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ABSTRACT

A crossing symmetric Regge representation for the invariant scattering amplitude is constructed which simultaneously exhibits all Regge poles in the three channels. It is assumed that the amplitude satisfies the Mandelstam representation, and that the usual Mandelstam-Sommerfeld-Watson transform exists. To achieve explicit crossing symmetry it is found necessary to work with the Legendre function of the second kind. Except for neglecting the influence of possible angular momentum cuts, the representation is exact for all s, t, and u, with no restriction on the location of the Regge poles. As an illustration of how it might be used in practice, the Chew-Jones formula for the amplitude of definite signature is derived in the strip approximation.

1. INTRODUCTION

In this paper we construct a crossing symmetric Regge representation for the invariant scattering amplitude with the assumption that $A(s,t,u)$ satisfies the Mandelstam representation, and that the amplitudes of definite signature have an "ordinary" Mandelstam-Sommerfeld-Watson (MSW) representation (see Section 2). Furthermore, for our approach to make sense, we must require that the Regge poles recede into the left half angular momentum plane above a certain energy. The final expression for the amplitude will be exact to the extent that we have neglected any angular momentum cuts, if they exist;[1] such neglect is often justified. Concerning the Gribov-Pomeranchuk singularities,[2] we shall assume that they are absent on the angular momentum sheet of interest, and thus will not contribute directly to the asymptotic behavior. For certain cases Mandelstam has shown this to be true.[1]

Khuri[3,4] has proposed a crossing symmetric Regge representation using power series expansions in s, t, and u. As a consequence of this technique, the "Regge terms" in his expression contain poles which have no physical meaning; Chew and Jones,[5] on the other hand, choose to work with the Legendre function of the first kind; their expression, however, is strictly correct only if none of the trajectories lies in the left half angular momentum plane. In this paper we propose to construct a representation for the amplitude with neither one of the just-mentioned drawbacks; in return we must assume that $A^{\pm}(s,t)$ has a MSW representation. Rather than working with $P_\ell(z)$, we choose to work with the Legendre function of the second kind, $Q_\ell(z)$, since $P_\ell(z)$ has the undesirable property of diverging for $\mathrm{Re}\,\ell < -1$, as $|z| \to \infty$.

Since the MSW transform plays a dominant role in our calculation, we shall devote the following section to its brief examination. In Section 3 we then discuss the analytic continuation of the "Regge term" to arbitrary complex values of its argument, and in Section 4 we finally construct the crossing symmetric Regge representation. We conclude with Section 5, where we use the representation to extend the Chew-Jones form of the new strip approximation[5] to include trajectories lying in the left half angular momentum plane; in particular we shall recover their expression if we limit ourselves to those poles lying in the right half angular momentum plane.

2. THE MANDELSTAM-SOMMERFELD-WATSON (MSW) TRANSFORM

It is well known that the presence of exchange forces requires us to work with the amplitudes of definite signature; it is these amplitudes, $A^{\pm}(s,t)$, which (we assume) have a MSW representation:[6,7]

$$A^{\pm}(s,t) = -\frac{i}{2}\int_{-L-i\infty}^{-L+i\infty} d\ell(2\ell+1)a^{\pm}(\ell,s)\,\frac{Q_{-\ell-1}(-z_s)}{\pi\cos\pi\ell}$$

$$+\sum_{\mathrm{Re}\,\alpha_j>-L}\beta_j(s)\,\frac{Q_{-\alpha_j(s)-1}(-z_s)}{\cos\pi\alpha_j(s)} + B^{\pm}(s,t) \quad , \qquad (2.1)$$

where the summation extends only over trajectories of a given ($\pm$) signature. Throughout this paper the subscript j will label a Regge trajectory of definite signature, and we shall suppress any ($\pm$) superscript on $\alpha_j(s)$ and $\beta_j(s)$ if what is meant is clear from the context in which the expression appears. Here z_s is defined in terms of s and t as follows:

$$z_s = 1 + t/2q_s^{\,2} \quad ,$$
$$s = 4(q_s^{\,2} + m^2) \quad ,$$

and $a^{\pm}(\ell,s)$ is given by the Froissart-Gribov formula,[8]

$$a^{\pm}(\ell,s) = \frac{1}{\pi}\int_{t_0}^{\infty}\frac{dt'}{2q_s^{\,2}}\,Q_\ell\!\left(1+\frac{t'}{2q_s^{\,2}}\right)[A_t(s,t') \pm A_u(s,\underline{u}')] \quad , \qquad (2.2)$$

for all ℓ for which the integral converges, and is determined otherwise by analytic continuation; $A_t(s,t)$ and $A_u(s,u)$ are the absorptive parts of $A(s,t,u)$ in the t and u channels respectively, with s,t, and u related by the equation $s + t + u = \sum_i m_i^2$, where m_i are the external particle masses; $\underline{u}$ (or $\underline{t}$) is obtained from u (or t) by making the substitution $z_s \to -z_s$; thus for the equal mass case $\underline{u} = t$ and $\underline{t} = u$. The quantities $\alpha_j(s)$ and $\beta_j(s)$ appearing in (2.1) are the jth Regge pole of ($\pm$) signature, and the residue of $(2\ell + 1)a^{\pm}(\ell,s)$ at the jth Regge pole, respectively. The last term in (2.1) is defined as follows:

$$B^{\pm}(s,t) = \frac{1}{\pi}\sum_{\ell=1}^{N}(-1)^{\ell}2\ell\left[a^{\pm}\left(\ell - \frac{1}{2}, s\right) - a^{\pm}\left(-\ell - \frac{1}{2}, s\right)\right]Q_{\ell-\frac{1}{2}}(-z_s)$$

$$+ \frac{1}{\pi}\sum_{\ell=N+1}^{\infty}(-1)^{\ell}2\ell a^{\pm}\left(\ell - \frac{1}{2}, s\right)Q_{\ell-\frac{1}{2}}(-z_s) ,$$

$$-N - \frac{3}{2} < L < -N - \frac{1}{2} . \qquad (2.3)$$

Let us consider formula (2.1); we notice that the second term seems to have poles at the half integers of $\alpha(s)$; such signularities must, of course, be absent in the full amplitude. If, however, we assume that the Mandelstam reflection symmetry holds [i.e., that $a^{\pm}(-\ell - \frac{1}{2}, s) = a^{\pm}(\ell - \frac{1}{2}, s)$ for ℓ integral], then one may readily verify, by letting $L \to \infty$ (thus extending the domain of analyticity in s of the second term in 2.1), that these poles will cancel pairwise in the sum, so that the sum itself has no such spurious singularities. The symmetry is known to hold for a large class of potential problems. In order to avoid these poles we shall assume henceforth that the partial-wave amplitudes satisfy the Mandelstam reflection symmetry; it then follows also that $A(s,t)$ is always dominated at large t by Regge poles.

One final remark should be added here; it is essential that the second term in (2.1), hereafter referred to as the "Regge term," be an analytic function of s and t; this means that we must choose L sufficiently large so that all Regge trajectories will lie to the right of the integration contour; in particular we shall take L to be infinite.

§3. ANALYTIC CONTINUATION OF THE "REGGE TERM"

Expression (2.1), with $Q_{-\alpha-1}(-z_s)$ defined on the conventional sheet cut from $z_s = -1$ to $+1$, and from $z_s = +\infty$ to $+1$, does not equal $A^{\pm}(s,t)$ for all s and t, as can be seen by comparing their respective analytic structures. In fact, it follows from the definition of $A^{\pm}(s,t)$,

$$A^{\pm}(s,t) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \, \frac{A_t(s,t')}{t' - t} \pm \frac{1}{\pi}\int_{u_0}^{\infty} du' \frac{A_u(s,u')}{u' - u} , \tag{3.1}$$

and from the dispersion relations for the absorptive parts $A_t(s,t)$ and $A_u(s,u)$,

$$\begin{aligned} A_t(s,t) &= \frac{1}{\pi}\int_{s_0}^{\infty} ds' \, \frac{\rho_{st}(s',t)}{s' - s} + \frac{1}{\pi}\int_{u_0}^{\infty} du' \frac{\rho_{tu}(t,u')}{u' - (\Sigma - s - t)} , \\ A_u(s,u) &= \frac{1}{\pi}\int_{s_0}^{\infty} ds' \, \frac{\rho_{su}(s',u)}{s' - s} + \frac{1}{\pi}\int_{t_0}^{\infty} dt' \, \frac{\rho_{tu}(t',u)}{t' - (\Sigma - s - u)} , \end{aligned} \tag{3.2}$$

that $A^{\pm}(s,t)$ is an analytic function of s and t with the s plane cut from threshold to $+\infty$ and from $\Sigma - u_0 - t_0$ to $-\infty$, and with the t plane cut from t_0 to infinity along the positive axis. As usual, s_0, t_0, and u_0 are the lowest thresholds in the s, t and u channels respectively, and $\Sigma = s + t + u = 4m^2$. The analytic structure of $a^{\pm}(\ell,s)$, which enters into the background integral, may be obtained from the Froissart-Gribov definition, Eq. (2.2). Writing $b^{\pm}(\ell,s) = a^{\pm}(\ell,s)/(q_s^2)^{\ell}$, we find that $b^{\pm}(\ell,s)$ is an analytic function of s except for a right-hand cut starting at s_0, and two left-hand cuts extending to $-\infty$ from $s_0 - t_0$ and from $\Sigma - t_0 - u_0$, respectively. It should be noticed that $(q_s^2)^{-\ell}Q_{\ell}(1 + t/2q_s^2)$ has no discontinuity for $-t/4 < q_s^2 < 0$.

Next we consider the "Regge term" of the MSW transform:

$$R_j(s,t) = \gamma_j(s)(q_s^2)^{\alpha_j(s)} \frac{Q_{-\alpha_j(s)-1}(-1-t/2q_s^2)}{\cos \pi\alpha_j(s)} . \qquad (3.3)$$

Here we have written $\beta_j(s) = \gamma_j(s)(q_s^2)^{\alpha_j(s)}$ where $\gamma_j(s)$ is the residue of $(2\ell + 1)b^{\sigma}(\ell,s)$ at $\ell = \alpha_j(s)$; σ is the signature of the trajectory $\alpha_j(s)$.[9] Equation (3.3) has the desired threshold cut in s plus a number of other cuts arising from the argument of the Legendre function and the factor $(q_s^2)^{\alpha}$; comparing the right- and left-hand sides of (2.1), we conclude that the latter cuts must be absent in the full amplitude. For s physical (i.e., $s > s_0$), we notice that the cuts in t of $R_j(s,t)$ are consistent with those of $A^{\pm}(s,t)$: a right-hand cut beginning at $t = 0$, and a finite left-hand cut extending from $t = 0$ to $t = -4q_s^2$; both cuts are seen to move with s. Since the Legendre function contains the entire t dependence, and since $\gamma_j(s)$ and $\alpha_j(s)$ are assumed to have only the right-hand threshold cut, we conclude from the foregoing analysis that the desired continuation of $R_j(s,t)$ must leave the RH s and t cuts fixed. Consider the expression.

$$(q_s^2)^{\alpha}\tilde{Q}_{-\alpha-1}\left(-1 - \frac{t}{2q_s^2}\right) = -\frac{\sin \pi\alpha}{\pi} \int_0^{\infty} \frac{dt'}{t' - t} (q_s^2)^{\alpha}Q_{-\alpha-1}\left(1 + \frac{t'}{2q_s^2}\right)$$

$$+ \frac{1}{2}\int_{-1}^{+1} \frac{dz'}{z' - \left(1 + \frac{t}{2q_s^2}\right)} (q_s^2)^{\alpha}P_{-\alpha-1}(-z') , \quad \mathrm{Re}\,\alpha < 0 ; \qquad (3.4)$$

for Re $\alpha \geqslant 0$ it is defined by analytic continuation. Except for cuts, the RHS defines an analytic function of s and t. Now for s physical, $\mathcal{Q}_{-\alpha-1}(-z_s) = Q_{-\alpha-1}(-z_s)$, since for $s > s_0$, (3.4) becomes the dispersion relation for the conventional Legendre function of the second kind, [for convenience we have multiplied both sides of the equation by the threshold factor $(q_s^2)^\alpha$]. It is clear that (3.4) is the desired continuation; its analytic structure in the t plane needs no comment. Concerning the cuts in s, we notice that the first integral on the RHS has a cut extending along the negative q_s^2 axis. The discontinuity across this cut is given by

$$\Delta_s \int_0^\infty \frac{dt'}{t' - t} (q_s^2)^\alpha Q_{-\alpha-1}\left(1 + \frac{t'}{2q_s^2}\right) = -i\pi(-q_s^2)^\alpha \int_0^{-4q_s^2} \frac{dt'}{t' - t} P_{-\alpha-1}\left(-1 - \frac{t'}{2q_s^2}\right)$$

$$-\infty < q_s^2 < 0 \quad . \tag{3.5}$$

Examination of the second integral, however, shows that it has a similar cut whose discontinuity is the negative of (3.5) [this is easily verified by using the relation $(q_s^2 + i\varepsilon)^\alpha - (q_s^2 - i\varepsilon)^\alpha = 2i(-q_s^2)^\alpha \sin \pi\alpha$]. In conclusion, we therefore find that (3.4) defines an analytic function of s and t, with the t plane cut from $-4q_s^2$ to $t = 0$, and from $t = 0$ to $+\infty$, and with the q_s^2 plane cut from $q_s^2 = 0$ to $+\infty$ and from $q_s^2 = -t/4$ to infinity in a radial direction. For future reference we state the formula for the analytically continued "Regge term:"

$$R_j(s,t) = -\beta_j(s) \frac{\tan \pi\alpha_j}{\pi} \int_0^\infty \frac{dt'}{t' - t} Q_{-\alpha_j-1}\left(1 + \frac{t'}{2q_s^2}\right)$$

$$+ \beta_j(s) \frac{1}{2 \cos \pi\alpha_j} \int_{-1}^{+1} \frac{dz'}{z' - \left(1 + \frac{1}{2q_s^2}\right)} P_{-\alpha_j-1}(-z') \ , \quad \alpha_j \equiv \alpha_j(s) \quad . \tag{3.6}$$

4. A CROSSING SYMMETRIC REGGE REPRESENTATION

Let us define the following set of variables:

$$z_s = 1 + t/2q_s^{\,2} = -1 - u/2q_s^{\,2} \quad , \tag{4.1a}$$

$$z_t = 1 + s/2q_t^{\,2} = -1 - u/2q_t^{\,2} \quad , \tag{4.1b}$$

$$z_u = 1 + s/2q_u^{\,2} = -1 - t/2q_u^{\,2} \quad ; \tag{4.1c}$$

$$x = 4(q_x^{\,2} + m^2) \quad , \tag{4.1d}$$

where x = s, t, or u; in the physical regions of the s, t, and u reactions, z_s, z_t, and z_u are the cosines of the respective c.m. (center of mass) scattering angles, and s, t, and u are the squares of the respective c.m. energies. We now write down three alternative expressions for the amplitude A(s,t,u) expressed in terms of the three possible pairs of independent variables: (s,z_s), (t,z_t), and (u,z_u); in fact, these expresions are the usual one-dimensional dispersion relations for A(s,t,u), with s, t, and u held fixed in turn:

$$A(s,z_s) = \frac{1}{\pi}\int_{t_0}^{\infty} dt' \, \frac{A_t(s,t')}{t' - t(s,z_s)} + \frac{1}{\pi}\int_{u_0}^{\infty} du' \, \frac{A_u(s,u')}{u' - u(s,z_s)}$$

$$= \frac{1}{2}\sum_{\sigma}[A^{\sigma}(s,z_s) + \xi_{\sigma}A^{\sigma}(s, -z_s)] \quad , \tag{4.2a}$$

$$A(t,z_t) = \frac{1}{\pi}\int_{s_0}^{\infty} ds' \, \frac{A_s(s',t)}{s' - s(t,z_t)} + \frac{1}{\pi}\int_{u_0}^{\infty} du' \, \frac{A_u(t,u')}{u' - u(t,z_t)}$$

$$= \frac{1}{2}\sum_{\eta}[A^{\eta}(t,z_t) + \xi_{\eta}A^{\eta}(t, -z_t)] \tag{4.2b}$$

$$A(u,z_u) = \frac{1}{\pi}\int_{s_0}^{\infty} ds' \frac{A_s(s',u)}{s' - s(u,z_u)} + \frac{1}{\pi}\int_{t_0}^{\infty} dt' \frac{A_t(t',u)}{t' - t(u,z_u)}$$

$$= \frac{1}{2}\sum_{\lambda} [A^{\lambda}(u,z_u) + \xi_{\lambda}A^{\lambda}(u, - z_u)] \quad . \tag{4.2c}$$

Here σ, η, and λ equal $(\pm)$ depending on the signature, and $\xi_{\pm} = \pm 1$. The expression below each of the dispersion relations is readily obtained from the definition of $A^{\pm}(x,z_x)$; we construct $A^{\pm}(x,z_x)$ by attaching a $(\pm)$ sign to the second integral in the dispersion relation for $A(x,z_x)$, and by substituting $-z_x$ for z_x in the integrand. Thus, for example, (4.2a) is seen to follow from (3.1). Let us make a partial-wave expansion of $A^{\pm}(x,z_x)$:

$$A^{\pm}(x,z_x) = \sum_{\ell} (2\ell + 1)a^{\pm}(\ell,x)P_{\ell}(z_x) \quad , \quad x = s,t,u \quad , \tag{4.3}$$

where z_x is given in terms of s, t, and u by Eqs. (4.1 a-d); performing a MSW transformation on this series we obtain

$$A^{\sigma}(x,z_x) = -\frac{i}{2}\int_{-\infty-i\infty}^{-\infty+i\infty} d\ell(2\ell + 1)a^{\sigma}(\ell,x) \frac{\widetilde{Q}_{-\ell-1}(-z_x)}{\pi \cos \pi\ell}$$

$$+ \sum_{j} \beta_j(x) \frac{\widetilde{Q}_{-\alpha_j(x)-1}(-z_x)}{\cos \pi\alpha_j(x)} \quad , \tag{4.4}$$

where $\widetilde{Q}_{\ell}(z)$ is defined by (3.4), and where the summation extends only over trajectories of a given signature. Next we consider the Mandelstam representation for the amplitude $A(s,t,u)$,[10]

$$A(s,t,u) = A_{12}(s,t) + A_{13}(s,u) + A_{23}(t,u) \quad , \tag{4.5}$$

where a typical term--say, $A_{12}(s,t)$--is given by

$$A_{12}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty} ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} \quad . \tag{4.6}$$

We leave it understood that the necessary subtractions have been made. Our program is to extract explicitly that part of the amplitude which has Regge-type asymptotic behavior.[11] Accordingly, we shall split the various integrals in (4.5) into parts whose domains of integration correspond to the various double spectral regions shown in Fig. 1. Thus, for example,

$$A_{12}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{s_1}\int_{t_0}^{t_1} ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{s_1}^{\infty}\int_{t_1}^{\infty} ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)}$$

$$+ [A_{12}^{s_1}(s,t) + A_{12}^{t_1}(s,t)] \quad , \tag{4.7}$$

where

$$A_{12}^{s_1}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{s_1}\int_{t_1}^{\infty} ds'dt' \frac{\rho_{st}(s',t')}{(s'-s)(t'-t)} \quad , \tag{4.8}$$

with a similar expression for $A_{12}^{t_1}(s,t)$; s_1 and t_1 are determined by the inequality[12]

$$\text{Re}\ \alpha_j(x) < -\frac{1}{2}, \quad \text{for} \quad x > x_1 \quad , \tag{4.9}$$

where $x = s, t$. Now $\rho_{st}(s,t)$ is bounded in regions 1' and 2' of Fig. 1; furthermore, for s and t in region 2', $\rho_{st}(s,t)$ vanishes faster than $x^{-1/2}$ for large $x (x = s \text{ or } t)$; it therefore follows that the first two integrals in (4.7) need no subtractions; we shall refer to them as "background integrals." A similar decomposition to (4.7) can be made for the amplitudes $A_{13}(s,u)$ and $A_{23}(t,u)$. In what follows we shall concentrate our attention on the contributions to $A(s,t,u)$ coming from strips 1 through 6 (see Fig. 1), since they will lead to Regge asymptotic behavior. We proceed as follows: to evaluate the contributions to the amplitude coming from strips 1 and 2, we compute the double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ from expression (4.2a), where $A^{\sigma}(s,z_s)$ is given by (4.4) with $x = s$. Similarly, to evaluate the contributions from strips 3 and 4, we compute the double spectral functions $\rho_{st}(s,t)$ and $\rho_{tu}(t,u)$ from expression (4.2b), where $A^{\eta}(t,z_t)$ is given by (4.4) with $x = t$. Finally, we obtain the contribution from strips 5 and 6, using expression (4.2c). It should be noticed that <u>either</u> $A^{\pm}(x,z_s)$ <u>or</u> $A^{\pm}(x, -z_x)$ contributes to the double spectral function in a given strip, but not both; this will become clearer in what follows. As an example we compute the contributions to the full amplitude coming from strips 1 and 2 of Fig. 1:

$$A^{s}(s,t) = A_{12}^{s_1}(s,t) + A_{13}^{s_1}(s,t) \quad . \tag{4.10}$$

The double spectral functions $\rho_{st}(s,t)$ and $\rho_{su}(s,u)$ may be computed from expressions (4.2a) and (4.4), i.e., from

$$A(s,z_s) = -\frac{i}{4}\sum_{\sigma}\int_{-\infty-i\infty}^{-\infty+i\infty} d\ell(2\ell+1)\frac{a^{\sigma}(\ell,s)}{\pi\cos\pi\ell}\left[\widetilde{Q}_{-\ell-1}(-z_s) + \xi_{\sigma}\widetilde{Q}_{-\ell-1}(z_s)\right]$$

$$+\frac{1}{2}\sum_{j}\frac{\beta_j(s)}{\cos\pi\alpha_j}\left[\widetilde{Q}_{-\alpha_j-1}\left(-1-\frac{t}{2q_s^{\,2}}\right)+\xi_j\widetilde{Q}_{-\alpha_j-1}\left(-1-\frac{u}{2q_s^{\,2}}\right)\right] ,$$

$$\alpha_j \equiv \alpha_j(s) \quad , \tag{4.11}$$

where we have written z_s explicitly in terms of s and t, and of s and u, using relation (4.1a); $\widetilde{Q}_{\ell}(z)$ is defined by (3.4). Notice that the summation index j runs over all Regge trajectories, and $\xi_j = \pm 1$ depending on the signature of the jth trajectory. From (4.11) we see that $\rho_{st}(s,t)$ gets a contribution only from the terms involving $\tilde{Q}_{-p-1}(-z_s)$, $p = \ell, \alpha$, since $\tilde{Q}_{-p-1}(z_s)$ has no right-hand cut in t for $s > s_0$. Similarly, $\rho_{su}(s,u)$ gets a contribution only from the terms involving $\tilde{Q}_{-p-1}(z_s)$. It is sufficient to evaluate explicitly, say $A_{12}^{s_1}(s,t)$; the contributions from the remaining strips may then be obtained in a similar way. The first step consists in splitting $A_{12}^{s_1}(s,t)$ into the following integrals:

$$A_{12}^{s_1}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{s_1}\int_{t_1}^{\infty} dt'dt' \frac{B_{st}{}^{s}(s',t')}{(s'-s)(t'-t)}$$

$$-\frac{1}{\pi^2}\int_{s_0}^{s_1}\int_{t_0}^{t_1} ds'dt' \frac{R_{st}{}^{s}(s',t')}{(s'-s)(t'-t)}$$

$$-\frac{1}{\pi^2}\int_{s_1}^{\infty}\int_{t_0}^{\infty} ds'dt' \frac{R_{st}{}^{s}(s',t')}{(s'-s)(t'-t)} + \sum_j R(\alpha_j(s);s,t) \quad , \qquad (4.12)$$

where

$$\sum_j R(\alpha_j(s);s,t) = \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty} ds'dt' \frac{R_{st}{}^{s}(s',t')}{(s'-s)(t'-t)} \quad , \qquad (4.13)$$

and where $B_{st}{}^{s}(s,t)$ and $R_{st}{}^{s}(s,t)$ denote the contributions to the double spectral function coming from the "background integral" and "Regge term" of (4.11) respectively. The reason for the notation in (4.13) will soon become apparent. If we assume that $\beta_j(s) \lesssim s^{-1/2}$ for $s \to \infty$, then it follows from the definition of s_1, Eq. (4.9), that the first three integrals of (4.12) need no subtractions; we therefore group them with the other "background terms." Expression (4.13) is the desired candidate which exhibits Regge asymptotic behavior and has only the right-hand threshold cuts in s and t. Now $R^s(s,t) = \frac{1}{2}\sum_j R_j(s,t)$ where $R_j(s,t)$ is given by (3.6); for $s > s_0$ only the first integral on the RHS of (3.6) has a right-hand cut in t; hence we obtain

$$\sum_j R(\alpha_j(s);s,t) = -\sum_j \frac{1}{2\pi i}\int_{s_0}^{\infty} \frac{ds'}{s'-s}\,\Delta_s\gamma_j(s')\,\frac{\tan \pi\alpha_j(s')}{2\pi}$$

$$\times\int_{t_0}^{\infty} \frac{dt'}{t'-t}\,(q_{s'}^2)^{\alpha_j(s')}\,Q_{-\alpha_j(s')-1}\left(1+\frac{t'}{2q_{s'}^2}\right) \quad . \qquad (4.14)$$

Next consider the expression

$$\widetilde{R}(\alpha_j(s);s,t) = -\gamma_j(s)\,\frac{\tan \pi\alpha_j}{2\pi}\int_{t_0}^{\infty} \frac{dt'}{t'-t}\,(q_s^2)^{\alpha_j(s)}\,Q_{-\alpha_j(s)-1}\left(1+\frac{t'}{2q_s^2}\right) \qquad (4.15)$$

As we have pointed out before, the integrand of (4.15) has no discontinuity in s for the argument of the Legendre function between -1 and $-\infty$. Hence (4.15) defines an analytic function in the s plane cut from threshold to $+\infty$, and from $s_0 - t_0$ to $-\infty$. The discontinuity across the left-hand cut is $+\frac{1}{2}\gamma(s)(-q_s^2)^{\alpha}\tan \pi\alpha \int_{t_0}^{-4q_s^2} dt'[P_{-\alpha-1}(-1-t'/2q_s^2)/t' - t]$, with $\alpha \equiv \alpha_j(s)$. Thus (4.15) is seen to be the contribution of the right-hand cut in s to the dispersion relation at fixed t for the function $R(\alpha_j(s);s,t)$. Hence we obtain

$$R(\alpha_j(s);s,t) = -\beta_j(s)\,\frac{\tan \pi\alpha_j(s)}{2\pi}\int_{t_0}^{\infty}\frac{dt'}{t'-t}\,Q_{-\alpha_j(s)-1}\left(1+\frac{t'}{2q_s^{\,2}}\right)$$

$$-\frac{1}{2}\int_{-\infty}^{s_0-t_0}\frac{ds'}{s'-s}\,\gamma_j(s')(-q_{s'}{}^2)^{\alpha_j(s')}\,\frac{\tan \pi\alpha_j(s')}{2\pi}$$

$$\times\int_{t_0}^{-4q_{s'}{}^2}\frac{dt'}{t'-t}\,P_{-\alpha_j(s')-1}\left(-1-\frac{t'}{2q_{s'}{}^2}\right) \quad , \qquad (4.16)$$

where

$$\beta_j(s) = (q_s^{\,2})^{\alpha_j(s)}\,\gamma_j(s) \quad .$$

The second term of (4.16) merely removes the left-hand cut in s of the first integral. The full contribution to $A^s(s,t)$, Eq. (4.10), which exhibits Regge behavior, is given by $\sum_j [R(\alpha_j(s);s,t) + \xi_j R(\alpha_j(s);s,u)]$. We remind the reader that the first integral appearing on the RHS of (4.16) is to be taken in the ordinary sense if it converges and is determined otherwise by analytic continuation.

The method we have used to evaluate $A_{12}^{s_1}(s,t)$ may be applied, of course, to the remaining strips. Collecting the various background terms, which we did not explicitly evaluate, we find that we can bring them to the form

$$B_{12}(s,t) = \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty} ds'dt'\,\frac{b_{st}(s',t')}{(s'-s)(t'-t)} \quad ,$$

using a method due to Khuri.[3] Here $b_{st}(s,t)$ is given as follows:

$$b_{st}(s,t) = \frac{1}{(2i)^2}\int_{-1/2-i\infty}^{-1/2+i\infty} d\mu \int_{-1/2-i\infty}^{-1/2+i\infty} d\nu \, C(\nu,\mu) s^{\nu} t^{\mu} \quad ,$$

where $C(\nu,\mu)$ is defined by

$$B_{12}(s,t) = \sum_{\mu,\nu} C(\nu,\mu) s^{\nu} t^{\mu} \quad ,$$

for s,t in the Mandelstam triangle. Since in practice $C(\nu,\mu)$ cannot be obtained explicitly, we shall omit giving its expression in terms of integrals over the double spectral functions, and shall limit ourselves to a statement of the final expression for the amplitude:[13]

$$\begin{aligned} A(s,t,u) = {} & \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{t_0}^{\infty} ds'dt' \frac{b_{st}(s',t')}{(s'-s)(t'-t)} + \frac{1}{\pi^2}\int_{s_0}^{\infty}\int_{u_0}^{\infty} ds'du' \frac{b_{su}(s',u')}{(s'-s)(u'-u)} \\ & + \frac{1}{\pi^2}\int_{t_0}^{\infty}\int_{u_0}^{\infty} dt'du' \frac{b_{tu}(t',u')}{(t'-t)(u'-u)} \\ & + \sum_j [R(\alpha_j(s);s,t) + \xi_j R(\alpha_j(s);s,u)] \\ & + \sum_j [R(\alpha_j(t);t,s) + \xi_j R(\alpha_j(t);t,u)] \\ & + \sum_j [R(\alpha_j(u);u,s) + \xi_j R(\alpha_j(u);u,t)] \quad . \end{aligned} \qquad (4.17)$$

Here $R(\alpha_j(s);s,t)$ is given by (4.16), with similar expressions for the other five "Regge functions." The "background terms" vanish at least as fast as $x^{-1/2}$ for large x (x = s,t, or u).

With the assumption that $\beta_j(x) \lesssim x^{-1/2}$ and that Re $\alpha_j(x) < -1/2$ for $x \to \infty$, one may readily verify that for large t and fixed s

$$A(s,t,u) \to \frac{1}{2}\sum_j [R_j(s,t) + \xi_j R_j(s,u)] + r(s,t,u) \quad , \tag{4.18}$$

where $R_j(s,t)$ is given by (3.6) and where $r(s,t,u) \sim t^N$, $N < -1/2$, [a similar relation to (3.6) holds for $R_j(s,u)$]. Furthermore, with the help of the relation $\Gamma(\frac{1}{2} + z)\Gamma(\frac{1}{2} - z)/\Gamma(z)\Gamma(1 - z) = \tan \pi z$ one may verify that $R_j(s,t) \to -\pi\beta_j(s)P_{\alpha_j(s)}(-1-t/2q_s^{\,2})/\sin \pi\alpha_j(s)$ for large t, if Re $\alpha_j(s) > -\frac{1}{2}$. We therefore find that A(s,t,u) does indeed have the Regge asymptotic behavior given by the usual Sommerfeld-Watson transform. With the assumption of the Mandelstam reflection symmetry, there presumably exists a further cancellation between the background terms and Regge terms, so that the amplitude is always dominated at large t and any s by the sum in (4.18).

5. DISCUSSION OF THE RESULTS

Expression (4.17) is the desired representation for the invariant amplitude with no subtractions needed in the background integrals, and with <u>all</u> Regge poles displayed in an explicit crossing symmetric way. To the extent that we ignored the possibility of cuts in the angular momentum plane, it is an exact expression, valid for all s,t, and u, and with no restrictions on the location of the Regge poles. The three background integrals and the collection of six Regge terms each separately statisfies the Mandelstam representation; furthermore, our assumption that the residue $\beta(x)$ vanishes faster than $x^{-1/2}$ for large $x(x = s,t,u)$, garantees the usual Regge-type asymptotic behavior of (4.17) in all three channel variables.

From the practical standpoint our expression seems to suffer from a disease, for the individual Regge functions have poles at the half integers of $\alpha_j(s)$; as we have pointed out in Section 1, these poles are absent in the sum. What this means in practice is that we must include the necessary Regge poles lying in the left half angular momentum plane to remove these spurious singularities. We wish to point out that the above difficulty may often be avoided. For reasons of comparison we make the following substitution in the first integral of (4.16):

$$(q_s^{\,2})^{\alpha} Q_{-\alpha-1}(1 + t/2q_s^{\,2}) = (-q_s^{\,2})^{\alpha} Q_{-\alpha-1}(-1- t/2q_s^{\,2}) \quad ;$$

although we had found it necessary to use $Q_\ell(z)$ instead of $P_\ell(z)$ in order to arrive at (4.17), we now reintroduce $P_\ell(z)$ with the help of the relation

$$Q_{-\ell-1}(z) = -\pi \operatorname{ctn} \pi\ell \, P_\ell(z) + Q_\ell(z) \quad .$$

A typical Regge term, say $R(\alpha_j(s);s,t)$ then takes the form

$$R(\alpha_j(s);s,t) = \frac{\pi}{2} \gamma_j(s)(-q_s^{\,2})^{\alpha_j} \frac{1}{\pi} \int_{t_0}^{\infty} \frac{dt'}{t' - t} P_{\alpha_j(s)}\left(-1- \frac{t'}{2q_s^{\,2}}\right)$$

$$+ R'(\alpha_j(s);s,t) \quad . \tag{5.1}$$

Except for the switch in the signs of $q_s^{\,2}$ and of the argument of the Legendre function, which was not necessary but convenient, the integral in (5.1) is essentially the conventional Regge term; furthermore we notice that this integral with t_0 replaced by t_1 is identical to the Chew-Jones definition of the Regge term, Eq. II-3 of reference 5.[14] The quantity $R'(\alpha_j(s);s,t)$, which now contains the undesirable half integral poles, plays the role of a background term as long as Re $\alpha_j(s) > -\frac{1}{2}$, ($R' < t^{-1/2}$ for large t);

this is often the domain of interest. Formula (5.1) also shows why (4.17) is valid regardless of the location of the Regge poles; the presence of the term $R'(\alpha_j(s);s,t)$, which competes with the first integral for $\mathrm{Re}\,\alpha_j(s) < -\frac{1}{2}$ (both terms behave like $t^{-\alpha-1}$ for large t), produces the necessary cancellation to insure the correct asymptotic behavior for all s. If in a calculation one wishes to go beyond the region $\mathrm{Re}\,\alpha_j(s) > -\frac{1}{2}$, then one will have to deal with the half integral poles. In general, these will be few. The explicit crossing symmetry of (4.17) allows us to keep an easy watch on the approximations being made in certain calculations. As an illustration we shall use expression (4.17) to derive the Chew-Jones formula for $A^{\pm}(s,t)$ in the strip approximation [Eqs. (III.7) and (III.9) of Ref. 5]. $A^{\pm}(s,t)$ had been defined by Eq. (3.1); if we denote by $A_B^{\pm}(s,t)$ and $A_R^{\pm}(s,t)$ the contributions to this amplitude coming from the background integrals and Regge terms of (4.17), respectively, then

$$A_B^{\pm}(s,t) = \frac{1}{\pi^2}\iint ds'dt' \frac{b_{st}^{\pm}(s',t')}{(s'-s)(t'-t)} , \qquad (5.2)$$

where

$$b_{st}^{\pm}(s,t) = \begin{cases} b_{st}(s,t) \pm b_{su}(s,\underline{u}) & \text{for} \quad s > s_0 \\ -b_{tu}(t,u) \mp b_{tu}(\underline{t},\underline{u}) & \text{for} \quad s < 0 , \end{cases}$$

and where

$$s + \underline{t} + \underline{u} = 4m^2 .$$

Here $\underline{u}$ and $\underline{t}$ are obtained from u and t respectively by letting $z_s \to -z_s$ in definition (4.1a); for the equal mass case $\underline{u} = t$, and $\underline{t} = u$. Formula (5.2) is readily obtained by using the dispersion relations for $A_t(s,t)$ and $A_u(s,u)$--i.e., Eq. (3.2)--and the definition of $A^{\pm}(s,t)$. To obtain

$A_R^{\pm}(s,t)$ we compute the contributions to the absorptive parts, $A_t(s,t)$ and $A_u(s,u)$, coming from the six Regge functions of (4.17); for $s > s_0$ we obtain

$$A_t(s,t) \to \sum_j R_t(\alpha_j(s);s,t) + \sum_j R_t(\alpha_j(t);t,s)$$

$$+ \sum_j \xi_j R_t(\alpha_j(t);t,u) + \sum_j \xi_j R_t(\alpha_j(u);u,t) \quad , \tag{5.3}$$

where u and t are related by $s + t + u = 4m^2$. The first two terms of (5.3), when substituted into (3.1), clearly yield

$$\sum_j R(\alpha_j(s);s,t) + \sum_j R(\alpha_j(t);t,s) \quad .$$

This follows from the analytic structure of $R(\alpha_j(x);x,y)$. The contribution of the third term in (5.3) to $A^{\pm}(s,t)$ may be rewritten in the following manner:

$$\sum_j \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt'}{t' - t}\, \xi_j R_t(\alpha_j(t');t',u') = \sum_j \xi_j R(\alpha_j(t);t,u)$$

$$- \sum_j \frac{1}{\pi}\int_{u_0}^{\infty} \frac{du'}{u' - u}\, \xi_j R_u(\alpha_j(t');t',u') \quad ,$$

where $s + u' + t' = 4m^2$; we have made a change of variables in the last integral. Hence we obtain

$$\frac{1}{\pi}\int_{t_0}^{\infty} dt' \, \frac{A_t(s,t')}{t' - t} = \sum_j R(\alpha_j(s);s,t) + \sum_j [R(\alpha_j(t);t,s)$$

$$+ \xi_j R(\alpha_j(t);t,u)] + \sum_j \frac{1}{\pi}\int_{t_0}^{\infty} \frac{dt'}{t' - t} \xi_j R_t(\alpha_j(u');u',t')$$

$$- \sum_j \frac{1}{\pi}\int_{u_0}^{\infty} \frac{du'}{u' - u} \xi_j R_u(\alpha_j(t');t',u') \quad . \qquad (5.4)$$

The second integral in (3.1) may be evaluated in a similar way. Hence $A^{\pm}(s,t)$ becomes

$$A^{\pm}(s,t) = A_B^{\pm}(s,t) + \sum_j [R(\alpha_j(s);s,t) \pm \xi_j R(\alpha_j(s);s,\underline{u})]$$

$$+ \sum_j [R(\alpha_j(t);t,s) + \xi_j R(\alpha_j(t);t,u)] \pm \sum_j [R(\alpha_j(\underline{u});\underline{u},s)$$

$$+ \xi_j R(\alpha_j(\underline{u});\underline{u},\underline{t})] \pm \sum_j \frac{1}{\pi}\int_{u_0}^{\infty} du' \left[\frac{1}{u' - \underline{u}} \mp \frac{1}{u' - u}\right] \xi_j R_u(\alpha_j(t');t',u')$$

$$+ \sum_j \frac{1}{\pi}\int_{t_0}^{\infty} dt' \left[\frac{1}{t' - t} \mp \frac{1}{t' - \underline{t}}\right] \xi_j R_t(\alpha_j(u');u',t') \quad . \qquad (5.5)$$

So far we have made no approximations. If we neglect the background contribution to $A^{\pm}(s,t)$--i.e., the quantity $A_B^{\pm}(s,t)$--and if we make the subsitution $u_0 \to u_1$ and $t_0 \to t_1$ in the limits of integration, then we obtain an approximate expression for $A^{\pm}(s,t)$ which is seen to be identical in form with the Chew-Jones formula, Eqs. (III.7) and (III.9) of Ref. 5; the two expressions differ only in the definition of the Regge functions.

As we have pointed out before, Chew and Jones define $R(\alpha_j(s);s,t)$ to be equal to the integral appearing in (5.1) with t_0 replaced by t_1. In view of the previous discussion of this expression, we see that there is no essential difference between (5.5) and the Chew-Jones formula if we worry only about those trajectories which stay in the right half angular momentum plane; however, we point out once more, that at least in principle, our definition of the Regge function allows us to include the effects of <u>all</u> Regge poles; but until we have more knowledge about the region Re $\alpha < -1/2$, this is of purely academic interest.

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FOOTNOTES AND REFERENCES

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2. V. N. Gribov and I. Ya. Pomeranchuk, Proceedings of the 1962 International Conference on High Energy Physics at CERN, p. 522.
3. N. N. Khuri, Phys. Rev. 132, 914 (1963).
4. R. J. Eden has constructed a modified representation by using Legendre transforms; see Phys. Rev. 132, 912 (1963).
5. Geoffrey F. Chew and C. Edward Jones, Phys. Rev. 135, B208 (1964).
6. See, for example, E. J. Squires, Complex Angular Momenta and Particle Physics, W. A. Benjamin, Inc., Publishers, p. 10.
7. To simplify the discussion we shall take all external masses to be equal; furthermore we will ignore spin complications.
8. See reference 6, p. 46.
9. Notice that the cut in q_s^2 of the function $(q_s^2)^{\alpha(s)}$ is not arbitrary, but is fixed by the choice of sheet for the Legendre function.
10. For simplicity we shall omit throughout this paper any bound-state pole terms in the Mandelstam representation and in any other dispersion relation, for these terms do not contribute to the final answer. The bound states will appear as poles in the Legendre function of the second kind.
11. The general spirit of the calculation is that of references 3 and 5.
12. With this definition of x_1 plus the assumption that $\beta_j(x) < x^{-1/2}$ for large x, our background terms will vanish faster than $y^{-1/2}$ for large y, where y stands for either variable of its argument.

13. By $\alpha_j(x)$ we mean the jth trajectory in the channel where x is the square of the c.m. energy. Each one of the summations extends over all Regge trajectories in a given channel.

14. Our reduced residue $\gamma_j(s)$ differs from their reduced residue by the factor $[2\alpha_j(s) + 1]$.

FIGURE CAPTIONS

Fig. 1 The Mandelstam diagram showing strips 1 through 6 which give rise to Regge asymptotic behavior; the remaining double spectral regions contribute only to the background terms.

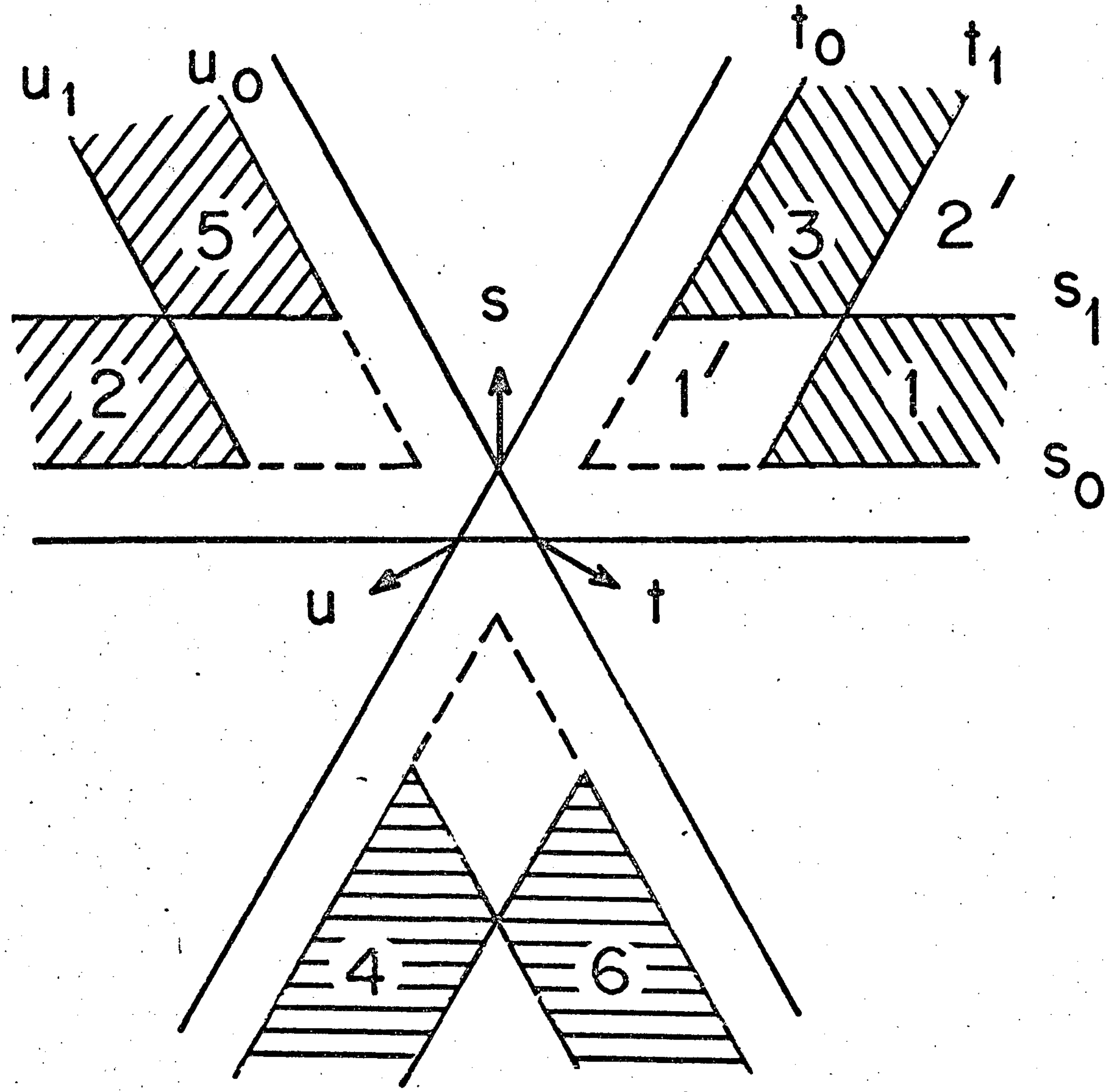

Fig. 1