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The Asymmetric Sandwich Theorem

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We discuss the asymmetric sandwich theorem, a generalization of the Hahn–Banach theorem. As applications, we derive various results on the existence of linear functionals that include bivariate, trivariate and quadrivariate generalizations of the Fenchel duality theorem. Most of the results are about affine functions defined on convex subsets of vector spaces, rather than linear functions defined on vector spaces. We consider both results that use a simple boundedness hypothesis (as in Rockafellar's version of the Fenchel duality theorem) and also results that use Baire's theorem (as in the Robinson–Attouch–Brezis version of the Fenchel duality theorem). This paper also contains some new results about metrizable topological vector spaces that are not necessarily locally convex.

1. Introduction

This paper is about the existence of linear functionals in various situations. The main results of Section 2 are the *asymmetric sandwich theorem* of Theorem 2.4 and the *sublevel set theorem* of Theorem 2.7. The asymmetric sandwich theorem is a straightforward extension of the Hahn–Banach theorem, and the sublevel set theorem is about the conjugate of a proper convex function defined on a CC space (a nonempty convex subset of some vector space).

The final result in Section 2, Corollary 2.9, is a technical result about the conjugate of a convex function defined as the infimum of another convex function over a variable set. We call this result a *trivariate existence theorem* because it uses three spaces: two CC spaces and a locally convex space. The best way of clarifying the interiority condition (10) that appears in Corollary 2.9 is to consider the case of two generalizations to infinite dimensional spaces of the Fenchel duality theorem for two convex functions f and g. The first, due to Rockafellar, assumes a local boundedness condition for g at some point where f is finite, but does not assume any global lower semicontinuity conditions for either function. Furthermore, the two functions are not treated symmetrically. The second, due to Robinson and Attouch–Brezis, treats the two functions in a symmetric fashion. Furthermore, two functions are assumed to be lower semicontinuous and the space complete.

Corollary 2.9 leads to further trivariate existence theorems: Theorem 3.1 in Section 3, and Theorem 5.1 in Section 5. At this point we will discuss Theorem 3.1, since our remarks about Theorem 5.1 are best postponed until after our consideration of Section 4. We give two consequences of Theorem 3.1: Corollary 3.2, an affine

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Fenchel duality theorem, and Theorem 3.4. We call Theorem 3.4 a *quadrivariate existence theorem* because it uses four spaces: two CC spaces, a vector space and a locally convex space. Corollary 3.2 extends the result of Rockafellar referred to above, and Theorem 3.4 shows how we can compute the conjugate of a convex function defined in terms of a function of four variables and two affine maps. Theorem 3.4 leads easily to Corollary 3.5. We call Corollary 3.5 a *bibivariate existence theorem* because it uses two pairs of two spaces. In it, we show how we can compute the conjugate of a convex function defined in terms of two functions of two variables and two affine maps. We give three consequences of Corollary 3.5: Corollaries 3.6–3.8. In Corollary 3.6, we show how we can compute the conjugate of a convex function defined in terms of a partial inf–convolution of two functions, and in Corollaries 3.7 and 3.8, we show how how we can compute the conjugate of a convex function defined in terms of a given convex function and two affine maps. All through the analysis that we have discussed so far, the conclusion is the existence of a *linear* functional satisfying certain properties. Corollary 3.8 is the first place in which we assume that one of the given maps is *linear* (rather than *affine*). The statement of Corollary 3.8 is also interesting in that it does not contain "+" or "−".

Section 4 is about (not necessarily locally convex) metrizable linear topological spaces. Lemma 4.1 and Lemma 4.2 are technical results, the second of which uses Baire's theorem. They lead to Theorem 4.3, which implies that, under the appropriate circumstances, the interiority condition (11) is equivalent to a much simpler condition. The statement of Theorem 4.3 is disarmingly simple given how much effort seems to be required to establish it. The automatic interiority result of Corollary 4.4 is immediate from Theorem 4.3.

In Section 5, we give applications of Corollary 4.4 to the existence of linear functionals. The trivariate existence theorem, Theorem 5.1 is the analog of Theorem 3.1, and the remaining results in Section 5 follow from Theorem 5.1 in much the same way that the results in Section 3 followed from Theorem 3.1. Corollary 5.2 extends the version of the Fenchel duality theorem due to Robinson and Attouch–Brezis that we have already mentioned. Theorem 5.3 is a second quadrivariate existence theorem, Corollary 5.4 is a second bibivariate existence theorem that extends a result of Simons, and the result on partial inf–convolutions that appears in Corollary 5.5 extends a result of Simons–Zălinescu. In Corollaries 5.6 and 5.7, we apply Corollary 5.4 to situations where the functions are defined on spaces of the form $E \times E^*$ and $F \times F^*$, where E and F are Banach spaces. Similar results are true in the context of Section 3, but they are less interesting since, in the situation in which these results are applied, the functions concerned are the Fitzpatrick functions of monotone multifunctions, which are known to be lower semicontinuous. Corollaries 5.8 and 5.9 also extend results that have been used recently in the study of maximally monotone multifunctions on nonreflexive Banach space. We refer the reader to [11,12] for more details of these applications.

We would like to express our sincere thanks to Constantin Zălinescu for reading through the first version of this paper, and making a number of suggestions that

have improved the exposition enormously.

2. The existence of linear functionals

All vector spaces in this paper are *real*.

Definition 2.1. We shall say that Z is a *convex combination space (CC space)* if Z is a nonempty convex subset of a vector space. Let Z and X be CC spaces and $B\colon Z \to X$. We say that B is *affine* if, for all $x, y \in Z$ and $\lambda \in]0,1[$, $B(\lambda x + (1-\lambda)y) = \lambda Bx + (1-\lambda)By$, and we write $\mathrm{aff}(Z,X)$ for the set of affine functions from Z into X. We write $Z^\flat$ for the set of affine functions from Z into $\mathbb{R}$, and so $Z^\flat = \mathrm{aff}(Z,\mathbb{R})$. If Z is a CC space, we write $\mathcal{PC}(Z)$ for the set of all convex functions $k\colon\ Z \to]-\infty,\infty]$ such that $\mathrm{dom}\, k \neq \emptyset$, where $\mathrm{dom}\, k$, the *effective domain* of k, is defined by

$$\mathrm{dom}\, k := \{z \in Z\colon\ k(z) \in \mathbb{R}\}.$$

(The "$\mathcal{P}$" stands for "proper", which is the adjective frequently used to denote the fact that a function is finite at at least one point.) If X is a vector space, we write X' for the set of linear functionals on X, the algebraic dual of X.

The main results of Section 2 are the *asymmetric sandwich theorem* of Theorem 2.4 and the *sublevel set theorem* of Theorem 2.7. In order to justify this nomenclature for Theorem 2.4, we state König's original result (see [3, Theorem 1.7, p. 112]), which can obviously be obtained from Theorem 2.4 by taking $Z = X$ and B to be the identity map. (*Sublinear* is defined in Definition 2.2.)

König's sandwich theorem. *Let X be a vector space, $S\colon\ X \to \mathbb{R}$ be sublinear, $k \in \mathcal{PC}(X)$ and $S \geq -k$ on X. Then there exists $x' \in X'$ such that $S \geq x' \geq -k$ on X.*

This theorem is *symmetric* because S and k are defined on the same set. By contrast, Theorem 2.4 is *asymmetric* because S and k are defined on the different sets X and Z.

Definition 2.2. Let X be a nontrivial vector space. We say that $S\colon\ X \to \mathbb{R}$ is *sublinear* if

$$S \text{ is } \textit{subadditive}\text{:} \qquad x_1, x_2 \in X \quad \Longrightarrow \quad S(x_1 + x_2) \leq S(x_1) + S(x_2)$$

and

$$S \text{ is } \textit{positively homogeneous}\text{:} \qquad x \in X \text{ and } \lambda > 0 \quad \Longrightarrow \quad S(\lambda x) = \lambda S(x).$$

We note that it follows automatically that $S(0) = 0$. Our results depend on the classical Hahn–Banach theorem for sublinear functionals, which we now state:

Lemma 2.3. *Let X be a vector space and $S\colon\ X \to \mathbb{R}$ be sublinear. Then there exists $x' \in X'$ such that $x' \leq S$ on E.*

Proof. See Kelly–Namioka, [2, 3.4, p. 21] for a proof using cones, Rudin, [7, Theorem 3.2, p. 56–57] for a proof using an extension by subspaces argument, and König, [3] and Simons, [8] for a proof using an ordering on sublinear functionals. □

We now come to the **asymmetric sandwich theorem**. Remark 2.5 contains several comments on this result.

Theorem 2.4. *Let X be a vector space, $S\colon\ X \to \mathbb{R}$ be sublinear, Z be a CC space, $k \in \mathcal{PC}(Z)$, $B \in \operatorname{aff}(Z, X)$ and $SB \geq -k$ on Z. Then there exists $x' \in X'$ such that $x' \leq S$ on X and $x'B \geq -k$ on Z.*

Proof. For all $x \in X$, let

$$T(x) := \inf_{z\in Z,\ \lambda>0}\,[S(x + \lambda Bz) + \lambda k(z)] \in [-\infty, \infty]. \qquad (1)$$

If $x \in X$, $z \in Z$ and $\lambda > 0$ then

$$S(x + \lambda Bz) + \lambda k(z) \geq S(x + \lambda Bz) - S(\lambda Bz) \geq -S(-x).$$

Taking the infimum over $z \in Z$ and $\lambda > 0$, $T(x) \geq -S(-x) > -\infty$. On the other hand, fix $z \in \operatorname{dom} k$. Let x be an arbitrary element of X. Then, for all $\lambda > 0$, $T(x) \leq S(x + \lambda Bz) + \lambda k(z) \leq S(x) + \lambda S(Bz) + \lambda k(z)$. Letting $\lambda \to 0$, $T(x) \leq S(x)$. Thus

$$T\colon\ X \to \mathbb{R} \quad \text{and} \quad T \leq S \text{ on } X.$$

We now show that T is subadditive. To this end, let $x_1,\ x_2 \in X$. Let $z_1,\ z_2 \in Z$ and $\lambda_1,\ \lambda_2 > 0$ be arbitrary. Write $x := x_1 + x_2$, and $z := (\lambda_1 z_1 + \lambda_2 z_2)/(\lambda_1 + \lambda_2)$. Then, since $\lambda_1 Bz_1 + \lambda_2 Bz_2 = (\lambda_1 + \lambda_2)Bz$ and $\lambda_1 k(z_1) + \lambda_2 k(z_2) \geq (\lambda_1 + \lambda_2)k(z)$,

$$\begin{aligned}
&[S(x_1 + \lambda_1 Bz_1) + \lambda_1 k(z_1)] + [S(x_2 + \lambda_2 Bz_2) + \lambda_2 k(z_2)]\\
&\qquad \geq S(x + \lambda_1 Bz_1 + \lambda_2 Bz_2) + \lambda_1 k(z_1) + \lambda_2 k(z_2)\\
&\qquad \geq S(x + (\lambda_1 + \lambda_2)Bz) + (\lambda_1 + \lambda_2)k(z) \geq T(x) = T(x_1 + x_2).
\end{aligned}$$

Taking the infimum over z_1, z_2, λ_1 and λ_2 gives $T(x_1) + T(x_2) \geq T(x_1 + x_2)$. Thus T is subadditive. It is easy to check that T is positively homogeneous, and so T is sublinear. From Lemma 2.3, there exists $x' \in X'$ such that $x' \leq T$ on X. Since $T \leq S$ on X, $x' \leq S$ on X, as required. Finally, let z be an arbitrary element of Z. Taking $\lambda = 1$ in (1), $k(z) = S(-Bz + Bz) + k(z) \geq T(-Bz) \geq x'(-Bz) = -(x'B)z$, hence $x'B \geq -k$ on X. This completes the proof of Theorem 2.4. □

Remark 2.5. It is worth pointing out that the definition of the *auxiliary sublinear functional*, T, used to prove Theorem 2.4 is "forced" in the sense that if $x' \in X'$, $x' \leq S$ on X and $x'B \geq -k$ on Z then, as the reader can easily verify, $x' \leq T$ on X.

It is easy to see that Theorem 2.4 follows from the *Hahn–Banach–Lagrange theorem* of [9, Theorem 2.9, p. 153] or [10, Theorem 1.11, p. 21]. On the other hand, Theorem 2.4 implies the Mazur-Orlicz theorem of [9, Lemma 2.4, p. 152] or [10, Lemma 1.6, p. 19], which in turn implies the Hahn–Banach–Lagrange theorem.

While we have presented Theorem 2.4 as a fairly direct consequence of the Hahn–Banach theorem, one can also establish it using an appropriate version of the Fenchel duality theorem. We will return to this issue in Remark 3.3.

We now define the *sublevel sets*, and also the *conjugate with respect to a real affine function*, of a proper convex function on a CC space.

Definition 2.6. Let Z be a CC space, $\Phi \in \mathcal{PC}(Z)$, $\gamma \in \mathbb{R}$ and $z^\flat \in Z^\flat$. Then we write $\sigma_\Phi(\gamma)$ for the sublevel set $\{z \in Z\colon\ \Phi z < \gamma\}$. The set $\sigma_\Phi(\gamma)$ is convex. We define $\Phi^*(z^\flat) := \sup_Z[z^\flat - \Phi] \in \,]{-\infty}, \infty]$.

The next result is the **sublevel set theorem**. If X is a locally convex space, we write X^* for the set of continuous linear functionals on X, the topological dual of X.

Theorem 2.7. *Let Z be a CC space, X be a locally convex space, $B \in \operatorname{aff}(Z, X)$, $\Phi \in \mathcal{PC}(Z)$,*

$$Y := \textstyle\bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Phi) \ \ \textit{be a linear subspace of } X, \tag{2}$$

and suppose that there exists $\gamma \in \mathbb{R}$ such that

$$0 \in \operatorname{int}_Y B\left(\sigma_\Phi(\gamma)\right). \tag{3}$$

Then

$$\min\nolimits_{x^* \in X^*} \Phi^*(x^*B) = -\inf \Phi\left(B^{-1}\{0\}\right). \tag{4}$$

Proof. From (3), there exists $z_0 \in B^{-1}\{0\} \cap \sigma_\Phi(\gamma)$. Then $\inf \Phi\left(B^{-1}\{0\}\right) \le \Phi z_0 < \gamma$, and so

$$\inf \Phi\left(B^{-1}\{0\}\right) < \gamma < \infty. \tag{5}$$

Let $x^* \in X^*$ and $z \in B^{-1}\{0\}$. Then

$$\Phi^*(x^*B) \ge x^*B(z) - \Phi z = x^*(Bz) - \Phi z = 0 - \Phi z = -\Phi z,$$

and so $\Phi^*(x^*B) \ge \sup\left[-\Phi\left(B^{-1}\{0\}\right)\right] = -\inf \Phi\left(B^{-1}\{0\}\right)$. So what we must prove for (4) is that

$$\text{there exists } x^* \in X^* \text{ such that } \Phi^*(x^*B) \le -\inf \Phi\left(B^{-1}\{0\}\right). \tag{6}$$

If $\inf \Phi\left(B^{-1}\{0\}\right) = -\infty$, the result is obvious with $x^* := 0$ so, using (5), we can and will suppose that $\inf \Phi\left(B^{-1}\{0\}\right) \in \mathbb{R}$. Define $k \in \mathcal{PC}(Z)$ by $k := \Phi - \inf \Phi\left(B^{-1}\{0\}\right)$. Since $\operatorname{dom} k = \operatorname{dom}\Phi$, (2) implies that $Y = \bigcup_{\lambda>0} \lambda B(\operatorname{dom} k)$. Let $\eta := \gamma - \inf \Phi\left(B^{-1}\{0\}\right)$. From (5) and (3), $\eta > 0$ and there exists a continuous seminorm S on X such that

$$\{y \in Y\colon\ Sy < 1\} \subset B\left(\sigma_k(\eta)\right). \tag{7}$$

From the definition of k,

$$z \in B^{-1}\{0\} \quad \Longrightarrow \quad k(z) \ge 0. \tag{8}$$

We now prove that

$$\eta SB \geq -k \ \text{ on } Z. \tag{9}$$

To this end, first let $z \in \operatorname{dom} k$. Let $\mu > S(Bz) \geq 0$. Then $-Bz/\mu \in Y$ and $S(-Bz/\mu) < 1$, and so (7) provides $\zeta \in \sigma_k(\eta)$ such that $-Bz/\mu = B\zeta$, from which $B\left((\mu\zeta + z)/(\mu+1)\right) = 0$. Thus, using (8) and the convexity of k,

$$0 \leq k\left((\mu\zeta + z)/(\mu+1)\right) \leq \left(\mu k(\zeta) + k(z)\right)/(\mu+1) < \left(\mu\eta + k(z)\right)/(\mu+1).$$

Letting $\mu \to S(Bz)$, we see that $0 \leq \eta S(Bz) + k(z) = (\eta SB + k)(z)$. Since this is trivially true if $z \in Z \setminus \operatorname{dom} k$, we have established (9). From Theorem 2.4, there exists $x' \in X'$ such that $x' \leq \eta S$ on X and $x'B \geq -k$ on Z. Now any linear functional dominated by ηS is continuous and so, writing $x^* = -x'$, $x^*B - k \leq 0$ on Z, that is to say, $x^*B - \Phi \leq -\inf \Phi\left(B^{-1}\{0\}\right)$ on Z. (6) follows easily from this. □

Remark 2.8. In this remark, we compare Theorem 2.7 with the *fundamental duality formula* of Zălinescu, [16, Theorem 2.7.1(i), pp. 113–114]. Let W and X be locally convex spaces, $\Phi \in \mathcal{PC}(W \times X)$, $\pi_X\colon\ W \times X \to X$ be defined by $\pi_X(w,x) := x$ and $\pi_X(\operatorname{dom}\Phi) \ni 0$. Let Y be the linear span of $\pi_X(\operatorname{dom}\Phi)$, and suppose that there exists $\gamma \in \mathbb{R}$ such that $0 \in \operatorname{int}_Y \pi_X\left(\sigma_\Phi(\gamma)\right)$. Then it is easily seen that the conditions of Theorem 2.7 are satisfied with $Z := W \times X$ and $B := \pi_X$. Now, $\pi_X^{-1}\{0\} = W \times \{0\}$ and, for all $x^* \in X^*$, $x^*\pi_X = (0, x^*) \in Z^*$. Thus Theorem 2.7 implies that $\min_{x^* \in X^*} \Phi^*(0,x^*) = -\inf \Phi\left(W \times \{0\}\right)$, which is exactly the conclusion of [16, Theorem 2.7.1(i)]. We now consider the reverse question of deducing Theorem 2.7 from [16, Theorem 2.7.1(i)]. Suppose first that W and X are locally convex spaces, $B \in \operatorname{aff}(W,X)$, $\Phi \in \mathcal{PC}(W)$, and (2) and (3) are satisfied. Define $\Psi \in \mathcal{PC}(W \times X)$ by

$$\Psi(w,x) = \begin{cases} \Phi(w) & (x = Bw); \\ \infty & (x \neq Bw). \end{cases}$$

Then $\pi_X(\operatorname{dom}\Psi) = B(\operatorname{dom}\Phi)$ and, if $\gamma \in \mathbb{R}$, $\pi_X\left(\sigma_\Psi(\gamma)\right) = B\left(\sigma_\Phi(\gamma)\right)$. But then, for all $x^* \in X^*$, $\Psi^*(0,x^*) = \Phi^*(x^*B)$ and $W \times \{0\} = B^{-1}\{0\}$, and so (4) follows from [16, Theorem 2.7.1(i)]. This establishes Theorem 2.7 in the special case when Z is a locally convex space. The general case when Z is a CC space can be deduced from the special case by a series of translations and extensions and using the finest locally convex topology.

Corollary 2.9 is our first **trivariate existence theorem**, in which the function h is defined as the infimum of Ψ over a variable set. Corollary 2.9 will be applied in Theorems 3.1 and 5.1.

Corollary 2.9. *Let Z and P be CC spaces, X be a locally convex space, $B \in \operatorname{aff}(Z,X)$, $A \in \operatorname{aff}(Z,P)$ and $\Psi \in \mathcal{PC}(Z)$. For all $p \in P$, let*

$$h(p) := \inf \Psi\left(A^{-1}\{p\} \cap B^{-1}\{0\}\right) > -\infty$$

and

$$Y := \bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Psi) \quad \textit{be a linear subspace of } X. \tag{10}$$

Let $p^\flat \in P^\flat$ *and* $\Phi := \Psi - p^\flat A \in \mathcal{PC}(Z)$*, and suppose that there exists* $\gamma \in \mathbb{R}$ *such that*

$$0 \in \operatorname{int}_Y B\left(\sigma_\Phi(\gamma)\right). \tag{11}$$

Then

$$h^*(p^\flat) = \min\nolimits_{x^*\in X^*} \Psi^*(p^\flat A + x^* B).$$

Proof. Clearly, $\operatorname{dom}\Phi = \operatorname{dom}\Psi$, and so (2) follows from (10). Of course, (3) is identical with (11). The result now follows from Theorem 2.7 since

$$\begin{aligned} h^*(p^\flat) &= \sup\left\{p^\flat(p) - \Psi z\colon\ p \in P,\ z \in A^{-1}\{p\} \cap B^{-1}\{0\}\right\} \\ &= \sup\left\{p^\flat A z - \Psi z\colon\ p \in P,\ z \in A^{-1}\{p\} \cap B^{-1}\{0\}\right\} \\ &= \sup\left\{p^\flat A z - \Psi z\colon\ z \in B^{-1}\{0\}\right\} \\ &= \sup\left[-\Phi\left(B^{-1}\{0\}\right)\right] = -\inf \Phi\left(B^{-1}\{0\}\right) \end{aligned}$$

and, for all $x^* \in X^*$,

$$\begin{aligned} \Phi^*(x^* B) &= \sup\nolimits_{z\in Z}\left[x^* B z - \Phi z\right] \\ &= \sup\nolimits_{z\in Z}\left[p^\flat A z + x^* B z - \Psi z\right] = \Psi^*(p^\flat A + x^* B). \end{aligned}$$

□

3. Results with a boundedness hypothesis

Theorem 3.1 is our second **trivariate existence theorem**, which should be compared with Theorem 5.1. There is an important difference between Corollary 2.9 and Theorem 3.1. In Corollary 2.9, the choice of the bound γ will normally depend on $p^\flat$, while in Theorem 3.1 the choice of the bound δ can be made independently of $p^\flat$. Theorem 3.1 will be used in Corollary 3.2 and Theorem 3.4.

Theorem 3.1. *Let* Z *and* P *be CC spaces,* X *be a locally convex space,* $B \in \operatorname{aff}(Z, X)$*,* $A \in \operatorname{aff}(Z, P)$*,* $\Psi \in \mathcal{PC}(Z)$ *and, for all* $p \in P$*,*

$$h(p) := \inf \Psi\left(A^{-1}\{p\} \cap B^{-1}\{0\}\right) > -\infty. \tag{12}$$

Suppose that there exist $z_0 \in Z$ *and* $\delta \in \mathbb{R}$ *such that*

$$0 \in \operatorname{int}_X B\left(A^{-1}\{Az_0\} \cap \sigma_\Psi(\delta)\right). \tag{13}$$

Then

$$p^\flat \in P^\flat \quad\Longrightarrow\quad h^*(p^\flat) = \min\nolimits_{x^*\in X^*} \Psi^*(p^\flat A + x^* B). \tag{14}$$

Proof. Let $x \in X$. If λ is sufficiently large then $x/\lambda \in B(\operatorname{dom}\Psi)$, from which $x \in \lambda B(\operatorname{dom}\Psi)$. Thus $\bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Psi) = X$, and (10) is satisfied. Furthermore, if $p^\flat \in P^\flat$, $\Phi := \Psi - p^\flat A \in \mathcal{PC}(Z)$, and $z \in A^{-1}\{Az_0\} \cap \sigma_\Psi(\delta)$ then, writing $\gamma := \delta - p^\flat A z_0$,

$$\Phi z = \Psi z - p^\flat A z = \Psi z - p^\flat A z_0 < \delta - p^\flat A z_0 = \gamma.$$

and so $A^{-1}\{Az_0\} \cap \sigma_\Psi(\delta) \subset \sigma_\Phi(\gamma)$. Consequently, (13) gives (11), and the result follows from Corollary 2.9. □

In our first result on **affine Fenchel duality**, Corollary 3.2, which should be compared with Corollary 5.2, we show how Theorem 3.1 leads to a result on the conjugate of a generalized sum of convex functions. These results can also be deduced from the more general results that follow from Theorem 3.4 – we have included them here because they provide a model for the somewhat more complex proof of Theorem 3.4. Corollary 3.2 generalizes the classical result of Rockafellar [5, Theorem 1]. In what follows, the product of CC spaces is understood to have the pointwise definition of the convex operation.

Corollary 3.2. *Let P be a CC space, X be a locally convex space, $C \in \mathrm{aff}(P,X)$, $f \in \mathcal{PC}(P)$ and $g \in \mathcal{PC}(X)$. Suppose that there exists $p_0 \in \mathrm{dom}\, f$ such that g is finitely bounded above in a neighborhood of Cp_0. Then*

$$p^\flat \in P^\flat \quad \Longrightarrow \quad (f+gC)^*(p^\flat) = \min\nolimits_{x^* \in X^*} \left[f^*(p^\flat - x^*C) + g^*(x^*)\right]. \tag{15}$$

Proof. Let $Z := P \times X$, and define $B \in \mathrm{aff}(Z,X)$ by $B(p,x) := x - Cp$, $A \in \mathrm{aff}(Z,P)$ by $A(p,x) := p$, and $\Psi \in \mathcal{PC}(Z)$ by $\Psi(p,x) := f(p) + g(x)$. If $p \in P$ then it is easy to see that $\Psi\left(A^{-1}\{p\} \cap B^{-1}\{0\}\right)$ is the singleton $\{(f+gC)(p)\}$ thus, in the notation of (12), $h = f + gC$, hence $h^* = (f+gC)^*$. Now let $p^\flat \in P^\flat$: then $p^\flat A + x^*B = (p^\flat - x^*C, x^*) \in Z^\flat$. By direct computation, for all $q^\flat \in P^\flat$, $\Psi^*(q^\flat, x^*) = f^*(q^\flat) + g^*(x^*)$, so the formula for $(f+gC)^*$ given in (15) reduces to the formula for h^* given in (14). Let $z_0 = (p_0, Cp_0) \in Z$. By hypothesis, there exists $\gamma \in \mathbb{R}$ such that if $y \in X$ is sufficiently small then $g(Cp_0 + y) < \gamma$, from which $\Psi(p_0, Cp_0 + y) = f(p_0) + g(Cp_0 + y) < f(p_0) + \gamma$. Since $(p_0, Cp_0 + y) \in A^{-1}\{Az_0\}$ and $B(p_0, Cp_0 + y) = y$, (13) is satisfied with $\delta := f(p_0) + \gamma$, and the result follows from Theorem 3.1. □

Remark 3.3. Let X be a vector space and $\mathcal{T}$ be the finest locally convex topology on X. Then every sublinear functional on X is finitely bounded above in a neighborhood of every element of X. Thus we can apply Corollary 3.2 with P, C, f and g replaced by Z, B, k and S (respectively), and $p^\flat := 0$, and obtain: *Let $S\colon\ X \to \mathbb{R}$ be sublinear, Z be a CC space, $B \in \mathrm{aff}(Z,X)$ and $k \in \mathcal{PC}(Z)$. Then* $(k + SB)^*(0) = \min_{x^* \in (X,\mathcal{T})^*} \left[k^*(-x^*B) + S^*(x^*)\right]$.

Suppose now that $SB \geq -k$ on Z. Then $(k + SB)^*(0) \geq 0$, and so there exists $x' \in X'$ such that $k^*(-x'B) + S^*(x') \leq 0$. In particular, $S^*(x') < \infty$, and since $x' - S$ is positively homogeneous, it follows that $x' \leq S$ on X and $S^*(x') = 0$. Thus $k^*(-x'B) \leq 0$, from which $x'B \geq -k$ on Z. Thus, in this somewhat indirect fashion, we obtain Theorem 2.4.

We now come to Theorem 3.4, our first **quadrivariate existence theorem**, which should be compared with Theorem 5.3. We have taken V to be a vector space because of the expression "$v - Du$" that appears in (16). The remaining results in this section all follow easily from Theorem 3.4.

Theorem 3.4. *Let U and W be CC spaces, V be a vector space, X be a locally convex space, $C \in \mathrm{aff}(W,X)$, $D \in \mathrm{aff}(U,V)$, $\Psi \in \mathcal{PC}(U \times V \times W \times X)$ and, for* *all* $(w,v) \in W \times V$,

$$h(w,v) := \inf_{u\in U} \Psi(u, v - Du, w, Cw) > -\infty. \tag{16}$$

Suppose that there exists $(u_0, v_0, w_0) \in U\times V\times W$ *such that* $\Psi(u_0, v_0, w_0, \cdot)$ *is finitely bounded above in a neighborhood of* Cw_0. *Then* $\big($*with* $(w^\flat, v^\flat)(w,v) := w^\flat w + v^\flat v\big)$,

$$(w^\flat, v^\flat) \in W^\flat \times V^\flat \quad\Longrightarrow\quad h^*(w^\flat, v^\flat) = \min_{x^*\in X^*} \Psi^*(v^\flat D, v^\flat, w^\flat - x^*C, x^*). \tag{17}$$

Proof. Let $Z := U \times V \times W \times X$ and $P := W \times V$, and define $B \in \mathrm{aff}(Z, X)$ by $B(u,v,w,x) := x - Cw$ and $A \in \mathrm{aff}(Z, P)$ by $A(u,v,w,x) := (w, v + Du)$. If now $(w,v) \in W\times V$ then we have $A^{-1}\{(w,v)\}\cap B^{-1}\{0\} = \{(u, v - Du, w, Cw)\colon\ u \in U\}$, thus the definition of h given in (16) reduces to the definition of h given in (12). Now let $(w^\flat, v^\flat) \in W^\flat \times V^\flat$: then $(w^\flat, v^\flat)A + x^*B = (v^\flat D, v^\flat, w^\flat - x^*C, x^*) \in U^\flat \times V^\flat \times W^\flat \times X^*$, and so the formula for h^* given in (17) reduces to the formula for h^* given in (14). Let $z_0 = (u_0, v_0, w_0, Cw_0) \in Z$. By hypothesis, there exists $\delta \in \mathbb{R}$ such that if $y \in X$ is sufficiently small then $\Psi(u_0, v_0, w_0, Cw_0 + y) < \delta$. Since $(u_0, v_0, w_0, Cw_0 + y) \in A^{-1}\{Az_0\}$ and $B(u_0, v_0, w_0, Cw_0 + y) = y$, (13) is satisfied, and the result follows from Theorem 3.1. □

Corollary 3.5, our first **bibivariate existence theorem**, should be compared with Corollary 5.4.

Corollary 3.5. *Let* U *and* W *be CC spaces,* V *be a vector space,* X *be a locally convex space,* $C \in \mathrm{aff}(W, X)$, $D \in \mathrm{aff}(U, V)$, $f \in \mathcal{PC}(W \times V)$, $g \in \mathcal{PC}(X \times U)$ *and, for all* $(w,v) \in W \times V$,

$$h(w,v) := \inf_{u\in U} [f(w, v - Du) + g(Cw, u)] > -\infty. \tag{18}$$

Define $\pi_W\colon\ W \times V \to W$ *by* $\pi_W(w,v) := w$ *and suppose that there exist* $w_0 \in \pi_W \mathrm{dom}\, f$ *and* $u_0 \in U$ *such that* $g(\cdot, u_0)$ *is finitely bounded above in a neighborhood of* Cw_0. *Then*

$$\left.\begin{aligned}&(w^\flat, v^\flat) \in W^\flat \times V^\flat \quad\Longrightarrow\\ &\qquad h^*(w^\flat, v^\flat) = \min_{x^*\in X^*} \left[f^*(w^\flat - x^*C, v^\flat) + g^*(x^*, v^\flat D)\right].\end{aligned}\right\} \tag{19}$$

Proof. Let $\Psi(u,v,w,x) := f(w,v) + g(x,u)$. Then the function h as defined in (18) reduces to the function h as defined in (16). Since $\Psi^*(u^\flat, v^\flat, w^\flat, x^*) = f^*(w^\flat, v^\flat) + g^*(x^*, u^\flat)$, the formula for h^* given in (19) reduces to the formula for h^* given in (17). The result is now immediate from Theorem 3.4. □

Corollary 3.6, our first result on **partial inf–convolutions**, which should be compared with Corollary 5.5, follows from Corollary 3.5 by taking $U = V$, $W = X$, and C and D identity maps.

Corollary 3.6. *Let* V *be a vector space,* X *be a locally convex space,* $f, g \in \mathcal{PC}(X \times V)$ *and, for all* $(x,v) \in X \times V$,

$$h(x,v) := \inf_{u\in V} [f(x, v - u) + g(x,u)] > -\infty.$$

Define $\pi_X\colon\ X \times V \to X$ by $\pi_X(x,v) := x$ and suppose that there exist $x_0 \in \pi_X \mathrm{dom}\, f$ and $v_0 \in V$ such that $g(\cdot, v_0)$ is finitely bounded above in a neighborhood of x_0. Then

$$(x^\flat, v^\flat) \in X^\flat \times V^\flat \quad\Longrightarrow\quad h^*(x^\flat, v^\flat) = \min_{x^* \in X^*} \left[f^*(x^\flat - x^*, v^\flat) + g^*(x^*, v^\flat)\right].$$

Corollaries 3.7 and 3.8 should be compared with Corollaries 5.8 and 5.9. The function f defined in (23) is an *indicator function*. Corollaries 3.7, 3.8, 5.8 and 5.9 are the only results in this paper that use indicator functions. The statements of Corollaries 3.7 and 3.8 are also interesting in that they do not contain "+" or "−".

Corollary 3.7. *Let U and W be CC spaces, V be a vector space, X be a locally convex space, $C \in \mathrm{aff}(W, X)$, $D \in \mathrm{aff}(U, V)$, $g \in \mathcal{PC}(X \times U)$ and, for all $(w, v) \in W \times V$,*

$$h(w, v) := \inf\{g(Cw, u)\colon\ u \in U,\ Du = v\} > -\infty. \tag{20}$$

Suppose that

$$y^\flat \in W^\flat \text{ and } \sup y^\flat(W) < \infty \quad\Longrightarrow\quad y^\flat = 0, \tag{21}$$

and there exists $(w_0, u_0) \in W \times U$ such that $g(\cdot, u_0)$ is finitely bounded above in a neighborhood of Cw_0. Then

$$\left.\begin{array}{l}(w^\flat, v') \in W^\flat \times V' \text{ and } h^*(w^\flat, v') < \infty \quad\Longrightarrow\\ \qquad h^*(w^\flat, v') = \min\left\{g^*\left(x^*, v'D\right):\ x^* \in X^*,\ x^*C = w^\flat\right\}.\end{array}\right\} \tag{22}$$

Proof. Define $f \in \mathcal{PC}(W \times V)$ by

$$f(w, v) = \begin{cases} 0 & (v = 0); \\ \infty & (v \neq 0). \end{cases} \tag{23}$$

The definition of h given in (20) clearly reduces to that given in (18). Let $(w^\flat, v') \in W^\flat \times V'$ and $h^*(w^\flat, v') < \infty$. Then Corollary 3.5 provides us with $x^* \in X^*$ such that $h^*(w^\flat, v') = f^*(w^\flat - x^*C, v') + g^*(x^*, v'D)$, and so $f^*(w^\flat - x^*C, v') < \infty$. Since $v'0 = 0$, for all $y^\flat \in W^\flat$, $f^*(y^\flat, v') = \sup_{(w,v) \in W \times V}\left[y^\flat w + v'v - f(w, v)\right] = \sup y^\flat(W)$. Thus (21) implies that $w^\flat = x^*C$ and, since $\pi_W \mathrm{dom}\, f = \pi_W\left(W \times \{0\}\right) = W$, the result follows from Corollary 3.5. □

Corollary 3.8. *Let U be a CC space, W and V be vector spaces, X be a locally convex space, $C\colon\ W \to X$ be linear, $D \in \mathrm{aff}(U, V)$, $g \in \mathcal{PC}(X \times U)$ and, for all $(w, v) \in W \times V$,*

$$h(w, v) := \inf\{g(Cw, u)\colon\ u \in U,\ Du = v\} > -\infty.$$

Suppose that there exists $(w_0, u_0) \in W \times U$ such that $g(\cdot, u_0)$ is finitely bounded above in a neighborhood of Cw_0. Then

$$\begin{array}{l}(w', v') \in W' \times V' \text{ and } h^*(w', v') < \infty \quad\Longrightarrow\\ \qquad h^*(w', v') = \min\left\{g^*\left(x^*, v'D\right):\ x^* \in X^*,\ x^*C = w'\right\}.\end{array}$$

Proof. This is immediate from Corollary 3.7, since $w' \in W' \Longrightarrow w' - x^*C \in W'$, and (21) is true if W is a vector space and $y^\flat \in W'$. $\square$

4. (F)–normed topological vector spaces

If M is a vector space, we say that a function $|\cdot|\colon\ M \to [0,\infty)$ is an *(F)–norm* if $|x| = 0 \iff x = 0$, for all $x, y \in M$, $|x+y| \le |x|+|y|$, and, for all $\lambda \in [-1,1]$ and $x \in X$, $|\lambda x| \le |x|$. If (M,d) is a metrizable topological vector space then it is well known that there exists an (F)–norm $|\cdot|$ on M such that the topology induced by d on M is identical to that induced on M by the metric $(x,y) \mapsto |x-y|$. (See [4, p. 163].) We will say that $(M, |\cdot|)$ is an *(F)–normed topological vector space* if M is a topological vector space, $|\cdot|$ is an (F)–norm on M, and the topology of M is identical to that induced on M by the metric $(x,y) \mapsto |x-y|$. We caution the reader that the sequences in M that are d–Cauchy are not necessarily the same as those that are $|\cdot|$–Cauchy. We say that $(Z, |\cdot|)$ is an *(F)–normed CC space* if there exists an (F)–normed topological vector space $(\widehat{Z}, \widehat{|\cdot|})$ such that Z is a nonempty convex subset of $\widehat{Z}$, and $|\cdot|$ is the restriction of $\widehat{|\cdot|}$ to Z.

We will need the following result, which seems to be more delicate than the corresponding result in the normed case. It is not true, even in the simplest case, that the map $(\lambda, z) \mapsto \lambda z$ is uniformly continuous: for all $\delta > 0$, $(\lambda+\delta)(\lambda+\delta) - \lambda\lambda \to \infty$ as $\lambda \to \infty$.

Lemma 4.1. *Let $(M, |\cdot|)$ be an (F)–normed topological vector space, $\{\alpha_n\}_{n\ge1}$ be a Cauchy sequence in $\mathbb{R}$, and $\{z_n\}_{n\ge1}$ be a Cauchy sequence in $(M, |\cdot|)$. Then $\{\alpha_n z_n\}_{n\ge1}$ is Cauchy in $(M, |\cdot|)$.*

Proof. Let $\alpha_0 = \lim_{n\to\infty} \alpha_n$. Let $\varepsilon > 0$. Since the map $(\lambda, z) \mapsto \lambda z$ is continuous at $(\alpha_0, 0)$, there exists $\delta > 0$ such that

$$|\lambda - \alpha_0| < \delta \text{ and } |z| < \delta \quad\Longrightarrow\quad |\lambda z| = |\lambda z - \alpha_0 0| < \varepsilon/3.$$

Since $\alpha_n \to \alpha_0$ in $\mathbb{R}$ and $\{z_n\}_{n\ge1}$ is Cauchy in $(M, |\cdot|)$, there exists $n_0 \ge 1$ such that

$$n \ge n_0 \quad\Longrightarrow\quad |\alpha_n - \alpha_0| < \delta \text{ and } |z_n - z_{n_0}| < \delta.$$

Since the map $\lambda \mapsto \lambda z_{n_0}$ is continuous at 0, there exists $n_1 \ge n_0$ such that

$$n \ge n_1 \quad\Longrightarrow\quad |(\alpha_n - \alpha_{n_1}) z_{n_0}| < \varepsilon/3.$$

Now let $n \ge n_1$. Then $|\alpha_n - \alpha_0| < \delta$, $|z_n - z_{n_0}| < \delta$, $|\alpha_{n_1} - \alpha_0| < \delta$ and $|z_{n_1} - z_{n_0}| < \delta$. Thus

$$\begin{aligned} |\alpha_n z_n - \alpha_{n_1} z_{n_1}| &= |\alpha_n(z_n - z_{n_0}) - \alpha_{n_1}(z_{n_1} - z_{n_0}) + (\alpha_n - \alpha_{n_1}) z_{n_0}| \\ &\le |\alpha_n(z_n - z_{n_0})| + |\alpha_{n_1}(z_{n_1} - z_{n_0})| + |(\alpha_n - \alpha_{n_1}) z_{n_0}| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

This completes the proof of Lemma 4.1. $\square$

Lemma 4.2. *Let $(Z, |\cdot|)$ be an (F)–normed CC space, X be an (F)–normed topological vector space, $B \in \operatorname{aff}(Z,X)$, $\Phi \in \mathcal{PC}(Z)$, $Y := \bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Phi)$ be a complete linear subspace of X, $\delta \in \mathbb{R}$ and $\delta > \inf \Phi\left(B^{-1}\{0\}\right)$. Then:*

(a) $Y = \bigcup_{i\geq 1} iB\left(\sigma_\Phi(\delta)\right)$.

(b) *If $q, k \geq 1$, let $R(q,k) := \{z \in \sigma_\Phi(\delta)\colon\ |(1/k)z| < 2^{-q}\}$. Let $q \geq 1$. Then there exists $k \geq 1$ such that $0 \in \operatorname{int}_Y \overline{B\left(R(q,k)\right)}$.*

(c) $0 \in \operatorname{int}_Y \overline{B\left(\sigma_\Phi(\delta)\right)}$.

Proof. (a) We can fix $z_0 \in \sigma_\Phi(\delta)$ so that $Bz_0 = 0$. If $y \in Y$, there exist $\lambda > 0$ and $\zeta \in \operatorname{dom}\Phi$ such that $y = \lambda B\zeta$. If $i \geq 1$ and $i > \lambda$ then, since $\Phi z_0 < \delta$, $\Phi\zeta \in \mathbb{R}$ and $\Phi\left((1-\lambda/i)z_0 + (\lambda/i)\zeta\right) \leq (1-\lambda/i)\Phi z_0 + (\lambda/i)\Phi\zeta$, we can choose i so large that $\Phi\left((1-\lambda/i)z_0 + (\lambda/i)\zeta\right) < \delta$, and so $(1-\lambda/i)z_0 + (\lambda/i)\zeta \in \sigma_\Phi(\delta)$. Then

$$y = \lambda B\zeta = iB\left((1-\lambda/i)z_0 + (\lambda/i)\zeta\right) \in iB\left(\sigma_\Phi(\delta)\right).$$

Since $B\left(\sigma_\Phi(\delta)\right) \subset B(\operatorname{dom}\Phi) \subset Y$, this completes the proof of (a).

(b) For all $z \in \sigma_\Phi(\delta)$, $|(1/k)z| \to 0$ as $k \to \infty$. Thus it follows from (a) that $Y = \bigcup_{i,m\geq 1} iB\left(R(q+1,m)\right)$, and so Baire's theorem provides us with $i, m \geq 1$ and $y_0 \in Y$ such that $y_0 \in \operatorname{int}_Y \overline{iB\left(R(q+1,m)\right)}$. Since $-y_0 \in Y$, there exist $j, n \geq 1$ and $z_2 \in R(q+1,n)$ such that $-y_0 = jBz_2$. Let $k := m \vee n$, $z_1 \in R(q+1,m)$, and write $z_3 := (iz_1 + jz_2)/(i+j)$. Then $im/\left((i+j)k\right) \leq 1$ and $jn/\left((i+j)k\right) \leq 1$, and so

$$\begin{aligned} |z_3/k| &= \left|\left[im/\left((i+j)k\right)\right](z_1/m) + \left[jn/\left((i+j)k\right)(z_2/n)\right]\right| \\ &\leq \left|\left[im/\left((i+j)k\right)\right](z_1/m)\right| + \left|\left[jn/\left((i+j)k\right)(z_2/n)\right]\right| \\ &\leq |z_1/m| + |z_2/n| < 2^{-q-1} + 2^{-q-1} = 2^{-q}. \end{aligned}$$

Consequently, $z_3 \in (R(q,k))$. Since $iBz_1 + jBz_2 = (i+j)Bz_3$, we have proved that $iB\left(R(q+1,m)\right) + jBz_2 \subset (i+j)B\left(R(q,k)\right)$. Thus

$$0 = y_0 - y_0 \in \operatorname{int}_Y \overline{iB\left(R(q+1,m)\right)} + jBz_2,$$

from which

$$0 \in \operatorname{int}_Y \overline{iB\left(R(q+1,m)\right) + jBz_2} \subset \operatorname{int}_Y \overline{(i+j)B\left(R(q,k)\right)}.$$

This gives (b).

(c) This is immediate from (b), since $R(q,k) \subset \sigma_\Phi(\delta)$. □

In the sequel, we write $\operatorname{caff}(Z,X)$ for $\{B \in \operatorname{aff}(Z,X)\colon\ B \text{ is continuous}\}$ and $\mathcal{PCLSC}(Z)$ for $\{f \in \mathcal{PC}(Z)\colon\ f \text{ is lower semicontinuous}\}$. Theorem 4.3 below is a considerable sharpening of the result proved in Rodrigues–Simons, [6, Lemma 1, pp. 1072–1073]. We note that we can make the constant α in (27) as close as we like to 1 by increasing the rate of growth of $\{k_j\}_{j\geq 1}$.

Theorem 4.3. *Let* $(Z, |\cdot|)$ *be a complete* (F)*–normed CC space,* X *be an* (F)*–normed topological vector space,* $B \in \operatorname{caff}(Z, X)$*,* $\Phi \in \mathcal{PCLSC}(Z)$*,* $Y := \bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Phi)$ *be a complete linear subspace of* X*, and* $\gamma \in \mathbb{R}$*. Then the conditions* (24)–(26) *are equivalent:*

$$0 \in \operatorname{int}_Y \overline{B(\sigma_\Phi(\gamma))} \tag{24}$$

$$0 \in B(\sigma_\Phi(\gamma)). \tag{25}$$

$$\gamma > \inf \Phi\left(B^{-1}\{0\}\right). \tag{26}$$

Proof. It is immediate that (24)$\Longrightarrow$(25). If (25) is true then there exists $z \in \sigma_\Phi(\gamma)$ such that $Bz = 0$, and so (26) is true.

Suppose, finally, that (26) is true. Choose $\delta \in \mathbb{R}$ so that $\inf \Phi(B^{-1}\{0\}) < \delta < \gamma$. Let $k_1 = 1$. From Lemma 4.2(b), for all $q \geq 2$, there exists $k_q \geq 2^{q-1}$ such that $0 \in \operatorname{int}_Y \overline{B(R(q, k_q))}$. For all $q \geq 1$, let $\eta_q := 1/k_q$. Write $\alpha := 1/\sum_{q=1}^{\infty} \eta_q \in]0, 1[$. We will prove that

$$\alpha\overline{B(\sigma_\Phi(\delta))} \subset B(\sigma_\Phi(\gamma)). \tag{27}$$

To this end, let $y \in \overline{B(\sigma_\Phi(\delta))} = \overline{\eta_1 B(\sigma_\Phi(\delta))}$. Let $\{V_q\}_{q\geq 1}$ be a base for the neighborhoods of 0 in X. Then there exists $z_1 \in \sigma_\Phi(\delta)$ such that $y - \eta_1 B z_1 \in \overline{\eta_2 B(R(2, k_2))} \cap V_1$. Similarly, there exists $z_2 \in R(2, k_2)$ such that $y - \eta_1 B z_1 - \eta_2 B z_2 \in \overline{\eta_3 B(R(3, k_3))} \cap V_2$. Continuing this process inductively, we end up with a sequence $\{z_q\}_{q\geq 1}$ such that,

$$\left.\begin{array}{r} \text{for all } q \geq 2, \quad z_q \in R(q, k_q) \subset \sigma_\Phi(\delta) \quad \text{and,} \\ \text{for all } m \geq 1, \quad y - \sum_{i=1}^{m} \eta_i B z_i \in V_m. \end{array}\right\} \tag{28}$$

In what follows, let $(\widehat{Z}, \widehat{|\cdot|})$ be an (F)–normed topological vector space such that $Z \subset \widehat{Z}$ and $|\cdot|$ is the restriction of $\widehat{|\cdot|}$ to Z. For all $n \geq 1$, let $s_n := \sum_{q=1}^{n} \eta_q z_q \in \widehat{Z}$ and $\alpha_n := 1/\sum_{q=1}^{n} \eta_q \in]0, 1[$. For all $q \geq 1$, $z_q \in Z$ and $\Phi(z_q) < \delta$ and so, for all $n \geq 1$,

$$\alpha_n s_n \in Z \quad \text{and} \quad \Phi(\alpha_n s_n) < \delta. \tag{29}$$

Let $1 \leq m < n$. Then (since $z_q \in R(q, k_q)$) $|s_n - s_m| = |\sum_{q=m+1}^{n} \eta_q z_q| \leq \sum_{q=m+1}^{n} |\eta_q z_q| = \sum_{q=m+1}^{n} |z_q/k_q| \leq \sum_{q=m+1}^{n} 2^{-q} < 2^{-m}$. Thus the sequence $\{s_n\}_{n\geq 1}$ is Cauchy in $\widehat{Z}$. Since the sequence $\{\alpha_n\}_{n\geq 1}$ is convergent in $\mathbb{R}$, Lemma 4.1 implies that the sequence $\{\alpha_n s_n\}_{n\geq 1}$ is Cauchy in Z, and so the completeness of Z implies that $\lim_{n\to\infty} \alpha_n s_n$ exists in Z. Write z_0 for this limit. Passing to the limit in (29) and using the lower semicontinuity of Φ, $\Phi z_0 \leq \delta$, from which $z_0 \in \sigma_\Phi(\gamma)$. Since B is continuous on Z, it follows from (28) and the observation that $\sum_{q=1}^{n} \alpha_n \eta_q = 1$ that

$$\begin{aligned} \alpha y &= (\lim_{n\to\infty} \alpha_n)\left(\lim_{n\to\infty} \sum_{q=1}^{n} \eta_q B z_q\right) = \lim_{n\to\infty}\left(\alpha_n \sum_{q=1}^{n} \eta_q B z_q\right) \\ &= \lim_{n\to\infty}\left(\sum_{q=1}^{n} \alpha_n \eta_q B z_q\right) = \lim_{n\to\infty} B\left(\sum_{q=1}^{n} \alpha_n \eta_q z_q\right) \\ &= \lim_{n\to\infty} B(\alpha_n s_n) = B z_0. \end{aligned}$$

Consequently, $\alpha y \in B\left(\sigma_\Phi(\gamma)\right)$, which gives (27). From Lemma 4.2(*c*), $0 \in \operatorname{int}_Y \overline{B\left(\sigma_\Phi(\delta)\right)}$, and (24) follows from (27). □

The final result in this section is about **automatic interiority**.

Corollary 4.4. *Let $(Z, |\cdot|)$ be a complete (F)–normed CC space, X be an (F)–normed topological vector space, $B \in \operatorname{caff}(Z, X)$, $\Phi \in \mathcal{PCLSC}(Z)$ and $Y := \bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Phi)$ be a complete linear subspace of X. Then there exists $\gamma \in \mathbb{R}$ such that $0 \in \operatorname{int}_Y B\left(\sigma_\Phi(\gamma)\right)$, that is to say, (11) is satisfied.*

Proof. Fix $z_0 \in \operatorname{dom}\Phi$ such that $Bz_0 = 0$, and let $\gamma > \Phi z_0$. Then $0 \in B\left(\sigma_\Phi(\gamma)\right)$, and the result follows from Theorem 4.3((25)$\Longrightarrow$(24)). □

5. Results that use completeness

Theorem 5.1 is our third **trivariate existence theorem**, which should be compared with Theorem 3.1. We write $Z^\sharp := \{z^\flat \in Z^\flat \colon\ z^\flat \text{ is continuous}\} = \operatorname{caff}(Z, \mathbb{R})$. We note that a *Fréchet space* is a complete (F)–normed *locally convex* topological vector space. In this connection, Banach's definition of space of *type (F)* (see [1, p. 35]) does not require either local convexity, or the continuity of the map $(\lambda, x) \mapsto \lambda x$.

Theorem 5.1. *Let Z be a complete (F)–normed CC space, P be a CC space, X be a Fréchet space, $B \in \operatorname{caff}(Z, X)$, $A \in \operatorname{aff}(Z, P)$, $\Psi \in \mathcal{PCLSC}(Z)$, $\bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Psi)$ be a closed linear subspace of X and, for all $p \in P$,*

$$h(p) := \inf \Psi\left(A^{-1}\{p\} \cap B^{-1}\{0\}\right) > -\infty. \tag{30}$$

Then

$$p^\flat \in P^\flat \text{ and } p^\flat A \in Z^\sharp \quad\Longrightarrow\quad h^*(p^\flat) = \min\nolimits_{x^* \in X^*} \Psi^*(p^\flat A + x^* B). \tag{31}$$

Proof. Let $p^\flat \in P^\flat$ and $p^\flat A \in Z^\sharp$. Let $\Phi := \Psi - p^\flat A \in \mathcal{PCLSC}(Z)$. Since $\operatorname{dom}\Phi = \operatorname{dom}\Psi$, $\bigcup_{\lambda>0} \lambda B(\operatorname{dom}\Phi)$ is a closed subspace of a Fréchet space, and thus complete. The result now follows from Corollaries 4.4 and 2.9. □

In Corollary 5.2 below, which should be compared with Corollary 3.2, we show how Theorem 5.1 leads to a new result on the conjugate of a generalized sum of convex functions. Corollary 5.2 is a generalization of the generalization of the Robinson–Attouch–Brezis theorem to Fréchet spaces that first appeared in Rodrigues–Simons, [6, Theorem 6, p. 1076]. A result similar to the latter, with slightly more restrictive hypotheses, had been established previously by Zălinescu in [15, Corollary 4, p. A91]. We also refer the reader to Zălinescu, [14, Corollary 2.2, p. 22 and Theorem 4.3, p. 32] for earlier results in this direction.

Corollary 5.2. *Let P be a complete (F)–normed CC space, X be a Fréchet space, $C \in \operatorname{caff}(P, X)$, $f \in \mathcal{PCLSC}(P)$, $g \in \mathcal{PCLSC}(X)$, and $\bigcup_{\lambda>0} \lambda\left[\operatorname{dom} g - C(\operatorname{dom} f)\right]$ be a closed linear subspace of X. Then*

$$p^\sharp \in P^\sharp \quad\Longrightarrow\quad (f + gC)^*(p^\sharp) = \min\nolimits_{x^* \in X^*} \left[f^*(p^\sharp - x^* C) + g^*(x^*)\right]. \tag{32}$$

Proof. Let $Z := P \times X$, and define $B \in \operatorname{caff}(Z, X)$ by $B(p, x) := x - Cp$, $A \in \operatorname{caff}(Z, P)$ by $A(p, x) := p$, and $\Psi \in \mathcal{PCLSC}(Z)$ by $\Psi(p, x) := f(p) + g(x)$. In the notation of (30), $h = f + gC$, hence $h^* = (f + gC)^*$. Now let $p^\sharp \in P^\sharp$: then $p^\sharp A + x^* B = (p^\sharp - x^* C, x^*) \in Z^\sharp$. By direct computation, for all $q^\sharp \in P^\sharp$, $\Psi^*(q^\sharp, x^*) = f^*(q^\sharp) + g^*(x^*)$, so the formula for $(f + gC)^*$ given in (32) reduces to the formula for h^* given in (31). Since $\operatorname{dom}\Psi = \operatorname{dom} f \times \operatorname{dom} g$ and $B(\operatorname{dom}\Psi) = \operatorname{dom} g - C(\operatorname{dom} f)$, the result is immediate from Theorem 5.1. $\square$

Theorem 5.3 is our second **quadrivariate existence theorem**, which should be compared with Theorem 3.4. The remaining results in this paper all follow from Theorem 5.3.

Theorem 5.3. *Let U and W be complete (F)–normed CC spaces, V be a complete (F)–normed topological vector space, X be a Fréchet space, $C \in \operatorname{caff}(W, X)$, $D \in \operatorname{caff}(U, V)$, $\Psi \in \mathcal{PCLSC}(U \times V \times W \times X)$, $\bigcup_{\lambda>0} \lambda \{x - Cw\colon\ (u, v, w, x) \in \operatorname{dom}\Psi\}$ be a closed linear subspace of X and, for all $(w, v) \in W \times V$,*

$$h(w, v) := \inf\nolimits_{u\in U} \Psi(u, v - Du, w, Cw) > -\infty. \tag{33}$$

Then

$$(w^\sharp, v^\sharp) \in W^\sharp \times V^\sharp \quad \Longrightarrow \quad h^*(w^\sharp, v^\sharp) = \min\nolimits_{x^*\in X^*} \Psi^*(v^\sharp D, v^\sharp, w^\sharp - x^* C, x^*). \tag{34}$$

Proof. Let $Z := U \times V \times W \times X$ and $P := W \times V$, and define $B \in \operatorname{caff}(Z, X)$ by $B(u, v, w, x) := x - Cw$ and $A \in \operatorname{caff}(Z, P)$ by $A(u, v, w, x) := (w, v + Du)$. If now $(w, v) \in W \times V$ then we have $A^{-1}\{(w, v)\} \cap B^{-1}\{0\} = \{(u, v - Du, w, Cw)\colon\ u \in U\}$, thus the definition of h given in (33) reduces to the definition of h given in (30). Now let $(w^\sharp, v^\sharp) \in W^\sharp \times V^\sharp$: then $(w^\sharp, v^\sharp)A + x^* B = (v^\sharp D, v^\sharp, w^\sharp - x^* C, x^*) \in U^\sharp \times V^\sharp \times W^\sharp \times X^*$, so the formula for h^* given in (34) reduces to the formula for h^* given in (31). Since $B(\operatorname{dom}\Psi) = \{x - Cw\colon\ (u, v, w, x) \in \operatorname{dom}\Psi\}$, the result follows from Theorem 5.1. $\square$

Corollary 5.4 is our second **bibivariate existence theorem**, which should be compared with Corollary 3.5. This generalizes the result for Banach spaces that first appeared in Simons, [11, Theorem 3, pp. 2–4].

Corollary 5.4. *Let U and W be complete (F)–normed CC spaces, V be a complete (F)–normed topological vector space, X be a Fréchet space, $C \in \operatorname{caff}(W, X)$, $D \in \operatorname{caff}(U, V)$, $f \in \mathcal{PCLSC}(W \times V)$, $g \in \mathcal{PCLSC}(X \times U)$ and, for all $(w, v) \in W \times V$,*

$$h(w, v) := \inf\nolimits_{u\in U} [f(w, v - Du) + g(Cw, u)] > -\infty. \tag{35}$$

Define $\pi_X\colon\ X \times U \to X$ and $\pi_W\colon\ W \times V \to W$ by $\pi_X(x, u) := x$ and $\pi_W(w, v) := w$. If $\bigcup_{\lambda>0} \lambda [\pi_X \operatorname{dom} g - C\pi_W \operatorname{dom} f]$ is a closed linear subspace of X then

$$\left.\begin{aligned}&(w^\sharp, v^\sharp) \in W^\sharp \times V^\sharp \quad \Longrightarrow \\ &\qquad h^*(w^\sharp, v^\sharp) = \min\nolimits_{x^*\in X^*} \left[f^*(w^\sharp - x^* C, v^\sharp) + g^*(x^*, v^\sharp D)\right].\end{aligned}\right\} \tag{36}$$

Proof. Let $\Psi(u, v, w, x) := f(w, v) + g(x, u)$. Then the definition of h given in (35) reduces to the definition of h given in (33). Now let $(w^\sharp, v^\sharp) \in W^\sharp \times V^\sharp$: then $\Psi^*(u^\sharp, v^\sharp, w^\sharp, x^*) = f^*(w^\sharp, v^\sharp) + g^*(x^*, u^\sharp)$, so the formula for h^* given in (36) reduces to the formula for h^* given in (34). Since $\{x - Cw\colon\ (u, v, w, x) \in \operatorname{dom}\Psi\} = \pi_X \operatorname{dom} g - C\pi_W \operatorname{dom} f$, the result now follows from Theorem 5.3. □

Corollary 5.5, our second result on **partial inf–convolutions**, which should be compared with Corollary 3.6, follows from Corollary 5.4 by taking $U = V$, $W = X$, and C and D to be identity maps. Corollary 5.5 generalizes the result for Banach spaces that first appeared in Simons–Zălinescu, [13, Theorem 4.2, pp. 9–10].

Corollary 5.5. *Let V be a complete (F)–normed topological vector space, X be a Fréchet space, $f, g \in \mathcal{PCLSC}(X \times V)$ and, for all $(x, v) \in X \times V$,*

$$h(x, v) := \inf\nolimits_{u \in U} [f(x, v - u) + g(x, u)] > -\infty.$$

Define $\pi_X\colon\ X \times V \to X$ by $\pi_X(x, v) := x$. If $\bigcup_{\lambda>0} \lambda\, [\pi_X \operatorname{dom} g - \pi_X \operatorname{dom} f]$ is a closed linear subspace of X then

$$(x^\sharp, v^\sharp) \in X^\sharp \times V^\sharp \quad\Longrightarrow \\ h^*(x^\sharp, v^\sharp) = \min\nolimits_{x^* \in X^*} \left[f^*(x^\sharp - x^*, v^\sharp) + g^*(x^*, v^\sharp)\right].$$

Corollary 5.6 is immediate from Corollary 5.4 with $U = F^*$, $V = E^*$, $W = E$ and $X = F$. Corollary 5.6 is a slight generalization of the result for Banach spaces that first appeared in [11, Theorem 5(a), pp. 4–5].

Corollary 5.6. *Let E and F be Banach spaces, $C \in \operatorname{caff}(E, F)$, $D \in \operatorname{caff}(F^*, E^*)$, $f \in \mathcal{PCLSC}(E \times E^*)$, $g \in \mathcal{PCLSC}(F \times F^*)$ and, for all $(x, x^*) \in E \times E^*$,*

$$h(x, x^*) := \inf\nolimits_{y^* \in F^*} [f(x, x^* - Dy^*) + g(Cw, y^*)] > -\infty.$$

Define $\pi_F\colon\ F \times F^* \to F$ and $\pi_E\colon\ E \times E^* \to E$ by $\pi_F(y, y^*) := y$ and $\pi_E(x, x^*) := x$. If $\bigcup_{\lambda>0} \lambda\, [\pi_F \operatorname{dom} g - C\pi_E \operatorname{dom} f]$ is a closed linear subspace of F then

$$(x^\sharp, v^\sharp) \in E^\sharp \times E^{*\sharp} \quad\Longrightarrow \\ h^*(x^\sharp, v^\sharp) = \min\nolimits_{y^* \in F^*} \left[f^*(x^\sharp - y^*C, v^\sharp) + g^*(y^*, v^\sharp D)\right].$$

Corollary 5.7 is immediate from Corollary 5.4 with $U = F$, $V = E$, $W = E^*$ and $X = F^*$, and changing the order of the arguments of f, g and h. Corollary 5.7 generalizes the result for Banach spaces that first appeared in [11, Theorem 5(b), pp. 4–5].

Corollary 5.7. *Let F and E be Banach spaces, $C \in \operatorname{caff}(E^*, F^*)$, $D \in \operatorname{caff}(F, E)$, $f \in \mathcal{PCLSC}(E \times E^*)$, $g \in \mathcal{PCLSC}(F \times F^*)$ and, for all $(x, x^*) \in E \times E^*$,*

$$h(x, x^*) := \inf\nolimits_{y \in F} [f(x - Dy, x^*) + g(y, Cx^*)] > -\infty.$$

Define $\pi_{F^*}\colon\ F\times F^*\to F^*$ *and* $\pi_{E^*}\colon\ E\times E^*\to E^*$ *by* $\pi_{F^*}(y,y^*):=y^*$ *and* $\pi_{E^*}(x,x^*):=x^*$. *If* $\bigcup_{\lambda>0}\lambda\left[\pi_{F^*}\operatorname{dom}g-C\pi_{E^*}\operatorname{dom}f\right]$ *is a closed linear subspace of* F^* *then*

$$(x^\sharp,w^\sharp)\in E^\sharp\times E^{*\sharp}\quad\Longrightarrow\quad h^*(x^\sharp,w^\sharp)=\min\nolimits_{y^{**}\in F^{**}}\left[f^*(x^\sharp,w^\sharp-y^{**}C)+g^*(x^\sharp D,y^{**})\right].$$

Corollaries 5.8 and 5.9, which should be compared with Corollaries 3.7 and 3.8, generalize the result for Banach spaces that first appeared in [11, Theorem 21, pp. 12–13]. This latter result also appeared in [12, Theorem 6], and was taken there as the starting point for proofs of a number of results that have already appeared in this paper, as well as many others which have applications to the theory of strongly representable multifunctions. We refer the reader to [12] for more details of these applications.

Corollary 5.8. *Let* U *and* W *be complete* (F)*–normed CC spaces,* V *be a complete* (F)*–normed topological vector space,* X *be a Fréchet space,* $C\in\operatorname{caff}(W,X)$, $D\in\operatorname{caff}(U,V)$, $g\in\mathcal{PCLSC}(X\times U)$ *and, for all* $(w,v)\in W\times V$,

$$h(w,v):=\inf\{g(Cw,u)\colon\ u\in U,\ Du=v\}>-\infty.$$

Suppose that

$$y^\sharp\in W^\sharp \text{ and } \sup y^\sharp(W)<\infty\quad\Longrightarrow\quad y^\sharp=0,$$

and define $\pi_X\colon\ X\times U\to X$ *by* $\pi_X(x,u):=x$. *If* $\bigcup_{\lambda>0}\lambda\left[\pi_X\operatorname{dom}g-C(W)\right]$ *is a closed linear subspace of* X *then*

$$(w^\sharp,v^*)\in W^\sharp\times V^* \text{ and } h^*(w^\sharp,v^*)<\infty\quad\Longrightarrow\quad h^*(w^\sharp,v^*)=\min\left\{g^*(x^*,v^*D)\colon\ x^*\in X^*,\ x^*C=w^\sharp\right\}.$$

Proof. We define $f\in\mathcal{PCLSC}(W\times V)$ as in (23), and $\pi_W\colon\ W\times V\to W$ by $\pi_W(w,v):=w$. Then $\pi_W\operatorname{dom}f=\pi_W(W\times\{0\})=W$. The rest of the proof now proceeds exactly as in Corollary 3.7, only using Corollary 5.4 instead of Corollary 3.5. □

Corollary 5.9. *Let* U *be a complete* (F)*–normed CC space,* W *and* V *be complete* (F)*–normed topological vector spaces,* X *be a Fréchet space,* $C\colon\ W\to X$ *be continuous and linear,* $D\in\operatorname{caff}(U,V)$, $g\in\mathcal{PCLSC}(X\times U)$ *and, for all* $(w,v)\in W\times V$,

$$h(w,v):=\inf\{g(Cw,u)\colon\ u\in U,\ Du=v\}>-\infty,$$

and define $\pi_X\colon\ X\times U\to X$ *by* $\pi_X(x,u):=x$. *If* $\bigcup_{\lambda>0}\lambda\left[\pi_X\operatorname{dom}g-C(W)\right]$ *is a closed linear subspace of* X *then*

$$(w^*,v^*)\in W^*\times V^* \text{ and } h^*(w^*,v^*)<\infty\quad\Longrightarrow\quad h^*(w^*,v^*)=\min\left\{g^*(x^*,v^*D)\colon\ x^*\in X^*,\ x^*C=w^*\right\}.$$

Proof. This follows from Corollary 5.9 in exactly the same way that Corollary 3.8 followed from Corollary 3.7. □

References

[1] S. Banach: Théorie des Opérations Linéaires, Chelsea, New York (1955).

[2] J. L. Kelley, I. Namioka et al.: Linear Topological Spaces, van Nostrand, Princeton (1963).

[3] H. König: On some basic theorems in convex analysis, in: Optimization and Operations Research (Bonn, 1979), B. Korte (ed.), North-Holland, Amsterdam (1982) 107–144.

[4] G. Köthe: Topological Vector Spaces I, Grundlehren der mathematischen Wissenschaften 159, Springer, Berlin (1969).

[5] R. T. Rockafellar: Extension of Fenchel's duality theorem for convex functions, Duke Math. J. 33 (1966) 81–89.

[6] B. Rodrigues, S. Simons: Conjugate functions and subdifferentials in non-normed situations for operators with complete graphs, Nonlinear Anal. 12 (1988) 1069–1078.

[7] W. Rudin: Functional Analysis, McGraw-Hill, New York (1973).

[8] S. Simons: Minimal sublinear functionals, Stud. Math. 37 (1970) 37–56.

[9] S. Simons: The Hahn–Banach–Lagrange theorem, Optimization 56 (2007) 149–169.

[10] S. Simons: From Hahn–Banach to Monotonicity, 2nd ed., Lecture Notes in Mathematics 1693, Springer, Berlin (2008).

[11] S. Simons: Quadrivariate versions of the Attouch-Brezis theorem and strong representability, arXiv:0809.0325.

[12] S. Simons: Quadrivariate existence theorems and strong representability, Optimization 60 (2011) 875–891.

[13] S. Simons, C. Zălinescu: Fenchel duality, Fitzpatrick functions and maximal monotonicity, J. Nonlinear Convex Anal. 6 (2005) 1–22.

[14] C. Zălinescu: Duality for vectorial nonconvex optimization by convexification and applications, An. Ştiinţ. Univ. Al. I. Cuza Iaşi, N. Ser., Secţ. Ia 29 (1983) 15–34.

[15] C. Zălinescu: Solvability results for sublinear functions and operators, Z. Oper. Res., Ser. A 31 (1987) 79–101.

[16] C. Zălinescu: Convex Analysis in General Vector Spaces, World Scientific, Singapore (2002).