$$

If $u_1, \ldots, u_{n-1}$ is an ordered basis of U, then we an talk about the orientation of a subset of the basis. We let u_{ij} denote the jth coordinate of the vector u_i. We have

$$w_u := \operatorname{sign} \det(\log(u_{ij})_{i,j=1}^{n-1}) = \pm 1.$$

For $\sigma \in S_{n-1}$, we let $v_{1,\sigma} = 1$ and

$$v_{i,\sigma} = \prod_{j<i} v_{\sigma(j)} \in U, \quad \text{for all } 2 \leq i \leq n.$$

To each σ, we associate the following constant:

$$w_\sigma = (-1)^{n-1} w_u \operatorname{sign} \det(v_{i,\sigma})_{i=1}^{n} \in \{0, \pm 1\}.$$

For the perturbation vector, we take $e_n = (0, \ldots, 0, 1) \in \mathbb{R}^n$. With this an explicit fundamental domain was given by Pierre Colmez:

Theorem 1.1.4 ([3], Lemme 2.2)**.** *Suppose that $w_\sigma = 1$ for all $\sigma \in S_{n-1}$, then*

$$\coprod_{\sigma \in S_{n-1}} C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma}) \tag{1.2}$$

is a fundamental domain for the action of U on $\mathbb{R}^n_{>0}$.

Remark 1.1.5. This fundamental domain may be more easily expressed using bar notation. We will represent it by:

$$\coprod_{\sigma \in S_{n-1}} C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma}) = \coprod_{\sigma \in S_{n-1}} C_{e_n}([v_{\sigma(1)} \mid \ldots \mid v_{\sigma(n-1)}]).$$

Note that $[x_1 \mid \cdots \mid x_n]$ denotes the bar notation from group cohomology. That is

$$[x_1 \mid \cdots \mid x_n] = (1, x_1, x_1 x_2, \ldots, x_1 x_2 \cdots x_n).$$

Definition 1.1.6 ([2], Definition 1.4)**.** *Let U be a finite index subgroup of E_+. If we have a formal linear combination $\sum_i a_i C_i$, where $a_i \in \mathbb{Z}$ and C_i is a simplicial cone for all i, then we say that $\mathcal{D} = \sum_i a_i C_i$ is a* **signed fundamental domain** *for the action of U on $\mathbb{R}^n_{>0}$ if the following identity holds:*

$$\sum_{u \in U} \sum_i a_i C_i(u \cdot x) = 1 \quad \text{for all } x \in \mathbb{R}^n_{>0}.$$

The characteristic function of $\mathcal{D}$ is $\mathbf{1}_{\mathcal{D}} = \sum_i a_i \mathbf{1}_{C_i}$.

The following theorem was proven by Diaz y Diaz and Friedman in 2014 and independently by Charollois, Dasgupta, and Greenberg in 2015:

Theorem 1.1.7 ([2], Theorem 1.5)**.** *The formal linear combination*

$$\sum_{\sigma \in S_{n-1}} w_\sigma C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})$$

is a signed fundamental domain for the action of U on $\mathbb{R}^n_{>0}$.

Now, we can define Shintani zeta functions. We will also introduce an integral version using a method of Cassou-Noguès.

Let F be a totally real number field of degree n. Further, let $G_F = \mathrm{Gal}(\overline{F}/F)$ and $\chi : G_F \to \overline{\mathbb{Q}}$ be a totally odd character, that is if σ_c is a complex conjugation, then $\chi(\sigma_c) = -1$. Let H be the fixed field of the kernel of χ. We fix a prime p and we let S be a finite set of primes which contains the Archimedean primes and which contains the set of primes of F above p. Suppose that $\mathfrak{p}$ is a prime that splits completely and is above p and $R = S - \{\mathfrak{p}\}$.

We make the following assumption before we define the Shintani zeta function:

Hypothesis 1.1.8. Let T be a finite set of primes satisfying the following conditions:

1. T is disjoint from S;

2. suppose that the primes of T have different residue characteristic from the primes of S.

Remark 1.1.9. We note that we will almost always take T to be a set containing a single prime λ. Later, we will give a further restriction on λ.

Definition 1.1.10. *Let $\mathfrak{f}$ be an integral ideal relatively prime to S and let $\mathfrak{b}$ be a factional ideal of F relatively prime to S and the prime to the residue characteristic of T, that is if $\mathfrak{q}$ divides $\mathfrak{b}$, then the residue characteristic of $\mathfrak{q}$ is not equal to the residue characteristic of any prime of T.*

Let $z \in \mathfrak{b}^{-1}$ such that $z \equiv 1 \pmod{\mathfrak{f}}$ and let C be a Shintani cone. Further, let U be a compact subset of $\mathcal{O}_{\mathfrak{p}}$, then for $\mathrm{Re}(s) > 1$, we have:

$$\zeta_R(\mathfrak{b}, C, U, s) = \mathrm{N}\,\mathfrak{b}^{-s} \sum_{\alpha} \mathrm{N}\,\alpha^{-s} \tag{1.3}$$

where the sum is taken over all $\alpha \in (\mathfrak{b}^{-1}\mathfrak{f} + z) \cap C \cap U$ and $(\alpha, R) = 1$. We call $\zeta_R(\mathfrak{b}, C, U, s)$ a **Shintani zeta function**.

If λ is a prime such that $\mathrm{N}\,\lambda = \ell \geq n + 2$, where ℓ is a rational prime and $T = \{\lambda\}$, then we define:

$$\zeta_{R,T}(\mathfrak{b}, C, U, s) = \zeta_R(\mathfrak{b}, C, U, s) - \ell^{1-s}\zeta_R(\mathfrak{b}\lambda^{-1}, C, U, s). \tag{1.4}$$

We call this an **integral Shintani zeta function**.

It is not immediately clear that $\zeta_{R,T}(\mathfrak{b}, C, U, s)$ has a meromorphic continuation, but this was proven in Proposition 1, in Shintani's 1976 paper, [9]. It is also not trivial that $\zeta_{R,T}(\mathfrak{b}, C, U, s)$ is integral, but this is a corollary to a proposition given by Dasgupta:

Proposition 1.1.11 ([4], Proposition 3.12)**.** *Suppose that T is given as in Definition 1.1.10 and C can be generated from $v_1, \ldots, v_r \in \mathcal{O}_F$ such that $v_i \notin \lambda$ for all $1 \le i \le r$, then*

$$\zeta_{R,T}(\mathfrak{b}, C, U, 0) \in \mathbb{Z}[1/\ell]$$

and the denominator of $\zeta_{R,T}(\mathfrak{b}, C, U, 0)$ is less than or equal to $\ell^{n/(\ell-1)}$.

If we have a signed fundamental domain, $\mathcal{D} = \sum_i a_i C_i$, then we have a zeta function:

$$\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, s) := \sum_i a_i \zeta_{R,T}(\mathfrak{b}, C_i, U, s).$$

By [4], Proposition 3.12, $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0) \in \mathbb{Z}[1/\ell]$. If we choose λ such that $\mathrm{N}\,\lambda \ge n+2$, then this proposition even implies that $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0) \in \mathbb{Z}$.

As $\zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0) \in \mathbb{Z}$, we obtain a $\mathbb{Z}$-valued measure for $\mathcal{O}_{\mathfrak{p}}$. We denote this measure by

$$\nu(\mathfrak{b}, \mathcal{D}, U) := \zeta_{R,T}(\mathfrak{b}, \mathcal{D}, U, 0).$$

Now, let $G_{\mathfrak{f}}$ denote the narrow ray class group of conductor $\mathfrak{f}$ and let $H_{\mathfrak{f}}$ be the narrow ray class field of conductor $\mathfrak{f}$. Let e be the order of $\mathfrak{p}$ in $G_{\mathfrak{f}}$ so that $\mathfrak{p}^e = (\pi)$. The element π is totally positive and congruent to 1 modulo $\mathfrak{f}$.

Let $E(\mathfrak{f})$ denote the totally positive units that are congruent to 1 modulo $\mathfrak{f}$. We are now ready to give Dasgupta's conjectural p-adic formula for the Gross-Stark unit.

Definition 1.1.12 ([4], Definition 3.18)**.** *Let $\mathcal{D}$ be a Shintani domain, then we have*

$$\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D}) = \epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot \pi^{\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\cdot \times\!\!\!\!\!\!\int_{\mathbf{O}_{\mathfrak{p}}} x\, d\nu(\mathfrak{b},\mathcal{D},x) \in F_{\mathfrak{p}}^{\times} \tag{1.5}$$

where

$$\epsilon(\mathfrak{b},\mathcal{D},\pi) = \prod_{\epsilon\in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})}$$

and $\mathbf{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}} - \pi\mathcal{O}_{\mathfrak{p}}$.

We note that $\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D})$ is not dependent on choice of π, as proven in [4], Proposition 3.19 and under certain conditions, it is independent of the choice of Shintani domain as proven in [4], Theorem 5.3.

Definition 1.1.13. *If $\mathcal{D}$ is a Shintani domain and $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}$ is a totally odd character, then we have*

$$\mathcal{U}_{\mathfrak{p},\chi} = \sum_{[\mathfrak{b}]\in G_{\mathfrak{f}}} \mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D}) \otimes \frac{\chi(\mathfrak{b})}{1-\chi(\lambda)\ell}.$$

1.2 Dasgupta and Spiess's Conjectural formula for the Gross p-adic Regulator

1.2.1 The Eisenstein Cocycle

Definition 1.2.1 ([5])**.** *Suppose that we have the following:*

- *a totally odd character* $\chi : \mathrm{Gal}(\overline{F}/F) \to \overline{\mathbb{Q}}$*;*

- *a rational prime* p*;*

- *a set of primes of* F *which are totally split and which divide* p*;*

- *the ring of* R*-integers of* F*,* $\mathcal{O}_{F,R}$*;*

- *a fractional ideal* $\mathfrak{b}$ *of* F *relatively prime to* S*;*

- *the* $\mathcal{O}_{F,R}$*-module generated by* $\mathfrak{b}$*,* $\mathfrak{b}_R = \mathfrak{b} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,R}$*;*

- *a compact open subset* U *of* $F_R := \prod_{\mathfrak{p}\in R} F_{\mathfrak{p}}$*;*

- *any union of simplicial cones* C *of* $\mathbb{R}^n_{>0}$*.*

Then a **Shintani L-function***, denoted by* $L(C, \chi, \mathfrak{b}, U, s)$ *is the meromorphic continuation of*

$$\sum_{\xi} \frac{\chi((\xi))}{\mathrm{N}\,\xi^s}$$

where the sum is taken over $\xi \in C \cap \mathfrak{b}_R^{-1}$*,* $\xi \in U$*, and* $(\xi, S \setminus R) = 1$*.*

Let λ *be a prime of* F*, such that no primes of* S *have the same residue characteristic of* λ*,* $\mathrm{N}\,\lambda = \ell$*, where* ℓ *is a rational prime, and* $\ell \geq n+2$*. Then the* **integral Shintani L-function** *is a twisted version of a Shintani* L*-function given by*

$$L_\lambda(C, \chi, \mathfrak{b}, U, 0) := L(C, \chi, \mathfrak{b}\lambda^{-1}, U, s) - \chi(\lambda)\ell^{1-s} L(C, \chi, \mathfrak{b}, U, s).$$

Remark 1.2.2. Shintani showed that the Shintani L-function has a meromorphic continuation to $\mathbb{C}$. Further, Cassou-Noguès showed that $L_\lambda(C, \chi, \mathfrak{b}, U, s)$ is integral at $s = 0$.

Definition 1.2.3. *Let k be the cyclotomic field generated by the image of χ. The space of k-valued distributions, the vector space of linear forms $C_c(F_R, k) \to k$, is denoted by* $\mathrm{Dist}(F_R, k)$*. There is an action of $F_R^\times$ on* $\mathrm{Dist}(F_R, k)$ *given by*

$$(x \cdot \mu)(U) = \mu(x^{-1}U).$$

Proposition 1.2.4 ([5], Proposition 2.1)**.** *If $x_1, \ldots, x_n$ are elements of $E_R^\times$ and x is the $n \times n$ matrix with columns given by the images of x_i, $1 \leq i \leq n$, in $\mathbb{R}^n_{>0}$, and U is a compact open subset of F_R, then we have a measure:*

$$\mu_{\chi,\mathfrak{b}}(x_1, \ldots, x_n)(U) := \mathrm{sgn}(x) L(C^*(x_1, \ldots, x_n), \chi, \mathfrak{b}, U, 0),$$

here $\mathrm{sgn}(x) := \mathrm{sign}(\det(x))$*. Further, the class defined by $\mu_{\chi,\mathfrak{b}}$ is a homogeneous $(n-1)$-cocycle, i.e.*

$$[\mu_{\chi,\mathfrak{b}}] \in H^{n-1}(E_R^\times, \mathrm{Dist}(F_R, k)).$$

Remark 1.2.5. The result of Cassou-Noguès shows that $U \mapsto L_\lambda(C, \chi, \mathfrak{b}, U, 0)$ is $\mathbb{Z}$-valued and so p-adically bounded, hence it defines a p-adic measure.

Let $\mathfrak{P}$ be the prime of k above p that corresponds to the embedding $k \hookrightarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}_p$, then we denote $k_{\mathfrak{P}}$ by K. Thus $U \mapsto L_\lambda(C, \chi, \mathfrak{b}, U, 0)$ gives a p-adic measure and so an element of $\mathrm{Meas}(F_R, K)$.

We define the function $(E_R^\times)^n \to \mathrm{Meas}(F_R, K)$ via

$$(E_R^\times)^n \ni (x_1, \ldots, x_n) \mapsto (U \mapsto \mathrm{sgn}(x) L_\lambda(C^*(x_1, \ldots, x_n), \chi, \mathfrak{b}, U, 0))$$

which we denote by $\mu_{\chi,\mathfrak{b},\lambda}$. From [5], Proposition 2.1, it follows that $\kappa_{\chi,\mathfrak{b},\lambda} := [\mu_{\chi,\mathfrak{b},\lambda}] \in H^{n-1}(E_R^\times, \mathrm{Meas}(F_R, K))$. This leads us to the next definition.

Definition 1.2.6. *The* **Eisenstein cocycle** *associated to* χ *and* λ *is given by*

$$\kappa_{\chi,\lambda} := \sum_{i=1}^{h} \frac{\chi(\mathfrak{b}_i)}{1-\chi(\lambda)\ell} \kappa_{\chi,\mathfrak{b}_i,\lambda} \in H^{n-1}(E_R^\times, \mathrm{Meas}(F_\mathfrak{p}, K))$$

where $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_h\}$ *is a set of integral ideals that are representatives of the elements of the narrow class group of* $\mathcal{O}_{F,R}$.

Proposition 1.2.7. *The element* $\kappa_{\chi,\lambda}$ *does not depend on the choice of representative of the narrow class group.*

Proof. This is a routine calculation, which we will quickly go through.

Suppose that $\{\mathfrak{b}_1, \ldots, \mathfrak{b}_h\}$ and $\{\mathfrak{a}_1, \ldots, \mathfrak{a}_h\}$ are two sets of representatives of the narrow ray class group numbered such that $\mathfrak{b}_i\mathfrak{a}_i^{-1} = (a_i)$ or $\mathfrak{b}_i = a_i\mathfrak{a}_i$, where a_i is totally positive and $a_i \in \mathcal{O}_{F,R}^\times$.

If $\xi \in \mathfrak{b}_i$, then $\xi = a_i\alpha_i$, where $\alpha_i \in \mathfrak{a}_i$. So, we have

$$\sum_{\xi} \frac{\chi(\xi)}{\mathrm{N}\,\xi^s} = \frac{\chi(a_i)}{\mathrm{N}\,a_i^s} \sum_{\alpha_i} \frac{\chi(\alpha_i)}{\mathrm{N}\,\alpha_i^s} = \frac{\chi(a_i)}{\mathrm{N}\,a_i^s} L(C, \chi, \mathfrak{a}_i, U, s).$$

Thus

$$L(C, \chi, \mathfrak{b}_i, U, 0) = \chi(a_i) L(C, \chi, \mathfrak{a}_i, U, 0).$$

This gives us

$$\sum_{i=1}^{h} \chi(\mathfrak{b}_i) L(C, \chi, \mathfrak{b}_i, U, 0) = \sum_{i=1} \chi(\mathfrak{b}_i)\chi(a_i) L(C, \chi, \mathfrak{a}_i, U, 0) = \sum_{i=1}^{h} \chi(\mathfrak{a}_i) L(C, \chi, \mathfrak{a}_i, U, 0)$$

A similar calculation shows that

$$\sum_{i=1}^{h} \chi(\mathfrak{b}_i) L(C, \chi, \mathfrak{b}_i\lambda^{-1}, U, 0) = \sum_{i=1} \chi(\mathfrak{b}_i)\chi(a_i) L(C, \chi, \mathfrak{a}_i, U, 0) = \sum_{i=1}^{h} \chi(\mathfrak{a}_i) L(C, \chi, \mathfrak{a}_i\lambda^{-1}, U, 0).$$

Therefore, $\kappa_{\chi,\lambda}$ is independent of the choice of set of representatives. □

1.2.2 Shintani Zeta Function Cocycle

We introduce a formulation of a cocycle based on the Shintani zeta function. The construction is similar to the construction of $\kappa_{\chi,\lambda}$, but there are key differences.

We recall the definition of a Shintani zeta function. If we are given an integral ideal $\mathfrak{f}$, a fractional ideal $\mathfrak{b}$ of F that is relatively prime to S, J, a subset of R, a union of simplicial cones C, a compact open subset U of $F_J := \prod_{\mathfrak{p}\in J} F_{\mathfrak{p}}$, then it is the meromorphic continuation of the summation

$$\zeta(C, \mathfrak{b}, U, s) = \mathrm{N}\,\mathfrak{b}^{-s} \sum_{\alpha} \mathrm{N}\,\alpha^{-s}, \quad \mathrm{Re}(s) > 1,$$

where the sum is taken over $\alpha \in \mathfrak{b}_J^{-1} := \mathfrak{b}^{-1} \otimes_{\mathcal{O}_F} \mathcal{O}_{F,J}$, $\alpha \in C$, $\alpha \in U$, $\alpha \equiv 1 \pmod{\mathfrak{f}}$, and $(\alpha, S \setminus R) = 1$. If we take λ as in the definition of the integral Shintani L-function, then we have a twisted Shintani zeta function

$$\zeta_\lambda(C, \mathfrak{b}, U, s) := \zeta(C, \mathfrak{b}, U, s) - \ell^{1-s}\zeta(C, \mathfrak{b}\lambda^{-1}, U, s)$$

which is integral at $s = 0$.

Now, as before let $\mathfrak{f}$ be the conductor of H/F and let $E(\mathfrak{f})$ denote the totally positive units which are congruent to 1 modulo $\mathfrak{f}$ and $E(\mathfrak{f})_J$ denote the group of totally positive J-units which are congruent to 1 modulo $\mathfrak{f}$.

Definition 1.2.8. *For $x_1, \ldots, x_n \in E(\mathfrak{f})_J$ and U a compact open subset of F_J, then we set*

$$\nu^J_{\mathfrak{b},\lambda}(x_1, \ldots, x_n)(U) := \mathrm{sgn}(x)\zeta_\lambda(C^*(x_1, \ldots, x_n), \mathfrak{b}, U, 0).$$

This element defines a class in $H^{n-1}(E(\mathfrak{f})_J, \mathrm{Meas}(F_J, K))$. *We denote this class by* $\omega^J_{\mathfrak{f},\mathfrak{b},\lambda} := [\nu^J_{\mathfrak{b},\lambda}] \in H^{n-1}(E(\mathfrak{f})_J, \mathrm{Meas}(F_J, K))$.

The **variant Eisenstein cocycle** *is defined as*

$$\omega^J_{\chi,\lambda} := \sum_{[\mathfrak{b}]\in G_{\mathfrak{f}}} \frac{\chi(\mathfrak{b})}{1-\chi(\lambda)\ell}\omega^J_{\mathfrak{f},\mathfrak{b},\lambda}.$$

Remark 1.2.9. The reason that $\omega^J_{\mathfrak{f},\mathfrak{b},\lambda} \in H^{n-1}(E(\mathfrak{f})_J, \mathrm{Meas}(F_J, K))$ is because $\zeta_\lambda(C, \mathfrak{b}, U, 0) \in \mathbb{Z}$ and so it is p-adically bounded and therefore a measure.

Further, we note that $\omega^J_{\mathfrak{f},\lambda}$ and $\kappa_{\chi,\lambda}$ are compatible in a sense that will be made more precise later. First, we need to introduce a specific cycle and cocycle.

1.2.3 Elements of $H^r(F_{\mathfrak{p}}^\times, C_c(F_{\mathfrak{p}}, K))$

We begin by introducing a continuous homomorphism $g : F_{\mathfrak{p}}^\times \to K$ and $f \in C_c(F_{\mathfrak{p}}, \mathbb{Z})$. Also, we have an action of $F_{\mathfrak{p}}^\times$ on $C_c(F_{\mathfrak{p}}, \mathbb{Z})$ given by $(a \cdot f)(x) = f(a^{-1}x)$, where $a \in F_{\mathfrak{p}}^\times$ and $x \in F_{\mathfrak{p}}$.

We use this action to extend g to a continuous function of $F_{\mathfrak{p}} \to K$. Specifically, given $a \in F_{\mathfrak{p}}^\times$, we take the function

$$\begin{aligned} F_{\mathfrak{p}} &\to K \\ x &\mapsto (f(x) - (a\cdot f)(x))g(x). \end{aligned}$$

This gives us a map

$$\begin{aligned} F_{\mathfrak{p}}^* &\to C_c(F_{\mathfrak{p}}, K) \\ a &\mapsto (a\cdot f)g(a) + (f - a\cdot f)g. \end{aligned}$$

We denote this by $z_{f,g} : F_{\mathfrak{p}}^{\times} \to C_c(F_{\mathfrak{p}}, K)$.

Proposition 1.2.10. *The class defined by $z_{f,g}$ is a 1-cocycle. That is*

$$[z_{f,g}] \in H^1(F_{\mathfrak{p}}^{\times}, C_c(F_{\mathfrak{p}}, K)).$$

Proof. We wish to show given $a, b \in F_{\mathfrak{p}}^{\times}$, we have

$$z_{f,g}(ab)(x) = z_{f,g}(a)(x) + a \cdot z_{f,g}(b)(x).$$

This is a straightforward calculation.

First, we have

$$z_{f,g}(ab)(x) = (ab \cdot f)(x)g(ab) + (f(x) - (ab \cdot f)(x))g(x)$$

by definition.

Next, we also have

$$\begin{aligned}
z_{f,g}(a)(x) + a \cdot z_{f,g}(b)(x) &= (a \cdot f)(x)g(a) + (f(x) - (a \cdot f)(x))g(x) \\
&\quad + a \cdot (b \cdot f)(x)g(b) + a \cdot ((f(x) - (b \cdot f)(x))g(x)) \\
&= (a \cdot f)(x)g(a) + (f(x) - (a \cdot f)(x))g(x) \\
&\quad + (ab \cdot f)(x)g(b) + (ab \cdot f)(x)g(a) \\
&\quad - (ab \cdot f)(x)g(a) + a \cdot ((f(x) - (b \cdot f)(x))g(x)) \\
&= (a \cdot f)(x)g(a) - (a \cdot f)(x)g(x) + f(x)g(x) \\
&\quad + (ab \cdot f)(x)g(ab) + (a \cdot f)(x)(a \cdot g)(x) \\
&\quad - (ab \cdot f)(x)g(a) - (ab \cdot f)(x)(a \cdot g)(x)
\end{aligned}$$

Finally, this gives us

$$\begin{aligned} z_{f,g}(a)(x) + a \cdot z_{f,g}(b)(x) &= (ab \cdot f)(x)g(ab) + f(x)g(x) - (ab \cdot f)(x)g(x) \\ &= z_{f,g}(ab)(x). \end{aligned}$$

□

Proposition 1.2.11. *If $f, f' \in C_c(F_\mathfrak{p}, \mathbb{Z})$ such that $f(0) = f'(0)$, then $[z_{f,g}] = [z_{f',g}]$.*

Proof. We calculate $z_{f,g}(a)(x) - z_{f',g}(a)(x)$ and we show that if $f(0) = f'(0)$, then this gives us a 1-coboundary.

We have

$$\begin{aligned} z_{f,g}(a)(x) - z_{f',g}(a)(x) &= (af)(x)g(a) + (f(x) - a \cdot f(x))g(x) \\ &\quad -(a \cdot f')(x)g(a) - (f'(x) - a \cdot f'(x))g(x) \\ &= (a \cdot f'(x) - a \cdot f(x))g(a^{-1}x) \\ &\quad -(f'(x) - f(x))g(x) \\ &= a \cdot ((f'(x) - f(x))g(x)) - (f'(x) - f(x))g(x). \end{aligned}$$

If $f(0) = f'(0)$, then $(f'(x) - f(x))g(x) \in C_c(F_\mathfrak{p}, K)$. Therefore, it is a 1-coboundary. □

Remark 1.2.12. This means we can choose an f that is amenable to calculation. To that end, we will choose f such that $f = 1_{\pi\mathcal{O}_\mathfrak{p}}$. This choice will simplify our calculations.

We will now construct an element of $H^r(F_R^\times, C_c(F_R, K))$, where $r = \#R$. Let

$R = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_r\}$ and let $J \subset R$, then we define $c_o, c_{\ell,J} \in H^r(F_{\mathfrak{p}}^\times, C_c(F_{\mathfrak{p}}, K))$ via:

$$\begin{aligned} c_o &= c_{o_{\mathfrak{p}_1}} \cup \cdots \cup c_{o_{\mathfrak{p}_r}} \\ c_{\ell,J} &= c_{g_{\mathfrak{p}_1}} \cup \cdots \cup c_{g_{\mathfrak{p}_r}}, \end{aligned}$$

where

$$g_i = \begin{cases} \ell_{\mathfrak{p}_i} & \text{if } i \in J \\ o_{\mathfrak{p}_i} & i \notin J. \end{cases}$$

1.2.4 Generators of $H_{n+r-1}(E_R^\times, \mathbb{Z})$ and $H_{n+d-1}(E(\mathfrak{f})_J, \mathbb{Z})$

We will provide a brief outline of the construction of $\vartheta \in H_{n+r-1}(E_R^\times, \mathbb{Z})$ and $\vartheta' \in H_{n+d-1}(E(\mathfrak{f})_J, \mathbb{Z})$, where $d = \#J$. For further details, see Spiess's Remark 2.1 in [10].

Let $\epsilon_1, \ldots, \epsilon_{n+r-1}$ be a $\mathbb{Z}$-basis for $E_R^\times$, then

$$\vartheta = \pm \sum_{\sigma \in S_{n+r-1}} \operatorname{sgn}(\sigma)[\epsilon_{\sigma(1)} \mid \cdots \mid \epsilon_{\sigma(n+r-1)}].$$

A similar construction is given for $\vartheta' \in H_{n+d-1}(E(\mathfrak{f})_J, \mathbb{Z})$. We take a $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_{n+d-1}$ for $E(\mathfrak{f})_J$. This gives us

$$\vartheta' = \pm \sum_{\sigma \in S_{n+d-1}} \operatorname{sgn}(\sigma)[\epsilon_{\sigma(1)} \mid \cdots \mid \epsilon_{\sigma(n+d-1)}].$$

1.2.5 Dasgupta and Spiess's Conjectural Formula for the Gross-Stark Units

We are now ready to state the conjecture of Dasgupta and Spiess:

Conjecture 1.2.13 ([5], Conjecture 1.2)**.** *We have*

$$-\frac{\ell_{\mathfrak{p}}(u_{\mathfrak{p},\chi})}{o_{\mathfrak{p}}(u_{\mathfrak{p},\chi})} = (-1)^d \frac{c_{\ell,\mathfrak{p}} \cap (\kappa_\chi \cap \vartheta)}{c_{o_{\mathfrak{p}}} \cap (\kappa_\chi \cap \vartheta)} \in K.$$

Proposition 1.2.14 ([5], Proposition 3.5)**.** *The element* $(-1)^d \frac{c_{\ell,\mathfrak{p}} \cap (\kappa_\chi \cap \vartheta)}{c_{o_{\mathfrak{p}}} \cap (\kappa_\chi \cap \vartheta)}$ *is independent of the choice of* λ*.*

Further, the Eisenstein cocycle $\kappa_{\chi,\lambda}$ is compatible with the variant Eisenstein cocycle $\omega^J_{\chi,\lambda}$ in the following sense:

Proposition 1.2.15 ([5], Proposition 3.6)**.** *Let* $\vartheta' \in H_{n+d-1}(E(\mathfrak{f})_J, \mathbb{Z})$ *be a generator, then*

$$(-1)^d \frac{c_{\ell,\mathfrak{p}} \cap (\kappa_\chi \cap \vartheta)}{c_{o_{\mathfrak{p}}} \cap (\kappa_\chi \cap \vartheta)} = (-1)^d \frac{c_{\ell_{\mathfrak{p}}} \cap (\omega^J_{\chi,\lambda} \cap \vartheta')}{c_{o_{\mathfrak{p}}} \cap (\omega^J_{\chi,\lambda} \cap \vartheta')}$$

Remark 1.2.16. We finish by noting that we will specialize to the case when $J = \{\mathfrak{p}\}$. Thus, we will drop the J from all notations, e.g. we will write $\omega_{\chi,\lambda}$ in place of $\omega^J_{\chi,\lambda}$ or $E(\mathfrak{f})_{\mathfrak{p}}$ in place of $E(\mathfrak{f})_J$.

Chapter 2

The General Case

2.1 Reduction of the Shintani domain

In this section, we set up the general case. That is, we will show that

$$c_g \cap (\kappa_{\chi,\lambda} \cap \vartheta) = g(\mathcal{U}_{\mathfrak{p},\chi})$$

where $g : F_{\mathfrak{p}}^* \to K$ is an arbitrary continuous homomorphism. We do this in several steps. First, we note that by [5], Proposition 3.6, we can calculate $c_g \cap (\omega_{\chi,\lambda} \cap \vartheta')$. Then, we reduce to the case

$$c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta') = g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})).$$

Next, we reduce to the case of a finite-index subgroup V of $E(\mathfrak{f})$ such that $w_\sigma = 1$ for all $\sigma \in S_{n-1}$, where w_σ was defined in Section 1.1. This subgroup is guaranteed to exist by a theorem of Pierre Colmez, which will be stated more precisely. Then, we will show that given a $\mathbb{Z}$-basis $\epsilon_1, \ldots, \epsilon_{n-1}$ of $V \subset E(\mathfrak{f})$, $\epsilon \in V$ with $\epsilon = \prod_i \epsilon_i^{m_i}$, and $\pi^{-1} \in \mathcal{D}$, then the following holds: if $\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D} \neq \emptyset$, then $m_i \in \{0, 1\}$ for $1 \leq i \leq n-1$. We remark that we can always take $\pi^{-1} \in \mathcal{D}$, because $\mathcal{D}$ is a fundamental domain. This means that we can multiply by a unit of $E(\mathfrak{f})$ to move π^{-1} into $\mathcal{D}$. Finally, we show that the conjecture is true if a certain identity of the measures is true.

Proposition 2.1.1. *If the following holds:*

$$c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta') = g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})),$$

then Conjecture 1.2.13 is true for $J = \{\mathfrak{p}\}$.

Proof. As

$$\omega_{\chi,\lambda} = \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell} \omega_{\mathfrak{f},\mathfrak{b},\lambda}$$

and

$$\mathcal{U}_{\mathfrak{p},\chi} = \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}) \otimes \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell}$$

then we have

$$\begin{aligned} c_g \cap (\omega_{\chi,\lambda} \cap \vartheta') &= \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell} c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta') \\ &= \sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell} g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})) \\ &= g\left(\sum_{[\mathfrak{b}] \in G_{\mathfrak{f}}} \frac{\chi(\mathfrak{b})}{1 - \chi(\lambda)\ell} \otimes \mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}) \right) \\ &= g\left(\mathcal{U}_{\mathfrak{p},\chi}\right). \end{aligned}$$

Therefore, Conjecture 1.2.13 is true for $J = \{\mathfrak{p}\}$. □

Next, we wish to show that we can work with a finite index subgroup of $E(\mathfrak{f})$ that has a nice Shintani domain. This subgroup is generated by elements $\epsilon_1, \ldots, \epsilon_{n-1}$ such that if $\sigma \in S_{n-1}$, then

$$\operatorname{sign}\det([\epsilon_{\sigma(1)} \mid \cdots \mid [\epsilon_{\sigma(n-1)}]) = \operatorname{sgn}(\sigma).$$

This is given by:

Theorem ([3], Lemme 2.1)**.** There exists a finite subgroup V of $E(\mathfrak{f})$, free of rank $n-1$ generated by $\epsilon_1, \ldots, \epsilon_{n-1} \in E(\mathfrak{f})$ such that if $\sigma \in S_{n-1}$, then

$$\operatorname{sign}\det([\epsilon_{\sigma(1)} \mid \cdots \mid [\epsilon_{\sigma(n-1)}]) = \operatorname{sgn}(\sigma). \tag{2.1}$$

Definition 2.1.2. *If a finite index subgroup V of $E(\mathfrak{f})$ satisfies Equation 2.1, then we call V a* **Colmez subgroup**.

Now, we are going to show that only specific units of $E(\mathfrak{f})$ have non-empty intersection with $\pi^{-1}\mathcal{D}$.

Lemma 2.1.3. *Let V be a finite index subgroup of $E(\mathfrak{f})$ and let $\epsilon_1, \ldots, \epsilon_{n-1}$ be a $\mathbb{Z}$-basis for V. Further, let $\mathcal{D}$ be a Shintani domain for V and $\pi^{-1} \in \mathcal{D}$, then for $\epsilon = \prod_i \epsilon_i^{m_i}$*

$$\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D} = \emptyset$$

unless $m_i \in \{0, 1\}$, $1 \le i \le n-1$.

Proof. This follows because $\mathcal{D}$ is a fundamental domain which has the form:

$$\sum_{\sigma \in S_{n-1}} w_\sigma C_{e_n}(v_{1,\sigma}, \ldots, v_{n,\sigma})$$

where $v_{1,\sigma}, \ldots, v_{n,\sigma}$ are given in Section 1.1. As $\pi^{-1} \in \mathcal{D}$, then we have $\pi^{-1} v_{i,\sigma} \in v_{i,\sigma}\mathcal{D}$ and $v_{i,\sigma} = \prod_{j=1}^{i-1} \epsilon_{\sigma(j)}$. So unless $m_i \in \{0,1\}$, we must have

$$\epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D} = \emptyset.$$

□

We now show that if Conjecture 1.2.13 is true for a finite index subgroup V, then it is true for $E(\mathfrak{f})$. As a consequence, we obtain as a corollary that if Conjecture 1.2.13 is true for a finite index subgroup V, then it is true for $E(\mathfrak{f})$

Proposition 2.1.4. *If Conjecture 1.2.13 is true for a subgroup V of $E(\mathfrak{f})$, then it is true for $E(\mathfrak{f})$.*

Specifically, if we have $\vartheta'_V \in H_n(V \oplus \langle \pi \rangle, \mathbb{Z})$ and $\vartheta' \in H_n(E(\mathfrak{f})_{\mathfrak{p}}, \mathbb{Z})$, $\omega \in H^{n-1}(E(\mathfrak{f}), \mathrm{Meas}(F_{\mathfrak{p}}, K))$, and $\omega_V \in H^{n-1}(V, \mathrm{Meas}(F_{\mathfrak{p}}, K))$. Then if

$$c_g \cap (\omega_V \cap \vartheta'_V) = g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}_V))$$

then

$$c_g \cap (\omega \cap \vartheta') = g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})).$$

Here, $\mathcal{D}_V$ is a Shintani domain for V and $\mathcal{D}$ is a Shintani domain for $E(\mathfrak{f})$

Proof. We mimic the proof of [2], Theorem 1.5 given by Charollois, Dasgupta, and Greenberg.

We have the following diagram:

$$\begin{array}{ccccc} H^{n-1}(V, \mathrm{Meas}(F_{\mathfrak{p}}, K)) & \times & H_n(V \oplus \langle \pi \rangle, \mathbb{Z}) & \xrightarrow{\cap} & H_1(E(\mathfrak{f}) \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, K)) \\ \mathrm{res}\uparrow & & \downarrow \mathrm{cores} & & \downarrow \mathrm{cores} \\ H^{n-1}(E(\mathfrak{f}), \mathrm{Meas}(F_{\mathfrak{p}}, K)) & \times & H_n(E(\mathfrak{f}) \oplus \langle \pi \rangle, \mathbb{Z}) & \xrightarrow{\cap} & H_1(E(\mathfrak{f}) \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, K)). \end{array}$$

We also have the commutative diagram:

$$\begin{array}{ccccc} H^1(F_{\mathfrak{p}}^{\times}, C_c(F_{\mathfrak{p}}, K)) & \times & H_1(V \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, K)) & \xrightarrow{\cap} & K \\ \uparrow{\scriptstyle \mathrm{res}} & & \downarrow{\scriptstyle \mathrm{cores}} & & \downarrow \\ H^1(E(\mathfrak{f}), C_c(F_{\mathfrak{p}}, K)) & \times & H_1(E(\mathfrak{f}) \oplus \langle \pi \rangle, \mathrm{Meas}(F_{\mathfrak{p}}, K)) & \xrightarrow{\cap} & K. \end{array}$$

These diagrams are commutative thanks to [1], pp. 112-114. Further, thanks to [1], pp. 112-114 and [1], Section 3, Prop. 9.5, we have the following identities:

$$\mathrm{cores}(\vartheta'_V) = [E(\mathfrak{f}) : V]\vartheta' \tag{2.2}$$

$$\mathrm{res}(\omega) = \omega_V. \tag{2.3}$$

As $\mathrm{cores}(\vartheta'_V) = [E(\mathfrak{f}) : V]\vartheta'_U$ and $\mathrm{res}(\omega) = \omega_V$, then we have

$$c_g \cap (\omega_V \cap \vartheta'_V) = [E(\mathfrak{f}) : V]c_g \cap (\omega \cap \vartheta').$$

So, we must show that

$$g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}_V)) = [E(\mathfrak{f}) : V]g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}))$$

or, alternatively, we may show the stronger equality

$$\mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D}_V) = \mathcal{U}_{\mathfrak{p}}(\mathfrak{b}, \lambda, \mathcal{D})^{[E(\mathfrak{f}):V]}.$$

By a result of Colmez in Section 2 of [3], we have $[E(\mathfrak{f}) : V]\zeta_\lambda(\mathfrak{b}, \mathcal{D}, U, s) = \zeta_\lambda(\mathfrak{b}, \mathcal{D}_V, U, s)$. In terms of our measure $[E(\mathfrak{f}) : V]\nu(\mathfrak{b}, \mathcal{D}, U) = \nu(\mathfrak{b}, \mathcal{D}_V, U)$ where $\nu(\mathfrak{b}, \mathcal{D}, U) = \zeta_\lambda(\mathfrak{b}, \mathcal{D}, U, 0)$ and $\nu(\mathfrak{b}, \mathcal{D}_V, U) = \zeta_\lambda(\mathfrak{b}, \mathcal{D}_V, U, 0)$.

This immediately implies that

$$\pi^{[E(\mathfrak{f}):V]\nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_{\mathfrak{p}})} = \pi^{\nu(\mathfrak{b}, \mathcal{D}_V, \mathcal{O}_{\mathfrak{p}})}$$

and

$$\left(\times\!\!\!\!\!\!\int_{\mathbf{O}_{\mathfrak{p}}} x\,d\nu(\mathfrak{b},\mathcal{D},x)\right)^{[E(\mathfrak{f}):V]} = \times\!\!\!\!\!\!\int_{\mathbf{O}_{\mathfrak{p}}} x\,d\nu(\mathfrak{b},\mathcal{D}_V,x).$$

To finish we must show that

$$\epsilon(\mathfrak{b},\mathcal{D},\pi)^{[E(\mathfrak{f}):V]} = \epsilon(\mathfrak{b},\mathcal{D}_V,\pi)$$

or

$$\prod_{\epsilon\in E(\mathfrak{f})} \epsilon^{[E(\mathfrak{f}):V]\nu(\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D})(\mathcal{O}_{\mathfrak{p}})} = \prod_{\epsilon\in V} \epsilon^{\nu(\epsilon\mathcal{D}_V\cap\pi^{-1}\mathcal{D}_V)(\mathcal{O}_{\mathfrak{p}})} \tag{2.4}$$

To that end, we first let $\epsilon_1,\ldots,\epsilon_{n-1}$ be a $\mathbb{Z}$-basis for $E(\mathfrak{f})$ and let V be a finite subgroup of $E(\mathfrak{f})$ with

$$E(\mathfrak{f})/V \cong \mathbb{Z}/b_1\mathbb{Z}\times\cdots\times\mathbb{Z}/b_{n-1}\mathbb{Z}.$$

We take a $\mathbb{Z}$-basis of V to be $\epsilon_1^{b_1},\ldots,\epsilon_{n-1}^{b_{n-1}}$. If $\mathcal{D}$ is a Shintani domain for $E(\mathfrak{f})$, then we take a Shintani domain for V given by

$$\mathcal{D}_V = \bigcup_{j_i} \epsilon_1^{j_1}\cdots\epsilon_{n-1}^{j_{n-1}}\mathcal{D},$$

where the union and product are taken over $0\le j_i\le b_i-1$ for $1\le i\le n-1$.

We calculate $\epsilon(\mathfrak{b},\mathcal{D}_V,\pi)$ and to that end, we have

$$\begin{aligned}
\epsilon(\mathfrak{b},\mathcal{D}_V,\pi) &= \prod_{\epsilon\in V} \epsilon^{\nu(\mathfrak{b},\epsilon\mathcal{D}_V\cap\pi^{-1}\mathcal{D}_V,\mathcal{O}_{\mathfrak{p}})} && (2.5)\\
&= \prod_{\mathbf{k}\in\mathbb{Z}^{n-1}} \left(\epsilon_1^{b_1k_1}\cdots\epsilon_{n-1}^{b_{n-1}k_{n-1}}\right)^{\nu(\mathfrak{b},\epsilon_1^{b_1k_1}\cdots\epsilon_{n-1}^{b_{n-1}k_{n-1}}\mathcal{D}_V\cap\pi^{-1}\mathcal{D}_V,\mathcal{O}_{\mathfrak{p}})} && (2.6)\\
&= \prod_{i=1}^{n-1}\prod_{\substack{k_j\in\{0,1\},j\ne i\\ k_i=1}} \epsilon_i^{b_i\nu(\mathfrak{b},\epsilon_1^{b_1k_1}\cdots\epsilon_{n-1}^{b_{n-1}k_{n-1}}\mathcal{D}_V\cap\pi^{-1}\mathcal{D}_V,\mathcal{O}_{\mathfrak{p}})}. && (2.7)
\end{aligned}$$

The last equality follows from the calculation used to derive Equation 2.19 in Conjecture 2.2.1.

On the other hand, we calculate $\epsilon(\mathfrak{b}, \mathcal{D}, \pi)$. We have

$$\begin{aligned}
\epsilon(\mathfrak{b}, \mathcal{D}, \pi) &= \prod_{\epsilon \in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b}, \epsilon\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} && (2.8) \\
&= \prod_{\mathbf{k} \in \mathbb{Z}^{n-1}} \left(\epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}}\right)^{\nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})} && (2.9) \\
&= \prod_{i=1}^{n-1} \prod_{\substack{k_j \in \{0,1\} \\ k_i = 1}} \epsilon_i^{\nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}})}. && (2.10)
\end{aligned}$$

Again, the last equality follows from the calculation used to derive Equation 2.19 in Conjecture 2.2.1.

Combining Equations 2.4, 2.7, and 2.10, we must show the following equality:

$$\begin{aligned}
& b_i \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \nu(\mathfrak{b}, \epsilon_1^{b_1 k_1} \cdots \epsilon_{n-1}^{b_{n-1} k_{n-1}} \mathcal{D}_V \cap \pi^{-1}\mathcal{D}_V, \mathcal{O}_{\mathfrak{p}}) && (2.11) \\
&= b_1 \cdots b_{n-1} \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_{\mathfrak{p}}) && (2.12)
\end{aligned}$$

We note that $\nu(\mathfrak{b}, \epsilon C, \mathcal{O}_{\mathfrak{p}}) = \nu(\mathfrak{b}, C, \mathcal{O}_{\mathfrak{p}})$, where C is a Shintani cone. This is due to the fact that $\mathrm{N}\,\epsilon = 1$.

We reduce the Equation 2.11. We have

$$b_i \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \nu(\mathfrak{b}, \epsilon_1^{b_1 k_1} \cdots \epsilon_{n-1}^{b_{n-1} k_{n-1}} \mathcal{D}_V \cap \pi^{-1} \mathcal{D}_V, \mathcal{O}_\mathfrak{p}) \tag{2.13}$$

$$= b_i \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \sum_{\substack{a_j = 0 \\ j \neq i}}^{m_j - 1} \nu(\mathfrak{b}, \epsilon_1^{a_1 + k_1} \cdots \epsilon_{n-1}^{a_{n-1} + k_{n-1}} \mathcal{D} \cap \pi^{-1} \epsilon_1^{a_1} \cdots \epsilon_{n-1}^{a_{n-1}} \mathcal{D}, \mathcal{O}_\mathfrak{p}) \tag{2.14}$$

$$= b_i \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \sum_{\substack{a_j = 0 \\ j \neq i}}^{m_j - 1} \nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}, \mathcal{O}_\mathfrak{p}) \tag{2.15}$$

$$= b_i \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \left(\prod_{j \neq i} b_j \right) \nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}, \mathcal{O}_\mathfrak{p}) \tag{2.16}$$

$$= b_1 \cdots b_{n-1} \sum_{\substack{k_j \in \{0,1\}, j \neq i \\ k_i = 1}} \nu(\mathfrak{b}, \epsilon_1^{k_1} \cdots \epsilon_{n-1}^{k_{n-1}} \mathcal{D} \cap \pi^{-1} \mathcal{D}, \mathcal{O}_\mathfrak{p}). \tag{2.17}$$

Thus, we see that we have equality and so the proposition holds. □

Corollary 2.1.5. *If Conjecture 1.2.13 is true for a Colmez subgroup V of $E(\mathfrak{f})$, then it is true for $E(\mathfrak{f})$.*

2.2 Explicit Calculation of $c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta')$

We calculate $c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta')$ explicitly. We do not make the assumption that we are working with a Colmez subgroup.

The explicit calculation of $c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta')$ is given below:

$$\begin{aligned}
c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta') &= c_g \cap \left(\omega_{\mathfrak{b},\lambda} \cap \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)[\theta_{\sigma(1)} \mid \cdots \mid \theta_{\sigma(n)}]\right) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) c_g \cap (\omega_{\mathfrak{b},\lambda} \cap [\theta_{\sigma(1)} \mid \cdots \mid \theta_{\sigma(n)}]) \\
&= \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \int_{F_\mathfrak{p}} c_g(\theta_{\sigma(1)}) d(\theta_{\sigma(1)} \nu_{\mathfrak{b},\lambda}([\theta_{\sigma(2)} \mid \cdots \mid \theta_{\sigma(n)}])) \\
&= -\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} c_g(\pi) d(\pi \nu_{\mathfrak{b},\lambda}([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}])) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} c_g(\epsilon_k) d(\epsilon_k \nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])) \\
&= -\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} \pi^{-1} c_g(\pi) d(\nu_{\mathfrak{b},\lambda}([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}])) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} \epsilon_k^{-1} c_g(\epsilon_k) d(\nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}]),
\end{aligned}$$

where the θ_i^k for $1 \le i \le n-1$ and $2 \le k \le n$ are given by $\theta_1^k = \pi$ for $2 \le k \le n$ and

$$\theta_i^k = \begin{cases} \epsilon_{i+1} & \text{for } i \ge k \\ \epsilon_i & \text{for } i < k. \end{cases}$$

The first three equalities follow from straightforward cap product calculations. The fourth equality follows, because we are splitting up the summation into the case where $\theta_{\sigma(1)} = \pi$ and the case where $\theta_{\sigma(1)} \neq \pi$. The fifth equality follows from the fact that the action of $E(\mathfrak{f})_\mathfrak{p}$ on a measure is given by $(x \cdot \mu)(U) = \mu(x^{-1}U)$ and the calculation:

$$\begin{aligned}
\int_{F_\mathfrak{p}} f(x)\, d(\theta\nu(x)) &= \lim_{\mathcal{V}} \sum_{V \in \mathcal{V}} f(x_V)\mu(\theta^{-1}V) = \lim_{\mathcal{V}} \sum_{V \in \mathcal{V}} f(\theta x_V)\mu(V) \\
&= \lim_{\mathcal{V}} \sum_{V \in \mathcal{V}} \theta^{-1} f(x_V)\mu(V) = \int_{F_\mathfrak{p}} \theta^{-1} f(x)\, d\mu(x),
\end{aligned}$$

where the limit is taken over covers of $F_{\mathfrak{p}}$ ordered by refinement.

We now calculate

$$\int_{F_{\mathfrak{p}}} \pi^{-1} c_g(\pi) d(\omega([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}]))$$

and

$$\int_{F_{\mathfrak{p}}} \epsilon_k^{-1} c_g(\epsilon_k) d(\omega([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}]).$$

To that end, we calculate $\pi^{-1}c_g(\pi)$ and $\epsilon_k^{-1}c_g(\epsilon_k)$ by calculating it for $z_{f,g}$, where $f = 1_{\pi\mathcal{O}_{\mathfrak{p}}}$. For $\pi^{-1}c_g(\pi)$, we have

$$\begin{aligned} \pi^{-1}c_g(\pi) &= \pi^{-1}(\pi(1_{\pi\mathcal{O}_{\mathfrak{p}}})) \cdot g(\pi) + \pi^{-1}(1_{\pi\mathcal{O}_{\mathfrak{p}}} - 1_{\pi^2\mathcal{O}_{\mathfrak{p}}}) \cdot g \\ &= 1_{\pi\mathcal{O}_{\mathfrak{p}}} \cdot g(\pi) + (1_{\mathcal{O}_{\mathfrak{p}}} - 1_{\pi\mathcal{O}_{\mathfrak{p}}}) \cdot g \\ &= 1_{\pi\mathcal{O}_{\mathfrak{p}}} \cdot g(\pi) + 1_{\mathbf{O}_{\mathfrak{p}}} \cdot g \end{aligned}$$

For $\epsilon_k^{-1}c_g(\epsilon_k)$, we have

$$\begin{aligned} \epsilon_k^{-1}c_g(\epsilon_k) &= \epsilon_k^{-1}(\epsilon_k 1_{\pi\mathcal{O}_{\mathfrak{p}}}) \cdot g(\epsilon_k) + \epsilon_k^{-1}(1_{\pi\mathcal{O}_{\mathfrak{p}}} - 1_{\epsilon_k\pi\mathcal{O}_{\mathfrak{p}}}) \cdot g \\ &= 1_{\pi\mathcal{O}_{\mathfrak{p}}} \cdot g(\epsilon_k) + \epsilon_k^{-1}(1_{\pi\mathcal{O}_{\mathfrak{p}}} - 1_{\pi\mathcal{O}_{\mathfrak{p}}}) \cdot g \\ &= 1_{\pi\mathcal{O}_{\mathfrak{p}}} \cdot g(\epsilon_k). \end{aligned}$$

Inputting these back into our cap product calculations gives us

$$\begin{aligned}
c_g \cap (\omega_{\mathfrak{b},\lambda} \cap \vartheta') &= -\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} \pi^{-1} c_g(\pi) d(\nu_{\mathfrak{b},\lambda}([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}])) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} \epsilon_k^{-1} c_g(\epsilon_k) d(\nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}]) \\
&= -\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} 1_{\pi\mathcal{O}_\mathfrak{p}}(x) g(\pi) \\
&\quad + 1_{\mathrm{O}_\mathfrak{p}}(x) g(x) d(\nu_{\mathfrak{b},\lambda}([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}]))(x) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \int_{F_\mathfrak{p}} 1_{\mathcal{O}_\mathfrak{p}}(x) g(\epsilon_k) d(\nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])(x) \\
&= -\int_{F_\mathfrak{p}} 1_{\pi\mathcal{O}_\mathfrak{p}}(x) g(\pi) + 1_{\mathrm{O}_\mathfrak{p}}(x) g(x)\, d\nu(\mathfrak{b}, \mathcal{D}, x) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) g(\epsilon_k) \nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])(\mathcal{O}_\mathfrak{p}) \\
&= -\nu(\mathfrak{b}, \mathcal{D}, \pi\mathcal{O}_\mathfrak{p}) g(\pi) - g\left(\times\!\!\!\!\!\!\int_{\mathrm{O}_\mathfrak{p}} x \, d\nu(\mathfrak{b}, \mathcal{D}, x) \right) \\
&\quad -\sum_{k=1}^{n-1} \sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) g(\epsilon_k) \nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])(\mathcal{O}_\mathfrak{p}).
\end{aligned}$$

The second equality follows, because we just input $\pi^{-1} c_g(\pi)$ and $\epsilon_k^{-1} c_g(\epsilon_k)$. The third equality follows from the fact that the Shintani domain for $E(\mathfrak{f})$ is

$$\sum_{\tau \in S_{n-1}} \operatorname{sgn}(\tau) \operatorname{sign} \det([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}]) c_{e_n}([\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}])$$

and plugging $-\sum_{\tau \in S_{n-1}} [\epsilon_{\tau(1)} \mid \cdots \mid \epsilon_{\tau(n-1)}]$ into $\nu_{\mathfrak{b},\lambda}$ gives us the first integral. The second integral comes from the elementary fact that $\int 1_U \, d\mu = \mu(U)$, where 1_U is the characteristic function of U. Finally, the fourth equality follows from the previous fact about $\int 1_U \, d\mu$ and the fact that $g : F_\mathfrak{p}^\times \to K$ is a continuous homomorphism.

On the other hand, we calculate $g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D}))$. This is

$$\begin{aligned} g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D})) &= g\left(\epsilon(\mathfrak{b},\mathcal{D},\pi)\cdot \pi^{\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\cdot \times\!\!\!\!\!\!\int_{\mathbf{O}\mathfrak{p}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \\ &= g(\epsilon(\mathfrak{b},\mathcal{D},\pi)) + g(\pi^{\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})}) + g\left(\times\!\!\!\!\!\!\int_{\mathbf{O}\mathfrak{p}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \\ &= g(\epsilon(\mathfrak{b},\mathcal{D},\pi)) + \nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})g(\pi) + g\left(\times\!\!\!\!\!\!\int_{\mathbf{O}\mathfrak{p}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \end{aligned}$$

We see that modulo a sign $c_g \cap (\omega_{\mathfrak{b},\lambda}\cap\vartheta') = g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D}))$ if

$$g(\epsilon(\mathfrak{b},\mathcal{D},\pi)) = \sum_{k=1}^{n-1}\sum_{\tau\in S_{n-1}} \operatorname{sgn}(\tau) g(\epsilon_k)\nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])(\mathcal{O}_{\mathfrak{p}}).$$

Further, we have

$$\begin{aligned} g(\epsilon(\mathfrak{b},\mathcal{D},\pi)) &= g\left(\prod_{\epsilon\in E(\mathfrak{f})} \epsilon^{\nu(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\right) \\ &= \sum_{\epsilon\in E(\mathfrak{f})} \nu(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})g(\epsilon) \\ &= \sum_{(k_1,\ldots,k_{n-1})\in\mathbb{Z}^{n-1}} \nu(\mathfrak{b},\epsilon_1^{k_1}\cdots\epsilon_{n-1}^{k_{n-1}}\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})g(\epsilon_1^{k_1}\cdots\epsilon_{n-1}^{k_{n-1}}) \\ &= \sum_{i=1}^{n-1} g(\epsilon_i) \sum_{\substack{m_j\in\{0,1\}, j\neq i \\ m_i=1}} \nu(\mathfrak{b},\epsilon_1^{m_1}\cdots\epsilon_{n-1}^{m_{n-1}}\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}}). \end{aligned}$$

This last equality is because of Proposition 2.1.3.

This leads us to the following conjecture that we will prove in the $n = 2$ and $n = 3$ case, but not for the general case.

Conjecture 2.2.1. *The following equality holds:*

$$\sum_{k=1}^{n-1}\sum_{\tau\in S_{n-1}} \operatorname{sgn}(\tau) g(\epsilon_k)\nu_{\mathfrak{b},\lambda}([\theta^k_{\tau(1)} \mid \cdots \mid \theta^k_{\tau(n-1)}])(\mathcal{O}_{\mathfrak{p}}) \tag{2.18}$$

$$= \sum_{i=1}^{n-1} g(\epsilon_i) \sum_{\substack{m_j\in\{0,1\}, j\neq i \\ m_i=1}} \nu(\mathfrak{b},\epsilon_1^{m_1}\cdots\epsilon_{n-1}^{m_{n-1}}\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}}). \tag{2.19}$$

Chapter 3

The $n = 2$ and $n = 3$ case

3.1 The $n = 2$ case

This particular case has already been proven by Dasgupta and Spiess. We follow their proof, but fill in some details. In this section, we prove it for $E(\mathfrak{f})$ and we do not make any assumptions that ϵ is in a finite index subgroup. We also let $\tau_1 : F \to \mathbb{R}$ and $\tau_2 : F \to \mathbb{R}$ be an ordering of embeddings of F into $\mathbb{R}$.

In the $n = 2$ case, we choose ϵ such that $\operatorname{sgn}(1, \epsilon) = 1$ (i.e. an ϵ such that $\epsilon^{\tau_2} > \epsilon^{\tau_1}$). Then $\mathcal{D} = C(1, \epsilon) \cup C(\epsilon)$.

We choose π such that $\pi \in \mathcal{D}$. We can do this as $\mathcal{D}$ is a fundamental domain for the action of ϵ and since $(\pi) = (\epsilon^n \pi)$. Thus we have

$$\vartheta' = [\pi \mid \epsilon] - [\epsilon \mid \pi].$$

As $\mathfrak{b}, \mathfrak{f}$, and λ are fixed, we will set $\omega = \omega_{\mathfrak{f},\mathfrak{b},\lambda}$. Calculating explicitly gives us

$$\begin{aligned}
c_g \cap (\omega_{\mathfrak{f},\mathfrak{b},\lambda} \cap \vartheta') &= c_g \cap (\omega \cap \vartheta') \\
&= c_g \cap (\omega \cap [\pi \mid \epsilon]) - c_g \cap (\omega \cap [\epsilon \mid \pi]) \\
&= c_g \cap ([\pi \otimes \pi\omega(1,\epsilon)) - c_g \cap ([\epsilon] \otimes \omega(1,\pi)) \\
&= \int_{F_\mathfrak{p}} c_g(\pi)(x)\, d(\pi\omega(1,\epsilon))(x) - \int_{F_\mathfrak{p}} c_g(\epsilon)(x)\, d(\pi\omega(1,\pi))(x) \\
&= \int_{F_\mathfrak{p}} \pi^{-1} c_g(\pi)(x)\, d\omega(1,\epsilon)(x) - \int_{F_\mathfrak{p}} \epsilon^{-1} c_g(\epsilon)(x)\, d\omega(1,\pi)(x).
\end{aligned}$$

To finish the calculation, we must first compute

$$\pi^{-1} c_g(\pi)(x) \quad \text{and} \quad \epsilon^{-1} c_g(\epsilon)(x).$$

For $\pi^{-1} c_g(\pi)(x)$, we have

$$\begin{aligned}
\pi^{-1} c_g(\pi)(x) &= \pi^{-1}((\pi \cdot 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(\pi) + (1_{\pi\mathcal{O}_\mathfrak{p}}(x) - \pi \cdot 1_{\pi\mathcal{O}_\mathfrak{p}})g(x)) \\
&= 1_{\pi\mathcal{O}_\mathfrak{p}}(x)g(\pi) + (1_{\mathcal{O}_\mathfrak{p}}(x) - 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(\pi x) \\
&= 1_{\pi\mathcal{O}_\mathfrak{p}}(x)g(\pi) + (1_{\mathcal{O}_\mathfrak{p}}(x) - 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(\pi) + (1_{\mathcal{O}_\mathfrak{p}}(x) - 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(x) \\
&= 1_{\mathcal{O}_\mathfrak{p}}(x)g(\pi) + (1_{\mathcal{O}_\mathfrak{p}}(x) - 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(x) \\
&= 1_{\mathcal{O}_\mathfrak{p}}(x)g(\pi) + (1_{\mathbf{O}_\mathfrak{p}}(x))g(x),
\end{aligned}$$

where $\mathbf{O}_\mathfrak{p} = \mathcal{O}_\mathfrak{p} - \pi\mathcal{O}_\mathfrak{p}$.

For $\epsilon^{-1} c_g(\epsilon)(x)$, we have

$$\begin{aligned}
\epsilon^{-1} c_g(\epsilon)(x) &= \epsilon^{-1}(\epsilon \cdot 1_{\pi\mathcal{O}_\mathfrak{p}}(x)g(\epsilon) + (1_{\pi\mathcal{O}_\mathfrak{p}} - \epsilon 1_{\pi\mathcal{O}_\mathfrak{p}}(x))g(x)) \\
&= 1_{\pi\mathcal{O}_\mathfrak{p}}(x)g(\epsilon).
\end{aligned}$$

So,

$$
\begin{aligned}
& \int_{F_\mathfrak{p}} \pi^{-1} c_g(\pi)(x)\, d\omega(1,\epsilon)(x) \\
- & \int_{F_\mathfrak{p}} \epsilon^{-1} c_g(\epsilon)(x)\, d\omega(1,\pi)(x) \\
= & \int_{F_\mathfrak{p}} 1_{\mathcal{O}_\mathfrak{p}}(x) g(\pi) + (1_{\mathbb{O}_\mathfrak{p}}(x)) g(x)\, d\omega(1,\epsilon)(x) \\
- & \int_{F_\mathfrak{p}} 1_{\pi\mathcal{O}_\mathfrak{p}}(x) g(\epsilon)\, d\omega(1,\pi)(x) \\
= & \; g(\pi) \operatorname{sign}\det(1,\epsilon) \nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_\mathfrak{p}) + \operatorname{sign}\det(1,\epsilon) g\left(\times\!\!\!\!\!\!\int_{\mathbb{O}_\mathfrak{p}} x\, d\nu(\mathfrak{b}, \mathcal{D}, x) \right) \\
+ & \; g(\epsilon) \operatorname{sign}\det(1,\pi) \nu(\mathfrak{b}, \mathcal{D}, \pi\mathcal{O}_\mathfrak{p}) \\
= & \; g(\pi) \nu(\mathfrak{b}, \mathcal{D}, \mathcal{O}_\mathfrak{p}) + g\left(\times\!\!\!\!\!\!\int_{\mathbb{O}_\mathfrak{p}} x\, d\nu(\mathfrak{b}, \mathcal{D}, x) \right) + g(\epsilon) \nu(\mathfrak{b}, \mathcal{D}, \pi\mathcal{O}_\mathfrak{p}).
\end{aligned}
$$

The above equalities are justified, because

$$\epsilon^{\tau_2} > \epsilon^{\tau_1} \quad \text{and} \quad \pi^{\tau_2} > \pi^{\tau_1},$$

so $\operatorname{sign}(1,\epsilon) = \operatorname{sign}(1,\pi) = 1$. We also have

$$\omega_{\mathfrak{f},\mathfrak{b},\lambda}(1,\epsilon) = [\nu_{\mathfrak{b},\lambda}(1,\epsilon)]$$

So, we have

$$
\begin{aligned}
\nu(\mathfrak{b}, C^*(1,\epsilon), U) &= \operatorname{sgn}(1,\epsilon) \zeta_\lambda(C^*(1,\epsilon), \mathfrak{b}, U, 0) \\
&= \zeta_\lambda(\mathcal{D}, \mathfrak{b}, U, 0) \\
&= \nu(\mathfrak{b}, \mathcal{D}, U).
\end{aligned}
$$

Now, we calculate

$$\begin{aligned}
g(\mathcal{U}_{\mathfrak{p}}(\mathfrak{b},\lambda,\mathcal{D})) &= g\left(\epsilon_{\mathfrak{b},\lambda,\mathcal{D},\pi}\cdot\pi^{\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\cdot \times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \\
&= g\left(\epsilon_{\mathfrak{b},\lambda,\mathcal{D},\pi}\right) + g\left(\pi^{\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\right) + g\left(\times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \\
&= g\left(\prod_{\epsilon\in E(\mathfrak{f})}\epsilon^{\nu(\mathfrak{b},\epsilon\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})}\right) + g(\pi)\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}}) \\
&+ g\left(\times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right) \\
&= g(\epsilon)\left(\sum_{n\in\mathbb{Z}}\nu(\mathfrak{b},\epsilon^n\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}})\right) + g(\pi)\nu(\mathfrak{b},\mathcal{D},\mathcal{O}_{\mathfrak{p}}) \\
&+ g\left(\times\!\!\!\!\!\!\int_{\mathbb{O}_{\mathfrak{p}}} x\, d\nu(\mathfrak{b},\mathcal{D},x)\right).
\end{aligned}$$

To finish up, we want to show that

$$\sum_{n\in\mathbb{Z}} n\nu(\mathfrak{b},\epsilon^n\mathcal{D}\cap\pi^{-1}\mathcal{D},\mathcal{O}_{\mathfrak{p}}) = -\nu_{\mathfrak{b},\lambda}(1,\pi)(\pi\mathcal{O}_{\mathfrak{p}}).$$

To that end, we must calculate

$$\epsilon^n\mathcal{D}\cap\pi^{-1}\mathcal{D}.$$

We have

$$\epsilon^n\mathcal{D} = \epsilon^n C^*(1,\epsilon) = C^*(\epsilon^n,\epsilon^{n+1})$$

and

$$\pi^{-1}\mathcal{D} = \pi^{-1}C^*(1,\epsilon) = C^*(\pi^{-1},\pi^{-1}\epsilon).$$

We claim that we have the following equality:

$$C^*(\epsilon^n, \epsilon^{n+1}) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = \begin{cases} C^*(1, \pi^{-1}\epsilon) & n = 0 \\ C^*(1, \pi^{-1}) & n = -1 \\ \emptyset & \text{otherwise.} \end{cases}$$

First, we need a couple lemmas:

Lemma 3.1.1. *If* $x_1 \neq x_2$ *and* $x_2^{\tau_2}/x_2^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$*, then*

$$\alpha \in C^*(x_1, x_2) = C(x_1, x_2) \cup C(x_2)$$

if and only if

$$\frac{x_2^{\tau_2}}{x_2^{\tau_1}} \geq \frac{\alpha^{\tau_2}}{\alpha^{\tau_1}} > \frac{x_1^{\tau_2}}{x_1^{\tau_1}}$$

Proof. Suppose that $\alpha \in C^*(x_1, x_2)$, then $\alpha = a_1 x_1 + a_2 x_2$, with $a_1 \geq 0$ and $a_2 > 0$ and $\alpha^{\tau_2} = a_1 x_1^{\tau_2} + a_2 x_2^{\tau_2}$. We show $x_2^{\tau_2}/x_2^{\tau_1} \geq \alpha^{\tau_2}/\alpha^{\tau_1}$. We have $x_2^{\tau_2}/x_2^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$ and $a_1 \geq 0$ and so $a_1 x_1^{\tau_1} x_2^{\tau_2} \geq a_1 x_1^{\tau_2} x_2^{\tau_1}$. We add $a_2 x_2^{\tau_1} x_2^{\tau_2}$ to both sides to get

$$a_1 x_1^{\tau_1} x_2^{\tau_2} + a_2 x_2^{\tau_1} x_2^{\tau_2} \geq a_1 x_1^{\tau_2} x_2^{\tau_1} + a_2 x_2^{\tau_1} x_2^{\tau_2}$$

which is

$$x_2^{\tau_2}(a_1 x_1^{\tau_1} + a_2 x_2^{\tau_1}) \geq x_2^{\tau_1}(a_1 x_1^{\tau_2} + a_2 x_2^{\tau_2})$$

or

$$x_2^{\tau_2} \alpha^{\tau_1} \geq \alpha^{\tau_2} x_2^{\tau_1}$$

which gives us the inequality $x_2^{\tau_2}/x_2^{\tau_1} \geq \alpha^{\tau_2}/\alpha^{\tau_1}$.

For $\alpha^{\tau_2}/\alpha^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$, from $x_2^{\tau_2}/x_2^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$ and $a_2 > 0$ we have

$$a_2 x_2^{\tau_2} x_1^{\tau_1} > a_2 x_2^{\tau_1} x_1^{\tau_2}$$

and adding $a_1 x_1^{\tau_2} x_1^{\tau_1}$ to both sides, we get

$$a_1 x_1^{\tau_2} x_1^{\tau_1} + a_2 x_2^{\tau_2} x_1^{\tau_1} > a_1 x_1^{\tau_2} x_1^{\tau_1} + a_2 x_2^{\tau_1} x_1^{\tau_2}$$

and so

$$x_1^{\tau_1}(a_1 x_1^{\tau_2} + a_2 x_2^{\tau_2}) > x_1^{\tau_2}(a_1 x_1^{\tau_1} + a_2 x_2^{\tau_1})$$

thus $x_1^{\tau_1}\alpha^{\tau_2} > x_1^{\tau_2}\alpha^{\tau_1}$, which gives us the inequality $\alpha^{\tau_2}/\alpha^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$.

Now, suppose that $x_2^{\tau_2}/x_2^{\tau_1} \geq \alpha^{\tau_2}/\alpha^{\tau_1} > x_1^{\tau_2}/x_1^{\tau_1}$. As x_1 and x_2 are not collinear by assumption, they form a basis of $\mathbb{R}^2$. So, $\alpha = a_1 x_1 + a_2 x_2$, which means we wish to show $a_1 \geq 0$ and $a_2 > 0$. This is a result of a similar calculation. □

By assumption $\pi \in \mathcal{D}$ and so $\epsilon^{\tau_2}/\epsilon^{\tau_1} \geq \pi^{\tau_2}/\pi^{\tau_1} > 1$. Thus, we see that $1 > (\pi^{-1})^{\tau_2}/(\pi^{-1})^{\tau_1} \geq (\epsilon^{-1})^{\tau_2}/(\epsilon^{-1})^{\tau_1}$. Thus, $\pi^{-1} \in C^*(1, \epsilon^{-1}) = \epsilon^{-1}\mathcal{D}$. This also shows that $\pi^{-1}\epsilon \in \mathcal{D}$.

We now want to show that

- $$C^*(1, \epsilon) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = C^*(1, \pi^{-1}\epsilon);$$

- $$C^*(\epsilon^{-1}, 1) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = C^*(1, \pi^{-1}\epsilon);$$

- $$C^*(\epsilon^n, \epsilon^{n+1}) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = \emptyset$$

  for $n \neq 0, -1$.

For the first one, $C^*(1, \pi^{-1}\epsilon) \subset C^*(1, \epsilon)$ and $C^*(1, \pi^{-1}\epsilon) \subset C^*(\pi^{-1}, \pi^{-1}\epsilon)$, because for the first one, $\pi^{-1}\epsilon \in \mathcal{D}$ and for the second one $1 \in C^*(\pi^{-1}, \pi^{-1}\epsilon)$ (using the lemma and $(\pi^{-1}\epsilon)^{\tau_2}/(\pi^{-1}\epsilon)^{\tau_1} \geq 1 > (\pi^{-1})^{\tau_2}/(\pi^{-1})^{\tau_1}$).

We use the lemma to prove the inverse inclusion. If $\alpha \in C^*(1, \epsilon) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon)$, then we have the two inequalities:

$$\begin{aligned} \epsilon^{\tau_2}/\epsilon^{\tau_1} &\geq \alpha^{\tau_2}/\alpha^{\tau_1} > 1 \\ (\pi^{-1}\epsilon)^{\tau_2}/(\pi^{-1}\epsilon)^{\tau_1} &\geq \alpha^{\tau_2}/\alpha^{\tau_1} > (\pi^{-1})^{\tau_2}/(\pi^{-1})^{\tau_1}. \end{aligned}$$

These two inequalities become the inequality

$$(\pi^{-1}\epsilon)^{\tau_2}/(\pi^{-1}\epsilon)^{\tau_1} \geq \alpha^{\tau_2}/\alpha^{\tau_1} > 1$$

and so $\alpha \in C^*(1, \pi^{-1}\epsilon)$.

We now want to show $C^*(1, \epsilon^{-1}) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = C^*(\pi^{-1}, 1)$. We have $C^*(\pi^{-1}, 1) \subset C^*(\pi^{-1}, \pi^{-1}\epsilon)$ as $1 \in C^*(\pi^{-1}, \pi^{-1}\epsilon)$, by using the lemma. We also have $C^*(\pi^{-1}, 1) \subset C^*(\epsilon^{-1}, 1)$, because $\pi^{-1} \in C^*(\epsilon^{-1}, 1)$.

We now show that $C^*(\epsilon^{-1}, 1) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) \subset C^*(\pi^{-1}, 1)$. If $\alpha \in C^*(\epsilon^{-1}, 1) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon)$, then we have the inequalities $1 \geq \alpha^{\tau_2}/\alpha^{\tau_1} > (\epsilon^{-1})^{\tau_2}/(\epsilon^{-1})^{\tau_1}$ and $(\pi^{-1}\epsilon)^{\tau_2}/(\pi^{-1}\epsilon)^{\tau_1} \geq \alpha^{\tau_2}/\alpha^{\tau_1} > (\pi^{-1})^{\tau_2}/(\pi^{-1})^{\tau_1}$. These inequalities reduce to

$$1 \geq \frac{\alpha^{\tau_2}}{\alpha^{\tau_1}} > \frac{(\pi^{-1})^{\tau_2}}{(\pi^{-1})^{\tau_1}},$$

thus $\alpha \in C^*(\pi^{-1}, 1)$.

Finally, $C^*(\epsilon^n, \epsilon^{n+1}) \cap C^*(\pi^{-1}, \pi^{-1}\epsilon) = \emptyset$ for $n \neq 0, -1$, because $\mathcal{D}$ is a fundamental domain and $\pi^{-1} \in \epsilon^{-1}\mathcal{D}$ and $\pi^{-1}\epsilon \in \mathcal{D}$.

We see that

$$\prod_{n \in \mathbb{Z}} \epsilon^{n\nu(\mathfrak{b}, \epsilon^n \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})} = \epsilon^{\sum_{n\in\mathbb{Z}} n\nu(\mathfrak{b}, \epsilon^n \mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})} = \epsilon^{-\nu(\mathfrak{b}, \epsilon^{-1}\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})}.$$

But, we have

$$\begin{aligned} -\nu(\mathfrak{b}, \epsilon^{-1}\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p}) &= -\nu(\mathfrak{b}, C^*(\pi^{-1}, 1), \mathcal{O}_\mathfrak{p}) \\ &= -\nu_{\mathfrak{b},\lambda}(1, \pi^{-1})(\mathcal{O}_\mathfrak{p}). \end{aligned}$$

By the 1-cocycle condition for ω, we have

$$0 = \nu(1, 1) = \nu(1, \pi\pi^{-1}) = \pi^{-1}\nu(1, \pi) + \nu(1, \pi^{-1}).$$

Thus $-\nu(1, \pi^{-1})(\mathcal{O}_\mathfrak{p}) = \pi^{-1}\nu(1, \pi)(\mathcal{O}_\mathfrak{p}) = \nu(1, \pi)(\pi\mathcal{O}_\mathfrak{p})$.

Putting this all together we see that

$$g(\epsilon_{\mathfrak{b},\lambda,\mathcal{D},\pi}) = g(\epsilon^{-\nu(\mathfrak{b}, \epsilon^{-1}\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p})}) = -g(\epsilon)\nu(\mathfrak{b}, \epsilon^{-1}\mathcal{D} \cap \pi^{-1}\mathcal{D}, \mathcal{O}_\mathfrak{p}) = \nu(1, \pi)(\pi\mathcal{O}_\mathfrak{p})g(\epsilon).$$

So, this gives us the desired equality:

$$c_g \cap (\omega_{\mathfrak{f},\mathfrak{b}} \cap \vartheta') = g(\mathcal{U}_\mathfrak{p}(\mathfrak{b}, \lambda, \mathcal{D})).$$

3.2 $n = 3$ case

We prove the case when $n = 3$. In this case, we prove a lemma that states that Conjecture 2.2.1 is true in the case $n = 3$. The validity of this conjecture then implies that Conjecture 1.2.13 is true in the case $n = 3$. To that end, we must show that the $\omega_{\mathfrak{b},\lambda}(x_1, \ldots, x_n)(\mathcal{O}_{\mathfrak{p}})$ are equal for various specific values of $x_1, \ldots, x_n$. This will be done by showing that certain Shintani cones are equivalent modulo the action of a unit. As the measures are invariant under the action of a unit, this means that the measures are equal.

Theorem 3.2.1. *Conjecture 1.2.13 is true when $n = 3$.*

Proof. The validity of this statement depends on whether or not Conjecture 2.2.1 is true for $n = 3$. This will be show in Lemma 3.2.2. □

Lemma 3.2.2. *Conjecture 2.2.1 is true when $n = 3$.*

Proof. By Corollary 2.1.5 we may suppose that we are working with a Colmez subgroup. Further, from the calculations in Section 2.2, we must show that the following equations are true:

$$\begin{aligned} & \sum_{n,m\in\mathbb{Z}} n\nu_{\mathfrak{b},\lambda,\epsilon_1^n\epsilon_2^m\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) \\ = \quad & -\operatorname{sgn}(1,\pi,\pi\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(1,\pi,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(1,\epsilon_2,\epsilon_2\pi)\nu_{\mathfrak{b},\lambda,C^*(1,\epsilon_2,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

and

$$\begin{aligned} & \sum_{n,m\in\mathbb{Z}} m\nu_{\mathfrak{b},\lambda,\epsilon_1^n\epsilon_2^m\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) \\ = \quad & \operatorname{sgn}(1,\pi,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(1,\pi,\pi\epsilon_1)}(\pi\mathcal{O}_{\mathfrak{p}}) - \operatorname{sgn}(1,\epsilon_1,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(1,\epsilon_1,\pi\epsilon_1)}(\pi\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

then we are finished for the $n = 3$ case.

Due to the action of π on the measure and the fact that the sgn function is invariant if we multiply any column by a positive number, we have

$$\begin{aligned} & -\operatorname{sgn}(1,\pi,\pi\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(1,\pi,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(1,\epsilon_2,\epsilon_2\pi)\nu_{\mathfrak{b},\lambda,C^*(1,\epsilon_2,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) \\ = & -\operatorname{sgn}(\pi^{-1},1,\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

and

$$\begin{aligned} & -\operatorname{sgn}(1,\pi,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(1,\pi,\pi\epsilon_1)}(\pi\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(1,\epsilon_1,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(1,\epsilon_1,\pi\epsilon_1)}(\pi\mathcal{O}_{\mathfrak{p}}) \\ = & \operatorname{sgn}(\pi^{-1},1,\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}) - \operatorname{sgn}(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

We suppose that $\pi^{-1} \in \mathcal{D}$. We also note in this case that $\epsilon_1^n\epsilon_2^m\mathcal{D} \cap \pi^{-1}\mathcal{D} = \emptyset$ unless $n = 0, 1$ and $m = 0, 1$ by Lemma 2.1.3. That means that we must prove

$$\begin{aligned} & \nu_{\mathfrak{b},\lambda,\epsilon_1\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) + \nu_{\mathfrak{b},\lambda,\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) \\ = & -\operatorname{sgn}(1,\pi,\pi\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}) \\ + & \operatorname{sgn}(1,\epsilon_2,\epsilon_2\pi)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}) \end{aligned}$$

and

$$\begin{aligned} & \nu_{\mathfrak{b},\lambda,\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) + \nu_{\mathfrak{b},\lambda,\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) \\ = & \operatorname{sgn}(1,\pi,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}) \\ - & \operatorname{sgn}(1,\epsilon_1,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}). \end{aligned}$$

Further, these two equalities are true if we can show that:

$$1_{\epsilon_1\mathcal{D}\cap\pi^{-1}\mathcal{D}} + 1_{\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}$$

and

$$1_{C^*(\pi^{-1},1,\epsilon_2)} + 1_{C^*(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)}$$

are equivalent modulo the action of a unit. We must also show that

$$1_{\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}} + 1_{\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}$$

$$1_{C^*(\pi^{-1},1,\epsilon_1)} + 1_{C^*(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)}$$

are equivalent modulo the action of a unit. This is because $\omega_{\mathfrak{b},\lambda}$ is invariant under the action of $E(\mathfrak{f})$. Additionally, this is because we are working with a Colmez subgroup and the fact that $\pi^{-1} \in \mathcal{D}$, thus

$$-\operatorname{sgn}(\pi^{-1}, 1, \epsilon_2) = \operatorname{sgn}(\pi^{-1}, \pi^{-1}\epsilon_2, \epsilon_2) = \operatorname{sgn}(\pi^{-1}, 1, \epsilon_1) = -\operatorname{sgn}(\pi^{-1}, \pi^{-1}\epsilon_1, \epsilon_1) = 1. \tag{3.1}$$

Determining this means that we have to determine what the generating cones of $\epsilon_1\mathcal{D} \cap \pi^{-1}\mathcal{D}$, $\epsilon_2\mathcal{D} \cap \pi^{-1}\mathcal{D}$, and $\epsilon_1\epsilon_2\mathcal{D} \cap \pi^{-1}\mathcal{D}$ are. To find these generating cones, we need to look at the intersection of two planes. This intersection will define a line and we can take the positive ray to define our Shintani cone. We have to look at the intersection of the planes:

- the plane defined by π^{-1} and $\pi^{-1}\epsilon_1$ and the plane defined by ϵ_1 and $\epsilon_1\epsilon_2$;
- the plane defined by $\epsilon_1\epsilon_2$ and $\epsilon_1^2\epsilon_2$ and the plane defined by $\pi^{-1}\epsilon_1$ and $\pi^{-1}\epsilon_1\epsilon_2$;
- the plane defined by $\pi^{-1}\epsilon_2$ and $\pi^{-1}\epsilon_1\epsilon_2$ and the plane defined by $\epsilon_1\epsilon_2$ and $\epsilon_1\epsilon_2^2$;
- the plane defined by π^{-1} and $\pi^{-1}\epsilon_2$ and the plane defined by ϵ_2 and $\epsilon_1\epsilon_2$;

- the plane defined by $\pi^{-1}\epsilon_1^{-1}$ and π^{-1} and the plane defined by 1 and ϵ_2;
- the plane defined by $\pi^{-1}\epsilon_2^{-1}$ and π^{-1} and the plane defined by 1 and ϵ_1.

If $\mathbf{a}, \mathbf{b}, \mathbf{c}$, and $\mathbf{d}$ are vectors in $\mathbb{R}^3$ and q, r, s, and t are real scalars, then we can find the line defined by the intersection of the planes $\mathbf{a}q + \mathbf{b}r = \mathbf{c}s + \mathbf{d}t$ by reducing the following matrix:

$$\begin{pmatrix} \mathbf{a} & \mathbf{b} & -\mathbf{c} & -\mathbf{d} \end{pmatrix}.$$

If $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$ are linearly independent, then the matrix reduces to

$$\begin{pmatrix} 1 & 0 & 0 & \frac{\det(-\mathbf{d},\mathbf{b},-\mathbf{c})}{\det(\mathbf{a},\mathbf{b},-\mathbf{c})} \\ 0 & 1 & 0 & -\frac{\det(\mathbf{a},-\mathbf{d},-\mathbf{c})}{\det(\mathbf{a},\mathbf{b},-\mathbf{c})} \\ 0 & 0 & 1 & \frac{\det(\mathbf{a},\mathbf{b},-\mathbf{d})}{\det(\mathbf{a},\mathbf{b},-\mathbf{c})} \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & 0 & 0 & -\frac{\det(\mathbf{b},\mathbf{c},\mathbf{d})}{\det(\mathbf{a},\mathbf{b},\mathbf{c})} \\ 0 & 1 & 0 & -\frac{\det(\mathbf{a},\mathbf{c},\mathbf{d})}{\det(\mathbf{a},\mathbf{b},\mathbf{c})} \\ 0 & 0 & 1 & \frac{\det(\mathbf{a},\mathbf{b},\mathbf{d})}{\det(\mathbf{a},\mathbf{b},\mathbf{c})} \end{pmatrix}.$$

So, the basis of the line defined by the intersection of the planes is given by $\det(\mathbf{b}, \mathbf{c}, \mathbf{d})\mathbf{a} + \det(\mathbf{a}, \mathbf{c}, \mathbf{d})\mathbf{b} = -\det(\mathbf{a}, \mathbf{b}, \mathbf{d})\mathbf{c} + \det(\mathbf{a}, \mathbf{b}, \mathbf{c})\mathbf{d}$.

With this result, we can determine the line that intersects the planes and hence a decomposition of the Shintani cone. So, with this, we get the following elements

- we have $\mathbf{a} = \pi^{-1}, \mathbf{b} = \pi^{-1}\epsilon_1, \mathbf{c} = \epsilon_1$, and $\mathbf{d} = \epsilon_1\epsilon_2$, so

$$\begin{aligned} \gamma_1 &= \det(\pi^{-1}\epsilon_1, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1} + \det(\pi^{-1}, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 \\ &= -\det(\pi^{-1}, \pi^{-1}\epsilon_1, \epsilon_1\epsilon_2)\epsilon_1 + \det(\pi^{-1}, \pi^{-1}\epsilon_1, \epsilon_1)\epsilon_1\epsilon_2; \end{aligned}$$

- we have $\mathbf{a} = \pi^{-1}\epsilon_1, \mathbf{b} = \pi^{-1}\epsilon_1\epsilon_2, \mathbf{c} = \epsilon_1\epsilon_2$, and $\mathbf{d} = \epsilon_1^2\epsilon_2$, so

$$\begin{aligned} \gamma_2 &= \det(\pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1}\epsilon_1, \epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\ &= -\det(\pi^{-1}\epsilon_1, \pi^{-1}\epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2)\epsilon_1\epsilon_2 + \det(\pi^{-1}\epsilon_1, \pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2)\epsilon_1^2\epsilon_2; \end{aligned}$$

- we have $\mathbf{a} = \pi^{-1}\epsilon_2, \mathbf{b} = \pi^{-1}\epsilon_1\epsilon_2, \mathbf{c} = \epsilon_1\epsilon_2$, and $\mathbf{d} = \epsilon_1\epsilon_2^2$, so

$$\begin{aligned}\gamma_3 &= \det(\pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_1\epsilon_2 \\ &= -\det(\pi^{-1}\epsilon_2, \pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\epsilon_1\epsilon_2 + \det(\pi^{-1}\epsilon_2, \pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2)\epsilon_1\epsilon_2^2;\end{aligned}$$

- we have $\mathbf{a} = \pi^{-1}, \mathbf{b} = \pi^{-1}\epsilon_2, \mathbf{c} = \epsilon_2$, and $\mathbf{d} = \epsilon_1\epsilon_2$, so

$$\begin{aligned}\gamma_4 &= \det(\pi^{-1}\epsilon_2, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1} + \det(\pi^{-1}, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_2 \\ &= -\det(\pi^{-1}, \pi^{-1}\epsilon_2, \epsilon_1\epsilon_2)\epsilon_2 + \det(\pi^{-1}, \pi^{-1}\epsilon_2, \epsilon_2)\epsilon_1\epsilon_2;\end{aligned}$$

- we have $\mathbf{a} = \pi^{-1}\epsilon_1^{-1}, \mathbf{b} = \pi^{-1}, \mathbf{c} = 1$, and $\mathbf{d} = \epsilon_2$, so

$$\begin{aligned}\gamma_5 &= \det(\pi^{-1}, 1, \epsilon_2)\pi^{-1}\epsilon_1^{-1} + \det(\pi^{-1}\epsilon_1^{-1}, 1, \epsilon_2)\pi^{-1} \\ &= -\det(\pi^{-1}\epsilon_1^{-1}, \pi^{-1}, \epsilon_2)1 + \det(\pi^{-1}\epsilon_1^{-1}, \pi^{-1}, 1)\epsilon_2;\end{aligned}$$

- we have $\mathbf{a} = \pi^{-1}\epsilon_2^{-1}, \mathbf{b} = \pi^{-1}, \mathbf{c} = 1$, and $\mathbf{d} = \epsilon_1$, so

$$\begin{aligned}\gamma_6 &= \det(\pi^{-1}, 1, \epsilon_1)\pi^{-1}\epsilon_2^{-1} + \det(\pi^{-1}\epsilon_2^{-1}, 1, \epsilon_1)\pi^{-1} \\ &= -\det(\pi^{-1}\epsilon_2^{-1}, \pi^{-1}, \epsilon_1)1 + \det(\pi^{-1}\epsilon_2^{-1}, \pi^{-1}, 1)\epsilon_1.\end{aligned}$$

Further, we note that these are the correct constants, because all of the various determinants are positive, due to Equation 3.1 and the fact that we are working with a Colmez subgroup. Thus all of the elements are in $\mathbb{R}^n_{>0}$.

We have the decomposition of the Shintani domains $(\epsilon_1\mathcal{D} \cap \pi^{-1}\mathcal{D}) \cup (\epsilon_1\epsilon_2\mathcal{D} \cap \pi^{-1}\mathcal{D})$ and $(\epsilon_2\mathcal{D} \cap \pi^{-1}\mathcal{D}) \cup (\epsilon_1\epsilon_2\mathcal{D} \cap \pi^{-1}\mathcal{D})$ given by:

- $1_{\epsilon_1\mathcal{D}\cap\pi^{-1}\mathcal{D}} = 1_{C(\gamma_1, \pi^{-1}\epsilon_1, \gamma_2)} + 1_{C(\gamma_1, \gamma_2, \epsilon_1\epsilon_2)}$;

- $1_{\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}} = 1_{C(\gamma_4,\pi^{-1}\epsilon_2,\gamma_3)} + 1_{C(\gamma_4,\epsilon_1\epsilon_2,\gamma_3)}$;

- $1_{\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}} = 1_{C(\epsilon_1\epsilon_2,\gamma_2,\pi^{-1}\epsilon_1\epsilon_2)} + 1_{C(\epsilon_1\epsilon_2,\gamma_3,\pi^{-1}\epsilon_1\epsilon_2)}$.

We have two sets of Shintani domains who have the following characteristic functions:

- $1_{C(\pi^{-1},1,\epsilon_2)} + 1_{C(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)}$ and

- $1_{C(\pi^{-1},1,\epsilon_1)} + 1_{C(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)}$.

These Shintani domains decompose into:

- $1_{C(1,\pi^{-1},\gamma_6)} + 1_{C(\pi^{-1},\gamma_6,\epsilon_1)} + 1_{C(\pi^{-1},\epsilon_2,\gamma_1)} + 1_{C(\epsilon_1,\gamma_1,\pi^{-1}\epsilon_1)}$, and

- $1_{C(1,\pi^{-1},\gamma_5)} + 1_{C(\pi^{-1},\gamma_5,\epsilon_2} + 1_{C(\pi^{-1},\gamma_4,\epsilon_2)} + 1_{C(\gamma_4,\epsilon_2,\pi^{-1}\epsilon_2)}$,

respectively.

We want to show that $(\epsilon_1\epsilon_2)\cdot 1_{C(1,\pi^{-1},\gamma_6)} = 1_{C(\epsilon_1\epsilon_2,\pi^{-1}\epsilon_1\epsilon_2,\gamma_2)}$, which amounts to showing that $\epsilon_1\epsilon_2\gamma_6 = \gamma_2$. We note that as $\operatorname{Nm}\epsilon = 1$, where ϵ is a totally positive unit, then we have $\det(x_1,x_2,x_3) = \det(\epsilon x_1,\epsilon x_2,\epsilon x_3)$. We have

$$\begin{aligned}
\epsilon_1\epsilon_2\gamma_6 &= \det(\pi^{-1},1,\epsilon_1)\pi^{-1}\epsilon_1 + \det(\pi^{-1}\epsilon_2^{-1},1,\epsilon_1)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_1\epsilon_2,\epsilon_1\epsilon_2,\epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1}\epsilon_1,\epsilon_1\epsilon_2,\epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_2,
\end{aligned}$$

as desired.

Next, we want to show that $\epsilon_2 \cdot 1_{C(\epsilon_1,\gamma_1,\pi^{-1}\epsilon_1)} = 1_{C(\epsilon_1\epsilon_2,\gamma_3,\pi^{-1}\epsilon_1\epsilon_2)}$. This means that we want to show that $\epsilon_2\gamma_1 = \gamma_3$. We have

$$\begin{aligned}
\epsilon_2\gamma_1 &= \det(\pi^{-1}\epsilon_1, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_3,
\end{aligned}$$

as desired.

We also wish to show that $\epsilon_2 \cdot 1_{C(\gamma_6,\pi^{-1},\gamma_1)} = 1_{C(\gamma_4,\pi^{-1}\epsilon_2,\gamma_3)}$ and $\epsilon_2 \cdot 1_{C(\gamma_6,\epsilon_1,\gamma_1)} = 1_{C(\gamma_4,\epsilon_1\epsilon_2,\gamma_3)}$. Thus, we would like to show that $\epsilon_2\gamma_6 = \gamma_4$ and $\epsilon_2\gamma_1 = \gamma_3$. For $\epsilon_2\gamma_6 = \gamma_4$, we have the following calculation:

$$\begin{aligned}
\epsilon_2\gamma_6 &= \det(\pi^{-1}, 1, \epsilon_1)\pi^{-1} + \det(\pi^{-1}\epsilon_2^{-1}, 1, \epsilon_1)\pi^{-1}\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_2, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1} + \det(\pi^{-1}, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_2 \\
&= \gamma_4,
\end{aligned}$$

as desired. For $\epsilon_2\gamma_1 = \gamma_3$, we have the following calculation:

$$\begin{aligned}
\epsilon_2\gamma_1 &= \det(\pi^{-1}\epsilon_1, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_3,
\end{aligned}$$

as desired.

This means that

$$\nu_{\mathfrak{b},\lambda,\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) + \nu_{\mathfrak{b},\lambda,\epsilon_1\epsilon_2\mathcal{D}\cap\pi^{-1}\mathcal{D}}(\mathcal{O}_{\mathfrak{p}}) =$$

$$-\operatorname{sgn}(1,\pi,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}) +$$

$$\operatorname{sgn}(1,\epsilon_1,\pi\epsilon_1)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_1,\epsilon_1)}(\mathcal{O}_{\mathfrak{p}}).$$

Now, we show other decomposition. To that end, we want to show that $(\epsilon_1\epsilon_2)\cdot 1_{C(1,\gamma_5,\pi^{-1})} = 1_{C(\epsilon_1\epsilon_2,\gamma_3,\pi^{-1}\epsilon_1\epsilon_2)}$. It suffices to show that $\epsilon_1\epsilon_2\gamma_5 = \gamma_3$. We have

$$\begin{aligned}
\epsilon_1\epsilon_2\gamma_5 &= \det(\pi^{-1},1,\epsilon_2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}\epsilon_1^{-1},1,\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_1\epsilon_2,\epsilon_1\epsilon_2,\epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_2 + \det(\pi^{-1}\epsilon_2,\epsilon_1\epsilon_2,\epsilon_1\epsilon_2^2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_3,
\end{aligned}$$

as desired.

Now, we want to show that $\epsilon_1\cdot 1_{C(\epsilon_2,\gamma_4,\pi^{-1}\epsilon_2)} = 1_{C(\epsilon_1\epsilon_2,\gamma_2,\pi^{-1}\epsilon_1\epsilon_2)}$. It suffices to show that $\epsilon_1\gamma_4 = \gamma_2$. We have

$$\begin{aligned}
\epsilon_1\gamma_4 &= \det(\pi^{-1}\epsilon_2,\epsilon_2,\epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1},\epsilon_2,\epsilon_1\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= -\det(\epsilon_1^2\epsilon_2,\epsilon_1\epsilon_2,\pi^{-1}\epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1}\epsilon_1,\epsilon_1\epsilon_2,\epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_2,
\end{aligned}$$

as desired.

We wish to show that $\epsilon_1\cdot 1_{C(\gamma_5,\pi^{-1},\gamma_4)} = 1_{C(\gamma_1,\pi^{-1},\gamma_2)}$ and $\epsilon_1\cdot 1_{C(\gamma_5,\epsilon_2,\gamma_4)} = 1_{C(\gamma_1,\epsilon_1\epsilon_2,\gamma_2)}$. This means that we need to show that $\epsilon_1\gamma_5 = \gamma_1$ and $\epsilon_1\gamma_4 = \gamma_2$. For

$\epsilon_1\gamma_5 = \gamma_1$, we have the following calculation:

$$\begin{aligned}
\epsilon_1\gamma_5 &= \det(\pi^{-1}, 1, \epsilon_2)\pi^{-1} + \det(\pi^{-1}\epsilon_1^{-1}, 1, \epsilon_2)\pi^{-1}\epsilon_1 \\
&= \det(\pi^{-1}\epsilon_1, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1} + \det(\pi^{-1}, \epsilon_1, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 \\
&= \gamma_1,
\end{aligned}$$

as desired. For $\epsilon_1\gamma_4 = \gamma_2$, we have the following calculation:

$$\begin{aligned}
\epsilon_1\gamma_4 &= \det(\pi^{-1}\epsilon_2, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1}, \epsilon_2, \epsilon_1\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\pi^{-1}\epsilon_1\epsilon_2, \epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1 + \det(\pi^{-1}\epsilon_1, \epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \det(\epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2, \pi^{-1}\epsilon_1\epsilon_2)\pi^{-1}\epsilon_1 + \det(\epsilon_1\epsilon_2, \epsilon_1^2\epsilon_2, \pi^{-1}\epsilon_1)\pi^{-1}\epsilon_1\epsilon_2 \\
&= \gamma_2,
\end{aligned}$$

as desired.

All this put together means that we have

$$-\operatorname{sgn}(1, \pi, \pi\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(1,\pi,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(1, \epsilon_2, \epsilon_2\pi)\nu_{\mathfrak{b},\lambda,C^*(1,\epsilon_2,\pi\epsilon_2)}(\pi\mathcal{O}_{\mathfrak{p}}) =$$

$$-\operatorname{sgn}(1, \pi, \pi\epsilon_2)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},1,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}) + \operatorname{sgn}(1, \epsilon_2, \epsilon_2\pi)\nu_{\mathfrak{b},\lambda,C^*(\pi^{-1},\pi^{-1}\epsilon_2,\epsilon_2)}(\mathcal{O}_{\mathfrak{p}}).$$

Therefore, we have shown the $n = 3$ case. □

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