Interactions Between Surface Gravity Wave Groups and Deep Stratification in the Ocean

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Abstract

Groups of surface gravity waves induce horizontally varying Stokes drift which drive convergence of water ahead of the group and divergence behind, thereby driving water downward in front of the group, and upward in the rear. This "Stokes pumping" creates a deep Eulerian return flow. We assess the impact of stable density stratification on the deep return flow. Our approach is to find solutions of the wave-averaged Boussinesq equations in two ($2D$) and three dimensions ($3D$) forced by Stokes pumping at the surface. We find that the shape of the return flow may be changed by vertical density stratification, and if the stratification is sufficiently strong, internal gravity waves may be radiated from the passing surface wave group. In the $2D$ case, the problem can be solved for arbitrary stratification profiles, however, we find that realistic stratification is too weak to produce internal waves. In the $3D$ case with constant stratification, internal waves are always emitted by the passing surface wave group.

1 Introduction

As surface gravity waves propagate, they induce flow in the direction of propagation at the crest of the waves, and backwards in the troughs. Following a neutrally buoyant particle, or parcel of water, one finds circular orbits, that do not completely close, but rather drift forward. The motion of these particles averaged over a wave period is called the Stokes drift. Mathematically, the Stokes drift is formed by the surface displacement (ζ) dotted into the gradient of the surface wave velocity ($\boldsymbol{u}$), and averaged over the surface wave period.

$$\boldsymbol{u}^S = \overline{\zeta \cdot \boldsymbol{\nabla} \boldsymbol{u}} \tag{1}$$

where the overbar denotes an average over the wave phase. The leading order solutions ζ_1 and $\boldsymbol{u}_1 = \boldsymbol{\nabla}\phi$ are well know (e.g. Phillips, 1977).

$$\zeta_1 = \tfrac{1}{2} a(\tilde{x}, 0) \exp\left[\mathrm{i}k\left(x - ct\right)\right] + c.c. \tag{2}$$

$$\phi = -\tfrac{1}{2}\mathrm{i}c\, a(\tilde{x}, z) \exp\left[\mathrm{i}k\left(x - ct\right) + kz\right] + c.c.\,, \tag{3}$$

If we insert the well known linear wave solution for ζ and $\boldsymbol{u}$ we find that the Stokes drift is second order in wave slope ($ak = \epsilon$)

$$\boldsymbol{u}^S = (ak)^2 c e^{2kz} \tag{4}$$

where a is the wave amplitude, k is the horizontal wavenumber, and c is the wave phase speed.

Now if we consider surface wave groups, the Stokes drift will vary in space and time, that is $a = a(x, t)$. Throughout this work we will use a gaussian envelope as an example, but the results apply more generally except where the gaussian functional form for $a(x,t)$ has been invoked. For an observer in a fixed position, as the wave group passes by, the Stokes drift starts out very small, and increases as the peak of the envelope (wave group) passes, decreasing thereafter. Since the Stokes drift represents the Lagrangian mass flux, there is then a convergence of mass flux in the front of the wave group, and a divergence behind. This surface convergence dipole drives downwelling in front of the group, and upwelling aft of the group. We will refer to this Stokes induced surface boundary condition as"Stokes pumping", and to the ensuing circulation as the Eulerian return flow. Such a return flow was first discussed by Longuet-Higgins and Stewart (1964), and later by McIntyre (1981) and Van Den Bremer and Taylor (2015). In particular, McIntyre (1981) noted that the interaction of the Eulerian return flow with stable stratification may incite internal waves.

Here, we examine the changes to the Eulerian return flow and the details of internal wave excitation by surface waves. Figure 1 shows the stream function for flow beneath the surface wave group in an unstratified case (similar to figure 2 in McIntyre (1981)), and in a stratified case in which internal waves are emitted. In figure 1 the flow is $2D$, however we consider $3D$ flows in the special case of constant stratification. Van Den Bremer and Taylor (2015) consider the unstratified $3D$ return flow, and show that it decays with depth as z^{-3} which, although more rapidly decaying than the $2D$ case (z^{-2}), is still algebraic so the return flow may be felt down to the bottom of the ocean. Therefore, in the $3D$ case, when the return flow may pass partially around the side of the wave group, it is still felt at the ocean floor.

Other approaches to coupling surface gravity waves to internal waves have been considered before. One approach is that of resonant triad interaction between two surface waves and one internal wave. Olbers and Eden (2016) present such a theory, and show that the energy flux to internal waves is much smaller than the energy flux to internal waves due to wind forcing. The surface wave group-internal wave interaction presented here is a special case of the triad interaction. In this case, the surface wave group includes two waves that are close in wavenumber k and $k+\delta k$, and to satisfy the resonance condition, the internal wave must have wavenumber δk. Although the work presented here is a special case of the triad interaction theories of Olbers and Eden (2016) and Olbers and Herterich (1979), it elucidates the connection between radiated internal waves and the eulerian return flow.

2 The Deep Return Flow

We begin from the Boussinesq equations

$$\boldsymbol{u}_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\boldsymbol{u} + \boldsymbol{\nabla} p = b\hat{\boldsymbol{z}}\,, \tag{5}$$

$$b_t + \boldsymbol{u}\cdot\boldsymbol{\nabla} b + wN^2 = 0\,, \tag{6}$$

$$\boldsymbol{\nabla}\cdot\boldsymbol{u} = 0\,. \tag{7}$$

where $\boldsymbol{u} = u\hat{\boldsymbol{x}} + v\hat{\boldsymbol{y}} + w\hat{\boldsymbol{z}}$, $b = g(\rho-\rho_0)\rho_0^{-1}$ is buoyancy, and $N=\sqrt{\partial_z b}$ is the buoyancy frequency. The surface boundary conditions, correct to second order in wave steepness ϵ, are

$$@z=0: \qquad \zeta_t + (u\zeta)_x = w\,, \text{and} \tag{8}$$

$$@z=0: \qquad p + \zeta p_z = g\zeta + N^2\tfrac{1}{2}\zeta^2\,. \tag{9}$$

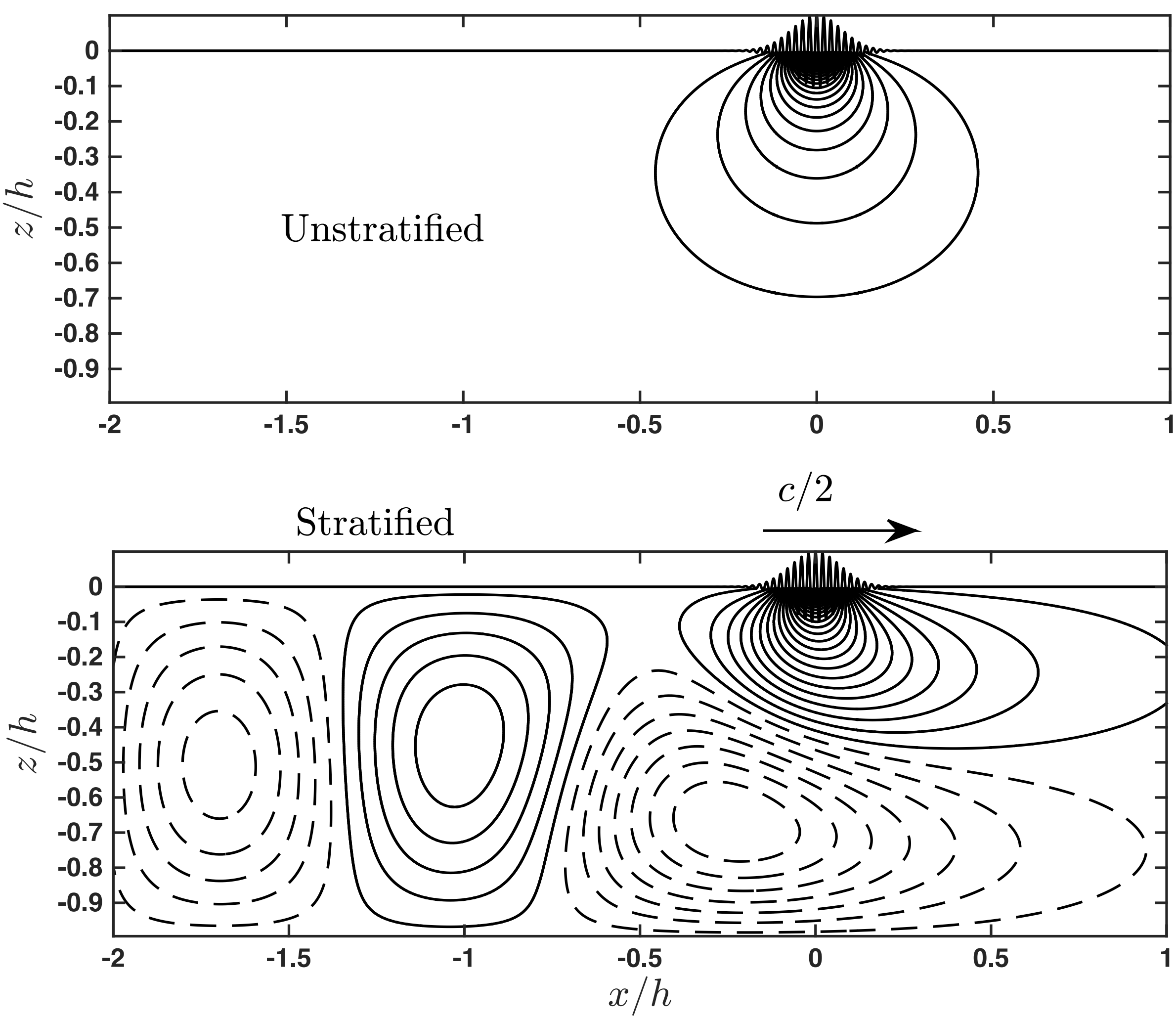

Figure 1: The stream function of the deep return flow beneath a gaussian wave group with an unstratified interior (top; $N = 0$), and with constant stratification (bottom; $N = 7 \times 10^{-3} s^{-1}$). The dashed contours indicate negative values of the stream function, and the contour interval is $0.05 m^2 s^{-1}$. In both panels, the surface wave group is moving to the right with speed $c/2$. The surface wave amplitude is exaggerated by orders of magnitude in order to visualize the group; in this illustration $\ell/h = 0.07$.

The bottom boundary condition is $w(x,-h,t) = 0$. We will only consider deep-water waves ($kh \gg 1$) so that this bottom boundary condition is only relevant to the Eulerian return flow. The leading order (ϵ^1) solution is that which yields the surface waves themselves as discussed in the introduction and references therein. Now we proceed with the second-order solution to reveal the form of the Eulerian return flow.

Retaining only the second-order terms and then applying a phase average yields the following momentum equations

$$\bar{u}_{2t} = \varpi_x\,, \qquad \bar{v}_{2t} = \varpi_y\,, \qquad \bar{w}_{2t} = -\varpi_z + \bar{b}_2\,, \qquad \bar{b}_{2t} = -wN^2\,, \tag{10}$$

where the second order pressure, ϖ, is defined as

$$\varpi \stackrel{\text{def}}{=} \bar{p}_2 + \tfrac{1}{2}\overline{|\boldsymbol{u}_1|^2}. \tag{11}$$

Combining (10) intro a single equation yields the vertical velocity equation for the return flow

$$\left[\partial_t^2 \Delta + N^2 \Delta_H\right] \overline{w} = 0, \tag{12}$$

where $\Delta = \partial_x^2 + \partial_y^2 + \partial_z^2$, and $\Delta_H = \partial_x^2 + \partial_y^2$. The second order surface boundary condition is then

$$@z = 0: \qquad \overline{w} = (\overline{u\zeta})_x\,, \tag{13}$$

where, following e.g. Longuet-Higgins and Stewart (1964), we have neglected the set down of the surface due to the passing group ($\bar{\zeta}_{2t}$), since in deep water, this term is much smaller than the remaining two. The term on the right hand side of (13) can be identified as the mass flux per unit length per unit density. Using the known leading order solutions we see that

$$M = \overline{u\zeta} = \int_{-\infty}^{0} u^{\mathrm{S}}\,\mathrm{d}z\,. \tag{14}$$

The bottom boundary condition remains $\overline{w} = 0$ at $z = -h$.

It will be convenient in the $2D$ case to consider the stream function defined by

$$\bar{u}_2 = -\psi_z\,, \qquad \bar{w}_2 = \psi_x\,. \tag{15}$$

Therefore the stream function equation for the return flow in $2D$ is

$$\left[\partial_t^2\left(\partial_x^2 + \partial_z^2\right) + N^2 \partial_x^2\right]\psi = 0\,. \tag{16}$$

Lastly, it will also prove convenient to move into the reference frame moving at the group speed $c/2$. We define the group frame coordinate as

$$\tilde{x} \stackrel{\text{def}}{=} x - \tfrac{1}{2}ct\,. \tag{17}$$

Making this transformation, and integrating twice in $\tilde{x}$, the $2D$ stream function equation becomes

$$\left[\partial_{\tilde{x}}^2 + \partial_z^2 + q_*^2\right]\psi = 0 \tag{18}$$

where $q_* = 2N/c$ is the critical wavenumber.

3 Finite Depth Solutions with Vertically Varying Stratification in $2D$

To solve (18) with vertically varying stratification we decompose the solution into vertical modes $\phi_n(z)$ which solve the following eigenvalue problem

$$\frac{\mathrm{d}^2\phi_n}{\mathrm{d}z^2} + \left(q_*^2 - q_n^2\right)\phi_n = 0\,. \tag{19}$$

Once this problem is solved for a particular value of q_*, we project ψ and (18) onto these modes which gives an equation for the modal amplitude ψ_n,

$$\left(\partial_{\tilde{x}}^2 + q_n^2\right)\psi_n = \frac{\phi_n' M}{h}\,, \tag{20}$$

where $\phi_n' = \mathrm{d}\phi_n/\mathrm{d}z$, evaluated at $z = 0$. (20) can be solved analytically with Green's functions. When $q_n^2 > 0$ we can see that the n^{th} mode is an internal wave with horizontal wavenumber q_n.

$$\psi_n(\tilde{x}) = -\frac{\phi_n'}{hq_n}\int_{\tilde{x}}^{\infty} M(x')\sin\left[q_n(\tilde{x} - x')\right]\,\mathrm{d}x'\,. \tag{21}$$

When $q_n^2 < 0$ the n^{th} mode is evanescent, and contributes to the Eulerian return flow discussed in the unstratified case.

$$\psi_n(\tilde{x}) = -\frac{\phi_n'}{2h\beta_n}\int_{-\infty}^{\infty} M(x')\mathrm{e}^{-\beta_n|\tilde{x}-x'|}\,\mathrm{d}x' \tag{22}$$

The exponent β is defined such that $\beta_n^2 = -q_n^2 > 0$. An example of ψ in uniform stratification is plotted in the lower panel of figure 1 using (21) - (22).

It is illustrative to consider the uniform stratification case. Although (19) applies to general stratification profiles $N(z)$, the problem is analytically solvable and tractable for constant N (recall $q_* = 2N/c$). In this case, $\phi_n(z) = sin(mz)$, where $m = n\pi/h$. From (19) we can then obtain the horizontal wavenumber of the mode n internal wave simply by inserting $\phi_n(z)$.

$$q_n^2 = q_*^2 - m^2 \tag{23}$$

Re-inserting the definition of q_* we see that this gives a resonance condition relating the surface wave group speed to the internal wave phase speed

$$\underbrace{\frac{N}{\sqrt{q_n^2 + m^2}}}_{\text{IW phase speed}} = \underbrace{\frac{c}{2}}_{\text{SW group speed}} \tag{24}$$

Now, assuming a large horizontal length scale ($q_n \to 0$) we see that this puts a lower bound on the stratification required to radiate internal waves.

$$N > \frac{n\pi}{2ch} \tag{25}$$

Although the resonance condition and bound on N have assumed constant stratification, several profiles of observed ocean stratification were tested with (21) - (22), and the lower bound was similar. We found that most regions of the ocean are not sufficiently stratified to produce emission of internal waves from passing surface gravity wave groups through this $2D$ mechanism. In section 4, we show that in $3D$, with constant stratification, internal waves are always radiated, at every vertical wavenumber.

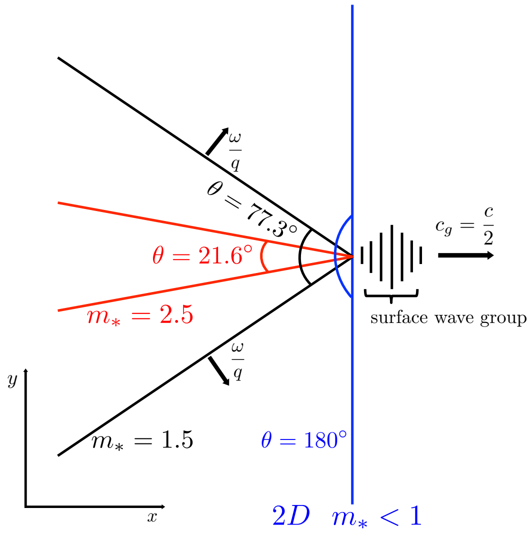

Figure 2: A schematic of the wake of internal waves behind the surface wave group as seen from above. The waves propagate to the right at the group speed $c/2$ while the internal waves propagate at oblique angles at their phase speed $\omega/|\boldsymbol{q}|$. The shape of the wake depends on the strength of the stratification and speed of the surface waves as in (35). Stronger stratification and slower surface waves imply wider wakes up to the $2D$ limit where $\theta = 180°$. The widest wake for a given stratification and surface wave group speed is created by the mode one internal wave. For a group of $8s$ waves, the stratification in most of the world's oceans puts m_* between 1.5 and 2.5.

4 The $3D$ Wake of Internal Waves

If the return flow, and therefore the radiated internal waves are allowed to be $3D$ the strict requirement of very fast internal waves (whose phase speeds match the surface wave group speed) need not be met. In $3D$ the internal waves can propagate obliquely to the surface wave propagation direction just as the waves of a boat wake propagate obliquely to the direction of propagation of the boat. A schematic of this situation is depicted in figure 2.

The increased complexity of the third dimension precludes solutions with variable stratification using the methods in section 3. Nevertheless, we can compute the shape of the wake, and the energy flux from surface to internal waves with uniform stratification. To compute the energy flux from surface to internal waves we solve (16) with constant N. To do so, we move to the wave group frame as in section 3, and we assume that the source of energy (the surface waves) is slowly.

$$w(x, y, 0, t) = \mathrm{e}^{\gamma t}\partial_x M(\tilde{x}, y) \tag{26}$$

γ is the slow growth rate which will later be taken to be infinitesimal. The radiative flux is defined as

$$R = \iint \varpi(x, y, 0, t)\, w(x, y, 0, t)\, \mathrm{d}x\mathrm{d}y\,, \tag{27}$$

The slow growth of the surface waves is assumed to avoid a singularity in the integrand of (27). This method is described in greater detail in Lighthill (1978). First we apply the same modal projection for constant N as in section 3 on (16) to obtain

$$\left[\partial_t^2\left(\partial_x^2 + \partial_y^2 - m_n^2\right) + N^2\left(\partial_x^2 + \partial_y^2\right)\right] w_n = \frac{\sqrt{2}}{h} m_n \partial_t^2 \partial_x\, \mathrm{e}^{\gamma t} M\,. \tag{28}$$

Next, we solve (28) via Fourier transform to find

$$\hat{w}_n = -\tfrac{\sqrt{2}}{h} m_n \frac{\mathrm{i}q(q+\mathrm{i}\eta)^2 \hat{M}(q,s)}{(q+\mathrm{i}\eta)^2(q^2+s^2+m_n^2) - q_*^2(q^2+s^2)}\,, \tag{29}$$

where $\eta 2\gamma/c$, and $\hat{(\cdot)}$ indicates Fourier transform in $\tilde{x}$ and y to spectral space in q and s respectively. Using this result, we can solve for the pressure ϖ at the surface by relating the two through the vertical momentum equation.

$$\hat{\varpi}_n = \tfrac{c}{2}\tfrac{\sqrt{2}}{h} \frac{q(q+\mathrm{i}\eta)\,[q_*^2 - (q+\mathrm{i}\eta)^2]\,\hat{M}(q,s)}{(q+\mathrm{i}\eta)^2(q^2+s^2+m_n^2) - q_*^2(q^2+s^2)} \tag{30}$$

Now we can insert (30) and the surface boundary condition into the radiation integral (27), and take the limit $\eta \to 0$ to find the radiation for each mode.

$$R_n = -4\pi \tfrac{1}{2}\tfrac{c}{2}\tfrac{2}{h} m_n^2 q_*^2 \iint_{(q,s)>0} \frac{q^2|\hat{M}|^2}{(q^2+s^2+m_n^2)^2}\,\delta(q - q_*\sin\vartheta_n)\,\frac{\mathrm{d}q\mathrm{d}s}{(2\pi)^2} \tag{31}$$

where

$$\sin\vartheta_n \overset{\text{def}}{=} \sqrt{\frac{q^2+s^2}{q^2+s^2+m_n^2}}\,, \qquad \text{and} \qquad \cos\vartheta_n \overset{\text{def}}{=} \sqrt{\frac{m_n}{q^2+s^2+m_n^2}} \tag{32}$$

Notice that the integrand of (31) is only nonzero along curves for which $q - q_*\sin\vartheta_n = 0$. These curves are the radiation conditions between the horizontal and vertical wavenumbers which define the shape of the wake (discussed in the next section). It is clear from these same curves (not shown) that $R_n > 0$ for all vertical mode numbers, and that the mode number determines the shape of the wake for that mode, and the magnitude of the radiation into that mode.

4.1 The Wake Angle

The internal waves in the wake must be traveling slower than the surface waves, and therefore they must propagate obliquely to the surface wave propagation direction. This implies that there is some maximum angle for the wake width given a stratification and surface wave group speed. This angle is defined as $\theta = 2\arctan(q/s)$. We can determine this angle as a function of stratification and surface wave group speed from the delta function in (31), which can be written more simply as

$$(Q^2+S^2+m_*^2) - (Q^2+S^2) = 0\,. \tag{33}$$

where $Q = q/q_*$, $S = s/q_*$, and $m_* = m_n/q_*$. Note that we are interested in the widest possible angle of the wake, so we will find where q/s is maximized. Plots (not shown) of (33) reveal that $\mathrm{d}q/\mathrm{d}s$ is maximized near $q = s = 0$. We take the derivative of (33) with respect to s to find that as $q, s \to 0$

$$\frac{\mathrm{d}q}{\mathrm{d}s} \approx \frac{1}{\sqrt{m_*^2 - 1}}\,. \tag{34}$$

Then the wake angle is given by

$$\theta = 2\arctan\left[\frac{1}{\sqrt{m_*^2-1}}\right]\,. \tag{35}$$

This angle is shown for a few values of m_* in figure 2. Note that θ becomes undefined when $m_* < 1$. This is the case in which the stratification is very strong, so the fastest radiated internal waves are moving at the surface wave group speed (i.e. the $2D$ limit). Choosing even smaller values of m_* (larger N or smaller $c/2$) does not change the wake angle beyond π, but rather allows for higher vertical modes to be radiated.

5 Conclusions

Groups of surface gravity waves produce a convergent Stokes drift in front of the group that drives water downward, and divergent Stokes drift aft of the group bringing water upward. This Stokes pumping drives a deep Eulerian return flow that decays algebraically with depth. The Eulerian return flow shakes the stable stratification below the mixed layer to produce internal gravity waves. In $2D$, in order to radiate internal gravity waves, the stratification must be very strong ($N > \pi/2ch$) so that the internal wave phase speed matches the surface wave group speed. Solutions for radiating internal waves in nonuniform stratification are obtained, however, the $2D$ case is unrealistic for the ocean due to the strong stratification requirement. In the $3D$ case the internal waves may radiate obliquely to the surface wave propagation direction, and therefore need not travel as quickly. This loosens the restriction on stratification. Instead, stronger stratification results in a wider v-shaped wake than weaker stratification. We have shown solutions for the internal wave radiation, and the shape of the wake in the $3D$ case with uniform stratification.

References

Lighthill, M. (1978). *Waves in Fluids*. Cambridge University Press.

Longuet-Higgins, M. S. and Stewart, R. (1964). Radiation stresses in water waves; a physical discussion, with applications. *Deep Sea Research*, 11(4):529–562.

McIntyre, M. (1981). On the 'wave momentum' myth. *Journal of Fluid Mechanics*, 106:331–347.

Olbers, D. J. and Eden, C. (2016). Revisiting the generation of internal waves by resonant interaction with surface waves. *Journal of Physical Oceanography*, 46(5).

Olbers, D. J. and Herterich, K. (1979). The spectral energy transfer from surface waves to internal waves. *Journal of Fluid Mechanics*, 92(02):349–379.

Phillips, O. M. (1977). *The Dynamics of the Upper Ocean*. Cambridge University Press.

Van Den Bremer, T. and Taylor, P. (2015). Estimates of lagrangian transport by surface gravity wave groups: the effects of finite depth and directionality. *Journal of Geophysical Research Oceans*, 120(4):2701–2722.