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Local Field Theory for Solitons

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Abstract

A method is developed for constructing the Lagrangian that describes the interaction of classical soliton solutions. We applied it to the Abelian Higgs model in (1 + 2) dimensions and Georgi-Glashow model in (1 + 3) dimensions, and various features of the relevant Lagrangians are investigated.

I. Introduction

A great deal of progress has been made in field theory by investigating classical solutions to field equations. These solutions fall into two categories: The instantons[1] of finite action in Euclidean space-time, and the solitons[2] of finite energy in the real space-time. Instantons determine the structure of vacuum[3], whereas the solitons correspond to particle states which are inaccessible through perturbation theory. Field theories with soliton solutions have been intensively studied, and methods have been developed for computing the lowest order (in $\hbar$) quantum corrections.[4] Although these methods have yielded a number of interesting results, they suffer from the following restrictions:

a) They are semi-classical in nature and they always involve an expansion in powers of Planck's constant. It would be nice to find alternative approximation methods or failing that, at least to formulate the problem in a manner independent of any approximation scheme.

b) Progress has been made in cases where the classical solution can be given in closed form. This is, in general, not possible except for the notable case of the two dimensional sine-Gordon equation.[5] Many soliton problems involving pair creation, scattering, etc. are not tractable (again excluding some two dimensional models).[6]

These difficulties would be eliminated if one could construct

a field theory governing the interaction of solitons. One could then treat problems involving creation and destruction of solitons. This has been achieved in the two dimensional sine-Gordon equation; where it is known that the theory can be transformed into the massive Thirring model.[7] The fermions of the Thirring theory are the solitons of the sine-Gordon theory, and they can be studied as Thirring quanta.

In this paper, we construct local field theories of solitons in 3 and 4 dimensions. We have applied our method to the known gauge theory soliton solutions of scalar quantum electrodynamics in (1 + 2) dimensions and the Georgi-Glashow[8] model in (1 + 3) dimensions.

The former are vortices[9] and the latter are static monopole solutions.[10] These models will be the subject matter of this paper.

Our approach to field theory will be through functional integration.[11] In particular, we shall focus on the contribution to the functional integral from topological configurations of the scalar Higgs field. Such a configuration will be called a "kink"; it carries a non-zero winding number in the appropriate dimensional space. The contribution of a kink will be equivalent to a "bare" or pointlike soliton. The kinks naturally form world lines, and the functional integral over all kinks reduces to a sum over all possible particle trajectories. We then have a Feynman "world line"[12] description of solitons which can easily be transcribed into the language of field theory[13], yielding a Lagrangian for solitons.

The final result has several unusual features. One such feature is the appearance of the inverse of the original coupling constant. For example, the monopole charge is inversely proportional to the coupling constant in the non-Abelian model. A similar inversion of the coupling constant takes place in the sine-Gordon and Thirring equivalence.(7) Another unexpected phenomenon is that these manifestly renormalizable theories lead to soliton Lagrangians which, on the surface, look non-renormalizable. However, this may be a case of deceptive appearances.

Finally, an intrigueing and attractive feature of our derivation is that it goes through even when there is no spontaneous symmetry breakdown in the original theory! In this case, there are clearly no classical solutions, and it is not clear what our soliton field theory represents. In the case of the Abelian Higgs model in (1 + 2) dimensions, we argue that the solitons always exist but they are confined in the absence of spontaneous symmetry breakdown. We end the paper with a speculative discussion about the connection between spontaneous symmetry breakdown and confinement in the non-Abelian model in (1 + 3) dimensions.

II. Topology of the Abelian Higgs Model in (1 + 2) Dimensions

The Lagrangian density of this model is

$$\mathcal{L} = -1/4\, F_{\mu\nu} F^{\mu\nu} + |\partial_\mu \phi + ieA_\mu \phi|^2 - \frac{\lambda^2}{4} (|\phi|^2 - h^2)^2, \tag{2.1}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, $\mu = 0,1,2$; metric = (+,-,-), and ϕ is the complex Higgs field. The standard generating functional is given by the following integral:

$$Z(J) = \int DA \int D\phi \ \exp\{i \int d^3x [\mathcal{L}(x) + J_{\mu\nu}(x)\, F^{\mu\nu}(x)]\}\ \delta(\partial_\mu A^\mu). \tag{2.2}$$

The source term for the ϕ field is omitted to keep the formulas from getting lengthy; however, if so desired, it can be reintroduced at any stage of the development. Also, for reasons of later convenience, we have introduced a source for the gauge invariant field strength $F_{\mu\nu}$, rather than the gauge dependent potential A_μ.

Finally, for the sake of definiteness, we have picked the Landau gauge, although any legitimate gauge choice would do.

We are interested in the topology of the Higgs field, ϕ. For simplicity, first consider the static case, where ϕ is time independent. Let $\vec{r} = (x_1, x_2)$ be a space point, and consider an infinitesimal closed curve around $\vec{r}$. If one makes a trip around this curve and arrives back at the starting point, the phase of ϕ has to change by $2\pi n$, where n is an integer. If n = 0, the point $\vec{r}$ is a regular point, and if $n \neq 0$, it is a kink of winding number n. A simple example of a kink of winding number n located at $\vec{r} = 0$ is

$$\phi = \text{const. } e^{in\theta}, \tag{2.3}$$

where $\tan\theta = \frac{x_2}{x_1}$.

If ϕ is time dependent, we still can use the above definition at a fixed time, and let the position of the kink be a function of time. Therefore, the position of the kink forms a trajectory in space-time, which can be conveniently parametrized by some (arbitrary) internal parameter, τ. By continuity, the winding number must be constant over the trajectory, and so the topological structure of the Higgs field can be characterized by giving the equations of the trajectories as functions of the internal parameters τ_ℓ as

$$x^\mu = \bar{x}^\mu_\ell(\tau_\ell), \tag{2.4}$$

along with the corresponding winding numbers n_ℓ, where $\ell = 1, 2, \cdots, N$, N = total number of kinks.

We have left the sense in which the trajectory is described

arbitrary. This could be fixed by, for example, taking τ an increasing function of time. Instead, however, it is best to allow for trajectories traveling backwards in time and to identify them with forward traveling trajectories with opposite sign of winding number.

The alert reader may have noticed that we have tacitly assumed the kink trajectories to be timelike. However, in the functional integral, space and light like trajectories must also be included, and for these trajectories there may be some problems in defining the winding number. A simple way out of these difficulties is to continue the functional integral (2.2) to Euclidean space and thus avoid any possible problems resulting from the Minkowski metric.

The gauge transformations which leave the Lagrangian invariant,

$$\phi \to e^{-i\Lambda}\phi,$$

$$A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \Lambda, \tag{2.5}$$

fall into two classes:

Any transformation which changes either the location of the kink trajectory or its winding number will be called a singular transformation, whereas the transformations which leave the topological properties of ϕ intact will be called regular transformations. An example of a singular transformation is the gauge transformation which straightens out the kink described by Eq. (2.3):

$$\phi \to e^{-in\theta}\phi$$

$$A_\mu \to A_\mu + \Delta A_\mu ,$$

$$\Delta A_\mu = \frac{1}{e}\,\partial_\mu \Lambda ,$$

$$\Delta A_0 = 0,\ \Delta A_1 = -\frac{n}{e}\,\frac{x_2}{x_1^2 + x_2^2} ,$$

$$\Delta A_2 = \frac{n}{e}\,\frac{x_1}{x_1^2 + x_2^2} . \tag{2.6}$$

Strictly speaking, the singular transformation given above is not a gauge transformation at the origin. In fact, ΔA_μ carries a singular tube of flux located at the origin:

$$\partial_1(\Delta A_2) - \partial_2(\Delta A_1) = 2\pi\,\frac{n}{e}\,\delta^2(\vec{r}), \tag{2.7}$$

where $\vec{r} = (x_1, x_2)$, $r = (x_1^2 + x_2^2)^{1/2}$.

This result is verified by the use of Stokes' theorem applied to a small circle around the origin in the (x_1, x_2) plane. Alternatively, to avoid the singularity at the origin, replace the second set of transformations in Eq. (2.6) by the following:

$$\Delta A_1 = -\frac{n}{e}\,f(r)\,\frac{x_2}{r^2} ,$$

$$\Delta A_2 = \frac{n}{e}\,f(r)\,\frac{x_1}{r^2} , \tag{2.8}$$

where $f(r)$ is zero at the origin and rapidly goes to one away from the origin; for example, take

$$f(r) = \frac{r^2}{r^2 + \epsilon^2} . \tag{2.9}$$

The original transformations (2.6) are now considered as the limit of (2.8) as $\epsilon \to 0$. Eq. (2.7) is now replaced by

$$\partial_1(\Delta A_2) - \partial_2(\Delta A_1) = \frac{n}{e}\,f'(r). \tag{2.10}$$

From (2.9), $f'(r)$ tends to 2π times a two dimensional delta function in the limit $\epsilon \to 0$, and we recover Eq. (2.7). Notice that since the transformation of ϕ is unchanged, the new transformations are still singular in the topological sense defined earlier. The only difference between Eq.'s (2.7) and (2.8) is that the latter smoothes out the delta function. Later on, this smoothing will be used to avoid at least temporarily the divergence difficulties resulting from the point-like structure of the bare solitons.

We have so far considered only time independent kink trajectories, but it is easy to generalize to arbitrary trajectories. Let $x^\mu \equiv \bar{x}^\mu(\tau)$ be a typical trajectory, and consider a fixed point on this trajectory corresponding to $\tau = \tau_0$. Given the tangent

$$\dot{\bar{x}}^\mu(\tau_0) \equiv \left(\frac{d}{d\tau}\,\bar{x}^\mu(\tau)\right)_{\tau = \tau_0}$$

at that point, we can rotate the coordinate system so that the tangent points in the time direction, and simply notice that (2.7) is valid in this coordinate system. The argument applies to any point on the trajectory, and the result can be written in a covariant form:

$$\Delta F_{\mu\nu} = \partial_\mu(\Delta A_\nu) - \partial_\nu(\Delta A_\mu)$$

$$= \frac{2\pi n}{e}\,\epsilon_{\mu\nu\lambda}\int d\tau\ \dot{\bar{x}}^\lambda(\tau)\ \delta^3[x - \bar{x}(\tau)], \tag{2.11}$$

where $\epsilon_{\mu\nu\lambda}$ is the complete antisymmetric tensor.

In the special case $\bar{x}^i=0$, $\bar{x}^0=\tau$, we recover Eq. (2.7). We also note that this result is invariant under non-singular gauge transformations, and so it does not rely on the detailed angular dependence given in Eq. (2-3), but only on the winding number n.

Eq. (2.11) is easily generalized to several kink trajectories $\bar{x}^\mu_\ell(\tau_\ell)$ with their winding numbers n_ℓ:

$$\Delta F_{\mu\nu} = \sum_\ell \frac{2\pi n_\ell}{e}\,\epsilon_{\mu\nu\lambda}\int d\tau_\ell\ \dot{\bar{x}}^\lambda_\ell(\tau_\ell)\ \delta^3[x - \bar{x}_\ell(\tau_\ell)], \tag{2.12}$$

where the integration is over the complete trajectory of the kink. If one wishes to avoid singular functions, one can replace the delta function on the right by a smooth function like (2.9) and finally consider the limit $\epsilon\to 0$. Finally, Eq. (2.12) is independent of the choice of parameter τ, as it should be.

III. Soliton Lagrangian From Abelian Higgs Model: The First Version

The functional integration over ϕ in Eq. (2.2) runs over all configurations containing arbitrary number of kinks with all possible winding numbers and trajectories. From now on, only kinks with winding numbers n=±1 will be considered in the functional integral over ϕ, and kinks with higher winding number will be assumed to form by the coalescing of kinks with winding number ±1. In any case, classical solitons with $|n|>1$ are known to be unstable. If they do exist quantum mechanically, they would probably emerge as bound states of fundamental solitons with n=±1.

In this section, our goal is to extract the contribution due to kinks from the functional integral. What remains is an integration over ϕ with no topological configurations. This can be achieved by means of a singular transformation of the type described in section 2, which "straightens out" all the kinks. This singular transformation is not a pure gauge transformation, and there is a contribution to flux given by Eq. (2.12) which has to be taken into account. It is this contribution that is usually missed in standard treatments;[14] for example, in going over to the "physical gauge" Im(ϕ)=0, the singular nature of the transformation involved is usually overlooked. Carrying out the singular transformation described above, we have the following result:

$$Z(J) = \sum_{N=0}^{\infty} \frac{1}{N!} \int DA \int \bar{D}\phi \int \prod_{\ell=1}^{N} D\bar{x}_\ell$$

$$\exp \{i \int d^3x [J_{\mu\nu}(x) F^{\mu\nu}(x) + \mathcal{L}(x;\bar{x}_1, \cdots, \bar{x}_N)]\}$$

$$\delta(\partial_\mu A^\mu), \qquad (3.1a)$$

where,

$$\mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N)$$

$$= -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} + |[\partial_\mu\phi(x) + ieA_\mu(x) + ie\Delta A_\mu(x;\bar{x}_1,\cdots,\bar{x}_N)]\phi(x)|^2$$

$$- \frac{\lambda^2}{4} (|\phi|^2 - h^2)^2,$$

and,

$$\partial_\mu(\Delta A_\nu) - \partial_\nu(\Delta A_\mu) = \Delta F_{\mu\nu}$$

$$= \sum_{\ell=1}^{N} \frac{2\pi}{e} \epsilon_{\mu\nu\lambda} \int d\tau_\ell \, \dot{\bar{x}}_\ell^\lambda(\tau_\ell) \, \delta^3[x - \bar{x}_\ell(\tau_\ell)]. \qquad (3.1b)$$

Let us explain the origin of various terms in this equation. In the functional integration over ϕ, the bar over D restricts the function space to fields without kinks, and therefore, the integration over ϕ has to be complemented by explicit functional integration over the kink trajectories $\bar{x}_\ell$, and summation over the total number of kinks N. The winding number n takes on the values ± 1, and if

trajectories traveling backwards in time are also included, n can be taken to be +1, as explained earlier. The factor of 1/N! avoids the overcounting problem due to the fact that kinks are indistinguishable. Finally, the contribution of kinks to the Lagrangian is given by (3.16).

The next step is to do the summation over the trajectories in closed form. This can best be accomplished by the Lagrange multiplier method: We introduce a Lagrange multiplier field $G_{\mu\nu}(x) = - G_{\nu\mu}(x)$ and another auxilliary field $B_\mu(x)$ and make use of the following identity:

$$\exp \{i \int d^3x \, \mathcal{L}(x,\bar{x}_1,\cdots,\bar{x}_N)\}$$

$$= \int DG \int DB \exp \{i \int d^3x [\mathcal{L}_B(x)$$

$$- G^{\mu\nu}(x) \, [\partial_\mu B_\nu(x) - \partial_\nu B_\mu(x)$$

$$- \sum_{\ell=1}^{N} \frac{2\pi}{e} \epsilon_{\mu\nu\lambda} \int d\tau_\ell \, \dot{\bar{x}}_\ell^\lambda(\tau_\ell) \, \delta^3[x - \bar{x}(\tau_\ell)]]]\}, \qquad (3.2)$$

where

$$\mathcal{L}_B = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda^2}{4} (|\phi|^2 - h^2)^2$$

$$+ |\partial_\mu\phi + ie(A_\mu + B_\mu)\phi|^2.$$

This identity can be verified by first integrating over G, picking up a functional delta function, and then integrating over B. Strictly speaking, there should be a normalization constant on the right hand side of the equation; however, this constant cancels when Green's functions are computed and so it is dropped for simplicity.

If Eq. (3.2) is substituted into Eq. (3.1a), it turns out that the integrations over the particle trajectories can be done with the help of the following equivalence theorem relating particle dynamics to field theory:

$$\sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{\ell=1}^{N} D\bar{x}_\ell \exp \{i \sum_{\ell=1}^{N} \int d\tau_\ell \, [m(\dot{\bar{x}}_\ell^2(\tau_\ell))^{1/2}$$

$$+ Q_\mu(\bar{x}_\ell(\tau_\ell)) \, \dot{\bar{x}}_\ell^\mu(\tau_\ell)]\}$$

$$= \int D\chi \exp \{i \int d^3x \, [-m^2|\chi(x)|^2$$

$$+ |\partial_\mu \chi(x) + iQ_\mu(x)\chi(x)|^2]\}, \qquad (3.3)$$

where $\chi(x)$ is a complex scalar field. The physical meaning of this equation is simple: Both sides are equal to the vacuum-vacuum transition amplitude of theory consisting of charged scalar particles of mass m in the presence of an external electromagnetic field $Q_\mu(x)$, and hence, they are equal to each other. The right hand side of the equation is the standard field theoretic functional integral for this process and needs no explanation. The left hand side is the quantum mechanical path integral over the trajectories of charged particles a la Feynman. The expression in the exponenet is the parametrization invariant form of the Lagrangian of a charged point

particle in an external electromagnetic field. To see this, let $\tau = t =$ time, making use of parametrization independence, and observe that

$$m[\dot{x}^2(\tau)]^{1/2} + Q_\mu[x(\tau)] \, \dot{x}^\mu(\tau)$$

$$\to m(1 - \vec{V}^2)^{1/2} + Q_0 - \vec{Q}\cdot\vec{V} \qquad (3.4)$$

where $\vec{V} = \frac{d}{dt} \vec{x}(t) =$ velocity.

For a full treatment of the correspondence between particle dynamics based on classical trajectories and field theory, the reader is referred to footnote (13). For the sake of completeness, we present a brief formal derivation of (3.3) in the appendix.

Upon substitution of (3.3) and (3.2) in (3.1), the following result is obtained:

$$Z(J) = \int DA \int \bar{D}\phi \int DG \int DB \exp \{i \int d^3x [J_{\mu\nu} F^{\mu\nu}$$

$$+ \mathcal{L}^{(\chi)}(x)]\} \, \delta(\partial_\mu A^\mu),$$

where,

$$\mathcal{L}^{(\chi)}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda^2}{4} (|\phi|^2 - h^2)^2$$

$$+ |\partial_\mu \phi + ie(A_\mu + B_\mu)\phi|^2 - 2G^{\mu\nu}(\partial_\mu B_\nu)$$

$$+ |\partial_\mu \chi + \frac{2\pi i}{e} \varepsilon_{\mu\nu\lambda} \, G^{\nu\lambda} \chi|^2. \qquad (3.5b)$$

This is the soliton Lagrangian promised earlier. As yet, it is not in a useful form, since the system has constraints. These constraints can be gotten rid of by integrating over one of the auxilliary fields, and depending on the choice of the auxilliary field, we obtain two Lagrangians different in appearance but completely equivalent in physical content. We present them in the next section.

IV. Soliton Lagrangian From Abelian Higgs Model: Later Versions

Let us define a new field V_μ by

$$A_\mu + B_\mu = V_\mu. \tag{4.1a}$$

Written in terms of V_μ and B_μ, the Lagrangian of Eq. (3.5b) with the source term becomes

$$\begin{aligned} J^{\mu\nu}(x)F_{\mu\nu}(x) + \mathcal{L}^{(\chi)}(x) = &- \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \\ &+ (\partial_\mu B_\nu - \partial_\nu B_\mu)(\partial^\mu V^\nu - G^{\mu\nu} - J^{\mu\nu}) \\ &- \frac{1}{4}(\partial_\mu V_\nu - \partial_\nu V_\mu)^2 + |\partial_\mu \phi + ie\, V_\mu \phi|^2 \\ &+ |\partial_\mu \chi + \frac{2\pi i}{e}\, \varepsilon_{\mu\nu\lambda}\, G^{\nu\lambda}\chi|^2 \\ &- \frac{\lambda^2}{4}(|\phi|^2 - h^2)^2. \end{aligned} \tag{4.1b}$$

This expression is quadratic in $H_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$, and therefore, the functional integral over $H_{\mu\nu}$ is a Gaussian and can be done easily. However, since

$$\varepsilon_{\mu\nu\lambda}\,(\partial_\lambda H_{\mu\nu}) = 0, \tag{4.2}$$

only that part of $G^{\mu\nu} + J^{\mu\nu}$ that satisfies

$$\varepsilon^{\mu\nu\lambda}\,(\partial_\lambda G_{\mu\nu} + \partial_\lambda J_{\mu\nu}) = 0 \tag{4.3}$$

will contribute. Note that Eq. (4.1) is invariant under gauge transformations

$$\chi \to e^{i\Lambda}\chi,$$

$$G_{\mu\nu} \to G_{\mu\nu} - \frac{e}{2\pi}\,\varepsilon_{\mu\nu\lambda}\,(\partial^\lambda \Lambda). \tag{4.4}$$

Hence, (4.3) can be imposed as a gauge condition. Since the external field $J_{\mu\nu}$ is at our disposal, we impose (4.3b) on both J and G separately for convenience, and carry out the integration over $H_{\mu\nu}$. Again, dropping an insignificant normalization constant, we obtain the following result:

$$Z(J) = \int DV \int \bar{D}\phi \int DG \, \exp\,\{i \int d^3x \;\mathcal{L}^{(\chi)}_{JG}(x)\}$$

$$\times\; \delta(\partial_\mu V^\mu)\;\delta(\varepsilon^{\mu\nu\lambda}\;\partial_\lambda G_{\mu\nu}), \tag{4.5a}$$

where,

$$\mathcal{L}^{(\chi)}_{JG}(x) = [1 + \frac{8\pi^2}{e^2}\,|\chi|^2]\;G_{\mu\nu}G^{\mu\nu}$$

$$+\; G^{\mu\nu}\,\{2J_{\mu\nu} + \frac{2\pi i}{e}\,\varepsilon_{\mu\nu\lambda}\,[\chi(\partial^\lambda\chi^*) - \chi^*(\partial^\lambda\chi)]$$

$$-\;(\partial_\mu V_\nu - \partial_\nu V_\mu)\} + J_{\mu\nu}J^{\mu\nu} - 2J^{\mu\nu}\,(\partial_\mu V_\nu)$$

$$+\;|\partial_\mu\phi + ieV_\mu\phi|^2 + |\partial_\mu\chi|^2$$

$$-\;\frac{\lambda^2}{4}\,(|\phi|^2 - h^2)^2. \tag{4.5b}$$

If the gauge constraint on $G_{\mu\nu}$ could be eliminated, the integration over this variable would be a Gaussian and easy to do. We eliminate the constraint by adding a Lagrange multiplier term to $\mathcal{L}^{(\chi)}_{JG}$:

$$\Delta\mathcal{L} = \varepsilon_{\mu\nu\lambda}\;G^{\mu\nu}\;(\partial^\lambda\psi), \tag{4.6}$$

where ψ is a real scalar field to be integrated over. With the help of this trick, the integration over G is done, and the Lagrangian is cast into the following final form:

$$Z(J) = \int DV \int \bar{D}\phi \int D\psi \, \exp\,\{i\int d^3x\;\mathcal{L}_1(x)\}$$

$$\times\; d\;\delta(\partial_\mu V^\mu), \tag{4.7a}$$

where,

$$\mathcal{L}_1 = -\frac{1}{4}\left[1 + \frac{8\pi^2}{e^2}|\chi|^2\right]^{-1}\{\partial_\mu V_\nu - \partial_\nu V_\mu$$

$$+ \frac{2\pi i}{e}\varepsilon_{\mu\nu\lambda}\left[\chi(\partial^\lambda \chi^*) - \chi^*(\partial^\lambda \chi)\right]$$

$$+ \varepsilon_{\mu\nu\lambda}(\partial^\lambda \psi) + 2J_{\mu\nu}\}^2 - J_{\mu\nu}J^{\mu\nu}$$

$$- 2J^{\mu\nu}(\partial_\mu V_\nu) + |\partial_\mu \chi|^2$$

$$+ |\partial_\mu \phi + ie V_\mu \phi|^2 - \frac{\lambda^2}{4}(|\phi|^2 - h^2)^2, \tag{4.7b}$$

and,

$$d = \prod_i \left[1 + \frac{8\pi^2}{e^2}|\chi(x_i)|^2\right]^{-3/2}. \tag{4.7c}$$

We now make some comments about Eq. (4.7):

a) The field $G_{\mu\nu}$ had to be scaled to eliminate the coefficient of the quadratic term in G before doing the functional integral. The factor d is the Jacobian that results from this scaling. The product is over all space time points, and so it is well-defined only for a lattice theory and becomes singular in the continuum limit. Such singular factors are well-known in the literature;[15] they contribute only to higher order terms (loops) and they are usually needed to eliminate other singular contributions.

b) The field ψ is unphysical and decouples on the mass shell.

It can be eliminated from the Lagrangian by a suitable gauge transformation on χ.

c) The ϕ integration is supposed to be over functions with no kinks. One way to ensure this is to transform from the gauge

$$\partial_\mu V^\mu = 0$$

to the physical gauge

$$\mathrm{Im}(\phi) = 0. \tag{4.8}$$

As a result, one picks up an additional Jacobian factor

$$d' = \prod_i |\phi(x_i)| \tag{4.9}$$

similar to (4.7c). Or, one could define a new field ϕ_t by

$$\phi = \phi_t + h \tag{4.10}$$

and imagine a perturbation expansion in powers of ϕ_t. Such an expansion is kink free, since any small fluctuation of ϕ around the average value h cannot produce any kinks.

d) If we translate the field ϕ as in Eq. (4.10), we find the standard Higgs result that the vector field V_μ has acquired a mass

$$m_V^2 = 2e^2h^2. \tag{4.11}$$

The theory has the unusual feature that the coupling of the vector field to the Higgs field is proportional to e, whereas its coupling to the soliton field χ is proportional to 1/e. Both e and its inverse are present in the Lagrangian, and any perturbation expansion in either direct or inverse powers of e seems out of question. However, a semiclassical expansion in the number of loops may still be all right.

e) Our derivation leading up to Eq. (4.7) nowhere made use of the equations of motion or, for that matter, of the existence of spontaneous symmetry breaking. Had we started with a Higgs self coupling given by

$$\Delta\mathcal{L} = -\frac{\lambda^2}{4}\left(|\phi|^2 + h^2\right)^2 \tag{4.12a}$$

instead of

$$\Delta\mathcal{L} = -\frac{\lambda^2}{4}\left(|\phi|^2 - h^2\right)^2 \tag{4.12b}$$

the vector meson, instead of acquiring the mass given by Eq. (4.11), would stay massless. Without spontaneous symmetry breaking there are no classical soliton solutions to the field equations. However, our derivation goes through, and we face the paradoxical situation of having a soliton field whereas semiclassically there is none. Our resolution to this paradox goes as follows: In the absence of symmetry breaking, the exchange of the zero mass vector meson gives rise to Coulomb interaction between charges, which, in three dimensions, grows logarithmically with distance. This interaction confines the electric charge of the solitons, and single soliton states cannot emerge as free particles. In the presence of spontaneous symmetry breaking, the vector meson acquires a finite mass, the long range confining interaction disappears, and the solitons are liberated.

f) Although we started with a renormalizable, in fact, super-renormalizable Lagrangian in 3 dimensions, we ended up with a Lagrangian that looks, at least superficially, non-renormalizable. Also, the external source $J_{\mu\nu}$ is coupled to a composite field in (4.7b), and there are additional singularities due to the singular Jacobian of Eq. (4.7c). It may be that these effects cancel each other to restore renormalizability. Notice that the singular graphs come from expanding in powers of 1/e, whereas graphs proportional to direct powers of e are still renormalizable. We shall return to this question again at the end of this section.

Now turn to Eq. (3.5b), and eliminate B_μ, instead of A_μ, in favor of V_μ through Eq. (4.1a). We also adopt the gauge $\mathrm{Im}(\phi) = 0$ instead of the Landau gauge, and let

$$2^{-1/2}\,\phi_r = \mathrm{Re}(\phi) - h. \tag{4.13}$$

The Lagrangian of (3.5b) becomes

$$\mathcal{L}^{(\chi)}(x) = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda^2}{4} \left(\frac{\phi_r^2}{2} + \frac{m_V}{e} \phi_r \right)^2$$

$$+ \frac{1}{2} (\partial_\mu \phi_r)^2 + \frac{1}{2} m_V^2 \left(1 + \frac{e}{m_V} \phi_r\right)^2 V_\mu^2$$

$$- G^{\mu\nu}(\partial_\mu V_\nu - \partial_\nu V_\mu - F_{\mu\nu})$$

$$+ \left|\partial_\mu \chi + \frac{2\pi i}{e} \varepsilon_{\mu\nu\lambda} G^{\nu\lambda} \chi\right|^2. \qquad (4.14)$$

The integral over V_μ is a Gaussian and can be done after the change of variable

$$V_\mu \to \left(1 + \frac{e}{m_V} \phi_r\right)^{-1} V_\mu \qquad (4.15)$$

which introduces a Jacobian given by

$$\prod_i \left[1 + \frac{e}{m_V} \phi_r (x_i)\right]^{-1} \equiv (d')^{-1}, \qquad (4.16)$$

where d', given by Eq. (4.9), is the singular factor that goes with the gauge $\mathrm{Im}(\phi) = 0$. These two factors cancel each other, and we have the following result which is free of singular factors:

$$Z(J) = \int DA \int D\phi_r \int D\chi \int DG \exp \{i \int d^3x \, [J_{\mu\nu} F^{\mu\nu}$$

$$+ \mathcal{L}_G(x)]\} \quad \delta(\partial_\mu A^\mu), \qquad (4.17a)$$

where,

$$\mathcal{L}_G = -\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{\lambda^2}{4} \left(\frac{\phi_r^2}{2} + \frac{m_V}{e} \phi_r \right)^2$$

$$+ \frac{1}{2} (\partial_\mu \phi_r)^2 + G^{\mu\nu}F_{\mu\nu}$$

$$- \frac{2}{m_V^2} \left(1 + \frac{e}{m_V} \phi_r\right)^{-2} (\partial_\mu G^{\mu\nu})^2$$

$$+ \left|\partial_\mu \chi + \frac{2\pi i}{e} \varepsilon_{\mu\nu\lambda} G^{\nu\lambda} \chi\right|^2. \qquad (4.17b)$$

The next step is to do the integration over $F_{\mu\nu}$, after choosing the gauge

$$\varepsilon_{\mu\nu\lambda} (\partial^\mu G^{\nu\lambda}) = 0. \qquad (4.18)$$

The result is written most conveniently in terms of a vector field W_μ dual to G_μ: Defining

$$G_{\mu\nu} = \frac{1}{2} m_V \varepsilon_{\mu\nu\lambda} W^\lambda, \qquad (4.19)$$

we have:

$$Z(J) = \int DW \int D\phi_r \int D\chi \;\; \exp \{i \int d^3x \, \mathcal{L}_2(x)\}$$

$$\times \, \delta(\partial_\mu W^\mu), \qquad (4.20a)$$

where,

$$\mathcal{L}_2 = \frac{1}{2}(\partial_\mu\phi_r)^2 - \frac{\lambda^2}{4}\left(\frac{\phi_r^2}{2} + \frac{m_V}{e}\phi_r\right)^2$$

$$- \frac{1}{4}\left(1 + \frac{e}{m_V}\phi_r\right)^{-2}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2$$

$$+ \frac{1}{2} m_V^2 W_\mu^2 + m_V\, \epsilon^{\mu\nu\lambda} J_{\mu\nu} W_\lambda + (J_{\mu\nu})^2$$

$$+ |\partial_\mu\chi + 2\pi i \frac{m_V}{e} W_\mu \chi|^2. \qquad (4.20b)$$

This is the second form of the soliton Lagrangian, and we comment upon it.

a) $\mathcal{L}_2$ is more suitable for investigating soliton-soliton interaction in the strong coupling limit $m_V/e^2 \ll 1$, whereas $\mathcal{L}_1$ is useful for investigating the interaction of the Higgs scalar in the weak coupling limit $e^2/m_V \ll 1$. We therefore have two complimentary pictures of the same basic interaction. To see this more clearly, let us "freeze" the field ϕ_r by letting $\lambda \to \infty$, in which limit one can set $\phi_r = 0$:

$$\mathcal{L}_2 \xrightarrow[\lambda\to\infty]{} -\frac{1}{4}(\partial_\mu W_\nu - \partial_\nu W_\mu)^2$$

$$+ \frac{1}{2} m_V^2 W_\mu^2 + m_V\, \epsilon^{\mu\nu\lambda} J_{\mu\nu} W_\lambda + J_{\mu\nu}J^{\mu\nu}$$

$$+ |\partial_\mu\chi + 2\pi i \frac{m_V}{e} W_\mu\chi|^2. \qquad (4.21)$$

This is the usual Lagrangian describing the interaction of a complex scalar field with a massive neutral vector meson, and perturbation theory can be used in the strong coupling limit discussed above. The field W_μ is the "dual" of the original field A_μ, as can be seen from the respective couplings of the external source $J_{\mu\nu}$ to A_μ and W_μ.

b) The mass of the soliton (χ particle) is zero at the level of tree graphs, and the lowest order (in $1/e$) loop contributions come from the graphs of Figures 1 and 2. If we denote the contributions of graphs 1 and 2 to the self-energy by $\Sigma_{1,2}(p)$, we have,

$$\Sigma_1(p) = -16\pi^2 \frac{m_V^2}{e^2} \int \frac{d^3k}{(2\pi)^3} \frac{p^2k^2 - (p\cdot k)^2}{k^2(p-k)^2(k^2 - m_V^2)},$$

$$\Sigma_2(p) = 4\pi^2 \frac{m_V^2}{e^2} \int \frac{d^3k}{(2\pi)^3} \frac{2}{k^2 - m_V^2}. \qquad (4.22)$$

The first contribution is finite, and the second one is linearly divergent. If we believe the derivation leading to Eq. (4.20), there is no counter term available to cancel this divergence! The only counter terms allowed are the ones already present in the original Lagrangian of Eq. (2.1), and the only infinite counter term is the mass counter term for the Higgs scalar, which does not cure the divergence problem mentioned above. In reaching this paradox, however, we have tacitly assumed that the Lagrangian given by Eq. (4.21) is not normal ordered with respect to any of the fields. On the basis of gauge invariance alone, we know that in the quartic term given by

$$\mathcal{L}_q = 4\pi^2 \frac{m_V^2}{e^2} V_\mu^2 |\chi|^2, \tag{4.22}$$

the factor $|\chi|^2$ should not be normal ordered. In fact, it is well known that the graph given by Fig. 4 is needed to cancel the divergence of the graph of Fig. 3. However, there is no such argument against normal ordering the factor $V_\mu V^\mu$ in Eq. (4.22); on the contrary, normal ordering would eliminate the divergent graph given by Fig. 2, and the lowest order contribution to the soliton mass would then be finite. We are therefore led to the conjecture that the factor V_μ^2 in the interaction term given by Eq. (4.22) should be normal ordered. However, such a prescription cannot be simply postulated; if true, it must follow naturally from the derivation of Eq. (4.20). Unfortunately our derivation has been too heuristic to answer such delicate questions about operator ordering. Strictly speaking, the functional integrals we are dealing with exist only on a space-time grid, and the limit when the grid size goes to zero requires careful analysis. We hope to carry out this program and settle the question of normal ordering in the future.

A final remark: If this model is imbedded in (1 + 3) dimensions, the soliton trajectories become surfaces traced out by strings, and our methods should be able to derive the dynamics of Nielsen-Olesen strings.

V. Monopoles in the Non-Abelian Model in (1 + 3) Dimensions

Our starting point is the Georgi-Glashow model in (1 + 3) dimensions:

$$\mathcal{L} = -\frac{1}{4}(G^i_{\mu\nu})^2 + \frac{1}{2}(\partial_\mu\phi^i + e\,\epsilon^{ijk} A^i_\mu \phi^k)^2 - \frac{\lambda^2}{4}\left[(\phi^i)^2 - h^2\right]^2, \tag{5.1}$$

where,

$$G^i_{\mu\nu} = \partial_\mu A^i_\nu - \partial_\nu A^i_\mu + e\,\epsilon^{ijk} A^j_\mu A^k_\nu .$$

The SU(2) (isospin) index i runs from 1 to 3, and the space-time indices μ and ν run from 0 to 3. Our metric is (+, -, -, -). The generating functional Z is given by the following:

$$Z(J) = \int DA \int D\phi \, \exp\{i\int d^4x\, [J(x)C(x) + \mathcal{L}(x)]\}\, \delta(\partial^\mu A^i_\mu)\Delta . \tag{5.2}$$

In general, C is a composite field built out of A and ϕ, and it can carry various indices, and Δ is the Faddeev-Popov factor.

We wish to transform from the Landau gauge to the physical

(Abelian) gauge, where the Higgs field $\phi^a(x)$ points along a fixed direction in isospin space, which, following the usual convention, will be taken to be a = 3. Such a transformation is singular since it destroys the topological structure of ϕ, which results from the presence of kinks similar to those given by Eq. (2.3). A static kink located at $\vec{x} = 0$ looks like

$$\phi^a = \pm \frac{x^a}{|\vec{x}|} |\phi|, \quad a = 1,2,3, \tag{5.3}$$

or it is related to the above form through non-singular gauge transformations. The sign in front is the sign of the "charge" of the kink. The static monopole solution[10] to the equations of motion has the topological structure of the kink described above, and the charge of the kink is proportional to the charge of the monopole. In general, we have to allow for arbitrary kink trajectories of the type given by Eq. (2.4), and the resulting complications are best handled by defining a topological (magnetic) current[16]:

$$k_\mu = \frac{1}{2e} \varepsilon_{\mu\nu\alpha\beta} \varepsilon^{abc} (\partial^\nu \hat{\phi}^a)(\partial^\alpha \hat{\phi}^b)(\partial^\beta \hat{\phi}^c), \tag{5.4}$$

where $\hat{\phi}^a = \phi^a (\phi^b \phi^b)^{-1/2} = \phi^a |\phi|^{-1}$. The topological current, k_μ, is invariant under non-singular gauge transformations and it is conserved:

$$\partial^\mu k_\mu = 0. \tag{5.5}$$

Furthermore, k_μ vanishes everywhere except on the trajectory of a kink. This can easily be verified by lining up ϕ^a along a fixed direction. This argument breaks down at the position of the kink, where there is a delta function contribution. Consider a static kink and integrate k_0 over a volume V around the position of this kink. It is shown in reference (16) that

$$\int_V d^3x \; k_0(x) = \pm \frac{4\pi}{e} . \tag{5.6}$$

In the general case, consider a set of kink trajectories

$$x^\mu = \bar{x}^\mu_\ell(\tau_\ell), \quad \ell = 1,2,\cdots,N, \tag{5.7}$$

with corresponding charges $n_\ell = \pm 1$. Then,

$$k^\mu(x) = \frac{4\pi}{e} \sum_{\ell=1}^{N} n_\ell \int d\tau \; \dot{\bar{x}}^\mu_\ell(\tau_\ell) \; \delta^4[x - \bar{x}_\ell(\tau_\ell)], \tag{5.8}$$

where $\dot{\bar{x}}(\tau) = \frac{d}{d\tau} \bar{x}(\tau)$ and the integral is over the trajectory of the kink. This equation summarizes all the relevant topological properties of the Higgs field.

Now transform into the physical gauge. Although this can be done directly, we proceed indirectly and define the following fields:

$$\phi \equiv |\phi| \equiv (\phi^a \phi^a)^{-1/2}, \tag{5.9a}$$

$$F_{\mu\nu} \equiv \hat{\phi}^a G^a_{\mu\nu} - \frac{1}{e} \varepsilon^{abc} \hat{\phi}^a (D_\mu \hat{\phi}^b)(D_\nu \hat{\phi}^c), \tag{5.9b}$$

$$H_\mu^a \equiv \frac{1}{e} \epsilon^{abc} \hat{\phi}^b (D_\mu \hat{\phi}^c), \qquad (5.9c)$$

where $D_\mu \hat{\phi}^a = \partial_\mu \hat{\phi}^a + e\, \epsilon^{abc} A_\mu^b \hat{\phi}^c$.

Transforming into the physical gauge is the same as expressing the Lagrangian in terms of the fields defined by Eq. (5.9), which is what makes these fields so useful. Since ϕ and $F_{\mu\nu}$, the electromagnetic tensor of 't Hooft,(10) are gauge invariant, they can, in the absence of kinks, be evaluated in the physical gauge

$$\phi^a(x) = \delta^{a3} \phi(x) \qquad (5.10)$$

without losss of generality. The result is

$$F_{\mu\nu} = \partial_\mu A_\nu^3 - \partial_\nu A_\mu^3. \qquad (5.11)$$

It then follows that A_μ^3 is the electromagnetic potential. If H_μ^a is also evaluated in the same gauge, one finds

$$H_\mu^3 = 0,\ H_\mu^{1,2} = A_\mu^{1,2}. \qquad (5.12)$$

There are only two independent isospin components of H_μ^a because

$$\phi^a H_\mu^a = 0. \qquad (5.13)$$

These transform homogeneously under the electromagnetic gauge

transformations (rotations around the 3rd direction in isospin space), and it is natural to regard them as the charged components of A_μ^a and define

$$A_\mu^\pm = \frac{A_\mu^1 \pm iA_\mu^2}{\sqrt{2}} = \frac{H_\mu^1 + iH_\mu^2}{\sqrt{2}}. \qquad (5.14)$$

Eq.'s (5.9) are then the components of A_μ^a expressed in a gauge covariant manner.

So far, we have assumed the absence of kinks. If they are present, Eq. (5.11) is no longer correct, and it must be replaced by the following gauge invariant identity(16):

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + N_{\mu\nu}, \qquad (5.15)$$

where,

$$A_\mu = \hat{\phi}^a A_\mu^a,$$

$$N_{\mu\nu} = \frac{1}{e} \epsilon^{abc} \hat{\phi}^a (\partial_\mu \hat{\phi}^b)(\partial_\nu \hat{\phi}^c).$$

In the absence of kinks, ϕ^a can be lined up along the 3rd direction; $N_{\mu\nu}$ vanishes and Eq. (5.11) is recovered. However, if kinks are present, $N_{\mu\nu}$ does not vanish, and instead the following equation holds:

$$\frac{1}{2} \epsilon_{\mu\nu\alpha\beta} (\partial^\nu N^{\alpha\beta}) = k_\mu. \qquad (5.16)$$

Combining this equation with Eq. (5.8), we arrive at that set of Maxwell equations whose source is the magnetic current k_μ:

$$\frac{1}{2}\,\varepsilon^{\mu\nu\alpha\beta}\,(\partial_\nu F_{\alpha\beta}) \equiv \partial_\nu \tilde{F}^{\mu\nu}$$

$$= k_\mu = \frac{4\pi}{e} \sum_{\ell=1}^{N} n_\ell \int d\tau_\ell\, \dot{\bar{x}}_\ell^\mu(\tau_\ell)\, \delta^4[x - \bar{x}_\ell(\tau_\ell)]. \qquad (5.17)$$

Notice that the magnetic current depends only on the trajectories of kinks (monopoles) and not on the detailed dynamics of the system. The other set of Maxwell equations whose source is the electric current j^μ can be derived by varying the Lagrangian (5.1) with respect to $A_\mu = A_\mu^3$:

$$\partial_\nu F^{\mu\nu} = j^\mu,$$

where,

$$\begin{aligned} j^\mu = ie\,\{&(A^-)^\nu(\partial^\mu A_\nu^+ + ieA^\mu A_\nu^+) \\ &- (A^+)^\nu(\partial^\mu A_\nu^- - ieA^\mu A_\nu^+) \\ &+ (A^-)^\mu(\partial^\nu A_\nu^+ + ieA^\nu A_\nu^+) \\ &- (A^+)^\mu(\partial^\nu A_\nu^- - ieA^\nu A_\nu^-) \\ &+ 2\,A_\nu^+[\partial^\nu(A^-)^\mu - ieA^\nu(A^-)^\mu] \\ &- 2\,A_\nu^-[\partial^\nu(A^+)^\mu + ieA^\nu(A^+)^\mu]\}. \end{aligned} \qquad (5.18)$$

Having both pairs of Maxwell equations at hand, we wish to construct a Lagrangian, which, when $A_\mu^\pm$ and ϕ are fixed and treated as external fields, will have the pair (5.17) and (5.18) as their classical equations of motion. Instead of this indirect approach, had we directly expressed (5.1) in terms of $A_\mu^\pm$ and A_μ, we would have gotten Eq. (5.18) correctly, but we would have missed the source term in Eq. (5.17).

The problem of constructing the Lagrangian for given electric and magnetic currents j_μ and k_μ has been solved by Schwinger[17] and by Zwanziger.[18] We shall follow Zwanziger's approach, in which one introduces another vector potential B_μ, in addition to A_μ, and $F_{\mu\nu}$ is expressed as follows:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + (n\cdot\partial)^{-1}\,(\varepsilon_{\mu\nu\alpha\beta}n^\alpha k^\beta)$$

$$= -\,\varepsilon_{\mu\nu\alpha\beta}\,(\partial^\alpha B^\beta) - (n\cdot\partial)^{-1}\,(n_\mu j_\nu - n_\nu j_\mu), \qquad (5.19)$$

where n is an arbitrary fixed four vector. The monopole Lagrangian can now be written as the sum of several terms:

$$\mathcal{L}(x;\bar{x}_1,\cdots,\bar{x}_N) = \mathcal{L}_\gamma + \mathcal{L}_e + \mathcal{L}_m + \mathcal{L}_c + \mathcal{L}_H. \qquad (5.20a)$$

$\mathcal{L}_e$ and $\mathcal{L}_m$ are the electric and magnetic parts of the interaction Lagrangian, and $\mathcal{L}_H$ is the Higgs Lagrangian.

The magnetic interaction is simple:

$$\mathcal{L}_m = k_\mu(x)\, B^\mu(x), \tag{5.20b}$$

where k_μ is given by Eq. (5.8). The electric interaction is a bit more complicated since the corresponding current given by (5.18) depends on A itself:

$$\begin{aligned}\mathcal{L}_e = &- [\partial_\mu A_\nu^+ + ieA_\mu A_\nu^+]\,[\partial^\mu (A^-)^\nu - ieA^\mu (A^-)^\nu] \\ &- (\partial^\mu A_\mu^+ + ieA^\mu A_\mu^+)\,(\partial^\nu A_\nu^- - ieA^\nu A_\nu^-) \\ &+ 2\,[\partial_\mu A_\nu^+ + ieA_\mu A_\nu^+]\,[\partial^\nu (A^-)^\mu - ieA^\nu (A^-)^\mu].\end{aligned} \tag{5.20c}$$

This term would be read off directly from the original Lagrangian (5.1). $\mathcal{L}_c$, the part of the Lagrangian that includes only the charged vector mesons, can also be read off from (5.1):

$$\begin{aligned}\mathcal{L}_C = &- \frac{1}{2}\,[\partial_\mu A_\nu^+ - \partial_\nu A_\mu^+]\,[\partial^\mu (A^-)^\nu - \partial^\nu (A^-)^\mu] \\ &+ \frac{e^2}{8}\,(A_\mu^+ A_\nu^- + A_\mu^- A_\nu^+)\,[(A^+)^\mu (A^-)^\nu + (A^-)^\mu (A^+)^\nu] \\ &- e^2\,[A_\mu^+ (A^-)^\mu]^2 + e^2\phi^2\, A_\mu^+ (A^\mu)^-.\end{aligned} \tag{5.20d}$$

Finally, $\mathcal{L}_\gamma$ is the free Lagrangian for the electromagnetic fields given by Zwanziger:

$$\begin{aligned}\mathcal{L}_\gamma = &- \frac{1}{8}\,(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 \\ &- \frac{1}{8}\,(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 \\ &- \frac{1}{4n^2}\,[n_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu + \epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha B_\beta)]^2 \\ &- \frac{1}{4n^2}\,[n_\mu(\partial^\mu B^\nu - \partial^\nu B^\mu - \epsilon^{\mu\nu\alpha\beta}\,\partial_\alpha A_\beta)]^2.\end{aligned} \tag{5.20e}$$

The generating functional is obtained by integrating over the fields ϕ, A_μ, B_μ, $A_\mu^\pm$, as well as $\bar{x}$'s, the trajectories of monopoles, in a manner similar to Eq. (3.1):

$$\begin{aligned}Z(J) = &\int D\phi \int DA \int DB \int DA^+ \int DA^- \;(d'') \\ &\sum_{N=0}^{\infty} \frac{1}{N!} \int \prod_{\ell=1}^{N} D\bar{x}_\ell \,\exp\{i\int d^4x[J(x)C(x) \\ &+ \mathcal{L}(x,\bar{x}_1,\cdots,\bar{x}_N)]\},\end{aligned} \tag{5.21a}$$

where d'' is the Faddeev-Popov factor of the physical gauge, similar to d and d′ of the previous section:

$$d'' = \prod_i\, [\phi^2(x_i)]. \tag{5.21b}$$

The integration over the monopole trajectories can be carried out with the help of Eq. (3.3):

$$Z(J) = \int D\phi \int DA \int DB \int DA^{+} \int DA^{-} \quad (d^{\cdot\cdot})$$

$$\exp\{i\int d^4x\,[J\cdot C + \mathcal{L}_\pi(x)]\}, \tag{5.22a}$$

where,

$$\mathcal{L}_\pi(x) = \mathcal{L}_\gamma + \mathcal{L}_e + \mathcal{L}_c$$

$$+ \left|\partial_\mu \pi + \frac{4\pi i}{e} B_\mu \pi\right|^2. \tag{5.22b}$$

In the above equation, the first three terms are the same as in Eq. (5.20), but the magnetic interaction gets replaced by the last term, where the field π is a complex scalar monopole field.

Eq.'s (5.20) and (5.22) express the final form of the action for monopole solitons of Georgi-Glashow model. Many of the comments made at the end of sections 3 and 4 also apply here. Renormalization and problems associated with the ordering of operators are left as open questions. There are also additional problems connected with Lorentz invariance due to the appearance of a fixed four vector n.

At the beginning of our investigation, we had hoped to establish a duality between electric and magnetic potentials suggested by various authors.[19] However, our final expression does not exhibit such a symmetry, at least not in any obvious fashion, and it remains to be seen whether there is still a hidden symmetry that has escaped us.

VI. Conclusions and Future Directions

In this paper, we have studied Abelian and non-Abelian gauge theories with Higgs mechanism which are known to possess non-trivial solutions (solitons). We have developed a method which enables one to construct a field theory of solitons and thereby bring out the hidden soliton content of the original theory. The method works even when there is no spontaneous symmetry breaking and no classical solution. In this case the solitons are probably confined. In the non-Abelian model in (1 + 3) dimensions, there is an exciting possibility suggested by Mandelstam and 't Hooft.[20] When the sign of h^2 in Eq. (5.1) is reversed, it may be that the monopole develops a tachyonic mass. This would result in a spontaneous violation of magnetic charge conservation and hence electric charge would be confined. To test this possibility, one has to compute the mass of the monopole in some approximation scheme like the expansion in the number of loops. Again, it is necessary to resolve the renormalization problem before attempting such a calculation.

Finally, we would like to stress that we have only considered kinks of the Higgs field. In non-Abelian gauge theories, the components of the vector potential A_μ^a themselves can have non trivial topological structure, and a Higgs field may not be needed. Again, this may be a promising line of future research.

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Appendix

To establish Eq. (3.3), one needs as a preliminary step to show the equivalence of the reparametrization invariant Lagrangian

$$\mathcal{L}_p(\bar{x}) = \int d\tau \; \left(m[\dot{\bar{x}}^2(\tau)]^{1/2} + Q_\mu(\bar{x})\dot{\bar{x}}^\mu(\tau)\right) \tag{A.1}$$

of a particle in an external field $Q_\mu(x)$ and the Feynman form for the same Lagrangian:

$$\mathcal{L}_f(\bar{x}) = \int d\tau \; \left(- \frac{1}{2} m^2 - \frac{1}{2} \dot{\bar{x}}^2(\tau) + Q_\mu[\bar{x}(\tau)] \; \dot{\bar{x}}^\mu(\tau)\right). \tag{A.2}$$

We note that since (A.1) is invariant under reparametrization, a definite choice of parameter τ must be made before it can be used on the left hand side of Eq. (3.3). A convenient way of doing this is to set

$$Z_1 = \int D\bar{x} \; \exp \{i \, \mathcal{L}_p(\bar{x})\} \; \delta\left([\dot{\bar{x}}^2(\tau)]^{1/2} - f(\tau)\right). \tag{A.3}$$

Here $f(\tau)$ is an arbitrary but fixed function of τ. In (A.3) we have neglected an overall constant inedepndent of the dynamical variables, as we have done throughout the paper. Since Z_1 is independent of f, we can "average" over f as follows:

$$Z_1 = N_1^{-1} \int Df \exp \{ -\frac{i}{2} \int d\tau \, [m + f(\tau)]^2 \}$$

$$\times \int D\bar{x} \exp [i\mathcal{L}_p(\bar{x})] \, \delta\left([\dot{\bar{x}}^2(\tau)]^{1/2} - f(\tau)\right), \qquad (A.4)$$

where,

$$N_1 = \int Df \exp \{ -\frac{i}{2} \int d\tau \, [m + f(\tau)]^2 \} \qquad (A.5)$$

is independent of m and will be dropped. Doing the integral over f in (A.4) with the help of the delta function, we get

$$Z_1 \cong \int D\bar{x} \exp [i \, \mathcal{L}_f(\bar{x})], \qquad (A.6)$$

which establishes the equivalence of (A.1) and (A.2).

We now consider the boundary conditions at $\tau = \pm\infty$ and the range of the parameter τ. Since there is no source for the soliton field, the $\bar{x}(\tau)$ will be taken to be closed trajectories. We can therefore impose the condition

$$\bar{x}_\mu(0) = \bar{x}_\mu(T), \qquad (A.7)$$

where 0 and T are the endpoints of the range of τ; i.e.,

$$0 \leqq \tau \leqq T. \qquad (A.8)$$

Since $\mathcal{L}_f$ is not reparametrization invariant, the functional integral over $\bar{x}$ must also include an integration over all possible values of T. These comments can be put together in the form of the following equation:

$$Z_1(Q) = \int_0^\infty \frac{dT}{T} \oint D\bar{x}$$

$$\exp \{i \int_0^T d\tau \left(-\frac{1}{2} \dot{\bar{x}}^2(\tau) - \frac{1}{2} m^2\right.$$

$$\left. + Q_\mu[\bar{x}(\tau)] \, \dot{\bar{x}}^\mu(\tau)\right)\}. \qquad (A.9)$$

The circle through the integral sign reminds us of the boundary condition $\bar{x}_\mu(0) = \bar{x}_\mu(T)$. The factor 1/T in the integration over T is needed to avoid overcounting. The point $\tau = 0$ is arbitrary and can be placed anywhere along the trajectory, and so the same geometrical curve is overcounted in a way proportional to its length T because of the arbitrariness of the location of the point $\tau = 0$. The divergence at T = 0 is not serious since it only contributes to the normalization constant; it can be eliminated by dividing Z_1 by its value at zero external field.

We have so far considered only a single trajectory; however, the sum over all trajectories indicated in Eq. (3.3) can easily be done:

$$Z = \sum_{N=0}^\infty \frac{1}{N!} \prod_{\ell=1}^N D\bar{x}_\ell \exp \{i \sum_{\ell=1}^N \mathcal{L}_p(\bar{x}_\ell)\}$$

$$= \exp [Z_1(Q)], \qquad (A.10)$$

where Z_1 is given by (A.9). The passage from Eq. (A.10) to field theory (or rather, the reverse) is given in reference 13, and for the sake of completeness, we give a very brief review of their derivation. One first passes from the Lagrangian form of (A.9) to the following Hamiltonian form:

$$Z_1 = \int_0^\infty \frac{dT}{T} \oint D\bar{x} \oint Dp$$

$$\exp \{i \int_0^T d\tau \, [p \cdot \dot{\bar{x}} + \frac{1}{2} (p - Q)^2 - m^2]\}, \qquad (A.11)$$

where $p_\mu(\tau)$ is the canonically conjugate variable to the position coordinate $\bar{x}_\mu(\tau)$. Feynman's fundamental result for quantum mechanics as an integral over classical paths tells us that

$$Z_1 = \int_0^\infty \frac{dT}{T} \, \mathrm{Tr} \, \{e^{-iHT}\}, \qquad (A.12)$$

where, with p and x now operators satisfying $[x^\mu, p^\nu] = ig^{\mu\nu}$,

$$H = \frac{1}{2} (p_\mu - Q_\mu)(p^\mu - Q^\mu) - \frac{1}{2} m^2. \qquad (A.13)$$

It is also well known that the boundary condition imposed by closed trajectories translates into a trace over the Hilbert space of the Hamiltonian H.

Up to a normalization, (A.12) gives

$$Z_1 = \mathrm{Tr} \, (\log H) \qquad (A.14)$$

which substituted into (A.10), leads to the result

$$Z(Q) = \exp \, [\mathrm{Tr}(\log H)] = \det \, (H). \qquad (A.15)$$

However, if one carries out the functional integration over the field χ on the right hand side of Eq. (3.3), upon identifying the partial derivative ∂_μ with ip_μ, one obtains precisely Eq. (A.15). Eq. (3.3) is therefore established.

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Figure Captions

Figure 1. A self-energy contribution to the vortex. The solid line represents a point vortex and the wavy line is the massive, W_μ.

Figure 2. A self-energy contribution to the vortex due to lack of normal ordering.

Figure 3. A contribution to the self-energy of the photon in scalar QED. The wavy line is the photon, and the dotted line is a charged boson.

Figure 4. A contribution to the self-energy of the photon in scalar QED due to lack of normal ordering. This graph cannot be thrown out since it combines with the graph of Figure 3 to form a gauge invariant result. Gauge invariance requires no normal ordering.

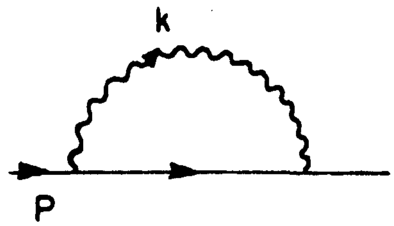

Figure 1

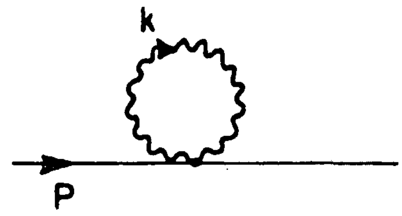

Figure 2

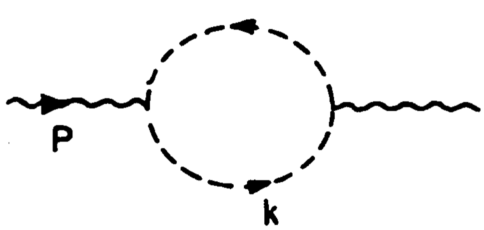

Figure 3

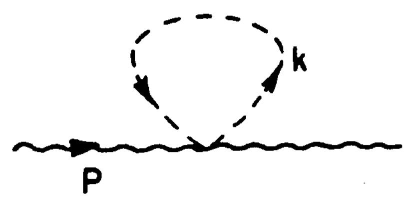

Figure 4

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