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NON-LINEAR σ-MODELS AND INTEGRATION METHODS

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NON-LINEAR σ-MODELS AND INTEGRATION METHODS[1]

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Abstract

We present a brief review of two integration methods – the inverse scattering problem methods and the Bäcklund transformations – for the U(n) principal σ-models.

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I. Introduction to Non-linear σ-Models [1-7]

Generally speaking, non-linear σ-models (or chiral models) are scalar fields, subject to a non-linear constraint causing them to take values within a homogeneous subspace G/H of a global symmetry group G. These models are well studied in the case of a Riemann symmetric space. Upon this space, the Lie algebra and the Maurer-Cartan form have the following G/H decomposition:

$$\begin{aligned} \mathcal{G} &= \mathcal{H} + \mathcal{N} \\ \tilde{\omega} &= \tilde{\omega}_{\mathcal{H}} + \tilde{\omega}_{\mathcal{N}} \end{aligned} \tag{1.1}$$

and the structure equations are

$$\begin{aligned} d\tilde{\omega}_{\mathcal{H}} + \tfrac{1}{2}[\tilde{\omega}_{\mathcal{H}}, \tilde{\omega}_{\mathcal{H}}] &= -\tfrac{1}{2}[\tilde{\omega}_{\mathcal{N}}, \tilde{\omega}_{\mathcal{N}}] \\ d\tilde{\omega}_{\mathcal{N}} + [\tilde{\omega}_{\mathcal{H}}, \tilde{\omega}_{\mathcal{N}}] &= 0 \end{aligned} \tag{1.2}$$

where $\mathcal{N}$ is the complement of $\mathcal{H}$. The form $\tilde{\omega}_{\mathcal{N}}$ may be used to define coframes on G/H:

$$\tilde{\theta}_{\mathcal{N}} = \tilde{\sigma}^* \tilde{\omega}_{\mathcal{N}} \tag{1.3}$$

for some local section σ : U → G, where U is an open set in G/H. The metric may be expressed as :

$$k = k(\tilde{\theta}, \tilde{\theta}) = k_{ij}\, \tilde{\theta}^i \tilde{\theta}^j \tag{1.4}$$

where k is the restriction to $\mathcal{N}$ of the Killing form on G. Similarly, defining locally

$$\omega = \tilde{\sigma}^* \tilde{\omega}_{\mathcal{H}} \tag{1.5}$$

we may pull-back the Maurer-Cartan form onto U:

$$\begin{aligned} \Omega &= d\omega + \tfrac{1}{2}[\omega, \omega] = -\tfrac{1}{2}[\tilde{\theta}, \tilde{\theta}] \\ T &= d\tilde{\theta} + [\omega, \tilde{\theta}] = 0 \end{aligned} \tag{1.6}$$

The first of these equations defines the curvature Ω in terms of form θ and the second implies that G/H has no torsion, i.e. the connection ω is Riemannian.

The non-linear σ-models form a class which has a strong analogy to gauge fields: they have similar geometrical properties, similar lagrangian and field equations, similar topological invariants and instantons. Recall that the gauge fields are defined as a 1-form ω over a manifold M with values in the Lie algebra of a compact group G and are described by a Lagrangian with action:

$$A = \int_M K(\Omega \wedge {}^*\Omega) \tag{1.7}$$

where

$$\Omega = d\omega + \tfrac{1}{2}[\omega, \omega] \tag{1.8}$$

is a 2-form over M with values in $\mathcal{G}$, K is a Killing metric, * is the Hodge dual with respect to some suitable metric on M. The field equations are derived by variation of (1.7):

$$D\,{}^*\Omega = d\,{}^*\Omega + [\omega, {}^*\Omega] = 0 \tag{1.9}$$

The Bianchi identities follow from (1.8):

$$D\,\Omega = d\,\Omega + [\omega, \Omega] = 0 \tag{1.10}$$

If M is compact and dim M = 4, the theory has the following topological invariant I:

$$I = \int_M K(\Omega \wedge \Omega) \tag{1.11}$$

which takes only integer values, the so-called "instanton numbers".

The σ-field is a smooth function σ : M → G/H on a manifold M with metric g. In terms of the pull-backs to M of $\tilde{\omega}$ and $\tilde{\theta}$, we have:

$$\begin{aligned} \omega^\sigma &= \sigma^* \tilde{\omega} \\ \theta^\sigma &= \sigma^* \tilde{\theta} \end{aligned} \tag{1.12}$$

The σ-field is described by a Lagrangian with action:

$$A = \int_M K(\theta^\sigma \wedge {}^*\theta^\sigma) \tag{1.13}$$

and the field equations are obtained by variation of (1.13):

$$D\,{}^*\theta^\sigma = d\,{}^*\theta^\sigma + [\omega^\sigma, {}^*\theta^\sigma] = 0 \tag{1.14}$$

We can already see the resemblance between the σ-field equations and the above gauge field equations. The analogue of the Bianchi identity (1.10) is the pull-back of the second equation of (1.6):

$$d\theta^\sigma + [\omega^\sigma, \theta^\sigma] = 0 \tag{1.15}$$

Note that the above equations being pullbacks are gauge invariant by construction. Under a change of gauge $\sigma \to R_h \cdot \sigma$, the quantities ω^σ and θ^σ transform as

$$\begin{aligned} \omega^\sigma &\longrightarrow Ad\, h^{-1} \theta^\sigma \\ \theta^\sigma &\longrightarrow Ad\, h^{-1} \omega^\sigma + h^{-1} dh \end{aligned} \tag{1.16}$$

Other formulations of σ-models are possible. One alternative version follows from the action expressed in terms of the metric tensor k on G:

$$A = \int_M k\,(d\theta \wedge {}^*d\theta) \tag{1.17}$$

where $d\theta$ is a differential form over M with values in the complement $\mathcal{N}$ of $\mathcal{H}$ in $\mathcal{G}$. Here, the expression of the action is explicitly G invariant.

II. Inverse Scattering Problem Method for σ-Models.

The method described here, due to Zakharov, Shabat and Mikhailov,[8-10] gives rise to a large number of non-linear integrable models, including σ-models. In particular, it is known[9] that all 2-dimensional σ-models are solvable by this method. We shall deal with U(n) principal σ-model g(t,x) which is an n × n complex matrix defined on 2-dimensional space-time t, x.

Consider a system of differential equations

$$d\psi + W\psi = 0 \tag{2.1}$$

where

$$W = U d\xi + V d\eta \tag{2.2}$$

is a differential form over a manifold M with values in U(n), $2\zeta = t - x$, $2\eta = t + x$ denote the "light-cone" variables and ψ, U, V are n × n complex matrix functions of ζ and η and a complex parameter λ. We assume that Eq. (2.1) has a compatible fundamental matrix of solutions. In terms of the components U and V, Eq. (2.1) may be written in the equivalent form as a system of partial differential equations

$$\psi_{,\xi} = U\psi \quad , \quad \psi_{,\eta} = V\psi \tag{2.3}$$

The Cartan integrability condition for Eq. (2.1) is:

$$dW + \frac{1}{2}[W, W] = 0 \tag{2.4}$$

or the equivalent relation

$$U_{,\eta} - V_{,\zeta} + [U, V] = 0 \tag{2.5}$$

which must hold for all values of λ.

Although the method is more general, we now specialize to U and V having respectively N_1 poles λ_1, $\lambda_2, \ldots, \lambda_{N_1}$ and N_2 poles $\mu_1, \mu_2, \ldots, \mu_{N_2}$ and we assume that all these poles are simple and with positions independent of ζ and η. Then, U and V may be written in the form:

$$\begin{aligned} U &= U_0 + \sum_{n=1}^{N_1} \frac{U_n}{\lambda - \lambda_n} \\ V &= V_0 + \sum_{n=1}^{N_2} \frac{V_n}{\lambda - \mu_n} \end{aligned} \tag{2.6}$$

where $\{U_i, V_i\}$ depend only on (ζ, η). Eq. (2.5) becomes:

$$\begin{aligned} &U_{0,\eta} - V_{0,\zeta} + [U_0, V_0] = 0 \\ &U_{n,\eta} + [U_n, R_n] = 0 \\ &V_{n,\zeta} + [V_n, T_n] = 0 \end{aligned} \tag{2.7}$$

where

$$\begin{aligned} R_n &= V_0 + \sum_{m=1}^{N_1} \frac{V_m}{\lambda_n - \mu_m} \\ T_n &= U_0 + \sum_{m=1}^{N_2} \frac{U_m}{\mu_n - \lambda_m} \end{aligned} \tag{2.8}$$

Assume now that U and V have a single pole $\lambda_1 = -1$ and $\mu_1 = 1$ respectively and set $U_1 = A$, $V_1 = -B$. Then, we are led to the system of equations:

$$\psi_{,\zeta} = \frac{A}{1+\lambda}\psi \quad , \quad \psi_{,\eta} = \frac{B}{1-\lambda}\psi \tag{2.9}$$

with the integrability condition:

$$A_{,\eta} - B_{,\zeta} + [A, B] = 0 \tag{2.10}$$

for a U(n) valued function g such that

$$A = g_{,\zeta}\, g^{-1} \quad , \quad B = g_{,\eta}\, g^{-1} \tag{2.11}$$

The sum gives

$$(g_{,\zeta}\, g^{-1})_{,\eta} + (g_{,\eta}\, g^{-1})_{,\zeta} = 0 \tag{2.12}$$

which defines the U(n) principal σ-model.[11]

Consider a particular solution $\{U_0, V_0, \psi_0\}$ is known for (2.3) and (2.5). The following procedure may be applied to construct new solutions with the help of the known one. Suppose a $n \times n$ matrix function $G(\lambda)$ is given which is regular on a closed contour Γ. The matrix Riemann problem is to find two matrix functions $\psi_1(\lambda)$ and

$\psi_2(\lambda)$, analytic on the closure respectively of the exterior D_1 and the interior D_2 of Γ, and such that on Γ

$$G(\lambda) = \psi_1(\lambda)\cdot\psi_2(\lambda) \tag{2.13}$$

If $\psi_1(\lambda)$, $\psi_2(\lambda)$ are required to be regular on D_1, D_2 respectively, this is the Regular Riemann Problem. If $\psi_1(\lambda)$, $\psi_2(\lambda)$ are allowed a finite number of isolated zeroes, this is the Riemann Problem with Zeroes.

Consider a given function $G_0(\lambda)$ on Γ. We define a new function

$$\begin{aligned} G(\lambda,\zeta,\eta) &= \psi_0(\lambda,\zeta,\eta)\, G_0(\lambda)\, \psi_0^{-1}(\lambda,\zeta,\eta) \\ &= \psi_1(\lambda,\zeta,\eta)\, \psi_2(\lambda,\zeta,\eta) \end{aligned} \tag{2.14}$$

Differentiating this relation with respect to ζ gives:

$$\begin{aligned} \psi_1^{-1}(U_0\psi_1 - \psi_{1,\zeta}) &= (\psi_{2,\zeta} + \psi_2 U_0) \equiv U \\ \psi_1^{-1}(V_0\psi_1 - \psi_{1,\zeta}) &= (\psi_{2,\zeta} + \psi_2 V_0) \equiv V \end{aligned} \tag{2.15}$$

which, if $\psi_1 + \psi_2$ are solutions to the Regular Riemann Problem, gives $\{U, V\}$ with the same holomorphic structure as U_0, V_0. Moreover, setting

$$\chi = \psi_1^{-1}\psi_0 \quad , \quad \tilde{\chi} = \psi_2\,\psi_0 \tag{2.16}$$

we obtain

$$\begin{aligned} \chi_{,\zeta} &= U\chi \quad , \quad \chi_{,\eta} = V\chi \quad \text{in } D_1 \\ \tilde{\chi}_{,\zeta} &= U\tilde{\chi} \quad , \quad \tilde{\chi}_{,\eta} = V\tilde{\chi} \quad \text{in } D_2 \end{aligned} \tag{2.17}$$

This is the new solution for Eq. (2.1) or Eq. (2.3). Hence, U and V satisfy (2.4) or (2.5).

For U and V of the form (2.6), we have the following expressions:

$$\begin{aligned} U_n &= g_n^{-1} U_n^0 g_n \quad , \quad g_n = \psi_1\big|_{\lambda=\lambda_n} \\ V_n &= p_n^{-1} V_n^0 p_n \quad , \quad p_n = \psi_1\big|_{\lambda=\mu_n} \\ U_0 &= g_0^{-1} U_0^0 g_0 - g_0^{-1} g_{0,\zeta} \quad , \\ & \qquad\qquad g_0 = \psi_2^{-1}\big|_{\lambda=\infty} \\ V_0 &= g_0^{-1} V_0^0 g_0 - g_0^{-1} g_{0,\eta} \end{aligned} \tag{2.18}$$

Notice that the change of normalization $\psi_1 \to \psi_1\, g^{-1}(\zeta, \eta)$, $\psi_2 \to \psi_2\, g^{-1}(\zeta, \eta)$ correspond to a change of gauge. If we are in a gauge with $U_0^0 = V_0^0 = 0$, the canonical normalization leads to $U_0 = V_0 = 0$. Thus, for the σ-model (2.9), we have:

$$A = q^{-1} A_0 q \quad , \quad q = \psi_1 |_{\lambda = -1}$$
$$B = p^{-1} B_0 p \quad , \quad p = \psi_2 |_{\lambda = +1} \tag{2.19}$$

and hence (A, B, ψ) satisfy (2.9), we have

$$g = \eta |_{\lambda = 0} = \psi_1^{-1} g_0 \tag{2.20}$$

where $g_0 = \psi_0|_{\lambda=0}$.

The above procedure of generating new solutions from an old one is equivalent to the traditional inverse scattering problem method.[8]

This method of generating new solutions by means of the Regular Riemann Problem does not lead to the determination of the whole set of solutions to the system (2.1) or (2.3). For this, it is necessary to resolve the Riemann Problem with Zeroes.

Consider a particular Riemann problem with two zeroes, for which:

$$G(\lambda) = 1 \tag{2.21}$$

and hence

$$\psi_2 = \psi_1^{-1} \tag{2.22}$$

The functions $\psi_1(\lambda,\zeta,\eta)$, $\psi_2(\lambda,\zeta,\eta)$ are thus rational with the poles of ψ_1 (resp. ψ_2) located at the zeroes of ψ_2 (resp. ψ_1). We now choose ψ_1 to have a simple zero at $\lambda = \lambda_0$ and ψ_2 to have one at $\lambda = \mu_0$. It then follows from (2.22) that:

$$\psi_1 = 1 - \frac{\lambda_0 - \mu_0}{\lambda - \mu_0} P$$
$$\psi_2 = 1 + \frac{\lambda_0 - \mu_0}{\lambda - \lambda_0} P \tag{2.23}$$

where $P^2 = P$ is a projection operator. Substituting (2.23) into (2.15), we have the following differential equations:

$$P(\partial_\zeta - U_0|_{\lambda_0})(1-P) = 0$$
$$P(\partial_\eta - V_0|_{\lambda_0})(1-P) = 0$$
$$(1-P)(\partial_\zeta - U_0|_{\mu_0})P = 0$$
$$(1-P)(\partial_\eta - V_0|_{\mu_0})P = 0 \tag{2.24}$$

If the solution $\psi_0(\lambda,\zeta,\eta)$ is known, these equations may be explicitly solved as follows. Let P be the projection operator determined by the two subspaces:

$$M = \mathrm{Im}\, P \quad , \quad N = \mathrm{Ker}\, P \tag{2.25}$$

defined by the conditions

$$(1-P)M = 0 \; , \quad PN = 0 \tag{2.26}$$

Taking some arbitrary initial values P_0, M_0, N_0, we see that (2.24) is solved by

$$M = \psi_0(\mu_0)\, M_0$$
$$N = \psi_0(\lambda_0)\, N_0 \tag{2.27}$$

which defines the projector operator P as

$$P = \psi_0(\mu_0) P_0 [\psi_0(\mu_0) P_0 + \psi_0(\lambda_0)(1 - P_0)]^{-1} \tag{2.28}$$

Substituting (2.28) into (2.24), using (2.3) for $\{U_0, V_0, \psi_0\}$ shows that (2.27) is a solution with initial values of $\psi_0(\mu_0)$, $\psi_0(\lambda_0)$ chosen as 1.

The solutions of the Riemann Problem with G = 1 with two zeroes of the functions ψ_1 and ψ_2 describe the collisions of soliton solutions, in particular, collisions of simple solitons as well as processes of "induced" decay of composite solitons in collision with simple ones.[9]

III. Bäcklund Transformations

The Bäcklund Transformations (henceforth BT) give a method of generating new hyperbolic n-submanifolds of R^{2n-1} from a given one.[12] Hence the BT should give rise to a method of generating a new solution of σ-model from a given one.

Consider the U(n) principal σ-model (2.12). The BT for this equation are given[13-15] by the following expression

$$g_{,\xi}\, g^{-1} - g_{0,\xi}\, g_0^{-1} = -\lambda (g g_0^{-1})_{,\xi}$$
$$g_{,\eta}\, g^{-1} - g_{0,\eta}\, g_0^{-1} = \lambda (g g_0^{-1})_{,\eta} \tag{3.1}$$

with the constraint

$$\lambda g g_0^{-1} + \bar{\lambda} g_0 g^{-1} = \lambda + \bar{\lambda} \tag{3.2}$$

where λ is a complex parameter.

It can be shown from Eqs. (3.1) that if g_0 is a solution of (2.12), so is g. Moreover, with given $g_0 \in U(n)$, the system (3.1) is solvable with $g \in U(n)$, and the non-linear constraint (3.2) is compatible with (3.1).[15] Hence Eqs. (3.1), (3.2) are indeed BT.

We shall not pursue the general analysis of the BT for (2.12), but rather consider a particular simple case (2.23). Our purpose now is to show that we may derive from Eq. (2.24) an equation which directly relates the new solution g to Eq. (2.12) from a known particular solution g_0.

The system of four Eqs. (2.24) is equivalent to the following system of two equations:

$$
\begin{aligned}
(1-P)\,U_0(\mu_0)\,P - P\,U_0(\lambda_0)\,(1-P) &= P_{,\xi} \\
(1-P)\,V_0(\mu_0)\,P - P\,V_0(\lambda_0)\,(1-P) &= P_{,\eta}
\end{aligned}
\tag{3.3}
$$

For the σ-model (2.9), we have:

$$
\begin{aligned}
U_0(\lambda_0) &= \frac{A}{1+\lambda_0}\,, \qquad U_0(\mu_0) = \frac{A}{1+\mu_0} \\
V_0(\lambda_0) &= \frac{B}{1+\lambda_0}\,, \qquad V_0(\mu_0) = \frac{B}{1+\mu_0}
\end{aligned}
\tag{3.4}
$$

where

$$
A = g_{0,\xi}\, g_0^{-1}\,, \qquad B = g_{0,\eta}\, g_0^{-1} \tag{3.5}
$$

and g_0 is the input solution. Define

$$
\begin{aligned}
Z &= \mu_0\, \Psi_1\big|_{\lambda_0} \\
&= \mu_0 + (\lambda_0 - \mu_0)\, P
\end{aligned}
\tag{3.6}
$$

The equation (3.3) reduces to the following:

$$
\begin{aligned}
Z_{,\xi} &= \frac{1}{(1+\mu_0)(1+\lambda_0)}\big[-\mu_0\lambda_0 A - ZA + \\
&\qquad + (1+\lambda_0+\mu_0)\, AZ - ZAZ\big] \\
Z_{,\eta} &= \frac{1}{(1-\mu_0)(1-\lambda_0)}\big[\mu_0\lambda_0 B - ZB + \\
&\qquad + (1-\mu_0-\lambda_0)\, BZ + ZBZ\big]
\end{aligned}
\tag{3.7}
$$

The condition $P^2 = P$ is equivalent to

$$
(Z-\mu_0)(Z-\lambda_0) = 0 \tag{3.8}
$$

Then, the new solution to the U(n) σ-model equations is, according to (2.17), given by

$$
g = \mu_0\, Z^{-1} g_0 \tag{3.9}
$$

Substituting into Eq. (3.7) and (3.8), after rearranging terms, we get

$$
\begin{aligned}
\mu_0\,(g^{-1} g_{,\xi} - g_0^{-1} g_{0,\xi}) &= -(g_0^{-1} g)_{,\xi} \\
\mu_0\,(g^{-1} g_{,\eta} - g_0^{-1} g_{0,\eta}) &= (g_0^{-1} g)_{,\eta}
\end{aligned}
\tag{3.10}
$$

with the constraint

$$
\mu_0\, g_0\, g^{-1} + \lambda_0\, g\, g_0^{-1} = (\mu_0 + \lambda_0)\, \mathbb{1} \tag{3.11}
$$

We may see directly from (3.10) that if g_0 is a solution to (2.12), so is g and vice versa. Thus, the system (3.10) provides a BT generating new solution from old one. The explicit solution is obtained by combining Eqs. (2.23), (2.28) with (3.6), which gives:

$$Z = [(\lambda_0 a_1 - \mu_0 a_2) Z_0 + \mu_0 \lambda_0 (a_2 - a_1)] \times [(a_1 - a_2) Z_0 - \mu_0 a_1 + \lambda_0 a_2]^{-1} \tag{3.12}$$

where

$$a_1 = \psi_0(\mu_0), \quad a_2 = \psi_0(\lambda_0) \tag{3.13}$$

satisfy

$$a_{1,\xi} = \frac{A a_1}{1+\mu_0}, \quad a_{1,\eta} = \frac{B a_1}{1-\mu_0}$$
$$a_{2,\xi} = \frac{A a_2}{1+\lambda_0}, \quad a_{2,\eta} = \frac{B a_2}{1-\lambda_0} \tag{3.14}$$

and $Z_0 = \mu_0 + (\lambda_0 - \mu_0) \mathcal{P}_0$

is some arbitrarily chosen initial value, satisfying the constraint

$$(Z_0 - \mu_0)(Z_0 - \lambda_0) = 0 \tag{3.15}$$

or, equivalently $P_0^2 = P_0$.

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