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Jean-Pierre Sursock

April 5, 1978

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MATHEMATICAL PROPERTIES OF ORDERED AMPLITUDE GRAPHS*

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April 5, 1978

ABSTRACT

This paper elaborates on some mathematical aspects of the ordered hadronic amplitudes. In particular, a systematic way of constructing the ordered Hilbert space is presented and it is shown to be the most general Hilbert space consistent with unitarity of the S-matrix. A number of topological properties of the ordered amplitude are analyzed and finally the ordered S-matrix is shown to be cluster decomposable and crossing symmetric.

1. Introduction

The S-matrix approach to the topological expansion of Veneziano¹ is made possible by invoking the notion of "order" first introduced by Chew et. al.² for mesonic amplitudes and later generalized to all hadronic amplitudes.³ The ordered S-matrix is defined in an ordered Hilbert space and obeys unitarity as does the physical S-matrix. This fact guarantees consistence between poles and asymptotic states. The attempt to get rid of the unobservable order by summing over all ordered amplitudes (planar amplitude) produces a breaking of the unitarity constraint. Unitarity is restored when all terms of the topological expansion are taken into account.

The objective of this paper is to discuss the mathematical framework of the material presented in Ref. 3.

2. The Concept of Order

The first step of the theory is to define a simpler S-matrix (in terms of analytical structure) in an unobservable Hilbert space, called the Ordered Hilbert Space (OHS). In this space, a particle, in addition to its mass, momentum and spin has a well-defined collection of "neighbors".* For instance, each meson has two neighbors, each baryon has three neighbors and so on. A more general channel

* The number of neighbors will turn out to be characterizable as the number of "quarks" and "antiquarks" that "build" a hadron.

defined by n particles is specified by the particles involved and by the specification of each particle neighbor (concept of order). For instance, in the simple case of a channel containing mesons only, the channel could be written as $|a\rangle = |M_1 \dots M_i \dots M_j \dots M_n\rangle$. Here, the order matters as it specifies the neighbors of each mesons M_i . This channel is, in general, different from the channel $|b\rangle = |M_1 \dots M_j \dots M_i \dots M_n\rangle$.

In either channel $|a\rangle$ or $|b\rangle$, M_1 and M_n have each one neighbor. However, considered in an amplitude, they will eventually have two neighbors also. For example, in the amplitude $\langle M_k \dots M_m | T | M_1 \dots M_n \rangle$ M_1 will have M_2 and M_m as neighbors and M_n will have M_k and M_{n-1} as neighbors. Such a channel $|a\rangle$ can be graphically represented in Fig. 1a and the amplitude in Fig. 1b. Thus, although all the neighbors of a given particle are not necessarily completely specified in a channel they are unambiguously defined when the particle is considered in a process (transition amplitude).

In the more general case, a particle may have many neighbors (e. g., Fig. 2). Apriori, any graph can represent a channel of the OHS. We shall however impose the following restrictions:

- (1) The graph must be connected.
- (2) The graph must not contain loops or tadpoles (i. e., edges leaving and returning to the same vertex).

(3) The graph should not include vertices of degree higher than three.* This requirement is discussed in more detail in Appendix C and is independently justified in Ref. 4. It does not preclude the existence of particles with more than three neighbors because a particle does not necessarily correspond to a vertex but can correspond to a whole graph. So, for instance, particles with four and five neighbors are shown on Fig. 3**

(4) Each edge of the graph representing a channel of the OHS must be distinguishable. As will be shown in Section 4, this requirement is crucial to obey the constraints of the ordered S-matrix

3. The Ordered Hilbert Space (OHS)

To construct the OHS we start from the cubic tree graphs (i.e., tree graphs such that all vertices are of degree three. Contrary to mathematician's practice, we drop the 1-vertices at the "tips" of the tree.). Examples are shown on Fig. 4.

All cubic tree graphs are allowed to represent ordered channels. In general, the topological structure of the tree graphs is sufficient to distinguish between the edges. However, some tree graphs exhibit a two- or a three-fold symmetry (e. g., Fig. 4b). It is possible to show (Appendix A) that in order to restore the distinguishability of every edge it is necessary and sufficient to orient and color the edges of the graph. The orientation is such that all edges are

* The degree of a vertex is the number of edges attached to it.

** A more complete discussion of exotic states is found in Ref. 3.

incoming toward a vertex or outgoing away from it. Then every edge is given a color index 1, 2 or 3 such that, at any 3-vertex, the three edges all have different color indices. This operation insures that no graph is symmetric.* An example is shown on Fig. 5.

A colored cubic tree graph is called a skeleton. All possible ordered states are built from skeletons by the following operation (shown on Fig. 6):

- (1) Select an edge a (Fig. 6a) with a color index i and cut it into two parts a_1 and a_2 (Fig. 6b).
- (2) Connect the two ends with two edges b and c with color indices j and k such that i, j and k are all different and with opposite orientation relative to a (Fig. 6c).

It is easy to see that the new edges (b and c) are still distinguishable. First they can be distinguished from one another because they have different color and second they can be distinguished from any other edge in the graph because they are "inserted" in the old edge a which was distinguishable by construction.

The two operations described above can be iterated at will on any edge or edges of the new graph thus constructed. An example is shown on Fig. 7 (where the arrows and color indices are omitted for clarity).

* This operation is called induced symmetry breaking.

Finally, any number of vertices of degree two (mesons) can be inserted in any edge of such graphs without altering its topology. Sometimes, however, a meson will be represented as the cubic graph of Fig. 6e. For the ordered S-matrix, the two notations are equivalent. It can be shown that the graphs thus constructed are the most general graphs consistent with unitarity. However, the proof will be postponed until the end of Section 4.

Thus, every graph of the OHS (allowed graph) can be related to a unique skeleton. The OHS is divided in disjoint subspaces called sectors. Each sector is defined by a skeleton and all graphs derived therefrom by the operations described earlier (Fig. 6).

4. The Ordered Amplitude Graphs (OAG)

We shall postulate that the sectors defined above are orthogonal. That is, the ordered S-matrix does not allow communication between channels belonging to different sectors. Thus, for a transition amplitude to be non-vanishing, the following requirements must be met:

- (1) Poincaré invariance
- (2) Conservation of internal quantum numbers
- (3) Conservation of order (i. e. sectors)

The transition amplitude between two states belonging to the same sector is represented by a graph such that all the neighbors of all particles are defined. An example is shown on Fig. 8.

The OAG is constructed by matching all^{*} the corresponding outer^{**} edges of each state. One can see now the importance of these edges being all distinguishable. If this were not the case, several matching would have been equally possible. This would be contrary to the principle that given the initial and final states, the transition amplitude is unambiguously determined.

The fact that every edge should be distinguishable, not only the outer edges, will result from the requirements of crossing symmetry (every edge can be an outer edge).

An important property of a unitary S-matrix is factorization. In this context it means that each cut separating the amplitude graph into two connected parts (bisection) defines a collection of poles belonging to the same sector (the cut edges) and two channels that should belong to the OHS. Several bisections of the amplitude graph of Fig. 8 are shown on Fig. 9. The cuts can all be thought of as planes cutting a sphere on which the amplitude graph is imbedded.^{***}

It is now possible to prove the statement made earlier that the most general channels allowed by unitarity are those defined in Section 3. We show in Appendix C that vertices of degree higher than 3 should be discarded and in the remainder we concentrate on

* This point will be discussed in Section 5.

** "Outer" or "dangling" edge refers to the peripheral edges of a tree graph (the "tips" of the tree).

*** Although, for the considerations discussed in this paper, we do not require such an imbedding.

graphs constructed with 3-vertices only (2-vertices can be added anywhere).

We first remark that the graph of Fig. 10 cannot be allowed in the OHS because the edges are not all distinguishable.* Then we show that any state with the same topology as Fig. 10 is inadmissible regardless of the coloring. Let's assume that we can use as many colors or indices as we wish to distinguish the edges and we start from Fig. 11a which we combine with itself to form the amplitude graph of Fig. 11b. This should always be possible. Next we cut (11b) as shown on Fig. 11c to obtain the state of Fig. 11d on the right hand side. This new state is made to combine with itself in a new amplitude shown on Fig. 11e which we cut again as shown on Fig. 11f. The resulting states are the unwanted states of Fig. 10. Thus, regardless of coloring and orientation, the graph of Fig. 10 should not appear in the OHS as a graph or as a subgraph.

The immediate consequence of this result is that any graph containing a subgraph of the type of Fig. 12a should not be allowed, because by combining this state with itself in an amplitude (Fig. 12b) we generate the unwanted graph of Fig. 10 as shown by the bisection in Fig. 12c. Finally, it is easy to see that a graph without subgraphs of the type (12a) and with only one outer edge

* The undesirable aspect of this graph was first noted by G. Weissmann.

is not colorable (Fig. 13) and therefore cannot be made consistent with unitarity.

This completes the proof that the only allowed states of the OHS are those constructed by the operations of Section 3 (Fig. 6).

In summary, an allowed channel graph has the following characteristics:

(1) It has more than one outer edge.

(2) Every cycle is connected to the set of outer edges by two and only two edges (Fig. 6c in contrast to Fig. 12a). These two edges will be called links. The vertices at which the links connect to the cycle will be called gates of the cycle and the disjoint paths of the cycle connecting the two gates will be called legs of the cycle.

In the remainder of this paper we shall use the notation $\langle g_f | g_i \rangle$ to denote the OAG obtained from the graph g_f and g_i corresponding to the states $|f\rangle$ and $|i\rangle$ respectively. γ_f and γ_i are the corresponding skeletons and we must have $\gamma_f = \gamma_i$ for $\langle g_f | g_i \rangle$ to be non-vanishing.

5. Topological Properties of the Ordered Amplitude Graphs

(1) Up to a homeomorphism, an Ordered Amplitude Graph (OAG) must have the topology of a regular cubic graph (i. e., all vertices are of degree three). In other words, all outer edges of the channel graphs must be connected as mentioned earlier. Failure to do so would result in a possibility of cutting the graph in

three parts when attempting to isolate certain ordered states.³ This is shown on Fig. 14. Inasmuch as a single quark would be associated with a one-vertex, one can see that the failure to reconcile graphs such as Fig. 14 with unitarity guarantees that quarks cannot be poles of the S-matrix and thus cannot be considered as asymptotic states (quark confinement).^{3*}

(2) The OAG are connected and planar (evident).

(3) Every cycle of the OAG has an even number of vertices.

This results from the fact that each vertex of the left hand skeleton has one counterpart on the right hand skeleton (cf. Fig. 8) and all cycles within each skeleton contains by construction of Section 3 an even number of vertices on either leg of the cycle. As discussed in Ref. 3, this property enables one to specify Fermi statistics for baryons.

(4) Therefore the OAG is bipartite (theorem of graph theory).⁵

That is, it is possible to divide the set of vertices of the graph into two disjoint parts A and B such that no vertex of A is adjacent to another vertex of A and similarly for B.

(5) If we assign a coefficient +1 to the vertices of A and -1 to the vertices of B, then, because of properties 3 and 4, the algebraic sum of these coefficients on any cycle is zero. This

* This argument was originally put forth by G. Weissman. A somewhat different argument for quark confinement is developed in Ref. 4.

is also true for the overall amplitude. If we associate these coefficients with the baryon number we automatically obtain baryon number conservation.

(6) Property 5 is equivalent to the possibility of coloring the edges of the OAG with three colors.*⁵ Furthermore, it is possible to assign the coefficients ± 1 to a vertex such that when the graph is imbedded on an orientable surface (e. g., sphere), the coefficient $+1$ is associated with the clockwise cyclic order of the color indices 1, 2 and 3 around the vertex and the coefficient -1 to the opposite orientation.⁵ Thus, if the OAG is imbedded on an orientable surface the edge orientation becomes redundant. The orientation of the color indices is associated with the charge conjugation operation.⁴ That is, an odd permutation of color indices transforms an amplitude into its charge conjugate.

(7) Existence and uniqueness of mates: In an OAG we define the mate of a vertex A as a vertex A' which could be reached from A by three disjoint paths (i. e., paths that have no edge in common). The fact that to each vertex of an OAG corresponds one and only one mate is a crucial result that will enable a

* This is also true for the faces of the OAG: namely, the faces of the OAG can be colored using three colors only such that no two adjacent faces are colored with the same color.

consistent topological expansion.⁸ We establish this result in Appendix B.

(8) Algebraic notation: Since each edge of the OAG is distinguishable let us relabel the edges with labels α_i . Then a vertex A is associated with three labels $\alpha_i, \alpha_j, \alpha_k$. We adopt the following conventions:

- (a) We write A with the α -indices as superscript or subscript according to whether the arrows are outgoing or incoming (or, if the graph is imbedded on an orientable surface, according to whether the color cyclic order around A is clockwise or anticlockwise).
- (b) The order of the indices is such that the first one applies to color index 1, the second to color index 2 and the third to color index 3.
- (c) Finally, we associate to each mate pair a number which helps identify the members of the pair. An example is shown on Fig. 15. One can see that each α -index should appear twice, once as a superscript and once as a subscript. The algebraic notation stresses the fact that only the connections between vertices are important.

If the subgraph (AB) of Fig. 15 represents one particle (a meson) then we could have used the more compact notation

$M_{\alpha_3}^{\alpha_1\alpha_2\alpha_3} = A_{\alpha_1\alpha_2\alpha_3}^{\alpha_1\alpha_2\alpha_3} B_{\alpha_1\alpha_2\alpha_3}^{\alpha_1\alpha_2\alpha_3}$. Similarly, if (AD) is a baryonium, we can write $E_{\alpha_4\alpha_5}^{\alpha_1\alpha_2} = A_{\alpha_1\alpha_2\alpha_3}^{\alpha_1\alpha_2\alpha_3} D_{\alpha_4\alpha_5\alpha_3}^{\alpha_1\alpha_2\alpha_3}$.

One can readily see the connection between the edges of the OAG and the quark structure usually associated to hadronic states. The concept of order thus implies the quark structure, including zero triality and quark confinement.^{2,3,4,7}

6. Unitarity and Cluster Decomposition

In this section we make no distinction between graphs representing a connected part T_{fi} and its complex conjugate T_{fi}^* .

As discussed in the introduction, the unitarity condition is expressed through the discontinuity relations in a given process. The process is specified by an initial state $|i\rangle$ and a final state $|f\rangle$. Let us denote the corresponding graphs g_i and g_f and the corresponding skeletons by γ_i and γ_f . Furthermore, for any intermediate state $|n\rangle$ to which we associate a graph g_n and a skeleton γ_n we can define two OAG which we denote $\langle g_f | g_n \rangle$ and $\langle g_n | g_i \rangle$. The product of the two ordered amplitudes, $\langle g_f | g_n \rangle \langle g_n | g_i \rangle$ will be non-zero only if $\gamma_i = \gamma_n = \gamma_f$. The result $\gamma_i = \gamma_f$ indicates that the product of two ordered

amplitudes can always be interpreted as part of the discontinuity of an ordered amplitude $\langle g_f | g_i \rangle$. This fact is essential for the consistency of the theory. It is a consequence of the requirement of conservation of order discussed earlier.

We now turn to Cluster Decomposition. Consider the two states $|i\rangle$ and $|f\rangle$ introduced above. Each graph g_i and g_f can be decomposed into n connected subgraphs g_{ij} and g_{fj} ; $j = 1, \dots, n$. and correspondingly, each skeleton into n connected subskeletons γ_{ij} and γ_{fj} . The following rules however must be obeyed by the decomposition:

(1) To each skeleton γ_{ij} there corresponds an identical skeleton γ_{fj} at the same relative location in the original tree. In other words, the two skeletons γ_i and γ_f must be decomposed identically.

(2) A subgraph representing a single particle must lie entirely in the same g_{ij} or g_{fj} .

We then define n ordered amplitudes $\langle g_{fj} | g_{ij} \rangle$ for each couple of graphs (g_{fj}, g_{ij}) separately. If one or more of these amplitudes vanish, the particular cluster decomposition is set equal to zero. An example of cluster decomposition is shown on Fig. 16.

Finally, we address the question of crossing. Consider an amplitude in the phase space region where $|a\rangle$ and $|b\rangle$ are the initial and final states respectively. This ordered amplitude is associated to an OAG $\langle g_b | g_a \rangle$. It is possible to analytically continue this amplitude to another region of phase space where $|c\rangle$ and $|d\rangle$ are the initial and final states respectively, provided $\langle g_d | g_c \rangle = \langle g_b | g_a \rangle$. In other words, given an OAG, the regions of phase space related by analytical continuation are those corresponding to all possible bisections of the OAG such that one connected graph thus obtained corresponds to an initial state and the other to a final state.

Conclusion

The objective of this paper was to expand on the mathematical aspects of the ordered hadronic amplitudes presented in Ref. 3. In particular we have presented a systematic way to construct the ordered Hilbert space of Ref. 3 and we have shown that it is the most general Hilbert space consistent with unitarity.

We have also discussed a number of topological properties of the ordered amplitude and, in particular, the concept of mates was exhibited and will turn out to be crucial in the topological expansion.⁸ Finally the ordered S-matrix turns out to be cluster decomposable and therefore crossing symmetric.

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APPENDIX A: INDUCED SYMMETRY BREAKING

To establish that cubic tree graphs have only three possible symmetries we recall Jordan and Sylvester's theorem:⁶ "Every tree has a center^{*} consisting of either one vertex or two adjacent vertices." Thus there are two cases to consider.

(a) There is only one central point. Since it is a 3-vertex, the symmetry requires either that the three branches of the point be identical yielding a 3-fold symmetry (e. g., Fig. A1) or two branches only are identical yielding a 2-fold symmetry (e. g., Fig. A2).

(b) There are two central points. In that case, the theorem states that the two points are connected by one edge: e . The only possible two-fold symmetry of a cubic tree graph is then obtained where the two subgraphs on either side of e are identical (e. g., Fig. A3).

In the case (a), the symmetry around the central point is manifestly broken by coloring the three edges of the central vertex with different colors and in the case (b), the symmetry is broken by giving to the edge e an orientation.

* To each vertex of a tree graph one can associate an integer corresponding to the (unique) path from that vertex to the most remote "tip" of the tree (relative to the vertex). A central vertex is a vertex with the smallest integer and the center of the tree graph is the set of all central vertices.

As we take a path on one (now well-defined) branch, away from the center of the tree graph and toward an outer vertex we encounter at each vertex a bifurcation. The choice at the bifurcation can be made well defined by the same coloring procedure. Thus, any edge of the graph can be specified by the colored path one must take to reach it with the center of the graph as starting point.

As an example, in Fig. A4, edge a is defined by the path that starts at the center of the graph c and follows the colored edge, 1, 2 and 3 successively.

APPENDIX B: EXISTENCE AND UNIQUENESS OF MATES

Existence: Given a vertex A in an ordered amplitude graph G we want to show that there exists another vertex A' such that A and A' are connected by three disjoint paths.

Since in a regular cubic graph each vertex is at least on two circuits (in fact on three circuits), A is on some circuit K . We bisect G such that K is entirely on one channel graph. Again, this is always possible because it suffices to isolate part of an edge of G as the bisection. Then by the construction of Section 3, A is necessarily the gate of some circuit C . Then let A' be the other gate of the same circuit C . It is easy to see that two of the three disjoint paths connecting A to A' are the two legs of the circuit C . The third path is obtained by following a path containing the links of C -- which always exists since the amplitude graph is regular.

Uniqueness: Consider $A'' \neq A'$. If A'' is on one leg of C then the path from A to A'' containing the links and the path containing the other leg are no longer disjoint. If A'' is not on C , then the paths containing the two legs have at least one link in common. Thus, no other vertex than A' can fulfill the definition of a mate.

Alternatively, it can be shown that two mates A and A' share the same three Edmond orbits (i. e., faces of G).

APPENDIX C: CONSTRUCTION OF HILBERT SPACE WITH n -VERTICES IN GENERAL

First we note that the sector displayed in Fig. C1 (two edges with different colors) would conflict with unitarity unless it does not communicate with any other state. This could be seen in the amplitude of Fig. C2 where a state with indistinguishable edges can be extracted.*

Now we discuss the possibility of admitting n -vertices ($n > 3$) in the graphs. This automatically requires at least n colors to distinguish the edges of a single vertex. There are two possibilities:

(a) The n -vertices are admitted concurrently with the 3-vertices. In this case, one can readily construct the unwanted state of Fig. C1 (e. g., Fig. C3).* Incidentally, Fig. C3 also shows that we cannot have more than three colors even if we restrict ourselves to 3-vertices.

(b) The 3-vertices are not admitted. There we have two subcases:

(b-1) n is even: In this case, one cannot construct skeletons** with an odd number of outer edges. This results from the fact that any skeleton is constructed with k

* This part of the argument is due to G. Weissmann.⁷

** For $n > 3$, skeleton graphs may include cycles as long as no more than two vertices reside on any cycle.

vertices. We thus have nk edges at our disposal but nk is even and every connection reduces that number by two. Hence the final number of outer edges is always even. Therefore the resulting Hilbert space is much more restrictive than the OHS (constructed with 3-vertices).

(b-2): n is odd (and $n > 3$). In this case it is easy to see that one cannot construct a skeleton graph with three outer edges. In fact, more generally, it is impossible to construct a skeleton graph with m outer edges and with $m < n$ if m and n are odd. This can be seen as follows: if k is the number of vertices and e the number of internal edges we have $kn - 2e = m$. On the other hand, if k is even, $e_{\max} = kn/2 - 1$ and if k is odd, $e_{\max} = (k-1)n/2$. The latter case is of interest here (m, n odd) and we have $m_{\min, \text{odd}} = kn - (k-1)n = n$. So the resulting Hilbert space is again more restrictive than the OHS.

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FIGURE CAPTIONS

- Fig. 1: (a) Mesonic ordered state. (b) mesonic amplitude
- Fig. 2: Generalization of the sequential order
- Fig. 3: Particle graph representing (a) a particle with four neighbors and (b) with five neighbors
- Fig. 4: Cubic tree graphs: basis of the ordered Hilbert space
- Fig. 5: Oriented and colored tree graph (skeleton)
- Fig. 6: Elementary operation generating all graphs of the ordered Hilbert space
- Fig. 7: Successive iterations of the operations of Fig. 6.
- Fig. 8: Combining states to obtain an ordered amplitude graph (OAG)
- Fig. 9: Bisections of ordered amplitude graphs
- Fig. 10: This graph is illegal because the diagonal edges are not distinguishables.
- Fig. 11: Constructing the graph of Fig. 10 starting with an arbitrary number of indices to distinguish between edges
- Fig. 12: (a) Prototype of inadmissible subgraph. (b) The result of its combination with itself and (c) showing the emergence of the unwanted graph of Fig. 10
- Fig. 13: Uncolorable graph with only one external vertex.
- Fig. 14: Possibility of cutting the OAG in three parts if 1-vertices are allowed

- Fig. 15: Algebraic notation for ordered amplitude graphs
- Fig. 16: Cluster decomposition of a transition amplitude
- Fig. A1: Cubic tree with one central vertex and 3-fold symmetry
- Fig. A2: Cubic tree with one central vertex and 2-fold symmetry
- Fig. A3: Cubic tree with two central vertices and 2-fold symmetry
- Fig. A4: Colored cubic tree of type (A-b) illustrating the fact that each edge (e.g., a) is distinguishable. Edge orientation should be superposed but is not needed here. The graph is drawn with the convention that the orientation of the vertices alternate on any path. C is the central vertex.
- Fig. C1: Undesirable sector defined by two outer edges of different colors.
- Fig. C2: Construction of a state with undistinguishable edges using a state belonging to the sector of C1
- Fig. C3: Construction of a state belonging to the sector C1 using 3-vertices and more than three colors.

Fig. 1a

Fig. 1b

Fig. 2

Fig. 3a

Fig. 3b

Fig. 4a

Fig. 4b

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Fig. 5

Fig. 6a

Fig. 6b

Fig. 6c

Fig. 7

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Fig. 8

Fig. 9

Fig. 10

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Fig. 11a

Fig. 11b

Fig. 11c

Fig. 11d

Fig. 11e

Fig. 11f

Fig. 12a

Fig. 12b

Fig. 12c

Fig. 13

XBL 784-713

Fig. 14

$$= \left\{ \begin{array}{cc} a_1 a_2 a_3 & a_5 a_1 a_6 \\ A(1) B(1) & C(2) D(2) \\ & a_1 a_2 a_6 & a_5 a_4 a_3 \end{array} \right\}$$

Fig. 15

$$= \begin{array}{c} \text{A} \text{---} \text{F} \\ \text{B} \text{---} \text{E} \\ \text{C} \text{---} \text{D} \end{array} + \begin{array}{c} \text{A} \text{---} \text{F} \\ \text{B} \text{---} \text{E} \\ \text{C} \text{---} \text{D} \end{array} + \begin{array}{c} \text{A} \text{---} \text{F} \\ \text{B} \text{---} \text{E} \\ \text{C} \text{---} \text{D} \end{array}$$

$$+ \begin{array}{c} \text{A} \text{---} \text{F} \\ \text{B} \text{---} \text{E} \\ \text{C} \text{---} \text{D} \end{array}$$

Fig. 16

XBL 784-712

Fig. A1

Fig. A2

Fig. A3

Fig. A4

Fig. C1

Fig. C2

Fig. C3

XBL 784-718

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