Title
Equivalence between Kuznetsov components of cubic fourfolds and Gushel-Mukai fourfolds

Author
Chen, Qingjing

Publication Date
2023

Peer reviewed|Thesis/dissertation

University of California
Santa Barbara

Equivalence between Kuznetsov components of cubic fourfolds and Gushel-Mukai fourfolds

A dissertation submitted in partial satisfaction
of the requirements for the degree

Doctor of Philosophy
in
Mathematics

by

Qingjing Chen

Committee in charge:

Professor Xiaolei Zhao, Chair
Professor Birge Huisgen-Zimmermann
Professor David Morrison

June 2023

The Dissertation of Qingjing Chen is approved.

Professor Birge Huisgen-Zimmermann

Professor David Morrison

Professor Xiaolei Zhao, Committee Chair

May 2023

Acknowledgements

I want to express my sincere gratitude and appreciation to my doctoral advisor Xiaolei Zhao, for his constant support with advices, smart ideas and for having encouraged me in moments I needed. I would also like to thank my doctoral committee as a whole for their time, consideration and feedback.

I want to thank Zhiyuan Li for hosting my visit in SCMS, Qizheng Yin for my visit in BICMR and Alex Perry for my visit in University of Michigan. They have provided me very warm hospitality and I also want to thank them for many interesting and inspiring discussions.

I am much indebted to Laura Pertusi, who shared many helpful ideas and answered, with great patience, my questions.

I owe Xianzhe Dai, Guofang Wei, Mihai Putinar, Darren Long, Daryl Cooper and Adebisi Agboola for very insightful courses and discussions that radically shifted my understanding of certain areas of mathematics. In the same vein, I would like to thank Yang Qiu, Ethan Robinett, Ashwin Trisal, Junrong Yan for introducing me to their research and for listening to mine.

I want to thank the staffs in our department, especially Medina Price, for their constant help. They have created a very warm and inclusive environment.

Bai, you are special, thank you for being there and always listen to me.

Lastly, I want to thank my parents and grandparents, thank you for all that you do for me.

I was partially supported by the grant NSF FRG DMS-2052665.

Curriculum Vitæ
Qingjing Chen

Education

2010 - 2014	B.Sc. (Hons) in Mathematical sciences, Nanyang Techonological University.
2017 - 2018	M.A. in Mathematics, University of California, Santa Barbara.
2018 - present	Ph.D. in Mathematics (Expected), University of California, Santa Barbara.

Reserch Interests

I am mainly interested in algebraic geometry; more specifically, in areas like derived category of varieties, Bridgeland stability and moduli spaces, hyperkahler geometry, etc.

Seminar Talks

Fall 2019	Moduli of sheaves via Quiver Representation, Reading seminar on moduli of sheaves, UC Santa Barbara.
Fall 2020	Homological Projective Duality, Learning seminar on algebraic geometry, UC Santa Barbara.
Summer 2021	Weil II, Learning seminar on algebraic geometry, UC Santa Barbara.
Summer 2022	Hodge numbers are not derived invariant in positive characteristic, Preprint Seminar, UC Santa Barbara.
Winter 2022	Kuznetsov components of some Fano fourfolds, Algebra seminar, Michigan State University.

Visiting Positions

Dec. 2020 - Feb. 2021	Shanghai Center for Mathematical Sciences, invited by Professor Ziyuan Li.
Mar. 2021 - Aug. 2021	Beijing International Center for Mathematical Research, invited by Professor Qizheng Yin.
Aug. 2022 - Dec. 2022	University of Michigan, invited by Professor Alexander Perry.

Fellowships and Awards

2018	Raymond L. Wilder Award for outstanding achievement as a first-year graduate student, UC Santa Barbara.
2021	NSF FRG DMS-2052665 (paritally supported)

2022	Outstanding Teaching Assistant Award, UC Santa Barbara.
2022	Graduate Division Dissertation Fellowship, UC Santa Barbara.

Teaching Experiences

MATH 34A, Calculus for Social Science.
MATH 4A, Linear Algebra with Applications.
MATH 4B, Differential Equations.
MATH 8, Transition to Higher Math.
MATH 6A, Vector Calculus I.
MATH 108A, Linear Algebra.
MATH 122A, Complex Analysis.

Abstract

Equivalence between Kuznetsov components of cubic fourfolds and Gushel-Mukai fourfolds

by

Qingjing Chen

There are two ways that certain Fano fourfolds (for example, cubic fourfolds and Gushel-Mukai fourfolds) can be associated with K3 surfaces. On the one hand, we can associate a K3 surface to the fourfold Hodge-theoretically, meaning the middle cohomology of the fourfold contains the middle cohomology of the K3 as a sub-Hodge structure; on the other hand, we can homologically associate a K3 to the fourfold, which is to require the Kuznetsov component of the fourfolds to be equivalent to the bounded derived of the K3. Conjecturally, one expect such K3 associations detect rationality of the fourfolds. It has been proved that for cubic fourfolds, these two types of K3 associations are equivalent, whereas for Gushel-Mukai fourfolds, Hodge-association of a K3 strictly implies homological association of a K3. We continue this line of study, instead of comparing fourfolds-K3 association we consider fourfolds-fourfolds association. We prove that, at least for a generic Gushel-Mukai fourfold in the Hodge-special loci, if it admits Hodge-associated cubic fourfolds, then it admits a homological-associated one, and vice versa.

Contents

Chapter 1

Introduction

An algebraic variety is said to be *rational* if it is birationally equivalent to a projective space, in other words these are the varieties that can be parametrized by rational maps. Studying the rationality of a variety is one of the major endeavors in algebraic geometry. Being vastly open for general algebraic varieties, it is natural to first narrow down the scope of investigation to smooth hypersurfaces in projective spaces. The rationality of hyperplane is trivial and is also completely known for quadrics, a smooth quadirc hypersurface $X \subset \mathbb{P}^n$ is rational if and only if X has a k-rational point, where k is the base field. Now if we just move one degree higher, namely to smooth cubic hypersurfaces, the situation immediately becomes much more intriguing. Using only elementary techniques, it is proved in early days that smooth cubic curves in $\mathbb{P}^2$ are not rational, and smooth cubic surfaces in $\mathbb{P}^3$ are rational. Rationality of smooth cubic hypersurface in $\mathbb{P}^4$ (known as cubic threefolds) remains untill 1972 when Clemens and Griffiths ([CG72]) proved the irrationality, by studying the intermediate Jacobians of cubic threefolds. Rationality of cubic hypersurfaces in $\mathbb{P}^5$ (known as cubic fourfolds) still remains open and is on the forefront of current research. Besides hypersurfaces in projective spaces, the rationality of more general Fano varieties, for example those given by suitable complete intersections in Grassmannians, like Gushel-Mukai varieties (see Definition 3.1.1), have also received much attentions.

Motivated by the investigation of irrationality of cubic threefolds by Clemens and Griffiths, it is expected that the rationality of certain Fano fourfolds, like cubic fourfolds and Gushel-Mukai fourfolds should be intricately related to some geometrically associated K3 surfaces. We give a brief account of associated K3 surfaces for these two kinds of fourfolds. It turns out that there are two ways one can associate a K3 surface to our fourfolds: one on the level of Hodge theory and the other on the level of derived category. We refer the reader to Section 3 & 4 for many details that we have to skip for the moment.

Cubic fourfolds A cubic fourfold is a smooth algebraic hypersurface of degree 3 in $\mathbb{P}^5$. It is possible to associate a K3 surface to a cubic fourfold Hodge-theoretically. In [Has00], Hassett introduced the notion of a *Hodge-special* cubic fourfold, which is a cubic fourfold X containing an algebraic surface $T \subset X$ not homologous to the square h_X^2 of the hyperplane class. The rank 2 sublattice $\langle h_X^2, T\rangle \subset \mathrm{H}^{2,2}(X;\mathbb{Z})$ is called a *labelling* on X. Special cubic fourfolds are parameterized by a countable union of irreducible divisors $\mathcal{C}_d$ in the moduli space $\mathcal{C}$ of smooth cubic fourfolds, indexed by the discriminant d of the labelling on the fourfolds. It can be shown that we must have $d > 6$ with $d \equiv 0$ or $2 \pmod 6$. The orthogonal complement of a labelling on X in $\mathrm{H}^4(X;\mathbb{Z})$ is called the *non-special cohomology* of the labelled cubic fourfold; it is a sublattice of $\mathrm{H}^4(X;\mathbb{Z})$ with signature $(19, 2)$ and carries a K3-type Hodge structure, so we can ask when this is in fact Hodge isometric to the (sign-reversed) primitive middle cohomology of an actual polarized K3 surface. Hassett proved that this happens exactly when the cubic fourfold is in the divisor $\mathcal{C}_d$ (meaning it has a labelling with discriminant

d) with the integer d satisfying the following numerical condition

$$(**)' \quad d \text{ is not divisible by } 4,\ 9$$
$$\text{and the only odd primes dividing } d \text{ are} \equiv 0 \text{ or } 1 \pmod 3.$$

In this case, the cubic fourfold is said to be *Hodge-associated* to a K3 surface. It is also conjectured by Hassett that the rational cubic fourfolds are exactly those admitting Hodge-associated K3 surfaces. Rationality of cubic fourfolds in $\mathcal{C}_d$ for the first few values of d satisfying $(**)'$ has been confirmed. Cubic fourfolds in $\mathcal{C}_{14}$ are known to be rational for a long time, it is in fact due to Fano (see [Fan43]); cubic fourfolds in $\mathcal{C}_{26}$, $\mathcal{C}_{38}$ and $\mathcal{C}_{42}$ are recently proved to be rational by Russo and Staglian?([RS19a] and [RS19b]) using classical projectve geometric arguments. We would like to point out that although cubic fourfolds not in divisor $\mathcal{C}_d$ are believed to be irrational, not a single provable example has been found at the moment this thesis is written!

We can also associate cubic fourfolds with K3 surfaces on the level of derived category. The bounded derived category $\mathrm{D}^b(X)$ of a cubic fourfold X always contains a special admissible subcategory $\mathcal{A}_X$, called the *Kuznetsov component* of X, defined as the right orthogonal of some exceptional sequences in $\mathrm{D}^b(X)$. The Kuznetsov components are often referred to as noncommutative K3 surfaces (or K3 categories), for they have the same Serre functor as the bounded derived categories of a K3 surfaces (shifting the degree by 2). When $\mathcal{A}_X$ is derived equivalent to the bounded derived category $\mathrm{D}^b(S)$ of some K3 surface S, we say that X is *homological-associated* to the K3 surface S. It is conjectured by Kuznetsov in [Kuz15] that X is rational if and only if X is homological-associated to a K3 surface.

Now we have two rationality conjectures of cubic fourfolds based on two notions of associating K3 surfaces, we should expect that these two notions to be equivalent. It is indeed the case for cubic fourfolds. This was first proved by Addinton and Thomas in [AT14], at least for cubic fourfolds general in the divisors $\mathcal{C}_d$, making use of some deformation theory of complexes (the fact that they are only able to prove this for general fourfolds in divisors is due to the limitation of deformation theory). Following the work of Bayer, Lahoz, Macrì, Nuer, Perry and Stellari in [BLM+19], making use of Bridgeland stability conditions on the Kuznetsov components of cubic fourfolds (see [BLMS17]), it is possible to bypass this deformation argument and conclude that these two notions of associating K3 surfaces are equivalent for all cubic fourfolds.

Unfortunately, the equivalence of K3 association for cubic fourfolds is deceptive. Actually we should *not* expect these two notions of associating K3 surfaces being equivalent in more general situations, as shown by the case of Gushel-Mukai fourfolds.

Gushel-Mukai fourfolds A Gushel-Mukai fourfold is the smooth 4-dimensional intersection

$$X = C\mathrm{Gr}(2, V_5) \cap \mathbb{P}^8 \cap Q.$$

This intersection happens in $\mathbb{P}^{10}$, where $C\mathrm{Gr}(2, V_5)$ is the projective cone over $\mathrm{Gr}(2, V_5) \subset \mathbb{P}^9$ in its PlÃŒcker embedding, $\mathbb{P}^8$ a linear subspace of $\mathbb{P}^{10}$ and Q a quadric hypersurface in $\mathbb{P}^8$. Since X is smooth, it does not contain the vertex ν of the cone $C\mathrm{Gr}(2, 5)$. Hence the projection from ν defines a regular map

$$\gamma_X : X \longrightarrow \mathrm{Gr}(2, 5),$$

called the *Gushel map*. Gushel-Mukai fourfolds are the only smooth Fano fourfolds of degree 10 and index 2 ([DK18a]); to be more precise, if (X, H) is a smooth polarized fourfold such that $\mathrm{H}^4 = 10$ with canonical bunlde $K_X = -2H$ and Picard group $\mathrm{Pic}(X) = \mathbb{Z}H$, then X is necessarily a Gushel-Mukai fourfold with the polarization H given by the restriction of the hyperplane class on $\mathbb{P}^{10}$.

Hodge-special Gushel-Mukai fourfolds are introduced, in a similar philosophy as Hassett, by Debarre, Iliev and Manivel in [DIM14]. These are Gushel-Mukai fourfolds X such that $\mathrm{H}^{2,2}(X;\mathbb{Z})$ contains a class not contained in $\gamma_X^*\mathrm{H}^4(Gr(2,5);\mathbb{Z})$; the rank 3 sublattice of $\mathrm{H}^{2,2}(X;\mathbb{Z})$ generated by this class and $\gamma_X^*\mathrm{H}^4(Gr(2,5);\mathbb{Z})$ is also called a labelling on X. Similar to the case of cubic fourfolds, it can be shown that such fourfolds are parameterized by a countable union of hypersurfaces $\mathcal{GM}_d$ in the moduli space $\mathcal{GM}$ of Gushel-Mukai fourfolds with d being the discriminant of the labelling, and we must have $d > 8$ and $d \equiv 0,\ 2$ or $4 \pmod 8$. Depending on the value of d, $\mathcal{GM}_d$ is either irreducible or is the union of two irreducible divisors. The orthogonal complement of a labelling on X is once again called the non-special cohomology and is a sublattice of $\mathrm{H}^4(X;\mathbb{Z})$ of signature $(19, 2)$. A Hodge special Gushel-Mukai fourfold whose non-special cohomology is Hodge isometric to the (sign-reversed) primitive middle cohomology of a polarized K3 surface S is said to be Hodge-associated with the K3 surface S. The Gushel-Mukai fourfold Hodge-associated with K3 surfaces are exactly those in the divisor $\mathcal{GM}_d$ with d satisfying the numerical condition:

$(**)$ d is not divisible by 8 and the only odd primes dividing d are $\equiv 1 \pmod 4$.

We can also conjecture about the rationality of Gushel-Mukai fourfolds with Hodge-associated K3 and this is already established for fourfolds in $\mathcal{GM}_{10}$ and $\mathcal{GM}_{20}$; the case for $d = 10$ is discussed in [DIM14] and the case for $d = 20$ is studied by Hoff and Staglian?([HS20]). Once again, no examples of Gushel-Mukai fourfolds have been proved to be irrational, although conjecturally one would like to say that very general Gushel-Mukai fourfolds (those not in any of the special divisors $\mathcal{G}_d$'s) should be irrational.

On the other hand, the bounded derived category $\mathrm{D}^b(X)$ of a Gushel-Mukai fourfold X also contains a Kuznetsov component $\mathcal{A}_X$, once again a K3 category. When $\mathcal{A}_X$ is derived equivalent to the bounded derived category of an actual K3 surface S, we also say that X is homological-associated to S. However, unlike the case of cubic fourfolds, homological and Hodge-theoretic associations of K3 surfaces are *not* equivalent for Gushel-Mukai fourfolds. In fact, by the results in [Pert17], [BLM+19] and [PPZ19], we can conclude that if a Gushel-Mukai fourfold X has a Hodge-associated K3 surface then it must have a homological-associated one; the converse, however, is shown to be false by explicit counterexample in [Pert17]. It is an interesting question to ask if Gushel-Mukai fourfolds with homological-associated K3 which have no Hodge-associated ones are irrational.

We give some more details about the "fourfold-K3" associations to illustrate some ideas that will persist in our work. Following [BLMS17], [BLM+19] and [PPZ19], we know that a cubic fourfold or a Gushel-Mukai fourfold X has a homological-associated K3 if and only if $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X;\mathbb{Z})$ contains a hyperbolic plane. Here $\tilde{\mathrm{H}}(\mathcal{A}_X;\mathbb{Z})$ is a weight-2 Hodge structure that can be associated to the K3 category $\mathcal{A}_X$; we call it the *Mukai lattice* of $\mathcal{A}_X$ and it generalizes the usual Mukai Hodge structure of K3 surfaces. On the other hand, we will see later that the data of a labelling on X is equivalent to the data of some rank 3 sublattice of $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X;\mathbb{Z})$ (necessarily of the same discriminant as the labelling). Therefore the condition of X being Hodge-associated to a K3 is equivalent to that $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X;\mathbb{Z})$ containing some rank 3 sublattice with discriminant d (satisfying $(**)$ if X is Gushel-Mukai fourfold or $(**)'$ if X is a cubic fourfold). Hence the comparison between these two notions of "fourfolds-K3" associations is reduced to comparing two conditions about sublattices of $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X;\mathbb{Z})$, which can be carried out by purely lattice theoretic computations (see [AT14, Theorem 3.1] and [Pert17, Theorem 3.6]). For cubic fourfolds, these two are equivalent and therefore Hodge-theoretic and homological association of K3 are equivalent for cubic fourfolds. Whereas for Gushel-Mukai fourfolds, $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X;\mathbb{Z})$ containing the aforementioned rank 3 sublattice always implies it containing a hyperbolic plane but not vice versa, therefore we only conclude that a Gushel-Mukai fourfold admitting Hodge-associated K3 also admits homological-associated one.

Now we have seen that, motivated by rationality question, Fano fourfolds with associated K3 surfaces have received much attention. However there is a lack of spotlight on "fourfolds-fourfolds" associations. To be more precise here, we can talk about homological or Hodge-association between cubic fourfolds and Gushel Mukai fourfolds, which simply means their Kuznetsov components being equivalent, or their non-special cohomologies being Hodge isometric. We do have the following numerical condition on d

$$(\dagger)\qquad \begin{gathered}(a)\text{either } d \equiv 2 \textit{ or } 20 \pmod{24}\\ \text{and the only odd primes dividing } d \text{ are} \equiv \pm 1 \pmod{12};\\ (b)\text{or } d \equiv 12 \textit{ or } 66 \pmod{72}\\ \text{and the only primes} \geq 5 \text{ dividing } d \text{ are} \equiv \pm 1 \pmod{12}.\end{gathered}$$

which is given in [DIM14], telling us that a general Gushel-Mukai fourfolds in $\mathcal{GM}_d$ will be Hodge-associated to some cubic fourfold in $\mathcal{C}_d$; yet we do not know much about the relation between homological and Hodge-associations in this case. Suggested by the case of "fourfolds-K3" comparison, we expect Hodge-association implies homological-association in the "fourfolds-fourfolds" comparison as well. We find out that this is indeed the case for fourfolds general in divisors.

Theorem 1.0.1. *Let $d > 8$ be an interger satisfying the numerical condition* ($\dagger$). *There is a non-empty Zariski open subset $\mathcal{U}_d$ of $\mathcal{GM}_d$ such that Gushel-Mukai* *fourfolds in $\mathcal{U}_d$ admit homological-associated ones; there is a non-empty Zariski open subset $\mathcal{V}_d \subset \mathcal{C}_d$ such that cubic fourfolds in $\mathcal{V}_d$ admit homological-associated Gushel-Mukai fourfolds.*

Notations and Conventions. Throughout we work over $\mathbb{C}$, the field of complex numbers, and by a variety we always mean a (not necessarily irreducible) reduced separated finite type scheme over $\mathbb{C}$. The set of $\mathbb{C}$-points on a variety can be naturally given the structure of a complex analytic space. When it is clear from the context, we will not stress whether we are using analytic or Zariski topology. The bounded derived category of coherent sheaves on a smooth projective variety X is denoted by $\mathrm{D}^b(X)$. When X is not necessarily smooth (for example, X being the total space of a smooth family over a singular base), we also need to consider the triangulated category $\mathrm{D}_{Perf}(X)$ of perfect complexes on X. We use a single capital Latin character, for example E instead of $\mathcal{E}^\bullet$, to denote an object (which is a complex) in $\mathrm{D}^b(X)$ or $\mathrm{D}_{Perf}(X)$. We also employ the common practice of writing a derived functor in the form of their underived counterpart whenever possible, for example for a proper morphism $f\colon X \longrightarrow Y$, we denote the derived pushforward Rf_* simply by f_*. Whenever we have a product $X \times Y$, we denote by $\pi_X\colon X \times Y \longrightarrow X$ and $\pi_Y\colon X \times Y \longrightarrow Y$ the two projection maps.

Chapter 2

Preliminaries

2.1 Basics of Lattice Theory

Definition 2.1.1. A *lattice* is a pair (L, b) consisting of a free abelian group L of finite rank and a non-degenerate symmetric bilinear form $b : L \times L \to \mathbb{Q}$ (called *intersection form*). The lattice (L, b) is *integral* if $b(x, y) \in \mathbb{Z}$ for all $x, y \in L$.

Suppose $\mathrm{rank}(L) = n$ and $\{e_1, \dots, e_n\}$ is a basis of L, the matrix $B = (b(e_i, e_j)) \in M_n(\mathbb{Q})$ is called the *intersection matrix* of the lattice (L, b). The followings are immediate consequences of the definition:

- The number of positive eigenvalues r_+ of B does not depend on the choice of the basis, the *signature* of (L, b), written $\mathrm{sign}(L)$, is the pair of integers (r_+, r_-), where $r_- := \mathrm{rank}(L) - r_+$ and is necessarily the number of negative eigenvalues of B. The lattice (L, b) is said to be *positive-definite* if $r_+ = \mathrm{rank}(L)$ and *negative-definite* if $r_- = \mathrm{rank}(L)$. It is *indefinite* if both $r_+ > 0$ and $r_- > 0$.

- The lattice (L, b) is integral if and only if its intersection matrix B (in any basis) is an integral matrix.

- The determinant $\det(B)$ of the intersection matrix B does not depend on the choice of the basis and is called the *discriminant* of the lattice (L, b), denoted by $\mathrm{disc}(L, b)$. Note the the discriminant of a positive-definite lattice must be positive. An integral lattice (L, b) is called *unimodular* if $\mathrm{disc}(L, b) = \pm 1$.

We often omit the intersection form b from the notation if it is understood from the context, and we wrtie $b(x, y)$ simply as $x \cdot y$, and $b(x, x)$ as x^2. Given two lattices L_1 and L_2, $L_1 \oplus L_2$ will always mean the *orthogonal* direct sum, meaning, if b_i is the bilinear form on $L_i, i = 1, 2$, the bilinear form b on $L_1 \oplus L_2$ is given by $b((x_1, x_2), (y_1, y_2)) = b_1(x_1, y_1) + b_2(x_2, y_2)$. We can always multiply the bilinear form b on a lattice L by a nonzero scalar $\lambda \in \mathbb{Q}$, we write the resulting lattice as $L(\lambda)$ (We sometimes write $L(-1)$ as $-L$).

Definition 2.1.2. An integral lattice L is called *even* if x^2 is an even integer for all $x \in L$, otherwise it is called *odd* (so an odd lattice L is one such that x^2 is an odd integer *for some* $x \in L$).

From now on, by a lattice, we will always mean an integral lattice. In fact, the only reason we want to allow the intersection form to take value in $\mathbb{Q}$ is because the intersection form on the dual lattice of an integral lattice (see later) takes value in $\mathbb{Q}$ rather than $\mathbb{Z}$.

Example. Let us introduce some examples that will be used later on. Up to isometry (see Definition 2.1.3), we can always describe a lattice by its intersection matrix in some basis.

1. We let $I_{m,n}$ be the lattice whose underlying group is $\mathbb{Z}^{m+n}$ and its intersection matrix with respect to the standard basis given by

$$\begin{pmatrix} I_m & 0 \\ 0 & -I_n \end{pmatrix}.$$

2. The hyperbolic plane U is the rank 2 lattice with intersection matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, it is an even unimodular lattice.

3. The lattice E_8 is the rank 8 positive-definite even unimodular lattice with intersection matrix

$$\begin{pmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{pmatrix}.$$

Up to isometry, this is the only positive-definite even unimodular lattice of rank 8.

4. We let A_1 be the rank 1 lattice with intersection matrix (2) and A_2 the rank 2 lattice with intersection matrix $\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$. Both are positive-definite.

As suggested by the notation, A_1, A_2 and E_8 are the root lattices of the root systems A_1, A_2 and E_8 respectively.

Definition 2.1.3. (1) An embedding of lattices $\phi\colon (L, b) \to (L', b')$ is an injective group homomorphism $\phi\colon L \to L'$ such that $\phi^* b' = b$ (meaning $b(x, y) = b'(\phi(x), \phi(y))$ for all $x, y \in M$), in this case we say L is a *sublattice* of L'. If in addition $L'/\phi(L)$ is a free abelian group, we call ϕ a *primitive* embedding and L a *primitive sublattice* of L'.

(2) If $\phi\colon M \to L$ is a bijective embedding of lattices, then ϕ is called an *isometry*, in this case M and L are said to be *isometric*. We denote by $O(L)$ the group of isometries from the lattice L to itself.

(3) Two lattice embeddings $\phi\colon M \to L$ and $\phi'\colon M \to L'$ of M are said to be *isomorphic* if there is an isometry $f : L \to L'$ such that $\phi' = f \circ \phi$.

If M is a sublattice of L, we often identify M with its image in L (so it is literally a subset of L) if the embedding map $\phi\colon M \to L$ is irrelevant in the situation.

The classification of indefinite unimodular lattice up to isometry is well-known (see [Mil73] or [Ser73]):

Theorem 2.1.4. *Let L be an indefinite unimodular lattice with signature (r_+, r_-):*

(1) If L is odd, then L is isometric to I_{r_+,r_-};

(2) If L is even , then we necessarily have $r_+ - r_- \equiv 0 \ (mod\ 8)$ and L is isometric to $U^{r_-} \oplus E_8^{\frac{r_+-r_-}{8}}$ if $r_+ \geq r_-$, or isometric to $U^{r_+} \oplus E_8(-1)^{\frac{r_--r_+}{8}}$ if $r_+ \leq r_-$.

In order to study the structures of lattices not necessarily unimodular, one need to consider the so called *discriminant form*. We start with introducing the *discriminant group* of the lattice, which measures how far the lattice is from being unimodular. Given a lattice (L, b), we denote by $b_{\mathbb{Q}}$ the $\mathbb{Q}$-linear extension of the intersection form b to the vector space $L \otimes_{\mathbb{Z}} \mathbb{Q}$. Note that L embeds into $L \otimes_{\mathbb{Z}} \mathbb{Q}$ via the map $x \mapsto x \otimes 1$.

Definition 2.1.5. Let (L, b) be an integral lattice, we define its dual lattice (L^*, b^*), where the underlying abelian group is given by

$$L^* = \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : b_{\mathbb{Q}}(x, y) \in \mathbb{Z}\ \forall x \in L\}$$

The symmetric pairing b^* is defined to be the restriction of $b_{\mathbb{Q}}$ to the subgroup L^*.

It is cleat that the image of the natural embedding $L \hookrightarrow L \otimes_{\mathbb{Z}} \mathbb{Q}$ is contained in L^*, hence we can make the following

Definition 2.1.6. The finite group $A_L := L^*/L$ is called the *discriminant group* of L.

Clearly L is unimodular if and only if $A_L = 0$ and more generally one can show that $|\mathrm{disc}(L)| = |A_L|$. The discriminant form is the bilinear form on A_L naturally induced by the bilinear form on L:

Definition 2.1.7. Let (L, b) be a lattice, b^* the bilinear form (which is $\mathbb{Q}$-valued) on the dual lattice L^* and A_L the discriminant group of L, we define the *discriminant form*

$$b_L \colon A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$$

by $b_L(x + L, y + L) = b^*(x, y) + \mathbb{Z}$ for $x, y \in L^*$; this is well-defined since by definition if x or $y \in L$, then $b^*(x, y) \in \mathbb{Z}$. The quadratic form associated to the bilinear form b_L is denoted by q_L. When L is even, q_L takes value in $\mathbb{Q}/2\mathbb{Z}$. We denote by $O(A_L, q_L)$, or simply $O(A_L)$ the group of automorphims of A_L preserving the discriminant form q_L.

The discriminant form and signature determine the genus of an even lattice. Two lattices are said to be in the same genus if they have isometric p–adic extensions for all p. In favorable cases, the genus of an indefinite even lattice contains only one isometry class and therefore in theses cases discrimnant form and signature uniquely determine the lattice up to isometry. We state two special cases that will be used later on:

Proposition 2.1.8. ([Nik79, Corollary 1.13.3 & 1.13.4]). *(1) An even indefinite lattice L is uniquely determined by its signature and discriminant form up to isometry provided that rank$L \geq \ell(A_L) + 2$. (The* length *of a finite abelian group S, denoted by $\ell(S)$, is defined to be the cardinality of the minimal generator of S, or equivalently, the number of invariant factors of S.)*

(2) Let L be an even indefinite lattice with signature (r_+, r_-), M be an even lattice with signature $(r_+ - 1, r_- - 1)$. If there is an isomorphism $A_M \simeq A_L$ identifying q_M with q_L, then L is isometric to $U \oplus M$.

Besides classification problem, discriminant form of a lattice M also play important roles in addressing the questions of existence and uniqueness of primitive embedding of M into some unimodular lattice, as well as the relation to the *orthogonal complement*.

Definition 2.1.9. Let M be a sublattice of L, the orthogonal complement of M in L, denoted by $M^\perp$ (or by$(M^\perp)_L$ if we want to emphasize that the orthogonal complement is taken in L), is the sublattice of L defined by

$$M^\perp = \{x \in L : (x, y) = 0 \ \forall y \in M\}.$$

The followings are straightforward consequences of the definitions.

1. $M^\perp$ is necessarily primitive, in fact, a sublattice M of L is primitive if and only if $(M^\perp)^\perp = M$;
2. L is an *overlattice* of $M \oplus M^\perp$ (meaning $M \oplus M^\perp$ is a sublattice of L and $L/(M \oplus M^\perp)$ is finite).
3. Any isometry of L to itself preserving M will also preserve $M^\perp$.

Remark 2.1.10. Given a sublattice $M \hookrightarrow L$, one cannot conclude that $L \simeq M \oplus M^\perp$, but there is one important exception: suppose the lattice L contains a hyperbolic plane U (not necessarily primitive), then L is given by $U \oplus U^{\perp}$. In fact, let e, f be the standard basis elements of U, then given any $x \in L$, we can write $x = x_0 + x'$, where $x_0 = (e \cdot x)f + (f \cdot x)e \in U$ and $x' = x - x_0 \in U^{\perp}$.

Given an isometry $\phi : (L, b) \to (L', b')$ of lattices, it naturally induces an isometry of dual lattices $(L^*, b^*) \to (L'^*, b'^*)$, which in turn induces an isomorphism between the discriminant groups $\bar{\phi} : A_L \to A_{L'}$ satisfying $\bar{\phi}^* b_{L'} = b_L$ (where b_L and $b_{L'}$ are the discriminant bilinear forms on L and L' resp.). The passage from ϕ to $\bar{\phi}$ is certainly functorial. In particular we have a group homomorphism

$$O(L, b) \to O(A_L, q_L),\ \phi \mapsto \bar{\phi}.$$

Theorem 2.1.11. ([Nik79, Proposition 1.6.1]). *(1) Given a primitive embedding of a lattice M into a unimodular lattice L, it induces canonically a commutative diagram of finite abelian groups*

$$\begin{tikzcd} & L/(M \oplus M^{\perp}) \arrow[dl, "\cong"'] \arrow[dr, "\cong"] & \\ A_M \arrow[rr, "\mu", "\cong"'] & & A_{M^{\perp}} \end{tikzcd} \tag{2.1.1}$$

with all arrows being isomorphisms such that $\mu^* b_{M^{\perp}} = -b_M$. (In particular, we will have $\ell(A_M) = \ell(A_{M^{\perp}})$ and $disc(M) = \pm disc(M^{\perp})$.)

(2) Conversely let M and K be lattices such that there is an isomorphism $\gamma\colon A_M \to A_K$ satisfying $\gamma^* b_K = -b_M$, then there exists a primitive embedding of M into a unimodular lattice L together with an isometry $\phi\colon M^{\perp} \to K$ such that $\gamma = \bar{\phi} \circ \mu$.

(3) Given two isomorphisms $\gamma_1, \gamma_2\colon A_M \to A_K$ such that $\gamma_i^* b_K = -b_M$, $i = 1, 2$, which determine primitive embeddings $f_i\colon M \hookrightarrow L_i$ together with isometries $\phi_i : (M^{\perp})_{L_i} \to K$, $i = 1, 2$, as in (2), suppose we also have an isometry $\psi \in O(M)$,

then there is an isometry $\Psi\colon L_1 \to L_2$ making the following diagram commutes

$$\begin{array}{ccc} L_1 & \xrightarrow{\Psi} & L_2 \\ {\scriptstyle f_1}\big\uparrow & & {\scriptstyle f_2}\big\uparrow \\ M & \xrightarrow{\psi} & M \end{array} \tag{2.1.2}$$

if and only if there is an isometry $\varphi \in O(K)$ making the following diagram commutes

$$\begin{array}{ccc} A_K & \xrightarrow{\bar{\varphi}} & A_K \\ {\scriptstyle \gamma_1}\big\uparrow & & {\scriptstyle \gamma_2}\big\uparrow \\ A_M & \xrightarrow{\bar{\psi}} & A_M \end{array} \tag{2.1.3}$$

In this case, we necessarily have the following commutative diagram

$$\begin{array}{ccc} K & \xrightarrow{\varphi} & K \\ {\scriptstyle \phi_1}\big\uparrow & & {\scriptstyle \phi_2}\big\uparrow \\ (M^\perp)_{L_1} & \xrightarrow{\Psi} & (M^\perp)_{L_2} \end{array} \tag{2.1.4}$$

Remark. In (3) of the previous proposition, the isometry Ψ is in fact uniquely determined by the data $\psi \in O(M)$, $\varphi \in O(K)$ and $\phi_i\colon A_M \tilde{\to} A_K$, $i = 1, 2$. Indeed these data uniquely determine the values of Ψ on $M \oplus (M^\perp)_{L_1}$ by the commutative diagrams (2.1.2) and (2.1.4) in (3), but this sublattice has finite index in L_1. Moreover, if we put $\psi = id_M$ and $\varphi = id_K$, (3) says that the primitive embedding of M into some unimodular lattice determined by some isomorphism $\gamma : A_M \to A_K$ with $\gamma^* b_K = -b_M$ as in (2) is unique up to isomorphism.

We will use this theorem in the forms of the following two corollaries.

Corollary 2.1.12. *Let $M \hookrightarrow L$ and $N \hookrightarrow L'$ be primitive embeddings of lattices into unimodular lattices.*

(a) If there is an isometry $M^\perp \simeq N^\perp$, then we have isomorphism of discriminant forms $(A_M, b_M) \simeq (A_N, b_N)$, and in particular we will have $disc(M) = \pm disc(N)$.

(b) Conversely, if $M^\perp$ and $N^\perp$ are even indefinite lattices of the same signature

and rank$(M^\perp) \geq \ell(A_M)+2$*, then an isomorphism* $(A_M, b_M) \simeq (A_N, b_N)$ *gives rise to an isometry* $M^\perp \simeq N^\perp$.

Proof. By (1) of Theorem 2.1.11, there is an isomorphism $(A_M, b_M) \simeq (A_N, b_N)$ if and only if there is an isomorphism $(A_{M^\perp}, b_{M^\perp}) \simeq (A_{N^\perp}, b_{N^\perp})$, the latter is induced by isometris $M^\perp \simeq N^\perp$; hence (a) is clear.

(b) follows from Theorem **??** (1) and the fact that $\ell(A_{M^\perp}) = \ell(A_M)$. $\square$

Corollary 2.1.13. *Let* $M \hookrightarrow L$ *be a primitive embedding of lattices where* L *is unimodular. Every isometry of* M *which induce trivial action on* A_M *is the restriction of a unique isometry of* L *which restricts to identity on* $M^\perp$.

Proof. Apply (3) of Theorem 2.1.11 to $K = M^\perp$ and $\gamma_1 = \gamma_2 = \mu \colon A_M \to A_{M^\perp}$. Suppose $\psi \in O(M)$ induces identity map $\bar{\psi} = id_{A_M}$ on A_M, then we can take $\varphi = id_K$ to be identity map, then diagram (2.1.3) commutes; hence there is a (necessarily unique) isometry $\Psi \in O(L)$ such that $\Psi|_M = \psi$ and $\Psi|_{M^\perp} = id$ $\square$

We will also need the following

Theorem 2.1.14. [Nik79. Theorem 1.14.4]. *Let* M *be an even lattice with signature* (t_+, t_-), *and* L *be an even unimodular lattice of signature* (s_+, s_-). *Suppose we have*

(i) $t_+ < s_+$,

(ii) $t_- < s_-$,

(iii) $\ell(A_M) \leq$ *rank*$(L) -$ *rank*$(M) - 2$.

Then there exists a unique primitive embedding (up to isomorphism) of M *in* L.

We will primarily use this theorem in the following way: let M be a primitive sublattice of an even unimodular lattice L satisfying (i)–(iii) above. Suppose that M' is another sublattice of L and there is an isometry $\varphi : M \to M'$, then there exists an isometry $\tilde{\varphi} : L \to L$ of lattices such that the following diagram commutes:

$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[d, hook] & M' \arrow[d, hook] \\ L \arrow[r, "\tilde{\varphi}"] & L \end{tikzcd} \tag{2.1.5}$$

2.2 Fourier-Mukai transform

For a thorough discussion of Fourier-Mukai transform, the reader is referred to the textbook [Huy06], we only collect a few points that will be used later on. Let us begin with the definition of Fourier-Mukai transform

Definition 2.2.1. Let X and Y be smooth projective vaireties, and we let $\pi_X \colon X \times Y \to X$ and $\pi_Y \colon X \times Y \to Y$ be the projection maps. The *Fourier-Mukai transform* with *kernel* $P \in \mathrm{D}^b(X \times Y)$ is the exact functor $\Phi_P \colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ defined by

$$\Phi_P(E) = \pi_{Y*}(P \otimes \pi_X^* E).$$

Let $\Phi_P \colon \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(Y)$ be a Fourier-Mukai transform, then both its left and right adjoint exist and are Fourier-Mukai transforms themselves. In fact suppose $\dim X = m$ and $\dim Y = n$ and we let $\tau \colon Y \times X \to X \times Y$ be the isomorphism interchanging the two factors, then

$$P_R := \tau^*(P^\vee \otimes \pi_X^* \omega_X[m]) \in \mathrm{D}^b(Y \times X)$$

is the kernel for the right-adjoint of Φ_P, and

$$P_L := \tau^*(P^\vee \otimes \pi_Y^* \omega_Y[n]) = P_R \otimes \pi_X^* \omega_X^{-1} \otimes \pi_Y^* \omega_Y[n-m]$$

is the kernel for the left adjoint of Φ_P.

Given two Fourier-Mukai transforms $\Phi_P\colon \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ and $\Phi_Q\colon \mathrm{D}^b(Y) \longrightarrow \mathrm{D}^b(Z)$, their composition $\Phi_Q \circ \Phi_P$ is still a Fourier-Mukai transform. In fact, define the *convolution* by

$$Q * P = \pi_{XZ*}(\pi_{YZ}^* Q \otimes \pi_{XY}^* P) \in \mathrm{D}^b(X \times Z),$$

we have $\Phi_Q \circ \Phi_P = \Phi_{Q*P}$.

We now discuss how the Fourier-Mukai transforms naturally induce maps between algebraic K-groups and between singular cohomologies of the varieties.

Let W be a smooth projective varieties, recall that its *algebraic K-group* $\mathrm{K}_{alg}(W)$ is defined as the Grothendieck group on $\mathrm{Coh}(W)$, the abelian category of coherent sheaves on W. Since W is smooth projective, coherent sheaves on W admit resolution by finite complexes of locally free sheaves, therefore we can take a generator of $\mathrm{K}_{alg}(W)$ the class of locally free sheaves on W. This allows us to define a natural ring structure on $\mathrm{K}_{alg}(W)$ given by linear extending tensor product on locally free sheaves; similarly, extending linearly the operation of dualizing a locally free sheaf on W, we obtain a dualization on $\mathrm{K}_{alg}(W)$. Morphism of varieties $f : W \to V$ naturally induces a pullback map $f^* : \mathrm{K}_{alg}(V) \to \mathrm{K}_{alg}(W)$, which is a ring homomorphism; there is also a pushforward $f_!$ defined for any proper morphism $f : W \to V$, given by $f_!([E]) = \sum(-1)^i[R^i f_*(E)]$ for any coherent sheaf E on W.

Let $e \in \mathrm{K}_{alg}(X \times Y)$, we can define the K-theoretic Fourier-Mukai transform with kernel e to be the group homomorphism $\Phi_e^K : \mathrm{K}_{alg}(X) \to \mathrm{K}_{alg}(Y)$ by

$$\Phi_e^K(f) = \pi_{Y!}(e \cdot (\pi_X^* f)).$$

For any object $F \in \mathrm{D}^b(W)$, we can assign its class $[F] \in \mathrm{K}_{alg}(W)$ given by $[F] = \sum(-1)^i[F^i] = \sum(-1)^i[\mathcal{H}^i(F)]$, it is straightforward to check that the operations of pullback, proper pushforward and dualization are compatible with the assignment $[\cdot] : \mathrm{D}^b(W) \to \mathrm{K}_{alg}(W)$.

- $[\pi^* E] = \pi^*[E]$ for any morphism f between smooth projective varieties;
- $[\pi_* E] = \pi_![E]$ for a proper morphism f (since we have assumed our varieties to be projective, all morphisms between them are necessarily proper);
- $[E^\vee] = [E]^\vee$ ($E^\vee := R\mathcal{H}om(E, \mathcal{O}_W)$ is the derived dual of the object $E \in \mathrm{D}^b(W)$).

Therefore, the Fourier-Mukai transform on the level of derived categories is compatible with the Fourier-Mukai transform on the level of algebraic K-theory That is to say, for any object $E \in \mathrm{D}^b(X \times Y)$ the following diagram commutes

$$\begin{tikzcd} \mathrm{D}^b(X) \arrow[r, "\Phi_E"] \arrow[d, "{[\cdot]}"'] & \mathrm{D}^b(Y) \arrow[d, "{[\cdot]}"] \\ \mathrm{K}_{alg}(X) \arrow[r, "\Phi^K_{[E]}"] & \mathrm{K}_{alg}(Y) \end{tikzcd} . \tag{2.2.1}$$

We will consider singular cohomologies of varieties. Let $e \in \mathrm{H}^*(X \times Y; \mathbb{Q})$, we can define the cohomological Fourier-Mukai transform $\Phi^H_e : \mathrm{H}^*(X;\mathbb{Q}) \to \mathrm{H}^*(Y;\mathbb{Q})$ to be the group homomorphism

$$\Phi^H_e(f) = \pi_{Y*}(e \cdot (\pi^*_X f)).$$

Here $\pi_{Y*} : \mathrm{H}^*(X \times Y; \mathbb{Q}) \to \mathrm{H}^*(Y;\mathbb{Q})$ is the Gysin-homomorphism associated to π_Y. The usual passage from K-theory to cohomology is achieved by the Chern character map, but due to the presence of a twisting by Todd class in the Grothendieck-Riemann-Roch formula, we need to use *Mukai vector* to relate the Fourier-Mukai transform on the level K-theory to that on the level of cohomology:

Definition 2.2.2. Let W be a smooth projective variety. The Mukai vector

$v : \mathrm{K}_{alg}(W) \to \mathrm{H}^*(W;\mathbb{Q})$ is given by

$$v(\cdot) = ch(\cdot)\sqrt{td(W)}.$$

Here $ch(\cdot)$ is the Chern character map $\mathrm{K}_{alg}(W) \to \mathrm{H}^*(W;\mathbb{Q})$ and $td(W)$ denotes the Todd class of W.

By the Grothendieck-Riemann-Roch formula, it is straightforward to check that for any class $e \in \mathrm{K}_{alg}(X \times Y)$, the following diagram commutes

$$\begin{array}{ccc} \mathrm{K}_{alg}(X) & \xrightarrow{\Phi_e^K} & \mathrm{K}_{alg}(Y) \\ {\scriptstyle v}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ \mathrm{H}^*(X,\mathbb{Q}) & \xrightarrow{\Phi_{v(e)}^H} & \mathrm{H}^*(Y,\mathbb{Q}) \end{array} . \tag{2.2.2}$$

From now on, we will write the induced map $\Phi_{[P]}^K$ and $\Phi_{v(P)}^H$ simply as Φ_P^K and Φ_P^H respectively. It should be mentioned that the passage from Φ_P to Φ_P^K and Φ_P^H are functorial, for example if $P \in \mathrm{D}^b(X \times Y)$ and $Q \in \mathrm{D}^b(Y \times Z)$, then the map between K-theories induced by$\Phi_Q \circ \Phi_P = \Phi_{Q*P}$ is given by $\Phi_Q^K \circ \Phi_P^K$.

We need to discuss how Fourier-Mukai transforms interact with some natural pairing defined on $\mathrm{D}^b(W)$, $\mathrm{K}_{alg}(W)$ and $\mathrm{H}^*(W;\mathbb{Q})$:

We can equip $\mathrm{D}^b(W)$ with a *Euler pairing*, give by $\chi(E,F) = \sum(-1)^i ext^i(E,F)$. Similarly an Euler pairing on $\mathrm{K}_{alg}(W)$ is give by $\chi(e,f) = \pi_!(e^\vee \cdot f)$, where $\pi : W \longrightarrow \{pt\}$ is the map to a point; it is a bilinear pairing and can be computed using the Hirzebruch-Riemann-Roch formula. Clearly the assignment $[\,] : \mathrm{D}^b(W) \longrightarrow \mathrm{K}_{alg}(W)$ preserves the Eular pairings on both sides. For $\mathrm{H}^*(W;\mathbb{Q})$, we need to consider the so called Mukai pairing:

Definition 2.2.3. Let $v = \sum v_j \in \bigoplus \mathrm{H}^j(W;\mathbb{Q})$, we let

$$v^\vee := \sum \sqrt{-1}^j v_j \in H^*(W;\mathbb{Q})$$

The *Mukai pairing* on $\mathrm{H}^*(W;\mathbb{Q})$ is the bilinear form

$$(v, v') := \int_W (v^\vee \cdot v') \cdot exp(\frac{c_1(W)}{2})$$

It should be noted that *a priori* neither the Euler pairing nor the Mukai pairing need to be symmetric. An important case when they are both symmetric is when W is an even dimensional Calabi-Yau manifold (for example when W is a K3 surface).

The Mukai vector $v : \mathrm{K}_{alg}(W) \to \mathrm{H}^*(W;\mathbb{Q})$ also preserves the bilinear pairings on both sides, meaning

$$\chi(e, f) = (v(e), v(f))$$

which is a easy consequence of Hirzebruch-Riemann-Roch formula.

When the Fourier-Mukai transform $\Phi_P : \mathrm{D}^b(X) \xrightarrow{\sim} \mathrm{D}^b(Y)$ is an equivalence, it is a straightforward computation to check that the induced isomorphisms

$$\begin{aligned} \Phi_P^K &: \mathrm{K}_{alg}(X) \xrightarrow{\sim} \mathrm{K}_{alg}(Y) \\ \Phi_P^H &: \mathrm{H}^*(X;\mathbb{Q}) \xrightarrow{\sim} \mathrm{H}^*(Y;\mathbb{Q}) \end{aligned}$$

are isometric with respect to these bilinear pairings.

We summarize these discussions in the form of the following

Proposition 2.2.4. *Let X and Y be smooth projective varieties and consider Fourier-Mukai transform $\Phi_P : D^b(X) \longrightarrow D^b(Y)$ associated to some $P \in D^b(X \times Y)$, then the following diagram commutes*

$$\begin{tikzcd} D^b(X) \arrow[r, "{[\ ]}"] \arrow[d, "\Phi_P"] & K_{alg}(X) \arrow[r, "v"] \arrow[d, "\Phi_P^K"] & H^*(X,\mathbb{Q}) \arrow[d, "\Phi_P^H"] \\ D^b(Y) \arrow[r, "{[\ ]}"] & K_{alg}(Y) \arrow[r, "v"] & H^*(Y,\mathbb{Q}) \end{tikzcd} . \qquad (2.2.3)$$

Moreover, the horizontal maps preserve various pairings on these spaces, and that is also the case for the vertical maps when Φ_P is an equivalence.

At last we state a criterion for the existence of first order deformation of Fourier-Mukai transform along some first order deformations of the varieties. To be more precise about what we mean by deforming a Fourier-Mukai transform, let $A_1 = \mathbb{C}[t]/(t^2)$ and $X_1 \to A_1$ and $Y_1 \to A_1$ be first order deformations of smooth projective varieties X and Y (so, for example, $X_1 \to A_1$ is a flat proper morphism of schemes such that $X_1 \times_{A_1} \mathrm{Spec}\mathbb{C} \simeq X$). Suppose we have a Fourier-Mukai transform $\Phi_P : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$ defined by some kernel $P \in \mathrm{D}^b(X \times Y)$, we seek for an object $P_1 \in \mathrm{D}_{Perf}(X_1 \times_{A_1} Y_1)$ whose derived restriction to $X \times Y$ is P. We can think of the Fourier-Mukai transform $\Phi_{P_1} : \mathrm{D}_{Perf}(X_1) \to \mathrm{D}_{Perf}(Y_1)$ as a first order deformation of $\Phi_P : \mathrm{D}^b(X) \to \mathrm{D}^b(Y)$.

The statement of the criterion involves the so called *Atiyah class* of a complex.

Definition 2.2.5. Let W be a smooth projective variety and $F \in \mathrm{D}^b(W)$ be a complex, the Atiyah class $At(F) \in \mathrm{Ext}^1_W(F, F \otimes \Omega_W)$ of F is defined to be the extension class corresponding the exact triangle

$$F \otimes \Omega_W \to \Phi_{\mathcal{O}_{2\Delta_W}}(F) \to F$$

induced by the obvious short exact sequence of Fourier-Mukai kernels

$$0 \to \Delta_* \Omega_W \to \mathcal{O}_{2\Delta_W} \to \mathcal{O}_{\Delta_W} \to 0$$

where $2\Delta_W$ is the subscheme of $W \times W$ defined by the square $\mathcal{I}^2_\Delta$ of the ideal sheaf of the diagonal $\Delta_W \subset W \times W$. The extension class

$$At_W \in \mathrm{Ext}^1_{W \times W}(\mathcal{O}_{\Delta_W}, \Delta_* \Omega_W)$$

given by the above short exact sequence is called the *universal Atiyah class.*

Now the criterion for the existence of first order deformation of Fourier-Mukai kernel is given by the following

Theorem 2.2.6. ([HMS09, Theorem 3.1] or [HT10, Corollary 3.4]) *Let $\kappa_X \in H^1(T_X)$ and $\kappa_Y \in H^1(T_Y)$ be the Kodaira-Spencer classes corresponding to the first order deformations $X_1 \to A_1$ and $Y_1 \to A_1$ of X and Y respectively, and we let $P \in D^b(X \times Y)$, then there exists a complex $P_1 \in D_{Perf}(X_1 \times_{A_1} Y_1)$ whose derived restriction to $X \times Y$ is isomorphic to P if and only if*

$$(\kappa_X, \kappa_Y) \circ At(P) = 0 \in Ext^2_{X\times Y}(P, P).$$

Remark. In fact, according to [HT10], the statement of the theorem should be valid if X and Y are defined over some complex Artinian space A. In this case the product $X \times Y$ is to be interpreted as $X \times_A Y$ and $P \in \mathrm{D}_{Perf}(X \times_A Y)$ a perfect complex. The two Kodaira-Spencer classes are to be interpreted as relative over A, meaning $\kappa_X \in \mathrm{H}^1(T_{X/A})$ and $\kappa_Y \in \mathrm{H}^1(T_{Y/A})$.

Digression on Topological K-theory

Topological K-theory is a subject of algebraic topology, we only use it to define a topological invariants for our K3 categories (see definition in 4.1), so we will only give a very brief account. For more detail on topological K-theory, the reader is referred to [AH61] and [AH62].

Let W be a finite CW complex, it has a *topological K-theory* $\mathrm{K}_{top}(W) = \mathrm{K}^0(W) \oplus \mathrm{K}^1(W)$, where

1. $\mathrm{K}^0(W) := \mathrm{K}(W)$ is the Grothendieck group of topological $\mathbb{C}$-vector bundles on W;

2. $\mathrm{K}^1(W) := \tilde{\mathrm{K}}(S(W_+))$, where W_+ is the disjoint union $W \sqcup \{x_0\}$, $S(-)$

is the toplogical suspension and $\tilde{\mathrm{K}}(-)$ is the kernal of the map $\mathrm{K}(-) \to \mathrm{K}(\text{base point}) = \mathbb{Z}$ induced by inclusion of the base point.

Similar to algebraic K-theory, $\mathrm{K}^0(W)$ becomes a commutative ring by extending tensor product between topological vector bundles; this ring structure extends to a $\mathbb{Z}_2$–graded commutative ring structure on $\mathrm{K}_{top}(W)$ (i.e. $\mathrm{K}^0(W) \cdot \mathrm{K}^1(W) \subset \mathrm{K}^1(W)$,$\mathrm{K}^1(W) \cdot \mathrm{K}^1(W) \subset \mathrm{K}^0(W)$ and $ab = (-1)^{ij}ba$ if $a \in \mathrm{K}^i(W)$ and $b \in \mathrm{K}^j(W)$). Any continous map $f : W \to V$ induces a $\mathbb{Z}_2$–graded ring homomorphism $f^* : \mathrm{K}_{top}(V) \to \mathrm{K}_{top}(W)$.

Apart from the natural pullback induced by continuous maps, topological K-theory has pushforward in some specific setting. Let $f : W \to V$ be a proper morphism between smooth projective varieties, viewed as a holomorphic map between compact complex manifolds, then there is a pushforward map $f_! : \mathrm{K}_{top}(W) \to \mathrm{K}_{top}(V)$ which is a $\mathbb{Z}_2$–graded abelian group homomorphism. This pushforward map is compatible with the pushforward $f_! : \mathrm{K}_{alg}(W) \to \mathrm{K}_{alg}(V)$ between algebraic K-groups defined earlier via the natural inclusion $\mathrm{K}_{alg}(\cdot) \hookrightarrow \mathrm{K}_{top}(\cdot)$ ([AH62]). Therefore, we can also define a (topological K-theoretic) Fourier-Mukai transform $\mathrm{K}_{top}(X) \to \mathrm{K}_{top}(Y)$ associated to any element $e \in \mathrm{K}_{top}(X \times Y)$ (by the same formula as in the case of algebraic K-theory) which restricts to the algebraic K-theoretic Fourier-Mukai transform by the inclusion $\mathrm{K}_{alg}(\cdot) \hookrightarrow \mathrm{K}_{top}(\cdot)$.

Given $\Phi_P : \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(Y)$ a Fourier-Mukai transform, we can think of the class $[P]$ as an element of $\mathrm{K}_{top}(X \times Y)$ via the inclusion $\mathrm{K}_{alg}(X \times Y) \hookrightarrow \mathrm{K}_{top}(X \times Y)$ and hence we get a induced map $\Phi_P^{\mathrm{K}_{top}} : \mathrm{K}_{top}(X) \longrightarrow \mathrm{K}_{top}(Y)$.

One can also define a *Euler pairing* χ on $\mathrm{K}_{top}(W)$ similarly as in the case of $\mathrm{K}_{alg}(W)$, which restricts to the Euler pairings on $\mathrm{K}_{alg}(W)$ discussed earlier. When Φ_P is a Fourier-Mukai equivalence, the induced isomorphism $\Phi_P^{\mathrm{K}_{top}}$ is also isometric with respect to Euler pairing.

There is also a Chern character map $ch\colon \mathrm{K}_{top}(W) \to \mathrm{H}^*(W;\mathbb{Q})$ defined on the topological K-theory satisfying the following properties:

Theorem 2.2.7. *(1) $ch\colon K_{top}(W) \otimes \mathbb{Q} \to H^*(W;\mathbb{Q})$ is a $\mathbb{Z}_2$-graded isomorphism*

(2) If $H^*(W,\mathbb{Z})$ is torsion free, then $ch : K_{top}(W) \to H^*(W;\mathbb{Q})$ is an injection and $K_{top}(W)$ is necessarily torsion free

Using this, we can also define the Mukai vector $v\colon \mathrm{K}_{top}(W) \longrightarrow \mathrm{H}^*(W;\mathbb{Q})$ on topological K-group by $v(\cdot) = ch(\cdot)\sqrt{td(W)}$ as well, which restricts to the Mukai vector defined on algebraic K-groups when W is an algebraic variety.

The Mukai vector $v\colon \mathrm{K}_{top}(W) \longrightarrow \mathrm{H}^*(W;\mathbb{Q})$ on topological K-group also preserves the bilinear pairings on both sides (Euler pairing on $\mathrm{K}_{top}(W)$ and Mukai pairing on $\mathrm{H}^*(W;\mathbb{Q})$)

Lastly, we observe that because $\sqrt{td(W)}$ is an invertible element contained in $\mathrm{H}^{even}(W;\mathbb{Q})$, both statements in Theorem 2.2.7 hold true if we replace the Chern character by the Mukai vector v.

2.3 Hochschild (co)homology

For a more comprehensive discussion of Hochschild (co)homologies of algebraic varieties, please see the expository articles [Căl03] and [Căl05]. For our purposes, we are mostly interested in the interaction between Hochschild (co)homologies and Fourier-Mukai transforms.

Let W be a smooth projective variety and we let $\Delta : W \to W \times W$ be the diagonal morphism, $\mathcal{O}_\Delta := \Delta_*\mathcal{O}_W$ be the structure sheaf of the diagonal, ω_W be the canonical bundle of W and $S_\Delta \in \mathrm{D}^b(W \times W)$ be the object $\Delta_*\omega_W[\dim W]$ (therefore S_Δ^{-1} means $\Delta_*\omega_W^{-1}[-\dim W]$)

Definition 2.3.1. The *Hochschild cohomology* $\mathrm{HH}^i(W)$ of W is defined as the vector space

$$\mathrm{Ext}^{i}_{W\times W}(\mathcal{O}_\Delta, \mathcal{O}_\Delta).$$

Its Hochschild homology $\mathrm{HH}_j(W)$ is defined as the vector space

$$\mathrm{Ext}^{-j}_{W\times W}(S_\Delta^{-1}, \mathcal{O}_\Delta)$$

Under the Yoneda pairing, The Hochschild cohomology $\mathrm{HH}^*(W)$ is naturally a graded commutative algebra and the Hochschild cohomology $\mathrm{HH}_*(W)$ a graded module over $\mathrm{HH}^*(W)$.

Let $\Phi_P\colon \mathrm{D}^b(X) \longrightarrow \mathrm{D}^b(Y)$ be a Fourier-Mukai transform between smooth projective varieties, we now describe its induced map on Hochschild homology $\Phi_P^{HH_*}\colon \mathrm{HH}_*(X) \longrightarrow \mathrm{HH}_*(Y)$:

The convolution of Fourier-Mukai kernels give us functors

$$\mathrm{D}^b(X\times X) \xrightarrow{P*} \mathrm{D}^b(X\times Y) \xrightarrow{*P_R} \mathrm{D}^b(Y\times Y).$$

Direct computation gives

$$(P * \mathcal{O}_{\Delta_X}) * P_R = P * P_R$$

and

$$(P * \Delta_{X*}\omega_X^{-1}) * P_R = P * P_L \otimes \pi_1^*\omega_Y^{-1}[m-n].$$

Therefore there is an induced map

$$\begin{aligned}
\mathrm{HH}_*(X) \quad &= \quad \mathrm{Ext}^{m-*}_{X\times X}(\Delta_{X*}\omega_X^{-1}, \mathcal{O}_{\Delta_X}) \\
&\to \mathrm{Ext}^{n-*}_{Y\times Y}(P * P_L \otimes \pi_1^*\omega_Y^{-1}, P * P_R) \\
&\to \mathrm{Ext}^{n-*}_{Y\times Y}(\mathcal{O}_{\Delta_Y} \otimes \pi_1^*\omega_Y^{-1}, \mathcal{O}_{\Delta_Y}) = \mathrm{HH}_*(Y)
\end{aligned}$$

Where the last map is given by composing with the morphisms

$$\mathcal{O}_{\Delta_Y} \xrightarrow{\eta} P * P_L, \quad P * P_R \xrightarrow{\epsilon} \mathcal{O}_{\Delta_Y}.$$

inducing the unit and counit in the adjunction. This is the desired induced map

$$\Phi_P^{HH_*} : \mathrm{HH}_*(X) \to \mathrm{HH}_*(Y).$$

When Φ_P is an equivalence, we also have an induced isomorphism

$$\Phi_P^{HH^*} : \mathrm{HH}^*(Y) \xrightarrow{\sim} \mathrm{HH}^*(X)$$

between the Hochschild cohomologies:

If Φ_P is an exact equivalence, then P_R is the Fourier-Mukai kernel of the inverse Φ_P^{-1}. Therefore

$$\mathrm{D}^b(X \times X) \xrightarrow{*P} \mathrm{D}^b(X \times Y)$$

gives an isomorphism

$$\mathrm{Ext}^*_{X \times Y}(P, P) \xrightarrow{P_R *} \mathrm{Ext}^*_{X \times X}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X})$$

with inverse given by

$$\mathrm{Ext}^*_{X \times X}(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \xrightarrow{*P} \mathrm{Ext}^*_{X \times Y}(P, P)$$

Similarly we have isomorphism

$$\mathrm{Ext}^*_{X \times X}(\mathcal{O}_{\Delta_Y}, \mathcal{O}_{\Delta_Y}) \xrightarrow{P*} \mathrm{Ext}^*_{X \times Y}(P, P)$$

The composition

$$\mathrm{Ext}^*_{Y\times Y}(\mathcal{O}_{\Delta_Y},\mathcal{O}_{\Delta_Y}) \xrightarrow{P_*} \mathrm{Ext}^*_{X\times Y}(P,P) \xrightarrow{P_{R*}} \mathrm{Ext}^*_{X\times X}(\mathcal{O}_{\Delta_X},\mathcal{O}_{\Delta_X})$$

is the desired isomorphism $\Phi_P^{HH^*} : \mathrm{HH}^*(Y) \xrightarrow{\sim} \mathrm{HH}^*(X)$.

The Hochschild structure $(\mathrm{HH}^*(W), \mathrm{HH}_*(W))$ associated to a smooth projective variety W is closely related to the *harmonic structure* on W, which consists of the spaces

$$\mathrm{HT}^i(W) := \bigoplus_{p+q=i} \mathrm{H}^p(\bigwedge^q T_W)$$

$$\mathrm{H}\Omega_j(W) := \bigoplus_{q-p=j} \mathrm{H}^p(\Omega^q_W)$$

Similar to the Hochschild structure, $\mathrm{HT}^*(W)$ is naturally a graded ring and $\mathrm{H}\Omega_*(W)$ a graded module over $\mathrm{HT}^*(W)$:

- To define the graded ring structure on $\mathrm{HT}^*(W)$, we can identify $\mathrm{H}^p(\bigwedge^q T_W)$ with the Dolbeault cohomology space $\mathrm{H}^{0,p}_{\bar\partial}(\bigwedge^q T_W)$, therefore the map

$$\begin{array}{rcl} \mathcal{A}^{0,p}(\bigwedge^q T_W) \otimes \mathcal{A}^{0,p'}(\bigwedge^{q'} T_W) & \longrightarrow & \mathcal{A}^{0,p+p'}(\bigwedge^{q+q'} T_W) \\ (\omega\otimes s, \omega'\otimes s') & \longmapsto & (\omega\wedge\omega')\otimes(s\wedge s') \end{array}$$

  naturally induces a map $\mathrm{HT}^i(W)\otimes \mathrm{H}T^{i'}(W) \to \mathrm{H}T^{i+i'}(W)$, turning $\mathrm{HT}^*(W)$ into a graded commutative algebra;

- The graded $\mathrm{HT}^*(W)$-module structure on $\mathrm{H}\Omega_*(W)$ can be defined in the following way: we identify $\mathrm{H}^p(\Omega^q_W)$ with the Dolbeault space $\mathrm{H}^{0,p}_{\bar\partial}(\Omega^q_W)$ and consider the map

$$\begin{array}{rcl} \mathcal{A}^{0,p}(\Omega^q_W) \otimes \mathcal{A}^{0,p'}(\bigwedge^{q'} T_W) & \longrightarrow & \mathcal{A}^{0,p+p'}(\bigwedge^{q-q'} T_W) \\ (\omega \otimes \eta, \omega' \otimes s') & \longmapsto & (\omega \wedge \omega') \otimes (s' \lrcorner \eta) \end{array}$$

which induces a map $\mathrm{H}T^i(W) \otimes \mathrm{H}\Omega_j(W) \to \mathrm{H}\Omega_{j-i}(W)$.

[Swa96] assets that the Hochschild-Kostant-Rosenberg isomorphism holds for any quasi-projective varieties, that is to say, we have isomorphisms between graded vector spaces

$$I^{HKR} \colon \mathrm{HH}^*(W) \xrightarrow{\cong} \mathrm{HT}^*(W),$$

$$I_{HKR} \colon \mathrm{HH}_*(W) \xrightarrow{\cong} \mathrm{H}\Omega_*(W).$$

Unfortunately these isomorphisms have some deficiencies. On the one hand, $(I^{HKR}, I_{HKR}) : (\mathrm{HH}^*(W), \mathrm{HH}_*(W)) \to (\mathrm{HT}^*(W), \mathrm{H}\Omega_*(W))$ does not preserve the ring-module structure on both sides we discussed above. On the other hand, the usual Hodge decomposition theorem allows us embed $\mathrm{H}\Omega_*(W)$ as a subspace of the singular cohomology $\mathrm{H}^*(W;\mathbb{C})$, therefore we can ask whether the cohomological Fourier-Mukai Φ^H_P restricts, via $\mathrm{HH}_*(\cdot) \xrightarrow{I_{HKR}} \mathrm{H}\Omega_*(\cdot) \subset \mathrm{H}(\cdot;\mathbb{C})$, to the induced map $\Phi^{HH_*}_P$; the answer is negative in general.

It turns out that the solution to both problems is to consider a twisted version of the HKR isomorphims, due to Kontsevich, they are given by the following isomorphisms of graded vector space

$$I^K \colon \mathrm{HH}^*(W) \xrightarrow{I^{HKR}} \mathrm{HT}^*(W) \xrightarrow{\lrcorner\sqrt{td(W)}^{-1}} \mathrm{HT}^*(W)$$

$$I_K \colon \mathrm{HH}_*(W) \xrightarrow{I_{HKR}} \mathrm{H}\Omega_*(W) \xrightarrow{\wedge\sqrt{td(W)}} \mathrm{H}\Omega_*(W)$$

and satisfy

Theorem 2.3.2. *(1)* ([CRV12]) *The following ring-module map*

$$(I^K, I_K)\colon (HH^*(W), HH_*(W)) \to (HT^*(W), H\Omega_*(W))$$

is an isomorphism of ring-module structures, meaning $I^K\colon HH^*(W) \to HT^*(W)$ *is a ring isomorphism and the following diagram commutes*

$$\begin{tikzcd} HH^i(W) \otimes HH_j(W) \arrow[r] \arrow[d, "I^K \otimes I_K"'] & HH_{j-i}(W) \arrow[d, "I_K"] \\ HT^i(W) \otimes H\Omega_j(W) \arrow[r] & H\Omega_{j-i}(W) \end{tikzcd} . \tag{2.3.1}$$

(2) ([MS09, Theorem 1.2]). *Let* $\Phi_P : D^b(X) \to D^b(Y)$ *be a Fourier-Mukai transform, then the following diagram commutes:*

$$\begin{tikzcd} HH_*(X) \arrow[r, "I_K"] \arrow[d, "\Phi_P^{HH_*}"'] & H\Omega_*(X) \subset H^*(X;\mathbb{C}) \arrow[d, "\Phi_{v(P)}^H"] \\ HH_*(Y) \arrow[r, "I_K"] & H\Omega_*(Y) \subset H^*(Y;\mathbb{C}) \end{tikzcd} . \tag{2.3.2}$$

Chapter 3

Hodge theory of Gushel-Mukai and Cubic Fourfold

3.1 Gushel-Mukai fourfolds

Definition 3.1.1. For $n \in \{2, 3, 4, 5, 6\}$, a *Gushel Mukai variety* X of dimension n (or Gushel-Mukai n-fold) is defined to be a smooth dimensionally transversal intersection

$$X = C\mathrm{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q,$$

where $C\mathrm{Gr}(2, V_5) \subset \mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5)$ is the projective cone of $\mathrm{Gr}(2, V_5)$ in $\mathbb{P}(\bigwedge^2 V_5)$ under the PlÃŒcker embedding; $\mathbb{P}(W) \subset \mathbb{P}(\mathbb{C} \oplus \bigwedge^2 V_5)$ is a linear section of dimension $n + 4$ and Q is a quadric hypersurface in $\mathbb{P}(W)$.

A Gushel-Mukai variety is naturally polarized by the restriction H of the hyperplane class on $\mathbb{P}(W)$. Gushel-Mukai variety X of dimension n is characterized by the following properties:

1. $H^n = 10$

2. $-K_X = (n - 2)H$

3. $\mathrm{Pic}(X) = \mathbb{Z}H$ when $n > 2$; if $n = 2$ then (X, H) is a Brill-Noether general K3 surface.

Any smooth projective polarized variety of dimension $2 \leq n \leq 6$ satisfying (1)-(3) above is necessarily a Gushel-Mukai variety (see [Gus82] and [Muk89], also see [DK18a] for a more general statement for not necessarily smooth Gushel-Mukai varieties).

The intersection $C\mathrm{Gr}(2, V_5) \cap \mathbb{P}(W) \cap Q$ that defines X does not contain the vertex of the cone since we assumed that X is smooth. Therefore the projection from the vertex defines a regular morphism of varieties

$$\gamma_X \colon X \longrightarrow \mathrm{Gr}(2, V_5)$$

called the *Gushel map*.

To discuss the moduli space of Gushel-Mukai varieties, we first define the appropriate moduli functor:

A family of (polarized) Gushel-Mukai varieties of dimension n over a $\mathbb{C}$-scheme S is a pair $(\pi_{\mathcal{X}} \colon \mathcal{X} \longrightarrow S, \mathcal{H})$, where $\pi_{\mathcal{X}} \colon \mathcal{X} \longrightarrow S$ is a smooth and proper morphism of relative dimension n and $\mathcal{H} \in \mathrm{Pic}_{\mathcal{X}/S}(S)$ is a $\pi_{\mathcal{X}}$-ample divisor class, such that for every geometric point s of S, the pair $(\mathcal{X}_s, \mathcal{H}_s)$ is a smooth polarized Gushel-Mukai variety of dimension n.

A morphism of families of Gushel-Mukai varieties is defined in the usual way. The assignment to every scheme S the groupoid $\mathfrak{GM}^n(S)$ of families of Gushel-Mukai n-folds over S gives us a category $\mathfrak{GM}^n$ fibered in groupoids, over the category of $\mathbb{C}$-schemes. We have the following

Proposition 3.1.2. ([KP18, Proposition A.2]). *For $n \in \{2, \ldots, 6\}$, the fibered category $\mathfrak{GM}^n$ is a smooth and irreducible Delign-Mumford stack of finite type over $\mathbb{C}$. It has dimension $25 - (5-n)(6-n)/2$.*

We are interested in the case $n = 4$, in this case we denote by $\mathcal{GM}$ the coarse moduli space of $\mathfrak{GM}^4$ and we will fix once and for all an etale presentation

$\mathcal{G} \longrightarrow \mathfrak{GM}^4$ by some smooth $\mathbb{C}$-scheme $\mathcal{G}$, there is a family of Gushel-Mukai fourfold $\mathcal{X}_U \longrightarrow \mathcal{G}$ determined by the map $\mathcal{G} \longrightarrow \mathfrak{GM}^4$. By construction all (polarized) Gushel-Mukai fourfold appears as a fiber of the family $\mathcal{X}_U$, hence $\mathcal{G}$ is a smooth parameter space parametrizing all Gushel-Mukai fourfold. By ause of language, we will call the family $\mathcal{X}_U \to \mathcal{G}$ the *universal family* of Gushel-Mukai fourfolds. Note that the natural map $\mathcal{G} \to \mathcal{GM}$ is an open map.

3.2 Period map and period domain

In this subsection we give a brief summary of the Hodge theory of Gushel-Mukai fourfolds and cubic fourfolds.

•Gushel-Mukai fourfolds

Let X be a Gushel-Mukai fourfold, its Hodge diamond is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 22 & & 1 & & 0 \end{array} . \tag{3.2.1}$$

The middle cohomology $\mathrm{H}^4(X;\mathbb{Z})$, equipped with the intersection pairing, is an odd unimodular lattice of signature $(22,2)$. Therefore it is isometric to $I_{22,2}$. The cohomology $\mathrm{H}^4(\mathrm{Gr}(2,5);\mathbb{Z})$ embeds in $\mathrm{H}^4(X;\mathbb{Z})$ as a rank 2 positive-definite primitive sublattice via the pullback γ_X^* by the Gushel map $\gamma_X \colon X \to \mathrm{Gr}(2,5)$; it has intersection matrix

$$\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$$

with respect to the basis $\{\sigma_{1,1}|_X, \sigma_2|_X\}$, where $\sigma_{1,1}$ and σ_2 are the Schubert cycles generating $\mathrm{H}^4(\mathrm{Gr}(2,5);\mathbb{Z})$. The sublattice

$$\mathrm{H}^4(X;\mathbb{Z})_{00} := \{x \in \mathrm{H}^4(X;\mathbb{Z}) : x \cdot \gamma_X^* \mathrm{H}^4(\mathrm{Gr}(2,5);\mathbb{Z}) = 0\}$$

is called the *vanishing cohomology* of X. Now to discuss period map and period domain for Gushel-Mukai fourfolds, one needs to choose some isometry of lattices $I_{22,2} \tilde{\to} \mathrm{H}^4(X;\mathbb{Z})$ so that the Hodge structures on $\mathrm{H}^4(X;\mathbb{Z})$ can be parameterized by points in the period domain. The subgroup $\gamma_X^* \mathrm{H}^4(\mathrm{Gr}(2,5);\mathbb{Z})$ serves as a lattice polarization of X and is fixed by the monodromy action (in any family of Gushel-Mukai fourfolds), therefore we need to fix this subgroup of $I_{22,2}$:

Let $(e_1, \ldots, e_{22}, f_1, f_2)$ be the standard basis of $I_{22,2}$, we fix the following distinguished rank 2 sublattice $\Lambda_G = \langle u, v \rangle$ of $I_{22,2}$ where $u = e_1 + e_2$ and $v = e_1 + \cdots + e_{22} - 3f_1 - 3f_2$; note that it has intersection matrix $\begin{pmatrix} 2 & 2 \\ 2 & 4 \end{pmatrix}$. It can be shown ([DIM14, Proposition 5.1]) that the orthogonal complement $\Lambda := \Lambda_G^{\perp}$ is an *even* lattice with signature $(20, 2)$ and is isometric to

$$E_8^2 \oplus U^2 \oplus A_1^2. \tag{3.2.2}$$

Definition 3.2.1. A *marking* on X is an isometry of lattices $f \colon I_{22,2} \xrightarrow{\sim} \mathrm{H}^4(X;\mathbb{Z})$ satisfying the condition $f(u) = \sigma_{1,1}|_X$ and $f(v) = \sigma_2|_X$.

Any marking f on X necessarily maps the even sublattice Λ of $I_{22,2}$ isometrically onto the vanishing cohomology $\mathrm{H}^4(X,\mathbb{Z})_{00}$. Consider the 20-dimensional complex manifold

$$\{v \in \mathbb{P}(\Lambda \otimes \mathbb{C}) | v \cdot v = 0,\ v \cdot \bar{v} < 0\}.$$

The group

$$\tilde{O}(\Lambda) = \{g \in O(I_{22,2}) : g|_{\Lambda_G} = id\}$$

acts properly discontinously ([Huy16, Chapter6, Remark1.10]) on $\Omega(\Lambda)$, the quotient $\mathcal{D} := \tilde{O}(\Lambda)\backslash\Omega(\Lambda)$ is called the *period domain* of Gushel-Mukai fourfolds. By a theorem of Baily and Borel ([BB66]), $\mathcal{D}$ is an irreducible quasi-projective variety.

Remark. We abuse the notation by confusing points in projective space with the vector spanning this point as a 1-dimensional subspace.

There is a *period map*

$$\wp \colon \mathcal{GM} \longrightarrow \mathcal{D},$$

which sends a fourfold $X \in \mathcal{GM}$ to the $\tilde{O}(\Lambda)$-orbit of the point $f^{-1}\mathrm{H}^{3,1}(X) \in \Omega(\Lambda)$, where f is any marking on X. Notice that a different choice of marking induces an isometry of $I_{22,2}$ fixing Λ_G, hence $\wp$ is well-defined. The argument of [Has00, Proposition 2.2.2] can be applied in this case and we can conclude that $\wp$, a prior only holomorphic, is in fact algebraically defined. Moreover, by [DK18b, Corollary 6.3], we can conclude that $\wp$ is a dominant morphism with irreducible fibers.

Remark 3.2.2. Local behavior of the period map is encoded by parallel transports of the local system of $\mathrm{H}^4(X,\mathbb{Z})$. We let $\pi\colon \mathcal{X} \to \Delta$ be a smooth family of Gushel-Mukai fourfolds over a complex disc $\Delta \subset \mathcal{G}$, then any choice of marking $f_0\colon I_{22,2} \xrightarrow{\sim} \mathrm{H}^4(X_0;\mathbb{Z})$ determines a *unique* lifting $\Delta \longrightarrow \Omega(\Lambda)$ of the map $\Delta \subset \mathcal{G} \xrightarrow{\wp} \mathcal{D} = \tilde{O}(\Lambda)\backslash\Omega(\Lambda)$. In fact, if we let $p_t\colon \mathrm{H}^4(X_t;\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(X_0;\mathbb{Z})$ be the parallel transport operator from t to 0 for the local system $R^4\pi_*\mathbb{Z}$, which is path-independent since Δ is a disc, then the map

$$\begin{array}{ccc} \Delta & \longrightarrow & \Omega(\Lambda) \\ t & \longmapsto & f_0^{-1}p_t\mathrm{H}^{3,1}(X_t) \end{array}$$

is the unique lifting of $\Delta \xrightarrow{\wp} \tilde{O}(\Lambda)\backslash\Omega(\Lambda)$ sending $0 \in \Delta$ to $f_0^{-1}\mathrm{H}^{3,1}(X_0) \in \Omega(\Lambda)$.

•Cubic fourfolds

Let Y be a Cubic fourfold, its Hodge diamond is

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 0 & & 0 & & & \\ & & 0 & & 1 & & 0 & & \\ & 0 & & 0 & & 0 & & 0 & \\ 0 & & 1 & & 21 & & 1 & & 0 \end{array} . \tag{3.2.3}$$

The middle cohomology $\mathrm{H}^4(Y;\mathbb{Z})$ is an odd unimodular lattice with signature $(21,2)$ and hence isomorphic to $I_{21,2}$. Let $h_Y \in \mathrm{H}^2(Y;\mathbb{Z})$ be the hyperplane class, the primitive cohomology is given by $\mathrm{H}^4(Y;\mathbb{Z})_0 = \langle h_Y^2\rangle^{\perp}$. Once again, we need to fix a distinguished element $\delta \in I_{21,2}$ with $\delta^2 = 3$. We choose it in the following way:

It is known that $I_{21,2}$ is isometric to

$$E_8^{\oplus 2} \oplus U^{\oplus 2} \oplus I_{3,0}.$$

We let $(\epsilon_1, \epsilon_2, \epsilon_3)$ be the standard basis of $I_{3,0}$ and take $\delta = \epsilon_1 + \epsilon_2 + \epsilon_3$. The orthogonal complement $\Lambda' := \langle\delta\rangle^{\perp}$ is an even lattice of signature $(20,2)$ isometric to

$$E_8^2 \oplus U^2 \oplus A_2,$$

where A_2 can be taken as the orthogonal complement of δ in $I_{3,0}$ and is spanned by $\epsilon_1 - \epsilon_2$ and $\epsilon_2 - \epsilon_3$. Similar to the case of Gushel-Mukai fourfolds, an isometry of lattices $f'\colon I_{21,2} \xrightarrow{\sim} \mathrm{H}^4(Y;\mathbb{Z})$ sending the distinguished element δ to square of the hyperplane class h_Y^2 is called a *marking* on Y. Under any marking, the lattice Λ' is mapped isometrically onto the primitive cohomology $\mathrm{H}^4(Y;\mathbb{Z})_0$

Similarly we consider the 20-dimensional complex manifold

$$\{v \in \mathbb{P}(\Lambda' \otimes \mathbb{C}) | v \cdot v = 0, \ v \cdot \bar{v} < 0\},$$

the group $\tilde{O}(\Lambda') = \{g \in O(I_{21,2}) : g(\delta) = \delta\}$ acts properly discontinously on Ω'; the quotient $\mathcal{D}' := \tilde{O}(\Lambda')\backslash\Omega(\Lambda')$ is the *period domain* of cubic fourfolds. By the same theorem of Baily and Borel, $\mathcal{D}'$ is an irreducible quasi-projective variety. Let $\mathcal{C}$ be the coarse moduli space of smooth cubic fourfolds ([Laz10]), we also have a well-defined period map

$$\wp' : \mathcal{C} \to \mathcal{D}'$$

sending any cubic fourfold Y to the $\tilde{O}(\Lambda')$-orbit of the point $f'^{-1}(\mathrm{H}^{3,1}(Y)) \in \Omega(\Lambda')$, where f' is any marking on Y. Moreover, by the Torelli theorem for cubic fourfold ([Vois86]), the period map $\wp'$ is an open immersion of analytic spaces; in fact it embeds $\mathcal{C}$ as a Zariski dense open subset of $\mathcal{D}'$ ([Has00, Proposition 2.2.2]).

3.3 Hodge special fourfolds

Definition 3.3.1. A Gushel-Mukai fourfold X is *Hodge special* if $\mathrm{H}^{2,2}(X;\mathbb{Z})$ contains a rank 3 primitive sublattice K containing $\gamma_X^*\mathrm{H}^4(\mathrm{Gr}(2,5),\mathbb{Z})$; the sublattice K will be called a *labelling* on X, we denote this sublattice by K_d if we want to emphasize that K has discriminant d; its orthogonal complement $K^\perp$ is called the *non-special cohomology* of X. Similarly, a cubic fourfold Y is *Hodge special* if $\mathrm{H}^{2,2}(Y;\mathbb{Z})$ contains a rank 2 primitive sublattice K' containing the square of the hyperplane class h_Y^2. Again K' is called a *labelling* on Y, we write K'_d to mean that K' has discriminant d; $K'^\perp$ is called the *non-special cohomology* of Y.

Sometimes we abuse the language and say "X has discriminant d", this just means that X has a labelling with discriminant d. It turns out that Hodge special

Gushel-Mukai fourfolds and Cubic fourfolds are parameterized by countable unions of hypersurfaces in their moduli spaces, indexed by the discriminants. We ellaborate this phenomenon:

•Gushel-Mukai fourfolds

Let L_d be a rank 3 positive definite primitive sublattice of $I_{22,2}$ of discriminant d containing the distinguished sublattice Λ_G. By [DIM14, Lemma 6.1], such lattices must have descriminant $d > 0$ and $d \equiv 0, 2, 4 \pmod 8$. Consider the hypersurface

$$\Omega(L_d^{\perp}) = \mathbb{P}(L_d^{\perp} \otimes \mathbb{C}) \cap \Omega(\Lambda)$$

in $\Omega(\Lambda)$. Let $\mathcal{D}_{L_d}$ be the image of $\Omega(L_d^{\perp})$ in $\mathcal{D}$ under the quotient map $\Omega(\Lambda) \to \mathcal{D}$. By [DIM14, Proposition 6.2], we have:

1. when $d \equiv 0 \pmod 4$, the set of rank 3 primitive sublattices $L_d \subset I_{22,2}$ containing Λ_G form a single $\tilde{O}(\Lambda)$-orbit; consequently$\mathcal{D}_{L_d}$is an irreducible divisor depending only on the integer d, hence we can denote $\mathcal{D}_d := \mathcal{D}_{L_d}$
2. when $d \equiv 2 \pmod 8$, the set of rank 3 primitive sublattices $L_d \subset I_{22,2}$ containing Λ_G is the union of two $\tilde{O}(\Lambda)$-orbits, interchanged by some involution $r_{\mathcal{D}} \in \tilde{O}(\Lambda)$; therefore$\mathcal{D}_{L_d}$ can be one of two irreducible divisors, we put $\mathcal{D}_d$ to be the union of them.

Now for any marking $f : I_{22,2} \to \mathrm{H}^4(X;\mathbb{Z})$, we have $f(L_d) \subset \mathrm{H}^{2,2}(X;\mathbb{Z})$ (hence defining a labelling with discriminant d on X) if and only if $f^{-1}(\mathrm{H}^{3,1}(X)) \subset L_d^{\perp} \otimes \mathbb{C}$. Therefore the hypersurface $\mathcal{GM}_d := \wp^{-1}(\mathcal{D}_d) \subset \mathcal{GM}$ parametrizes all special Gushel-Mukai fourfolds admitting labelling with discriminant d . By [DIM14, Theorem 8.1], $\mathcal{GM}_d$ is non-empty when $d > 8$, in fact $\mathrm{im}\wp \cap \mathcal{D}_d$ is a dense open subset of $\mathcal{D}_d$ for all $d > 8$. We would like to point out that in [DIM14], it is conjectured that the image of the period map $\wp$ is exactly the complement of $\mathcal{D}_2 \cup \mathcal{D}_4 \cup \mathcal{D}_8$.

Remark. It turns out that $\mathcal{GM}_d$ is also irreducible when $d \equiv 0 \pmod 4$ and is the union of two irreducible hypersurfaces when $d \equiv 2 \pmod 8$. Recall that the period map $\wp : \mathcal{GM} \longrightarrow \mathcal{D}$ has irreducible fibers, hence the restriction $\wp : \mathcal{GM}_d \longrightarrow \mathcal{D}_d$ is an open map to an irreducible (or union of two irreducible) variety with irreducible fibers. By standard general topology, $\mathcal{GM}_d$ is either an irreducible or union of two irreducible hypersurfaces in $\mathcal{GM}$.

•Cubic fourfolds

Let N_d be a rank 2 positive definite primitive sublattice of $I_{21,2}$ of discriminant d containing the distinguished element δ. By [Has00, Theorem 1.0.1], we must have $d > 0$ and $d \equiv 0, 2 \pmod 6$. Consider the hypersurface

$$\Omega(N_d^{\perp}) = \mathbb{P}(N_d^{\perp} \otimes \mathbb{C}) \cap \Omega(\Lambda')$$

in $\Omega(\Lambda')$. Let $\mathcal{D}'_d$ be the image of $\Omega(N_d^{\perp})$ in $\mathcal{D}'$under the quotient map $\Omega(\Lambda') \to \mathcal{D}'$. By [Has00, Thereom 3.1.2 and Proposition 3.2.4], the set of all such sublattices N_d with a fixed discriminant form a single $\tilde{O}(\Lambda')$-orbit, therefore $\mathcal{D}'_d$ depend only on the discriminant d and is an irreducible divisor in $\mathcal{D}'$. Similar to the case of Gushel-Mukai fourfolds, for any marking $f : I_{21,2} \to \mathrm{H}^4(Y, \mathbb{Z})$, $f(N_d) \subset \mathrm{H}^{2,2}(Y, \mathbb{Z})$ if and only if $f^{-1}(\mathrm{H}^{3,1}(Y)) \subset N_d^{\perp} \otimes \mathbb{C}$; therefore $\mathcal{C}_d := \mathcal{C} \cap \mathcal{D}'_d$ is the irreducible divisor in the moduli space $\mathcal{C}$ parametrizing special cubic fourfolds with discriminant d . Moreover by [Has00, Theorem 1.0.1], $\mathcal{C}_d$ is nonempty when $d > 6$. In fact it can be shown that the image of the period map $\wp' : \mathcal{C} \to \mathcal{D}'$ is exactly the complement of $\mathcal{D}'_2 \cup \mathcal{D}'_6$ (see [Laz10] and [Loo09]).

3.4 Hodge-association

Observe that the non-special cohomologies of both special Gushel-Mukai and cubic fourfolds have signature $(19, 2)$, and both carry polarized Hodge structures induced from that on the middle cohomologies, hence we can make the following

Definition 3.4.1. A labelled Gushel-Mukai fourfold (X, K_d) and a polarized K3 surface (S, ℓ) are said to be *Hodge-associated* if there is a Hodge isometry (up to a shift of weight) $K_d^{\perp} \simeq H^2_{\mathrm{prim}}(S, \mathbb{Z})(-1)$. Similarly, a labelled cubic fourfold (Y, K'_d) and a polarized K3 surface (S, ℓ) are said to be Hodge-associated if there is a Hodge isometry $K_d'^{\perp} \simeq H^2_{\mathrm{prim}}(S, \mathbb{Z})(-1)$.

Notice that in both cases, the degree ℓ^2 of the polarized K3 surface S is necessarily equal to d (Corollary 2.1.12(a)). Hodge-association between fourfolds and K3 surfaces are characterized by some numerical conditions on the discriminant d.

Proposition 3.4.2. ([DIM14], Proposition 6.5). *A labelled Gushel Mukai fourfold (X, K_d) is Hodge-associated to some K3 surface if and only if d satisfies the following numerical condition*

$$(**) \quad d \textit{ is not divisible by } 8$$

$$\textit{and the only odd primes dividing } d \textit{ are} \equiv 1 \ (\textit{mod } 4).$$

. ([Has00], Theorem 1.0.2). *A labelled cubic fourfold (Y, K'_d) is Hodge-associated* *to some K3 surface if and only if d satisfies the following numerical condition*

$$(**)' \quad d \textit{ is not divisible by } 4,\ 9$$

$$\textit{and the only odd primes dividing } d \textit{ are} \equiv 0 \textit{ or } 1 \ (\textit{mod } 3).$$

.

We are primarily interested in the comparison between Gushel-Mukai fourfolds and cubic fourfolds.

Definition 3.4.3. A labelled Gushel-Mukai fourfold (X, K_d) and a labelled cubic fourfold (Y, K'_d) are said to be Hodge-associated if there is a Hodge isometry between their non-special cohomologies $K_d^{\perp} \simeq K_d'^{\perp}$.

Hodge-association between Gushel-Mukai and cubic fourfolds is also determined by some numerical condition on the discriminants:

Proposition 3.4.4. ([DIM14], Proposition 6.6). *Let (X, K_d) be a labelled Gushel-Mukai fourfold generic in the divisor $\mathcal{G}_d$, there is labelled cubic fourfold (Y, K'_d) (necessarily of the same discriminant) together with a Hodge isometry $K_d^{\perp} \simeq K_d'^{\perp}$ between non-special cohomologies if and only if d satisfies the following numerical condition:*

$$(\dagger) \quad \begin{array}{c} (a) \textit{either } d \equiv 2 \textit{ or } 20 \ (\textit{mod } 24) \\ \textit{and the only odd primes dividing } d \textit{ are} \equiv \pm 1 \ (\textit{mod } 12); \\ (b) \textit{or } d \equiv 12 \textit{ or } 66 \ (\textit{mod } 72) \\ \textit{and the only primes} \geq 5 \textit{ dividing } d \textit{ are} \equiv \pm 1 \ (\textit{mod } 12). \end{array}$$

It may seem as a surprise at a first glance that the Hodge-association between special fourfolds and K3 surfaces as well as between the special fourfolds are characterized only by the discriminants of the labellings. In fact both Propositions 3.4.2 and 3.4.4 boil down to some lattice theoretic (hence numerical) arguments, together with the fact that the period maps are dominant. To elaborate this, we give the proof of Proposition 3.4.4.

We first prove the following lattice theoretic statement:

Lemma 3.4.5. *Let $L_d \subset I_{22,2}$ be any rank 3 positive definite primitive sublattice containing the distinguished sublattice Λ_G and $N_d \subset I_{21,2}$ any positive definite primitive sublattice containing the distinguished element δ. Then there is an isometry of lattices $L_d^{\perp} \simeq N_d^{\perp}$ (they necessarily have the same discriminant) if and only if d satisfies the numerical condition (†).*

Proof. First of all, recall that in order for the sublattices L_d and N_d to exist at the first place we must have $d \equiv 0, 2, 4 \pmod 8$ and simultaneously $d \equiv 0, 2 \pmod 6$, this means $d = 24d' + e$ with $e \in \{0, 2, 8, 12, 18, 20\}$.

By Corollary 2.1.12(a), if $L_d^{\perp}$ is isometric to $N_d^{\perp}$, then there exists an isomorphism $(A_{L_d}, q_{L_d}) \simeq (A_{N_d}, q_{N_d})$ of the discriminant forms. The discriminant groups are computed in both cases:

(1) For A_{L_d} ([DIM14, Proposition 6.5]):

–$A_{L_d} \simeq (\mathbb{Z}/2\mathbb{Z})^2 \times (\mathbb{Z}/(d/4)\mathbb{Z})$ when $d \equiv 0 \pmod 8$,

–$A_{L_d} \simeq \mathbb{Z}/d\mathbb{Z}$ when $d \equiv 2 \pmod 8$, the isomorphism can be chosen so that $b_{L_d}(1,1) = \frac{d+8}{2d} \pmod{\mathbb{Z}}$,

–$A_{L_d} \simeq \mathbb{Z}/d\mathbb{Z}$ when $d \equiv 4 \pmod 8$, then isomorphism may be chosen so that

$$b_{L_d}(1,1) = \frac{d+2}{2d} \pmod{\mathbb{Z}}.$$

(2) For $A_{L'_d}$ ([H1, Proposition 3.2.5]):

–$A_{N_d} \simeq (\mathbb{Z}/3\mathbb{Z}) \times (\mathbb{Z}/(d/3)\mathbb{Z})$ when $d \equiv 0 \pmod 6$, the isomorphism can be chosen so that $b_{N_d}(1,1) = \frac{3-2d}{3d} \pmod{2\mathbb{Z}}$,

$-A_{N_d} \simeq \mathbb{Z}/d\mathbb{Z}$ when $d \equiv 2 \pmod 6$, then isomorphism can be chosen so that $b_{N_d}(1,1) = \frac{1-2d}{3d} \pmod{2\mathbb{Z}}$.

Comparing the two list, we can exclude $e = 0$ or 8; furthermore, if $e = 12$ or 18, we need $d/3$ coprime to 3, this forces $d' \not\equiv 1 \pmod 3$ when $e = 12$ and $d' \not\equiv 0 \pmod 3$ when $e = 18$. In all these cases, the disciminant group is isomorphic to $\mathbb{Z}/d\mathbb{Z}$.

When $e = 2$, then $d \equiv 2$ (mod8) and $d \equiv 2 \pmod 6$, therefore the discriminant form are conjugate if and only of $\frac{d+8}{2d} \equiv n^2 \frac{1-2d}{3d} \pmod{\mathbb{Z}}$ where n is some integer prime to d, which is in turn equivalent, since 3 is invertible modulo d in this case, to that $\frac{d}{2} + 12 \equiv 3\frac{d+8}{2} \equiv n^2 \pmod d$. This is equivalent to saying that $12d' + 13$is a square in $\mathbb{Z}/d\mathbb{Z}$; in view of the decomposition $\mathbb{Z}/d\mathbb{Z} \simeq (\mathbb{Z}/(12d'+1)\mathbb{Z}) \times (\mathbb{Z}/2\mathbb{Z})$, $12d' + 13$ is a square in $\mathbb{Z}/d\mathbb{Z}$ if and only if 3 is a square in $\mathbb{Z}/(12d'+1)\mathbb{Z}$, which by quadratic reciprocity, equivalent to that the only odd primes dividing d are $\equiv \pm 1 \pmod{12}$.

When $e = 20$, then $d \equiv 4 \pmod 8$ and $d \equiv 2 \pmod 6$, therefore the discriminant form are conjugate if and only if $\frac{d+2}{2d} \equiv n^2 \frac{1-2d}{3d} \pmod{\mathbb{Z}}$ for some integer n prime to d, which is equivalent to $\frac{d}{2} + 3 \equiv n^2 \pmod d$, but this is once again equivalent to that $12d' + 13$ is a square in $\mathbb{Z}/d\mathbb{Z}$. Hence we get the same condition as in the previous case.

When $e = 12$, then $d \equiv 4 \pmod 8$ and $d \equiv 0 \pmod 6$, therefore we need $\frac{d+2}{2d} \equiv n^2 \frac{3-2d}{3d} \pmod{\mathbb{Z}}$ for some integer n prime to d, which is equivalent to $12d' + 7 \equiv n^2(-16d' - 5) \pmod d$. This condition is equivalent to:

i) $12d' + 7 \equiv n^2(-16d' - 5) \pmod 3$ and

ii) $12d' + 7 \equiv n^2(-16d' - 5) \pmod 4$ and

iii) $12d' + 7 \equiv n^2(-16d' - 5) \pmod{2d' + 1}$.

i) is equivalent to $1 - d'$ being a nonzero square modulo 3, which is equivalent to that $3|d'$. ii) is a tautology. iii). is equivalent to $1 \equiv 3n^2 \pmod{2d'+1}$, which is in turn equivalent to say that 3 is square modulo $d/12$, hence by quadratic reciprocity again it is equivalent to that any odd primes of $d/12$ is $\equiv \pm 1 \pmod{12}$. Combining all these, and recall that in case we have $9 \nmid d$, the desired condition is equivalent to

$$d \equiv 12 \pmod{72} \text{ and the only primes} \geq 5 \text{ dividing } d \text{ are } \equiv \pm 1 \pmod{12}.$$

Lastly, when $e = 18$, then $d \equiv 2 \pmod 8$ and $d \equiv 0 \pmod 6$, therefore we need $\frac{d+8}{2d} \equiv n^2 \frac{3-2d}{3d} \pmod{\mathbb{Z}}$ for some integer n prime to d, which is equivalent to $12d' + 13 \equiv n^2(-16d' - 9) \pmod d$. Following an argument similar to the last case, one eventually conclude that this is equivalent to

$$d \equiv 66 \pmod{72} \text{ and the only primes} \geq 5 \text{ dividing } d \text{ are } \equiv \pm 1 \pmod{12}.$$

For the converse, the above computation above shows that when d satisfies the condition (†), we have isomorphism of discriminant forms $(A_{L_d}, q_{L_d}) \simeq (A_{N_d}, q_{N_d})$. Moreover, $L_d^{\perp}$ and $N_d^{\perp}$ are even indefinite lattice with rank 21, the discriminant group of L_d (resp. N_d) has length at most 2; thus by Corollary 2.1.12 (b), $L_d^{\perp}$ is isometric to $N_d^{\perp}$. □

Proof of Proposition 3.4.4. Let (X, K_d) be a labelled Gushel-Mukai fourfold, choose any marking $f\colon I_{22,2} \to \mathrm{H}^4(X, \mathbb{Z})$, then $L_d := f^{-1}K_d$ is a primitive rank 3 sublattice of $I_{22,2}$ containing Λ_G and $f^{-1}\mathrm{H}^{3,1}(X) \in \Omega(L_d^{\perp})$; if furthermore d satisfies the condition (†), then by the above Lemma there is an isometry of lattices $\mu : L_d^{\perp} \to N_d^{\perp}$ where N_d is *any* rank 2 primitive sublattice of $I_{21,2}$ containing δ; we will still denote by μ the induced isomorphism $\Omega(L_d^{\perp}) \simeq \Omega(N_d^{\perp})$. Now since (X, K_d) is generic in $\mathcal{G}_d$, we may assume that $\mu(f^{-1}\mathrm{H}^{3,1}(X)) \in \Omega(N_d^{\perp})$ is in the image of the period map for cubic fourfolds, that is to say there exists a cubic fourfold Y and a marking $f'\colon I_{21,2} \to \mathrm{H}^4(Y, \mathbb{Z})$ such that $f'^{-1}\mathrm{H}^{3,1}(Y) =$

$\mu(f^{-1}\mathrm{H}^{3,1}(X))$. Then $K'_d = f'(N_d) \subset \mathrm{H}^{2,2}(Y,\mathbb{Z})$ is a labelling on Y and

$$f'\mu f^{-1}\colon K_d^{\perp} \tilde{\to} K_d'^{\perp}$$

is a Hodge isometry of non-special cohomologies. Conversely, if (X, K_d) and (Y, K'_d) are labelled Gushel-Mukai and cubic fourfold respectively, then choose any markings f for X and f' for Y, then we have isometry of the lattices $(f^{-1}K_d)^{\perp} \simeq (f'^{-1}N_d)^{\perp}$, hence by the Lemma again, d must satisfy the condition (†).

□

3.5 Cubic fourfolds in $\mathcal{C}_d$ with associated K3 form a Zariski dense subset

In this subsection we prove that in each Hassett divisor $\mathcal{C}_d$ parameterizing special cubic fourfolds, those with Hodge-associated K3 surfaces form a Zariski dense subset of $\mathcal{C}_d$. The proof is based on some studies by Yang & Yu ([YY20] and [YY23]) concerning the intersection of the Hassett divisors $\mathcal{C}_d$ in $\mathcal{C}$. Let's first introduce a notation, for any positive definite primitive sublattice $M \subset I_{21,2}$ containing the distinguished element δ, we consider the submanifold of $\Omega(\Lambda')$:

$$\Omega(M^{\perp}) := \mathbb{P}(M^{\perp} \otimes \mathbb{C}) \cap \Omega(\Lambda'),$$

define $\mathcal{D}'_M$ to be image of $\Omega(M^{\perp})$ in the period domain $\mathcal{D}'$ under the period map $\wp'\colon \mathcal{C} \longrightarrow \mathcal{D}'$. We consider the subvariety $\mathcal{C}_M \subset \mathcal{C}$ given by $\wp'^{-1}(\mathcal{D}'_M)$, which is a closed irreducible subvariety of $\mathcal{C}$ of codimension $\mathrm{rank}(M) - 1$ (The argument for the irreducibility of the Hassett divisors $\mathcal{C}_d$ applies in this case and gives us the irreducibilty of $\mathcal{C}_M$, see [Has00, Theorem 3.1.2]). For example, if $N_d \subset I_{21,2}$ is a rank 2 positive-definite primitive sublattice of discriminant d containing the distinguished element δ, then the irreducible hypersurface $\mathcal{C}_{N_d}$ is nothing but the Hassett divisor $\mathcal{C}_d$.

The following is a consequence of the definition

Lemma 3.5.1. *Let M and N be positive definite primitive sublattices of $I_{21,2}$ containing δ, then $\mathcal{D}'_M \subset \mathcal{D}'_N$ if and only if there is a primitive embedding $\psi : N \hookrightarrow M$ such that $\psi(\delta) = \delta$.*

Proof. If there is a primitive embedding $N \hookrightarrow M$ fixing the distinguished element δ, then it is clear that $\mathcal{D}'_M \subset \mathcal{D}'_N$.

For the converse, we first prove that it is always possible to find a point $v \in \Omega(M^\perp)$ such that $v^\perp \cap I_{21,2} = M$:

Given any $v \in \Omega(M^\perp)$ we must have $v^\perp \cap I_{21,2} \supseteq M$. If $v^\perp \cap I_{21,2} \supsetneqq M$, then we can find a nonzero lattice point $x \in v^\perp \cap I_{21,2}$ such that $x \perp M$; we consider a linear combination

$$w = v - \left(\frac{\mu^2 x^2}{2v \cdot \bar{v}}\right) \bar{v} + \mu x$$

for some nonzero *real* number μ. Note that $x^2 > 0$ since x is necessarily contained in a positive definite subspace of $I_{21,2} \otimes \mathbb{R}$. Now it is straightforward to check:

1) $w \cdot w = -\frac{\mu^2 x^2}{v \cdot \bar{v}} v \cdot \bar{v} + \mu^2 x^2 = 0$;

2) $w \cdot \bar{w} = v \cdot \bar{v} + \left(\frac{\mu^4}{4} + \mu^2\right) x^2$, hence we can choose μ sufficiently small so that $w \cdot \bar{w} < 0$;

3) for any $y \in M$, we have $y \cdot v = y \cdot \bar{v} = y \cdot x = 0$ hence $y \cdot w = y \cdot v - \frac{\mu^2 x^2}{2v \cdot \bar{v}} y \cdot \bar{v} + \mu y \cdot x = 0$.

So we have constructed a new element $w \in \Omega(M^\perp)$ with $w^\perp \cap I_{21,2} \supseteq M$. We now show that $w^\perp \cap I_{21,2} \subsetneqq v^\perp \cap I_{21,2}$:

Given any $z \in w^\perp \cap I_{21,2}$, we have

$$z \cdot w = z \cdot v - \frac{\mu^2 x^2}{2v \cdot \bar{v}} z \cdot \bar{v} + \mu z \cdot x = 0,$$

in particular $z \cdot v - \frac{\mu^2 x^2}{2v\cdot\bar{v}} z \cdot \bar{v}$ must have zero imaginary part ($\mu z \cdot x$ is real since we choose μ to be real); but $z \cdot v$ and $z \cdot \bar{v}$ are complex conjugate to each other, so if $z \cdot v \neq 0$ we must have $\frac{\mu^2 x^2}{2v\cdot\bar{v}} = 1$ which is not possible ($x^2 > 0$ while $v \cdot \bar{v} < 0$). Thus we must have $z \cdot v = 0$, namely $z \in v^\perp \cap I_{21,2}$. Lastly, we notice that $x \cdot w = -\mu x^2 \neq 0$, hence $x \notin w^\perp \cap I_{21,2}$. This shows that $w^\perp \cap I_{21,2} \subsetneqq v^\perp \cap I_{21,2}$. Also notice that $w^\perp \cap I_{21,2}$ is a primitive sublattice of $v^\perp \cap I_{21,2}$, hence $w^\perp \cap I_{21,2} \subsetneqq v^\perp \cap I_{21,2}$ imlies that rank of $w^\perp \cap I_{21,2}$ is strictly smaller than of $v^\perp \cap I_{21,2}$.

Therefore, by induction on the rank of $v^\perp \cap I_{21,2}$, we can always find a $v \in \Omega(M^\perp)$ such that $v^\perp \cap I_{21,2} = M$. Now the $\tilde{O}(\Lambda')$-orbit of this v is a point in $\mathcal{D}'_M$, hence a point in $\mathcal{D}'{}_N$; meaning there is an isometry $g \in \tilde{O}(\Lambda')$ such that $g(v)^\perp \cap I_{21,2} \supset N$. Thus $\psi = g^{-1}|_N \colon N \longrightarrow M$ is a primitive embedding mapping δ to δ. $\square$

The following is a useful criterion for the non-emptiness of $\mathcal{C}_M$

Lemma 3.5.2. ([YY20], Lemma 6). *The subvariety $\mathcal{C}_M$ defined by the positive definite primitive sublattice $M \subset I_{21,2}$ containing δ is non-empty if and only if there is no $r \in M$ such that $r^2 = 2$ (i.e., M does not represent 2).*

Proof. Since $\mathcal{C}_M = \wp'^{-1}(\mathcal{D}'_M)$, it is non-empty if and only if $\mathcal{D}'_M \not\subseteq \mathcal{D}'_2 \cup \mathcal{D}'_6$ (we know that $\mathcal{D}'_2 \cup \mathcal{D}'_6$ is the complement of the image of $\wp'$). Recall that the rank 2 positive-definite primitive sublattices $N_d \subset I_{21,2}$ of a fixed discriminant d form a single $\tilde{O}(\Lambda')$-orbit, hence we can put $\mathcal{D}'_2 = \mathcal{D}'_{N_2}$ and $\mathcal{D}'_6 = \mathcal{D}'_{N_6}$ where $N_2 = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$ and $N_6 = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$; more explicitly we can take $N_2 = \langle \delta, \epsilon_1 \rangle$ and $N_6 = \langle \delta, \epsilon_1 - \epsilon_2 \rangle$ (recall that $\epsilon_1, \epsilon_2, \epsilon_3$ is the standard basis of $I_{3,0}$).

By the last lemma, $\mathcal{D}'_M \subset \mathcal{D}'_2 \cup \mathcal{D}'_6$ if and only if $M \supset N_2$ or $M \supset N_6$.

If $M \supset N_2$ or $M \supset N_6$, then M represents 2, since both N_2 and N_6 do. Conversely, if $r \in M$ such that $r^2 = 2$, then the intersection matrix of $K = \langle \delta, r \rangle$

is given by

$$\begin{pmatrix} 3 & a \\ a & 2 \end{pmatrix}$$

where $a = \delta \cdot r$. Since K is positive definite, we must have $a^2 < 6$ and thus $a = 0, 1, 2$. We cannot have $a = 1$, otherwise we will have $(\delta - 3r) \cdot \delta = 0$ hence $(\delta - 3r) \in \Lambda'$ but meanwhile $(\delta - 3r)^2 = 15$, contradicting that Λ' is even. If $a = 0$, then K is isometric to N_6; if $a = 2$, then $\langle \delta, \delta - r \rangle$ is isometric to N_2. $\square$

Combining the previous two lemmas we have

Proposition 3.5.3. ([YY23], Lemma 7.1). *Let M_i, $i = 1, 2$ be two positive definite primitive sublattices of $I_{21,2}$ of rank$(M_i) \geq 2$ containing the distinguished element δ such that $\mathcal{C}_{M_i} \neq \emptyset$. Then $\mathcal{C}_{M_1} \subset \mathcal{C}_{M_2}$ if and only of there exists a primitive embedding $\phi \colon M_2 \hookrightarrow M_1$ such that $\phi(\delta) = \delta$.*

We also have the following statement about the intersection of two Hassett divisors.

Proposition 3.5.4. ([YY20], Theorem 7). *Given any two Hassett divisors $\mathcal{C}_{d_1}$ and $\mathcal{C}_{d_2}$ parameterizing special cubic fourfolds with discriminant d_1 and d_2 respectively, their intersection $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is non-empty; one of the irreducible component of $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$ is necessarily given by the irreducible subvariety $\mathcal{C}_M$ for some rank 3 positive definite primitive sublattice $M \subset I_{21,2}$ containing δ, with disc$(M) = \frac{d_1 d_2}{3}$ or $\frac{d_1 d_2 - 1}{3}$ depending on the values of d_1 and d_2.*

Proof. Recall that $d_1, d_2 > 6$ and $\equiv 0$ or $2 \pmod 6$, so we divide the proof into three cases. Let's only discuss the case $d_1 \equiv d_2 \equiv 0 \pmod 6$. The other two cases are completely similar and covered in detail in ([YY20]). We let (g_1, h_1) and (g_2, h_2) be the standard basis of the two copies of hyperbolic plane $U \subset I_{21,2}$.

Suppose $d_1 = 6n_1$ and $d_2 = 6n_2$ and $n_1, n_2 \geq 2$. Consider the rank 3 lattice

$$M = \langle \delta, g_1 + n_1 h_1, g_2 + n_2 h_2 \rangle \subset I_{21,2}$$

It is straightforward to check that M is a primitive sublattice and has intersection matrix

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2n_1 & 0 \\ 0 & 0 & 2n_2 \end{pmatrix}$$

In addition, given any nonzero $r = x\delta + y(g_1 + n_1 h_1) + z(g_2 + n_2 h_2) \in M$, we have

$$r^2 = 3x^2 + 2n_1 y^2 + 2n_2 z^2 \geq 3$$

since $n_1, n_2 \geq 2$ and at least one of the integers x, y, z is nonzero. Hence by Lemma 3.5.2 $\mathcal{C}_M$ is nonempty and is an irreducible subvariety of $\mathcal{C}$ of codimension 2. Moreover, M contains primitive sublattice $\langle \delta, g_i + n_i h_i \rangle$ with discriminant d_i, $i = 1, 2$, hence by Lemma 3.5.3, $\mathcal{C}_M \subset \mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$; it is necessarily one of the irreducible components of $\mathcal{C}_{d_1} \cap \mathcal{C}_{d_2}$.

The case of $d_1 \equiv d_2 \equiv 2 \pmod 6$ will give us a rank 3 lattice with discriminant $(d_1 d_2 - 1)/3$. □

Proposition 3.5.5. *Let $\{d_0, d_1, d_2, \ldots\}$ be an infinite collection of integers> 6 and each $d_i \equiv 0, 2 \pmod 6$. Then $\bigcup_{n\geq 1}(\mathcal{C}_{d_0} \cap \mathcal{C}_{d_n})$ is a dense subset of C_{d_0} in Zariski topology.*

Proof. Assume on the contrary there is a Zariski closed subset $V \subset C_{d_0}$ such that

$$V \supset \bigcup_{n\geq 1} (\mathcal{C}_{d_0} \cap \mathcal{C}_{d_n}).$$

One of the irreducible component of V must contain infinitely many $\mathcal{C}_{d_n}$, hence we may assume V is an irreducible hypersurface in $\mathcal{C}_{d_0}$. By Proposition 3.5.4, one of the irreducible components of $\mathcal{C}_{d_0} \cap \mathcal{C}_{d_n}$ is given by the hypersurface in $\mathcal{C}_{d_0}$ of the form $\mathcal{C}_{M_n}$ for some rank 3 primitive lattice M_n with discriminant given by either $d_0 d_n/3$ or $(d_0 d_n - 1)/3$. But then we are forced to have

$$V = \mathcal{C}_{M_1} = \mathcal{C}_{M_2} = \mathcal{C}_{M_3} = \cdots$$

This is a contradiction by Proposition 3.5.3. $\square$

In particular, the union

$$\bigcup_{d'} (\mathcal{C}_d \cap \mathcal{C}_{d'}),$$

where d' run over all natural numbers satisfying $(**)'$ is Zariski dense in $\mathcal{C}_d$. Recall that cubic fourfolds in each $\mathcal{C}_{d'}$ has Hodge-associated K3 surfaces, this shows that in each $\mathcal{C}_d$, cubic fourfolds with Hodge-associated K3 form a Zariski dense subset.

3.6 Non-special cohomologies in family of fourfolds

The construction we do in this subsection is essential for what comes later, so we include more details. First we need to make sense of a "continuous family of labellings on a family of fourfolds". We do this for Gushel-Mukai fourfolds first. Fix a primitive rank 3 positive definite sublattice $L_d \subset I_{22,2}$ of discriminant d, containing the distinguished sublattice Λ_G, and consider the subgroup $H(L_d) \subset \tilde{O}(\Lambda)$ given by

$$H(L_d) := \{g \in \tilde{O}(\Lambda) | g|_{L_d} = id\}.$$

Define the quotient space $\mathcal{D}^{lab}_{L_d} := H(L_d)\backslash\Omega(L_d^{\perp})$, which is a quasi-projective normal variety by Baily and Borel again. We also define $\mathcal{G}^{lab}_{L_d} := \mathcal{D}^{lab}_{L_d} \times_{\mathcal{D}} \mathcal{G}$. By definition a typical element of $\mathcal{G}^{lab}_{L_d}$ is a pair $(X, [\omega]_{H(L_d)})$ consisting of a Gushel-

Mukai fourfold X and the $H(L_d)$-orbit $[\omega]_{H(L_d)}$ of some point $\omega \in \Omega(L_d^\perp)$ such that its $\tilde{O}(\Lambda)$-orbit $[\omega]_{\tilde{O}(\Lambda)} = \wp(X) \in \mathcal{D}_d$. Equivalently, we can think of it as a pair $(X, [f]_{H(L_d)})$ where $f\colon I_{22,2} \to \mathrm{H}^4(X, \mathbb{Z})$ is a marking on X and $[f]_{H(L_d)}$ its $H(L_d)$-orbit (the group $H(L_d)$ acts on the set of all markings on X by the restriction of the action of $\tilde{O}(\Lambda)$) such that $[f^{-1}\mathrm{H}^{3,1}(X)]_{H(L_d)} = [\omega]_{H(L_d)} \in \mathcal{D}_{L_d}^{lab}$. Therefore the element $(X, [\omega]_{H(L_d)})$ determines a well-defined labelling $K_{[\omega]} := f(L_d) \subset \mathrm{H}^{2,2}(X, \mathbb{Z})$ on X.

Let $\pi\colon \mathcal{X}^{lab} \to \mathcal{G}_{L_d}^{lab}$ be the pullback of the universal family $\mathcal{X}_U$ of Gushel-Mukai fourfolds by the map $\mathcal{G}_{L_d}^{lab} \to \mathcal{G}$. The fiber of $\mathcal{X}^{lab}$ over the point $(X, [\omega]) \in \mathcal{G}_{L_d}^{lab}$ is just the fourfold X and we can think of the sublattices $K_{[\omega]}$ as giving a continuous family of labellings on the fibers of $\mathcal{X}^{lab}$; we denote by $\mathcal{K}^\perp$ the local system of non-special cohomologies on the family $\mathcal{X}^{lab} \to \mathcal{G}_{L_d}^{lab}$, meaning the stalk of $\mathcal{K}^\perp$ at the point $(X, [\omega])$ is just the non-special cohomology $K_{[\omega]}^\perp$ determined by the labelling $K_{[\omega]}$ on X. We have the following

Lemma 3.6.1. *The inclusion maps $\mathcal{K}^\perp \subset (R^4\pi_*\mathbb{Z})_{00} \subset R^4\pi_*\mathbb{Z}$ are inclusions of variations of Hodge structures.*

Here, $(R^4\pi_*\mathbb{Z})_{00}$ denotes the local system of vanishing cohomologies.

Proof. The inclusion maps restricts to identity map on $\mathrm{H}^{3,1}$ of each fibers, so we only need to show that these are inclusions of local systems. It is clear that $(R^4\pi_*\mathbb{Z})_{00} \subset R^4\pi_*\mathbb{Z}$ is an inclusion of local systems. Now we only need to show that $\mathcal{K}^\perp \subset R^4\pi_*\mathbb{Z}$ is an inclusion of local systems, meaning, the parallel transports of $R^4\pi_*\mathbb{Z}$ restricts to that of $\mathcal{K}^\perp$. The statement is local, hence we consider instead a family $\pi\colon \mathcal{X} \longrightarrow \Delta$ of Gushel-Mukai fourfolds, being the restriction of $\mathcal{X}^{lab}$ to some complex disc $\Delta \subset \mathcal{G}$. So each point $t \in \Delta$ can be considered as a pair $(X_t, [\omega_t])$, where X_t is just the fiber of $\mathcal{X}$ at t and $[\omega_t]$ is an $H(L_d)$-orbit of some period point $\omega_t \in \Omega(L_d^\perp)$. Fix a choice of ω_0 we can think of $t \mapsto \omega_t$ as a map

$\Delta \longrightarrow \Omega(L_d^{\perp})$ lifting the period map $\Delta \longrightarrow \mathcal{D}_{L_d}^{lab} = H(L_d)\backslash\Omega(L_d^{\perp})$, therefore we have a commutative diagram

$$\begin{tikzcd} & \Omega(L_d^{\perp}) \arrow[r, hook] \arrow[d] & \Omega(\Lambda) \arrow[d] \\ \Delta \arrow[ur, "\omega_t"] \arrow[r] & \mathcal{D}_{L_d}^{lab} \arrow[r] & \mathcal{D} \end{tikzcd} \tag{3.6.1}$$

But then by the discussion in Remark 3.2.2, we must have $\omega_t = f_0^{-1}p_t\mathrm{H}^{3,1}(X_t)$, where $f_0\colon I_{22,2} \to \mathrm{H}^4(X_0,\mathbb{Z})$ is some suitable marking on X_0 such that $\omega_0 = f_0^{-1}\mathrm{H}^{3,1}(X_0)$ and $p_t\colon \mathrm{H}^4(X_t,\mathbb{Z}) \to \mathrm{H}^4(X_0,\mathbb{Z})$ is the parallel transport operator for the local system $R^4\pi_*\mathbb{Z}$ on Δ. From there, we see that the labelling K_{ω_t} on X_t is given by $p_t^{-1}f_0(L_d)$, hence p_t maps $K_{\omega_t}^{\perp} = p_t^{-1}f_0(L_d^{\perp})$ isometrically onto $K_{\omega_0}^{\perp} = f_0(L_d^{\perp})$. $\square$

Similar construction can be performed for cubic fourfolds. We fix a primitive rank 2 positive definite sublattice $N_d \subset I_{21,2}$ of discriminant d, containing the distinguished element δ, and consider the subgroup $H(N_d) := \{g \in \tilde{O}(\Lambda') | g|_{N_d} = id\}$ and define $\mathcal{D}_{N_d}^{lab} := H(N_d)\backslash\Omega(N_d^{\perp})$, which is again a quasi-projective normal variety. Define $\mathcal{C}_{N_d}^{lab} := \mathcal{D}_{N_d}^{lab} \times_{\mathcal{D}'} \mathcal{C}$, just as in the case of Gushel-Mukai fourfolds, each point of $\mathcal{C}_{N_d}^{lab}$ may be identified with a pair $(Y,[\omega'])$ where $[\omega']$ is the $H(N_d)$-orbit of some point $\omega' \in \Omega(N_d^{\perp})$ and it induces a well-defined labelling $K_{[\omega']} \subset \mathrm{H}^{2,2}(Y,\mathbb{Z})$ on Y. There is also a family of cubic fourfolds $\pi' : \mathcal{Y}^{lab} \to \mathcal{C}_{N_d}^{lab}$. Although we don't have a universal family to pullback since $\mathcal{C}$ is only a coarse moduli space, nevertheless $\mathcal{C}_{N_d}^{lab}$ still carries a smooth family of cubic fourfolds, which can be obtained by gluing local deformations via local Torelli theorem for cubic fourfolds (cf. [AT14, Proposition 5.2]). The fiber of $\mathcal{Y}^{lab}$ at the point $(Y,[\omega'])$ is just the fourfold Y and $K_{[\omega']}$ gives us continuous family of labellings on the fibers of $\mathcal{Y}^{lab}$. If we denote by $\mathcal{K}'^{\perp}$ the local system of non-special cohomologies and $(R^4\pi'_*\mathbb{Z})_0$ the local system of primitive cohomologies, then $\mathcal{K}'^{\perp} \subset (R^4\pi'_*\mathbb{Z})_0 \subset$

$R^4\pi'_*\mathbb{Z}$ are inclusions of variation of Hodge structures.

Now if the integer d satisfies the numerical condition (†) as in Proposition 3.4.4, then there is an isometry of lattices $\mu\colon L_d^\perp \longrightarrow N_d^\perp$ by Lemma 3.4.5. It turns out that this isometry extends to a map from $\mathcal{G}_{L_d}^{lab}$ to $\mathcal{C}_{N_d}^{lab}$ which identifies the variation of Hodge structures (at least locally) on the local systems of non-special cohomologies:

Proposition 3.6.2. *The isometry of lattices $\mu\colon L_d^\perp \xrightarrow{\sim} N_d^\perp$ induces a rational map*

$$\nu\colon \mathcal{G}_{L_d}^{lab} \dashrightarrow \mathcal{C}_{N_d}^{lab},$$

which is generically a smooth submersion. Moreover, given any point

$$t_0 = (X_0, [\omega_0]) \in \mathcal{G}_{L_d}^{lab}$$

and put

$$\nu(t_0) = (Y, [\mu(\omega_0)]) \in \mathcal{C}_{N_d}^{lab},$$

there is a Hodge isometry $\phi_0\colon K_{[\omega_0]}^\perp \xrightarrow{\sim} K_{[\mu(\omega_0)]}^\perp$ between non-special cohomologies which remains Hodge isometries under parallel transport; more precisely, if

$$p_\gamma\colon K_{[\omega_t]}^\perp \xrightarrow{\sim} K_{[\omega_0]}^\perp$$

is the parallel transport of $\mathcal{K}^\perp$ along a path γ from $t = (X_t, [\omega_t])$ to t_0 and

$$p'_{\nu(\gamma)}\colon K_{[\mu(\omega_t)]}^\perp \xrightarrow{\sim} K_{[\mu(\omega_0)]}^\perp$$

the parallel transport of $\mathcal{K}'^\perp$ along the image path $\nu(\gamma)$, then

$$p'^{-1}_{\nu(\gamma)}\phi_0 p_\gamma\colon K_{[\omega_t]}^\perp \xrightarrow{\sim} K_{[\mu(\omega_t)]}^\perp$$

is still a Hodge isometry.

Proof. Since both L_d and N_d are primitive sublattices (of $I_{22,2}$ and $I_{21,2}$ respectively), so we have $(L_d^{\perp})^{\perp} = L_d$ and $(N_d^{\perp})^{\perp} = N_d$, hence by Corollary 2.1.13, $H(L_d)$ acts on $\Omega(L_d^{\perp})$ by self-isometries of $L_d^{\perp}$ inducing identity on the discriminant group $A_{L_d^{\perp}}$, and similarly for the action of $H(N_d)$ on $\Omega(N_d^{\perp})$; thus the isometry $\mu\colon L_d^{\perp} \simeq N_d^{\perp}$ induces an isomorphism of varieties $\mathcal{D}_{L_d}^{lab} \simeq \mathcal{D}_{N_d}^{lab}$. The period map $\wp\colon \mathcal{G} \to \mathcal{D}$ of Gushel-Mukai fourfolds is a dominant smooth submersion and the period map $\wp'$ of cubic fourfold embeds $\mathcal{C}$ as a Zariski dense open subset of $\mathcal{D}'$, hence

$$\nu\colon \mathcal{G}_{L_d}^{lab} \xrightarrow{\wp} \mathcal{D}_{L_d}^{lab} \simeq \mathcal{D}_{N_d}^{lab} \xrightarrow{\wp'^{-1}} \mathcal{C}_{N_d}^{lab}$$

can be defined on some Zariski open U subset of $\mathcal{G}_{L_d}^{lab}$ and is a smooth submersion on that (The period map $\wp\colon \mathcal{G} \longrightarrow \mathcal{D}$ is a smooth submersion).

For the second part, let $f_0\colon I_{22,2} \longrightarrow \mathrm{H}^4(X_0,\mathbb{Z})$ be a marking on X_0 such that $f_0^{-1}\mathrm{H}^{3,1}(X_0) = \omega_0$ (or equivalently $K_{[\omega_0]} = f_0(L_d)$) and $f_0'\colon I_{21,2} \longrightarrow \mathrm{H}^4(Y_{\nu(0)},\mathbb{Z})$ be a marking on $Y_{\nu(0)}$ such that $f_0'^{-1}\mathrm{H}^{3,1}(Y_{\nu(0)}) = \mu(\omega_0)$ (or equivalently $K_{[\mu(\omega_0)]} = f_0'(N_d)$),

$$\phi_0\colon K_{[\omega_0]}^{\perp} = f_0(L_d^{\perp}) \overset{f_0^{-1}}{\cong} L_d^{\perp} \overset{\mu}{\cong} N_d^{\perp} \overset{f_0'}{\cong} f_0'(N_d^{\perp}) = K_{[\mu(\omega_0)]}^{\perp}$$

is a Hodge isomtery. To check that ϕ_0 remains Hodge isometry under parallel transport (which is a local statement), we consider $t = (X_t, [\omega_t])$ in some analytic neighborhood Δ of $t_0 \in T$, let $p_t\colon \mathrm{H}^4(X_t,\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(X_0,\mathbb{Z})$ be parallel transport along some path from t to 0 inside Δ, and $p'_{\nu(t)}\colon \mathrm{H}^4(Y_{\nu(t)},\mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(Y_{\nu(0)},\mathbb{Z})$ from $\nu(t)$ to $\nu(0)$ inside $\Delta' := \nu(\Delta)$ (assuming Δ and Δ' are contractible so these parallel transport operators are path-independent). Then we have the following commutative diagram

$$\begin{tikzcd}
t \arrow[r, mapsto] & f_0^{-1}p_t\mathrm{H}^{3,1}(X_t) \\
\Delta \arrow[r, "\omega_t"] \arrow[d, "\nu"'] & \Omega(L_d^{\perp}) \arrow[d, "\mu"] \\
\Delta' \arrow[r, "\omega'_s"] & \Omega(N_d^{\perp}) \\
s \arrow[r, mapsto] & f_0'^{-1}p'_s\mathrm{H}^{3,1}(Y_s)
\end{tikzcd} \tag{3.6.2}$$

which shows that the composition

$$p'^{-1}_{\nu(t)}\phi_0 p_t \colon K^{\perp}_{\omega_t} = p_t^{-1}f_0(L_d^{\perp}) \cong L_d^{\perp} \overset{\mu}{\cong} N_d^{\perp} \cong p'^{-1}_{\nu(t)}f'_0(N_d^{\perp}) = K^{\perp}_{\mu(\omega_t)}$$

maps $\mathrm{H}^{3,1}(X_t)$ into $\mathrm{H}^{3,1}(Y_{\nu(t)})$, hence a Hodge isometry. □

Remark 3.6.3. Later on we will consider deformations over smooth spaces, which requires us to lift the above rational map to some smooth coverings of $\mathcal{G}^{lab}_{L_d}$ and $\mathcal{C}^{lab}_{N_d}$. To do so, observe that $H(L_d)$ is arithmetically defined, hence by [Huy16, Chapter 6], we can find a torsion free subgroup $\Gamma \subset H(L_d)$ of finite index and the quotient $\overline{\mathcal{D}^{lab}_{L_d}} := \Gamma\backslash\Omega(L_d^{\perp})$ will be a smooth quasi-projective variery, and there is a finite covering $\overline{\mathcal{D}^{lab}_{L_d}} \longrightarrow \mathcal{D}^{lab}_{L_d}$; so if we put $\overline{\mathcal{G}^{lab}_{L_d}} := \overline{\mathcal{D}^{lab}_{L_d}} \times_{\mathcal{D}} \mathcal{G}$, it is a smooth finite covering $\overline{\mathcal{G}^{lab}_{L_d}} \longrightarrow \mathcal{G}^{lab}_{L_d}$, and we define $\overline{\mathcal{C}^{lab}_{N_d}}$ in a similar fashion. Then when d satisfies (†), the rational map ν lifts to a rational map $\nu \colon \overline{\mathcal{G}^{lab}_{L_d}} \dashrightarrow \overline{\mathcal{C}^{lab}_{N_d}}$ which is still generically a smooth submersion. The statements of Lemma 3.6.1 and 3.6.2 hold true if we replace $\mathcal{G}^{lab}_{L_d}$ and $\mathcal{C}^{lab}_{N_d}$ by $\overline{\mathcal{G}^{lab}_{L_d}}$ and $\overline{\mathcal{C}^{lab}_{N_d}}$ respectively, with the families of fourfolds replaced by their pullbacks to these smooth base.

Chapter 4

K3 Category and Mukai Lattice

4.1 K3 category and Kuznetsov components

Definition 4.1.1. We define a *K3 category* to be an admissible subcategory $\mathcal{T}$ of the bounded derived category of some smooth projective variety such that the Serre functor $S_{\mathcal{T}}$ of $\mathcal{T}$ is given by [2] (shifting by 2).

The most basic example of a K3 category is the bounded derived category $\mathrm{D}^b(S)$ of a K3 surface S. More nontrivial examples of K3 categories can be constructed as the so called Kuznetsov components of the bounded derived categories of some Fano fourfolds. To define it, let us first recall the notion of an exceptional sequence:

Let W be a smooth projective variety, an ordered collection of objects

$$(E_1, E_2, \cdots, E_m) \subset \mathrm{D}^b(W)$$

is called an *exceptional sequence* if

$$\begin{cases} \mathrm{RHom}(E_i, E_i) = \mathbb{C}[0], \\ \mathrm{RHom}(E_i, E_j) = 0, \text{ for all } i > j. \end{cases}$$

Definition 4.1.2. Let W be a smooth projective variety and let $(E_1, E_2, \cdots, E_m)$

be an exceptional sequence in $\mathrm{D}^b(W)$, then the right orthogonal complement

$$\mathcal{A}_W := \langle E_1, E_2, \cdots, E_m \rangle^{\perp},$$

which is an admissible subcategory of $\mathrm{D}^b(W)$, is called the *Kuznetsov component* of W.

From now on, W will always be a Gushel-Mukai fourfold or a cubic fourfold. In this case, $\mathrm{D}^b(W)$ naturally contains some exceptional sequences ([Kuz10], [KP18]).

- If W is a Gushel-Mukai fourfold, then

$$(\mathcal{O}_W, \mathcal{U}_W^{\vee}, \mathcal{O}_W(1), \mathcal{U}_W^{\vee}(1)) \tag{4.1.1}$$

  is an exceptional sequence in $\mathrm{D}^b(W)$. Here $\mathcal{U}_W$ is the pullback to W of the tautological (sub)bundle on $\mathrm{Gr}(2,5)$ by the Gushel map $\gamma_W : W \to \mathrm{Gr}(2,5)$.

- If W is a cubic fourfold, then

$$(\mathcal{O}_W, \mathcal{O}_W(1), \mathcal{O}_W(2)) \tag{4.1.2}$$

  is an exceptional sequence in $\mathrm{D}^b(W)$.

In both cases, the Kuznetsov components $\mathcal{A}_W$ have Serre functor given by [2], i.e. they are K3 categories. Hence we can make the following definitions.

Definition 4.1.3. . Let W be a Gushel-Mukai fourfold or a cubic fourfold, we say that W is homological-associated to the K3 surface S if there is an exact equivalence $\mathcal{A}_W \simeq \mathrm{D}^b(S)$.

Similarly if X is a Gushel-Mukai fourfold and Y a cubic fourfold, we say that X is homological-associated to Y is there is an exact equivalence $\mathcal{A}_X \simeq \mathcal{A}_Y$.

The followings are some explicit examples of homological associations:

- If X is a Gushel-Mukai forufold containing a quintic del pezzo surface, $\mathcal{A}_X \simeq \mathrm{D}^b(S)$ for some K3 surface ([KP18, Theorem 1.2]).
- If Y is a Pfaffian cubic fourfold, then $\mathcal{A}_Y \simeq \mathrm{D}^b(S)$ for some K3 surface S (essentially due to [Kuz06, Theorem 2]).

In both cases, the K3 surfaces can be concretely constructed by some classical projective geometric argument.

- [KP18, Theorem 1.3]. If X is a generic Gushel-Mukai fourfold containing a plane of type $Gr(2,3)$, then there is a cubic fourfold Y such that $\mathcal{A}_X \simeq \mathcal{A}_Y$. The cubic fourfold Y can be obtained by performing some birational changes on X.

Later in this section, we will see that homological association between fourfolds and K3 surfaces is closely related to their Hodge associations and exists in much greater abundance. For the moment, we will content ourselves to just point out the fact that all the exact equivalences involved in defining homological associations are of Fourier-Mukai type. For the "fourfold-K3 association", this just follows from a well-known result due to Orlov [Or03], which states that the fully-faithful exact functor $\mathrm{D}^b(S) \to \mathrm{D}^b(W)$ which identifies $\mathrm{D}^b(S)$ with $\mathcal{A}_W \subset \mathrm{D}^b(W)$ must be a Fourier-Mukai transform. For "fourfolds-fourfolds association", we first need to make sense what is meant by a Fourier-Mukai transform between admissible subcategories:

Definition 4.1.4. Let W and W' be any two smooth projective varieties, suppose we have admissible subcategories $\mathcal{T} \overset{\iota}{\hookrightarrow} \mathrm{D}^b(W)$ and $\mathcal{T}' \overset{\iota'}{\hookrightarrow} \mathrm{D}^b(W')$. An exact functor $\Phi \colon \mathcal{T} \to \mathcal{T}'$ is of *Fourier-Mukai type* if there is a Fourier-Mukai transform $\Phi_{\mathcal{P}} \colon \mathrm{D}^b(W) \to \mathrm{D}^b(W')$ such that

$$\Phi_{\mathcal{P}} = \iota' \circ \Phi \circ \delta$$

where $\delta : \mathrm{D}^b(W) \to \mathcal{T}$ is the left adjoint of the inclusion functor ι.

Exact equivalences between Kuznetsov components are of Fourier-Mukai type, this is due to the following recent result:

Theorem 4.1.5. ([LPZ22], Theorem 1.3). *Let W and W' be Gushel-Mukai fourfolds or cubic fourfolds, then any fully faithful exact functor $\mathcal{A}_W \longrightarrow \mathcal{A}_{W'}$ is of Fourier-Mukai type.*

4.2 Mukai Lattices of K3 surfaces

The Mukai lattice is first introduced by Mukai for ordinary K3 surfaces, we briefly review the construction.

Let S be a K3 surface, its Mukai lattice $\tilde{\mathrm{H}}(S,\mathbb{Z})$ is the weight-2 integral Hodge structure whose underlying abelian group is the full singular chomology group

$$\mathrm{H}^*(S,\mathbb{Z}) = \mathrm{H}^0(S,\mathbb{Z}) \oplus \mathrm{H}^2(S,\mathbb{Z}) \oplus \mathrm{H}^4(S,\mathbb{Z})$$

with Hodge decomposition given by

$$\begin{aligned} \tilde{\mathrm{H}}^{2,0}(S) &= \mathrm{H}^{2,0}(S) \\ \widetilde{\mathrm{H}}^{1,1}(S) &= \mathrm{H}^{0,0}(S) \oplus \mathrm{H}^{1,1}(S) \oplus \mathrm{H}^{2,2}(S) \\ \tilde{\mathrm{H}}^{0,2}(S) &= \mathrm{H}^{0,2}(S) \end{aligned}$$

The intersection pairing on $\tilde{\mathrm{H}}(S,\mathbb{Z})$ is given by the *Mukai pairing*:

$$(c, c') = \int_S (c_0 c'_4 - c_2 c'_2 + c_4 c'_0)$$

It is well-known that the middle cohomology $\mathrm{H}^2(S,\mathbb{Z})$, under the usual intersection paring $\int_S c_2 c'_2$, is isometric to the abstract lattice $U^3 \oplus E_8(-1)^2$; therefore, $\tilde{\mathrm{H}}(S,\mathbb{Z})$, under the Mukai pairing, is given by $U^4 \oplus E_8^2$. The Mukai vector

$$v\colon \mathrm{K}_{top}(S) \to \tilde{\mathrm{H}}(S, \mathbb{Z})$$

is an isomorphism (over $\mathbb{Z}$) for K3 surfaces, and hence an isometry of lattices (when $\mathrm{K}_{top}(S)$ is equipped with the Euler pairing).

The Mukai Hodge structure allows us to formulate the Torelli theorem for derived equivalence between K3 surfaces. Given any exact equivalence

$$\Phi : \mathrm{D}^b(S) \to \mathrm{D}^b(S')$$

between derived categories of K3 surfaces (which is necessarily of Fourier-Mukai type), it induces an isomorphism on cohomologies:

$$\Phi^H : \mathrm{H}^*(S, \mathbb{Q}) \to \mathrm{H}^*(S', \mathbb{Q}).$$

It turns out that for K3 surfaces, Φ^H maps $\mathrm{H}^*(S, \mathbb{Z})$ isomorphically onto $\mathrm{H}^*(S', \mathbb{Z})$ and it is straightforwad to check that it induces a Hodge isomtery $\tilde{\mathrm{H}}(S, \mathbb{Z}) \to \tilde{\mathrm{H}}(S', \mathbb{Z})$ between the Mukai lattices.

The converse is also true under certain circumstances. To elaborate on this, first of all the Mukai lattice $\tilde{\mathrm{H}}(S, \mathbb{Z})$ of K3 surface S has signature $(20, 4)$, thus we can talk about an *orientation on its four negative directions*, which is defined to be a choice of an orientation on a negative-definite 4-dimesional vector subspace of $\tilde{\mathrm{H}}(S, \mathbb{R})$; two such oriented subspaces V_1 and V_2 are considered to give rise to the same orientation on the negative directions of $\tilde{\mathrm{H}}(S, \mathbb{Z})$ if the vector space isomorphism given by natural inclusion followed by orthogonal projection

$$V_1 \hookrightarrow \tilde{\mathrm{H}}(S, \mathbb{R}) \to V_2$$

is orientation preserving. Given any Kähler class $\alpha \in \mathrm{H}^{1,1}(S)$ and fix a generator σ of $\mathrm{H}^{2,0}(S) \simeq \mathbb{C}$, the 4-dimensional subspace of $\tilde{\mathrm{H}}(S, \mathbb{R})$ spanned by the classes $\mathrm{Re}(\sigma), \mathrm{Im}(\sigma), 1-\frac{\alpha^2}{2}$ and α is negative-definite, therefore it determines an orientation on the negative directions of $\tilde{\mathrm{H}}(S, \mathbb{Z})$; since the Kähler cone of S is connected and rescaling σ won't change the orientation on the plane spanned by $\mathrm{Re}(\sigma)$ and $\mathrm{Im}(\sigma)$, this orientation is independent of the choice of Kähler class α and the generator σ, and we choose it to be the *natural orientation* on the four negative directions of $\tilde{\mathrm{H}}(S, \mathbb{Z})$. It should be clear what is meant by an isometry $\tilde{\mathrm{H}}(S, \mathbb{Z}) \simeq \tilde{\mathrm{H}}(S', \mathbb{Z})$ to preserve the natural orientation.

Theorem 4.2.1. ([Huy06], Corollary 10.13). *Suppose $\phi\colon \tilde{H}(S, \mathbb{Z}) \to \tilde{H}(S', \mathbb{Z})$ is a Hodge isometry respecting the natural orientation on the four negative directions, then there exists an exact equivalence $\Phi\colon D^b(S) \to D^b(S')$ with $\Phi^H = \phi$.*

4.3 Mukai Lattices of K3 categories

We wish to define a similar weight-2 integral Hodge structure for the K3 category $\mathcal{A}_W$. Suggested by the situation for ordinary K3 surfaces, we make use of the topological K-theory and Mukai vectors. $\mathrm{H}^*(W, \mathbb{Z})$ is torsion free, thus $\mathrm{K}_{top}(W)$ is also torsion free and the Mukai vector $v\colon \mathrm{K}_{top}(W) \longrightarrow \mathrm{H}^*(W, \mathbb{Q})$ is injective and induces an isomorphism of $\mathbb{C}$-vector spaces.

Definition 4.3.1. The *Mukai lattice* $\widetilde{\mathrm{H}}(\mathcal{A}_W, \mathbb{Z})$ for the Kuznetsov components $\mathcal{A}_W$ is the lattice on the free abelian group (it is torsion free because $\mathrm{K}_{top}(W)$ is)

$$\mathrm{K}_{top}(\mathcal{A}_W) := \{\kappa \in \mathrm{K}_{top}(W) |\ \chi([E_i], \kappa),\ i = 1, \ldots, m\}$$

endowed with the Euler pairing (we will soon find out that the Euler pairing on $\mathrm{K}_{top}(\mathcal{A}_W)$ is indeed symmetric), where $\{E_1, \ldots, E_m\}$ is the exceptional sequence

(4.1.1) for Gushel-Mukai fourfold and (4.1.2) for cubic fourfold. We define the *Mukai Hodge structure* on $\widetilde{\mathrm{H}}(\mathcal{A}_W, \mathbb{Z})$ to be the following weight-2 Hodge structure given by:

$$\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_W) := v^{-1}(\mathrm{H}^{3,1}(W)),$$

$$\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W) := v^{-1}(\bigoplus_{p=0}^{4} \mathrm{H}^{p,p}(W)),$$

$$\widetilde{\mathrm{H}}^{0,2}(\mathcal{A}_W) := v^{-1}(\mathrm{H}^{1,3}(W)).$$

We also denote $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W, \mathbb{Z}) := \mathrm{K}_{top}(\mathcal{A}_W) \cap \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W)$.

Observe that the Mukai vector $v(E_i)$ of the exceptional objects are all of Hodge-type, consequently $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_W)$ is the one dimensional subspace of $\mathrm{K}_{top}(W) \otimes \mathbb{C}$ that is mapped isomorphically onto $\mathrm{H}^{3,1}(W)$ by v. In particular, the Mukai Hodge structure is K3 type.

Similar to the case of usual K3 surfaces, an equivalence of Kuznetsov components should induce a Hodge isometry of Mukai lattices.

Proposition 4.3.2. ([Huy17]) *(1) Let W be a Gushel-Mukai fourfold or a cubic fourfold, any exact equivalence $\mathcal{A}_W \simeq D^b(S)$, where S is a K3 surface, induces a Hodge isometry*

$$\widetilde{H}(\mathcal{A}_W, \mathbb{Z}) \simeq \widetilde{H}(S, \mathbb{Z}).$$

(2) Let W and W' be Gushel-Mukai or cubic fourfolds, any exact equivlance $\mathcal{A}_W \simeq \mathcal{A}'_W$ between the Kuznetsov components induces a Hodge isometry

$$\widetilde{H}(\mathcal{A}_W, \mathbb{Z}) \simeq \widetilde{H}(\mathcal{A}_{W'}, \mathbb{Z}).$$

Proof. We give a brief account of (2), (1) is completely similar. Given an exact equivalence $\Phi\colon \mathcal{A}_W \xrightarrow{\sim} \mathcal{A}'_W$, we know that it must be of Fourier-Mukai type, meaning there is a Fourier-Mukai transform $\Phi_P\colon \mathrm{D}^b(W) \longrightarrow \mathrm{D}^b(W')$ which factors as

$$\mathrm{D}^b(W) \to \mathcal{A}_W \xrightarrow{\Phi} \mathcal{A}_{W'} \hookrightarrow \mathrm{D}^b(W').$$

Then it is straightforward to check that the induced map $\Phi_P^K\colon \mathrm{K}_{top}(W) \longrightarrow \mathrm{K}_{top}(W')$ brings $\mathrm{K}_{top}(\mathcal{A}_W)$ isometrically onto $\mathrm{K}_{top}(\mathcal{A}_{W'})$, hence we have an isometry of lattice $\tilde{\mathrm{H}}(\mathcal{A}_W, \mathbb{Z}) \simeq \tilde{\mathrm{H}}(\mathcal{A}_{W'}, \mathbb{Z})$. To see that it is an Hodge isometry we only need to show that Φ_P^K brings $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_W) = v^{-1}\mathrm{H}^{3,1}(W)$ into $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_{W'}) = v^{-1}\mathrm{H}^{3,1}(W')$, but this simply follows from that Φ_P^K is compatible, via v, with Φ_P^H, and Φ_P^H maps $\mathrm{H}^{3,1}(W)$ into $\mathrm{H}^{3,1}(W')$. □

We know that there exists Gushel-Mukai (resp. cubic) fourfolds homological-associated to K3 surfaces, and that all Gushel-Mukai (resp. cubic) fourfolds are deformation equivalent, therefore we have

Corollary 4.3.3. *Let W be an arbitrary Gushel-Mukai or cubic fourfold, $\tilde{H}(\mathcal{A}_W, \mathbb{Z})$ is isometric to the abstract lattice $U^4 \oplus E_8^2$. (in particular, the Euler pairing on $\mathrm{K}_{top}(\mathcal{A}_W)$ happens to be symmetric).*

Remark. We are using the fact that $\mathrm{K}_{top}(\mathcal{A}_W)$ is a deformation invariant, in fact it can be shown that the parallel transports of $\mathrm{K}_{top}(W)$ when W vary in a family preserves the subgroup $\mathrm{K}_{top}(\mathcal{A}_W)$. We will discuss this in more detail in the next subsection.

There is a copy of hyperbolic plane U primitively contained in $\widetilde{\mathrm{H}}^{1,1}(S, \mathbb{Z})$, hence by the previous Proposition again if the fourfold W is homological-associated to a K3 surface, then $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W, \mathbb{Z})$ necessarily contains a copy of U. This turns out to be a sufficient condition for homological association with K3 surfaces:

Theorem 4.3.4. ([BLM+19] amd [PPZ19]). *Let W be a Gushel-Mukai fourfold or a cubic fourfold, if $\widetilde{H}^{1,1}(\mathcal{A}_W, \mathbb{Z})$ contains primitive sublattice isometric to U, then there exists a K3 surface S such that $\mathcal{A}_W \simeq D^b(S)$.*

This theorem allows us to compare Hodge association and homological association between the fourfolds and K3 surfaces. Notice that it essentially says that a fourfold W being homological-associated to a K3 surface is detected by its Mukai lattice $\tilde{\mathrm{H}}(\mathcal{A}_W, \mathbb{Z})$, hence to compare to Hodge association, it is desirable to reinterpret the condition for a fourfold W being Hodge-special using its Mukai lattice.

In order to do so, we notice that $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W, \mathbb{Z})$ always contain some special rank 2 sublattice ([AT14] and [Pert17]):

- If X is a cubic fourfold, we take $\lambda_1 = \mathrm{pr}([\mathcal{O}_{line}(1)])$ and $\lambda_2 = \mathrm{pr}([\mathcal{O}_{line}(2)])$ where pr is the projection map from $\mathrm{K}_{top}(X)$ into $\mathrm{K}_{top}(\mathcal{A}_X)$, and compute the Euler pairing one has $\langle \lambda_1, \lambda_2 \rangle = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} = -A_2$. Notice that $\langle \lambda_1, \lambda_2 \rangle$ is a primitive sublattice of $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X, \mathbb{Z})$.

- If Y is a Gushel-Mukai fourfold, there also exists some special classes $\kappa_1, \kappa_2 \in \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_Y, \mathbb{Z})$ (We avoid giving their definitions here, since they are somewhat involved) and we have $\langle \kappa_1, \kappa_2 \rangle = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix} = -A_1^{\oplus 2}$.

Proposition 4.3.5. *(1) For a Gushel-Mukai fourfold X* ([Pert17], Proposition 3.2) *the Mukai vector $v : K_{top}(\mathcal{A}_X) \to H^*(X, \mathbb{Z})$ induces a Hodge isometry (up to a shift of bidegree)*

$$\langle \kappa_1, \kappa_2 \rangle^{\perp} \simeq H^4(X, \mathbb{Z})_{00}.$$

Moreover it maps the sublattice

$$\langle \kappa_1, \kappa_2, \zeta_1, \cdots, \zeta_n \rangle^{\perp}$$

isometrically onto the sublattice

$$\langle \gamma_X^*\sigma_{1,1}, \gamma_X^*\sigma_2, c_2(\zeta_1), \cdots, c_2(\zeta_n)\rangle^{\perp}$$

for any objects $\zeta_1, \cdots, \zeta_n$ *in* $K_{top}(\mathcal{A}_X)$.

Moreover, c_2 *induces an isomorphism of groups*

$$\bar{c}_2\colon \frac{\widetilde{H}(\mathcal{A}_X, \mathbb{Z})}{\langle \kappa_1, \kappa_2\rangle} \longrightarrow \frac{H^4(X,\mathbb{Z})}{\langle \gamma_X^*\sigma_{1,1}, \gamma_X^*\sigma_2\rangle}$$

identifying $\frac{\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})}{\langle \kappa_1,\kappa_2\rangle}$ *with* $\frac{H^{2,2}(X,\mathbb{Z})}{\langle \gamma_X^*\sigma_{1,1},\gamma_X^*\sigma_2\rangle}$.

(2) For a cubic fourfold Y ([AT14], Propostion 2.3 & Proposition 2.4) *the Mukai vector* $v : K_{top}(\mathcal{A}_Y) \to H^*(Y,\mathbb{Z})$ *induces a Hodge isometry*

$$\langle \lambda_1, \lambda_2\rangle^{\perp} \simeq H^4(Y,\mathbb{Z})_0$$

Which maps $\langle \lambda_1, \lambda_2, \xi_1, \cdots, \xi_n\rangle^{\perp}$ *isometrically onto* $\langle h^2, c_2(\xi_1), \cdots, c_2(\xi_n)\rangle^{\perp}$*for any objects* $\xi_1, \cdots, \xi_n$ *in* $K_{top}(\mathcal{A}_Y)$.

Moreover, c_2 *induces an isomorphism of groups*

$$\bar{c}_2\colon \frac{\widetilde{H}(\mathcal{A}_Y, \mathbb{Z})}{\langle \lambda_1, \lambda_2\rangle} \longrightarrow \frac{H^4(Y,\mathbb{Z})}{\langle h^2\rangle}$$

identifying $\frac{\widetilde{H}^{1,1}(\mathcal{A}_Y,\mathbb{Z})}{\langle \lambda_1,\lambda_2\rangle}$ *with* $\frac{H^{2,2}(Y,\mathbb{Z})}{\langle h^2\rangle}$.

One of the main take away of the Proposition is that the inverse of the Mukai vector gives us an inclusion of Hodge structures $v^{-1}\colon \mathrm{H}^4(X,\mathbb{Z})_{00} \hookrightarrow \tilde{\mathrm{H}}(\mathcal{A}_X,\mathbb{Z})$ (and $v^{-1}\colon \mathrm{H}^4(Y,\mathbb{Z})_0 \hookrightarrow \widetilde{\mathrm{H}}(\mathcal{A}_Y,\mathbb{Z})$ for cubic fourfold Y), hence we can translate the data of a labelling on a fourfold X to its Mukai lattice:

Corollary 4.3.6. *A labelling* $K_d \subset H^{2,2}(X,\mathbb{Z})$ *on a Gushel-Mukai fourfold* X *is equivalent to a non-degenerate rank 3 primitive sublattice* $M_d \subset \widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$ *of* *signature* $(1,2)$ *containing* κ_1 *and* κ_2*, the non-special cohomology* $K_d^\perp \subset H^4(X,\mathbb{Z})_{00}$ *can be identified, by the Mukai vector, with the sublattice* $M_d^\perp \subset \widetilde{H}(\mathcal{A}_X,\mathbb{Z})$*.*

Similarly, a labelling $K_d' \subset H^{2,2}(Y,\mathbb{Z})$ *on cubic fourfold* Y *is equivalent to a non-degenerate rank 3 primitive sublattice* $M_d' \subset \widetilde{H}^{1,1}(\mathcal{A}_Y,\mathbb{Z})$ *of signature* $(1,2)$ *containing* λ_1,λ_2 *and the non-speical cohomology is identified with* $M_d'^\perp \subset \widetilde{H}(\mathcal{A}_Y,\mathbb{Z})$ *by Mukai vector.*

Now the situation is clear: suppose we have a Gushel-Mukai or a cubic fourfold W, on the one hand from the discussion above we know that W being Hodge-associated to a K3 surface is equivalent to that $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W,\mathbb{Z})$ containing certain rank 3 sublattice with disriminant d satisfying the numerical conditions in Proposition 3.4.2; on the other hand, Theorem 4.3.4 tells us that W being homological-associated to some K3 surface if and only if $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W,\mathbb{Z})$ contains a copy of U. Therefore, comparing Hodge and homological asscociation between fourfolds and K3 surfaces is reduced to a purely lattice theoretic questions, and along this line we have

Theorem 4.3.7. *(1)* ([AT14, Theorem 3.1]). *Let* Y *be a cubic fourfold,* $\widetilde{H}^{1,1}(\mathcal{A}_Y,\mathbb{Z})$ *contains a primitive sublattice* M_d' *of discriminant* d *satisfying the numerical condition* $(**)'$ *as in Proposition 3.4.2 if and only if contains a copy of hyperbolic plane* U*.*

(2) ([Pert17, Theorem 3.6]). *Let* X *be a Gushel-Mukai fourfold, if* $\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$ *contains a primitive sublattice* M_d *of discriminant* d *satisfying the numerical condition* $(**)$ *as in Propostion 3.4.2, then* $\widetilde{H}^{1,1}(\mathcal{A}_X,\mathbb{Z})$ *contains a copy of* U*; the converse holds if in addition we assume one of the following:*

- $H^{2,2}(X,\mathbb{Z})$ *has rank 3, or*
- *There is an element* τ *in the hyperbolic plane such that the sublattice* $\langle\kappa_1,\kappa_2,\tau\rangle$ *has discriminant* $d \equiv 2$ *or* 4 *(mod* 8*).*

In particular we know that a cubic fourfold is homological-associated to K3 if and only if it has a Hodge theoretically associated one. For a Gushel-Mukai fourfold, it is homological-associated to a K3 surface if it is Hodge-associated to one, the converse holds at least for Gushel-Mukai fourfolds very general in divisors.

Remark. There exists Gushel-Mukai fourfolds with homological-associated K3 which cannot have Hodge-associated K3 (see [Pert17, 3.3]), therefore homological association of a K3 to Gushel-Mukai does not imply Hodge-theoretic association of a K3 in general.

Before proceeding, we make the following important observation that under the embedding given by the inverse of Mukai vectors, any Hodge isometry between non-special cohomologies extends to a Hodge isometry between Mukai lattices.

Proposition 4.3.8. *Let (X, K_d) be a special Gushel-Mukai fourfold and (Y, K'_d) be a special cubic fourfold, then any Hodge isometry $\phi : K_d^{\perp} \to K_d'^{\perp}$ extends to a Hodge isometry $\tilde{\phi}\colon \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \to \widetilde{H}(\mathcal{A}_Y, \mathbb{Z})$, in the sense that the following diagram commutes*

$$\begin{tikzcd} K_d \arrow[r, "\phi"] \arrow[d, hook, "v^{-1}"] & K'_d \arrow[d, hook, "v^{-1}"] \\ \widetilde{H}(\mathcal{A}_X, \mathbb{Z}) \arrow[r, "\tilde{\phi}"] & \widetilde{H}(\mathcal{A}_Y, \mathbb{Z}) \end{tikzcd} \tag{4.3.1}$$

Proof. Because the Mukai vector v identifies $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_X)$ with $\mathrm{H}^{3,1}(X)$ and $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_Y)$ with $\mathrm{H}^{3,1}(Y)$, any isometry of lattices $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ extending a Hodge isometry $K_d^{\perp} \simeq K_d'^{\perp}$ via v^{-1} will map $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_X)$ into $\widetilde{\mathrm{H}}^{2,0}(\mathcal{A}_Y)$, therefore is already a Hodge isometry. Thus we only need to show that there is a isometry of lattices $\tilde{\phi}$: $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ making the above diagram commutes. Now by corollary 4.3.3, both $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ and $\widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ are isometric to the even unimodular lattice $U^4 \oplus E_8^2$. So, we are reduced to show that an isometry $K_d^{\perp} \simeq K_d'^{\perp}$ extends, via some primitive embeddings $K_d^\perp, K_d'^\perp \hookrightarrow U^4 \oplus E_8^2$, to an isometry of $U^4 \oplus E_8^2$. This is where the of Nikulin (Theorem 2.1.14) comes in, notice that both $K_d^\perp$ and $K_d'^\perp$ are even (there are sublattices of even lattices) and $K_d^\perp \simeq K_d'^\perp$ implies that d satisfies the condition (†), therefore we must gave $d \equiv 2 \pmod 8$, then by the proof of Proposition 3.4.4, we know that $A_{K_d^\perp} \simeq A_{K_d} \simeq \mathbb{Z}/d\mathbb{Z}$, hence $\ell(A_{K_d^\perp}) = 1$, thus by Theorem 2.1.14 there is an isometry of lattice $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ extending $K_d^\perp \simeq K_d'^\perp$. □

Remark 4.3.9. It should be noted that the Hodge isomtery $\tilde{\phi}$ extending $\phi : K_d^\perp \simeq K_d'^\perp$ as in the last Proposition need not be unique. Indeed, using the notation as in the discussion after Proposition 4.3.5, the orthogonal complement of $K_d^\perp$ inside $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ is the primitive sublattice M_d of signature $(1, 2)$, we can change the sign of $\tilde{\phi}$ on this sublattice, the result is still an isometry extending ϕ. The ability of doing so will be important to us at the end.

4.4 Mukai lattices in family of fourfolds

We only lay out the detail in the case of Gushel-Mukai fourfolds, the case of cubic fourfolds follows similarly. Let $\pi : \mathcal{X} \longrightarrow T$ be a smooth family of Gushel-Mukai fourfolds, denote by $\mathcal{K}_{top}$ the local system of topological K-groups on the fibers of π and $\tilde{P}_\gamma : \mathrm{K}_{top}(X_1) \xrightarrow{\sim} \mathrm{K}_{top}(X_2)$ be the parallel transport operator along some continuous path γ in T, we first observe that $\tilde{P}_\gamma$ maps $\mathrm{K}_{top}(\mathcal{A}_{X_1})$ into $\mathrm{K}_{top}(\mathcal{A}_{X_2})$. To make sense of this, at least locally (assuming T is affine for example), $\mathrm{D}^b(\mathcal{X})$ has a semi-orthogonal decomposition which restricts to each fiber $X_t := \pi^{-1}(t)$ to the defining semi-orthogonal decomposition ([Kuz11])

$$\langle \mathcal{A}_{X_t}, \mathcal{O}_{X_t}, \mathcal{U}_{X_t}^\vee, \mathcal{O}_{X_t}(1), \mathcal{U}_{X_t}^\vee(1) \rangle$$

of the Kuznetsov component $\mathcal{A}_{X_t}$. Therefore the elements $[\mathcal{O}_{X_t}], [\mathcal{U}^\vee_{X_t}], [\mathcal{O}_{X_t}(1)]$, $[\mathcal{U}^\vee_{X_t}(1)]$, whose right orthogonal respect to the Euler pairing defines $\mathrm{K}_{top}(\mathcal{A}_{X_t})$, are restriction of classes in $\mathrm{K}_{top}(\mathcal{X})$.

We will always denote by $\mathcal{M}$ the local system of Mukai lattices on the fibers of any given family of Gushel-Mukai fourfolds; and by $\mathcal{M}'$ the local system of Mukai lattices on the fibers of any given family of cubic fourfolds. We will never need to deal with Mukai lattices for two different families of Gushel-Mukai (or cubic) fourfolds at the same time, so this shouldn't cause any confusion. Both $\mathcal{M}, \mathcal{M}'$ are local subsystems of the local systems of topological K-groups.

Proposition 4.3.5 gives us an embedding of Hodge structures $v^{-1} : \mathrm{H}^4(X, \mathbb{Z})_{00} \hookrightarrow \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ for any Gushel-Mukai fourfold X. Our first goal is to show that this extends to an inclusion of variation of Hodge structures $(R^4\pi_*\mathbb{Z})_{00} \hookrightarrow \mathcal{M}$ when we have a family of Gushel-Mukai fourfolds $\pi : \mathcal{X} \longrightarrow T$.

First we observe that if we let $P_\gamma : \mathrm{H}^4(X_1, \mathbb{Z}) \xrightarrow{\sim} \mathrm{H}^4(X_2, \mathbb{Z})$ be the parallel transport for the local system $R^4\pi_*\mathbb{Z}$ of singular cohomologies along some path γ and $\tilde{P}_\gamma$ be the parallel transport operator for the local system $\mathcal{K}_{top}$ of topological K-groups, then the following diagram commutes

$$\begin{array}{ccc} \mathrm{H}^4(X_1, \mathbb{C}) & \xrightarrow{P_\gamma} & \mathrm{H}^4(X_2, \mathbb{C}) \\ {\scriptstyle v}\uparrow & & {\scriptstyle v}\uparrow \\ \mathrm{K}_{top}(X_1) & \xrightarrow{\tilde{P}_\gamma} & \mathrm{K}_{top}(X_2) \end{array} \tag{4.4.1}$$

where the vertical maps v are Mukai vectors. This is simply because that both parallel transport operators are induced by a diffeomorphism $X_1 \simeq X_2$ and the Mukai vector is clearly functorial under diffeomorphisms. Together with the fact that $(R^4\pi_*\mathbb{Z})_{00} \subset R^4\pi_*\mathbb{Z}$ is an inclusion of local systems we have proved the following

Proposition 4.4.1. *If $\pi : \mathcal{X} \longrightarrow T$ is a family of Gushel-Mukai fourfolds, then* *the inverse of Mukai vector defines an embedding of variations of Hodge structures $(R^4\pi_*\mathbb{Z})_{00} \hookrightarrow \mathcal{M}$. Similarly if $\pi' : \mathcal{Y} \longrightarrow S$ is a family of cubic fourfolds, then the inverse of Mukai vector defines an embedding of variations of Hodge structures $(R^4\pi'_*\mathbb{Z})_0 \hookrightarrow \mathcal{M}'$.*

Recall that in 3.6 we defined the family $\pi : \mathcal{X}^{lab} \longrightarrow \mathcal{G}^{lab}_{L_d}$ of labelled Gushel-Mukai fourfolds and the family $\pi' : \mathcal{Y}^{lab} \longrightarrow \mathcal{C}^{lab}_{L'_d}$ of labelled cubic fourfolds, also recall that in this case we have inclusion of variations of Hodge structures $\mathcal{K}^\perp \subset (R^4\pi_*\mathbb{Z})_{00}$ and $\mathcal{K}'^\perp \subset (R^4\pi'_*\mathbb{Z})_0$ respectively. Hence on these families of fourfolds we have

Corollary 4.4.2. *For the family $\pi : \mathcal{X}^{lab} \longrightarrow \mathcal{G}^{lab}_{L_d}$ of labelled Gushel-Mukai fourfolds, the inverse of Mukai vector determines an embedding of variation of Hodge structures $\mathcal{K}^\perp \hookrightarrow \mathcal{M}$. Similar, for the family $\pi' : \mathcal{Y}^{lab} \longrightarrow \mathcal{C}^{lab}_{L'_d}$ of labelled cubic fourfolds the inverse of Mukai vector gives an embedding of variation of Hodge structures $\mathcal{K}'^\perp \hookrightarrow \mathcal{M}'$.*

In particular, we know that the parallel transports of Mukai lattices restrict to the parallel transports of non-special cohomologies on these families of labelled fourfolds.

Now Proposition 3.6.2 tells us that if the discriminant d satisfies the numerical condition (†) then we have an isometry of lattices $\mu : L_d^\perp \longrightarrow N_d^\perp$ inducing a rational map $\nu : \mathcal{G}^{lab}_{L_d} \dashrightarrow \mathcal{C}^{lab}_{N_d}$, and at each point $t = (X, [\omega]) \in \mathcal{G}^{lab}_{L_d}$ and $\nu(t) = (Y, [\mu(\omega)]) \in \mathcal{C}^{lab}_{N_d}$ we have a Hodge isometry $\phi : K^\perp_{[\omega]} \xrightarrow{\sim} K^\perp_{[\mu(\omega)]}$, which, by Proposition 4.3.8, extends to a Hodge isometry $\tilde{\phi} : \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \simeq \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$. Recall that ϕ remains a Hodge isometry under parallel translations, we now show that this is also the case for $\tilde{\phi}$.

Proposition 4.4.3. *Let $\tilde{\phi} : \widetilde{H}(\mathcal{A}_{X_0}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{Y_{\nu(0)}}, \mathbb{Z})$ be a Hodge isometry extending the Hodge isometry $\phi_0 : K^\perp_{[\omega_0]} \xrightarrow{\sim} K^\perp_{[\mu(\omega_0)]}$ as in Propostion 4.3.8. Let*

γ be any continuous path in (a suitable open subset of) $\mathcal{G}^{lab}_{L_d}$ from $(X_t, [\omega_t])$ to $(X_0, [\omega_0])$ and $\nu(\gamma)$ the image path in $\mathcal{C}^{lab}_{L_d}$ from $(Y_{\nu(t)}, [\mu(\omega_t)])$ to $(Y_{\nu(0)}, [\mu(\omega_0)])$, and let $\tilde{P}_\gamma$ and $\tilde{P}'_{\nu(\gamma)}$ be the parallel transport along these paths for the local systems of Mukai lattices, then

$$\tilde{P}'^{-1}_{\nu(\gamma)}\tilde{\phi}_0\tilde{P}_\gamma : \widetilde{H}(\mathcal{A}_{X_t}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{Y_{\nu(t)}}, \mathbb{Z})$$

is still a Hodge isometry.

Proof. Since parallel transports for the local system of Mukai lattices restricts to that of the local system of non-special cohomologies (Proposition 4.4.2), we see that $\tilde{P}'^{-1}_{\nu(\gamma)}\tilde{\phi}_0\tilde{P}_\gamma$ is an isometry of lattice extending the Hodge isometry $p'^{-1}_{\nu(\gamma)}\phi_0 p_\gamma : K^\perp_{[\omega_t]} \xrightarrow{\sim} K^\perp_{[\mu(\omega_t)]}$, hence by the fact that the Mukai vector v identifies $\mathrm{H}^{3,1}$ with $\widetilde{\mathrm{H}}^{2,0}$ again, $\tilde{P}'^{-1}_{\nu(\gamma)}\tilde{\phi}_0\tilde{P}_\gamma$ must be a Hodge isometry. □

Remark 4.4.4. Everything discussed in this subsection remains true if we replace the base spaces $\mathcal{G}^{lab}_{L_d}$ by $\overline{\mathcal{G}^{lab}_{L_d}}$ and $\mathcal{C}^{lab}_{N_d}$ by $\overline{\mathcal{C}^{lab}_{N_d}}$ respectively.

Chapter 5

Comparing Hodge- and homological-association between fourfolds

In this section, we compare Hodge-association and homological-association between Gushel-Mukai and cubic fourfolds, which is our Theorem 1.0.1.

As we have seen in 4.3, the comparison between Hodge and homological association of K3 surfaces to our fourfolds has being well-understood; it essentially reduces to the problem of comparing two lattice theoretic conditions on $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_W, \mathbb{Z})$. Unfortunately, this strategy breaks down when implemented to comparing Hodge and homological association between Gushel-Mukai and cubic fourfolds. The difficulty is that there is no lattice theoretic condition on $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X, \mathbb{Z})$ (X being a Gushel-Mukai fourfold) which implies that $\mathcal{A}_X \simeq \mathcal{A}_Y$ for some cubic fourfold Y, or vice versa. Therefore we retreat to the argument used by Addington & Thomas in [AT14], which develops a way to deform Fourier-Mukai equivalence. We state this deformation result, in the following form due to Huybrechts (see [Huy17, Proposition 5.1])\.

Theorem 5.0.1. *Let $\pi : \mathcal{X} \longrightarrow T$ be a smooth family of Gushel-Mukai fourfolds and $\pi' : \mathcal{Y} \longrightarrow S$ be a smooth family of cubic fourfolds, and there is a smooth submersion $f : T \longrightarrow S$ between the base spaces. Assume $\Phi : \mathcal{A}_{X_0} \xrightarrow{\sim} \mathcal{A}_{Y_{f(0)}}$*

is a Fourier-Mukai equivalence between the Kuznetsov components and let ϕ_0 : $\widetilde{H}(\mathcal{A}_{X_0}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{Y_{f(0)}}, \mathbb{Z})$ *be the induced Hodge isometry on the Mukai lattices. Suppose* ϕ *remains a Hodge isometry* $\phi_t : \widetilde{H}(\mathcal{A}_{X_t}, \mathbb{Z}) \xrightarrow{\sim} \widetilde{H}(\mathcal{A}_{Y_{f(t)}}, \mathbb{Z})$ *under parallel transport for all* $t \in T$*. Then there is a Zariski open subset* $U \subset T$ *such that for all* $t \in U$*, there is a Fourier-Mukai equivalence* $\Phi_t : \mathcal{A}_{X_t} \longrightarrow \mathcal{A}_{Y_{f(t)}}$.

We will explain the proof of this Theorem in 5.2, let us first use this to give a proof of Theorem 1.0.1.

5.1 Proof of Theorem 1.0.1

In this subsection, we prove

Theorem 5.1.1. (Theorem 1.0.1) *Let* $d > 8$ *be an interger satisfying the numerical condition* (†)*. There is a non-empty Zariski open subset* $\mathcal{U}_d$ *of* $\mathcal{G}_d$ *such that Gushel-Mukai fourfolds in* $\mathcal{U}_d$ *admit homological-associated cubic fourfolds; there is a non-empty Zariski open subset* $\mathcal{V}_d \subset \mathcal{C}_d$ *such that cubic fourfolds in* $\mathcal{V}_d$ *admit homological-associated Gushel-Mukai fourfolds.*

Proof. Let $T \subset \overline{\mathcal{G}_{L_d}^{lab}}$ be the domain of ν and $T' = \nu(T) \subset \overline{\mathcal{C}_{L_d}^{lab}}$. Let $\mathcal{X} \longrightarrow T$ be the family of labelled Gushel-Mukai fourfolds given by the restriction of the family $\pi : \mathcal{X}^{lab} \longrightarrow \overline{\mathcal{G}_{L_d}^{lab}}$ and $\mathcal{Y} \longrightarrow T'$ be the family of labelled cubic fourfolds given by the restriction of $\pi' : \mathcal{Y}^{lab} \rightarrow \overline{\mathcal{C}_{L_d}^{lab}}$. By our discussion in 3.6 and 4.4, we know that any point $t \in T$ can be viewed as a pair $(X, [\omega])$ where X is the fiber $\mathcal{X}_t$ and $[\omega] \in \mathcal{D}_{L_d}^{lab}$ gives rise a to labelling $K_{[\omega]}$ on X and $\nu(t)$ is given by $(Y, [\mu(\omega)])$ with Y the unique cubic fourfold determined by $[\mu(\omega)] \in \mathcal{D}_{L_d'}^{lab}$ and is the fiber $\mathcal{Y}_{f(t)}$, $[\mu(\omega)]$ determines a labelling $K_{[\mu(\omega)]}$ on Y and there is a Hodge isometry $\phi : K_{[\omega]}^{\perp} \xrightarrow{\sim} K_{[\mu(\omega)]}^{\perp}$ between the non-special cohomologies that extends to a Hodge isometry $\tilde{\phi} : \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ between Mukai lattices which, under parallel transports, remains a Hodge isometry. So we only need to show, for suitable choices of the point $t = (X, [\omega])$, the Hodge isometry ϕ is induced by an exact equivalence, then by Theorem 5.0.1, this equivalence deforms to a Zariski open subset $\mathcal{U} \subset \overline{\mathcal{G}^{lab}_{L_d}}$. Then we just let $\mathcal{U}_d$ be the image of $\mathcal{U}$ under the covering map $\overline{\mathcal{G}^{lab}_{L_d}} \longrightarrow \mathcal{G}_d$ and let $\mathcal{V}_d$ be the image of $\nu(\mathcal{U}) \subset \overline{\mathcal{C}^{lab}_{L_d}}$ under $\overline{\mathcal{C}^{lab}_{L_d}} \longrightarrow \mathcal{C}_d$.

It turns out that in order for the Hodge isometry $\tilde{\phi} : \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z})$ to be induced by an exact equivalence, it is sufficient that $\mathcal{A}_Y$ is exact equivalent to $\mathrm{D}^b(S')$ for some K3 surface S', meaning Y homological-associated to S'. To see this we let $\Delta' : \mathcal{A}_Y \xrightarrow{\sim} \mathrm{D}^b(S')$ be an exact equivalence, recall that it induces a Hodge isometry $\delta' : \widetilde{\mathrm{H}}(\mathcal{A}_Y, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S', \mathbb{Z})$. Now the composition $\delta'\tilde{\phi} : \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S', \mathbb{Z})$ is a Hodge isometry and it shows that $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X, \mathbb{Z})$ contains a copy of U, therefore by Theorem 4.3.4 there is an exact equivalence $\Delta : \mathcal{A}_X \xrightarrow{\sim} \mathrm{D}^b(S)$ for some K3 surface S with an induced Hodge isometry $\delta : \widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S, \mathbb{Z})$. Hence we have Hodge isometry $\delta'\tilde{\phi}\delta^{-1} : \widetilde{\mathrm{H}}(S, \mathbb{Z}) \xrightarrow{\sim} \widetilde{\mathrm{H}}(S', \mathbb{Z})$ between Mukai lattices of K3 surfaces, we now show that we can assume it preserves the natural orientation on the four negative directions.

If $\delta'\tilde{\phi}\delta^{-1}$ does not preserve the natural orientation, we perform the following modification on $\tilde{\phi}$: recall that $\tilde{\phi}$ restricts to the Hodge isometry $\phi : K^{\perp}_{[\omega]} \xrightarrow{\sim} K^{\perp}_{[\mu(\omega)]}$ between the non-special cohomologies. By Corollary 4.3.6, the orthogonal complement of $K^{\perp}_{[\omega]}$ inside $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ is given by a certain rank 3 sublattice M, it turns out that the hyperbolic plane U contained in $\widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X, \mathbb{Z})$ in this case can be constructed inside M (see the proof of Theorem 3.1 of [AT14]). This shows that $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ can be written as $U \oplus U^{\perp}$ ($\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$ agrees with $U \oplus U^{\perp}$ by Proposition 2.1.8 (2)) with the sublattice $K^{\perp}_{[\omega]}$ contained in $U^{\perp}$. Now we can modify $\tilde{\phi}$ by changing its sign on the copy of U in $\widetilde{\mathrm{H}}(\mathcal{A}_X, \mathbb{Z})$, the resulting isometry still restricts to ϕ on $K^{\perp}_{[\omega]}$. This changes the sign of $\tilde{\phi}$ on exactly one negative direction.

Hence by Theorem 4.2.1, $\delta'\tilde{\phi}\delta^{-1}$ is induced by some derived equivalence

$$\Psi : \mathrm{D}^b(S) \xrightarrow{\sim} \mathrm{D}^b(S').$$

Then we see that $\tilde{\phi}$ is induced by $\Delta'^{-1}\Psi\Delta : \mathcal{A}_X \xrightarrow{\sim} \mathcal{A}_Y$.

Lastly, to see that we can choose the point t so that the resulting cubic fourfold Y has homological associated K3, we notice that the image of T' in $\mathcal{C}_d$ under the map $\overline{\mathcal{C}_{N_d}^{lab}} \longrightarrow \mathcal{C}_d$ is a Zariski open subset of $\mathcal{C}_d$, which, by Proposition 3.5.5, contains a Hodge-associated, therefore homological-associated K3 surface. This shows that we can choose $t \in T$ at the beginning such that the resulting cubic fourfold Y corresponding to the point $\nu(t)$ has a homological-associated K3. $\square$

5.2 Deformation of equivalence of categories

We dedicate this subsection to the proof of Theorem 5.0.1. This result is first proved by Addington & Thomas (see [AT14]) for the case of deforming an exact functor $\mathrm{D}^b(S) \longrightarrow \mathrm{D}^b(Y)$ where S is a K3 surface and Y a cubic fourfold, the version for deforming equivalence between Kuznetsov compoents is due to Huybrecht ([Huy17]). We will skip some details which will be verbatim repetition of discussions in [AT14] and [Huy17], but we will point out the details of some key steps.

First of all, we can pullback the family $\pi' : \mathcal{Y} \longrightarrow S$ by the map f, so we will assume that both families are defined on the same base space T and f is the identity map. By the assumption, the compostion $\mathrm{D}^b(X_0) \to \mathcal{A}_{X_0} \xrightarrow{\Phi} \mathcal{A}_{Y_0} \hookrightarrow \mathrm{D}^b(Y_0)$ is given by a Fourier-Mukai transform Φ_{P_0} for some kernel $P_0 \in \mathrm{D}^b(X_0 \times Y_0)$. As in [Huy17, Theorem 5.1], the condition that a Fourier-Mukai transform $\mathrm{D}^b(X_t) \longrightarrow \mathrm{D}^b(Y_t)$ factor through an equivalence $\mathcal{A}_{X_t} \simeq \mathcal{A}_{Y_t}$ is a Zariski open condition, therefore it is sufficient to deform the functor Φ_P, which amounts to deform the Fourier-Mukai kernel P_0. This is achieved in two steps, first we make use of Theorem 2.2.6 to show that under our hypothesis on the induced Hodge isometry on Mukai lattices there is no obstruction to first order deformation of P_0 (along the appropriate first order deformation of X_0 and Y_0), then we can handle extension to higher orders by a so called T^1-lifting argument,

•**First order**

For simplicity, we set $X = X_0$, $Y = Y_0$ and $P = P_0$. Let $\kappa_X \in \mathrm{H}^1(T_X)$ and $\kappa_Y \in \mathrm{H}^1(T_Y)$ be the Kodaira-Spencer classes given by some fixed tangent vector $v \in T_0T$, we aim at proving $(\kappa_X, \kappa_Y) \circ At(P) = 0$ (Theorem 2.2.6).

Lemma 5.2.1. *The following diagram commutes*

$$\begin{array}{ccc} HH_2(X) & \xrightarrow[\cong]{\Phi_P^{HH_*}} & HH_2(Y) \\ {\scriptstyle \kappa_X \lrcorner}\downarrow & & \downarrow{\scriptstyle \kappa_Y \lrcorner} \\ HH_0(X) & \xrightarrow{\Phi_P^{HH_*}} & HH_0(Y) \end{array} \tag{5.2.1}$$

Before proving this, we need to explain why $\Phi_P^{HH_*} : \mathrm{HH}_2(X) \to \mathrm{HH}_2(Y)$ is an isomorphism. In fact, it is possible to define Hochschild homologies for admissible subcategories of bounded derived categories of smooth projective varieties which behaves functorailly and they are additive in semi-orthogonal decomposition (see [MS19]). Then $\Phi_P^{HH_*}$ factors as

$$\mathrm{HH}_2(X) \twoheadrightarrow \mathrm{HH}_2(\mathcal{A}_X) \xrightarrow[\cong]{\Phi^{HH_*}} \mathrm{HH}_2(\mathcal{A}_Y) \hookrightarrow \mathrm{HH}_2(Y)$$

with Φ^{HH_*} being an isomorphism since Φ is an equivalence; furthermore, subcategory generated by an exceptional object is equivalent to that of a point, hence it has trivial HH_2, therefore both the projections (left or right adjoint of the inclusion functor) $\mathrm{HH}_2(X) \twoheadrightarrow \mathrm{HH}_2(\mathcal{A}_X)$ and the inclusions $\mathrm{HH}_2(\mathcal{A}_Y) \hookrightarrow \mathrm{HH}_2(Y)$ are isomorphisms.

Proof. Recall from 2.3, we have the twisted HKR isomorphism $I_K : HH_* \xrightarrow{\sim} H\Omega_*$ which is compatible with the maps induced by Fourier-Mukai transfomations. Meanwhile in our case, we have $\mathrm{H}\Omega_2(X) = \mathrm{H}^{3,1}(X) \cong \mathbb{C}$ and $\mathrm{H}\Omega_2(Y) = \mathrm{H}^{3,1}(Y) \cong \mathbb{C}$. Hence we only need to prove the following diagram commutes:

$$\begin{tikzcd} \mathrm{H}^{3,1}(X) \arrow[r, "\Phi_P^H", "\cong"'] \arrow[d, "\kappa_X \lrcorner"'] & \mathrm{H}^{3,1}(Y) \arrow[d, "\kappa_Y \lrcorner"] \\ \bigoplus \mathrm{H}^{p,p}(X) \arrow[r, "\Phi_P^H"] & \bigoplus \mathrm{H}^{p,p}(Y) \end{tikzcd} \tag{5.2.2}$$

Our hypothesis that ϕ_0 remains Hodge isomtery under parallel transport implies, at least in some (analytic) neighborhood of $0 \in T$, $\mathcal{M} \simeq \mathcal{M}'$ as variation of Hodge structures; by Griffith transversality and the fact that $(R^4\pi_*\mathbb{Z})_{00}$ (resp. $(R^4\pi'_*\mathbb{Z})_0$) is embedded a sub-variation of Hodge structures of $\mathcal{M}$ (resp. $\mathcal{M}'$), we have the following commutative diagram:

$$\begin{tikzcd} \mathrm{H}^{3,1}(X) \arrow[r, "\Phi_P^H", "\cong"'] \arrow[d, "\kappa_X \lrcorner"'] & \mathrm{H}^{3,1}(Y) \arrow[d, "\kappa_Y \lrcorner"] \\ \mathrm{H}^{2,2}(X)_{00} \arrow[d, hook, "v^{-1}"'] & \mathrm{H}^{2,2}(Y)_0 \arrow[d, hook, "v^{-1}"] \\ \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X) \arrow[r, "\phi_0"] & \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_Y) \end{tikzcd} \tag{5.2.3}$$

Now recall that ϕ_0 is given by the restriction of the map $\Phi_P^K : \mathrm{K}_{top}(X) \to \mathrm{K}_{top}(Y)$ to $\mathrm{K}_{top}(\mathcal{A}_X)$; we also know that Φ_P^K and Φ_P^H are compatible via Mukai vector v. Hence we also have the commutative diagram:

$$\begin{tikzcd} \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_X) \arrow[r, "\phi_0"] \arrow[d, "v"'] & \widetilde{\mathrm{H}}^{1,1}(\mathcal{A}_Y) \arrow[d, "v"] \\ \bigoplus \mathrm{H}^{p,p}(X) \arrow[r, "\Phi_P^H"] & \bigoplus \mathrm{H}^{p,p}(Y) \end{tikzcd} \tag{5.2.4}$$

Combine the last two diagrams, we're done. □

In order to effectively compute the obstruction class $(\kappa_X, \kappa_Y) \circ At(P)$, we need to split the Atiyah class $At(P)$:

The direct sum split $\Omega_{X\times Y} = \pi_X^*\Omega_X \oplus \pi_Y^*\Omega_Y$ induces a split $\mathrm{Ext}^1(P, P\otimes \Omega_{X\times Y}) \simeq \mathrm{Ext}^1(P, P\otimes\pi_X^*\Omega_X)\oplus\mathrm{Ext}^1(P, P\otimes\pi_Y^*\Omega_Y)$, hence we can write $At(P) = (At_X(P), At_Y(P))$ for $At_X(P)\in\mathrm{Ext}^1(P, P\otimes\pi_X^*\Omega_X)$ and $At_Y(P)\in\mathrm{Ext}^1(P, P\otimes \pi_Y^*\Omega_Y)$, they are called the partial Atiyah classes.

Lemma 5.2.2. [AT14, Lemma 7.2 & 7.3]). *$At_X(P)$ corresponds to the exact triangle*

$$P * (\Delta_{X*}\Omega_X) \longrightarrow P * \mathcal{O}_{2\Delta_X} \longrightarrow P * \mathcal{O}_{\Delta_X}$$

$At_Y(P)$ corresponds to the exact triangle

$$(\Delta_{Y*}\Omega_Y) * P \longrightarrow \mathcal{O}_{2\Delta_Y} * P \longrightarrow \mathcal{O}_{\Delta_Y} * P$$

Remark. Notice that we indeed have $P*(\Delta_{X*}\Omega_X)\simeq P\otimes\pi_X^*\Omega_X$ and $P*\mathcal{O}_{\Delta_X}\simeq P$ as objects of $\mathrm{D}^b(X\times Y)$ and $(\Delta_{Y*}\Omega_Y)*P\simeq P\otimes\pi_Y^*\Omega_Y$ and $\mathcal{O}_{\Delta_Y}*P\simeq P$ as objects of $\mathrm{D}^b(X\times Y)$.

Observe that under the spliting $At(P) = (At_X(P), At_Y(P))$, we have

$$(\kappa_X, \kappa_Y)\circ At(P) = \pi_X^*\kappa_X\circ At_X(P) + \pi_Y^*\kappa_Y\circ At_Y(P).$$

So we are reduced to prove that $\pi_X^*\kappa_X\circ At_X(P) = -\pi_Y^*\kappa_Y\circ At_Y(P)$. To do so, we need to analyze the splitting of the Atiyah class

$$At(\mathcal{O}_{\Delta_X})\in\mathrm{Ext}^1(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}\otimes\Omega_{X\times X})$$

into its partial classes. We write $At(\mathcal{O}_{\Delta_X}) = (At_1(\mathcal{O}_{\Delta_X}), At_2(\mathcal{O}_{\Delta_X}))$ where

$$At_i(\mathcal{O}_{\Delta_X})\in\mathrm{Ext}^1(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}\otimes\pi_i^*\Omega_X),\ i=1,2,$$

are the partial classes of $At(\mathcal{O}_{\Delta_X})$. It turns out that under the natural identification $\mathcal{O}_{\Delta_X} \otimes \pi_1^*\Omega_X \simeq \mathcal{O}_{\Delta_X} \otimes \pi_2^*\Omega_X \simeq \Delta_{X*}\Omega_X$, we have $At_1(\mathcal{O}_{\Delta_X}) = At_X = -At_2(\mathcal{O}_{\Delta_X})$ where $At_X \in \mathrm{Ext}^1(\mathcal{O}_{\Delta_X}, \Delta_{X*}\Omega_X)$ is the universal Atiyah class of X ([AT14, Corollary 7.5]). The same holds true on Y.

The next observation we can make is that ([AT14, Proof of Theorem 7.1])

Lemma 5.2.3. *Under the twisted HKR isomorphism $I^K : HH^2(X) \to HT^2(X)$, the class $\pi_1^*\kappa_X \circ At_1(\mathcal{O}_{\Delta_X}) \in Ext^2(\mathcal{O}_{\Delta_X}, \mathcal{O}_{\Delta_X}) = HH^2(X)$ is mapped to $\kappa_X \in H^1(T_X) \subset HT^2(X)$. Similarly, $I^K : HH^2(Y) \to HT^2(Y)$ maps $-\pi_2^*\kappa_Y \circ At_2(\mathcal{O}_{\Delta_Y}) = \pi_1^*\kappa_Y \circ At_1(\mathcal{O}_{\Delta_Y})$ to κ_Y.*

Combining Lemma 5.2.1, 5.2.2 and 5.2.3, we claim the existence of the following commutative diagram (recall notations in 2.3):

$$
\begin{tikzcd}
\mathrm{Ext}^{-2}_{X\times Y}(P * S^{-1}_{\Delta_X}, P) \arrow[rr] \arrow[dd, "\pi_X^*\kappa_X \circ At_X(P)"'] & & \mathrm{Ext}^{-2}_{X\times Y}(S^{-1}_{\Delta_Y} * P, P) \arrow[dd, "-\pi_Y^*\kappa_Y \circ At_Y(P)"' near start] & \\
& \mathrm{HH}^2(X) \arrow[ul, "P*"'] \arrow[rr, "\Phi_P^{HH_*}" near start] \arrow[dd, "\pi_1^*\kappa_X \circ At_1(\mathcal{O}_{\Delta_X})" near start] & & \mathrm{HH}^2(Y) \arrow[ul, "*P"'] \arrow[dd, "-\pi_2^*\kappa_Y \circ At_2(\mathcal{O}_{\Delta_Y})"] \\
\mathrm{Ext}^{0}_{X\times Y}(P * S^{-1}_{\Delta_X}, P) \arrow[rr] & & \mathrm{Ext}^{0}_{X\times Y}(S^{-1}_{\Delta_Y} * P, P) & \\
& \mathrm{HH}^0(X) \arrow[ul, "P*"'] \arrow[rr, "\Phi_P^{HH_*}"'] & & \mathrm{HH}^0(Y) \arrow[ul, "*P"']
\end{tikzcd}
\tag{5.2.5}
$$

The front square being commutative follows from Lemma 5.2.1 and 5.2.3; the left and right squares are commutative by Lemma 5.2.2; the top and bottom squares exist and are commutative for the simple reason that the map $\Phi_P^{HH_*} : HH_j(X) \to HH_j(Y)$ factor through (see the description of $\Phi_P^{HH_*}$ in 2.3)

$$P* : HH_j(X) = \mathrm{Ext}^{-j}_{X\times X}(S^{-1}_{\Delta_X}, \mathcal{O}_{\Delta_X}) \longrightarrow \mathrm{Ext}^{-j}_{X\times Y}(P * S^{-1}_{\Delta_X}, P)$$

So we are left with the commutativity of the rear square, we only need to show that both $\mathrm{HH}_2(X) \xrightarrow{P*} \mathrm{Ext}^{-2}_{X\times Y}(P*S^{-1}_{\Delta_X}, P)$ and $\mathrm{HH}_2(Y) \xrightarrow{*P} \mathrm{Ext}^{-2}_{X\times Y}(S^{-1}_{\Delta_Y} * P, P)$ are isomorphisms. In fact, as we mentioned before, it is possible to define Hochschild homology of the Kuznetsov components, it turns out that (see [Kuz09, Lemma 7.4]) $\mathrm{Ext}^{-2}_{X\times Y}(P*S^{-1}_{\Delta_X}, P) \simeq \mathrm{HH}_2(\mathcal{A}_X)$ and $\mathrm{Ext}^{-2}_{X\times Y}(S^{-1}_{\Delta_Y}*P, P) \simeq \mathrm{HH}_2(\mathcal{A}_Y)$ are nothing but just HH_2 of the Kuznetsov components, moreover the maps $P*$ and $*P$ play the role of projections $\mathrm{HH}_2(X) \twoheadrightarrow \mathrm{HH}_2(\mathcal{A}_X)$ and $\mathrm{HH}_2(Y) \twoheadrightarrow \mathrm{HH}_2(\mathcal{A}_Y)$, which, as remarked earilier, are isomorphims.

Hence the rear square is commutative, but that translates to $\pi_X^*\kappa_X \circ At_X(P) = -\pi_Y^*\kappa_Y \circ At_Y(P)$ as classes in $\mathrm{Ext}^2_{X\times Y}(P, P)$. This completes the first order step.

Remark 5.2.4. We should point out that everything discussed in this step remains valid if we X and Y are defined, instead, over some complex Artinian space A, and the product $X \times Y$ is understood to be the fiber product over A. The Kodaira Spencer classes κ_X and κ_Y are to be interpreted as relative classes over A, meaning $\kappa_X \in \mathrm{H}^1(T_{X/A})$ for example, and they dictates some "side-way" first order deformations $X_1 \longrightarrow A \times A_1$ and $Y_1 \longrightarrow A \times A_1$ of our A-schemes X and Y. Both lemma 5.2.2 and 5.2.3 carry over with no change, therefore if we insist the commutativity of the diagram

$$\begin{tikzcd} \mathrm{HH}_2(X) \arrow[r, "\Phi_P^{HH_*}", "\cong"'] \arrow[d, "\kappa_X \lrcorner"'] & \mathrm{HH}_2(Y) \arrow[d, "\kappa_Y \lrcorner"] \\ \mathrm{HH}_0(X) \arrow[r, "\Phi_P^{HH_*}"] & \mathrm{HH}_0(Y) \end{tikzcd} \tag{5.2.6}$$

as in Lemma 5.2.1. Then we will still have $(\kappa_X, \kappa_Y) \circ At(P) = 0$ and therefore the complex $P \in \mathrm{D}_{Perf}(X\times Y)$ deforms "side-way" to a complex $\tilde{P} \in \mathrm{D}_{Perf}(X_1 \times_{A\times A_1} Y_1)$ (see the Remark after Theorem 2.2.6) .

•Higher order

We will be brief in this part and refer all the details to [AT14, 7.2]. Consider the following Artinian spaces

$$\begin{aligned} A_n &= \mathrm{Spec}\mathbb{C}[t]/(t^{n+1}) \\ B_n &= A_n \times A_1 = \mathrm{Spec}\mathbb{C}[x,y]/(x^{n+1}, y^2) \end{aligned}$$

and we consider the map $p_n : B_n \longrightarrow A_n$ given by $t \longmapsto x + y$, and let $\iota_n : A_n \longrightarrow A_{n+1}$ and $j_n : A_n \longrightarrow B_n$ be the natural inclusions.

$$\begin{tikzcd} A_{n-1} \arrow[r, "\iota_{n-1}"] \arrow[d, "j_{n-1}"] \arrow[dd, bend right=60, "\iota_{n-1}"'] & A_n \arrow[d, "j_n"'] \arrow[dd, bend left=60, "\iota_n"] \\ B_{n-1} \arrow[r, "\iota_n \times id"] \arrow[d, "p_{n-1}"] & B_n \arrow[d, "p_n"'] \\ A_n \arrow[r, "\iota_n"] & A_{n+1} \end{tikzcd} \tag{5.2.7}$$

Now suppose we have a smooth family $\mathcal{A}_{n+1} \longrightarrow A_{n+1}$, which base change to $\mathcal{A}_n \longrightarrow A_n$ by ι_n and $\mathcal{B}_n \longrightarrow B_n$ by p_n. We want to extend a perfect complex $P_n \in \mathrm{D}_{Perf}(\mathcal{A}_n)$ to $P_{n+1} \in \mathrm{D}_{Perf}(\mathcal{A}_n)$ so that $\iota_n^* P_{n+1} = P_n$. The T^1-lifting for complexes of sheaves asserts the following

Proposition 5.2.5. ([AT14 Proposition 7.6]). *Suppose there exists a complex*

$$\tilde{P}_{n+1} \in D_{Perf}(\mathcal{B}_n)$$

such that $j_n^* \tilde{P}_{n+1} = P_n$ ($\tilde{P}_{n+1}$ is a "side way" first order deformation of P_n) and also satisfies

$$(\iota_{n-1} \times id)^* \tilde{P}_{n+1} = p_{n-1}^* P_n$$

Then there exists $P_{n+1} \in D_{Perf}(\mathcal{A}_n)$ such that $\iota_n^* P_{n+1} = P_n$.

In our case, we consider a formal curve $A_\infty := \mathrm{Spec}\mathbb{C}[[t]] \longrightarrow T$ centered at $0 \in T$, let $X_\infty \longrightarrow A_\infty$ and $Y_\infty \longrightarrow A_\infty$ the base change to A_∞ of the families $\mathcal{X}$ and $\mathcal{Y}$ and we denote X_n and Y_n their restriction to A_n, the tangent vector

v of the formal curve determines, for each n, a relative Kodaira Spencer class $\kappa_{X_n} \in \mathrm{H}^1(T_{X_n/A_n})$ and $\kappa_{Y_n} \in \mathrm{H}^1(T_{Y_n/A_n})$ (including the case $n = \infty$). For $n = 0$, we have a complex $P_0 \in \mathrm{D}^b(X_0 \times Y_0)$ inducing some Fourier-Mukai transform Φ_{P_0}. As in [AT14], we can consider some natural identifications

$$\mathrm{H}^*_{dR}(X_\infty/A_\infty) \cong H^*(X_0) \otimes \mathbb{C}[[t]], \quad \mathrm{H}^*_{dR}(Y_\infty/A_\infty) \cong H^*(Y_0) \otimes \mathbb{C}[[t]]$$

This will give us the commutative diagram

$$\begin{tikzcd} \mathrm{H}^4_{dR}(X_\infty/A_\infty) \arrow[r, "\Phi^H_{P_0} \otimes 1_{\mathbb{C}[[t]]}"] \arrow[d, "\kappa_{X_\infty}"'] & \mathrm{H}^4_{dR}(Y_\infty/A_\infty) \arrow[d, "\kappa_{Y_\infty}"] \\ \mathrm{H}^2_{dR}(X_\infty/A_\infty) \arrow[r, "\Phi^H_{P_0} \otimes 1_{\mathbb{C}[[t]]}"] & \mathrm{H}^2_{dR}(Y_\infty/A_\infty) \end{tikzcd} \tag{5.2.8}$$

Suppose now we have constructed inductively a complex $P_j \in \mathrm{D}_{Perf}(X_j \times_{A_j} Y_j)$ for all $1 \leq j \leq n$ such that P_j restricts to P_{j-1}, then the above diagram commutes implies that

$$\begin{tikzcd} \mathrm{HH}^2(X_n/A_n) \arrow[r, "\Phi^{HH_*}_{P_n}"] \arrow[d, "\kappa_{X_n}"'] & \mathrm{HH}^2(Y_n/A_n) \arrow[d, "\kappa_{Y_n}"] \\ \mathrm{HH}^0(X_n/A_n) \arrow[r, "\Phi^{HH_*}_{P_n}"] & \mathrm{HH}^0(Y_n/A_n) \end{tikzcd} \tag{5.2.9}$$

is commutative. Thus, as we discussed in Remark 5.2.4, P_n deforms to a complex $\tilde{P}_{n+1} \in \mathrm{D}_{Perf}(X_n^{(1)} \times_{A_n \times A_1} Y_n^{(1)})$, where $X_n^{(1)}$ and $Y_n^{(1)}$ are the "side-way" first order deformations of X_n and Y_n over $A_n \times A_1$ given by κ_{X_n} and κ_{Y_n} . Meanwhile, one has $\mathrm{Ext}^1_{X_n \times_{A_n} Y_n}(P_n, P_n) = 0$ ($\mathrm{Ext}^1_{X_n \times_{A_n} Y_n}(P_n, P_n)$ computes the first *Hochschild cohomology* of a family of K3 categories over A_n, which vanishes), this implies the side-way extension $\tilde{P}_{n+1}$ is unique (One heuristic way to convince us this is that the condition

$$\mathrm{Ext}^1_{X_n \times_{A_n} Y_n}(P_n, P_n) = 0$$

says that P_n is *rigid* on the A_n-scheme $X_n \times_{A_n} Y_n$, thus if X_n and Y_n undergo trivial deformation, meaning κ_{X_n} and $\kappa_{Y_n} = 0$, $\tilde{P}_{n+1}$ must be the trivial deformation of P_n as well). Now notice that both $(\iota_{n-1} \times id)^* \tilde{P}_{n+1}$ and $p_{n-1}^* P_n$ restricts to P_{n-1} by $j_{n-1} \colon A_{n-1} \hookrightarrow B_{n-1}$, hence by uniqueness they must be the same. Thus Proposition 5.2.5 applies and gives us a $P_{n+1} \in \mathrm{D}_{Perf}(X_{n+1} \times_{A_{n+1}} Y_{n+1})$ which restricts to P_n.

This proves that the complex P_0 can be unobstructedly deformed in any tangent direction $v \in T_0 T$, by standard deformation theory P_0 can be deformed to the formal neighborhood $\widehat{Z} := \mathrm{Spec}\widehat{\mathcal{O}_{T,0}}$ to give us a complex $P_{\widehat{Z}} \in \mathrm{D}_{Perf}(\mathcal{X} \times_{\widehat{Z}} \mathcal{Y})$. The rest follows from the existence of a locally finitely presented Artin stack $\mathcal{S}$ parametrizing complexes with no negative self-Exts in the fibers of $\mathcal{X} \times_T \mathcal{Y} \longrightarrow T$. Since having no negative self-Exts is an open condition satisfied by P_0, the complex $P_{\widehat{Z}}$ defines a classifying map $(\widehat{Z}, 0) \longrightarrow (\mathcal{S}, P_0)$. Once again since $\mathrm{Ext}^1_{X_0 \times Y_0}(P_0, P_0) = 0$, i.e. P_0 is rigid, and is an open condition, this classifying map is in fact the formal neighborhood of P_0 in $\mathcal{S}$. Now since $\mathcal{S}$ is locally finitely presented, we can find a smooth pointed scheme $(Z, 0)$ mapping to $(\mathcal{S}, P_0)$ taking the formal neighborhood of $0 \in Z$ isomorphically onto $\widehat{Z}$. Shrink Z if necessary, we can ensure that the map $(Z, 0) \longrightarrow (T, 0)$ is an open immersion and gives the desired Zariski neighborhood of 0 in T over which P_0 deforms.

Bibliography

[AH61] Atiyah, M., Hirzebruch, F., Vector bundles and homogeneous spaces, in *Proc. Sympos. Pure Math., Vol. III*, 7–38, Amer. Math. Soc., Providence, 1961.

[AH62] Atiyah, M., Hirzebruch, F., The Riemann-Roch theorem for analytic embeddings, *Topology* 1 (1962), 151–166.

[AT14] Addington N., Thomas, R., Hodge theory and derived categories of cubic fourfolds, *Duke Math. J.* 163 (2014), 1885–1927.

[BB66] Baily, W., Borel, A., Compactification of arithmetic quotients of bounded symmetric domains. *Ann. of Math. (2)*, 84:442–528, 1966. 82, 103

[BLM+19] Bayer, A., Lahoz, M., Macrì E., Nuer, H., Perry, A., Stellari, P., Stability conditions in family, eprint arXiv:1902.08184v1.

[BLMS17] Bayer, A., Lahoz, M., Macrì E., Stellari, P., Stability conditions on Kuznetsov components, eprint arXiv:1703.10839v1.

[Căl03] Căldăraru, A., The Mukai pairing. I. The Hochschild structure, eprint arXiv:math/0308079v2.

[Căl05] Căldăraru, A., The Mukai pairing. II. The Hochschild-Kostant-Rosenberg isomorphism, *Adv. Math.* 194 (2005), 34–66.

[CG72] Clemens, C., Griffiths, P., The intermediate Jacobian of the cubic threefold. *Ann. of Math. (2)*, 95:281–356, 1972.

[CRV12] Calaque, D., Rossi, C., Van den Bergh, M., Căldăraru's conjecture and Tsygan's formality. *Ann. of Math. (2)*, 176(2):865{923, 2012. Also arXiv:0904.4890.

[DIM14] Debarre, O., Iliev, A., Manivel, L., Special prime Fano fourfolds of degree 10 and index 2, in *Recent Advances in Algebraic Geometry*, 123–155.

[DK18a] Debarre, O., Kuznetsov, A., Gushel–Mukai varieties: classification and birationalities, *Algebr. Geom.* 5 (2018), 15–76.

[DK18b] Debarre, O., Kuznetsov, A., Gushel–Mukai varieties: moduli, eprint arXiv:1812.09186.

[Dol96] Dolgachev, I., Mirror symmetry for lattice polarized K3 surfaces. *J. Math. Sci.*, 81(3):2599–2630, 1996. Algebraic geometry, 4.

[Fan43] Fano, G., Sulle forme cubiche dello spazio a cinque dimensioni contenenti rigate razionali del 4 ◦ ordine, Comment. *Math. Helv.* 15 (1943), no. 1, 71–80.

[Gus82] Gushel, N.P., Fano varieties of genus 6 (in Russian), *Izv. Ross. Akad. Nauk Ser. Mat.* 46 (1982), 1159–1174, 1343. English transl.: *Izv. Math.* 21 (1983), 445–459.

[Has00] Hassett, B., Special cubic fourfolds, *Compos. Math.* 120 (2000), 1–23.

[Huy06] Huybrechts, D., *Fourier-Mukai transform in algebraic geometry*, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2006.

[Huy16] Huybrechts, D., *Lectures on K3 surfaces*, Cambridge Studies in Advanced Mathematics, Cambridge University Press, Cambridge, 2016.

[Huy17] Huybrechts, D., The K3 category of a cubic fourfold, *Compos. Math.* 153 (2017), 586–620.

[HS20] Hoff, M., Staglianò G., New examples of rational Gushel-Mukai fourfolds, eprint arXiv:1910.12838.

[KP18] Kuznetsov, A., Perry, A., Derived categories of Gushel-Mukai varieties, *Compositio Math.* 154 (2018), 1362-1406.

[Kuz06] Kuznetsov, A., Homological projective duality for Grassmannians of lines, eprint arXiv:math/0610957v1.

[Kuz10] Kuznetsov, A., Derived categories of cubic fourfolds, in *Cohomological and geometric approaches to rationality problems*, 219–243, Progr. Math. 282, Birkh¨auser, Boston, 2010.

[Kuz11] Kuznetsov, A., Base change for semiorthogonal decompositions, *Compos. Math.* 147 (2011), 852–876.

[Kuz15] Kuznetsov, A., Derived categories view on rationality problems. Lecture notes for the CIME–CIRM summer school, Levico Terme, 2015. arXiv:1509.09115.

[Laz10] Laza, R., The moduli space of cubic fourfolds via the period map. *Ann. of Math. (2)*, 172(1):673–711, 2010.

[Loo09] Looijenga, E., The period map for cubic fourfolds, *Invent. Math.*, 177 (2009), 213–233.

[LPZ22] Li, C., Pertusi, L., Zhao, X., Derived categories of hearts on Kuznetsov components, eprint arXiv:2203.13864v1.

[MS09] Macrì E., Stellari, P., Infinitesimal derived Torelli theorem for K3 surfaces, (Appendix by S. Mehrotra), *Int. Math. Res. Not. IMRN* (2009), 3190–3220.

[Muk89] Mukai, S., Biregular Classification of Fano 3-Folds and Fano Manifolds of Coindex 3, *Proc. Natl. Acad. Sci. USA* 86 (1989), 3000–3002.

[Nik79] Nikulin, V.V., Integral symmetric bilinear forms and some of their applications, *Izv. Akad. Nauk SSSR Ser. Mat.* 43 (1979), 111–177, 238 (Russian); translation in Math USSR Izvestija 14 (1980), 103–167.

[Or03] Orlov, D. Derived categories of coherent sheaves and equivalences between them, *Russian Math. Surveys*, 58 (2003), 3, 89-172

[Pert17] Pertusi, L., On the double EPW sextic associated to a Gushel-Mukai fourfold, to appear in: *J. London Math. Soc.*, eprint arXiv:1709.02144v1.

[PPZ19] Perry, A., Pertusi, L., Zhao, X., Stability conditions and moduli spaces for Kuznetsov components of Gushel-Mukai varieties, eprint arXiv:1912.06935v1

[RS19a] Russo, F., Staglianò G., Congruences of 5-secant conics and the rationality of some admissible cubic fourfolds, *Duke Math. J.* 168 (2019), no. 5, 849–865.

[RS19b] Russo, F., Staglianò G., Trisecant Flops, their associated K3 surfaces and the rationality of some Fano fourfolds, eprint arxiv:1909.01263.

[Swa96] Swan, R., Hochschild cohomology of quasi-projective schemes, *J. Pure Appl. Algebra* 110 (1996), 57–80.

[Vois86] Voisin, C.: Théorèm de Torelli pour les cubiques de $\mathbb{P}^5$, *Invent. Math.* 86 (1986), 577–601.

[YY20] Yang, S., Yu, X., Rational cubic fourfolds in Hassett divisors. *Comptes Rendus. Mathématique,* Volume 358 (2020)

[YY23] Yang, S., Yu, X., On lattice polarizable cubic fourfolds. *Res Math Sci* 10, 2 (2023).