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THE SPATIAL EVOLUTION OF TRAFFIC UNDER THE TWO WAVE SPEED ASSUMPTION: A SHORTCUT PROCEDURE AND SOME OBSERVATIONS*

by

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Abstract

This paper describes the behavior of traffic in a homogeneous highway according to the hydrodynamic theory, in the special case where the flow-density relationship is triangular; i.e. when only two wave velocities exist. It presents an exact formula that predicts the vehicle that would be found at position x at time t, given the locations of all the vehicles at time zero. The formula, which does not require identification of the vehicle positions at intermediate times, automatically accounts for the creation and dissipation of any shocks. It can be used to calculate system performance measures such as the flow, speed and density at any future point in time-space and the vehicle travel times. The paper also introduces two graphical procedures. The first one identifies all the vehicle positions along the highway for any fixed t, and the second one identifies the traffic state on all the points in time-space. The second procedure can also be applied to highways that are inhomogeneous in space (e.g. including lane drops) and/or time (e.g. including traffic lights), if suitable boundary conditions between adjoining homogeneous sections can be identified. An example involving a moving lane drop, as would occur behind a snow plow, is given.

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1. INTRODUCTION

Luke (1972) and Newell (1991) have proposed related methods to solve the partial differential equations of the hydrodynamic model of traffic flow studied by Lighthill and Witham (1955) and Richards (1956) (LWR). Luke's paper was devoted to a geological erosion problem which is also governed by these equations; the variables denoting the ground slope and its elevation in Luke's model correspond to the traffic density and its integral in Newell's.

The new methods can be applied to general equations of state between flow and density. They differ from the original LWR technique in that they predict the cumulative vehicle count (ground elevation) at a point rather than the traffic density (ground slope) at the point. The Luke-Newell modification, based on the method of characteristics, eliminates the difficulties inherent in the treatment of shocks. In Luke's method the initial conditions are specified by giving the distribution of the vehicles along the highway at time zero; the method predicts future distributions, expressed as the cumulative count of vehicles as a function of distance for a given instant. Newell's method predicts the cumulative count of vehicles as a function time at a fixed point in space. The cumulative temporal counts at two locations on both sides of the point of interest are taken as initial conditions since these are the conditions most likely to be encountered in the field.

Newell (1991) also proposes a streamlined procedure for the important case where the equation of state is triangular as in Fig.

1. Here we will see how Newell's streamlined approach can be applied when the initial condition is the cumulative vehicle count along the highway. For this initial condition, care must be exercised to treat properly the discontinuity in the equation of state at the optimum density (i.e. the density for maximum flow k_o), and this will be explained. An extension to inhomogeneous highways, possibly including time-varying conditions, will also be presented.

The procedure can be applied to some of the problems that are addressed by difference equations in Daganzo (1993). Although the new method is faster (since it can predict the future in time steps of any size) and exact, it is more difficult to incorporate into a network model. The exact method can be used to solve practical problems by hand and to verify the correctness of computer procedures.

The paper is organized as follows. Section 2 describes how the flows, densities and the vehicle number at any point on the highway can be identified after a short while (at time $\epsilon \rightarrow 0$) from the information at time zero. Section 3 then extends the results to finite time jumps; it develops a simple formula that predicts the vehicle number at location x at time t. The formula readily yields flows, densities, speeds and travel times anywhere, anytime. Based on this formula, the section also presents two simple graphical methods that can be used to: (i) predict the vehicle count along the highway at any fixed time, and (ii) construct a time-space diagram displaying the shock trajectories and the traffic state everywhere. The second procedure is applied to the moving bottleneck problem studied in Gazis and Herman (1992). The paper concludes with a brief discussion.

2. INFINITESSIMAL TIME JUMPS

We assume that vehicles are numbered along a highway in order of increasing x (the direction of flow) and define a continuous, non-decreasing, piece-wise differentiable function $N(x,t)$, which gives the vehicle number (label) observed at position x at time t. Flow is assumed to happen in the direction of increasing x. This three dimensional representation of traffic flow was first proposed by Makigami et al (1971). Note that where the derivatives exist, N_x represents the vehicular density, k, and N_t the negative of the flow, $-q$.[1]

The curve on the left side of the diagram in Fig. 2 represents a set of initial conditions, $N(x,0)$. The two tangent lines to the curve have slope k_o, so that our diagram represents a highway with two sections of light traffic (with $k<k_o$) separated by an intermediate section of dense traffic (with $k \geq k_o$) — possibly the aftereffect of an incident in the process of dissipation.

We now show how $N(x,\epsilon)$ can be obtained from $N(x,0)$ with the method of characteristics if the relation of Fig. 1 holds. For the most part, the logic is as in Luke (1972) and Newell (1991).

[1] We use in this paper the subscript notation for partial derivatives. Note that Makigami et al (1971) numbered the vehicles in order of increasing t (decreasing x). Our convention is as in Luke (1971).

The right side of the diagram on Fig. 2 contains three families of parallel straight lines, termed characteristics. According to the theory, the line passing through point $(x,0)$ must have a slope (velocity) equal to the slope of the tangent line to the q(k) curve of Fig. 1 for $k=k(x,0)$. In our case, the slope *is* v_f whenever $k<k_0$ and $-w$ whenever, $k>k_0$. The characteristics are not defined in wedges such as the shaded region behind segment **IH;** in these regions $k=k_o$ and $q = q_{max}$. This creates a difficulty that needs to be addressed.[2]

We note that at time ε four distinct sections of space can be distinguished, containing points that are intersected by either:

(i) one characteristic with slope v_f,
(ii) one characteristic with slope $-w$,
(iii) two characteristics, or
(iv) **no** characteristic.

We first determine the vehicle number at time t in cases (i-iii) by following the vehicle count along the characteristics as in Luke (1972) and Newell (1991). Regions with no characteristics (case iv) are examined subsequently.

Because the flow and density are constant along the characteristics, it is trivial to identify the vehicle number prevailing at any position, x, for cases (i) or (ii). The result for case (i) is:

2 The method of characteristics would yield no prediction where there are no characteristics. Of course, predictions could be created by smoothing out the vertex of the q-k curve at k_o. This suggests that a solution for the triangular case might be constructed by examining the limit of the smoothed predictions as the degree of smoothing is reduced. The result of this process should be the same as that of our more direct arguments, below.

Case (i): $N(x,\epsilon) = N(x-v_f\epsilon,0)$ **(1a)**

because the wave speed is equal to the car speed and no vehicle trajectories cross the characteristic. Similarly, we find for **case** (ii),

Case (ii): $N(x,\epsilon) = N(x+w\epsilon,0)-k_o\delta$. **(1b)**

where

$$6 = \epsilon(v_f+w). \quad (2)$$

Equation (lb) holds because the difference in the vehicle labels **at** both ends of the characteristic ($N(x,\epsilon)$ and $N(x+w\epsilon,0)$) must be equal to the number of vehicle trajectories that cross the characteristic. This number is $\epsilon(wk+q)$, which can be seen to equal $\epsilon(k_ov_f+k_ow) = k_o\delta$ from the geometry of Fig.1. According to **this** relationship, the change in the count is independent of x and k, and is the same for all the characteristics.

For case (iii), the count is the maximum of the **two** possible counts. Thus,

Case (iii): $N(x,\epsilon) = \max\{N(x-v_f\epsilon,0)\ ,\ N(x-v_f\epsilon+\delta,0)-k_o\delta\}$. (Ic)

This maximum relationship was first noted in Luke (1972). It is given in that reference without formal proof, based on physical considerations. Newell (1991) makes a similar statement based **on** the observation that the function $N(x,t)$ is a continuous ruled surface, and choosing the minimum would ensure continuity[3]. (Whether the minimum or maximum should be chosen depends on the definition of $N(x,t)$ and the direction of flow.)

A more direct explanation of (1c) can be obtained from examination of segment **KJ** (corresponding to case iii) of Fig. 2. Note that in this segment the downstream count increases at rate $k<k_o$ and the upstream count at rate $k>k_o$.[4] At **G** the upstream and downstream counts must match, by continuity of $N(x,t)$. It is then clear that (1c) holds because from **K** to **G** the actual count (from downstream) is higher, and from **G** to **J** the actual count (from upstream) is higher too. Clearly then, the location, x, of the shock at time ϵ satisfies:

$$N(x-v_f\epsilon,0) = N(x-v_f\epsilon+\delta,0)-k_o\delta. \tag{3}$$

For case (iv) the result is less obvious. The appendix shows that:

$$\text{Case (iv):} \quad N(x,\epsilon) = \max_z\{N(x-v_f\epsilon+z,0)-k_oz: \; 0\leq z\leq\delta\} \tag{1d}$$

We show below that (1d) holds for all four cases and that we can therefore write generally:

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3 Although $k(x,t)$ is discontinuous along a shock, $N(x,t)$ is not.

4 Because the difference in the counts at the ends of a backward moving characteristic is constant across characteristics, the upstream count will increase with x at the same rate as it does at the point from which the pertinent backward moving characteristic emanates; for backward moving characteristics, this rate is greater than k_o.

$$N(x,\epsilon) = \max_z\{N(x-v_f\epsilon+z,0)-k_o z: 0\le z\le\delta\}, \tag{4a}$$

or equivalently,

$$N(x+v_f\epsilon,\epsilon) = \max_z\{N(x+z,0)-k_o z: 0\le z\le\delta\}. \tag{4b}$$

with 6 given by (2).

To show that (4) is true we note that the argument of (4a) increases with z at a rate $k(x-v_f\epsilon+z,0)-k_o$, which is greater than zero whenever $k>k_o$ and negative if $k<k_o$. In case (i) $k<k_o$ and the argument decreases; therefore the argument is maximized for $z = 0$, which yields (la). In case (ii) the argument increases and it is maximized for $z=\delta$, which corresponds to (lb). In case (iv) the argument decreases and then increases; therefore it must be maximized at one of the extreme points as stated in (ld).

3. FINITE TIME JUMPS AND GRAPHICAL INTERPRETATION

In what follows we view $N(x,\epsilon)$ as the result of applying a differential operator T_ϵ to the function $N(x,0)$. We first show in this section that the differential operator T, satisfies the composition property: $T_\epsilon * T_{\epsilon'} = T_{\epsilon+\epsilon'}$. Since $N(x,t)$ is the result of repeated application of the differential operator, this establishes that: $N(x,t) = T_t * N(x,0)$ for finite t. The result is a simple formula. Two graphical constructions related to the formula will then be presented.

To establish the composition property note that $T_{\epsilon'} * T_\epsilon * N(x,0)$ equals:

$$T_{\epsilon'}*N(x+v_f\epsilon,\epsilon) = \max_{z'}\{N(x+v_f\epsilon+z',\epsilon)-k_o z' : 0\leq z'\leq\delta'\}$$

where δ' is given by (2) with $E = \epsilon'$. On substituting (4b) into the above, we find:

$$T_{\epsilon'}*T_\epsilon*N(x,0) = \max_{z'}\{\max_z[N(x+z+z',0)-k_o z]-k_o z'\}.$$

for $0\leq z\leq\delta$ and $0\leq z'\leq\delta'$. Because the term $k_o z'$ is not dependent on z, the RHS can be expressed as $\max_{z'}\{\max_z[N(x+z+z',0)-k_o z-k_o z']\}$, which is the maximum of a function of $z+z'$. As a result, if (z^*,z'^*) is a feasible optimum solution to the maximization problem, the $(z^*+ z'^*,0)$ is also an optimum solution. Because the latter satisfies the constraint $0\leq z+z'\leq\delta+\delta'$, we can write:

$$T_{\epsilon'}*T_\epsilon*N(x,0) = \max_{z'+z}\{N(x+z+z',0)-k_o(z+z'):0\leq z+z'\leq\delta+\delta'\}$$

and because $\delta+\delta'$ is the result of substituting E for $\epsilon+\epsilon'$ in (2), the right side is the definition of $T_{\epsilon'+\epsilon}*N(x,0)$, which concludes our proof.

An exact formula

The above allows us to write:

$$N(x,t) = \max_z\{N(x-v_f t+z,0)-k_o z:\ 0\leq z\leq\delta\}, \tag{5a}$$

where

$$\delta = t(v_f+w). \tag{5b}$$

From an algorithmic point of view, the complexity of this formula increases with t because the length of the search interval increases with t. A simplification is given below.

We note that the argument of (5a) increases when the initial density at $y = x - v_f + z$ is greater than k_o; the argument remains constant when said density equals k_o and decreases when it is smaller than k_o. It follows that if the maximum is an interior point, it must satisfy $k(y,0) = k_o$. Furthermore, because the argument of (5a) is constant in the intervals where $k(y,0)=k_o$, the interior points of such intervals can be eliminated from further consideration. Eliminating local minima too, we find that the only y's needed to identify an interior maximum are those where the initial density switches from $k>k_o$ to $k\le k_o$ and from $k\ge k_o$ to $k<k_o$. We let X denote the set of such points, defined for the whole highway.

Now, Eq. (5a) can be rewritten concisely if we denote by $X(x,t)$ the set of candidate points corresponding to (x,t). This set includes the intersection of X and the interval $(x-v_f t, x+wt)$, and the two end points of this interval. The expression is:

$$N(x,t) = \max_y \{N(y,0) - k_o(y-x+v_f t) : y \in X(x,t)\}. \qquad (6)$$

In practical applications the set X will be finite and can be defined a priori; the set $X(x,t)$ should only include a small number of easily identified points. Therefore, numerical calculation of (6) requires an effort comparable with the difference equation in

Daganzo (1993).[5] Equation (6), however, does not require $t \to 0$.

As explained in Makigami et al (1972) knowledge of $N(x,t)$ suffices to describe the movement of all the vehicles and the evolution of traffic. For example, if viewed as a function of x for a given t, Eq.(6) gives cumulative spatial counts; the derivative of this function yields the density profile at t. Viewed as a function of t for a given x, Eq.(6) gives the vehicle number as a function of position; the negative of the derivative of this function is the flow profile at a given point; the inverse of the function gives the times at which individual vehicles pass x, which identifies travel times. Vehicle trajectories are lines where $N(x,t)$ is constant.

Graphical construction of the density profile

First note that (5a) can be rewritten as:

$$N(x+v_f t,t) = \max_z\{N(x+z,0)-k_o z:\ 0 \le z \le \delta\}. \qquad (7)$$

For a fixed t, the term $N(x+z,0)-k_o z$ appearing in *(7)* defines a translation of the curve $N(x,0)$ by a horizontal amount $-z$ and a vertical amount $-k_o z$. Note that the slope of the shift vector is the optimum density. Thus, according to (5a), $N(x+v_f t,t)$ is the upper envelope of all the shifted curves with $0 \le z \le \delta = t(v_f+w)$. Because all the shifts have the same slope, the upper envelope is

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5 The cell transmission approach is based on the recursion:

$$q(x,\epsilon) = \min\{q_{max},\ v_f k(x-v_f\epsilon,0),\ w[k_j-k(x+w\epsilon,0)]\}$$

which is also satisfied by all the points (x,ϵ) of Fig. 2.

easy to identify, as shown in Fig.3.

From the original curve, draw the maximal shift vectors (i.e. with $z=t(v_f+w)$) from several points on the original curve, making sure to include any vectors that are tangent to the curve. Following the ends of the vectors, draw the maximally shifted curve. Notice that points in the region swept by the original curve as it is shifted along the vectors cannot be **on** the upper envelope, and that the upper envelope is the upper boundary **of** this region. **As** can be seen from Fig. 3, this upper boundary (curve ABCD) is a composite curve defined by portions of the original and maximally shifted curves and by the tangent vectors at points of concavity.

Because Fig. 3 uses a moving coordinate system with speed v_f, the horizontal shift between the original and final curves at any N represents the extra distance that vehicle **N** could have traveled in time t if it had been allowed to travel at the free-flow speed.

Like Eq.(6), this construction can identify future states in one single step without any intermediate,calculations. The location of shocks is identified by breaks in the slope of the upper envelope, as happens at point C in the figure.

<u>Graphical construction in time-space</u>

If one is interested in obtaining the shock path in time-space, an alternative construction can be used. Consider points **F** and **D** of Fig.2, whose characteristics meet at the shock (point **G**). According to **Eq.** (3), the difference in the vehicle number at these two points is $k_o\delta$, where δ is the distance separating the points. Therefore, the slope of the straight line passing through points **C** and **A** must be k_o (viewing **x** as the abscissa). The following construction can be used as a result: Identify a tangent line of slope k_o to the curve $N(x,0)$ at a point where the density is increasing (such as point **B** in the Figure). Slide the tangent in the direction of increasing **N** to identify intersection points (such as **A** and *C)* and corresponding highway positions (such as points **D** and F). The characteristics emanating from **D** and **F** intersect on the shock. **A** gradual shift in the tangent line identifies the shock path.

Application to a movinq bottleneck

The method that has been described can be applied to highways that are piecewise homogeneous, and where conditions change with time (as when traffic lights are in operation). The trick is to identify the proper boundary conditions that will hold at the edges of the regions of time-space where the highway can be considered to be homogeneous. **As** an example we consider the moving bottleneck problem formulated in Gazis and Herman **(1992)**.

The problem arises if a snowplow moving at speed $v_s<v_f$ is widening a highway while traffic flows past into the narrow section ahead. Figure 4a displays the diagrams for the equation of state in the wide and narrow sections. It is assumed that the wave speeds are the same in both **sections.**[6]

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[6] **A** graphical construction could be developed without this condition, but the presentation would be considerably longer.

A boundary condition follows from the vehicle conservation relation along the snowplow trajectory; the trajectory is displayed in Fig. 4b. If we let superscripts "u" and "d" denote the traffic conditions immediately upstream and downstream of the snowplow, the relation is: $(q^u - q^d)/(k^u - k^d) = v_s$. This means that the only feasible traffic states immediately upstream of the snowplow are on the darkened lines of Fig. 4a.

The initial conditions to our problem are given by a feasible $N(x,0)$, an instance of which is depicted in Fig. 4b. Feasibility means that the slope of $N(x,0)$ never exceeds the jam density corresponding to the specific x, and that the density immediately upstream of the snowplow is the abscissa of a point in the darkened lines of Fig. 4a.

The time space construction described in the previous subsection can be used in each of the homogeneous highway sections away from the snowplow, provided that the characteristics don't cross the snowplow trajectory. One must make sure that the lines in the N vs. x diagram have the appropriate slope: either $k_o{}^u$ or $k_o{}^d$.

If the snowplow crosses a characteristic, modifications may be needed. In developing these, we recognize that the characteristics remain straight as they cross the snowplow trajectory because the wave speeds are the same on both sides.

We first consider the intersection of a forward moving characteristic. Because the vehicle count remains constant along both types of forward moving characteristics, no changes are induced by the crossing; the construction is identical as if we were treating the downstream section.

On the other hand, if a backward moving characteristic is crossed, as depicted by line **BDG** in Fig. 4b, changes are needed. In this case the count decreases at different rates along the characteristic on both sides of the snowplow. Because the rates are proportional to k_o^u and k_o^d the following construction can be used: Starting with a downstream point such as **A**, identify point **B**, draw its characteristic and identify the point of intersection with the snowplow trajectory, **D**. In addition, draw a line of slope k_o^u through **A** (viewing **x** as the abscissa) and identify point C (with the same **x** as point **D**). From *c*, draw a line of slope k_o^u to identify points **E** and F. The (forward) characteristic through F, intersects the line **BD** at the shock (point **G**).

Figure 4b depicts the trajectory of the shock. Points between **H** and **I** are unaffected by conditions downstream from the snowplow; hence, the unmodified construction for the upstream section can be used to locate them. Points beyond **I** are affected by the snowplow and this causes the shock trajectory to bend sharply; the procedure just described would be used to locate these points. Once the shock trajectory intersects the snowplow's, and bends sharply once more, the unmodified procedure for the downstream section can be used; conditions upstream of the snowplow are irrelevant from then on.

4. DISCUSSION

It may be possible to extend the technique described here to more general equations of state, but the result is not likely to lead to an easy to implement graphical procedure. It may also be of purely academic interest, since some of the properties of traffic flow exhibited by the LWR model with general equations of state are less realistic than with Fig. 1. For example, Newell (1993) shows that the general hydrodynamic solution to the moving bottleneck problem posed by Gazis and Herman (1992) exhibits properties not observed in practice. The odd behavior, however, is a peculiarity of the general $q(k)$ relation; it does not arise with only two wave speeds.

Considerations arising from the moving bottleneck phenomena are relevant to our future work. Building on the results just presented, we are planning to develop a hydrodynamic theory of traffic flow upstream of freeway diverges (such as off-ramps and splits) recognizing that the traffic stream is composed of two different vehicle types that will bunch on different sides of the freeway. When an off-ramp is congested, the back-up may essentially force through vehicles to behave as if they were traveling on a narrower freeway. Because the length of the narrowed section grows and dissipates, through vehicles should behave as if they were passing a moving bottleneck. only sensible forms of the equations of state should be considered.

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APPENDIX
Proof of Eq. (1d)

Consider first point "Q" of Fig. 5a. We assume momentarily that any line passing through "Q" with slope s, $-w \le s \le v_f$, is contained entirely in the shaded region with $k=k_o$. Said lines can be identified by the non-negative distance $z \ge \delta$ shown in the figure. Because the number of vehicle trajectories crossing SQ equals the number crossing SP, the count obtained at "Q" by following the line is $N(x-v_f\epsilon+z,0)-k_oz$. This is the argument of (1d), which in this case equals the count at P, $N(x-v_f\epsilon,0)$. It is therefore constant in the range $[0,\delta]$, and (1d) holds.

Equation (1d) will be true in general if we can show that it also holds if the line with slope $-w$ crosses the upper boundary of the region with $k=k_o$ and/or the line with slope v_f crosses the lower boundary. The former can only happen if the upper boundary is a backward moving characteristic, and the latter if the lower boundary is a backward moving characteristic. Figure 5b displays the case where both exceptions arise for points such as Q. To establish (1d) it suffices to observe that if the lower boundary is intersected then the argument of (1d) increases from P to I (where $k>k_o$) and that similarly, if the lower boundary is intersected then the argument of (1d) decreases from H to R (where $k<k_o$).

Note that the search for the maximum in Eq.(4) can be restricted to points P, R and to the extremes of an interval with $k=k_o$ if part of one such interval is included between P and R.

LIST OF FIGURES

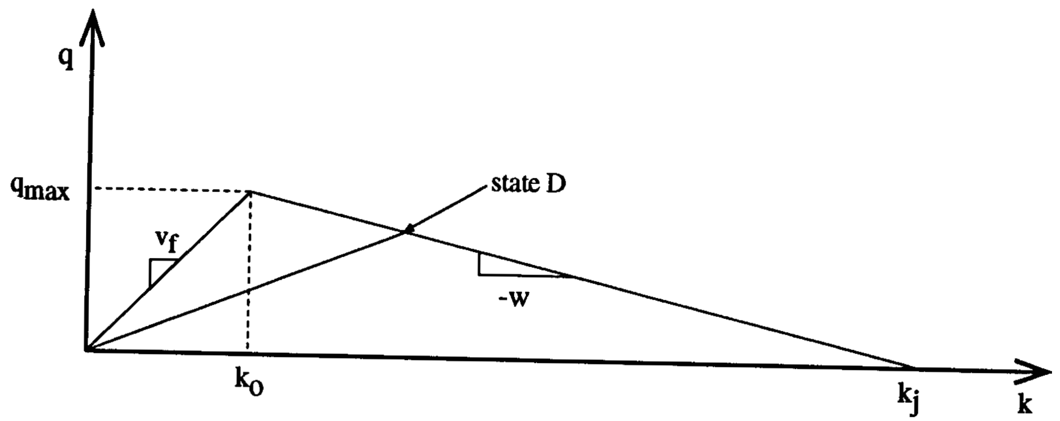

Figure 1. Triangular flow-density relationship.

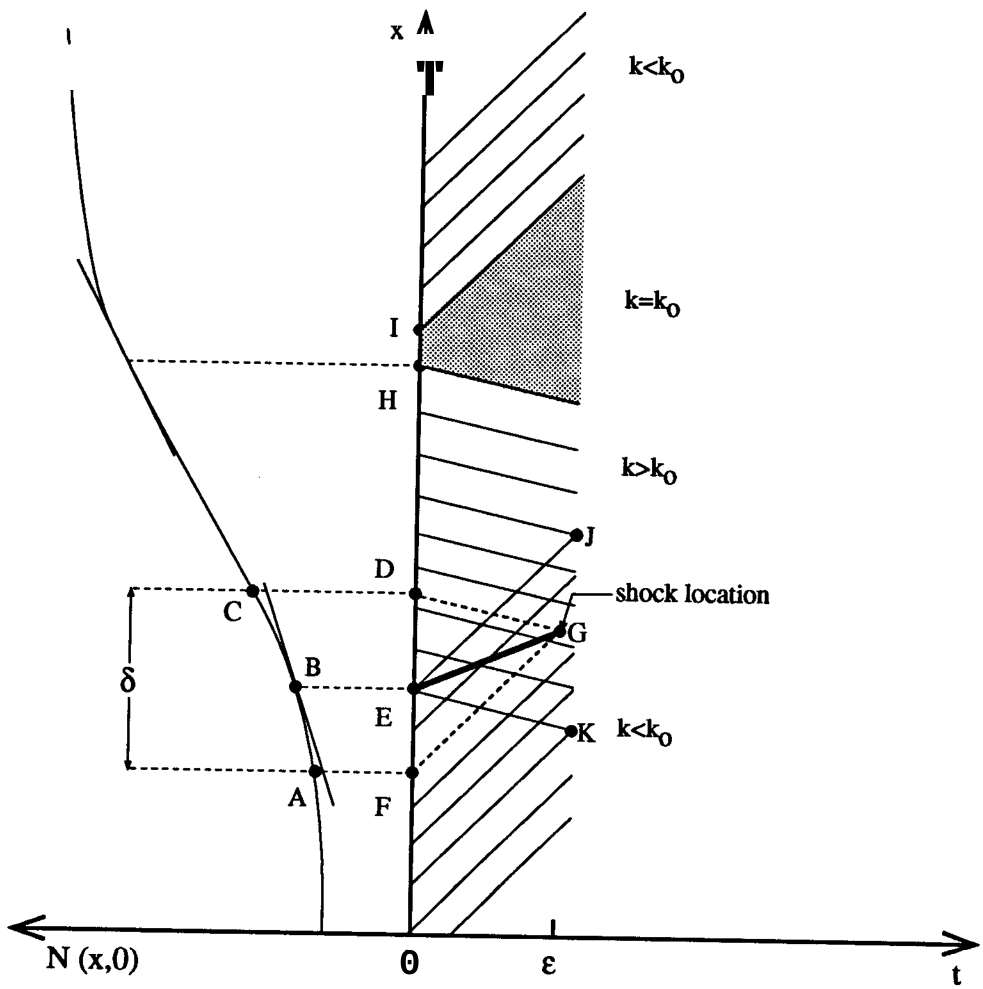

Figure 2. Traffic evolution over time ε : Map of characteristics and cumulative density count.

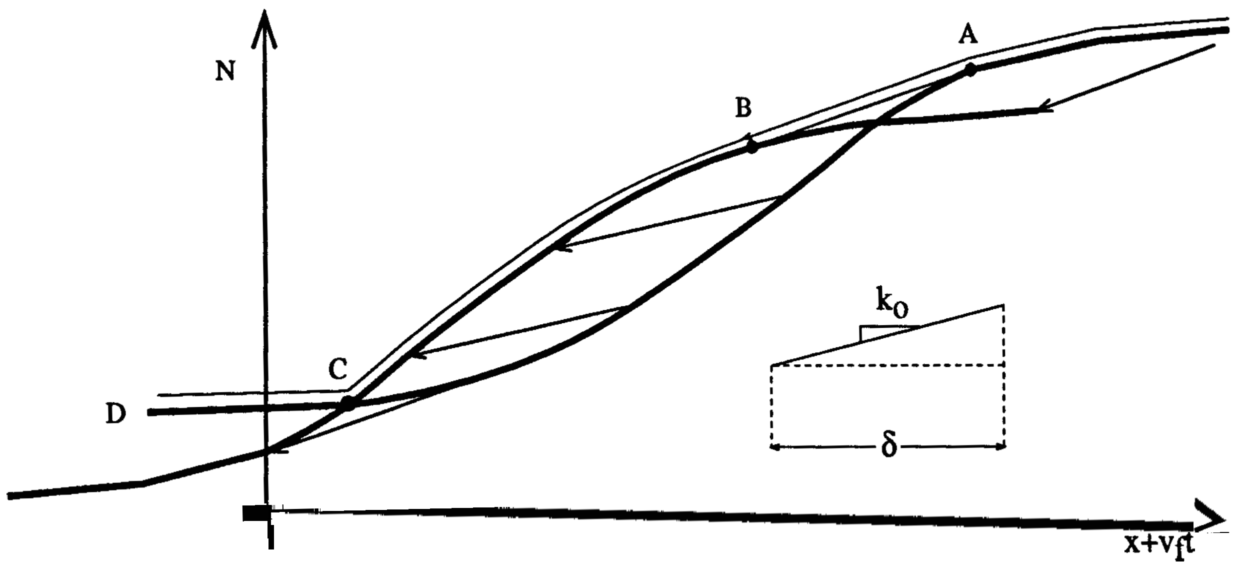

Figure 3. Cumulative density count evolution in a finite time.

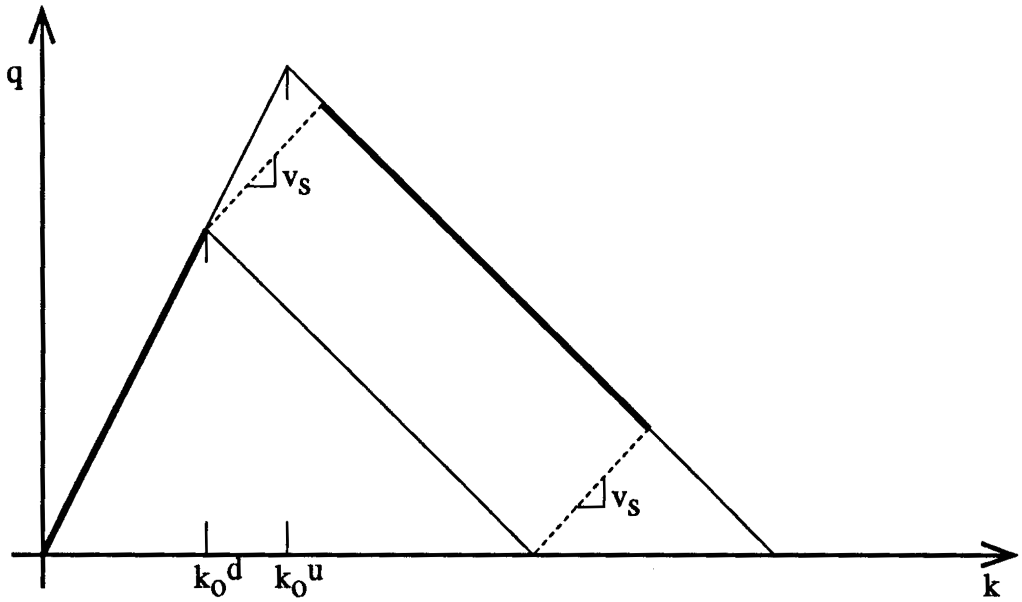

Figure 4A

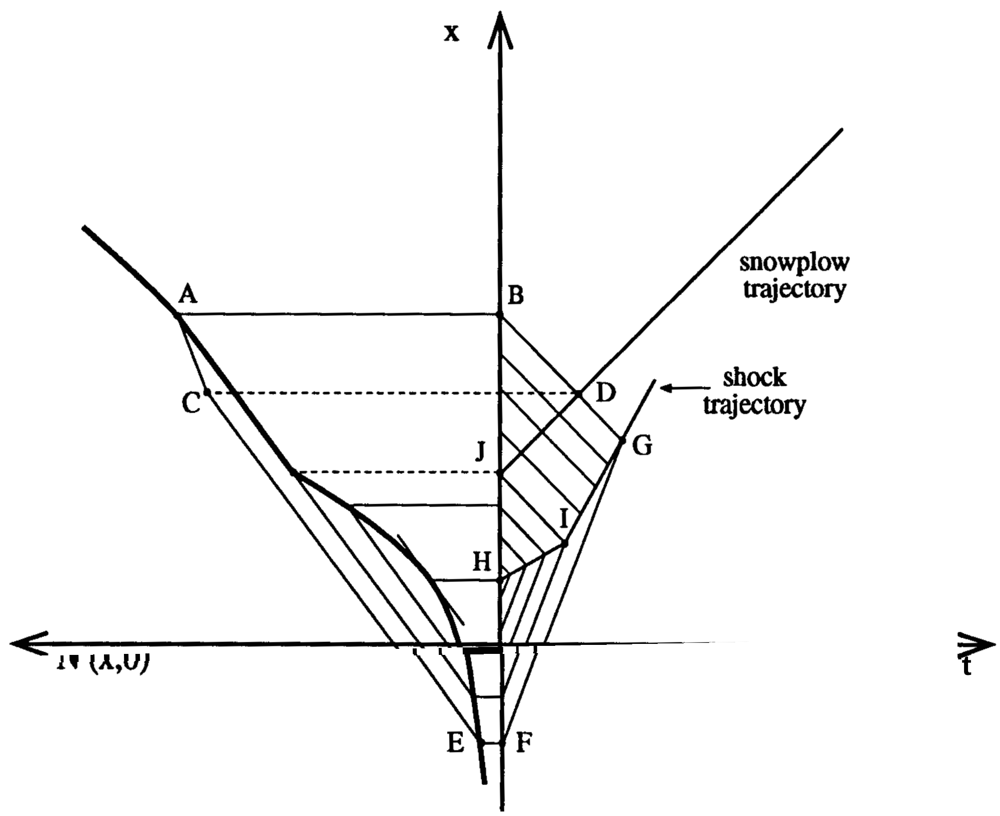

Figure 4B

Figure 4. Geometrical construction for the moving bottleneck problem.

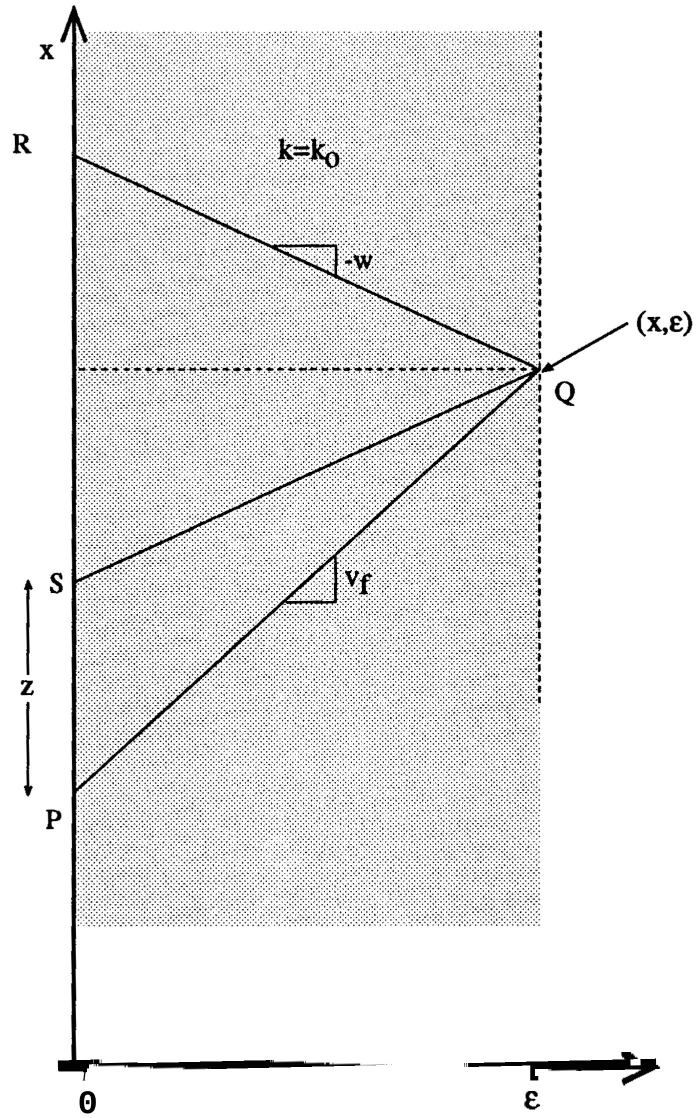

Figure 5A

Figure 5. Wedges of constant optimum density.

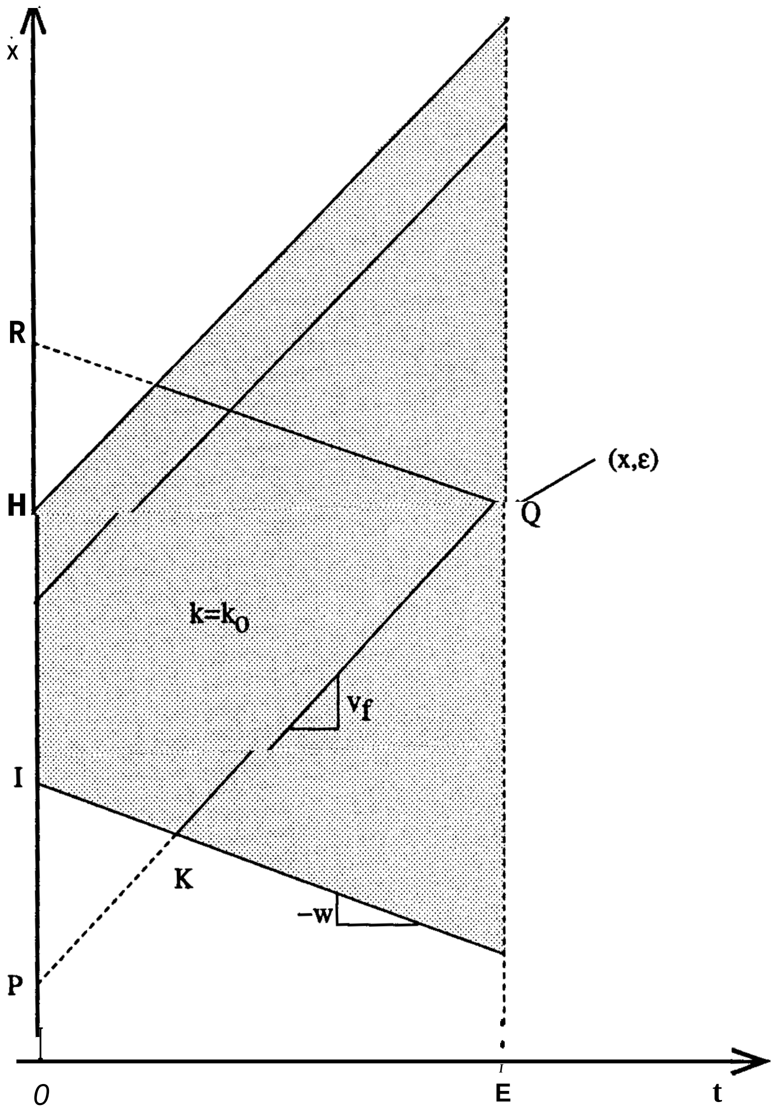

Figure 5B

Figure 5. Wedges of constant optimum density.