Title
Connectivity, graph minors, and subgraph multiplicity

Author
Eppstein, David

Publication Date
1992-01-10

Peer reviewed

Connectivity, Graph Minors, and Subgraph Multiplicity

David Eppstein

Department of Information and Computer Science
University of California, Irvine, CA 92717

Tech. Report 92-06

January 10, 1992

Abstract

It is well known that any planar graph contains at most $O(n)$ complete subgraphs. We extend this to an exact characterization: G occurs $O(n)$ times as a subgraph of any planar graph, if and only if G is three-connected. Even more generally, G occurs $O(n)$ times as a subgraph of the $K_{b,c}$-free graphs, $b \geq c$, if and only if G is c-connected; G occurs $O(n)$ times as a subgraph of the K_a-free graphs if and only if G is $(a-1)$-connected. Our results use a simple Ramsey-theoretic lemma that may be of independent interest.

1 Introduction

It follows from the sparsity of planar graphs that each such graph contains at most $O(n)$ complete subgraphs K_3 and K_4 [6]. All cliques in a planar graph can be listed by an algorithm with $O(n)$ worst-case time complexity [3, 4, 6]. Enumeration of subgraphs has a number of uses, including a recent application in testing inscribability [5].

These results naturally raise the question of determining which other planar graphs occur $O(n)$ times as subgraphs of planar graphs. A necessary condition is that the subgraph G be 3-connected: otherwise, G can be split into two parts A and B by a separating pair of vertices, and by connecting many copies of A and B at the same two vertices, we find a family of planar graphs in which G occurs $\Omega(n^2)$ times as a subgraph.

Our main result is that this is also a sufficient condition. Every three-connected planar graph can only occur $O(n)$ times as a subgraph of other planar graphs.

We extend this to minor-closed families of graphs. As is well known, the planar graphs are exactly those for which neither $K_{3,3}$ nor K_5 is a minor, and our main result for planar graphs appears as a corollary of this generalization. We first investigate graphs for which some complete bipartite graph $K_{b,c}$ (with $b \geq c$) is not a minor. We show that the subgraphs occurring $O(n)$ times in this family of graphs are exactly the c-connected graphs. We extend this to other minor-closed families of graphs; in particular the subgraphs occurring $O(n)$ times in the K_a-free graphs are exactly the $(a-1)$-connected graphs. Thus in the planar case it is the absense of $K_{3,3}$ rather than K_5 which is decisive in our characterization.

Our proofs use a simple Ramsey-theoretic lemma that may be of independent interest. In English, it states that if a set is formed as the union of a large number of k-tuples, then we can find a smaller set of tuples for which each member of the set always appears in the same tuple position. In the language of graph theory, if many copies of a subgraph occur in a larger graph, we can find a smaller number of copies of the subgraph, so that among those copies each vertex of the larger graph appears in only one way as a vertex of the subgraph.

Finally, we discuss algorithmic enumeration of subgraphs. In some cases, we are able to make our results constructive, so that the copies of a given subgraph can be enumerated in $O(n)$ time, matching the previous results for cliques in planar graphs.

2 A Ramsey-theoretic lemma

Given a set S, denote the ordered k-tuples of members of S by S^k. We do not allow any tuple to contain the same element in more than one position. Each tuple can be interpreted as a function from the integers $1 \ldots k$ onto the set S, such that for any tuple t, and $1 \leq i < j \leq k$, $t(i) \neq t(j)$.

Suppose we have a collection T of k-tuples, $T \subset S^k$. We say T is *coherent* if each member of S appears in at most a single position of the k-tuples. In other words, for any distinct tuples t and t', and for all integers $1 \leq i < i' \leq k$, $t(i) \neq t'(i')$.

As we now show, any set of tuples has a large coherent subset.

Lemma 1. *Suppose we are given a set T containing at least $k!^2 m^k$ k-tuples. Then there is a coherent subset $T' \subset T$ containing at least m tuples.*

Proof: We use induction on k. Any set of 1-tuples is coherent.

First assume some element $x \in S$ is contained in at least $k!^2 m^{k-1}/k$ of the k-tuples. Then there is a collection of $(k-1)!^2 m^{k-1}$ of those tuples for which x always appears in some particular position, say position k. By ignoring this shared position, we construct a collection of $(k-1)$-tuples which by induction must have a coherent subcollection of size m. Restoring the shared positions gives our coherent subset T'.

Otherwise, each element is in fewer than $k!^2 m^{k-1}/k$ of the tuples. Then we can repeatedly pick a tuple t, add it to T', and throw away the tuples that have an element in common with t. By the time we run out of tuples to pick, we will have added m tuples to T'. $\square$

3 Complete bipartite minors

We first show that not too many copies of a highly connected subgraph can share the same low-degree vertex, unless the graph contains a large complete bipartite graph as a minor. The bound on the number of times the subgraph can occur will then follow from the sparseness of minor-free graphs.

Lemma 2. *Let G be a c-connected graph with k vertices. Suppose that $k!^2 b^{k^2} d^{ck^2}$ copies of G are subgraphs of some larger graph H, and that some vertex x (with degree d in H) is contained in each of these copies of G. Then H contains a complete bipartite graph $K_{b,c}$ as a minor.*

Proof: By Lemma 1, we can find $b^k d^{ck}$ copies of G such that each vertex of H is used in at most one way as a vertex of G. We throw away the remaining copies of G.

We now have a one-to-many correspondence between the vertices of G and those of H. Suppose each vertex of G corresponds to fewer than bd^c different vertices of H. Each copy of G could then be found by choosing one such vertex of H for each of the k vertices of G, so there would be fewer than $(bd^c)^k$ possible copies of G in the coherent set, a contradiction. Therefore we can find some vertex y of G that corresponds to at least $\binom{d}{c}(b-1)+1 < bd^c$ vertices of H.

Since G is c-connected, we can find c vertex-disjoint paths in G from y to x. Partition the remaining copies of G into equivalence classes according to the edges in H adjacent to x used by these paths. There are $\binom{d}{c}$ such classes, so some class uses at least b copies of vertex y. Unless $b = 1$, y will not be adjacent to x in G; if $b = 1$ the lemma is trivially solved by the neighbors of any vertex. We select the copies of G in this class, and throw away the remaining copies.

We then form a minor of H by removing all edges not corresponding to portions of these vertex-disjoint paths, and contracting all remaining edges except those adjacent to x or to copies of y. Because the paths are vertex-disjoint in G, and because we have selected a coherent collection of subgraphs, the images of the paths in H are also vertex-disjoint. Thus the contraction process described above finds a minor in which x is connected to c vertices, which are also each connected to $b-1$ copies of y. This gives us $K_{b,c}$ as a minor of H. □

Theorem 1. *Let $F_{b,c}$ be the family of graphs in which $K_{b,c}$ does not occur as a minor, and let $G \in F_{b,c}$ be c-connected. Then there is some constant $\kappa = \kappa(G)$ such that, for any n-vertex graph $H \in F_{b,c}$, there can be at most κn copies of G as a subgraph in H.*

Proof: Since $F_{b,c}$ is a minor-free family of graphs, the graphs in $F_{b,c}$ are sparse (any such graph has $O(n)$ edges). Thus we can find a vertex x in H which has degree at most some constant $d = O(1)$. By Lemma 2, there are at most $O(1)$ copies of G as a subgraph of H containing vertex x. Let κ denote this number of copies. Then by induction there are at most $\kappa(n-1)$ copies of G in the graph H' formed by removing x from H. Thus there are at most κn copies in all. □

4 Other forbidden minors

We now extend this characterization to more general minor-closed families of graphs.

Theorem 2. *Let S be a set of c-connected graphs containing some complete bipartite graph $K_{b,c}$. Denote by F_S the family of graphs not containing any member of S as a minor. Then a graph $G \in F_S$ occurs $O(n)$ times as a subgraph of members of F_S if and only if G is c-connected.*

Proof: If G is c-connected, the result follows from Theorem 1 and the fact that F_S is a subset of $F_{b,c}$.

If G is in F_S but not c-connected, it can be separated by some $(c-1)$-tuple of vertices. Let A and B be two components of the separated graph. Without loss of generality, A, B, and $G-A-B$ are all connected (otherwise we could find a smaller separating set). Then we can form a family of graphs in which G occurs $\Omega(n^2)$ times, by connecting many copies of A and B to the same $(c-1)$-tuple. Any c-connected minor M of a graph in this family can contain vertices from only one of A, B, or $G-A-B$. Thus M consists of a minor of A, B, or $G-A-B$, together with possibly some extra edges connecting vertices in the $(c-1)$-tuple. But any such graph is also a minor of G, and hence cannot be in S. □

Corollary 1. *Let F_a be the family of graphs in which the complete graph K_a does not occur as a minor. Then a graph $G \in F_a$ occurs $O(n)$ times as a subgraph of members of F_a if and only if G is $(a-1)$-connected.*

Proof: Let $c = a-1$, let $b = \binom{c}{2}+1$, and let $S = \{F_a, F_{b,c}\}$. Then K_a is a minor of $K_{b,c}$, so $F_a = F_S$ and the result follows from Theorem 2. □

Corollary 2. *An outerplanar graph G occurs $O(n)$ times as a subgraph of all outerplanar graphs if and only if G is biconnected.*

Proof: Apply Theorem 2, with $S = \{K_{2,3}, K_4\}$. □

Corollary 3. *A planar graph G occurs $O(n)$ times as a subgraph of all planar graphs if and only if G is 3-connected.*

Proof: Apply Theorem 2, with $S = \{K_{3,3}, K_5\}$. □

5 Algorithms

We now discuss algorithms for enumerating the occurrences of a given subgraph. Clearly, for any fixed subgraph G, there is a polynomial time algorithm for enumerating its occurrences: if there are k vertices in G, simply test all $\binom{n}{k} = O(n^k)$ possible choises for those vertices. We are interested in situations in which the occurrences can be listed more efficiently, in linear time (matching our bounds on the occurrences of G).

Such algorithms were already known for enumerating copies of the complete graphs K_3 and K_4 as subgraphs of planar graphs. More generally, in any family of graphs with bounded arboricity, all clique subgraphs can be listed in linear time [3, 4].

The family of $K_{1,b}$-free graphs is also relatively easy. Since these graphs have maximum vertex degree $b-1$, there are fewer than b^k vertices within distance k of any given vertex. The occurrences of any connected subgraph G can therefore be enumerated in time $O(b^k n) = O(n)$.

The first interesting cases are the families of $K_{2,b}$-free graphs. Theorem 1 shows that in any such family, the subgraphs appearing $O(n)$ times are the biconnected graphs. The biconnected $K_{2,2}$-free graphs are simply the triangles, which can be enumerated using the clique algorithms cited above. We next consider $K_{2,3}$-free graphs. The biconnected components of these graphs are either outerplanar graphs, or copies of K_4. Thus we can enumerate any biconnected subgraph of such a graph, if only we can do so for outerplanar graphs.

Theorem 3. *Let G be biconnected. Then there is a linear-time algorithm which enumerates the occurrences of G in any outerplanar graph.*

Proof: Any biconnected outerplanar graph is Hamiltonian, and adding extra edges to a graph only makes it easier to enumerate its occurrences, so we can assume without loss of generality that G is a simple cycle C_k. Let H be an outerplanar graph with n vertices, for which we are to enumerate the occurrences of G as a subgraph.

Any occurrence of G consists of the vertices in a union of cycles of H. To find an occurrence of G, we proceed as follows. We start by choosing a single cycle in H. Then as long as the current subgraph is a cycle shorter than G, we choose an edge of the subgraph and replace it by the adjacent cycle. There are at most k edges to be replaced at each step, and each step increases the cycle length by at least one, so there are $O(k^k)$ ways to augment the initially chosen cycle into an occurrence of G. Thus the total time to enumerate the occurrences of G is $O(k^k n) = O(n)$. □

Corollary 4. *Let G be biconnected. Then there is a linear-time algorithm which enumerates the occurrences of G in any $K_{2,3}$-free graph.* □

Corollary 5. *Let G be a wheel W_k. Then there is a linear-time algorithm which enumerates the occurrences of G in any $K_{1,2,3}$-free graph, and in particular in any $K_{3,3}$-free or planar graph.*

Proof: W_k consists of a single "hub" vertex connected to each vertex of a cycle C_k. Let H be a $K_{1,2,3}$-free graph. We independently enumerate, for each vertex x of H, the occurrences of W_k for which x corresponds to the hub. This can be done by finding all occurrences of C_k in the neighbors of x. But the induced graph of these neighbors must be $K_{2,3}$-free, as any $K_{2,3}$ minor together with the hub vertex would form a $K_{1,2,3}$ in the original graph. So by Corollary 4 we can find all such occurrences in time linear in the number of neighbors. The time to find all occurrences of W_k is proportional to the sum of the degrees of each vertex of H, which is $O(n)$. □

This result generalizes the enumeration of cliques in planar graphs, since the only such cliques that can occur are K_3 and K_4, which can alternately be interpreted as wheels W_2 and W_3.

We would like to be able to extend our results further, at least to the enumeration of all 3-connected planar subgraphs. We have been unable to achieve such a result. However Corollary 5 is promising, as wheels figure prominently in Tutte's characterization of triconnected graphs [7].

6 Conclusions

We have described a number of circumstances in which some subgraph occurs $O(n)$ times in the graphs of a minor-closed family. We have also described algorithms that, in a more limited class of situations, can efficiently enumerate the occurrences of a given subgraph.

A number of directions for possible generalization suggest themselves. First, we would like to extend our algorithms so that we can enumerate the occurrences of a subgraph G whenever G is guaranteed to have $O(n)$ occurrences. This would entail a new, constructive proof of Theorem 1.

Second, we would like to characterize the subgraphs occurring few times in minor-closed families other than those described in Theorem 2. In particular, it seems the forbidden minors for graphs of bounded genus include graphs that are not 3-connected, so our characterization does not apply to these families.

Third, perhaps we should investigate families of graphs that are not closed under minors. Any family of sparse graphs (more precisely, graphs with bounded arboricity) will contain only $O(n)$ cliques. But perhaps some other subgraphs occur few times. For instance, what are the subgraphs occurring few times in the k-page graphs [2]?

Finally, it might be interesting to characterize graphs occurring a number of times which is a non-linear function of n. For the families for which Theorem 2 applies, we can demonstrate a gap in the possible such functions: any subgraph G occurs $\Omega(n)$ times, and the theorem tells us that G either occurs $O(n)$ or $\Omega(n^2)$ times. For other families, there may be graphs that occur only $O(1)$ times (indeed, if F excludes a minor consisting of k disjoint copies of G, then G occurs at most $k-1$ times in graphs of family F). If G is a tree with k leaves (other than a single edge), then it occurs $\Theta(n^k)$ times as a subgraph in other trees. Perhaps for planar graphs, a similar characterization can be described in terms of the tree of triconnected components. For sparse graphs (graphs with $O(n)$ edges) there can be at most $O(n^{3/2})$ triangles [1], and more generally $O(n^{c/2})$ complete subgraphs K_c. Are there minor-closed families, and subgraphs in those families, such that the function describing the number of occurrences is similarly nonpolynomial?

References

[1] M. Bern and D. Eppstein. Mesh generation and optimal triangulation. In *Euclidean Geometry and Computers*, D.A. Du and F.K. Hwang, eds. World Scientific, to appear.

[2] F. Bernhart and B. Kainen. The book thickness of a graph. *J. Comb. Th. B* 27 (1979) 320–331.

[3] N. Chiba and T. Nishizeki. Arboricity and subgraph listing algorithms. *SIAM J. Comput.* 14 (1985) 210–223.

[4] M. Chrobak and D. Eppstein. Planar orientations with low out-degree and compaction of adjacency matrices. *Theor. Comput. Sci.* 86 (1991) 243–266.

[5] M.B. Dillencourt and W.D. Smith. A linear-time algorithm for testing the inscribability of trivalent polyhedra. Manuscript, 1991.

[6] C. Papadimitriou and M. Yannakakis. The clique problem for planar graphs. *Inform. Process Lett.* 13 (1981) 131–133.

[7] W.T. Tutte. A theory of 3-connected graphs. *Proc. Koninklijke Nederlandse Akademie van Wetenschappen, Ser. A* 64 (1961) 441–455.