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Representation Stability of Consistent Sequences of GL(n, Fq)-Modules

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Representation Stability of Consistent Sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-modules

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I am grateful to my advisor, without whose help, I would not have been here.

To my family for all their support.

ABSTRACT OF THE DISSERTATION

Representation Stability of Consistent Sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-modules

by

John Patrick Watterlond

Doctor of Philosophy, Graduate Program in Mathematics
University of California, Riverside, December 2016
Dr. Wee Liang Gan, Chairperson

We define a notion of representation stability for consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-modules. We prove that the consistent sequence obtained from a finitely generated VI-module or a finitely generated VIC-module is representation stable.

Contents

Introduction.

In this paper we study consistent sequences of representations of $\mathrm{GL}(n, \mathbf{F}_q)$, the general linear groups over the field $\mathbf{F}_q$ with q elements. We introduce a notion of representation stability for consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations and show that finitely generated VI/VIC-modules produce consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations that are representation stable. These results, and their consequences, are the $\mathrm{GL}(n, \mathbf{F}_q)$-analog of the work on FI-modules and consistent sequences of representations of the symmetric groups S_n, developed by Church, Ellenberg, and Farb in [1] which we now discuss.

Suppose we have an increasing chain of groups

$$G_0 \subset G_1 \subset G_2 \subset \cdots$$

and a sequence

$$V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} \cdots$$

where each V_n is a representation of G_n and each φ_n is a linear map. We denote this sequence by $(V_n, \varphi_n)_{n=0}^{\infty}$.

We say the sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence of G_n-representations if for every non-negative integer n and every $g \in G_n$ the following diagram commutes

$$\begin{array}{ccc} V_n & \xrightarrow{\varphi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle g} \\ V_n & \xrightarrow[\varphi_n]{} & V_{n+1} \end{array}$$

(where g acts on V_{n+1} by considering it as an element of G_{n+1}). i.e. if for every n, $\varphi_n : V_n \to \mathrm{Res}_{G_n}^{G_{n+1}}(V_{n+1})$ is a morphism of G_n-representations.

Representation stability for a consistent sequence of G_n-representations is a set of conditions (injectivity, surjectivity, multiplicity) concerning the asymptotic behavior of the G_n-representations. A consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ of G_n-representations satisfies the injectivity and surjectivity conditions of representation stability if there exists a non-negative integer N such that for all $n \geq N$:

1. Injectivity: the map $\varphi_n : V_n \to V_{n+1}$ is injective.

2. Surjectivity: The span of the G_{n+1}-orbit of $\varphi_n(V_n)$ is all of V_{n+1}.

To define the multiplicity condition, more information is needed about the chain

$$G_0 \subset G_1 \subset G_2 \subset \cdots .$$

In particular, we need a way to get an irreducible G_{n+1}-representation from an irreducible G_n-representation.

For the family of symmetric groups S_n, Church and Farb (see [3]) start with a consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ of S_n-representations and decompose each V_n into irreducibles as

$$V_n = \bigoplus_{|\lambda| \leq n} c_{\lambda,n} \, \mathrm{L}(\lambda[n])$$

where λ is a partition of size at most n, $\lambda[n]$ is the padded partition of n corresponding to λ, and $\mathrm{L}(\lambda[n])$ is the irreducible S_n-representation corresponding to $\lambda[n]$. Now a consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ of S_n-representations satisfies the multiplicity condition of representation stability if there exists a non-negative integer N such that for all $n \geq N$, in the decomposition

$$V_n = \bigoplus_{|\lambda| \leq n} c_{\lambda,n} \, \mathrm{L}(\lambda[n]),$$

the multiplicities $c_{\lambda,n}$ are independent of n, for all λ.

As a consequence we have that if a consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ of S_n-representations satisfies multiplicity condition of representation stability and each V_n is finite dimensional, then the dimension of V_n is eventually polynomial in n.

In [1], Church, Ellenberg, and Farb used certain FI-modules to produce consistent sequences of S_n-representations that are representation stable. An FI-module is a functor $\mathrm{FI} \to \mathbf{k}\text{-mod}$, where FI is the category with finite sets as objects and injections as morphisms. If V is an FI-module then one can produce a consistent sequence, $(V_n, \varphi_n)_{n=0}^{\infty}$, of S_n-representations by taking $V_n = V(\{1, 2, \ldots, n\})$ and taking $\varphi_n : V_n \to V_{n+1}$ to be the image of the standard inclusion $\{1, 2, \ldots, n\} \hookrightarrow \{1, 2, \ldots, n+1\}$. Church, Ellenberg, and Farb showed that a consistent sequence of S_n-representations coming from a finitely generated FI-module is representation stable. To prove this result, they use a clever argument with branching rules and invariants/coinvariants.

One consequence of this result is another proof of Murnaghan's Theorem: For any pair of partitions λ and μ there exists a finite set S of partitions ν and a set of non-negative integers $g^{\nu}_{\lambda,\mu}$ such that for all sufficiently large n

$$\mathrm{L}(\lambda[n]) \otimes \mathrm{L}(\mu[n]) = \bigoplus_{\nu \in S} g^{\nu}_{\lambda,\mu}\, \mathrm{L}(\nu[n]).$$

In this paper we introduce a notion of representation stability for consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations. As the discussion above indicates, the task will be to define a multiplicity condition of representation stability. We choose a definition analogous to the one for consistent sequences of S_n-representations. With this definition we show that if a consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ of $\mathrm{GL}(n, \mathbf{F}_q)$-representations satisfies multiplicity stability and each V_n is finite dimensional, then the dimension of V_n is eventually polynomial in q^n (q is the number of elements in $\mathbf{F}_q$). We use VI- and VIC-modules to produce consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations. Similar to Church, Ellenberg, and Farb, we show that finitely generated VI/VIC-modules produce consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations which are representation stable.

A VI-module is a functor VI $\to$ $\mathbf{k}$-mod where VI is the category with finite dimensional vector spaces over $\mathbf{F}_q$ as objects and linear injections as morphisms. A VIC-module is a functor VIC $\to$ $\mathbf{k}$-mod where VIC is the category with finite dimensional vector spaces over $\mathbf{F}_q$ as objects and whose morphisms are pairs consisting of a linear injection and a subspace complement to the image of that injection.

Our proof that a finitely generated VI/VIC-module produces consistent sequence of $\mathrm{GL}(n, \mathbf{F}_q)$-representations which is representation stable follows an argument similar to Church, Ellenberg, and Farb using branching rules and invariants/coinvariants. One notable difference with our argument is that we do not prove any results for *when* stabilization occurs, whereas Church, Ellenberg, and Farb do.

Chapter Zero contains all of the preliminary material of our paper. We introduce notation and review the representation theory of symmetric groups and the finite general linear groups.

In Chapter One we give our definition of representation stability for consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations. We introduce the notions of weak-stability and weight for a consistent sequence of $\mathrm{GL}(n, \mathbf{F}_q)$-representations. We prove that a consistent sequence of $\mathrm{GL}(n, \mathbf{F}_q)$-representations which has finite weight and is weakly-stable satisfies the injectivity and multiplicity conditions of representation stability.

In Chapters Two and Three we prove that finitely generated VI/VIC-modules produce consistent sequences of $\mathrm{GL}(n, \mathbf{F}_q)$-representations that have finite weight and are weakly-stable. Moreover we prove that such consistent sequences also satisfy the surjectivity condition of representation stability. We also prove a $\mathrm{GL}(n, \mathbf{F}_q)$-analog of Murnaghan's Theorem.

Chapter 0

Preliminaries.

This chapter introduces most of the notation and definitions we will use. Almost all of the terminology for categories and representations is standard and is included for completeness. Of particular importance is the parameterization of irreducible representations of the general linear groups over finite fields. This parameterization will be used in the following Chapter to define multiplicity stability for consistent sequences of representations of these groups.

Starting Notation.

We let $\mathbf{k}$ stand for an algebraically closed field with characteristic zero. We let $\mathbf{Z}$ stand for the ring of integers and $\mathbf{Z}_{\geq 0}$ the integers which are greater than or equal to zero. The category of $\mathbf{k}$-modules is denoted by $\mathbf{k}$-mod. We denote the symmetric groups by S_n. We use $\mathbf{F}_q$ to denote the finite field with q elements. We let $G_n = \mathrm{GL}(n, \mathbf{F}_q)$, the group of invertible $n \times n$ matrices with entries in $\mathbf{F}_q$. For a group G, we let $\mathbf{k}G$ denote the free

$\mathbf{k}$-module with the elements of G as a $\mathbf{k}$-basis and with multiplication on the basis elements by group multiplication.

Categories.

Let $\mathscr{A}$ be a category. Let A and A' be objects in $\mathscr{A}$. We let $\mathrm{Hom}_{\mathscr{A}}(A, A')$ denote the morphisms in $\mathscr{A}$ from A to A'.

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Let $V : \mathscr{A} \to \mathscr{B}$ be a functor. For each object A of $\mathscr{A}$ there is an object $V(A)$ of $\mathscr{B}$. For each morphism $f : A \to A'$ between objects in $\mathscr{A}$ there is a morphism $V(f) : V(A) \to V(A')$ between objects in $\mathscr{B}$. Let $W : \mathscr{A} \to \mathscr{B}$ be another functor. A natural transformation of functors, $\alpha : V \to W$, is a collection of morphisms $\alpha_A : V(A) \to W(A)$ for each object A of $\mathscr{A}$ with the property that: for every morphism $f : A \to A'$ between objects in $\mathscr{A}$ the diagram

$$\begin{array}{ccc} V(A) & \xrightarrow{V(f)} & V(A') \\ \downarrow{\scriptstyle \alpha_A} & & \downarrow{\scriptstyle \alpha_{A'}} \\ W(A) & \xrightarrow[W(f)]{} & W(A') \end{array}$$

commutes.

Let $\mathscr{S}$ be a small category. We denote the category of functors

$$\mathscr{S} \longrightarrow \mathbf{k}\text{-mod}$$

by $\mathbf{k}\mathscr{S}$-Mod. The objects of $\mathbf{k}\mathscr{S}$-Mod are functors $\mathscr{S} \to \mathbf{k}\mathscr{S}$-Mod. The morphisms of $\mathbf{k}\mathscr{S}$-Mod are natural transformations of functors. Objects of $\mathbf{k}\mathscr{S}$-Mod are called $\mathbf{k}\mathscr{S}$-Modules.

The notions submodule, kernel, image, and direct sum for $\mathbf{k}\mathscr{S}$-Modules are defined object-wise. For example, let U and V be $\mathbf{k}\mathscr{S}$-Modules. Suppose the following two conditions hold:

1. For all objects S of $\mathscr{S}$, $U(S)$ is a $\mathbf{k}$-submodule of $V(S)$.

2. For all morphisms $f : S \to S'$ of between objects in $\mathscr{S}$, the $\mathbf{k}$-module morphism $V(f) : V(S) \to V(S')$ restricts to the $\mathbf{k}$-module morphism $U(f) : U(S) \to U(S')$.

Then U is a $\mathbf{k}\mathscr{S}$-submodule of V.

We have the following useful fact:

Fact 1. *A sequence $0 \to U \to V \to W \to 0$ of $\mathbf{k}\mathscr{S}$-Modules is exact if and only if for every object S of $\mathscr{S}$ the sequence of $\mathbf{k}$-modules*

$$0 \to U(S) \to V(S) \to W(S) \to 0$$

is exact.

Let V be a $\mathbf{k}\mathscr{S}$-Module. Let $\Sigma \subset \sqcup_{S \in \mathscr{S}} V(S)$. We define $\text{span}_V(\Sigma)$ to be the minimal $\mathbf{k}\mathscr{S}$-submodule of V containing each element of Σ. If $\text{span}_V(\Sigma) = V$ and Σ is a finite set then we say V is finitely generated.

Representations of Finite Groups Over k.

In this section we recall standard notation and definitions from the representation theory of finite groups over an algebraically closed field of characteristic zero, most of which is found in Fulton and Harris [6].

Let G be a group and let V be a $\mathbf{k}$-module. We say that V is a representation of G over $\mathbf{k}$ if V is a $\mathbf{k}G$-module.

Let G be a group and let V and W be two $\mathbf{k}G$-modules. A $\mathbf{k}$-module morphism $\varphi : V \longrightarrow W$ is a morphism of $\mathbf{k}G$-modules if $g.\varphi(v) = \varphi(g.v)$ for all $v \in V$ and $g \in G$.

Let V be a $\mathbf{k}G$-module and $W \subset V$. We say that W is a subrepresentation of V if W is a $\mathbf{k}G$-submodule of V. If V is a non-zero $\mathbf{k}G$-module and has no proper non-trivial $\mathbf{k}G$-submodules, then we say that V is an irreducible representation of G over $\mathbf{k}$. We let $\mathrm{Irr}_{\mathbf{k}}(G)$ denote a set of representatives of isomorphism classes of finite dimensional irreducible representations of G over $\mathbf{k}$.

If V and V' are $\mathbf{k}G$-modules, then the $\mathbf{k}$-module $V \oplus V'$ is a $\mathbf{k}G$-module with action $g.(v, v') := (g.v, g.v')$ for all $g \in G$, $v \in V$ and $v' \in V'$. Any finite dimensional representation of G is a direct sum of irreducible representations of G. Therefore any finite dimensional representation V of G can be decomposed into a direct sum of irreducible representations:

$$V = \bigoplus_{L \in \mathrm{Irr}_{\mathbf{k}}(G)} c_L L$$

where for each $L \in \mathrm{Irr}_{\mathbf{k}}(G)$, c_L is the multiplicity of L in G. In other words, c_L is how many times L appears as a direct summand of V. If $c_L \neq 0$ then we say that L is a constituent of V.

Let V and V' be $\mathbf{k}G$-modules. The $\mathbf{k}$-module $V \otimes_{\mathbf{k}} V'$ is a $\mathbf{k}G$-module via the action $g.(v \otimes v') := g.v \otimes g.v'$ for $g \in G$, $v \in V$ and $v' \in V'$.

Let $H < G$. Let W be a $\mathbf{k}H$-module and V a $\mathbf{k}G$-module. We define $\mathrm{Ind}_H^G(W) := \mathbf{k}G \otimes_{\mathbf{k}H} W$. Since $\mathbf{k}H$ is a subalgebra of $\mathbf{k}G$, it follows that $\mathrm{Ind}_H^G(W)$ is a $\mathbf{k}G$-module. We call $\mathrm{Ind}_H^G(W)$ the induction of W from H to G. We define $\mathrm{Res}_H^G(V)$ to be the $\mathbf{k}H$-module obtained from V by restricting the $\mathbf{k}G$-module action to $\mathbf{k}H$. We call $\mathrm{Res}_H^G(V)$ the restriction of V from G to H.

Given a $\mathbf{k}$-module V, we let $\mathrm{triv}_G(V)$ denote the $\mathbf{k}G$-module given by: $g.v = v$ for all $g \in G$ and $v \in V$. Viewing $\mathbf{k}$ as a module over itself, we can form the $\mathbf{k}G$-module $\mathrm{triv}_G(\mathbf{k})$. We call $\mathrm{triv}_G(\mathbf{k})$ the trivial representation of G over $\mathbf{k}$.

Let G be a group and let V be a $\mathbf{k}G$-module. The $\mathbf{k}$-modules V^G and V_G are defined as

$$V^G := \{v \in V : gv = v \text{ for all } g \in G\}$$

and

$$V_G := \frac{V}{\mathrm{span}_{\mathbf{k}}\{gv - v : g \in G, v \in V\}}$$

We call V^G the G-invariants of V and V_G the G-coinvariants of V. If G is a finite group then as $\mathbf{k}$-modules $V_G \simeq V^G \simeq V \otimes_G \mathbf{k}$.

It will be useful to think of $\mathrm{triv}_G(\,-\,)$, $(\,-\,)^G$ and $(\,-\,)_G$ as functors. We do this in the following way:

1. $\text{triv}_G(\,-\,) : \mathbf{k}\text{-mod} \to \mathbf{k}G\text{-mod}$ is defined on objects by $A \mapsto \text{triv}_G(A)$ and defined on morphisms by $(A \xrightarrow{f} B) \mapsto (\text{triv}_G(A) \xrightarrow{f} \text{triv}_G(B))$.

2. $(\,-\,)^G : \mathbf{k}G\text{-mod} \to \mathbf{k}\text{-mod}$ is defined on objects by $A \mapsto A^G$ and defined on morphisms by $(A \xrightarrow{f} B) \mapsto (A^G \xrightarrow{f} B^G)$.

3. $(\,-\,)_G : \mathbf{k}G\text{-mod} \to \mathbf{k}\text{-mod}$ is defined on objects by $A \mapsto A_G$ and defined on morphisms by $(A \xrightarrow{f} B) \mapsto (A_G \xrightarrow{\overline{f}} B_G)$, where $\overline{f}$ is the map on quotients induced by f.

In the Chapters that follow we will apply the 'taking invariants/coinvariants' functor to exact sequences of $\mathbf{k}G$-modules. The following Lemma tells us that the result is an exact sequence of $\mathbf{k}$-modules.

Lemma 2. *Let G be a group.*

1. *$(\,-\,)^G$ is right-adjoint to $\text{triv}_G(\,-\,)$ (hence $(\,-\,)^G$ is left-exact).*

2. *$(\,-\,)_G$ is left-adjoint to $\text{triv}_G(\,-\,)$ (hence $(\,-\,)_G$ is right-exact).*

Proof. For the first assertion consider

$$\text{Hom}_{\mathbf{k}G\text{-mod}}(\text{triv}_G(A), B) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \text{Hom}_{\mathbf{k}\text{-mod}}(A, B^G)$$

Given $\alpha \in \text{Hom}_{\mathbf{k}G\text{-mod}}(\text{triv}_G(A), B)$, set $\varphi(\alpha) := \alpha$. Since $\alpha : \text{triv}_G(A) \to B$ is a morphism of $\mathbf{k}G$-modules, $\text{im}(\alpha) \subset B^G$ and φ is well-defined. Given $\beta \in \text{Hom}_{\mathbf{k}\text{-mod}}(A, B^G)$, set $\psi(\beta) := \beta$. Since $\text{im}(\beta) \subset B^G$ and $g.a = a$ for all $a \in A$ and $g \in G$, we conclude β is a morphism of $\mathbf{k}G$-modules and ψ is well-defined. By construction φ and ψ are inverses. This proves the first assertion.

For the second assertion consider

$$\mathrm{Hom}_{\mathbf{k}\text{-mod}}(A_G, B) \underset{\psi}{\overset{\varphi}{\rightleftarrows}} \mathrm{Hom}_{\mathbf{k}G\text{-mod}}(A, \mathrm{triv}_G(B))$$

Given $\alpha \in \mathrm{Hom}_{\mathbf{k}\text{-mod}}(A_G, B)$ set $\varphi(\alpha) := \widetilde{\alpha}$ where $\widetilde{\alpha}$ is the composition $A \to A_G \overset{\alpha}{\to} B$. Since $\widetilde{\alpha}$ is a morphism of $\mathbf{k}G$-modules, φ is well-defined. Given $\beta \in \mathrm{Hom}_{\mathbf{k}G\text{-mod}}(A, \mathrm{triv}_G(B))$, $\mathrm{span}_{\mathbf{k}}\{g.a - a : a \in A, g \in G\} \subset \ker(\beta)$. Therefore there exists a well-defined $\mathbf{k}$-module morphism $\overline{\beta} : A_G \to B$. We set $\psi(\beta) := \overline{\beta}$. Since φ and ψ are inverses, this completes the proof. □

We end this section with a well known fact that we state in order to reference later.

Fact 3. *Let G be a group and let X be a non-empty set. Suppose that G acts transitively on X. Let $x_0 \in X$ be any element and let $\mathrm{Stab}_G(x_0)$ denote the stabilizer of x_0 under the action of G. Then there exists a bijection*

$$G/\mathrm{Stab}_G(x_0) \longrightarrow X.$$

Moreover, if $\mathbf{k}X$ denotes the free $\mathbf{k}$-module with basis indexed by X, then $\mathbf{k}X$ is a $\mathbf{k}G$-module and

$$\mathbf{k}X \simeq \mathbf{k}G \otimes_{\mathbf{k}\mathrm{Stab}_G(x_0)} \mathrm{triv}_{\mathrm{Stab}_G(x_0)}(\mathbf{k})$$

as $\mathbf{k}G$-modules.

Partitions.

A partition, λ, is a sequence of non-negative integers $(\lambda_1, \lambda_2, \lambda_3, \dots)$ such that $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$ and such that only finitely many λ_i are non-zero. For a partition λ we define $|\lambda| := \sum_i \lambda_i$ and call $|\lambda|$ the size of λ. A partition of $n \in \mathbf{Z}_{\geq 0}$ is a partition λ of size n i.e. such that $|\lambda| = n$. We let $\mathscr{P}_n$ denote the partitions of size n and $\mathscr{P} = \cup_{n \geq 0} \mathscr{P}_n$. A partition λ corresponds to a Young diagram where the number of boxes in row i is λ_i. We use the terms 'partition of n' and 'Young diagram with n boxes' interchangeably.

We will use $\varnothing$ to denote the empty-partition $()$, it corresponds to the Young diagram with zero boxes.

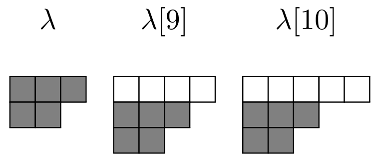
The hook length of a box in a Young diagram is the number of boxes directly below or to the right of the box, including the box once.

Suppose we have a partition λ and an integer $n \in \mathbf{Z}_{\geq 0}$. If $n \geq |\lambda| + \lambda_1$ we define the partition $\lambda[n]$ as

$$\lambda[n] := (n - |\lambda|, \lambda_1, \lambda_2, \lambda_3, \dots).$$

Since $|\lambda[n]| = n$, $\lambda[n]$ is a partition of n. For any partition μ of $n \in \mathbf{Z}_{\geq 0}$, there exists a unique partition λ such that $\mu = \lambda[n]$. In terms of Young diagrams, $\lambda[n]$ is the Young diagram obtained from λ by adding one box to the first $n - |\lambda|$ columns. Equivalently, $\lambda[n]$ is the Young diagram obtained from λ by adding a row to the top of λ with $n - |\lambda|$ boxes. i.e. with enough boxes to get a Young diagram from λ with n boxes.

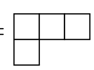
Example 4.



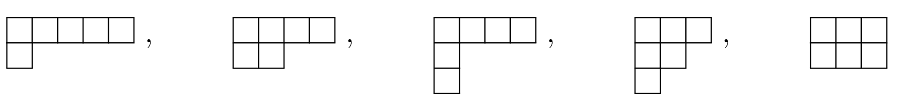
Here the gray boxes form the partition λ*. Since* $|\lambda| = 5$*, to construct* $\lambda[9]$*, we need to add a row to the top of the Young diagram of* λ *containing* 4 *boxes. Observe that one can get* $\lambda[10]$ *from* $\lambda[9]$ *by adding one box to the top row of* $\lambda[9]$*.*

Suppose λ and μ are partitions and $n \in \mathbf{Z}_{\geq 0}$. We write $\mu \sim \lambda + n$ if μ can be obtained from λ by adding one box to n distinct columns of λ (including empty columns). Similarly, we write $\mu \sim \lambda - n$ if μ can be obtained from λ by removing one box from n distinct columns of λ.

Example 5. *Consider the partition* λ

$\lambda =$  .

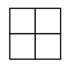
The set of all partitions μ *such that* $\mu \sim \lambda + 2$ *is:*



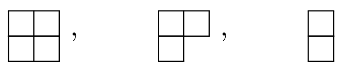
The set of all partitions μ *such that* $\mu \sim \lambda - 2$ *is:*

Suppose λ and μ are partitions. We write $\mu \leftarrow \lambda$ (equivalently $\lambda \to \mu$) if μ is obtained from λ by removing at most one box from each row.

Example 6. *Consider the partition* λ

$\lambda =$  .

The set of all partitions μ *such that* $\mu \leftarrow \lambda$ *is*



Next we have the definition of what we call a 'zigzag diagram'. A zigzag diagram will be a finite ordered list of partitions with certain properties. In the section that follows we introduce an analog where instead of partitions we use certain partition valued functions. We include the concept of zigzag diagrams of partitions to make the later concept easier to understand.

Definition 7. *Fix* $m, n \in \mathbf{Z}_{\geq 0}$ *with* $n \geq m$*. For* $\mu \in \mathscr{P}_n$ *and* $\nu \in \mathscr{P}_{n-m}$ *we define* $zz(\nu, \mu)$ *to be*

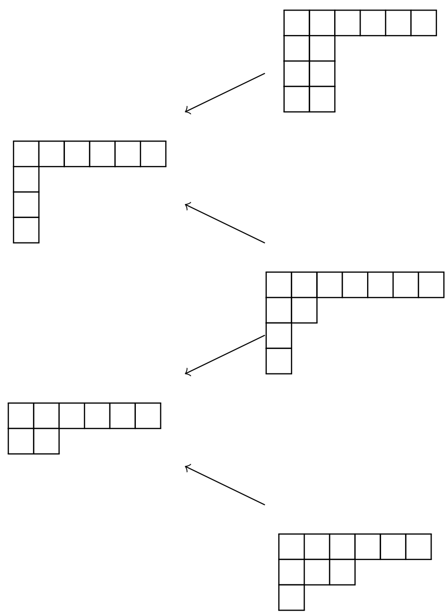
$$\left\{ \begin{array}{c} (\lambda^{(1)}, \ldots, \lambda^{(m)}, \mu^{(0)} = \nu, \mu^{(1)}, \ldots, \mu^{(m)} = \mu) \in \mathscr{P}^{2m+1} : \\ |\mu^{(s)}| = n - m + s, \quad s = 0, 1, \ldots, m \\ \textit{and} \quad \mu^{(k-1)} \to \lambda^{(k)} \leftarrow \mu^{(k)}, \quad k = 1, \ldots, m \end{array} \right\}.$$

We call elements of $zz(\nu, \mu)$ *zigzag diagrams.*

The following example clarifies the definition of zigzag diagrams.

Example 8. *Consider the partitions ν and μ*

Then partitions in the following diagram form a zigzag diagram from ν to μ i.e. an element of $zz(\nu, \mu)$ with the partitions on the right being $\mu^{(0)} = \nu, \mu^{(1)}, \mu^{(2)} = \mu$ (starting from the bottom) and the partitions on the left being $\lambda^{(1)}, \lambda^{(2)}$ (starting from the bottom).



The partitions corresponding to $\mu^{(i)}$ (on the right) increase in size at each step from the bottom, going up by one each time. The partitions corresponding to $\lambda^{(i)}$ (on the left) can be any size as long as they fit into the diagram.

Partition Valued Functions.

In this section we introduce the definitions that will be used to parametrize the irreducible representations of G_n, namely certain partition valued functions. We also introduce the partition valued analog of zigzag diagrams from the previous section. These partition valued analogs to zigzag diagrams are related to multiplicities when restricting a representation of G_n to a representation of $G_m < G_n$.

Let $\mathscr{C}$ be a set along with a distinguished element $\iota \in \mathscr{C}$ and a function $\mathrm{d} : \mathscr{C} \to \mathbf{Z}_{\geq 0}$ such that $\mathrm{d}(\iota) = 1$.

Let $\underline{\lambda} : \mathscr{C} \to \mathscr{P}$ be a partition valued function. We define

$$|\underline{\lambda}| := \sum_{\rho \in \mathscr{C}} \mathrm{d}(\rho) |\underline{\lambda}(\rho)|.$$

For each $n \in \mathbf{Z}_{\geq 0}$ we define $C(\mathscr{C}, \mathscr{P})_n := \{\underline{\lambda} : \mathscr{C} \to \mathscr{P} \mid |\underline{\lambda}| = n\}$ and we define

$$C(\mathscr{P}, \mathscr{C}) := \bigcup_{n \geq 0} C(\mathscr{C}, \mathscr{P})_n.$$

Suppose we have $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ and $n \geq |\underline{\lambda}| + \underline{\lambda}(\iota)_1$. Then we define $\underline{\lambda}[n] \in C(\mathscr{C}, \mathscr{P})$ by

$$\underline{\lambda}[n](\rho) := \begin{cases} (n - |\underline{\lambda}|, \underline{\lambda}(\iota)_1, \underline{\lambda}(\iota)_2, \underline{\lambda}(\iota)_3, \dots), & \rho = \iota \\ \underline{\lambda}(\rho), & \rho \neq \iota \end{cases}$$

Note that $\underline{\lambda}[n](\iota) = \underline{\lambda}(\iota)[n - |\underline{\lambda}| + |\underline{\lambda}(\iota)|]$ and

$$\begin{aligned}
|\underline{\lambda}[n]| &= |\underline{\lambda}[n](\iota)| + \sum_{\rho \in \mathscr{C} \setminus \{\iota\}} \mathrm{d}(\rho) |\underline{\lambda}(\rho)| \\
&= n - |\underline{\lambda}| + |\underline{\lambda}(\iota)| + \sum_{\rho \in \mathscr{C} \setminus \{\iota\}} \mathrm{d}(\rho) |\underline{\lambda}(\rho)| \\
&= n - |\underline{\lambda}| + |\underline{\lambda}| \\
&= n
\end{aligned}$$

i.e. $|\underline{\lambda}[n]| = n$. For any $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})_n$, there is a unique $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ such that $\underline{\mu} = \underline{\lambda}[n]$.

For any two $\underline{\lambda}, \underline{\mu} \in C(\mathscr{C}, \mathscr{P})$ and non-negative integer n, we write $\underline{\mu} \sim \underline{\lambda} + n$ if

$$\underline{\mu}(\iota) \sim \underline{\lambda}(\iota) + n \text{ and } \underline{\mu}(\rho) = \underline{\lambda}(\rho) \text{ for } \rho \in \mathscr{C} \setminus \{\iota\}.$$

Similarly we write $\underline{\mu} \sim \underline{\lambda} - n$ if

$$\underline{\mu}(\iota) \sim \underline{\lambda}(\iota) - n \text{ and } \underline{\mu}(\rho) = \underline{\lambda}(\rho) \text{ for } \rho \in \mathscr{C} \setminus \{\iota\}.$$

Given $\underline{\mu}, \underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ we write $\underline{\mu} \leftarrow \underline{\lambda}$ (equivalently $\underline{\lambda} \rightarrow \underline{\mu}$) if $\underline{\mu}(\rho) \leftarrow \underline{\lambda}(\rho)$ for all $\rho \in \mathscr{C}$.

Given $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})$ we define $\underline{\mu}[+1] \in C(\mathscr{C}, \mathscr{P})$ as

$$\underline{\mu}[+1](\rho) := \begin{cases} (\underline{\mu}(\iota)_1 + 1, \underline{\mu}(\iota)_2, \underline{\mu}(\iota)_3, \dots), & \rho = \iota \\ \underline{\mu}(\rho), & \rho \neq \iota \end{cases}.$$

If $\underline{\mu}(\iota)_1 - \underline{\mu}(\iota)_2 \neq 0$ then we define $\underline{\mu}[-1] \in C(\mathscr{C}, \mathscr{P})$ as

$$\underline{\mu}[-1](\rho) := \begin{cases} (\underline{\mu}(\iota)_1 - 1, \underline{\mu}(\iota)_2, \underline{\mu}(\iota)_3, \dots), & \rho = \iota \\ \\ \underline{\mu}(\rho), & \rho \neq \iota \end{cases}.$$

Remark 9. *Let $\underline{\mu}, \underline{\mu}' \in C(\mathscr{C}, \mathscr{P})$. If $\underline{\mu} \leftarrow \underline{\mu}'$, then $\underline{\mu}[+1] \leftarrow \underline{\mu}'[+1]$. If $\underline{\mu}[-1]$ and $\underline{\mu}'[-1]$ are both defined, then $\underline{\mu} \leftarrow \underline{\mu}'$ implies $\underline{\mu}[-1] \leftarrow \underline{\mu}'[-1]$.*

We now come to the partition valued function analog of zigzag diagrams from the previous section.

Definition 10. *Fix $m, n \in \mathbf{Z}_{\geq 0}$ with $n \geq m$. For $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})_n$ and $\underline{\nu} \in C(\mathscr{C}, \mathscr{P})_{n-m}$ we define $zz(\underline{\nu}, \underline{\mu})$ to be*

$$\left\{ \begin{array}{c} (\underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(m)}, \underline{\mu}^{(0)} = \underline{\nu}, \underline{\mu}^{(1)}, \dots, \underline{\mu}^{(m)} = \underline{\mu}) \in C(\mathscr{P}, \mathscr{C})^{2m+1} : \\ |\underline{\mu}^{(s)}| = n - m + s, \quad s = 0, 1, \dots, m \\ \text{and} \quad \underline{\mu}^{(k-1)} \to \underline{\lambda}^{(k)} \leftarrow \underline{\mu}^{(k)}, \quad k = 1, \dots, m \end{array} \right\}.$$

We call elements of $zz(\underline{\nu}, \underline{\mu})$ zigzag diagrams.

Remark 11. *Let $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})_n$ and $\underline{\nu} \in C(\mathscr{C}, \mathscr{P})_{n-m}$. By combinatorics we have*

$$|zz(\underline{\nu}, \underline{\mu})| = \sum_{\underline{\mu}' \in C(\mathscr{C}, \mathscr{P})_k} |zz(\underline{\mu}', \underline{\mu})||zz(\underline{\nu}, \underline{\mu}')|$$

for any $k = n - m + 1, \dots, n - 1$.

Representations of S_n.

In this section we set notation and collect facts for the representation theory of the symmetric groups over $\mathbf{k}$.

For each $n \in \mathbf{Z}_{\geq 0}$ let $\mathbf{Z}\mathrm{Irr}_{\mathbf{k}}(S_n)$ denote the free $\mathbf{Z}$-module with basis indexed by $\mathrm{Irr}_{\mathbf{k}}(S_n)$. We define the $\mathbf{Z}$-module $A(S_*)$ as

$$A(S_*) := \bigoplus_{n \geq 0} \mathbf{Z}\mathrm{Irr}_{\mathbf{k}}(S_n).$$

Given $\pi \in \mathrm{Irr}_{\mathbf{k}}(S_n)$ and $\pi' \in \mathrm{Irr}_{\mathbf{k}}(S_{n'})$ define

$$\pi \circ \pi' := \mathrm{Ind}_{S_n \times S_{n'}}^{S_{n+n'}} (\pi \otimes_{\mathbf{k}} \pi')$$

where $S_n \times S_{n'}$ is the subgroup of $S_{n+n'}$ given by

$$S_n \times S_{n'} := \{\sigma \in S_{n+n'} : \sigma|_{\{1,2,\ldots,n\}} \in S_n \quad \text{and} \quad \sigma|_{\{n+1,n+2,\ldots,n+n'\}} \in S_{n'}\}.$$

We call this operation multiplication. We can extend this multiplication linearly to $A(S_*)$. In this way, $A(S_*)$ is a commutative algebra.

Fact 12. *(Parametrization of Irreducible Representations). There is a bijective correspondence*

$$\mathrm{Irr}_{\mathbf{k}}(S_n) \leftrightarrow \mathscr{P}_n$$

under which $\mathrm{triv}_{S_n}(\mathbf{k})$ *corresponds to the Young diagram consisting of a single row with* n *boxes.*

For a partition λ of n we let $\mathrm{L}(\lambda) \in \mathrm{Irr}_{\mathbf{k}}(S_n)$ denote the corresponding irreducible representation of S_n.

An advantage of Fact 12 is that calculations involving irreducible representations of symmetric groups can be translated into combinatorics of partitions and Young diagrams. We will use the following branching rule:

Fact 13. *(Pieri's Rule). Let $k \in \mathbf{Z}_{\geq 0}$ and let $\lambda \in \mathscr{P}$. Then*

$$\mathrm{L}(\lambda) \circ \mathrm{triv}_{S_k}(\mathbf{k}) = \bigoplus_{\nu \sim \lambda + k} \mathrm{L}(\nu)$$

where the sum is over all $\nu \in \mathscr{P}_{|\lambda|+k}$ such that $\nu \sim \lambda + k$.

Stuff About G_n.

Before describing the representation theory of the finite general linear groups we list notation we will use to discuss the groups G_n. We conclude the section with a miscellaneous Lemma.

For $n \in \mathbf{Z}_{\geq 0}$ and $g \in G_n$, we let $\hat{g}$ denote the image of g under the standard inclusion $G_n \hookrightarrow G_{n+1}$.

Let $r, s \in \mathbf{Z}_{\geq 0}$. We define $H_{r,s}$, $P_{r,s}$ and $U_{r,s}$ to be the following subgroups of G_{r+s}:

$$H_{r,s} := \left\{ \begin{pmatrix} I_r & * \\ & G_s \end{pmatrix} \in G_{r+s} \right\}, \qquad P_{r,s} := \left\{ \begin{pmatrix} G_r & * \\ & G_s \end{pmatrix} \in G_{r+s} \right\},$$

$$U_{r,s} := \left\{ \begin{pmatrix} I_r & * \\ & I_s \end{pmatrix} \in G_{r+s} \right\}$$

where a block containing G_n means any element of G_n, a block containing I_n denotes the $n \times n$ identity matrix, a block with ▨ denotes any (possibly non-square) matrix with entries in $\mathbf{F}_q$ and an empty block denotes a (possibly non-square) matrix with all entries equal to zero.

Convention 14. *Let $r, s \in \mathbf{Z}_{\geq 0}$. If we wish to consider G_r as a subgroup of G_{r+s} as*

$$G_r = \left\{ \begin{pmatrix} \boxed{G_r} & \\ & \boxed{I_s} \end{pmatrix} \in G_{r+s} \right\}$$

we say $G_r < G_{r+s}$. If we wish to consider G_s as a subgroup of G_{r+s} as

$$G_s = \left\{ \begin{pmatrix} \boxed{I_r} & \\ & \boxed{G_s} \end{pmatrix} \in G_{r+s} \right\}$$

we say $G_s < G_{r+s}$.

In Chapter 3 it will be convenient to use an alternative notation to refer to the subgroup $G_r < G_{r+s}$. We let $H'_{r,s} < G_{r+s}$ denote the subgroup $G_r < G_{r+s}$. i.e.

$$H'_{r,s} = \left\{ \begin{pmatrix} \boxed{G_r} & \\ & \boxed{I_s} \end{pmatrix} \in G_{r+s} \right\}.$$

Fact 15. *Let $r, s \in \mathbf{Z}_{\geq 0}$. Then*

1. *$U_{r,s} < H_{r,s}$,*

2. *$H_{r,s} = U_{r,s} G_s$ (where $G_s < G_{r+s}$ as in Convention 14),*

3. *$G_r \times G_s \simeq P_{r,s}/U_{r,s}$ (where $G_r < G_{r+s}$ and $G_s < G_{r+s}$ as in Convention 14).*

Lemma 16. *Let $a, m, n \in \mathbf{Z}_{\geq 0}$ with $n > a + m + \min\{a, m\}$. For every $g \in G_{n+1}$, there exists an $h_1 \in H'_{a,n+1-a}$ and there exists an $h_2 \in H'_{m,n+1-m}$ such that $h_2 g h_1 \in G_n$. i.e. the last row and column of g are all zero except for the bottom right corner which is 1.*

Proof. Column operations on the last $(n+1) - a$ columns of g is given by multiplication on the right by a suitable element of $H'_{a,n+1-a}$. Row operations on the last $(n + 1) - m$ rows of g is given by multiplication on the left by a suitable element of $H'_{m,n+1-m}$.

By row and column operations there exists an $h \in H'_{a,n+1-a}$ and an $h' \in H'_{m,n+1-m}$ such that the (r, s)-entry of $h'gh$ is zero if:

$$r \leq m \text{ and } s > a + m \qquad \text{or} \qquad r > a + m \text{ and } s \leq a$$

$$h'gh = \left(\begin{array}{c:c:c} & A' & 0 \\ \hdashline A & & \\ \hdashline 0 & & \end{array}\right)$$

Next we claim that there exist $r, s > a+m$ such that the (r, s)-entry of $h'gh$ is non-zero. To see this, consider the submatrices A and A' in the diagram of $h'gh$. They have column rank (equivalently row rank) of at most a and m respectively. If the claim is false, the rank of $h'gh$ would be bounded above by

$$\min\{a + m + a, a + m + m\} = a + m + \min\{a, m\}.$$

However, the rank of $h'gh$ is $n + 1$ and $n + 1 > a + m + \min\{a, m\}$, hence the claim must be true.

We now use row and column operations to move this non-zero entry to the $(n+1, n+1)$-entry and turn it into 1. Then do row and column operations to make the remaining entries in the last row and column into zeros. □

Representations of G_n.

Let us informally describe the parameterization of irreducible representations of G_n. The $\mathbf{Z}$-module $A(G_*)$ is an algebra by using parabolic induction as multiplication. For each irreducible cuspidal ρ there is a subalgebra $A(\rho) \subset A(G_*)$, each of which is isomorphic to $A(S_*)$. The algebra $A(G_*)$ is isomorphic to the tensor product of these $A(\rho)$. From this, irreducible representations correspond to certain functions from the cuspidals $\mathscr{C}$ to partitions $\mathscr{P}$. This section describes this parameterization in more detail and contains the branching rules we will use as well as some miscellaneous results. We use Zelevinsky [17] as the reference for the facts in this section.

For each $n \in \mathbf{Z}_{\geq 0}$ let $\mathbf{Z}\mathrm{Irr}_{\mathbf{k}}(G_n)$ be the free $\mathbf{Z}$-module with basis indexed by $\mathrm{Irr}_{\mathbf{k}}(G_n)$. Define the $\mathbf{Z}$-module $A(G_*)$ to be

$$A(G_*) := \bigoplus_{n \geq 0} \mathbf{Z}\mathrm{Irr}_{\mathbf{k}}(G_n).$$

We can define a multiplication on $A(G_*)$ which we now describe. Let $n, n' \in \mathbf{Z}_{\geq 0}$ and suppose that π and π' are $\mathbf{k}G_n$- and $\mathbf{k}G_{n'}$-modules respectively. Since $\pi \otimes_{\mathbf{k}} \pi'$ is a $\mathbf{k}[G_n \times G_{n'}]$-module, it follows that $\pi \otimes \pi'$ is a $\mathbf{k}[P_{n,n'}/U_{n,n'}]$-module (recall $G_n \times G_{n'} \simeq P_{n,n'}/U_{n,n'}$). We consider $\pi \otimes \pi'$ as a $\mathbf{k}P_{n,n'}$-module on which $U_{n,n'}$ acts trivially. We can induce from

$P_{n,n'}$ to $G_{n+n'}$ to get a $\mathbf{k}G_{n+n'}$-module which we call $\pi \circ \pi'$. i.e.

$$\pi \circ \pi' := \mathrm{Ind}_{P_{n,n'}}^{G_{n+n'}} (\pi \otimes_{\mathbf{k}} \pi')$$

This process is called parabolic induction.

Fact 17. *The* $\mathbf{Z}$*-module* $A(G_*)$ *is a commutative algebra with multiplication given on irreducible representations by* $\circ$ *and extended linearly.*

A representation of G_n is cuspidal if it contains no non-zero vectors invariant under some non-trivial subgroup $U_{r,s}$ with $r + s = n$. For each $n \geq 0$, let $\mathscr{C}_n$ be the cuspidal representations (up to isomorphism) of G_n and let $\mathscr{C} := \sqcup_{n \geq 0} \mathscr{C}_n$. If $\rho \in \mathscr{C}_n$ we say $\mathrm{d}(\rho) := n$. The irreducible representation $\iota := \mathrm{triv}_{G_1}(\mathbf{k})$ is cuspidal.

Given $\rho \in \mathscr{C}$ we define the subalgebra $A(\rho) \subset A(G_*)$ as

$$A(\rho) := \mathbf{Z}\{\pi\colon \pi \text{ is a constituent of } \rho^{\circ k} \text{ for some } k \in \mathbf{Z}_{\geq 0}\}$$

The subalgebra $A(\iota)$ contains $\mathrm{triv}_{G_n}(\mathbf{k})$ for all n. We call representations in $A(\iota)$ unipotent representations.

Convention 18. *Recall the definitions and notation for* $C(\mathscr{C}, \mathscr{P})$ *from previous sections. The definition of* $C(\mathscr{C}, \mathscr{P})$ *depends on* $\mathscr{C}$, ι *and* d. *From now on* $C(\mathscr{C}, \mathscr{P})$ *and all associated definitions are defined using* $\mathscr{C}$ *for the cuspidal representations,* ι *for the trivial representation of* G_1 *and* $\mathrm{d} : \mathscr{C} \to \mathbf{Z}_{\geq 0}$ *for the function given by* $\mathrm{d}(\rho) = n$ *if* ρ *is an irreducible* $\mathbf{k}G_n$*-module.*

Fact 19. *The algebra $A(G_*)$ decomposes into the tensor product*

$$\bigotimes_{\rho \in \mathscr{C}} A(\rho),$$

the inductive limit of the finite tensor products

$$\bigotimes_{\rho \in S} A(\rho)$$

as S ranges over all finite subsets of $\mathscr{C}$.

The previous Fact is from Zelevinsky [17], Section 9.3.

Hence any irreducible representation of G_n over $\mathbf{k}$ decomposes uniquely (up to permutation of factors) into the product of constituents of powers of cuspidals.

Fact 20. *For all $\rho \in \mathscr{C}$ there exists an isomorphism*

$$\varphi_\rho : A(\rho) \to A(S_*)$$

of algebras. The isomorphism

$$\varphi_\iota : A(\iota) \to A(S_*)$$

satisfies

$$\varphi_\iota(\mathrm{triv}_{G_n}(\mathbf{k})) = \mathrm{triv}_{S_n}(\mathbf{k})$$

for all $n \in \mathbf{Z}_{\geq 0}$.

Let $n \in \mathbf{Z}_{\geq 0}$ and let $\pi \in \mathrm{Irr}_{\mathbf{k}}(G_n)$. By Fact 19

$$\pi = \pi_{\rho_{i_1}} \circ \pi_{\rho_{i_2}} \circ \cdots \circ \pi_{\rho_{i_k}}$$

where ρ_{i_j} are distinct cuspidals and $\pi_{\rho_{i_j}} \in A(\rho_{i_j})$ for all $j = 1, \ldots, k$. By Fact 20, π uniquely corresponds to $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})_n$ where

$$\underline{\lambda}(\rho_{i_j}) = \varphi_{\rho_{i_j}}(\pi_{\rho_{i_j}}) \in \mathscr{P}$$

and $\underline{\lambda}(\rho) = \varnothing$ for $\rho \in \mathscr{C} \setminus \{\rho_{i_1}, \rho_{i_2}, \ldots, \rho_{i_k}\}$.

We now have a parameterization of the irreducible representations of G_n.

Fact 21. *(Parametrization of Irreducible Representations). There is a bijective correspondence*

$$Irr_{\mathbf{k}}(G_n) \leftrightarrow C(\mathscr{C}, \mathscr{P})_n.$$

Given $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})_n$ we let $\mathrm{L}(\underline{\lambda}) \in \mathrm{Irr}_{\mathbf{k}}(G_n)$ denote the corresponding irreducible representation of G_n. Every representation V of G_n can be decomposed as:

$$V = \bigoplus_{\underline{\lambda}} c_{\underline{\lambda},n} \, \mathrm{L}(\underline{\lambda}[n])$$

where the sum is over all $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ for which $\underline{\lambda}[n]$ is defined. We will use this notation frequently, summing over different sets, but in all cases $c_{\underline{\lambda},n}$ is the multiplicity of $\mathrm{L}(\underline{\lambda}[n])$.

Lemma 22. *(Pieri's Rule for G_n) Let $n \in \mathbf{Z}_{\geq 0}$ and $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. Then*

$$\mathrm{L}(\underline{\lambda}) \circ \mathrm{triv}_{G_n}(\mathbf{k}) = \bigoplus_{\underline{\nu} \sim \underline{\lambda}+n} \mathrm{L}(\underline{\nu}).$$

The previous Lemma follows directly from the Pieri rule for symmetric groups found in Chapter One of **??**.

Fact 23. *(Branching Rule for G_n) Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})_n$ and $\underline{\lambda}' \in C(\mathscr{C}, \mathscr{P})_{n-1}$. The multiplicity of $\mathrm{L}(\underline{\lambda}')$ in the restriction of $\mathrm{L}(\underline{\lambda})$ to G_{n-1} is the number of diagrams*

$$\underline{\lambda}' \to \underline{\mu} \leftarrow \underline{\lambda}$$

that can be constructed.

i.e. The multiplicity of $\mathrm{L}(\underline{\lambda}')$ in the restriction of $\mathrm{L}(\underline{\lambda})$ to G_{n-1} is $|zz(\underline{\lambda}', \underline{\lambda})|$.

The previous Fact can be found in Zelevinsky [17], Section 13.8. Zelevinsky remarks that result was first obtained by Thoma [14].

Lemma 24. *Fix $m, n \in \mathbf{Z}_{\geq 0}$ with $n \geq m$. Let $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})_n$ and $\underline{\nu} \in C(\mathscr{C}, \mathscr{P})_{n-m}$. The multiplicity of $\mathrm{L}(\underline{\nu})$ in $\mathrm{Res}^{G_n}_{G_{n-m}} \mathrm{L}(\underline{\mu})$ is $|zz(\underline{\nu}, \underline{\mu})|$.*

Proof. We do induction on m. The base case is Fact 23.

Suppose $m > 1$. Then

$$\begin{aligned}
\operatorname{Res}^{G_n}_{G_{n-m}} \mathrm{L}(\underline{\mu}) &= \operatorname{Res}^{G_{n-1}}_{G_{n-m}} \operatorname{Res}^{G_n}_{G_{n-1}} \mathrm{L}(\underline{\mu}) \\
&= \operatorname{Res}^{G_n}_{G_{n-m}} \bigoplus_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-1}} |zz(\underline{\mu}', \underline{\mu})| \, \mathrm{L}(\underline{\mu}') \\
&= \bigoplus_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-1}} |zz(\underline{\mu}', \underline{\mu})| \operatorname{Res}^{G_n}_{G_{n-m}} \mathrm{L}(\underline{\mu}') \\
&= \bigoplus_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-1}} |zz(\underline{\mu}', \underline{\mu})| \left(\bigoplus_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-m}} |zz(\underline{\nu}', \underline{\mu}')| \, \mathrm{L}(\underline{\nu}') \right) \\
&= \bigoplus_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-m}} \left(\sum_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-1}} |zz(\underline{\mu}', \underline{\mu})||zz(\underline{\nu}', \underline{\mu}')| \right) \mathrm{L}(\underline{\nu}')
\end{aligned}$$

Therefore by Fact 11 the multiplicity of $\mathrm{L}(\underline{\nu})$ is

$$\sum_{\underline{\mu}' \in C(\mathscr{C},\mathscr{P})_{n-1}} |zz(\underline{\mu}', \underline{\mu})||zz(\underline{\nu}, \underline{\mu}')| = |zz(\underline{\nu}, \underline{\mu})|$$

□

Lemma 25. *Fix $a \in \mathbf{Z}_{\geq 0}$. Let $n \in \mathbf{Z}_{\geq 0}$ be such that $n+1 > 3a$. Let $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})_{n+1}$. Let*

$$(\underline{\lambda}^{(1)}, \ldots, \underline{\lambda}^{(a)}, \underline{\mu}^{(0)} = \operatorname{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}^{(1)}, \ldots, \underline{\mu}^{(a)} = \underline{\mu}) \in zz(\operatorname{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}).$$

Then for all $\underline{\nu} \in \{\underline{\lambda}^{(1)}, \ldots, \underline{\lambda}^{(a)}, \underline{\mu}^{(0)}, \ldots, \underline{\mu}^{(a)}\}$, $\underline{\nu}(\iota)_1 - \underline{\nu}(\iota)_2 \neq 0$.

Proof. Suppose

$$(\underline{\lambda}^{(1)}, \ldots, \underline{\lambda}^{(a)}, \underline{\mu}^{(0)} = \operatorname{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}^{(1)}, \ldots, \underline{\mu}^{(a)} = \underline{\mu}) \in zz(\operatorname{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}).$$

For $k = 1, \ldots, a$,

$$\underline{\mu}^{(k-1)} \to \underline{\lambda}^{(k)} \leftarrow \underline{\mu}^{(k)}. \qquad (\star)$$

Therefore if the lemma is true for $\underline{\mu}^{(0)}, \ldots, \underline{\mu}^{(a)}$ then it is true for $\underline{\lambda}^{(1)}, \ldots, \underline{\lambda}^{(a)}$.

From $(\star)$ we have

$$\underline{\mu}^{(k)}(\iota)_1 - \underline{\mu}^{(k)}(\iota)_2 \geq \underline{\mu}^{(k-1)}(\iota)_1 - \underline{\mu}^{(k-1)}(\iota)_2 - 2$$

for $k = 1, \ldots, a$.

Applying the previous inequality and using

$$\underline{\mu}^{(0)}(\iota)_1 - \underline{\mu}^{(0)}(\iota)_2 = n + 1 - a$$

we have

$$\begin{aligned}
\underline{\mu}^{(a)}(\iota)_1 - \underline{\mu}^{(a)}(\iota)_2 &\geq \underline{\mu}^{(a-1)}(\iota)_1 - \underline{\mu}^{(a-1)}(\iota)_2 - 2 \\
&\geq \underline{\mu}^{(a-2)}(\iota)_1 - \underline{\mu}^{(a-2)}(\iota)_2 - 4 \\
&\vdots \\
&\geq \underline{\mu}^{(0)}(\iota)_1 - \underline{\mu}^{(0)}(\iota)_2 - 2a \\
&= n + 1 - 3a
\end{aligned}$$

from which we conclude

$$\underline{\mu}^{(a-i)}(\iota)_1 - \underline{\mu}^{(a-i)}(\iota)_2 \geq n + 1 - 3a + 2i \geq n + 1 - 3a > 0$$

for $i = 0, 1, \dots, a$ which proves the lemma for $\underline{\mu}^{(0)}, \dots, \underline{\mu}^{(a)}$ and hence for $\underline{\lambda}^{(1)}, \dots, \underline{\lambda}^{(a)}$. $\square$

Lemma 26. *Fix $a \in \mathbf{Z}_{\geq 0}$. Let $n \in \mathbf{Z}_{\geq 0}$ be such that $n+1 > 3a$. Then*

$$|zz(\mathrm{triv}_{G_{n-a}}(\mathbf{k}), \underline{\mu}[n])| = |zz(\mathrm{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}[n+1])|.$$

Proof. Define

$$zz(\mathrm{triv}_{G_{n-a}}(\mathbf{k}), \underline{\mu}[n]) \underset{[-1]}{\overset{[+1]}{\rightleftarrows}} zz(\mathrm{triv}_{G_{n+1-a}}(\mathbf{k}), \underline{\mu}[n+1])$$

where $[+1]$ is $\underline{\nu} \mapsto \underline{\nu}[+1]$ in each component and $[-1]$ is $\underline{\nu}[-1] \leftarrow \underline{\nu}$ in each component.

By Lemma 25 both maps are well-defined. Moreover they are inverses hence we get the result. $\square$

Lemma 27. *Let $a, n \in \mathbf{Z}_{\geq 0}$ with $a < n$. Consider $G_a < G_{a+(n-a)}$ and $G_{n-a} < G_{a+(n-a)}$ (as in Convention 14). Consider the $\mathbf{k}$-modules $\mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k})$ and $\mathbf{k}G_n \otimes_{\mathbf{k}P_{a,n-a}} (\mathbf{k}G_a \otimes_{\mathbf{k}} \mathrm{triv}_{G_{n-a}}(\mathbf{k}))$ as $\mathbf{k}G_n$-modules by acting $\mathbf{k}G_n$. Then*

$$\mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k}) \simeq \mathbf{k}G_n \otimes_{\mathbf{k}P_{a,n-a}} (\mathbf{k}G_a \otimes_{\mathbf{k}} \mathrm{triv}_{G_{n-a}}(\mathbf{k}))$$

as $\mathbf{k}G_n$-modules.

Proof. Consider the map of $\mathbf{k}$-modules

$$\begin{array}{rcl} \mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k}) & \xrightarrow{\varphi} & \mathbf{k}G_n \otimes_{\mathbf{k}P_{a,n-a}} \left(\mathbf{k}G_a \otimes_{\mathbf{k}} \mathrm{triv}_{G_{n-a}}(\mathbf{k})\right) \\ g \otimes 1 & \mapsto & g \otimes (I_a \otimes 1) \end{array}$$

To show that φ is a well-defined map of $\mathbf{k}$-modules we need to show that $\varphi((h.g)\otimes 1) = \varphi(g\otimes(h.1))$ for all $g \in G_n$ and $h \in H_{a,n-a}$. Since $g \otimes (h.1) = g \otimes 1$ for all $g \in G_n$ and all $h \in H_{a,n-a}$ and since $H_{a,n-a} \subset P_{a,n-a}$ it follows that φ is a well-defined $\mathbf{k}$-module morphism. Moreover G_n acts on $\mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k})$ and $\mathbf{k}G_n \otimes_{\mathbf{k}P_{a,n-a}} \big(\mathbf{k}G_a \otimes_{\mathbf{k}} \mathrm{triv}_{G_{n-a}}(\mathbf{k})\big)$ by acting on $\mathbf{k}G_n$. Therefore $g'.\varphi(g\otimes 1) = \varphi(g'.(g\otimes 1))$ for all $g, g' \in G_n$ and so φ is a morphism of $\mathbf{k}G_n$-modules.

Next consider the map of $\mathbf{k}$-modules

$$\begin{array}{ccc} \mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k}) & \stackrel{\psi}{\longleftarrow} & \mathbf{k}G_n \otimes_{\mathbf{k}P_{a,n-a}} \big(\mathbf{k}G_a \otimes_{\mathbf{k}} \mathrm{triv}_{G_{n-a}}(\mathbf{k})\big) \\ (g'\hat{g}) \otimes 1 & \leftarrow\!\shortmid & g' \otimes (g \otimes 1) \end{array}$$

where $g' \in G_n$, $g \in G_a < G_{a+(n-a)}$ and $\hat{g} \in G_n$ is g viewed as an element of G_n. To show that ψ is a well-defined map of $\mathbf{k}$-modules we need to show $\psi((p.g')\otimes(g\otimes 1)) = \psi(g'\otimes(p.(g\otimes 1)))$ for all $g' \in G_n$, $g \in G_a$ and $p \in P_{a,n-a}$. But this follows since $P_{a,n-a} = G_a H_{a,n-a}$. As with φ, ψ is a morphism of $\mathbf{k}G_n$-modules.

By construction φ and ψ are inverses. □

Dimension of Irreducible Representations of G_n.

In this section we prove that the dimension of $\mathrm{L}(\lambda[n])$ is eventually polynomial in q^n. The statements about the dimension of irreducible $\mathbf{k}G_n$-modules are found in Zelevinsky [17].

For $n \geq 1$ let

$$\Phi_n(q) := \prod_{i=1}^{n}(q^i - 1).$$

Given a partition $\lambda = (\lambda_1, \dots)$, set

$$\varepsilon := \sum_{i \geq 1}(i-1)\lambda_i.$$

Let $h(x)$ be the hook-length at box x in the Young Diagram of λ and let

$$\Psi_\lambda(q) := q^{\varepsilon(\lambda)} \prod_{x \in \lambda} \left(q^{h(x)} - 1\right)^{-1}$$

where '$x \in \lambda$' should be interpreted as 'box x in the Young Diagram of λ'.

With this notation we can state the the dimension of irreducible $\mathbf{k}G_n$-modules; the proof can be found in Zelevinsky [17], Section 11.10.

Fact 28. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. Then*

$$\dim\left(\mathrm{L}(\underline{\lambda})\right) = \Phi_n(q) \prod_{\rho \in \mathscr{C}} \Psi_{\underline{\lambda}(\rho)}\left(q^{\mathrm{d}(\rho)}\right).$$

Now we show that for a fixed $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$, that the dimension of $\mathrm{L}(\underline{\lambda}[n])$ is eventually polynomial in q^n.

Lemma 29. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. There exists a rational polynomial $P \in \mathbf{Q}[T]$ and there exists an $N \in \mathbf{Z}$ such that $\dim(\mathrm{L}(\underline{\lambda}[n])) = P(q^n)$ for all $n \geq N$.*

Proof. Let $N = |\underline{\lambda}| + \underline{\lambda}(\iota)_1$.

In the formula for $\dim(\mathrm{L}(\underline{\lambda}[n]))$, the only parts that depend on n are $\Phi_n(q)$ and $\Psi_{\underline{\lambda}[n](\iota)}(q)$. There are $n - |\underline{\lambda}|$ hooks in the first row of $\underline{\lambda}[n](\iota)$ they are

$$n - r_1, n - r_2, \ldots, n - r_{\underline{\lambda}(\iota)_1}, n - N, \ldots, 2, 1$$

for some $r_1, \ldots, r_{\underline{\lambda}(\iota)_1} < N$ which are independent of n. Let $s_1 < s_2 < \cdots < s_{|\underline{\lambda}|}$ be integers such that

$$\{r_1, \ldots, r_{\underline{\lambda}(\iota)_1}\} \sqcup \{s_1, \ldots, s_{|\underline{\lambda}|}\} = \{0, 1, \ldots, N-1\}$$

Then

$$\dim \mathrm{L}(\underline{\lambda}[n]) = c(q^{n-s_1} - 1)(q^{n-s_2} - 1) \cdots (q^{n-s_{|\underline{\lambda}|}} - 1)$$

for some $c \in \mathbf{Q}$ which is independent of n. So the result follows by choosing P to be the polynomial

$$P(T) = c(q^{-s_1}T - 1) \cdots (q^{-s_{|\underline{\lambda}|}}T - 1)$$

of degree $|\underline{\lambda}|$. □

Invariants and Coinvariants of $\mathbf{k}G_n$-modules.

Lemma 30. *Let $r, s \in \mathbf{Z}_{\geq 0}$. Let V be a $\mathbf{k}G_{r+s}$-module. Consider $G_r < G_{r+s}$ and $G_s < G_{r+s}$ (as in Convention 14). Then we have the following:*

1. *$V_{H_{r,s}}$, $V^{H_{r,s}}$ and $(V^{U_{r,s}})^{G_s}$ are all $\mathbf{k}G_r$-modules.*

2. *$V^{U_{r,s}}$ is a $\mathbf{k}[G_r \times G_s]$-module.*

3. $V_{H_{r,s}} \simeq V^{H_{r,s}} \simeq (V^{U_{r,s}})^{G_s}$ *as* $\mathbf{k}G_r$*-modules.*

Proof.

1. First consider $V^{H_{r,s}}$. As $\mathbf{k}$-modules $V^{H_{r,s}} \subset V$. Given $v \in V^{H_{r,s}}$ and $g \in G_r < G_{r+s}$ define $g.v$ to be the $\mathbf{k}G_{r+s}$-action on V. Since $g^{-1}hg \in H_{r,s}$ for all $g \in G_r$ and $h \in H_{r,s}$, we conclude $g^{-1}hgv = v$ for all $g \in G_r < G_{r+s}$, $h \in H_{r,s}$ and $v \in V^{H_{r,s}}$. Therefore the $\mathbf{k}G_r$-action on $V^{H_{r,s}}$ is well-defined.

   Next we consider $V_{H_{r,s}}$. If $g \in G_r < G_{r+s}$, $h \in H_{r,s}$ and $v \in V$ then $ghv - v$ and $gv - v$ are both in $\{hv - v : h \in H_{r,s}, v \in V\}$ hence

   $$g(hv - v) = (ghv - v) - (gv - v) \in \mathrm{span}_{\mathbf{k}}\{hv - v : h \in H_{r,s}, v \in V\}.$$

   Therefore if $g \in G_{r+s}$ and $\overline{v} = v + \mathrm{span}_{\mathbf{k}}\{hv - v : h \in H_{r,s}, v \in V\}$, then $g.\overline{v} := \overline{gv} = gv + \mathrm{span}_{\mathbf{k}}\{hv - v : h \in H_{r,s}, v \in V\}$ (where gv is the $\mathbf{k}G_{r+s}$-action on V) is a well-defined $\mathbf{k}G_r$-action on $V_{H_{r,s}}$.

   Lastly we consider $(V^{U_{r,s}})^{G_s}$. As $\mathbf{k}$-modules $(V^{U_{r,s}})^{G_s} \subset V$. Given $v \in (V^{U_{r,s}})^{G_s}$ and $g \in G_r < G_{r+s}$ define $g.v$ to be the $\mathbf{k}G_{r+s}$-action on V. To show that $gv \in (V^{U_{r,s}})^{G_s}$ we need to show that $gv \in V^{U_{r,s}}$ and $g'gv = gv$ for all $g' \in G_s$. If $g \in G_r < G_{r+s}$, $g' \in G_s < G_{r+s}$ and $u \in U_{r,s}$ then $g^{-1}ug \in U_{r,s}$ and $g^{-1}g'g \in G_s < G_{r+s}$. This shows $gv \in (V^{U_{r,s}})^{G_s}$ and therefore the $\mathbf{k}G_r$-action on $(V^{U_{r,s}})^{G_s}$ is well-defined.

2. As $\mathbf{k}$-modules $V^{U_{r,s}} \subset V$. Given $v \in V^{U_{r,s}}$ and $g \in G_r \times G_s < G_{r+s}$ define $g.v$ to be the $\mathbf{k}G_{r+s}$-action on V. If $g \in G_r \times G_s < G_{r+s}$ and $u \in U_{r,s}$, then $g^{-1}ug \in U_{r,s}$. Therefore the $\mathbf{k}[G_r \times G_s]$-action on $V^{U_{r,s}}$ is well-defined.

3. The $\mathbf{k}$-module isomorphism $V_{H_{r,s}} \simeq V^{H_{r,s}}$ is s morphism of $\mathbf{k}G_r$-modules.

   To see $(V^{U_{r,s}})^{G_s} \simeq V^{H_{r,s}}$ as $\mathbf{k}G_r$-modules observe that as $\mathbf{k}$-modules both $(V^{U_{r,s}})^{G_s}$ and $V^{H_{r,s}}$ are contained in V and their $\mathbf{k}G_r$-actions are given by the $\mathbf{k}G_{r+s}$-action on V. Since $H_{r,s} = U_{r,s}G_s$, $G_s < H_{r,s}$ and $U_{r,s} < H_{r,s}$ it follows that $(V^{U_{r,s}})^{G_s}$ and $V^{H_{r,s}}$ are equal as sets and therefore are isomorphic as $\mathbf{k}G_r$-modules.

□

Next we have a version of Frobenius reciprocity for parabolic induction.

Fact 31. *Let $n, n' \in \mathbf{Z}_{\geq 0}$. Let V be a $\mathbf{k}G_{n+n'}$-module and $\pi \otimes \pi'$ a $\mathbf{k}[G_n \times G_{n'}]$-module. Then*

$$\mathrm{Hom}_{\mathbf{k}G_{n+n'}}(V, \pi \circ \pi') \simeq \mathrm{Hom}_{\mathbf{k}[G_n \times G_{n'}]}(V^{U_{n,n'}}, \pi \otimes \pi')$$

The next Lemma will be a crucial tool in the sections that follow.

Lemma 32. *Let $r, s \in \mathbf{Z}_{\geq 0}$ and $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})_{r+s}$. Then*

$$\mathrm{L}(\underline{\lambda})_{H_{r,s}} \simeq \bigoplus_{\underline{\nu} \sim \underline{\lambda} - s} (\mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{triv}_{G_s}(\mathbf{k}))$$

as $\mathbf{k}G_r$-modules.

Proof. By Lemma 30 it suffices to show this for $(\mathrm{L}(\underline{\lambda})^{U_{r,s}})^{G_s}$ in place of $\mathrm{L}(\underline{\lambda})_{H_{r,s}}$. Moreover $\mathrm{L}(\underline{\lambda})^{U_{r,s}}$ is a $\mathbf{k}[G_r \times G_s]$-module where the action is the G_{r+s}-action on $\mathrm{L}(\underline{\lambda})$ restricted to $G_r \times G_s$. As a $G_r \times G_s$-module, $\mathrm{L}(\underline{\lambda})^{U_{r,s}}$ decomposes as

$$\mathrm{L}(\underline{\lambda})^{U_{r,s}} = \bigoplus_{\substack{|\underline{\nu}|=r \\ |\underline{\nu}'|=s}} c_{\underline{\nu},\underline{\nu}'} \, \mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\nu}') \qquad (\star)$$

We will take $G_s \simeq \{I_r\} \times G_s$-invariants of both sides. As $\mathbf{k}$-modules we have

$$\begin{aligned}\big(\mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\nu}')\big)^{\{I_r\}\times G_s} &= \mathrm{L}(\underline{\nu})^{\{I_r\}} \otimes_{\mathbf{k}} \mathrm{L}(\underline{\nu}')^{G_s} \\ &= \begin{cases} 0 & \mathrm{L}(\underline{\nu}') \not\simeq \mathrm{triv}_{G_s}(\mathbf{k}) \\ \mathrm{triv}_{G_s}(\mathbf{k}) & \mathrm{L}(\underline{\nu}') \simeq \mathrm{triv}_{G_s}(\mathbf{k}) \end{cases}\end{aligned}$$

for all $\mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\nu}')$. Hence as $\mathbf{k}$-modules Therefore

$$(\mathrm{L}(\underline{\lambda})^{U_{r,s}})^{G_s} = \bigoplus_{|\underline{\nu}|=r} c_{\underline{\nu},\mathrm{triv}_{G_s}(\mathbf{k})}\, \mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{triv}_{G_s}(\mathbf{k}) \tag{$\star\star$}$$

The G_r-action on both sides of ($\star\star$) comes from $G_r \times G_s$-actions on $\mathrm{L}(\underline{\lambda})^{U_{r,s}}$ and on $\mathrm{L}(\underline{\nu}) \otimes_{\mathbf{k}} \mathrm{triv}_{G_s}(\mathbf{k})$. These $G_r \times G_s$-actions are the same from ($\star$).

Using Lemma 31

$$c_{\underline{\nu},\mathrm{triv}_{G_s}(\mathbf{k})} = \dim \mathrm{Hom}_{G_r\times G_s}(\mathrm{L}(\underline{\lambda})^{U_{r,s}}, \mathrm{L}(\underline{\nu})\otimes \mathrm{triv}_{G_s}(\mathbf{k})) = \dim \mathrm{Hom}_{G_{r+s}}(\mathrm{L}(\underline{\lambda}), \mathrm{L}(\underline{\nu})\circ \mathrm{triv}_{G_s}(\mathbf{k}))$$

But by Pieri, $\mathrm{L}(\underline{\nu}) \circ \mathrm{triv}_{G_s}(\mathbf{k}) = \oplus_{\underline{\mu}\sim\nu+s} \mathrm{L}(\underline{\mu})$, hence $c_{\underline{\nu},\mathrm{triv}_{G_s}(\mathbf{k})}$ is either 0 or 1. It is 1 if and only if $1 = c_{\underline{\nu},\mathrm{triv}_{G_s}(\mathbf{k})} = \dim \mathrm{Hom}_{G_{r+s}}(\mathrm{L}(\underline{\lambda}), \mathrm{L}(\underline{\mu}))$ for some $\underline{\mu} \sim \underline{\nu} + s$ (i.e. $\mathrm{L}(\underline{\lambda}) = \mathrm{L}(\underline{\mu})$). And so $\underline{\lambda} \sim \underline{\nu} + s$ or equivalently, $\underline{\nu} \sim \underline{\lambda} - s$. □

Chapter 1

Consistent Sequences and Representation Stability.

Consistent Sequences.

Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a sequence of $\mathbf{k}G_n$-modules V_n and $\mathbf{k}$-module morphisms φ_n

$$V_0 \xrightarrow{\varphi_0} V_1 \xrightarrow{\varphi_1} V_2 \xrightarrow{\varphi_2} V_3 \xrightarrow{\varphi_3} \cdots \tag{1.1}$$

We say $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules if for every $n \in \mathbf{Z}_{\geq 0}$ and every $g \in G_n$ the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\varphi_n} & V_{n+1} \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \hat{g}} \\ V_n & \xrightarrow[\varphi_n]{} & V_{n+1} \end{array}$$

commutes.

Example 33. *Fix $a \in \mathbf{Z}_{\geq 0}$. For each $n \in \mathbf{Z}_{\geq 0}$ define $M'(a)_n$ as*

$$M'(a)_n := \begin{cases} 0 & n \leq a \\ \mathrm{Ind}_{G_{n-a}}^{G_n} \mathrm{triv}_{G_{n-a}}(\mathbf{k}) & n > a \end{cases}$$

For each $n \in \mathbf{Z}_{\geq 0}$ with $n \leq a$, define $\varphi_n : M'(a)_n \to M'(a)_{n+1}$ as the zero map. For $n \in \mathbf{Z}_{\geq 0}$ with $n > a$ define $\varphi_n : M'(a)_n \to M'(a)_{n+1}$ for $g \in G_n$ and $u \in \mathbf{k}$ by

$$\begin{aligned} \mathbf{k}G_n \otimes_{\mathbf{k}G_{n-a}} \mathrm{triv}_{G_{n-a}}(\mathbf{k}) &\to \mathbf{k}G_{n+1} \otimes_{\mathbf{k}G_{n+1-a}} \mathrm{triv}_{G_{n+1-a}}(\mathbf{k}) \\ g \otimes u &\mapsto \hat{g} \otimes u \end{aligned}$$

and extend linearly to all of $\mathbf{k}G_n \otimes_{\mathbf{k}G_{n-a}} \mathrm{triv}_{G_{n-a}}(\mathbf{k})$. For all $n \in \mathbf{Z}_{\geq 0}$ and all $g, g' \in G_n$ observe that $\widehat{gg'} = \hat{g}\hat{g'}$. Therefore $(M'(a)_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules.

Representation Stability.

Definition 34. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. We say that $(V_n, \varphi_n)_{n=0}^{\infty}$ is representation stable if there exists an $N \in \mathbf{Z}$ such that for all $n \geq N$, the following three conditions hold:*

(RS1) Injectivity: the map $\varphi_n : V_n \to V_{n+1}$ is injective.

(RS2) Surjectivity: the span of the G_{n+1}-orbit of $\varphi_n(V_n)$ is all of V_{n+1}.

(RS3) Multiplicities: there is a decomposition of V_n as a $\mathbf{k}G_n$-module

$$V_n = \bigoplus_{\underline{\lambda}} c_{\underline{\lambda},n} \, \mathrm{L}(\underline{\lambda}[n])$$

where the multiplicities $c_{\underline{\lambda},n} \leq \infty$ do not depend on n for all $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$.

Remark 35.

1. *Suppose there exists an $N \in \mathbf{Z}$ such that for all $n \geq N$, RS1 holds. Then we say $(V_n, \varphi_n)_{n=0}^{\infty}$ satisfies RS1. Similarly for RS2 and RS3.*

2. *The conclusion of RS3 is that there exists some finite set $\Omega \subset C(\mathscr{C}, \mathscr{P})$ such that for all $n \geq N$:*

   (a) V_n decomposes as a $\mathbf{k}G_n$-module into a sum with constituents $\mathrm{L}(\underline{\lambda}[n])$ with $\underline{\lambda} \in \Omega$.

   (b) $c_{\underline{\lambda},N} = c_{\underline{\lambda},n}$ for all $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$.

Example 36. *Fix $a \in \mathbf{Z}_{\geq 0}$. The consistent sequence $(M'(a)_n, \varphi_n)_{n=0}^{\infty}$ in Example 33 satisfies RS3.*

Proof. This follows from Lemmas 25 and 26. □

Lemma 37. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. Suppose that for each $n \in \mathbf{Z}_{\geq 0}$, $\dim(V_n) < \infty$. If $(V_n, \varphi_n)_{n=0}^{\infty}$ satisfies RS3 (in particular if $(V_n, \varphi_n)_{n=0}^{\infty}$ is representation stable) then there exists $P \in \mathbf{Q}[T]$ and there exists $M \in \mathbf{Z}$ such that for all $n \geq M$, $\dim(V_n) = P(q^n)$.*

Proof. Since the consistent sequence $(V_n, \varphi_n)_{n=0}^{\infty}$ satisfies RS3 there exists an $N \in \mathbf{Z}$ and a finite subset $\Omega \subset C(\mathscr{C}, \mathscr{P})$ such that for all $n \geq N$:

$$V_n = \bigoplus_{\underline{\lambda} \in \Omega} c_{\underline{\lambda},N}\, \mathrm{L}(\underline{\lambda}[n]). \qquad (\star)$$

Moreover each $c_{\underline{\lambda},N}$ is finite for each $\underline{\lambda} \in \Omega$ because $\dim(V_n)$ is finite.

By Lemma 29, for each $\underline{\lambda} \in \Omega$ there is an integer $N_{\underline{\lambda}}$ and a polynomial $P_{\underline{\lambda}} \in \mathbf{Q}[T]$ such that for all $n \geq N_{\underline{\lambda}}$, $\dim(\mathrm{L}(\underline{\lambda}[n])) = P_{\underline{\lambda}}(q^n)$. Set $P(T) = \sum_{\underline{\lambda} \in \Omega} c_{\underline{\lambda},N} P_{\underline{\lambda}}(T)$, $M' = \max_{\underline{\lambda} \in \Omega}\{N_{\underline{\lambda}}\}$ and $M = \max\{N, M'\}$. Then for all $n \in \mathbf{Z}_{\geq 0}$ such that $n \geq M$, $\dim(V_n) = P(q^n)$. □

Corollary 38. *Fix $a \in \mathbf{Z}_{\geq 0}$. Consider the consistent sequence $(M'(a)_n, \varphi_n)_{n=0}^{\infty}$ in Example 33. For n sufficiently large the dimension of $M'(a)_n$ is polynomial in q^n. i.e. there exists a polynomial $P \in \mathbf{Q}[T]$ such that for n sufficiently large*

$$\dim\left(\mathrm{Ind}_{G_{n-a}}^{G_n} \mathrm{triv}_{G_{n-a}}(\mathbf{k})\right) = P(q^n).$$

Weak-Stability of Consistent Sequences.

Recall that if $r, s \in \mathbf{Z}_{\geq 0}$ and V is a $\mathbf{k}G_{r+s}$-module then $V_{H_{r,s}}$ is a $\mathbf{k}G_r$-module. Hence if $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules and $m \in \mathbf{Z}_{\geq 0}$ is fixed, then for all $n > m$, $(V_n)_{H_{m,n-m}}$ is a $\mathbf{k}G_m$-module. The G_r-action on V is the G_{r+s}-action on V restricted to $G_r < G_{r+s}$ (here we are using Convention 14).

Lemma 39. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. Fix $m \in \mathbf{Z}_{\geq 0}$. For* *every $n \in \mathbf{Z}_{\geq 0}$ such that $n > m$ the map $\varphi_n : V_n \to V_{n+1}$ descends to a well-defined map of $\mathbf{k}G_m$-modules:*

$$\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \longrightarrow (V_{n+1})_{H_{m,n+1-m}}.$$

Proof. Consider the composition of $\mathbf{k}$-module morphisms

$$V_n \xrightarrow{\varphi_n} V_{n+1} \xrightarrow{\pi} \frac{V_{n+1}}{\mathrm{span}_{\mathbf{k}}\{h'v' - v' : h' \in H_{m,n+1-m}, v' \in V_{n+1}\}}$$

where π is the standard projection map. Since $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence, for all $h \in H_{m,n-m}$ and all $v \in V_n$

$$\varphi_n(hv - v) = \hat{h}\varphi_n(v) - \varphi_n(v) \in \mathrm{span}_{\mathbf{k}}\{h'v' - v' : h' \in H_{m,n+1-m}, v' \in V_{n+1}\}.$$

Therefore $hv - v \in \ker(\pi \circ \varphi_n)$ and φ_n descends to a well-defined map of $\mathbf{k}$-modules:

$$\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \longrightarrow (V_{n+1})_{H_{m,n+1-m}}.$$

If $g \in G_m$ and $v \in V_n$ then

$$\begin{aligned}
\varphi_{m,n-m}(g\overline{v}) &= \varphi_{m,n-m}(\overline{gv}) \\
&= \pi(\varphi_n(gv)) \\
&= \pi(\hat{g}\varphi_n(v)) \\
&= \hat{g}\pi(\varphi_n(v)) \\
&= \hat{g}\varphi_{m,n-m}(\overline{v}).
\end{aligned}$$

Since g and $\hat{g}$ represent the same element of G_m we conclude that $\varphi_{m,n-m}$ is a morphism of $\mathbf{k}G_m$-modules. □

Definition 40. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. We say that $(V_n, \varphi_n)_{n=0}^{\infty}$ is weakly-stable if for each $m \in \mathbf{Z}_{\geq 0}$, there exists an $s(m) \in \mathbf{Z}_{\geq 0}$ such that for each $n - m \geq s(m)$, the map*

$$\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \longrightarrow (V_{n+1})_{H_{m,n+1-m}}.$$

is an isomorphism of $\mathbf{k}$-modules (equivalently of $\mathbf{k}G_m$-modules).

Remark 41. *Suppose $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules and $\dim(V_n) < \infty$ for each $n \in \mathbf{Z}_{\geq 0}$. Then to show $\varphi_{m,n-m}$ is bijective for n sufficiently large, it suffices to show that $\varphi_{m,n-m}$ is surjective for n sufficiently large.*

Lemma 42. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. Fix $m \in \mathbf{Z}_{\geq 0}$. For every $n \in \mathbf{Z}_{\geq 0}$ such that $n > m$ the map $\varphi_n : V_n \to V_{n+1}$ descends to a well-defined map of $\mathbf{k}G_m$-modules:*

$$\varphi'_{m,n-m} : (V_n)_{H'_{m,n-m}} \to (V_{n+1})_{H'_{m,n+1-m}}.$$

Moreover the diagram

$$\begin{array}{ccc}
(V_n)_{H'_{m,n-m}} & \xrightarrow{\varphi'_{m,n-m}} & (V_{n+1})_{H'_{m,n+1-m}} \\
\downarrow & & \downarrow \\
(V_n)_{H_{m,n-m}} & \xrightarrow{\varphi_{m,n-m}} & (V_{n+1})_{H_{m,n+1-m}}
\end{array}$$

commutes.

Proof. Consider the composition of $\mathbf{k}$-module morphisms

$$V_n \xrightarrow{\varphi_n} V_{n+1} \xrightarrow{\pi} \frac{V_{n+1}}{\mathrm{span}_{\mathbf{k}}\{h'v' - v' : h' \in H'_{m,n+1-m}, v' \in V_{n+1}\}}$$

where π is the standard projection map. Since $(V_n, \varphi_n)_{n=0}^{\infty}$ is a consistent sequence, for all $h \in H'_{m,n-m}$ and all $v \in V_n$

$$\varphi_n(hv - v) = \hat{h}\varphi_n(v) - \varphi_n(v) \in \mathrm{span}_{\mathbf{k}}\{h'v' - v' : h' \in H'_{m,n+1-m}, v' \in V_{n+1}\}.$$

Therefore $hv - v \in \ker(\pi \circ \varphi_n)$ and φ_n descends to a well-defined map of $\mathbf{k}$-modules:

$$\varphi'_{m,n-m} : (V_n)_{H'_{m,n-m}} \longrightarrow (V_{n+1})_{H'_{m,n+1-m}}.$$

If $g \in G_m$ and $v \in V_n$ then

$$\begin{aligned}
\varphi_{m,n-m}(g\overline{v}) &= \varphi_{m,n-m}(\overline{gv}) \\
&= \pi(\varphi_n(gv)) \\
&= \pi(\hat{g}\varphi_n(v)) \\
&= \hat{g}\pi(\varphi_n(v)) \\
&= \hat{g}\varphi_{m,n-m}(\overline{v}).
\end{aligned}$$

Since g and $\hat{g}$ represent the same element of G_m we conclude that $\varphi'_{m,n-m}$ is a morphism of $\mathbf{k}G_m$-modules.

The maps $\varphi_{m,n-m}$ and $\varphi'_{m,n-m}$ are constructed in the same way, so the diagram commutes. $\square$

Weight of a Consistent Sequence.

Definition 43. *Let* $(V_n, \varphi_n)_{n=0}^{\infty}$ *be a consistent sequence of* $\mathbf{k}G_n$*-modules. Decompose each* V_n

$$V_n = \bigoplus_{\underline{\lambda}} c_{\underline{\lambda},n} \, \mathrm{L}(\underline{\lambda}[n])$$

where the sum is over all $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ *such that* $\underline{\lambda}[n]$ *is defined. If*

$$\max\{|\underline{\lambda}| : c_{\underline{\lambda},k} \neq 0 \quad \textit{for some } k \in \mathbf{Z}_{\geq 0}\} < \infty$$

then we say the weight of $(V_n, \varphi_n)_{n=0}^{\infty}$ *is finite or that* $(V_n, \varphi_n)_{n=0}^{\infty}$ *is weight-bounded. If* $(V_n, \varphi_n)_{n=0}^{\infty}$ *is weight-bounded we call*

$$\max\{|\underline{\lambda}| : c_{\underline{\lambda},k} \neq 0 \quad \textit{for some } k \in \mathbf{Z}_{\geq 0}\}$$

the weight of $(V_n, \varphi_n)_{n=0}^{\infty}$.

Remark 44.

1. *If* $(V_n, \varphi_n)_{n=0}^{\infty}$ *is a consistent sequence with finite weight then there exists a finite subset* $\Omega \subset C(\mathscr{C}, \mathscr{P})$ *such that every* V_n *can be built from* $\mathrm{L}(\underline{\lambda}[n])$ *with* $\underline{\lambda} \in \Omega$.

2. *Conversely, if there exists a finite subset* $\Omega \subset C(\mathscr{C}, \mathscr{P})$ *such that every* V_n *can be*

built from $\mathrm{L}(\underline{\lambda}[n])$ *with* $\underline{\lambda} \in \Omega$*, then* $(V_n, \varphi_n)_{n=0}^{\infty}$ *has finite weight. In particular if* $(V_n, \varphi_n)_{n=0}^{\infty}$ *satisfies RS3, then* $(V_n, \varphi_n)_{n=0}^{\infty}$ *has finite weight.*

Corollary 45. *Fix* $a \in \mathbf{Z}_{\geq 0}$*. The consistent sequence* $(M'(a)_n, \varphi_n)_{n=0}^{\infty}$ *in Example 33 is weight-bounded.*

Proof. The consistent sequence $(M'(a)_n, \varphi_n)_{n=0}^{\infty}$ satisfies RS3 by Example 36, hence is weight-bounded. □

Weakly-Stable and Weight-Bounded Consistent Sequences.

In this section we show that consistent sequences which are weakly-stable and weight-bounded satisfy RS1 and RS3.

Theorem 46. *Let* $(V_n, \varphi_n)_{n=0}^{\infty}$ *be a consistent sequence of* $\mathbf{k}G_n$*-modules. If* $(V_n, \varphi_n)_{n=0}^{\infty}$ *is weakly-stable and weight-bounded then* $(V_n, \varphi_n)_{n=0}^{\infty}$ *satisfies RS1.*

Proof. Let a be the weight of $(V_n, \varphi_n)_{n=0}^{\infty}$. Let $s(a) \in \mathbf{Z}$ be such that for all $n - a \geq s(a)$,

$$\varphi_{a,n-a} : (V_n)_{H_{a,n-a}} \longrightarrow (V_{n+1})_{H_{a,n+1-a}}$$

is an isomorphism. We claim that for all $n \geq a + s(a)$, $\varphi_n : V_n \to V_{n+1}$ is injective.

Let $n \geq a + s(a)$ and let K_n be the kernel of $\varphi_n : V_n \to V_{n+1}$. For each n, K_n is a $\mathbf{k}G_n$-module by restricting the G_n-action on V_n to K_n. We will show that $K_n = 0$.

Since $\varphi_{a,n-a}$ is an isomorphism and $(K_n)_{H_{a,n-a}} \subset \ker(\varphi_{a,n-a})$, we conclude that $(K_n)_{H_{a,n-a}} = 0$.

Suppose that $K_n \neq 0$. Then K_n contains an irreducible subrepresentation $\mathrm{L}(\underline{\lambda}[n])$ for some $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. Moreover $|\underline{\lambda}| \leq a$ since K_n is a $\mathbf{k}G_n$-submodule of V_n. The number of columns in $\underline{\lambda}[n](\iota)$ is $n - |\underline{\lambda}|$ which is greater than or equal to $n - a$. Therefore there exists a $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})$ such that

$$\underline{\mu} \sim \underline{\lambda} - (n - a).$$

By Lemma 32 this implies $(K_n)_{H_{a,n-a}} \neq 0$ which is a contradiction. $\square$

Theorem 47. *Let $(V_n, \varphi_n)_{n=0}^{\infty}$ be a consistent sequence of $\mathbf{k}G_n$-modules. If $(V_n, \varphi_n)_{n=0}^{\infty}$ is weakly-stable and weight-bounded then $(V_n, \varphi_n)_{n=0}^{\infty}$ satisfies RS3.*

Proof. Let a be the weight of $(V_n, \varphi_n)_{n=0}^{\infty}$. Since $(V_n, \varphi_n)_{n=0}^{\infty}$ is weakly-stable, for each $m \leq a$ there exists $s(m)$ such that

$$\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \longrightarrow (V_{n+1})_{H_{m,n+1-m}}$$

is an isomorphism for all $n - m \geq s(m)$. Let $s := \max_{0 \leq m \leq a}\{s(m)\}$. Then if $m \leq a$ and $n - m \geq s$, $\varphi_{m,n-m}$ is an isomorphism of $\mathbf{k}G_m$-modules.

For every $n \in \mathbf{Z}_{\geq 0}$ and every $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ such that $n \geq |\underline{\lambda}| + \underline{\lambda}(\iota)_1$, let $c_{\underline{\lambda},n}$ denote the multiplicity of $\mathrm{L}(\underline{\lambda}[n])$ in V_n. Then for each $n \in \mathbf{Z}_{\geq 0}$, V_n decomposes as a $\mathbf{k}G_n$-module as

$$V_n = \bigoplus_{|\underline{\lambda}| \leq a} c_{\underline{\lambda},n}\, \mathrm{L}(\underline{\lambda}[n]).$$

Set $N := \max\{a + s, 2a\}$. We claim that for $n \geq N$ and $|\underline{\lambda}| \leq a$ that $c_{\underline{\lambda},n} = c_{\underline{\lambda},n+1}$.

Let $n \geq N$. We use induction on $|\underline{\lambda}|$. Suppose the claim is true for all $|\underline{\lambda}| < m$. We will show that for $|\underline{\lambda}| = m$ that $c_{\underline{\lambda},n} = c_{\underline{\lambda},n+1}$.

By Lemma 32

$$(V_n)_{H_{m,n-m}} = \bigoplus_{|\underline{\lambda}| \leq a} c_{\underline{\lambda},n} \left(\bigoplus_{\underline{\mu} \sim \underline{\lambda}[n]-(n-m)} \mathrm{L}(\underline{\mu}) \right)$$

Recalling that the number of columns in $\underline{\lambda}[n](\iota)$ is $n - |\underline{\lambda}|$ we make two observations:

- If $|\underline{\lambda}| > m$, then $n - |\underline{\lambda}| < n - m$. In this case there are no $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})$ such that $\underline{\mu} \sim \underline{\lambda}[n] - (n-m)$.

- If $|\underline{\lambda}| = m$, then $n - |\underline{\lambda}| = n - m$. In this case there is exactly one $\underline{\mu} \in C(\mathscr{C}, \mathscr{P})$ such that $\underline{\mu} \sim \underline{\lambda}[n] - (n-m)$: $\underline{\mu} = \underline{\lambda}$.

Therefore

$$(V_n)_{H_{m,n-m}} = \left(\bigoplus_{|\underline{\lambda}| < m} c_{\underline{\lambda},n} \left(\bigoplus_{\underline{\mu} \sim \underline{\lambda}[n]-(n-m)} \mathrm{L}(\underline{\mu}) \right) \right) \oplus \left(\bigoplus_{|\underline{\lambda}| = m} c_{\underline{\lambda},n} \, \mathrm{L}(\underline{\lambda}) \right).$$

Since $n - m \geq N - a \geq s$, $\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \to (V_{n+1})_{H_{m,n+1-m}}$ is an isomorphism of $\mathbf{k}G_m$-modules. Moreover, if $|\underline{\lambda}| < m$ then

$$\{\underline{\mu} : \underline{\mu} \sim \underline{\lambda}[n] - (n-m)\} = \{\underline{\mu} : \underline{\mu} \sim \underline{\lambda}[n+1] - (n+1-m)\}.$$

Combining these these facts and the induction hypothesis we have

$$\bigoplus_{|\underline{\lambda}| = m} c_{\underline{\lambda},n} \, \mathrm{L}(\underline{\lambda}) \simeq \bigoplus_{|\underline{\lambda}| = m} c_{\underline{\lambda},n+1} \, \mathrm{L}(\underline{\lambda})$$

as $\mathbf{k}G_m$-modules. This implies that for all $|\underline{\lambda}| = m$, $c_{\underline{\lambda},n} = c_{\underline{\lambda},n+1}$ which completes the induction argument. $\square$

Chapter 2

VI-modules.

In this chapter we introduce VI-modules. Each VI-module produces a consistent sequence of $\mathbf{k}G_n$-modules. We show that the consistent sequence associated to a finitely generated VI-module is weakly-stable and weight-bounded and therefore is representation stable.

To prove this result we first show it holds true for VI-modules of the form $V_n = \mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{F}_q^a, \mathbf{F}_q^n)$ (for a fixed integer a). To finish the proof we use exactness of invariants/coinvariants and that every finitely generated VI-module is the image of a direct sum of VI-modules of the form $V_n = \mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{F}_q^a, \mathbf{F}_q^n)$ (for various integers a).

We conclude this chapter by looking at the tensor product of VI-modules. The similarity between the parameterization of irreducible $\mathbf{k}S_n$-modules and the parameterization of irreducible $\mathbf{k}G_n$-modules suggests a G_n-version of Murnaghan's theorem. We use tensor products of VI-modules to prove this G_n-analog.

The Category VI.

Let VI be the category whose objects are $\{\mathbf{F}_q^n : n \in \mathbf{Z}_{\geq 0}\}$ and whose morphisms are linear injections. For every $n \in \mathbf{Z}_{\geq 0}$ we let $\mathbf{n} := \mathbf{F}_q^n$. So for example, instead of writing '$\mathrm{Hom}_{\mathrm{VI}}(\mathbf{F}_q^m, \mathbf{F}_q^n)$' we write '$\mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$'.

Fact 48. *For every $n \in \mathbf{Z}_{\geq 0}$, $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{n}, \mathbf{n}) \simeq G_n$.*

Given $n \in \mathbf{Z}_{\geq 0}$ we let $I_{n,n+1} : \mathbf{F}_q^n \hookrightarrow \mathbf{F}_q^{n+1}$ denote the standard inclusion. We let $I_{m,n} : \mathbf{F}_q^m \hookrightarrow \mathbf{F}_q^n$ denote the composition

$$I_{n-1,n} \circ I_{n-2,n-1} \circ \cdots \circ I_{m,m+1}.$$

Fact 49. *Let $m, n \in \mathbf{Z}_{\geq 0}$ with $m \leq n$. Then G_n acts transitively on $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$ by post-composition and the stabilizer of $I_{m,n}$ under this action is $H_{m,n-m}$.*

The Category VI-mod.

Let VI-mod be the category **k**VI-Mod. i.e. VI-mod is the category of functors

$$\mathrm{VI} \longrightarrow \mathbf{k}\text{-mod}.$$

The objects of VI-mod are called VI-modules. The morphisms of VI-mod are called VI-module morphisms.

Let V be a VI-module. For every $n \in \mathbf{Z}_{\geq 0}$ there is a **k**-module $V(\mathbf{n})$. For every $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$ there is a **k**-module morphism $V(f) : V(\mathbf{m}) \to V(\mathbf{n})$. To clear up notation we always write 'V_n' in place of '$V(\mathbf{n})$' and we sometimes write 'f_*' in place of '$V(f)$'.

If W is a VI-module and V is a submodule of W we say that V is a VI-submodule of W. If V is a VI-module and $\Sigma \subset \sqcup_{n\geq 0} V_n$, then the VI-submodule $\mathrm{span}_V(\Sigma)$ of V is

$$\mathrm{span}_V(\Sigma)_n = \mathrm{span}\{v \in V_n : v = f_*(\sigma) \text{ for some } \sigma \in \Sigma \cap V_a \text{ and } f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n})\}$$

for each $n \in \mathbf{Z}_{\geq 0}$.

The following two facts tell us that the collection of $\mathbf{k}$-modules $\{V_n\}_{n\in\mathbf{Z}_{\geq 0}}$ from a VI-module V form a consistent sequence, $(V_n, \varphi_n)_{n=0}^{\infty}$, of $\mathbf{k}G_n$-modules. The inclusions $\mathbf{n} \hookrightarrow \mathbf{n+1}$ are maps in VI, and through V induce the $\mathbf{k}$-module morphisms $\varphi_n : V_n \to V_{n+1}$.

Fact 50. *Let V be a VI-module. For every $n \in \mathbf{Z}_{\geq 0}$, V_n is a $\mathbf{k}G_n$-module. The G_n-action on V_n is given by $g.v = g_*(v)$ for $g \in G_n$ and $v \in V_n$.*

Fact 51. *Let V be a VI-module. Then $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules. We call $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ the consistent sequence induced from the VI-module V.*

The maps $\varphi_{m,n-m}$ in Definition 40 of weak-stability commute with VI-module morphisms as we now show.

Lemma 52. *Let V and W be VI-modules and let $\psi : V \longrightarrow W$ be a morphism of VI-modules. Fix $m \in \mathbf{Z}_{\geq 0}$. For all $n \in \mathbf{Z}$ such that $n \geq m$ the $\mathbf{k}$-module morphism $\psi_n : V_n \to$*

W_n descends to a map $\overline{\psi}_n : (V_n)_{H_{m,n-m}} \to (W_n)_{H_{m,n-m}}$. Moreover the diagram

$$\begin{array}{ccc} (V_n)_{H_{m,n-m}} & \xrightarrow{\overline{\psi}_n} & (W_n)_{H_{m,n-m}} \\ \downarrow{\scriptstyle \varphi_{m,n-m}} & & \downarrow{\scriptstyle \varphi_{m,n-m}} \\ (V_{n+1})_{H_{m,n+1-m}} & \xrightarrow{\overline{\psi}_{n+1}} & (W_{n+1})_{H_{m,n+1-m}} \end{array} \tag{$\star$}$$

commutes.

Proof. Consider the composition of **k**-module morphisms

$$V_n \xrightarrow{\psi_n} W_n \xrightarrow{\pi} \frac{W_n}{\operatorname{span}_{\mathbf{k}}\{hw - w : h \in H_{m,n-m}, w \in W_n\}} \tag{$\dagger$}$$

where π is the standard projection map.

For all $h \in H_{m,n-m}$ the diagram

$$\begin{array}{ccc} V_n & \xrightarrow{\psi_n} & W_n \\ \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle h} \\ V_n & \xrightarrow{\psi_n} & W_n \end{array}$$

commutes. So for all $h \in H_{m,n-m}$ and all $v \in V_n$

$$\psi_n(hv - v) = h\psi_n(v) - \psi_n(v) \in \operatorname{span}_{\mathbf{k}}\{hw - w : h \in H_{m,n-m}, w \in W_n\}.$$

Therefore ($\dagger$) leads to the well-defined map $\overline{\psi}_n$.

The vertical maps in ($\star$) are constructed from $V(I_{n,n+1})$ and $W(I_{n,n+1})$ and ψ commutes with both of these. All the quotients in ($\star$) are well-defined and all the maps are well-defined. By a direct calculation ($\star$) commutes. $\square$

Lemma 53. *Let V and W be VI-modules. Suppose that the induced consistent sequences $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ and $(W_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ are weakly-stable and weight-bounded. Then the consistent sequence $(V_n \oplus W_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ induced by the VI-module $V \oplus W$ is weakly-stable and weight-bounded.*

Proof. Fix $m \in \mathbf{Z}_{\geq 0}$. Let $s_V(m)$ and $s_W(m)$ be integers such that for all $n - m \geq s_V(m)$, $\varphi_{m,n-m} : (V_n)_{H_{m,n-m}} \to (V_{n+1})_{H_{m,n-m}}$ is an isomorphism and for all $n - m \geq s_W(m)$, $\varphi_{m,n-m} : (W_n)_{H_{m,n-m}} \to (W_{n+1})_{H_{m,n-m}}$ is an isomorphism.

The diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & (V_n)_{H_{m,n-m}} & \longrightarrow & (V_n \oplus W_n)_{H_{m,n-m}} & \longrightarrow & (W_n)_{H_{m,n-m}} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \varphi_{m,n-m}} & & \downarrow{\scriptstyle \varphi_{m,n-m}} & & \downarrow{\scriptstyle \varphi_{m,n-m}} & & \\
0 & \longrightarrow & (V_{n+1})_{H_{m,n+1-m}} & \longrightarrow & (V_{n+1} \oplus W_{n+1})_{H_{m,n+1-m}} & \longrightarrow & (W_{n+1})_{H_{m,n+1-m}} & \longrightarrow & 0
\end{array}$$

is commutative and has exact rows because $\varphi_{m,n-m}$ commutes with VI-module morphisms (Lemma 52) and taking invariants is exact (Lemma 2). If $n - m \geq \max\{s_V(m), s_W(m)\}$, the outer vertical maps are isomorphisms and hence the middle vertical map is an isomorphism. Therefore $(V_n \oplus W_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable.

Since constituents of $V_n \oplus W_n$ are sums of constituents of V_n and W_n it follows that $(V_n \oplus W_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weight-bounded. $\square$

The VI-module $M(a)$.

Fix $a \in \mathbf{Z}_{\geq 0}$. Let $M(a)$ be the VI-module defined by $M(a)_n := \mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n})$, the free $\mathbf{k}$-module with basis indexed by $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n})$ and for $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$, $f_* : M(a)_m \to M(a)_n$ the map given by post-composition on $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{m})$ and extended linearly.

By Fact 3 and Lemma 27, for $n > a$,

$$M(a)_n \simeq \mathbf{k}G_n \otimes_{\mathbf{k}H_{a,n-a}} \mathrm{triv}_{H_{a,n-a}}(\mathbf{k}) \simeq \mathbf{k}G_a \circ \mathrm{triv}_{G_{n-a}}(\mathbf{k})$$

as $\mathbf{k}G_n$-modules.

Lemma 54. *Fix $a \in \mathbf{Z}_{\geq 0}$. The consistent sequence $(M(a)_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ induced by the VI-module $M(a)$ is weakly-stable and weight-bounded.*

Proof. First we show weak-stability. Fix $m \in \mathbf{Z}_{\geq 0}$. We claim that for all $n \in \mathbf{Z}_{\geq 0}$ such that $n - m > 0$, $\varphi_{m,n-m}$ is surjective. By Remark 41 this implies weak-stability.

Write the elements of $M(a)_{n+1}$ as matrices of rank a. For every such matrix A, there exists an $h \in H_{m,n+1-m}$ such that the last row of hA is zero. The element hA lies in the image of $(I_{n,n+1})_* : M(a)_n \to M(a)_{n+1}$. This shows $\varphi_{m,n-m}$ is surjective and hence $(M(a)_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable.

Next we show weight-bounded. Suppose $\mathrm{L}(\underline{\mu})$ is a constituent of $M(a)_n$. Then by Lemma 22, there exists $\underline{\nu} \in C(\mathscr{C}, \mathscr{P})_a$ such that $\underline{\mu} \sim \underline{\nu} + (n - a)$. Hence the number of columns of $\underline{\mu}(\iota)$ is at least $n - a$. If $\underline{\mu} = \underline{\lambda}[n]$, then the number of columns of $\underline{\mu}(\iota)$ is exactly $n - |\underline{\lambda}|$. Hence $n - |\underline{\lambda}| \geq n - a$ which implies $|\underline{\lambda}| \leq a$. This shows $(M(a)_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weight-bounded. $\square$

The previous lemma told us that the VI-module $M(a)$ produces a consistent sequence of $\mathbf{k}G_n$-modules which is weakly-stable and weight bounded, hence satisfies RS1 and RS3. The next lemma tells us that the consistent sequence induced by $M(a)$ also satisfies RS2.

Lemma 55. *Fix $a \in \mathbf{Z}_{\geq 0}$. The consistent sequence $(M(a)_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ induced by the VI-module $M(a)$ satisfies RS2.*

Proof. Let $n \in \mathbf{Z}_{\geq 0}$ and suppose $n > a$. There exists an $\alpha \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1})$ such that α is in the image of

$$(I_{n,n+1})_* : \mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n}) \to \mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1}).$$

Since G_{n+1} acts transitively on $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1})$, the G_{n+1}-orbit of α is $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1})$. Therefore the G_{n+1}-orbit of the image of $(I_{n,n+1})_*$ contains $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1})$. Hence the span of the G_{n+1}-orbit of the image of $(I_{n,n+1})_*$ is $\mathbf{k}\mathrm{Hom}_{\mathrm{VI}}(\mathbf{a}, \mathbf{n+1})$. □

Finitely Generated VI-modules.

Suppose M is VI-module which induces a consistent sequence that is weakly-stable and weight-bounded. If V is a VI-module and there exists a surjection of VI-modules $M \twoheadrightarrow V$ then it is easy to see that the consistent sequence induced by V is itself weakly-stable and weight-bounded. The next Lemma tells us that if V is finitely generated then there is a VI-module M and a surjection of VI-modules $M \twoheadrightarrow V$, moreover this VI-module M is a sum of VI-modules of the form $M(a)$ and hence induces a consistent sequence which is weakly-stable and weight-bounded.

Lemma 56. *Let V be a VI-module. Then V is finitely generated if and only if there exists* *a surjection of VI-modules*

$$M(d_1) \oplus M(d_2) \oplus \cdots \oplus M(d_k) \twoheadrightarrow V$$

Proof. First we observe that the image of any VI-module morphism

$$M(d_1) \oplus \cdots \oplus M(d_k) \xrightarrow{\varphi} V$$

is $\mathrm{span}_V(\{\varphi_{d_1}(I_{d_1}), \ldots, \varphi_{d_k}(I_{d_k})\})$ since each $M(d_i)$ is generated as a VI-module by $I_{d_i} \in M(d_i)_{d_i}$.

Now suppose there exists a surjection

$$\varphi : M(d_1) \oplus \cdots \oplus M(d_k) \twoheadrightarrow V.$$

Then V is generated by $\{\varphi_{d_1}(I_{d_1}), \ldots, \varphi_{d_k}(I_{d_k})\}$ hence is finitely generated.

Conversely suppose that V is generated by $v_1, \ldots, v_k$ with $v_i \in V_{d_i}$. Let $n \in \mathbf{Z}_{\geq 0}$ and let

$$\varphi_n : M(d_1)_n \oplus \cdots \oplus M(d_k)_n \to V_n$$

be the map given on basis elements $f \in M(d_i)_n$ by $\varphi_n(f) = f_*(v_i)$ and extended linearly. The maps $\{\varphi_n\}_{n \in \mathbf{Z}_{\geq 0}}$ form a VI-module morphism $M(d_1) \oplus \cdots \oplus M(d_k) \to V$ whose image is $\mathrm{span}_V(\{v_1, \ldots, v_k\}) = V$. □

Corollary 57. *Let V and W be VI-modules. Suppose there exists a surjection of VI-modules*

$$V \twoheadrightarrow W.$$

If V is finitely generated then W is finitely generated.

We are now ready to show that finitely generated VI-modules produce consistent sequences which are representation stable. We prove that finitely generated VI-modules produce consistent sequences which are weakly-stable and weight-bounded, this establishes RS1 and RS3. All of the proofs rely on the VI-modules $M(a)$ and the surjection in Lemma 56. We collect the relevant results into the following Fact.

Fact 58. *Let V be a finitely generated VI-module. There exists a VI-module M and a surjection of VI-modules $M \twoheadrightarrow V$. Moreover the induced consistent sequence $(M_n, (I_{n,n+1})_*)_{n=0}^{\infty}$*

- *is weakly-stable,*
- *is weight-bounded,*
- *satisfies RS2.*

Lemma 59. *Let V be a finitely generated VI-module. Then the induced consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable.*

Proof. By Fact 58 there exists a VI-module M and a surjection of VI-modules $M \overset{\pi}{\twoheadrightarrow} V$. Moreover, the induced consistent sequence $(M_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable.

Fix $m \in \mathbf{Z}_{\geq 0}$. Let $s_M(m) \in \mathbf{Z}_{\geq 0}$ be such that for all $n - m \geq s_M(m)$

$$\varphi_{m,n-m} : (M_n)_{H_{m,n-m}} \to (M_{n+1})_{H_{m,n+1-m}}$$

is an isomorphism. For $n \in \mathbf{Z}_{\geq 0}$ such that $n - m \geq s_M(m)$ consider

$$\begin{array}{ccc} (M_n)_{H_{m,n-m}} & \xrightarrow{\varphi_{m,n-m}} & (M_{n+1})_{H_{m,n+1-m}} \\ \downarrow & & \downarrow \\ (V_n)_{H_{m,n-m}} & \xrightarrow[\varphi_{m,n-m}]{} & (V_{n+1})_{H_{m,n+1-m}} \end{array}$$

The vertical maps are surjections because π_n and π_{n+1} are. The top horizontal map is surjective because $n - m \geq s_M(m)$. The diagram commutes because $\varphi_{m,n-m}$ commutes with VI-module morphisms. Therefore the bottom horizontal map is surjective for $n - m \geq s_M(m)$. By Remark 41 this implies the consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable. $\square$

Lemma 60. *Let V be a finitely generated VI-module. Then the induced consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weight-bounded.*

Proof. By Fact 58 there exists a VI-module M and a surjection of VI-modules $M \overset{\pi}{\twoheadrightarrow} V$. Moreover, the induced consistent sequence $(M_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weight-bounded.

The surjection implies that constituents of V_n come from constituents of M_n. Therefore $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weight-bounded. $\square$

Lemma 61. *Let V be a finitely generated VI-module. Then the induced consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ satisfies RS2.*

Proof. By Fact 58 there exists a VI-module M and a surjection of VI-modules $M \overset{\pi}{\twoheadrightarrow} V$. Moreover, the induced consistent sequence $(M_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ satisfies RS2. Let $N \in \mathbf{Z}_{\geq 0}$ be such that for all $n \geq N$, the span of the G_{n+1}-orbit of the image of $(I_{n,n+1})_* : M_n \to M_{n+1}$ is M_{n+1}.

It follows that for $n \geq N$, the span of the G_{n+1}-orbit of the image of $(I_{n,n+1})_* : V_n \to V_{n+1}$ is V_{n+1}. □

Theorem 62. *Let V be a finitely generated VI-module. Then the induced consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is representation stable.*

Proof. We need to show that the consistent sequence $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ satisfies RS1, RS2, and RS3.

By Lemma 59 and Lemma 60, $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is weakly-stable and weight-bounded. Hence by Theorem 46 and Theorem 47, $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ satisfies RS1 and RS3.

By Lemma 61, $(V_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ satisfies RS2. □

Tensor Products of VI-modules.

In this Section we prove a G_n-analog of Murnaghan's theorem.

Let V and W be VI-modules. The VI-module $V \otimes W$ is defined on objects by $(V \otimes W)_n := V_n \otimes_{\mathbf{k}} W_n$. For $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$, $(V \otimes W)(f) := V(f) \otimes W(f)$. Note that for $g \in G_n$ and $v \otimes w \in V_n \otimes_{\mathbf{k}} W_n$ we have $g_*(v \otimes w) = (g_* v) \otimes (g_* w)$. Hence for each $n \in \mathbf{Z}_{\geq 0}$, the $\mathbf{k}G_n$-module $(V \otimes W)_n$ is the tensor product of the $\mathbf{k}G_n$-modules V_n and W_n.

Our first task is to prove that the tensor product of finitely generated VI-modules is finitely generated.

Lemma 63. *Let $a_1, a_2, n \in \mathbf{Z}_{\geq 0}$. If $n \geq a_1 + a_2$, then for all*

$$(f_1, f_2) \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{n}) \times \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_2}, \mathbf{n})$$

there exists a

$$(g_1, g_2) \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{a_1} + \mathbf{a_2}) \times \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_2}, \mathbf{a_1} + \mathbf{a_2})$$

and there exists an $h \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1} + \mathbf{a_2}, \mathbf{n})$ such that

$$(f_1, f_2) = (h \circ g_1, h \circ g_2).$$

Proof. Given (f_1, f_2), set $g_1 \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{a_1} + \mathbf{a_2})$ to be I_{a_1, a_1+a_2}.

Next we construct h column by column. Set the first a_1 columns of h to be the a_1 columns of f_1. For each column C of f_2, if C is in the column span of h, do nothing. If C is not in the column span of h, include it as a column of h. After doing this for all columns of f_2, if h needs more columns, add standard basis vectors that are not in the span of the columns of h.

Since each column of f_2 is in the span of the columns of h, it follows that there is a matrix g_1 (consisting of scalars needed to write columns of f_2 as linear combinations of columns of h) such that $hg_2 = f_2$. We conclude that g_2 is injective because h and f_2 are.

By construction

$$(f_1, f_2) = (h \circ g_1, h \circ g_2).$$

□

Lemma 64. *Fix $a_1, a_2 \in \mathbf{Z}_{\geq 0}$. The VI-module $M(a_1) \otimes M(a_2)$ is finitely generated.*

Proof. Let $n \geq a_1 + a_2$. By definition

$$(M(a_1) \otimes M(a_2))_n = \mathrm{span}_{\mathbf{k}}\{(f_1, f_2) \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{n}) \times \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_2}, \mathbf{n})\}$$

By Lemma 63, for every

$$(f_1, f_2) \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{n}) \times \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_2}, \mathbf{n})$$

there exists a

$$(g_1, g_2) \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1}, \mathbf{a_1} + \mathbf{a_2}) \times \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_2}, \mathbf{a_1} + \mathbf{a_2})$$

and there exists an $h \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{a_1} + \mathbf{a_2}, \mathbf{n})$ such that

$$(f_1, f_2) = (h \circ g_1, h \circ g_2).$$

Since $h_*(g_1, g_2) := (h \circ g_1, h \circ g_2)$ we conclude that $M(a_1) \otimes M(a_2)$ is finitely generated. $\square$

Lemma 65. *Let V and W be finitely generated VI-modules. The VI-module $V \otimes W$ is finitely generated.*

Proof. The VI-modules V and W are quotients of VI-modules of the form $M(d_1) \oplus \cdots \oplus M(d_k)$. Since VI-modules of the form $M(a_1) \otimes M(a_2)$ are finitely generated, there exists a finitely generated VI-module M and a surjection $M \twoheadrightarrow V \otimes W$. Therefore $V \otimes W$ is finitely generated. $\square$

We now show that for each $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$ there exists a VI-module $\mathbf{L}(\underline{\lambda})$ which is finitely generated and such that $\mathbf{L}(\underline{\lambda})_n = \mathrm{L}(\underline{\lambda}[n])$ (for n large enough). Then for $\underline{\mu}, \underline{\mu}' \in C(\mathscr{C}, \mathscr{P})$, there exists a finitely generated VI-module $\mathbf{L}(\underline{\mu}) \otimes \mathbf{L}(\underline{\mu}')$ and for n large enough, $(\mathbf{L}(\underline{\mu}) \otimes \mathbf{L}(\underline{\mu}'))_n = \mathrm{L}(\underline{\mu}[n]) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\mu}'[n])$ which will lead us to our G_n-analog of Murnaghan's theorem.

We introduce a functor M that assigns a VI-module to a $\mathbf{k}G_a$-module. The VI-module $\mathbf{L}(\underline{\lambda})$ will be a VI-submodule of the VI-module $M(\mathrm{L}(\underline{\lambda}))$.

Fix $a \in \mathbf{Z}_{\geq 0}$. We define the functor M as

$$\begin{aligned} M : G_a\text{-Rep} &\longrightarrow \text{VI-mod} \\ W_a &\mapsto M(a) \otimes_{\mathbf{k}G_a} W_a \\ (V_a \xrightarrow{\varphi} W_a) &\mapsto (M(a) \otimes V_a \xrightarrow{\mathrm{id} \otimes \varphi} M(a) \otimes W_a) \end{aligned}$$

where $M(a) \otimes_{\mathbf{k}G_a} W_a$ is the VI-module given by $(M(a) \otimes_{\mathbf{k}G_a} W_a)_n := M(a)_n \otimes_{\mathbf{k}G_a} W_a$.

Fact 66. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. By Pieri:*

$$M(\mathrm{L}(\underline{\lambda}))_n = \begin{cases} 0 & n < |\underline{\lambda}| \\ \bigoplus_{\underline{\nu} \sim \underline{\lambda} + (n - |\underline{\lambda}|)} \mathrm{L}(\underline{\nu}) & n \geq |\underline{\lambda}| \end{cases}$$

as a $\mathbf{k}G_n$-module.

Lemma 67. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. There exists a VI-submodule, W, of $M(\mathrm{L}(\underline{\lambda}))$ such that*

$$W_n = \begin{cases} \mathrm{L}(\underline{\lambda}[n]) & n \geq |\underline{\lambda}| + \underline{\lambda}(\iota)_1 \\ 0 & \text{otherwise} \end{cases}$$

Proof. For each $n \in \mathbf{Z}_{\geq 0}$, W_n as defined above is a $\mathbf{k}G_n$ submodule of $M(\mathrm{L}(\underline{\lambda}))_n$ (Fact 66). To show the collection $\{W_n\}_{n \in \mathbf{Z}_{\geq 0}}$ forms a VI-submodule of $M(\mathrm{L}(\underline{\lambda}))$ we need to show that for all $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$ that $f_* : M(\mathrm{L}(\underline{\lambda}))_m \to M(\mathrm{L}(\underline{\lambda}))_n$ satisfies $f_*(W_m) \subset W_n$. Any $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$ is a composition of morphisms in $\mathrm{Hom}_{\mathrm{VI}}(\mathbf{k}, \mathbf{k+1})$ for $k = m, m+1, \ldots, n-1$. Therefore it suffices to consider the case of $m = n-1$.

Let n be such that $n - 1 \geq |\underline{\lambda}| + \underline{\lambda}(\iota)_1$. We first show that for $I := I_{n-1,n} \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{n-1}, \mathbf{n})$ that $I_*(W_{n-1}) \subset W_n$. Since $(M(\mathrm{L}(\underline{\lambda}))_n, (I_{n,n+1})_*)_{n=0}^{\infty}$ is a consistent sequence, it follows that for all $g \in G_{n-1}$ the diagram

$$\begin{array}{ccc} M(\mathrm{L}(\underline{\lambda}))_{n-1} & \xrightarrow{I_*} & M(\mathrm{L}(\underline{\lambda}))_n \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \hat{g}} \\ M(\mathrm{L}(\underline{\lambda}))_{n-1} & \xrightarrow[I_*]{} & M(\mathrm{L}(\underline{\lambda}))_n \end{array}$$

commutes. Moreover since I_* is injective, $I_*(W_{n-1}) \subset \mathrm{Res}^{G_n}_{G_{n-1}} M(\mathrm{L}(\underline{\lambda}))_n$ is isomorphic to W_{n-1} as $\mathbf{k}G_{n-1}$-modules. Therefore it suffices to show that $W_{n-1} \subset \mathrm{Res}^{G_n}_{G_{n-1}} W_n$.

Restricting the $\mathbf{k}G_n$-module

$$M(\mathrm{L}(\underline{\lambda}))_n = W_n \oplus \left(\bigoplus_{\substack{\underline{\nu} \sim \underline{\lambda} + (n - |\underline{\lambda}|) \\ \underline{\nu} \neq \underline{\lambda}[n]}} \mathrm{L}(\underline{\nu}) \right)$$

we have

$$\mathrm{Res}^{G_n}_{G_{n-1}} M(\mathrm{L}(\underline{\lambda}))_n = \mathrm{Res}^{G_n}_{G_{n-1}}(W_n) \oplus \left(\bigoplus_{\substack{\underline{\nu}\sim\underline{\lambda}+(n-|\underline{\lambda}|)\\ \underline{\nu}\neq\underline{\lambda}[n]}} \mathrm{Res}^{G_n}_{G_{n-1}} \mathrm{L}(\underline{\nu}) \right)$$

We claim that W_{n-1} does not appear in $\mathrm{Res}^{G_n}_{G_{n-1}} \mathrm{L}(\underline{\nu})$ for $\underline{\nu} \sim \underline{\lambda} + (n - |\underline{\lambda}|)$ and $\underline{\nu} \neq \underline{\lambda}[n]$. This would imply that $W_{n-1} \subset \mathrm{Res}^{G_n}_{G_{n-1}} W_n$.

The multiplicity of $W_{n-1} = \mathrm{L}(\underline{\lambda}[n-1])$ in $\mathrm{Res}^{G_n}_{G_{n-1}} \mathrm{L}(\underline{\nu})$ is the number of $\underline{\eta} \in C(\mathscr{C}, \mathscr{P})$ such that $\underline{\lambda}[n-1] \to \underline{\eta} \leftarrow \underline{\nu}$. Any partition $\eta \leftarrow \underline{\nu}(\iota)$ has at least $n - |\underline{\lambda}|$ boxes in the first row and any partition $\eta \leftarrow \underline{\lambda}[n-1](\iota)$ has exactly $n - |\underline{\lambda}| - 1$ or $n - |\underline{\lambda}| - 2$ boxes in the first row. There are no partitions μ such that $\underline{\lambda}[n-1](\iota) \to \eta \leftarrow \underline{\nu}(\iota)$, so there are no $\underline{\eta} \in C(\mathscr{C}, \mathscr{P})$ such that $\underline{\lambda}[n-1] \to \underline{\eta} \leftarrow \underline{\nu}$.

To complete the proof consider any $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{n-1}, \mathbf{n})$. There exists some $g \in G_n$ such that $f = gI$. Therefore

$$f_*(W_{n-1}) = g_* I_*(W_{n-1}) \subset g_*(W_n) \subset W_n.$$

□

Definition 68. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. Let $\mathbf{L}(\underline{\lambda})$ be the VI-module W in Lemma 67 such that*

$$\mathbf{L}(\underline{\lambda})_n = \begin{cases} \mathrm{L}(\underline{\lambda}[n]) & n \geq |\underline{\lambda}| + \underline{\lambda}(\iota)_1 \\ 0 & \text{otherwise} \end{cases}$$

Lemma 69. *Let $\underline{\lambda} \in C(\mathscr{C}, \mathscr{P})$. The VI-module $\mathbf{L}(\underline{\lambda})$ is finitely generated.*

Proof. Set $d = |\underline{\lambda}| + \underline{\lambda}(\iota)_1$. We claim that $\mathrm{span}_{\mathbf{L}(\underline{\lambda})}(\mathbf{L}(\underline{\lambda})_d) = \mathbf{L}(\underline{\lambda})$.

Let $m, n \in \mathbf{Z}_{\geq 0}$. For any $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$

$$f_* : M(\mathrm{L}(\underline{\lambda}))_m \longrightarrow M(\mathrm{L}(\underline{\lambda}))_n$$

is injective. Suppose $n \geq d$. Then for any $f \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{d}, \mathbf{n})$

$$f_* : \mathbf{L}(\underline{\lambda})_d \longrightarrow \mathbf{L}(\underline{\lambda})_n$$

is injective. Since $\mathbf{L}(\underline{\lambda})_d \neq 0$, $f_*(\mathbf{L}(\underline{\lambda})_d) \neq 0$ and hence $\mathrm{span}_{\mathbf{L}(\underline{\lambda})}(\mathbf{L}(\underline{\lambda})_d)_n \neq 0$. But $\mathrm{span}_{\mathbf{L}(\underline{\lambda})}(\mathbf{L}(\underline{\lambda})_d)_n$ is then a non-zero sub-representation of an irreducible representation, $\mathbf{L}(\underline{\lambda})_n$, so $\mathrm{span}_{\mathbf{L}(\underline{\lambda})}(\mathbf{L}(\underline{\lambda})_d)_n = \mathbf{L}(\underline{\lambda})_n$.

Since $\mathbf{L}(\underline{\lambda})_d$ is a finite dimensional vector space, this implies $\mathbf{L}(\underline{\lambda})$ is finitely generated.

□

Remark 70. *The category of VI-modules is Noetherian. Therefore one can prove the previous Lemma by noting that $\mathbf{L}(\underline{\lambda})$ is a VI-submodule of the finitely generated VI-module $M(\mathrm{L}(\underline{\lambda}))$.*

We are now ready to state our G_n-analog of Murnaghan's theorem.

Theorem 71. *Let $\underline{\mu}, \underline{\mu}' \in C(\mathscr{C}, \mathscr{P})$. For $n \in \mathbf{Z}_{\geq 0}$ sufficiently large, the $\mathbf{k}G_n$-module $\mathrm{L}(\underline{\mu}[n]) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\mu}'[n])$ decomposes as*

$$\mathrm{L}(\underline{\mu}[n]) \otimes_{\mathbf{k}} \mathrm{L}(\underline{\mu}'[n]) = \bigoplus_{\underline{\lambda}} c(\underline{\mu}, \underline{\mu}')_{\underline{\lambda},n}\, \mathrm{L}(\underline{\lambda}[n])$$

where the multiplicities $c(\underline{\mu}, \underline{\mu}')_{\underline{\lambda},n}$ are independent of n.

Proof. The VI-modules $\mathbf{L}(\underline{\mu})$ and $\mathbf{L}(\underline{\mu}')$ are finitely generated. Hence by Lemma 65 the consistent sequence induced by the VI-module $\mathbf{L}(\underline{\mu}) \otimes \mathbf{L}(\underline{\mu}')$ is finitely generated and hence satisfies RS3. Since $(\mathbf{L}(\underline{\mu}) \otimes \mathbf{L}(\underline{\mu}'))_n = \mathbf{L}(\underline{\mu})_n \otimes \mathbf{L}(\underline{\mu}')_n$ we get the desired result. $\square$

Chapter 3

VIC-modules.

The Category VIC.

Let VIC be the category whose objects are $\{\mathbf{F}_q^n : n \in \mathbf{Z}_{\geq 0}\}$ and whose morphisms are pairs (α, U) where α is a linear injection and U is complementary to the image of α. More explicitly, (α, U) is a morphism in $\mathrm{Hom}_{\mathrm{VIC}}(\mathbf{m}, \mathbf{n})$ if $\alpha \in \mathrm{Hom}_{\mathrm{VI}}(\mathbf{m}, \mathbf{n})$ and $\mathbf{n} = \alpha(\mathbf{m}) \oplus U$. Composition of morphisms (α, U) and (α', U') is defined as

$$(\alpha, U) \circ (\alpha', U') = (\alpha \circ \alpha', U + \alpha(U')).$$

Fact 72. *For every $n \in \mathbf{Z}_{\geq 0}$, $\mathrm{Hom}_{\mathrm{VIC}}(\mathbf{n}, \mathbf{n}) \simeq G_n$.*

Fact 73. *Let $m, n \in \mathbf{Z}_{\geq 0}$ with $m \leq n$. Then G_n acts transitively on $\mathrm{Hom}_{\mathrm{VIC}}(\mathbf{m}, \mathbf{n})$ and the stabilizer of $(I_{m,n}, \mathrm{span}\{e_{m+1}, \ldots, e_n\})$ under this action is $H'_{m,n-m}$.*

The Category VIC-mod.

Let VIC-mod be the category **k**VIC-Mod. The objects of VIC-mod are called VIC-modules. The morphisms of VIC-mod are called VIC-module morphisms.

Let V be a VIC-module. For every $n \in \mathbf{Z}_{\geq 0}$ there is a **k**-module $V(\mathbf{n})$. For every $(\alpha, U) \in \mathrm{Hom}_{\mathrm{VIC}}(\mathbf{m}, \mathbf{n})$ there is a **k**-module morphism $V((\alpha, U)) : V(\mathbf{m}) \to V(\mathbf{n})$. To clear up notation we always write 'V_n' in place of '$V(\mathbf{n})$' and we sometimes write '$(\alpha, U)_*$' in place of '$V((\alpha, U))$'.

If W is a VIC-module and V is a submodule of W we say that V is a VIC-submodule of W. If V is a VIC-module and $\Sigma \subset \sqcup_{n \geq 0} V_n$, then the VIC-submodule $\mathrm{span}_V(\Sigma)$ of V is

$$\mathrm{span}_V(\Sigma)_n = \mathrm{span}\{v \in V_n : v = (\alpha, U)_*(\sigma) \text{ for some } \sigma \in \Sigma \cap V_a \text{ and } (\alpha, U) \in \mathrm{Hom}_{\mathrm{VIC}}(\mathbf{a}, \mathbf{n})\}$$

for each $n \in \mathbf{Z}_{\geq 0}$.

Fact 74. *Let V be a VIC-module. For every $n \in \mathbf{Z}_{\geq 0}$, V_n is a $\mathbf{k}G_n$-module. The G_n-action on V_n is given by $g.v = g_*(v) = g_* v$ for $g \in G_n$ and $v \in V_n$.*

Fact 75. *Let V be a VIC-module. Then $(V_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is a consistent sequence of $\mathbf{k}G_n$-modules. We call $(V_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ the consistent sequence induced from the VIC-module V.*

Fact 76. *Let V and W be VIC-modules and let $\psi : V \longrightarrow W$ be a morphism of VIC-modules. Fix $m \in \mathbf{Z}_{\geq 0}$. For all integers $n \geq m$ the $\mathbf{k}$-module morphism $\psi_n : V_n \to W_n$*

descends to a map $\overline{\psi}_n : (V_n)_{H_{m,n-m}} \to (W_n)_{H_{m,n-m}}$*. Moreover the diagram*

$$\begin{array}{ccc} (V_n)_{H_{m,n-m}} & \xrightarrow{\overline{\psi}_n} & (W_n)_{H_{m,n-m}} \\ \downarrow{\scriptstyle \varphi_{m,n-m}} & & \downarrow{\scriptstyle \varphi_{m,n-m}} \\ (V_{n+1})_{H_{m,n+1-m}} & \xrightarrow{\overline{\psi}_{n+1}} & (W_{n+1})_{H_{m,n+1-m}} \end{array} \qquad (\star)$$

commutes.

We refer to the previous fact by saying that $\varphi_{m,n-m}$ commutes with VIC-module morphisms.

Lemma 77. *Let* V *and* W *be VIC-modules. Suppose that the induced consistent sequences*

$$(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty} \qquad \textit{and} \qquad (W_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$$

are weakly-stable and weight-bounded. Then the consistent sequence

$$(V_n \oplus W_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$$

induced by the VIC-module $V \oplus W$ *is weakly-stable and weight-bounded.*

Proof. This proof follows the same argument as the analogous statement about VI-modules.

□

The VIC-module $M'(a)$.

Fix $a \in \mathbf{Z}_{\geq 0}$. Let $M'(a)$ be the VIC-module defined by $M'(a)_n := \mathbf{k}\mathrm{Hom}_{\mathrm{VIC}}(\mathbf{a}, \mathbf{n})$, the free $\mathbf{k}$-module with basis indexed by $\mathrm{Hom}_{\mathrm{VIC}}(\mathbf{a}, \mathbf{n})$ and for $(\alpha, U) \in \mathrm{Hom}_{\mathrm{VIC}}(\mathbf{m}, \mathbf{n})$, $(\alpha, U)_* : M'(a)_m \to M'(a)_n$ the map given by post-composition (in VIC) and extended linearly.

By Facts 3 and 73 we have

$$M'(a)_n \simeq \mathrm{Ind}_{G_{n-a}}^{G_n}(\mathrm{triv}_{G_{n-a}}(\mathbf{k}))$$

as $\mathbf{k}G_n$-modules for all $n > a$. Consequently the consistent sequence

$$(M'(a)_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$$

induced by $M'(a)$ is weight-bounded (Lemma 45).

Lemma 78. *Fix $a \in \mathbf{Z}_{\geq 0}$. For all $m \in \mathbf{Z}_{\geq 0}$ there exists an $s'(m) \in \mathbf{Z}_{\geq 0}$ such that for all $n - m \geq s'(m)$*

$$\varphi'_{m,n-m} : \big(M'(a)_n\big)_{H'_{m,n-m}} \longrightarrow \big(M'(a)_{n+1}\big)_{H'_{m,n+1-m}}$$

is an isomorphism.

Proof. Let $n \in \mathbf{Z}_{\geq 0}$ be such that $n > a + m + \min\{a, m\}$. For all $g \in G_{n+1}$ there exists an $h_1 \in H'_{a,n+1-a}$ and there exists an $h_2 \in H'_{m,n+1-m}$ such that $h_2 g h_1 \in G_n < G_{n+1}$ (Lemma 16). This $h_2 g h_1$ is a representative of g in $(M'(a)_{n+1})_{H'_{m,n+1-m}}$ and is in the image of $\varphi'_{m,n-m}$. Therefore if $n-m > a+\min\{a,m\}$ then $\varphi'_{m,n-m}$ is surjective hence is an isomorphism. $\square$

Lemma 79. *Fix $a \in \mathbf{Z}_{\geq 0}$. The consistent sequence $(M'(a)_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ induced by the VIC-module $M'(a)$ is weakly-stable and weight-bounded.*

Proof. First we show weak-stability. Fix $m \in \mathbf{Z}_{\geq 0}$. Let $s'(m) \in \mathbf{Z}_{\geq 0}$ be such that for all $n \in \mathbf{Z}_{\geq 0}$ such that $n-m \geq s'(m)$

$$\varphi'_{m,n-m} : \left(M'(a)_n\right)_{H'_{m,n-m}} \longrightarrow \left(M'(a)_{n+1}\right)_{H'_{m,n+1-m}}$$

is an isomorphism.

For all $n \in \mathbf{Z}_{\geq 0}$ such that $n-m \geq s'(m)$ consider the diagram

$$\begin{array}{ccc} (M'(a)_n)_{H'_{m,n-m}} & \xrightarrow{\varphi'_{m,n-m}} & (M'(a)_{n+1})_{H'_{m,n+1-m}} \\ \downarrow & & \downarrow \\ (M'(a)_n)_{H_{m,n-m}} & \xrightarrow{\varphi_{m,n-m}} & (M'(a)_{n+1})_{H_{m,n+1-m}} \end{array}$$

The top horizontal map is an isomorphism because $n-m \geq s'(m)$. The two vertical maps are surjections because for all $r, s \in \mathbf{Z}_{\geq 0}$, $H'_{r,s} < H_{r,s}$. Finally the diagram commutes by Lemma 42. These facts imply that the bottom horizontal map is surjective, hence $(M'(a)_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weakly-stable.

Weight-boundedness was discussed before Lemma 78. $\square$

Finitely Generated VIC-modules.

Remark 80. *There is a natural functor from VIC to VI. Therefore a VI-module is also a VIC-module. In particular finitely generated VI-modules are finitely generated VIC-modules.*

Lemma 81. *Let V be a VIC-module. Then V is finitely generated if and only if there exists a surjection of VIC-modules*

$$M'(d_1) \oplus M'(d_2) \oplus \cdots \oplus M'(d_r) \twoheadrightarrow V.$$

Proof. First we observe that the image of any VI-module morphism

$$M'(d_1) \oplus \cdots \oplus M'(d_k) \xrightarrow{\varphi} V$$

is $\mathrm{span}_V(\{\varphi_{d_1}(I_{d_1}), \ldots, \varphi_{d_k}(I_{d_k})\})$ since each $M'(d_i)$ is generated by $I_{d_i} \in M'(d_i)_{d_i}$.

Now suppose there exists a surjection

$$\varphi : M'(d_1) \oplus \cdots \oplus M'(d_k) \twoheadrightarrow V.$$

Then V is generated by $\{\varphi_{d_1}(I_{d_1}), \ldots, \varphi_{d_k}(I_{d_k})\}$ hence is finitely generated.

Conversely suppose that V is generated by $v_1, \ldots, v_k$ with $v_i \in V_{d_i}$. Let $n \in \mathbf{Z}_{\geq 0}$ and let

$$\varphi_n : M'(d_1)_n \oplus \cdots \oplus M'(d_k)_n \to V_n$$

be the map given on basis elements $(\alpha, U) \in M(d_i)_n$ by $\varphi_n((\alpha, U)) = (\alpha, U)_*(v_i)$ and extended linearly. The maps $\{\varphi_n\}_{n\in\mathbf{Z}_{\geq 0}}$ form a VI-module morphism $M'(d_1)\oplus\cdots\oplus M'(d_k) \to V$ whose image is $\mathrm{span}_V(\{v_1,\ldots,v_k\}) = V$. ☐

Fact 82. *Let V be a finitely generated VIC-module. There exists a VIC-module M' and a surjection of VIC-modules $M' \twoheadrightarrow V$. Moreover the induced consistent sequence*

$$(M'_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$$

- *is weakly-stable,*
- *is weight-bounded,*
- *satisfies RS2.*

Lemma 83. *Let V be a finitely generated VIC-module. Then the induced consistent sequence $(V_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weakly-stable.*

Proof. By Fact 82 there exists a VIC-module M' and a surjection of VIC-modules $M' \overset{\pi}{\twoheadrightarrow} V$. Moreover, the induced consistent sequence $(M'_n, (I_{n,n+1}, \mathrm{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weakly-stable.

Fix $m \in \mathbf{Z}_{\geq 0}$. Let $s_{M'}(m) \in \mathbf{Z}_{\geq 0}$ be such that for all $n - m \geq s_{M'}(m)$

$$\varphi_{m,n-m} : (M'_n)_{H_{m,n-m}} \to (M'_{n+1})_{H_{m,n+1-m}}$$

is an isomorphism. For $n \in \mathbf{Z}_{\geq 0}$ such that $n - m \geq s_{M'}(m)$ consider

$$\begin{array}{ccc} (M'_n)_{H_{m,n-m}} & \xrightarrow{\varphi_{m,n-m}} & (M'_{n+1})_{H_{m,n+1-m}} \\ \downarrow & & \downarrow \\ (V_n)_{H_{m,n-m}} & \xrightarrow[\varphi_{m,n-m}]{} & (V_{n+1})_{H_{m,n+1-m}} \end{array}$$

The vertical maps are surjections because π_n and π_{n+1} are. The top horizontal map is surjective because $n - m \geq s_{M'}(m)$. The diagram commutes because $\varphi_{m,n-m}$ commutes with VIC-module morphisms. By Remark 41 this implies the consistent sequence $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weakly-stable. □

Lemma 84. *Let V be a finitely generated VIC-module. Then the induced consistent sequence $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weight-bounded.*

Proof. By Fact 82 there exists a VIC-module M' and a surjection of VIC-modules $M' \overset{\pi}{\twoheadrightarrow} V$. Moreover, the induced consistent sequence $(M'_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weight-bounded.

The surjection implies that constituents of V_n come from constituents of M'_n. Therefore $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weight-bounded. □

From the two previous Lemmas we conclude that the induced consistent sequences from finitely generated VIC-modules satisfy RS1 and RS3. The next Lemma shows they also satisfy RS2.

Lemma 85. *Let V be a finitely generated VIC-module. Then the induced consistent sequence $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ satisfies RS2.*

Proof. This proof follows the same argument as the analogous statement about VI-modules. □

Theorem 86. *Let V be a finitely generated VIC-module. Then the induced consistent sequence $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is representation stable.*

Proof. We need to show that the consistent sequence $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ satisfies RS1, RS2, and RS3.

By Lemma 83 and Lemma 84, $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ is weakly-stable and weight-bounded. Hence by Theorem 46 and Theorem 47, $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ satisfies RS1 and RS3.

By Lemma 85, $(V_n, (I_{n,n+1}, \text{span}\{e_{n+1}\})_*)_{n=0}^{\infty}$ satisfies RS2. □

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