Title

Common variance fractional factorial designs and their optimality to identify a class of models

Journal
Journal of Statistical Planning and Inference, 143(10)

ISSN
0378-3758

Authors
Ghosh, Subir
Flores, Analisa

Publication Date
2013-10-01

DOI
10.1016/j.jspi.2013.06.008

Peer reviewed

Journal of Statistical Planning and Inference 143 (2013) 1807–1815

Contents lists available at SciVerse ScienceDirect

Journal of Statistical Planning and Inference

journal homepage: www.elsevier.com/locate/jspi

Common variance fractional factorial designs and their optimality to identify a class of models

Subir Ghosh *, Analisa Flores

University of California, Riverside, CA 92521-0138, USA

ARTICLE INFO

Article history:
Received 17 November 2012
Received in revised form
13 June 2013
Accepted 14 June 2013
Available online 25 June 2013

Keywords:
Balanced arrays
Constant variance
Factorial designs
Interaction effects
Model identification
Optimum designs

ABSTRACT

Fractional factorial designs with n treatments for 2^m factorial experiments are considered to identify a class of $\binom{m}{2}$ models with the common parameters representing the general mean and the main effects while the uncommon parameter in each model represents a two factor interaction. A new property $P_g(v_1, \ldots, v_g)$ of designs is introduced in this context to least squares estimate the uncommon parameters in g groups of models so that the estimates of v_i such parameters in the ith group have a common variance (CV), where g is an integer satisfying $1 \le g \le \binom{m}{2}$, $i = 1, \ldots, g$, $v_1 + \cdots + v_g = \binom{m}{2}$. The property $P_1(v_1)$ is desirable to have for the fractional factorial designs to identify the $\binom{m}{2}$ models. The concept of CV designs having the property $P_1(v_1)$ is introduced for the model identification. Several series of CV designs for general m and n are presented. For fixed values of n and m, $D_{n,m}$ represents the class of all fractional factorial CV designs having the property $P_1(v_1)$. CV designs in $D_{n,m}$ have possible unequal values for the common variance. The smaller the common variance, the better the CV designs for the model identification. The concept of optimum common variance (OPTCV) design having the smallest common variance in $D_{n,m}$ is also introduced. This paper presents some OPTCV designs.

1. Introduction

Fractional factorial experiments are used for scientific experiments to investigate the dependence of a response variable on a number of factors. Not all but a few factors may interact with each other. Sufficient evidence may not be available in advance to identify these interacting factors. The objective of this paper is to determine the efficient fractional factorial plans for this purpose.

In the study of dependence of a response variable on m factors each at two levels, the possible models considered are

$$M_i: \quad E(\mathbf{y}) = \mathbf{j}_n\beta_0 + \mathbf{X}_1\boldsymbol{\beta}_1 + \mathbf{X}_{2i}\beta_{2i}, \quad \text{Var}(\mathbf{y}) = \sigma^2\mathbf{I}, \quad i = 1, \ldots, \binom{m}{2}, \tag{1}$$

where $\mathbf{y}$ is a column vector of n observations on the response variable; $\mathbf{j}_n$ $(n \times 1)$ is a column vector with all elements unity; β_0 is an unknown parameter representing the general mean; $\mathbf{X}_1$ $(n \times m)$ and $\mathbf{X}_{2i}$ $(n \times 1)$ are matrices known from the design; $\boldsymbol{\beta}_1$ $(m \times 1)$ is the vector of fixed unknown parameters representing the main effects; β_{2i} is a fixed unknown parameter

* Corresponding author. Tel.: +1 9518273781.
E-mail address: subir.ghosh@ucr.edu (S. Ghosh).

0378-3758/$ - see front matter © 2013 Elsevier B.V. All rights reserved.
http://dx.doi.org/10.1016/j.jspi.2013.06.008

representing the ith two-factor interaction effect; and σ^2 is an unknown parameter. It is assumed for the identification of all models in (1) that

$$\text{Rank}[\mathbf{j}_n, \mathbf{X}_1, \mathbf{X}_{2i}] = m+2, \quad \text{and} \quad n \geq m+2, \quad i = 1, \ldots, \binom{m}{2}. \tag{2}$$

The models in (1) with the conditions in (2) can be expressed as

$$\text{M}_i : \text{E}(\mathbf{y}) = \mathbf{X}_{(i)}\boldsymbol{\beta}_{(i)}, \quad \text{Var}(\mathbf{y}) = \sigma^2\mathbf{I}, \quad \text{Rank}[\mathbf{X}_{(i)}] = m+2, \quad n \geq m+2, \quad i = 1, \ldots, \binom{m}{2},$$

where $\boldsymbol{\beta}_{(i)} = (\beta_0, \boldsymbol{\beta}'_1, \beta_{2i})'$ and $\mathbf{X}_{(i)} = [\mathbf{j}_n \vdots \mathbf{X}_1 \vdots \mathbf{X}_{2i}]$. The least squares estimator of $\boldsymbol{\beta}_{(i)}$ and its variance are

$$\widehat{\boldsymbol{\beta}}_{(i)} = (\mathbf{X}_{(i)}'\mathbf{X}_{(i)})^{-1}\mathbf{X}'_{(i)}\mathbf{y}, \quad \text{Var}(\widehat{\boldsymbol{\beta}}_{(i)}) = \sigma^2(\mathbf{X}'_{(i)}\mathbf{X}_{(i)})^{-1}. \tag{3}$$

The rank condition in (2) is equivalent to $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}| > 0$ and it requires $n \geq m+2$. Denote $\mathbf{X}_{01} = [\mathbf{j}_n \vdots \mathbf{X}_1]$. Observe that

$$\frac{1}{\sigma^2}\,\text{Var}(\widehat{\beta}_{2i}) = \frac{|\mathbf{X}'_{01}\mathbf{X}_{01}|}{|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}|}.$$

Clearly $\text{Var}(\widehat{\beta}_{2i})$ is finite if and only if $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}| > 0$.

The $\binom{m}{2}$ models in (1) have the common parameters β_0 and $\boldsymbol{\beta}_1$. Two models M_i and $\text{M}_{i'}$ have the uncommon parameters as β_{2i} and $\beta_{2i'}$. The uncommon parameters play a critical role in discriminating between two models (Srivastava, 1975, 1977). Note that $\text{Var}(\widehat{\beta}_{2i})$ is the last diagonal element of $\text{Var}(\widehat{\boldsymbol{\beta}}_{(i)})$ in (3). When we do not have any *a priori* information about the uncommon parameters in the possible models, we may want to estimate them with equal precision or equivalently having a common variance (CV) for $\text{Var}(\widehat{\beta}_{2i}), i = 1, \ldots, \binom{m}{2}$.

Definition 1. A design is said to be a CV design if the variances $\text{Var}(\widehat{\beta}_{2i}),\ i = 1, \ldots, \binom{m}{2}$, in (3) are all equal.

The notion of CV design is a new concept. A good CV design should minimize this common variance to achieve the efficient model discrimination. We now introduce the concept of optimum CV (OPTCV) designs to minimize this common variance. For fixed values of n and m, we consider the class of all possible CV designs $D_{n,m}$.

Definition 2. A design d^* in $D_{n,m}$ is said to be an OPTCV design if it provides the smallest common value of $\text{Var}(\widehat{\beta}_{2i})$ for $i = 1, \ldots, \binom{m}{2}$.

While there are many optimum designs using different optimality criteria (Atkinson et al., 2007; Fedorov, 1972; Kiefer, 1959; Läuter, 1974; Pukelsheim, 1993; Srivastava, 1977), the notion of OPTCV design is again a new concept. By *design* we mean a fraction of all possible 2^m treatments for m factors each at two levels in a completely randomized design with equal or unequal replications of treatments. The role of design is twofold: First to achieve the common variance for the uncommon parameters and then to minimize this common value of their variance. The goal of this paper is to characterize the two-fold role of design and then present such designs for some values of m and n.

Srivastava (1977) introduced the optimality criterion functions AD, AT, AMCR, GD, GT, and GMCR for designs to identify and discriminate a class of models. The criterion functions AD, AT, and AMCR are the arithmetic means of determinants, traces, and maximum characteristic roots of $\text{Var}(\widehat{\boldsymbol{\beta}}_{(i)})$ in (3). The criterion functions GD, GT, and GMCR are the geometric means of the same. Shirakura and Ohnishi (1985) considered AD optimal designs for a 2^m factorial experiment, Ghosh and Tian (2006) presented optimum designs with respect to the criterion functions AD, AT, AMCR, GD, GT, and GMCR. The new criterion function OPTCV in this paper is based on $\text{Var}(\widehat{\beta}_{2i})$ which is only the last diagonal element of $\text{Var}(\widehat{\boldsymbol{\beta}}_{(i)})$.

Although the paper introduces the new property $\text{P}_g(v_1, \ldots, v_g)$ for a positive integer g, the CV designs have the property $\text{P}_1(v_1)$ as can be seen from Definition 1 and the OPTCV designs are all CV designs with a special optimality property given in Definition 2. The designs presented in this paper are all for 2^m factorial experiments because of the property $\text{P}_1(v_1)$. For a general s^m factorial experiment with the m factors each at s levels ($s \geq 3$) or an even more general $s_1^{m_1} \times \ldots \times s_f^{m_f}$ experiment with at least one $s_i \geq 3$, the property $\text{P}_g(v_1, \ldots, v_g)$ with $g > 1$ is needed and consequently demands a special attention particularly for obtaining the optimal designs which is beyond the scope of this paper. The characterizations of CV and OPTCV designs given in this paper provide the general construction methods ready to be checked by a computer.

There are some clear advantages of considering the models in (1) as possible models over the model with all interactions terms present or even the models with the subsets of interactions present. The identification of and discrimination among the bigger models than the models in (1) require more treatments, increasing the size of the experiment as well as the cost and time of the investigation. While fitting the models in (1), a few interactions may emerge as much stronger than others. By fitting a model that includes only the stronger interactions and screening out the other weaker interactions, these interactions can be further investigated.

The paper is organized as follows. Section 2 introduces the property $\text{P}_g(v_1, \ldots, v_g)$ of designs. Section 3 presents the characterizations of property $\text{P}_1(v_1)$ and CV designs. Section 4 with its four subsections obtains the CV and OPTCV designs. Section 5 draws conclusions on the findings of the paper.

2. Property $\mathbf{P}_g(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_g)$

The $\mathrm{Var}(\widehat{\beta_{2i}})$ for $i = 1, \ldots, \binom{m}{2}$ may or may not be all identical for a design. We denote the number of distinct $\mathrm{Var}(\widehat{\beta_{2i}})$ by g, $1 \le g \le \binom{m}{2}$. When $g=1$, the variances are all equal which is a desirable characteristic of a design for the model discrimination and giving a CV design from Definition 1. However, not all designs will have this desirable characteristic. In view of this realization, we first present a new property of designs.

Definition 3. A design is said to have the property $\mathrm{P}_g(v_1, \ldots, v_g)$ for a positive integer g if the $\binom{m}{2}$ models in (1) can be divided into g groups so that, for the uth group consisting of v_u models, $u = 1, \ldots g$, the expressions of $\mathrm{Var}(\widehat{\beta_{2i}})$ are all identical to each other but the common expressions of variance are different for the u_1^{th} and u_2^{th} groups, $u_1, u_2 = 1, \ldots g, u_1 \neq u_2$.

In the above definition $v_1 + \cdots + v_g = \binom{m}{2}$. When $g=1$, $v_1 = \binom{m}{2}$ and $\mathrm{Var}(\widehat{\beta_{2i}})$=constant for $i = 1, \ldots, \binom{m}{2}$.

Example 1. The design d_1 which is the design $\mathrm{D}_{4.1}$ for $m=4$ and $n=8$ in Ghosh and Tian (2006), has the property $\mathrm{P}_2(3,3)$ with the common variance in the two groups 0.136 σ^2 and 0.188 σ^2, respectively.

$$\mathrm{d}_1 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

Example 2. The design d_2 for $m=4$ and $n=8$ is an orthogonal array of strength 3. Denoting the four factors by A, B, C, and D, the defining relation of this fraction is expressed as ABCD=−I.

$$\mathrm{d}_2 = \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \\ -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & 1 \\ 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

The design d_2 has the property $\mathrm{P}_1(6)$ with the common variance 0.125 σ^2. Out of 4954 designs with $n=8$ satisfying the conditions (2), the only non-isomorphic design d_2 has the same value of $\mathrm{Var}(\widehat{\beta_{2i}})$ for $i = 1, \ldots, 6$.

3. Characterizations of property $\mathrm{P}_1(v_1)$ and CV designs

It follows from Definitions 1 and 3 that a CV design has the property $\mathrm{P}_1(v_1)$ and vice versa. We denote the determinant of the matrix $\mathbf{X'}_{(i)}\mathbf{X}_{(i)}$ by $|\mathbf{X'}_{(i)}\mathbf{X}_{(i)}|$. The results below are true.

Theorem 1. *A design has the property* $\mathrm{P}_1(v_1)$ *and is a CV design if and only if* $|\mathbf{X'}_{(i)}\mathbf{X}_{(i)}| = \text{constant}$ *for* $i = 1, \ldots, \binom{m}{2}$.

Corollary 1. *A design has the property* $\mathrm{P}_1(v_1)$ *and is a CV design if the* $\binom{m}{2}$ *vectors of eigenvalues of* $\mathbf{X'}_{(i)}\mathbf{X}_{(i)}$ *for* $i = 1, \ldots, \binom{m}{2}$ *are all identical.*

Example 3. For $m=5$ and $n=7$, the design d_3 given below is the design D9.2 in Ghosh and Tian (2006).

$$\mathrm{d}_3 = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & 1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & 1 \\ 1 & -1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

For the design d_3, the vectors of eigenvalues of $\mathbf{X}'_{(i)}\mathbf{X}_{(i)}$ are (16, 16, 4, 4, 4, 4, 1) for $i=1$ and (16, 10.8706742, 9.5102687, 4, 4, 4, 0.6190571) for $i=2,\ldots,10$ but interestingly $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}|=65,536$ for $i=1,\ldots,10$. The design d_3 therefore has the property $P_1(v_1)$ and is a CV design by Theorem 1 although the condition of Corollary 1 does not hold.

Example 4. The design d_4 given below is the design D14 for $m=5$ and $n=12$ in Ghosh and Tian (2006).

$$d_4=\begin{bmatrix} -1 & -1 & -1 & -1 & 1\\ 1 & 1 & -1 & -1 & -1\\ 1 & -1 & 1 & -1 & -1\\ 1 & -1 & -1 & 1 & -1\\ -1 & 1 & 1 & -1 & -1\\ -1 & 1 & -1 & 1 & -1\\ -1 & -1 & 1 & 1 & -1\\ 1 & 1 & 1 & 1 & -1\\ 1 & 1 & 1 & -1 & 1\\ 1 & 1 & -1 & 1 & 1\\ 1 & -1 & 1 & 1 & 1\\ -1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The 10 vectors of eigenvalues of $\mathbf{X}'_{(i)}\mathbf{X}_{(i)}$ for $i=1,\ldots,10$ are all identical to (16.8989, 16, 12, 12, 12, 8, 7.1010) with the common value of $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}|$ as 26,542,080. The conditions of both Theorem 1 and Corollary 1 hold for the design d_4 and therefore it is a CV design.

The complement of a design d, denoted by $\overline{d}$, is obtained by interchanging the high and low levels of m factors or in other words $\overline{d}=-d$.

Corollary 2. *A design d has the property* $P_1(v_1)$ *and is a CV design if and only if its complement* $\overline{d}$ *has the same property and is also a CV design.*

We now present the definition of a balanced array with 2 symbols, n rows, m columns, strength t, and index set $(\mu_0^t,\mu_1^t,\ldots,\mu_t^t)$ (Srivastava and Chopra, 1973). Let $\mathbf{b}$ be a $(1\times t)$ row vector with elements −1 and 1, and wt($\mathbf{b}$) be the number of 1's in $\mathbf{b}$. For an $(n\times t)$ matrix $\mathbf{T_0}$ with elements −1 and 1, we define $\lambda(\mathbf{b},\mathbf{T_0})$ as the number of times $\mathbf{b}$ appears as a row of $\mathbf{T_0}$.

Definition 4. A balanced array with 2 symbols, n rows, m columns, strength t, and index set $(\mu_0^t,\mu_1^t,\ldots,\mu_t^t)$ is an $(n\times m)$ matrix $\mathbf{T}$ with elements −1 and 1 such that for every $(n\times t)$ sub-matrix $\mathbf{T_0}$ of $\mathbf{T}$ and every row vector $\mathbf{b}$ of $\mathbf{T_0}$ with wt($\mathbf{b}$)$=k$, we have $\lambda(\mathbf{b},\mathbf{T_0})=\mu_k^t$.

It can be seen that $n=\binom{t}{0}\mu_0^t+\binom{t}{1}\mu_1^t+\cdots+\binom{t}{t}\mu_t^t$. A balanced array is said to be of "full strength" if $t=m$ (Srivastava and Chopra, 1973). A balanced array becomes an orthogonal array when $\mu_0^t=\mu_1^t=\cdots=\mu_t^t$. A balanced array exists for all values of n but an orthogonal array exists only for special values of n. For a given n, there are many possible balanced arrays.

Corollary 3. *A design which is a balanced array of full strength and satisfies the rank conditions (2), has the property* $P_1(v_1)$ *and is a CV design.*

Corollary 4. *A design which is a balanced array of strength 3 and satisfies the rank conditions (2), has the property* $P_1(v_1)$ *and is a CV design.*

Example 5. For $m=4$ and $n=9$, we consider the design d_5 which is the design D5 in Ghosh and Tian (2006).

$$d_5=\begin{bmatrix} 1 & 1 & 1 & 1\\ 1 & -1 & -1 & -1\\ -1 & 1 & -1 & -1\\ -1 & -1 & 1 & -1\\ -1 & -1 & -1 & 1\\ 1 & 1 & 1 & -1\\ 1 & 1 & -1 & 1\\ 1 & -1 & 1 & 1\\ -1 & 1 & 1 & 1 \end{bmatrix}.$$

The design d_5 is a balanced array of full strength. Consequently $\mathrm{Var}(\widehat{\beta_{2i}})=0.116\sigma^2$, for $i=1,\ldots,6$.

An orthogonal array of strength 3 is a special balanced array of strength 3 and therefore has the property $P_1(v_1)$ and is a CV design. Example 2 presents such an orthogonal array of strength 3 which is a CV design.

It can be seen from Rao (1973) and Ghosh et al. (2007) that

$$\frac{1}{\sigma^2}\,\mathrm{Var}(\widehat{\beta}_{2i}) = [n - \mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}]^{-1}, \quad i=1,\ldots,\binom{m}{2}, \tag{4}$$

where $\mathbf{P}_{01} = \mathbf{X}_{01}(\mathbf{X}'_{01}\mathbf{X}_{01})^{-1}\mathbf{X}'_{01}$ is a projection matrix. The result below immediately follows from (4).

Theorem 2. *A design has the property* $P_1(v_1)$ *and is a CV design if and only if* $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i} = \text{constant}$ *for* $i=1,\ldots,\binom{m}{2}$.

4. CV and OPTCV designs

It follows from Theorem 2 that an OPTCV design maximizes the common value of $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}|$ or equivalently minimizes the common value of $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}$. In (4) $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i} \geq 0$ because $\mathbf{P}_{01}$ is an idempotent matrix and hence positive semi definite. Consequently

$$\frac{1}{\sigma^2}\,\mathrm{Var}(\widehat{\beta}_{2i}) \geq \frac{1}{n}, \quad i=1,\ldots,\binom{m}{2}. \tag{5}$$

Let $\mathbf{J}_n$ be an $(n \text{ x } n)$ matrix with all elements unity and $\mathbf{W} = (\mathbf{X}'_1\mathbf{X}_1 - (1/n)\mathbf{X}'_1\mathbf{J}_n\mathbf{X}_1)^{-1}$. It can be checked (Rao, 1973) that

$$(\mathbf{X}'_{01}\mathbf{X}_{01})^{-1} = \begin{bmatrix} \frac{1}{n} + \frac{1}{n^2}\mathbf{j}'_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{j}_n & -\frac{1}{n}\mathbf{j}'_n\mathbf{X}_1\mathbf{W} \\ -\frac{1}{n}\mathbf{W}\mathbf{X}'_1\mathbf{j}_n & \mathbf{W} \end{bmatrix},$$

$$\mathbf{P}_{01} = \frac{1}{n}\mathbf{J}_n + \frac{1}{n^2}\mathbf{J}_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{J}_n - \frac{1}{n}\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{J}_n - \frac{1}{n}\mathbf{J}_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1 + \mathbf{X}_1\mathbf{W}\mathbf{X}'_1. \tag{6}$$

The matrix $\mathbf{P}_{01}$ is idempotent. Observing that $\mathbf{J}_n\mathbf{J}_n = n\mathbf{J}_n$, we get $\mathbf{P}_{01}\mathbf{J}_n = \mathbf{J}_n\mathbf{P}_{01} = \mathbf{J}_n$. Consequently, the matrix $(\mathbf{P}_{01} - (1/n)\mathbf{J}_n)$ is also idempotent. It can be seen from (4) that for $i=1,\ldots,\binom{m}{2}$

$$\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\beta}_{2i}) = \left[n - \frac{1}{n}\mathbf{X}'_{2i}\mathbf{J}_n\mathbf{X}_{2i} - \mathbf{X}'_{2i}\left(\mathbf{P}_{01} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i}\right]^{-1} = \left[n - \frac{1}{n}(\mathbf{X}'_{2i}\mathbf{j}_n)^2 - \mathbf{X}'_{2i}\left(\mathbf{P}_{01} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i}\right]^{-1}. \tag{7}$$

In (7) $(\mathbf{X}'_{2i}\mathbf{j}_n)^2 \geq 0$ and $\mathbf{X}'_{2i}(\mathbf{P}_{01} - (1/n)\mathbf{J}_n)\mathbf{X}_{2i} \geq 0$. Hence $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i} = 0$ if and only if $\mathbf{X}'_{2i}\mathbf{j}_n = \mathbf{X}'_{2i}(\mathbf{P}_{01} - (1/n)\mathbf{J}_n)\mathbf{X}_{2i} = 0$. We have

$$\begin{aligned}\mathbf{X}'_{2i}\left(\mathbf{P}_{01} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i} &= \left(\frac{\mathbf{X}'_{2i}\mathbf{j}_n}{n}\right)^2 \mathbf{j}'_n X_1\mathbf{W}\mathbf{X}'_1\mathbf{j}_n - 2\left(\frac{\mathbf{X}'_{2i}\mathbf{j}_n}{n}\right)\mathbf{j}'_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{X}_{2i} + \mathbf{X}'_{2i}\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{X}_{2i} \\ &= \left[\left(\frac{\mathbf{X}'_{2i}\mathbf{j}_n}{n}\right)\mathbf{j}'_n\mathbf{X}_1 - \mathbf{X}'_{2i}\mathbf{X}_1\right]\mathbf{W}\left[\left(\frac{\mathbf{X}'_{2i}\mathbf{j}_n}{n}\right)\mathbf{j}'_n\mathbf{X}_1 - \mathbf{X}'_{2i}\mathbf{X}_1\right]'.\end{aligned} \tag{8}$$

Using the properties of idempotent matrices and observing that $\mathbf{X}'_{2i}\mathbf{X}_{2i} = n$, we get

$$\begin{aligned}&0 \leq \mathbf{X}'_{2i}\left(\mathbf{P}_{01} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i} < n - \frac{1}{n}(\mathbf{X}'_{2i}\mathbf{j}_n)^2 \leq n, \\ &\quad 0 < n - \mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i} \leq \mathrm{Min}\left(n - \frac{1}{n}(\mathbf{X}'_{2i}\mathbf{j}_n)^2, n - \mathbf{X}'_{2i}\left(\mathbf{P}_{01} - \frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i}\right) \leq n.\end{aligned} \tag{9}$$

We consider the following situations for $i=1,\ldots,\binom{m}{2}$ in the subsections below:

(i) $\mathbf{X}'_{2i}\mathbf{X}_1$ and $\mathbf{X}'_{2i}\mathbf{j}_n$ are not dependent on i,
(ii) $\mathbf{X}'_{2i}\mathbf{X}_1$ is dependent on i but $\mathbf{X}'_{2i}\mathbf{j}_n$ may or may not be dependent on i.

When (i) is true, $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i} = \text{constant}$ for $i=1,\ldots,\binom{m}{2}$. Hence, by Theorem 2, the design has the property $P_1(v_1)$ and is a CV design. For (ii), $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}$ may or may not change with i.

4.1. $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ *and* $\mathbf{X}'_{2i}\mathbf{j}_n = c$ *for* $i=1,\ldots,\binom{m}{2}$

The assumptions $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ and $\mathbf{X}'_{2i}\mathbf{j}_n = c$ for $i=1,\ldots,\binom{m}{2}$ in this subsection make both $\mathbf{X}'_{2i}\mathbf{X}_1$ and $\mathbf{X}'_{2i}\mathbf{j}_n$ not dependent on i. The constant c is an integer. We first consider the situation $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ for $i=1,\ldots,\binom{m}{2}$ to get the result below.

Theorem 3. *If* $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ *for* $i=1,\ldots,\binom{m}{2}$, *then* $\mathbf{j}'_n\mathbf{X}_1 = \mathbf{0}'$ *and* $\mathbf{X}'_{2i}(\mathbf{P}_{01} - (1/n)\mathbf{J}_n)\mathbf{X}_{2i} = 0$ *for* $i=1,\ldots,\binom{m}{2}$.

Proof. The proof of $\mathbf{j}'_n\mathbf{X}_1 = \mathbf{0}'$ follows by observing the columns of $\mathbf{X}_1$ represent the main effects, the column $\mathbf{X}_{2i}$ represents a 2-factor interaction effect and is obtained by element-by-element product of two columns of $\mathbf{X}_1$ for two factors of the interaction effect. The proof of $\mathbf{X}'_{2i}(\mathbf{P}_{01} - (1/n)\mathbf{J}_n)\mathbf{X}_{2i} = 0$ follows from (6) by considering $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ and $\mathbf{j}'_n\mathbf{X}_1 = \mathbf{0}'$. □

We next consider the situation $\mathbf{X}'_{2i}\mathbf{j}_n = c$ for $i=1,\ldots,\binom{m}{2}$. Observe that

$$c = n-2(\text{the number of } -1 \text{ in } \mathbf{X}_{2i}). \tag{10}$$

The number of −1 in $\mathbf{X}_{2i}$ cannot be n or 0 to satisfy the rank condition in (2). Hence

$$-n+2 \le c \le n-2. \tag{11}$$

When $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ and $\mathbf{X}'_{2i}\mathbf{j}_n = c$ for $i=1,\ldots,\binom{m}{2}$, we get

$$\frac{1}{\sigma^2}\,\mathrm{Var}(\widehat{\beta}_{2i}) = \frac{1}{n-\dfrac{c^2}{n}}, \quad i=1,\ldots,\binom{m}{2}, \tag{12}$$

which becomes $1/n$ or in other words the equality in (5) when $c=0$. We now have the result:

Theorem 4. *If for some values of n and m the class of designs $D_{n,m}$ contains a design d^* satisfying $\mathbf{X}'_{2i}\mathbf{j}_n = 0$ and $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ for $i=1,\ldots,\binom{m}{2}$, then the design d^* is a CV design having $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/n$ for $i=1,\ldots,\binom{m}{2}$ and is an OPTCV design in $D_{n,m}$.*

Proof. The proof is clear from (5) to (7). □

When $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ for $i=1,\ldots,\binom{m}{2}$ and $c\neq 0$, the result below is useful for obtaining CV-optimum designs in $D_{n,m}$ for different values of n and m.

Theorem 5. *If for some n and m the class of designs $D_{n,m}$ contains a design d^* satisfying $\mathbf{X}'_{2i}\mathbf{j}_n = c\neq 0$, $\mathbf{j}'_n\mathbf{X}_1 = \mathbf{0}'$, and $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ for $i=1,\ldots,\binom{m}{2}$ and a constant c not depending on i, then the design d^* satisfies $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/(n-c^2/n)$ for $i=1,\ldots,\binom{m}{2}$ and is an OPTCV design in $D_{n,m}$ when the smallest value of $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}$ is c^2/n.*

Proof. The proof follows from (6), (7) and (12). □

The result below uses the properties of orthogonal array (OA) to determine the OPTCV designs.

Theorem 6. *If for some n and m the class of designs $D_{n,m}$ contains one or more OAs of strength t (≥3), then all the OAs of strength t (≥3) are OPTCV designs in $D_{n,m}$.*

Proof. Observe that $|\mathbf{X}'_{(i)}\mathbf{X}_{(i)}| = n^{m+2}$ and $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/n$, $i=1,\ldots,\binom{m}{2}$, for the OAs of strength t (≥3). The rest is clear from (5). □

The OAs of strength 3 satisfy $\mathbf{X}'_{2i}\mathbf{j}_n = 0$ and $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ for $i=1,\ldots,\binom{m}{2}$. Hence, Theorem 6 also follows from Theorem 4. As an application of Theorem 6, the design d_2 in Example 2 is an OPTCV design in $D_{8,4}$. The class of designs $D_{16,5}$ for $m=5$ and $n=16$ with factors A, B, C, D, and E contains the strength 4 OA designs with the defining relation ABCDE=−I and its complement as well as the strength 3 OA designs with their respective defining relation ABCD=−I, ABCE=−I, ABDE=−I, ACDE=−I, BCDE=−I. All these designs are OPTCV designs.

A design d is said to be a fold-over design if $\mathrm{d}\equiv\overline{\mathrm{d}}$ in the sense that all the treatments in d and $\overline{\mathrm{d}}$ are identical. Consider the fold-over designs $\mathrm{d}^{(2m)}$ and $\mathrm{d}^{(2m+2)}$ with m factors as columns and n treatments as rows where $n=2m$ and $2m+2$ respectively

$$\begin{aligned}\mathrm{d}^{(2m)} &= [(2\mathbf{I}_m-\mathbf{J}_m)\,\vdots\,(-2\mathbf{I}_m+\mathbf{J}_m)]',\\ \mathrm{d}^{(2m+2)} &= [\mathbf{j}_m\,\vdots\,-\mathbf{j}_m\,\vdots\,(2\mathbf{I}_m-\mathbf{J}_m)\,\vdots\,(-2\mathbf{I}_m+\mathbf{J}_m)]'.\end{aligned} \tag{13}$$

The designs $\mathrm{d}^{(2m)}$ and $\mathrm{d}^{(2m+2)}$ are balanced arrays of full strength and also OAs of strength one for all m. For $m=4$, $\mathrm{d}^{(2m)} = \mathrm{d}_2$ in Example 2 and is an OA of strength 3. We have for $\mathrm{d}^{(2m)}$, (i) $\mathbf{X}'_{2i}\mathbf{j}_n = c = 2(m-4)$, (ii) $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$, and (iii) $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/(n-c^2/n) = m/16(m-2)$ for $i=1,\ldots,\binom{m}{2}$. Thus $c=0$ when $m=4$, $c<0$ when $m<4$, and $c>0$ when $m>4$. For $m=3$ and 5, we have $|c|=2$ and for $m>5$, $|c|>2$. Clearly, $|c|$ has its minimum value 2 for $m=3$ and 5. The design $\mathrm{d}^{(2m+2)}$ for $m=3$ is a complete factorial with all distinct 8 treatments present and an OA of strength 3.

For $\mathrm{d}^{(2m+2)}$, (i) $\mathbf{X}'_{2i}\mathbf{j}_n = c = 2(m-3)$, (ii) $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$, and (iii) $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/(n-c^2/n) = (m+1)/16(m-1)$ for $i=1,\ldots,\binom{m}{2}$. The $|c|$ has its minimum value 0 for $m=3$, and 2 for $m=4$.

Theorem 7. *The designs $\mathrm{d}^{(2m)}$ and $\mathrm{d}^{(2m+2)}$ in (13) have the property $\mathrm{P}_1(v_1)$ and are CV designs. The $\mathrm{d}^{(2m)}$ when $m=4$ and $\mathrm{d}^{(2m+2)}$ when $m=3$ are OPTCV designs in $D_{n,m}$. The design $\mathrm{d}^{(2m)}$ for $m=3,5$ and the design $\mathrm{d}^{(2m+2)}$ for $m=4$ are OPTCV in the subclass of $D_{n,m}$ containing designs satisfying $\mathbf{X}'_{2i}\mathbf{X}_1 = \mathbf{0}'$ and $\mathbf{X}'_{2i}\mathbf{j}_n = c$ for $i=1,\ldots,\binom{m}{2}$ and are also OPTCV in $D_{n,m}$ if their common values of $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})$ for $i=1,\ldots,\binom{m}{2}$ are still the smallest for some values of m.*

Proof. When $m=4$, we have $c=0$ and therefore the optimality of $\mathrm{d}^{(2m)}$ follows from Theorem 4 or Theorem 6. Similarly the optimality of $\mathrm{d}^{(2m+2)}$ holds for $m=3$. We have the minimum value of $|c|=2$ for $m=3,5$ in $\mathrm{d}^{(2m)}$ and $m=4$ in $\mathrm{d}^{(2m+2)}$ and moreover $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 1/(n-c^2/n)$ for $i=1,\ldots,\binom{m}{2}$. Thus the designs are OPTCV by Theorem 5 for $m=3,5$ for the first and $m=4$ for the second. The rest is clear. □

Observation 1. The $\mathrm{d}^{(2m)}$ is an OPTCV design in $D_{n,m}$ for $m=3,5$ and the $\mathrm{d}^{(2m+2)}$ is an OPTCV design in $D_{n,m}$ for $m=4,5$.

All the results presented in Observations starting from Observation 1 are supported by the computer check. The design $d_1^{(12)}$ with the treatment $(-1,-1,-1,-1,1)$, three treatments in $(\mathbf{j}_3 \vdots (2\mathbf{I}_3-\mathbf{J}_3) \vdots -\mathbf{j}_3)$, three treatments in $(-\mathbf{j}_3 \vdots (-2\mathbf{I}_3+\mathbf{J}_3) \vdots -\mathbf{j}_3)$, and five treatments in $(-2\mathbf{I}_5+\mathbf{J}_5)$ is not OPTCV for $m=5$ with $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})=0.096875$ but is a competitor of the OPTCV design $d^{(2m+2)}$ for $m=5$ with $(1/\sigma^2)\mathrm{Var}(\widehat{\beta}_{2i})=0.09375$. The design $d_1^{(12)}$ is same as d_4 in Example 4.

4.2. $\mathbf{X}'_{2i}\mathbf{X}_1=\mathbf{j}'_n\mathbf{X}_1$ *and* $\mathbf{X}'_{2i}\mathbf{j}_n=c$ *for* $i=1,\ldots,\binom{m}{2}$

The assumptions $\mathbf{X}'_{2i}\mathbf{X}_1=\mathbf{j}'_n\mathbf{X}_1$ and $\mathbf{X}'_{2i}\mathbf{j}_n=c$ for $i=1,\ldots,\binom{m}{2}$ make both $\mathbf{X}'_{2i}\mathbf{X}_1$ and $\mathbf{X}'_{2i}\mathbf{j}_n$ not dependent on i. We get from (8)

$$\mathbf{X}'_{2i}\left(\mathbf{P}_{01}-\frac{1}{n}\mathbf{J}_n\right)\mathbf{X}_{2i}=\left(1-\frac{c}{n}\right)^2\mathbf{j}'_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{j}_n. \tag{14}$$

Hence

$$\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\beta}_{2i})=\frac{1}{n-\dfrac{c^2}{n}-\left(1-\dfrac{c}{n}\right)^2\mathbf{j}'_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{j}_n},\quad i=1,\ldots,\binom{m}{2}, \tag{15}$$

which becomes (12) when $\mathbf{j}'_n\mathbf{X}_1=\mathbf{0}'$.

Consider the design $d^{(2m+1)}$ below with m factors as columns and $n=2m+1$ treatments as rows and is obtained from $d^{(2m)}$ in (12) by augmenting the treatment $\mathbf{j}'_m$

$$d^{(2m+1)}=[\mathbf{j}_m \vdots (2\mathbf{I}_m-\mathbf{J}_m) \vdots (-2\mathbf{I}_m+\mathbf{J}_m)]'. \tag{16}$$

The $d^{(2m+1)}$ for $m=4$ is exactly the design d_5 in Example 5. The design $d^{(2m+1)}$ and its complement $\overline{d}^{(2m+1)}$ are balanced arrays of full strength but not OAs for all m. We have $\mathbf{X}'_{2i}\mathbf{X}_1=\mathbf{j}'_n\mathbf{X}_1$ and equals to $\mathbf{j}'_m$ for $d^{(2m+1)}$ and $-\mathbf{j}'_m$ for $\overline{d}^{(2m+1)}$. Both designs give $c=2m-7$ and $\mathbf{j}'_n\mathbf{X}_1\mathbf{W}\mathbf{X}'_1\mathbf{j}_n=m/m(n-(1/n)-8)+8$. Therefore from (15) we get for $i=1,\ldots,\binom{m}{2}$

$$\frac{1}{\sigma^2}\,\mathrm{Var}(\widehat{\beta}_{2i})=\frac{1}{n-\frac{c^2}{n}-\left(1-\dfrac{c}{n}\right)^2\dfrac{m}{m\left(n-\dfrac{1}{n}-8\right)+8}}. \tag{17}$$

The design $d^{(2m+1)}$ in (16) has the property $P_1(v_1)$ and is a CV design.

Observation 2. The $d^{(2m+1)}$ and $\overline{d}^{(2m+1)}$ are CV designs and OPTCV designs in $D_{n,m}$ for $m=3,4,5$.

4.3. $\mathbf{X}'_{2i}\mathbf{X}_1$ *is dependent on* i

In this subsection $\mathbf{X}'_{2i}\mathbf{X}_1$ is dependent on i and $\mathbf{X}'_{2i}\mathbf{j}_n$ may or may not be dependent on i. Consider the design $d^{(m+2)}$ below with m factors as columns and $n=m+2$ treatments as rows

$$d^{(m+2)}=[\mathbf{j}_m \vdots -\mathbf{j}_m \vdots (-2\mathbf{I}_m+\mathbf{J}_m)]'. \tag{18}$$

The design $d^{(m+2)}$ is a balanced array of full strength but not an OA for $m\geq 3$. We have $\mathbf{X}'_{2i}\mathbf{X}_1\neq\mathbf{j}'_n\mathbf{X}_1=(m-2)\mathbf{j}'_m$, $c=m-2$, and $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}=(m-2)(m^2+3m+6)/(m^2-m+2)$. Therefore from (4) we get for $i=1,\ldots,\binom{m}{2}$

$$\frac{1}{\sigma^2}\,\mathrm{Var}(\widehat{\beta}_{2i})=\frac{m^2-m+2}{16}. \tag{19}$$

The design $d^{(m+2)}$ in (18) has the property $P_1(v_1)$ and is a CV design.

Observation 3. The $d^{(m+2)}$ is a CV design and an OPTCV design in $D_{n,m}$ for $m=3$ and 4 but is not an OPTCV design in $D_{n,m}$ for $m=5$.

Define a vector $\mathbf{f}$ as $(1,1,-1,-1,-1)'$. We present the design $d_1^{(7)}$ below

$$d_1^{(7)}=[-\mathbf{j}_5 \vdots \mathbf{f} \vdots (-2\mathbf{I}_5+\mathbf{J}_5)]'. \tag{20}$$

The $d_1^{(7)}$ is same as d_3 in Example 3. Two distinct values of $\mathbf{X}'_{2i}\mathbf{j}_5$ for $i=1,\ldots,10$ are 3 and 1 respectively. The $\mathbf{X}'_{2i}\mathbf{X}_1$ is dependent on i. However, $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}=\frac{54}{10}$ for all i. Consequently, $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})=\frac{10}{16}=0.625$ for $d_1^{(7)}$ but its value is $\frac{22}{16}=1.375$ for $d^{(7)}$ when $m=5$ in (19).

Observation 4. The $d_1^{(7)}$ is a CV design and an OPTCV design in $D_{7,5}$.

Consider the design $d^{(\binom{m}{2}+1)}$ with the treatment $-\mathbf{j}'_m$ as well as $\binom{m}{2}$ treatments with $+1$ for two factors and -1 for $(m-2)$ factors. The $d^{(\binom{m}{2}+1)}$ has the property $P_1(v_1)$ and a CV design. It can be seen that $\mathbf{X}'_{2i}\mathbf{P}_{01}\mathbf{X}_{2i}=\frac{10}{6}$ for $m=4$ and 3 for $m=5$. Consequently, $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})=\frac{6}{32}=0.1875$ for $m=4$ and $\frac{1}{8}=0.125$ for $m=5$.

Observation 5. The $d^{(\binom{m}{2}+1)}$ is a CV design and an OPTCV design in $D_{n,m}$ for $m=4$ and is not an OPTCV design for $m=5$.

Recall from Observation 2 that $d^{(2m+1)}$ is an OPTCV design for $m=5$ and $n=11$ with $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 11 \times 71/7912 = 0.09871$ which is smaller than $\frac{1}{8} = 0.125$ for $d^{(1+\binom{m}{2})}$ when $m=5$ with the same $n=11$.

Consider the design $d^{(\binom{m}{2}+m)}$ with $\binom{m}{2}$ treatments with $+1$ for two factors and -1 for $(m-2)$ factors as well as treatments in $(-2\mathbf{I}_m + \mathbf{J}_m)$. The $d^{(\binom{m}{2}+m)}$ has the property $P_1(v_1)$. It can be checked that $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = \frac{1}{8} = 0.125$ for $m=4$ and 0.069 for $m=5$.

Observation 6. The $d^{(\binom{m}{2}+m)}$ is a CV design and an OPTCV design in $D_{n,m}$ for $m=5$ and is not an OPTCV design for $m=4$.

We have from Observation 1 that the design $d^{(2m+2)}$ is OPTCV or $m=4$ with $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = \frac{5}{48} = 0.104$ which is smaller than $\frac{1}{8} = 0.125$ for $d^{(\binom{m}{2}+m)}$ when $m=4$ with the same $n=10$.

4.4. More designs

Consider the designs $d^{(3m+1)}$, $d^{(3m)}$, and $d^{(3m-1)}$ consisting of treatments in $d^{(2m+2)}$, $\overline{d}^{(2m+1)}$, and $d^{(2m)}$ respectively as well as the $(m-1)$ treatments as rows in

$$[\mathbf{j}_{(m-1)} \vdots (2\mathbf{I}_{(m-1)} - \mathbf{J}_{(m-1)})].$$

Observation 7. The $d^{(3m+1)}$, $d^{(3m)}$, and $d^{(3m-1)}$ are CV designs and are OPTCV designs in $D_{n,m}$ for $m=4$ but are not OPTCV designs for $m=5$.

When $m=4$, $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})$=0.084, 0.094, and 0.100 respectively for $d^{(3m+1)}$, $d^{(3m)}$, and $d^{(3m-1)}$.

Consider the designs $d^{(\binom{m}{2}+2m+2)}$, $d^{(\binom{m}{2}+2m+1)}$, and $d^{(\binom{m}{2}+2m)}$ for $m=4$ consisting of treatments in $d^{(2m+2)}$, $\overline{d}^{(2m+1)}$, and $d^{(2m)}$ respectively as well as the $\binom{m}{2} = 6$ treatments with $+1$ for two factors and -1 for $(m-2)$ factors.

Observation 8. The $d^{(\binom{m}{2}+2m+2)}$, $d^{(\binom{m}{2}+2m+1)}$, and $d^{(\binom{m}{2}+2m)}$ for $m=4$ are CV design and OPTCV designs in $D_{n,4}$ with $n=16, 15$, and 14 respectively.

The $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i})$=0.063, 0.069, and 0.073 for $d^{(16)}$, $d^{(15)}$, and $d^{(14)}$ respectively.

Consider the design $d_1^{(8)}$ below for $m=5$ with eight treatments as rows

$$d_1^{(8)} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & -1 & -1 & -1 \\ -1 & -1 & 1 & -1 & -1 \\ -1 & 1 & -1 & -1 & 1 \\ -1 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 \end{bmatrix}.$$

Observation 9. The $d_1^{(8)}$ for $m=5$ is a CV design and an OPTCV design in $D_{8,5}$ with $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 0.375$.

Consider the design $d^{(\binom{m}{2}+m+1)}$ consisting of treatments in $d^{(\binom{m}{2}+m)}$ and the treatment $-\mathbf{j}'_m$.

Observation 10. The $d^{(\binom{m}{2}+m+1)}$ is a CV design and an OPTCV design in $D_{n,m}$ for $m=5$ with $(1/\sigma^2)\,\mathrm{Var}(\widehat{\beta}_{2i}) = 0.063$ but not an OPTCV design for $m=4$.

The treatments in $d^{(\binom{m}{2}+m+1)}$ for $m=5$ satisfy the defining relation ABCDE=–I and therefore the part of Observation 10 for $m=5$ follows from Theorem 6. The class of designs $D_{n,m}$ could be empty for some values of n and m.

Observation 11. For $m=5$ and $1 \le n \le 16$, there is no CV design when $n = 9, 13$ and 14.

The results in Observations 1–11 are summarized in Table 1.

5. Concluding remarks

In the identification of a class of models for factorial experiments using the fractional factorial designs, a new property $P_1(v_1)$ of designs is proposed in terms of variances of the least squares estimators of the uncommon parameters. The notion of CV design with the property $P_1(v_1)$ is introduced. Several series of CV designs are given for general m and n. The notion of OPTCV design is also introduced within the class $D_{n,m}$ of all CV designs having the property $P_1(v_1)$. The paper presents some OPTCV designs. Table 1 displays designs for $3 \le m \le 5$ that are ready to use in real experiments. The optimum design $d_4 = d_1^{(12)}$ for $m=5$ and $n=12$ in Ghosh and Tian (2006) using different optimality criteria is not an OPTCV design. However the optimum designs $d_2 = d^{(2m)}$ for $m=4$ and $n=8$, $d_3 = d_1^{(7)}$ for $m=5$ and $n=7$, and $d_5 = d^{(2m+1)}$ for $m=4$ and $n=9$ in Ghosh and

Table 1
The OPTCV designs for (m=3 and n=5, …,8), (m=4 and n=6, …, 16), and (m=5 and n=7, 8, 10, 11, 12, 15, 16).

m	n	$\frac{1}{\sigma^2}\mathrm{Var}(\widehat{\beta}_{2i})$	Design	Source
3	5	0.500	$d^{(m+2)}$	Observation 3
	6	0.188	$d^{(2m)}$	Observation 1
	7	0.167	$d^{(2m+1)}$	Observation 2
	8	0.125	$d^{(2m+2)}$	Theorems 6 and 7
4	6	0.875	$d^{(m+2)}$	Observation 3
	7	0.1875	$d^{(\binom{m}{2}+1)}$	Observation 5
	8	0.125	$d^{(2m)} = d_2$	Theorem 7
	9	0.116	$d^{(2m+1)} = d_5$	Observation 2
	10	0.104	$d^{(2m+2)}$	Observation 1
	11	0.100	$d^{(3m-1)}$	Observation 7
	12	0.094	$d^{(3m)}$	Observation 7
	13	0.084	$d^{(3m+1)}$	Observation 7
	14	0.073	$d^{(14)}$	Observation 8
	15	0.069	$d^{(15)}$	Observation 8
	16	0.063	$d^{(16)}$	Observation 8
5	7	0.625	$d_1^{(7)} = d_3$	Observation 4
	8	0.375	$d_1^{(8)}$	Observation 9
	10	0.104	$d^{(2m)}$	Observation 1
	11	0.099	$d^{(2m+1)}$	Observation 2
	12	0.094	$d^{(2m+2)}$	Observation 1
	15	0.069	$d^{(\binom{m}{2}+m)}$	Observation 6
	16	0.063	$d^{(\binom{m}{2}+m+1)}$	Observations 10 and 6

Tian (2006) are OPTCV designs. The possibility of $D_{n,m}$ being empty for some values of m and n is demonstrated in Observation 11.

Acknowledgments

The authors are thankful to the two reviewers for their careful reading of the earlier version of this paper and their valuable comments.

References

Atkinson, A.C., Donev, A.N., Tobias, R.D., 2007. Optimum Experimental Designs, with SAS. Oxford University Press, New York.
Fedorov, V.V., 1972. Theory of Optimal Experiments. Academic Press, New York.
Ghosh, S., Tian, Y., 2006. Optimum two level fractional factorial plans for model identification, and discrimination. Journal of Multivariate Analysis 97, 1437–1450.
Ghosh, S., Deng, H., Luan, L., 2007. Error sum of squares and power comparisons for model search, identification, and discrimination. Journal of Statistical Planning and Inference 137, 3475–3482.
Kiefer, J., 1959. Optimum experimental designs (with discussion). Journal of Royal Statistical Society, Series B 21, 272–319.
Läuter, E., 1974. Experimental design in a class of models. Mathematische Operationsforschung und Statistik. Series Statistics 5, 379–398.
Pukelsheim, F., 1993. Optimum Design of Experiments. Wiley, New York.
Rao, C.R., 1973. Linear Statistical Inference and Its Applications. Wiley, New York.
Shirakura, T., Ohnishi, T., 1985. Search designs for 2^m factorials derived from balanced arrays of strength 2(l+1) and AD-optimal search designs. Journal of Statistical Planning and Inference 11, 247–258.
Srivastava, J.N., 1975. Designs for searching non-negligible effects. In: Srivastava, J.N. (Ed.), In a Survey of Statistical Design and Linear Models. North-Holland, Amsterdam, pp. 507–520.
Srivastava, J.N., 1977. Optimal search designs. In: Gupta, S.S., Moore, D.S. (Eds.), Proceedings of the International Symposium on Decision Theory. Academic Press, New York, pp. 375–409.
Srivastava, J.N., Chopra, D.V., 1973. Balanced arrays and orthogonal arrays. In: Srivastava, J.N. (Ed.), A Survey of Combinatorial Theory. North-Holland, Amsterdam, pp. 411–428.