Title
Skew Calabi–Yau algebras and homological identities

Authors
Reyes, Manuel
Rogalski, Daniel
Zhang, James J

Publication Date
2014-10-01

DOI
10.1016/j.aim.2014.07.010

SKEW CALABI-YAU ALGEBRAS AND HOMOLOGICAL IDENTITIES

MANUEL REYES, DANIEL ROGALSKI, AND JAMES J. ZHANG

Abstract. A skew Calabi-Yau algebra is a generalization of a Calabi-Yau algebra which allows for a non-trivial Nakayama automorphism. We prove three homological identities about the Nakayama automorphism and give several applications. The identities we prove show (i) how the Nakayama automorphism of a smash product algebra $A\#H$ is related to the Nakayama automorphisms of a graded skew Calabi-Yau algebra A and a finite-dimensional Hopf algebra H that acts on it; (ii) how the Nakayama automorphism of a graded twist of A is related to the Nakayama automorphism of A; and (iii) that the Nakayama automorphism of a skew Calabi-Yau algebra A has trivial homological determinant in case A is noetherian, connected graded, and Koszul.

0. Introduction

While the Calabi-Yau property originated in geometry, it now has incarnations in the realm of algebra that seem to be of growing importance. Calabi-Yau triangulated categories were introduced by Kontsevich [Ko] in 1998. See [Ke] for an introductory survey about Calabi-Yau triangulated categories. Calabi-Yau algebras were introduced by Ginzburg [Gi] in 2006 as a noncommutative version of coordinate rings of Calabi-Yau varieties. Since the late 1990s, the study of Calabi-Yau categories and algebras has been related to a large number of other topics such as quivers with superpotentials, DG algebras, cluster algebras and categories, string theory and conformal field theory, noncommutative crepant resolutions, and Donaldson-Thomas invariants. Some fundamental questions in the area were answered by Van den Bergh recently in [VdB3].

One known method for constructing a noncommutative Calabi-Yau algebra is to form the smash product $A\#H$ of a Calabi-Yau algebra A with a Calabi-Yau Hopf algebra H that acts nicely on A. This phenomenon has been studied quite broadly; for instance, see [BSW], [Fa, Section 3], [IR, Section 3], [LM], [LiWZ], [WZhu], and [YuZ1, YuZ2]. One of the most general results in this direction [LiWZ] states that under technical hypotheses on Calabi-Yau algebras A and H, $A\#H$ is Calabi-Yau if and only if the homological determinant of the H-action on A is trivial. However, the smash product $A\#H$ may be Calabi-Yau even when A is not Calabi-Yau; see, for instance, Example 1.6 below. In order to explain this phenomenon, it is natural to look into a broader class of interesting algebras, called *skew Calabi-Yau algebras*

Date: February 15, 2013.

2000 *Mathematics Subject Classification.* Primary 16E65, 18G20, Secondary 16E40, 16W50, 16S35, 16S38.

Key words and phrases. Calabi-Yau algebra, Skew Calabi-Yau algebra, Artin-Schelter regular, Artin-Schelter Gorenstein, Nakayama automorphism, homological determinant, homological identity.

in this paper, which are a generalization of Ginzburg's Calabi-Yau algebras. While Ginzburg originally defined the Calabi-Yau property for DG-algebras [Gi, Definition 3.2.3], we only consider the non-DG case in this paper.

We will employ the following notation. Let A be an algebra over a fixed commutative base field k. The unmarked tensor $\otimes$ always means $\otimes_k$. Let M be an A-bimodule, and let μ, ν be algebra automorphisms of A. Then ${}^{\mu}M^{\nu}$ denotes the induced A-bimodule such that ${}^{\mu}M^{\nu} = M$ as a k-space, and where

$$a \circledast m \circledast b = \mu(a)m\nu(b)$$

for all $a, b \in A$ and all $m \in {}^{\mu}M^{\nu}(= M)$. Let A^e denote the enveloping algebra $A \otimes A^{op}$, where A^{op} is the opposite ring of A. An A-bimodule can be identified with a left A^e-module naturally, or with a right A^e-module since A^e is isomorphic to $(A^e)^{op}$. We usually work with left modules, unless otherwise stated. For example, $\mathrm{Hom}_A(-,-)$ and $\mathrm{Ext}^d_A(-,-)$ are defined for left A-modules. A right A-module is identified with a left A^{op}-module. Let B-Mod be the category of left B-modules for a ring B.

Definition 0.1. Let A be an algebra over k.

(a) A is called *skew Calabi-Yau* (or *skew CY*, for short) if
  (i) A is homologically smooth, that is, A has a projective resolution in the category A^e-Mod that has finite length and such that each term in the projective resolution is finitely generated, and
  (ii) there is an integer d and an algebra automorphism μ of A such that

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d \\ {}^1A^{\mu} & i = d, \end{cases} \tag{E0.1.1}$$

  as A-bimodules, where 1 denotes the identity map of A.

(b) If (E0.1.1) holds for some algebra automorphism μ of A, then μ is called the *Nakayama automorphism* of A, and is usually denoted by μ_A. It is not hard to see that μ_A (if it exists) is unique up to inner automorphisms of A (see Lemma 1.7 below).

(c) [Gi, Definition 3.2.3] We call A *Calabi-Yau* (or CY, for short) if A is skew Calabi-Yau and μ_A is inner (or equivalently, μ_A can be chosen to be the identity map after changing the generator of the bimodule ${}^1A^{\mu}$).

There is some variation in the literature concerning the exact definition of (skew) CY algebras. Ginzburg included in his definition of a CY algebra [Gi, Definition 3.2.3] the condition that the A^e-projective resolution of A is self-dual, but Van den Bergh has shown that this is satisfied automatically [VdB3, Appendix C]. We are also not the first to study skew CY algebras; the notion has been called twisted CY in several recent papers (see the beginning of Section 1 for more discussion).

We are primarily interested here in the special case of graded algebras A. If A is $\mathbb{Z}^w$-graded, Definition 0.1 should be made in the category of $\mathbb{Z}^w$-graded modules and (E0.1.1) should be replaced by

$$\mathrm{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d, \\ {}^1A^{\mu}(\mathfrak{l}) & i = d, \end{cases} \tag{E1.0.1}$$

where $\mathfrak{l} \in \mathbb{Z}^w$ and ${}^1A^\mu(\mathfrak{l})$ is the shift of ${}^1A^\mu$ by degree $\mathfrak{l}$, and here $\mathfrak{l}$ is called the *Artin Schelter (AS) index*. In this case, the Nakayama automorphism μ is a $\mathbb{Z}^w$-graded algebra automorphism of A.

For simplicity of exposition, *in the remainder of this introduction A is a noetherian, connected (that is, $A_0 = k$) $\mathbb{N}$-graded skew CY algebra.* Equivalently (as we will note in Lemma 1.2), A is a noetherian Artin-Schelter regular algebra. In this case μ_A is unique since there are no non-trivial inner automorphisms of A. Suppose further that A is a left H-module algebra for some Hopf algebra H, where each graded piece A_i is a left H-module. Let $A\#H$ be the corresponding smash product algebra; we review the definition in Section 2, but we note here that it is the same as the skew group algebra $A \rtimes G$ in the important special case that $H = kG$ is the group algebra of a group G acting by automorphisms on A. There is also a well-established theory of homological determinant of a Hopf algebra action [JoZ, JiZ, KKZ] that determines a map $\operatorname{hdet} : H \to k$; this is reviewed in Section 3.

Our first result provides an identity for the Nakayama automorphism of $A\#H$ in terms of those of A and H, along with the homological determinant hdet of the action. It helps to explain how the smash product $A\#H$ may become Calabi-Yau even when A is only skew Calabi-Yau. The idea is that even if μ_A is not an inner automorphism, $\mu_{A\#H}$ may become inner; see also Corollary 0.6 below.

Theorem 0.2. *Let H be a finite dimensional Hopf algebra acting on a noetherian connected graded skew CY algebra A, such that each A_i is a left H-module and A is a left H-module algebra. Then*

$$\mu_{A\#H} = \mu_A\#(\mu_H \circ \Xi^l_{\operatorname{hdet}}), \tag{HI1}$$

where hdet *is the homological determinant of the H-action on A, and $\Xi^l_{\operatorname{hdet}}$ is the corresponding left winding automorphism (see Section 1).*

Theorem 0.2 will be proved in Section 4. In the special case where $\mu_A = Id_A$ and $\mu_H = Id_H$, Theorem 0.2 partially recovers some results in a number of papers [LM, Fa, WZhu, LiWZ]. A natural question is if Theorem 0.2 holds when H is infinite dimensional or when A is ungraded. When H is an involutory CY Hopf algebra and A is an N-Koszul CY algebra, the question was answered in [LiWZ, Theorem 2.12], but in general the question is open.

By the term *homological identity* which we use in the title of the paper, we mean an equation involving several invariants, at least one of which is defined homologically. Equation (HI1) is of course an example, and we note that other homological identities containing the Nakayama automorphism with interesting applications have appeared in [BZ, Theorem 0.3] and [CWZ, Theorem 0.1].

Next, we prove a homological identity which shows how the Nakayama automorphism changes under a graded twist in the sense of [Zh]. Let σ be a graded algebra automorphism of A and let $A^{\tilde{\sigma}}$ denote the left graded twist of A associated to the twisting system $\tilde{\sigma} := \{\sigma^n \mid n \in \mathbb{Z}\}$. Recall that $A^{\tilde{\sigma}}$ is an algebra with the same underlying graded vector space as A, but with new product $a * b = \sigma^{||b||}(a)b$ for homogeneous elements a, b, where $|| \, b \, ||$ indicates the degree of the element b. Then it is easy to check using the properties of graded twists that A is skew CY if and only if $A^{\tilde{\sigma}}$ is (see Theorem 5.4 below). For a nonzero scalar c, define a graded algebra automorphism ξ_c of A by $\xi_c(a) = c^{||a||}a$ for all homogeneous elements $a \in A$.

Theorem 0.3. *Let A be a noetherian connected graded skew CY algebra, and let $\mathfrak{l}$ be the AS index of A. Then*

$$\mu_{A^{\tilde{\sigma}}} = \mu_A \circ \sigma^{\mathfrak{l}} \circ \xi^{-1}_{\mathrm{hdet}(\sigma)}. \tag{HI2}$$

Note here that hdet is defined using the natural action of the group of graded automorphisms $H = \mathrm{Aut}_{\mathbb{Z}}(A)$ on A. Theorem 0.3 will be proved in Section 5. The result has many applications; for example, if one wants to prove a result about the Nakayama automorphism, one can often perform a graded twist to reduce to the case of an algebra with a simpler Nakayama automorphism (for example, see Lemma 6.2 below). Similarly, in some cases one can twist a skew CY algebra to obtain a CY one, and so this gives a method of producing more CY algebras; see Section 7 for examples.

Another goal of this paper is to demonstrate a strong connection between the Nakayama automorphism μ_A and the homological determinant hdet. This is already evidenced by identities (HI1), (HI2) and [CWZ, Theorem 0.1], and it is reinforced by the next result.

Theorem 0.4. *Let A be a noetherian connected graded Koszul skew CY algebra. Then*

$$\mathrm{hdet}(\mu_A) = 1. \tag{HI3}$$

In fact, in all examples of graded skew CY algebras we have observed, $\mathrm{hdet}(\mu_A) = 1$, and we conjecture that this is true in much wider generality. In particular, one may consider AS Gorenstein rings, which satisfy (E0.1.1) but for which the homological smoothness condition is replaced by the weaker assumption that A has finite injective dimension. In fact, homological identities (HI1) and (HI2) are true in the AS Gorenstein case; we give the more general versions in the body of the paper. We conjecture that (HI3) also holds for all connected graded AS Gorenstein rings (Conjecture 6.4).

The above homological identities have some immediate consequences. The following result may be useful in further study of the homological determinant.

Corollary 0.5. *Let A be a connected graded skew CY algebra, and assume that Conjecture 6.4 holds. For every $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(A)$, then*

$$\mathrm{hdet}\,\varphi = (\mu_{(A[t;\varphi])}(t))\, t^{-1},$$

where $A[t;\varphi]$ is the corresponding skew polynomial ring.

The next result provides several methods for constructing CY algebras starting with skew CY algebras.

Corollary 0.6. *Let A be connected graded skew CY and let $\langle \mu_A \rangle$ be the subgroup of $\mathrm{Aut}_{\mathbb{Z}}(A)$ generated by μ_A. Assume that $\mathrm{hdet}\,\mu_A = 1$.*

(a) *The skew polynomial ring $A[t;\mu_A]$ is CY.*
(b) *The skew group algebra $A \rtimes \langle \mu_A \rangle$ is CY.*
(c) *Suppose that μ_A has finite order. Then the fixed subring $A^{\langle \mu_A \rangle}$ is AS Gorenstein with trivial Nakayama automorphism.*

Using homological identity (HI2), one may study the Nakayama automorphisms of the entire family of graded twists of a given skew CY algebra A. Let $o(\mu_A)$ be the order of the Nakayama automorphism μ_A. In most cases, one can twist A to produce an algebra with Nakayama automorphism of finite order.

Corollary 0.7. *Assume that the base field k is algebraically closed of characteristic zero. Let A be connected graded skew CY with* $\operatorname{hdet}\mu_A = 1$*. Then there is a* $\sigma \in \operatorname{Aut}_{\mathbb{Z}}(A)$ *such that* $o(\mu_{A^{\tilde\sigma}}) < \infty$.

The paper is organized as follows. Section 1 discusses various sources of examples of skew CY algebras. Section 2 contains some general results on Hopf algebra actions on algebras and bimodules, as well as smash products. Section 3 includes a study of the local cohomology of a smash product involving a finitely graded algebra, as well as a discussion of the homological determinant for Hopf actions on certain generalized AS Gorenstein algebras. Sections 4–6 respectively contain the proofs of the homological identities (HI1)–(HI3). Section 7 is devoted to applications of these homological identities and includes proofs of Corollaries 0.5–0.7. In addition, some questions and conjectures are presented in Sections 4, 6, and 7.

We plan to study more results about the order of the Nakayama automorphism in a second paper [RRZ]. In particular, there is an example of a skew CY algebra A such that $A^{\tilde\sigma}$ is not CY for any graded algebra automorphism σ [RRZ]. Thus

$$\min\{o(\mu_{A^{\tilde\sigma}}) \mid \sigma \in \operatorname{Aut}_{\mathbb{Z}}(A)\}$$

is a non-trivial numerical invariant of A. We hope this invariant might be helpful in the project of classifying Artin-Schelter (AS) regular algebras of global dimension four. For example, if A is AS regular of dimension 4 and generated by 2 elements in degree 1, then, up to a graded twist, we can show that μ_A is of the form ξ_c where $c^7 = 1$ [RRZ].

To close, we note that it would be useful to develop a more general theory of skew CY algebras parallel to that of CY algebras. For example, much work on CY algebras has focused on showing how these arise in many important cases from potentials. While in this paper we do not pursue this point of view, we note that the theory of potentials has been generalized to the skew CY setting; for example, "twisted superpotentials" are discussed in [BSW, Section 2.2]. It would be interesting to work out some basic properties of such generalized potentials and relate them to our results in this paper. We also mention that skew Calabi-Yau categories have been considered by Keller, who suggests that "conservative functors" may serve as a good replacement for the identity functor in the definition of Calabi-Yau categories.

Acknowledgments. The authors thank Raf Bocklandt, Bernhard Keller, Travis Schedler, Paul Smith, Michael Wemyss, and Quanshui Wu for useful conversations. We thank Bernhard Keller for sharing his ideas on defining skew Calabi-Yau categories. We are also grateful to Chelsea Walton and Amnon Yekutieli for helpful comments on a draft of this paper. Reyes was partially supported by a University of California President's Postdoctoral Fellowship during this project. Rogalski was partially supported by the US National Science Foundation grants DMS-0900981 and DMS-1201572. Zhang was partially supported by the US National Science Foundation grant DMS-0855743.

1. Examples of skew CY algebras

In this section, we set the scene by pointing out a number of known examples of skew CY algebras. We also review some definitions and terminology which we will need in the sequel. The skew CY property for noncommutative algebras has been studied under other names for many years, even before the definitions of CY algebras and categories. If A is Frobenius (finite dimensional, but not necessarily homologically smooth), then the automorphism μ_A as defined in (E0.1.1) is the Nakayama automorphism of A in the classical sense. Similar ideas were considered in [VdB1, YeZ1] for graded or filtered algebras when rigid dualizing complexes were studied in the late 1990s. The skew CY property was called "rigid Gorenstein" in [BZ, Definition 4.4], was called "twisted Calabi-Yau" in [BSW] (the word "twisted" was also used in the work [DV]), and mentioned in talks by several other people with possibly other names during the last few years. As mentioned in the introduction, we prefer the term "skew Calabi-Yau", since we will use the word "twisted" already in our study of graded twists of algebras.

The first major examples of skew CY algebras are the Artin-Schelter regular algebras. The following definition was introduced by Artin and Schelter in [ASc].

Definition 1.1. A connected graded algebra A is called *Artin-Schelter Gorenstein* (or *AS Gorenstein*, for short) if the following conditions hold:

(a) A has finite injective dimension $d < \infty$ on both sides,
(b) $\operatorname{Ext}^i_A(k, A) = \operatorname{Ext}^i_{A^{op}}(k, A) = 0$ for all $i \neq d$ where $k = A/A_{\geq 1}$, and
(c) $\operatorname{Ext}^d_A(k, A) \cong k(\mathfrak{l})$ and $\operatorname{Ext}^d_{A^{op}}(k, A) \cong k(\mathfrak{l})$ for some integer $\mathfrak{l}$.

The integer $\mathfrak{l}$ is called the *AS index.* If moreover

(d) A has (graded) finite global dimension d,

then A is called *Artin-Schelter regular* (or AS regular, for short).

In the original definition of Artin and Schelter, A is required to have finite Gelfand-Kirillov dimension, but this condition is sometimes omitted. It is useful to not require it here, since CY algebras of infinite GK-dimension are also of interest to those working on the subject. It is known that if A is AS regular, then the AS index of A is positive [SteZh, Proposition 3.1]. If A is only AS Gorenstein, the AS index of A could be zero or negative.

It is important to point out that the skew CY and AS regular properties coincide for connected graded algebras. The following lemma is well-known (at least in the case when A is noetherian).

Lemma 1.2. *Let A be a connected graded algebra. Then A is skew CY (in the graded sense) if and only if it is AS regular.*

Proof. If A is AS regular, then A is homologically smooth by [YeZ2, Propositions 4.4(c) and 4.5(a)] and satisfies (E1.0.1) by [YeZ2, Proposition 4.5(b)]. Thus A is skew CY.

Conversely, we assume that A is skew CY. By [YeZ2, Lemma 4.3(b)], the (left and right) global dimension of A is equal to the projective dimension of ${}_{A^e}A$, which is finite by the homological smoothness of A. We now work in the bounded derived category of left A^e-modules. Let $U = \operatorname{Ext}^d_{A^e}(A, A^e) = {}^1A^{\mu}(\mathfrak{l})$ as in (E1.0.1). Since A is homologically smooth, by [VdB2, Theorem 1],

$$\operatorname{RHom}_{A^e}(A, N) \cong U[-d] \otimes^L_{A^e} N \tag{E1.2.1}$$

for any left A^e-module N, where $[n]$ means complex shift (or suspension) by degree n. Let σ be any graded automorphism of A and let $P^\bullet$ be a graded projective resolution of ${}^1A^{\sigma}$ as a left A^e-module. Let $k = A/A_{\geq 1}$; since A and k are naturally (A, A)-bimodules, $A \otimes k$ is an (A^e, A^e)-bimodule, with outer left A^e-action and

inner right A^e-action. Then as elements of the derived category, we have
$(A\otimes k)\otimes^L_{A^e} {}^1A^\sigma = (A\otimes k)\otimes_{A^e} P^\bullet \cong A\otimes_A P^\bullet\otimes_A k \cong P^\bullet \otimes_A k\cong {}^1A^\sigma\otimes_A k\cong k.$
Similarly, in the derived category we have

$$^1A^\sigma \otimes^L_{A^e}(A\otimes k)\cong k.$$

Now, using the above equations, hom-tensor adjunction and (E1.2.1), we have

$$\begin{aligned}\mathrm{RHom}_A(k,A) &\cong \mathrm{RHom}_A((A\otimes k)\otimes^L_{A^e} A, A)\cong \mathrm{RHom}_{A^e}(A,\mathrm{RHom}_A((A\otimes k),A))\\ &\cong \mathrm{RHom}_{A^e}(A, A\otimes k)\cong U[-d]\otimes^L_{A^e}(A\otimes k)\\ &\cong ({}^1A^\mu(\mathfrak{l})[-d])\ \otimes^L_{A^e}(A\otimes k)\cong k(\mathfrak{l})[-d]\end{aligned}$$

which means that $\mathrm{Ext}^i_A(k,A)=\begin{cases}0 & \text{if } i\neq d\\ k(\mathfrak{l}) & \text{if } i=d\end{cases}$. By symmetry, a similar statement is true for $\mathrm{Ext}^i_{A^{op}}(k,A)$. Therefore A is AS regular. □

By the previous result, the theory of skew CY algebras encompasses all of the theory of AS regular algebras (even those of infinite GK-dimension). In fact, it is much more general still, as it applies to many other types of algebras, including ungraded ones. For example, one may consider Hopf algebras which are skew CY, and we discuss this case next. We first review a few facts about Hopf algebras; some more review will be found in the next section. We recommend [Mo] as a basic reference for the theory of Hopf algebras.

Throughout this paper H will stand for a Hopf algebra (H,m,u,Δ,ϵ) over k, with bijective antipode S. This assumption on S is automatic if H is finite dimensional over k, or if H is a group algebra kG, and these are two important cases of interest. We use Sweedler notation, so for example we write $\Delta(h)=\sum h_1\otimes h_2$, and further we often even omit the $\sum$ from expressions. For an algebra homomorphism $f: H\to k$, the right winding automorphism of H associated to f is defined to be

$$\Xi^r_f : h\to \sum h_1 f(h_2)$$

for all $h\in H$. The left winding automorphism Ξ^l_f of H associated to f is defined similarly, and it is well-known that both Ξ^l_f and Ξ^r_f are algebra automorphisms of H. Recall that the k-linear dual $H^*=\mathrm{Hom}_k(H,k)$ is an algebra with product given by the convolution $f\star g$, where $f\star g(h)=\sum f(h_1)g(h_2)$. Let $f: H\to k$ be an algebra map. Then the functions $f: H\to k$ and $f\circ S: H\to k$ are easily proved to be inverses in the algebra H^*, and using this, one sees that Ξ^r_f and $\Xi^r_{f\circ S}$ are inverse automorphisms of H. This also implies that $f\circ S$ and $f\circ S^2$ are also inverses in H^*, and thus

$$f\circ S^2 = f. \tag{E1.2.2}$$

As another consequence, it is easy to check that any winding automorphism commutes with S^2.

A standard tool in the theory of the finite-dimensional Hopf algebras H is the notion of integrals [Mo, Chapter 2]. The theory of integrals has been extended to infinite-dimensional AS-Gorenstein Hopf algebras H in [LuWZ2]. The left homological integral of such an H is defined to be the 1-dimensional H-bimodule $\int^l = \mathrm{Ext}^d_H({}_Hk, {}_HH)$, where $d=\mathrm{injdim}(H)$. Picking $0\neq \mathfrak{u}\in\int^l$, then $\mathfrak{u}$ is an invariant for the left H-action, that is, $h\mathfrak{u}=\epsilon(h)\mathfrak{u}$ for all $h\in H$, but the right action gives an algebra map $\eta : H \to k$ defined by $\mathfrak{u} \cdot h = \eta(h)\mathfrak{u}$ which may be nontrivial. By an abuse of notation we write $\int^l = \eta$, so that one has the corresponding winding automorphisms $\Xi^r_{\int^l}$ and $\Xi^l_{\int^l}$. One may define the right integral $\int^r = \mathrm{Ext}^d_H(k_H, H_H)$ analogously; the left action on it gives an algebra map $\int^r : H \to k$ which is known to satisfy $\int^r = \int^l \circ S$. Thus $\Xi^r_{\int^r}$ and $\Xi^r_{\int^l}$ are inverses in the group of algebra automorphisms of H, and similarly for the left winding automorphisms.

Lemma 1.3. *Let H be a noetherian Hopf algebra. Then H is AS regular in the sense of* [BZ, Definition 1.2] *if and only if H is skew CY. A Nakayama automorphism of such a Hopf algebra H is given by $S^{-2} \circ \Xi^r_{\int^l}$. (Alternatively, $S^2 \circ \Xi^l_{\int^l}$ is also a Nakayama automorphism of H.)*

Proof. By [BZ, Lemma 5.2(c)], if H is AS regular, then H is homologically smooth. Then H is skew CY, and each of the given formulas is a Nakayama automorphism, by [BZ, Theorems 0.2 and 0.3] and the discussion in [BZ, Section 4]. Conversely, by [LuWZ1, Theorem 2.3], a noetherian skew CY Hopf algebra is AS regular. $\square$

The same argument as in the second half of the proof of Lemma 1.2 shows that a non-noetherian skew CY Hopf algebra is AS regular. It is plausible that the converse is true, but this is unknown. Brown has conjectured that every noetherian Hopf algebra is AS Gorenstein [Br, Question E]. If Brown's conjecture holds, then Lemma 1.3 (together with [LuWZ1, Theorem 2.3]) implies that the class of noetherian Hopf algebras with finite global dimension is equal to the class of noetherian skew CY Hopf algebras. In any case, Hopf algebras clearly give another large class of examples of skew CY algebras.

An interesting special case of both of the classes of examples introduced so far is the class of enveloping algebras of graded Lie algebras. Note that there are many examples of graded Lie algebras, for example the Heisenberg algebras.

Example 1.4. Let $\mathfrak{g}$ be a finite dimensional positively graded Lie algebra. Then the universal enveloping algebra $U(\mathfrak{g})$ is noetherian, connected graded, AS regular, and CY, of global dimension $d = \dim_k \mathfrak{g}$.

Proof. This is quite well-known. For instance, the CY property of $U(\mathfrak{g})$ follows from [Ye, Theorem A]. We give a short proof based on Lemma 1.3.

Write $U = U(\mathfrak{g})$. Since $\mathfrak{g}$ is positively graded, the Hopf algebra U is connected graded. Since $\dim_k \mathfrak{g} = d < \infty$, the algebra U is noetherian. It is well-known that U is in addition an AS regular, Auslander regular and Cohen-Macaulay domain of global dimension d. See, for example, [FV, Section 2]. Now by Lemma 1.3, the algebra U is skew CY. Since U is connected graded and the maximal graded ideal of U is $\mathfrak{m} := \ker \epsilon$, we have $\mathrm{Ext}^d_U(k, U) \cong k = U/\mathfrak{m}$ as U-bimodules. This implies that the left integral of U is trivial and hence $\Xi^r_{\int^l}$ is the identity map. Since U is cocommutative, S^2 is the identity map. By Lemma 1.3, μ_U is the identity and hence U is CY. $\square$

The Nakayama automorphism of a skew CY algebra is not in general easy to compute. For a Frobenius algebra, it is the same as the classical Nakayama automorphism and is related to the corresponding bilinear form (see the end of Section 3). We have seen that there is a formula for the Nakayama automorphism in the case of Hopf algebras (Lemma 1.3) which reduces the problem to calculating the homological integral. If the algebra A is a connected graded Koszul algebra, then the Nakayama automorphism of A can be determined if one knows the Nakayama automorphism of the Koszul dual $A^!$, which is Frobenius, since the actions of the Nakayama automorphisms on the degree 1 pieces of A and $A^!$ are dual up to sign [VdB1, Theorem 9.2]. This also works in the more general case of N-Koszul algebras [BM, Theorem 6.3].

The aim of the rest of the paper is to give formulas which help to compute the Nakayama automorphism by showing how the Nakayama automorphism changes under some common constructions. We give one simple result of this type right now. Recall that an element $z \in A$ is called normal if (i) there is a $\tau \in \operatorname{Aut}(A)$ (the group of algebra automorphisms of A) such that $za = \tau(a)z$ for all $a \in A$, and (ii) $\tau(z) = z$. It is clear that, if z is a nonzerodivisor, then (ii) follows from (i). Let σ be in $\operatorname{Aut}(A)$. A normal element z is called σ-normal if it is a σ-eigenvector, namely, $\sigma(z) = cz$ for some $c \in k^\times := k \setminus \{0\}$.

There is a strong connection between the Nakayama automorphism and the theory of dualizing complexes, which we need in the next proof and a few others, especially in Lemma 3.5 below. These instances are few enough that we do not review the theory of dualizing complexes here; the reader can find the basic definitions and results in the papers we reference.

Lemma 1.5. *Let A be noetherian connected graded AS Gorenstein algebra and let z be a homogeneous μ_A-normal nonzerodivisor of positive degree. Let τ be the element in $\operatorname{Aut}_{\mathbb{Z}}(A)$ such that $za = \tau(a)z$ for all $a \in A$. Then $\mu_{A/(z)}$ is equal to $\mu_A \circ \tau$ restricted to $A/(z)$.*

Proof. Let $d = \operatorname{injdim} A$. Considering A as an ungraded algebra, ${}^1A^\nu[d]$ is the rigid dualizing complex over A, where ν^{-1} is the Nakayama automorphism of A, by [YeZ1, Proposition 6.18(2)] (see also Lemma 3.5 below and [YeZ2, Proposition 4.5(b,c)]). The rigid dualizing complex over $A/(z)$ is equal to $\operatorname{RHom}_A(A/(z), {}^1A^\nu[d])$ by [YeZ1, Theorem 3.2 and Proposition 3.9]. A computation shows that

$$\operatorname{Ext}^1_A(A/(z), {}^1A^\nu) = \operatorname{Ext}^1_A(A/(z), A)^\nu = \{{}^1(A/(z))^{\tau^{-1}}\}^\nu = (A/(z))^{\tau^{-1}\circ\nu},$$

as left A^e-modules, and that $\operatorname{Ext}^i(A/(z), A) = 0$ for $i \neq 1$. Thus

$$\operatorname{RHom}_A(A/(z), {}^1A^\nu[d]) = \operatorname{Ext}^1_A(A/(z), {}^1A^\nu)[d-1] = (A/(z))^{\tau^{-1}\circ\nu}[d-1].$$

Thus the Nakayama automorphism of $A/(z)$ is $\mu_A \circ \tau$ restricted to $A/(z)$. □

Recall from the introduction that when A is a $\mathbb{Z}$-graded algebra, for any $0 \neq c \in k$ we have the graded automorphism $\xi_c : A \to A$ with $\xi_c(a) = c^{||a||}a$ for all homogeneous elements a, where $||\ \ ||$ denotes the degree of homogeneous elements. A $\mathbb{Z}$-graded skew CY algebra with $\mu_A = \xi_c$ is called c-Nakayama. In a sense, graded c-Nakayama algebras are closer to CY algebras than more general skew CY algebras. We now give an explicit example which was one of the motivating examples for our work in this paper calculating the Nakayama automorphism of a smash product. It is a simple example of a ring A which is skew CY only, but which becomes CY after passing to a skew group algebra.

Example 1.6. Let A be the k-algebra generated by x and y of degree 1 and subject to the relations

$$x^2y = yx^2, \quad y^2x = xy^2.$$

This is the down-up algebra $A(0,1,0)$ [KK, p. 465], which is AS regular algebra of type S_1, with AS index 4 [ASc]. Let $z = xy - yx$, then one checks that $xz = -zx$ and $yz = -zy$. Hence z is normal and $zf = \xi_{-1}(f)z$ for all $f \in A$.

Let $B = A/(z)$. Then $\mu_B = Id_B$, as $B \cong k[x,y]$ is commutative. Since A is $\mathbb{Z}^2$-graded with $\deg(x) = (1,0)$ and $\deg(y) = (0,1)$, μ_A must be $\mathbb{Z}^2$-graded. Hence μ_A sends x to ax and y to by for some $a, b \in k^\times$. Consequently, $\mu_A(z) = \lambda z$ for some $\lambda \in k^\times$. Applying Lemma 1.5 (viewing A as $\mathbb{N}$-graded), $\mu_B = (\xi_{-1} \circ \mu_A)\mid_B$. Thus $\mu_A = \xi_{-1}$ when restricted to $A/(z)$. Since both A and $A/(z)$ are generated by x and y, it follows that $\mu_A = \xi_{-1}$, or in other words A is (-1)-Nakayama (hence not CY).

Consider the group $G = \{1, \xi_{-1}\}$ acting on A naturally. We claim that for $H = kG$, the smash product $A\#H$ (or equivalently, the skew group algebra $A \rtimes G$) is CY. For those familiar with theory of superpotentials, one way to see this is to show that $A \rtimes G$ is isomorphic to a factor algebra of the path algebra kQ, where Q is the McKay quiver corresponding to the action of G on A_1; see [BSW, Section 3]. In the case at hand, Q is the quiver given in [Boc, Section 5.3] with vertices $\{1,2\}$, two arrows a_1, a_2 from 1 to 2, and two arrows a_3, a_4 from 2 to 1. The algebra $A \rtimes G$ is the factor algebra of kQ given by the cubic relations given by taking cyclic partial derivatives of the superpotential $W = a_1a_3a_2a_4 + a_3a_1a_4a_2$. This algebra is shown to be CY of dimension 3 in [Boc, Theorem 5.4].

We have not given full details of the argument above, because one of the goals of our paper was to seek a deeper reason why a skew group algebra of a non CY algebra might become CY. This is achieved by Corollary 0.6(b), which shows that $A \rtimes G$ must be CY since G is the cyclic subgroup generated by the Nakayama automorphism.

The current example is also interesting in the context of our main theorem on the Nakayama automorphisms of graded twists. Let $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(A)$ be defined by $\sigma(x) = x$ and $\sigma(y) = iy$ where $i^2 = -1$. By [KK, Theorem 1.5], $\mathrm{hdet}\,\sigma = i^2 = -1$. Since $\sigma^4 = 1$, by Theorem 0.3 the graded twist $A^{\tilde{\sigma}}$ is CY. One may check that the algebra $A^{\tilde{\sigma}}$ is generated by x and y subject to the relations $x^2y = -yx^2$ and $y^2x = -xy^2$. One may also use Lemma 1.5 together with a direct computation to show that $A^{\tilde{\sigma}}$ is CY.

If A is a $\mathbb{Z}^w$-graded algebra, we will let $\mathrm{Aut}_{\mathbb{Z}^w}(A)$ denote the group of all graded algebra automorphisms of A. Starting from Section 2, we will consider non-connected graded skew CY algebras. In this case the Nakayama automorphism may not be unique, as we see in the following standard lemma whose proof we leave to the reader.

Lemma 1.7. *Let A be a $\mathbb{Z}^w$-graded algebra and M be a graded A-bimodule. Suppose that every homogeneous left-invertible (respectively, right-invertible) element of A is invertible. Let ν, σ denote elements in $\mathrm{Aut}_{\mathbb{Z}^w}(A)$.*

- (a) *M is isomorphic to ${}^1A^\nu$, for some ν, if and only if $M_A \cong A_A$ and ${}_AM \cong {}_AA$.*
- (b) *Suppose $M \cong {}^1A^\nu$. Then an element $\mathfrak{e} \in M$ is a generator for ${}_AM$ if and only if it is a generator for M_A.*
- (c) *Let $\mathfrak{e}$ be a generator of ${}^1A^\nu$. Replacing $\mathfrak{e}$ to $\mathfrak{e}f$ for some invertible element f changes ν to $\nu \circ \eta_f$, where η_f is the inner automorphism $a \mapsto faf^{-1}$ for all $a \in A$.*
- (d) *${}^1A^\nu$ is isomorphic to ${}^1A^\sigma$ if and only if $\nu\sigma^{-1}$ is an inner automorphism.*

Of course in most common situations, for example if A is $\mathbb{N}$-graded and A_0 is finite dimensional, or if A is a domain, then every homogeneous left-invertible (respectively, right-invertible) element is invertible. In the situation of Lemma 1.7, if ${}_AM = A\mathfrak{e} \cong_A A$ and $M_A = \mathfrak{e}A \cong A_A$, then we call $\mathfrak{e}$ a *generator* of the A-bimodule M.

2. Hopf actions on bimodules and smash products

In this section we collect some preliminary material about (bi)-modules over smash products and dualization. Some of this material is well-known, but we have tried to make our presentation relatively self-contained. A reader who is less familiar with Hopf algebras may wish to think primarily about the case of group algebras kG on a first reading, since the results of Sections 2-4 are still non-trivial in that case.

We maintain the assumptions about Hopf algebras already mentioned in the previous sections. Let A be a left H-module algebra; by definition [Mo, Definition 4.1.1], this means that A is a left H-module such that

$$h(ab) = \sum h_1(a)\ h_2(b) \quad \text{and} \quad h(1_A) = \epsilon(h)1_A$$

for all $h \in H$, and all $a, b \in A$. Following [Mo, Definition 4.1.3], the left smash product algebra $A\#H$ is defined as follows. As a k-space, $A\#H = A \otimes H$, and the multiplication of $A\#H$ is given by

$$(a\#g)(b\#h) = \sum a\ g_1(b)\#g_2h$$

for all $g, h \in H$ and $a, b \in A$. We identify H with a subalgebra of $A\#H$ via the map $i_H : h \to 1\#h$ for all $h \in H$, and identify A with a subalgebra of $A\#H$ via the map $i_A : a \to a\#1$ for all $a \in A$. We note the following useful formula which determines how an element of A and an element of H move past each other:

$$a\#h = (a\#1)(1\#h) = (1\#h_2)(S^{-1}(h_1)(a)\#1). \tag{E2.0.1}$$

Let $B = A\#H$. Identifying H and A with subalgebras of B, any left B-module is both a left A and left H-module. It is easy to check that a k-vector space M which is both a left A and left H-module has an induced left B-module structure restricting to the given ones on both A and H if and only if the condition

$$h(am) = h_1(a)h_2(m)$$

is satisfied for all $h \in H, a \in A, m \in M$. As a special case, if A is a left H-module such that $h(1_A) = \epsilon(h)1_A$, then A is a left H-module algebra if and only if the left A-module and the left H-module structures on A induce a left $A\#H$-module structure on A.

One can also construct a right-handed version of smash product. We say that A is a right H-module algebra if it is a right H-module satisfying

$$(ab)^h = a^{h_2}b^{h_1} \quad \text{and} \quad (1_A)^h = \epsilon(h)1_A$$

for all $h \in H$ and $a, b \in A$. While one could instead define a right H-module algebra by the more natural-seeming rule $(ab)^h = a^{h_1}b^{h_2}$, the convention we have chosen will allow us to more easily relate left and right smash products below. Note also that our convention is to write all right H-actions in the exponent to avoid notational confusion. If A is a right H-module algebra, then the right smash product $H\#A$ is defined to be the tensor product $H\otimes A$ with multiplication $(h\#a)(g\#b) = hg_2\#a^{g_1}b$.

(We use the same symbol # for the left and the right smash products.) As on the left, a k-vector space is a right $H\#A$-module if and only if it is a right H and right A-module satisfying $(ma)^h = m^{h_2}a^{h_1}$ for all $m \in M, a \in A$, and $h \in H$.

Recall that in a Hopf algebra, the antipode always satisfies the formulas

(E2.0.2) $$(S \otimes S) \circ \Delta = \tau \circ \Delta \circ S \quad \text{and} \quad \epsilon \circ S = \epsilon,$$

where $\tau : H \otimes H \to H \otimes H$ is the coordinate switch map $(g \otimes h) \mapsto (h \otimes g)$ [Mo, Proposition 1.5.10]. Suppose that A is a left H-module algebra. We may make it into a right H-module algebra by defining $a^h = S^{-1}(h)(a)$ for all $a \in A, h \in H$; this is easy to check using (E2.0.2). We always use this fixed convention for making a left H-module algebra into a right one, since with it we have the following nice property.

Lemma 2.1. *Let A be a left H-module algebra, and make it into a right H-module algebra via $a^h = S^{-1}(h)(a)$. Then there is an algebra isomorphism $\Psi : A\#H \cong H\#A$ given by the formula $a\#h \mapsto h_2\#a^{h_1}$, with inverse Ψ^{-1} having the formula $g\#b \mapsto g_1(b)\#g_2$.*

Proof. We have

$$\begin{aligned}\Psi((a\#g)(b\#h)) &= \Psi(ag_1(b)\#g_2h) = g_3h_2\#(ag_1(b))^{g_2h_1} \\ &= g_4h_3\#(a^{g_3h_2}b^{S(g_1)g_2h_1}) = g_2h_3\#a^{g_1h_2}b^{h_1},\end{aligned}$$

while

$$\Psi(a\#g)\Psi(b\#h) = (g_2\#a^{g_1})(h_2\#b^{h_1}) = g_2h_3\#a^{g_1h_2}b^{h_1}.$$

Thus Ψ is a homomorphism of algebras. A dual proof shows that the map $\Psi^{-1} : g\#b \mapsto g_1(b)\#g_2$ is a homomorphism $H\#A \to A\#H$.

Note that both the formulas for Ψ and Ψ^{-1} act as the identity on the subalgebras identified with H and A. Thus it is obvious that $\Psi\Psi^{-1} = \Psi^{-1}\Psi = 1$, since both compositions are clearly trivial when restricted to these subalgebras. □

From now on we identify the left and right smash products $A\#H$ and $H\#A$ via the isomorphism of the previous lemma. In particular, note that if M is a k-vector space with left A and H-actions, then the condition $h(am) = h_1(a)h_2(m)$ is equivalent to M being either a left $A\#H$ or a left $H\#A$-module which restricts to the given actions of A and H. Similarly, the condition $(ma)^h = m^{h_2}a^{h_1}$ is equivalent to being either a right $A\#H$ or a right $H\#A$-module.

One of the main purposes of this section is to discuss actions of Hopf algebras on bimodules, where clearly one would like to require the Hopf action to interact with the bimodule structure in some nice way. The twist by S^i we allow in the following definition gives a slightly more general notion than what we have seen discussed in the literature. We will see below that the extra generality will be useful to describe the dual of a bimodule with a Hopf action (Proposition 2.7).

Definition 2.2. Let A be a left H-module algebra. If M is an A-bimodule with a left H-action satisfying

(E2.2.1) $$h(amb) = h_1(a)h_2(m)S^i(h_3)(b)$$

for all $h \in H$, $a, b \in A$, $m \in M$ and some fixed even integer i, then M is called an *H_{S^i}-equivariant A-bimodule.* When $i = 0$, then M is simply called an *H-equivariant A-bimodule.*

Remark 2.3. Being an H-equivariant A-bimodule is equivalent to being a left module over a certain algebra $A^e \rtimes H$ introduced by Kaygun [Kay, Definition 3.1, Lemma 3.3]. As a vector space, $A^e \rtimes H = A \otimes A \otimes H$, with product given by the formula

$$(a_1 \otimes a_1' \otimes h)(a_2 \otimes a_2' \otimes g) = a_1 h_1(a_2) \otimes h_3(a_2')a_1' \otimes h_2 g,$$

for all $a_1 \otimes a_1' \otimes h,\ a_2 \otimes a_2' \otimes g \in A^e \rtimes H$. More specifically, if M is an H-equivariant bimodule, one may check that it is an $A^e \rtimes H$-module via $(a \otimes a' \otimes h) \cdot m = ah(m)a'$.

Lemma 2.4. *Let M be an A-bimodule with a left and right H-action related by*

$$m^h = S^{-i-1}(h)(m) \tag{E2.4.1}$$

for all $m \in M, h \in H$, and some even integer i. Then M is an H_{S^i}-equivariant A-bimodule if and only if M is a left $A\#H$-module and a right $A\#H$-module under the given actions of A and H.

Proof. We have already seen that M being a left $A\#H$-module is equivalent to the condition $h(am) = h_1(a)h_2(m)$ for all $h \in H, a \in A, m \in M$, and that being a right $A\#H \cong H\#A$-module is equivalent to $(mb)^h = m^{h_2}b^{h_1}$ for all $h \in H$, $b \in A$, $m \in M$. If M is an H_{S^i}-equivariant A-bimodule, then in particular we have $h(am) = h_1(a)h_2(m)$ and

$$\begin{aligned}(mb)^h &= S^{-i-1}(h)(mb) = S^{-i-1}(h)_1(m)S^i(S^{-i-1}(h)_2)(b)\\ &= S^{-i-1}(h_2)(m)S^{-1}(h_1)(b) && \text{using (E2.0.2)}\\ &= m^{h_2}b^{h_1},\end{aligned}$$

so that M is a left and right $A\#H$-module. The converse is similar. □

Given any H-equivariant A-bimodule M, one can define an $A\#H$-bimodule $M\#H$ with left and right $A\#H$ actions given by formulas analogous to the multiplication in $A\#H$; the proof that it is an $A\#H$-bimodule is virtually the same as the proof that the multiplication of $A\#H$ is associative. See, for example, [LiWZ, Lemma 2.5(1)]. We will need a generalization of this construction where we allow the bimodule to be H_{S^i}-equivariant and we smash with a twist of H by an automorphism.

Lemma 2.5. *Let $B = A\#H$ for some left H-module algebra A. Let M be an H_{S^i}-equivariant A-bimodule, and make M into a right H-module as in* (E2.4.1). *Suppose that $\sigma : H \to H$ is an algebra automorphism of H.*

(a) *Choose a left and right generator $\mathfrak{u}$ of the H-bimodule ${}^{\sigma}H^1$, such that $h\mathfrak{u} = \mathfrak{u}\sigma(h)$. Let $M\#^{\sigma}H^1$ be the tensor product $M \otimes {}^{\sigma}H^1$, with left B-action*

$$(a\#g)(m\#\mathfrak{u}h) = ag_1(m)\#g_2\mathfrak{u}h$$

and right B-action

$$(m\#\mathfrak{u}h)(b\#k) = mh_1(b)\#\mathfrak{u}h_2k.$$

If σ satisfies the property

$$\Delta \circ \sigma = (S^i \otimes \sigma) \circ \Delta, \tag{E2.5.1}$$

then $M\#^{\sigma}H^1$ is a B-bimodule.

(b) *Choose a left and right generator* $\mathfrak{u}$ *of the* H*-bimodule* ${}^1H^\sigma$*, such that* $\sigma(h)\mathfrak{u} = \mathfrak{u}h$*. Let* ${}^1H^\sigma \# M$ *be the tensor product* ${}^1H^\sigma \otimes M$*, with left* $B \cong H\#A$*-action*

$$(g\#a)(h\mathfrak{u}\#m) = gh_2\mathfrak{u}\#a^{h_1}m$$

and right B*-action*

$$(h\mathfrak{u}\#m)(k\#b) = h\mathfrak{u}k_2\#m^{k_1}b.$$

If σ *satisfies the property*

$$\Delta \circ \sigma = (S^{-i} \otimes \sigma) \circ \Delta, \tag{E2.5.2}$$

then ${}^1H^\sigma\#M$ *is a* B*-bimodule.*

(c) (E2.5.1) *holds for any automorphism of the form* $\sigma = S^i \circ \Xi^r_\gamma$*, where* $\gamma : H \to k$ *is an algebra map. Similarly,* (E2.5.2) *holds for any automorphism of the form* $S^{-i} \circ \Xi^r_\gamma$*.*

Proof. (a) We leave the proof of this part to the reader, since it is similar to but a bit simpler than the proof of part (b).

(b) It is straightforward to see that ${}^1H^\sigma\#M$ is a left and right B-module, and so we check carefully only that ${}^1H^\sigma\#M$ is a B-bimodule. Note first that

$$(am)^k = S^{-i-1}(k)(am) = S^{-i-1}(k_2)(a)S^{-i-1}(k_1)(m) = a^{S^{-i}(k_2)}m^{k_1}.$$

Now we have

$$\begin{aligned}[(g\#a)(h\mathfrak{u}\#m)](k\#b) &= (gh_2\mathfrak{u}\#a^{h_1}m)(k\#b) = (gh_2\mathfrak{u}k_2\#(a^{h_1}m)^{k_1}b)\\ &= gh_2\sigma(k_3)\mathfrak{u}\#a^{h_1S^{-i}(k_2)}m^{k_1}b,\end{aligned}$$

while

$$\begin{aligned}(g\#a)[(h\mathfrak{u}\#m)(k\#b)] &= (g\#a)(h\mathfrak{u}k_2\#m^{k_1}b) = g(h\sigma(k_2))_2\mathfrak{u}\#a^{(h\sigma(k_2))_1}m^{k_1}b\\ &= gh_2\sigma(k_2)_2\mathfrak{u}\#a^{h_1\sigma(k_2)_1}m^{k_1}b.\end{aligned}$$

Using the hypothesis that $\Delta \circ \sigma = (S^{-i} \otimes \sigma) \circ \Delta$, applying this formula to k_2 we have $\Delta(\sigma(k_2)) = S^{-i}(k_2) \otimes \sigma(k_3)$, which implies that the two expressions above are the same.

(c) If $\sigma = S^i \circ \Xi^r_\gamma$, then

$$\begin{aligned}\Delta(\sigma(h)) &= \Delta(S^i(h_1)S^i(\gamma(h_2))) = \Delta(S^i(h_1)(\gamma(h_2)))\\ &= S^i(h_1) \otimes S^i(h_2)\gamma(h_3) = S^i(h_1) \otimes \sigma(h_2)\\ &= (S^i \otimes \sigma) \circ \Delta(h)\end{aligned}$$

for all $h \in H$. The other calculation is analogous. $\square$

The next goal is to discuss the behavior of the constructions above under k-linear duals. First, we recall the structure of the dual $H^* = \operatorname{Hom}_k(H, k)$ of a finite-dimensional Hopf algebra H.

Lemma 2.6. *Let* H *be a finite-dimensional Hopf algebra. Then as* H*-bimodules,* $H^* \cong {}^1H^\sigma$ *where* $\sigma = \mu_H^{-1} = S^2 \circ \Xi^r_{\int^r}$*.*

Proof. Let $0 \neq \mathfrak{u} \in H^*$ be a left integral of the Hopf algebra H^* [Mo, Definition 2.1.1]. Then by definition, in the algebra H^* we have $g\mathfrak{u} = \epsilon^*(g)\mathfrak{u}$ for all $g \in H^*$, where here ϵ^* is the counit of H^*, given by $\epsilon^*(f) = f(1_H)$. It is easy to check that the fact that $\mathfrak{u}$ is a left integral is equivalent to

$$\sum h_1 \mathfrak{u}(h_2) = \mathfrak{u}(h) 1_H \tag{E2.6.1}$$

for all $h \in H$ [DNR, Remark 5.1.2]. By [Mo, Theorem 2.1.3(3) and its proof], $\mathfrak{u}$ is a generator of the left and right H-module H^*, with $H\mathfrak{u} \cong H$ as left H-modules and $\mathfrak{u}H \cong H$ as right H-modules. Therefore, by Lemma 1.7 there is an algebra automorphism σ of H and an isomorphism $\phi : H^* \to {}^1H^\sigma$ of H-bimodules with $\phi(h\mathfrak{u}) = h$; in other words, we have the formula $\sigma(h)\mathfrak{u} = \mathfrak{u}h$ for $h \in H$. The left-handed version of the computation in the proof of [FMS, Lemma 1.5] now shows that $\sigma = S^2 \circ \Xi^r_{\int^r} = S^2 \circ \Xi^r_{\int^l \circ S}$ (the formula (E2.6.1) is needed in the computation).

Since H is finite-dimensional, it is also Frobenius, and so σ^{-1} is its Nakayama automorphism μ_H (see the discussion at the end of Section 3). In fact, then the formula for σ also follows from Lemma 1.3, since

$$(S^2 \circ \Xi^r_{\int^l \circ S})^{-1} = (\Xi^r_{\int^l \circ S})^{-1} \circ S^{-2} = \Xi^r_{\int^l} \circ S^{-2} = S^{-2} \circ \Xi^r_{\int^l}$$

(see the discussion of winding automorphisms and integrals in Section 1). □

In our applications of this section below, usually A will be an $\mathbb{N}$-graded algebra $A = \bigoplus_{i \geq 0} A_i$ which is locally finite ($\dim_k A_i < \infty$ for all i) and which is a left H-module algebra, where the action of H respects the grading in the sense that each A_i is a left H-submodule. We then say that A is a *graded left H-module algebra*. More generally, a *graded H_{S^i}-equivariant A-bimodule* will be a $\mathbb{Z}$-graded A-bimodule M satisfying Definition 2.2, such that the action of H respects the grading. In the next result, we see why we defined both a left and right-sided version of the bimodule smash construction in Lemma 2.5: taking a k-linear dual naturally changes from one of the versions to the other. When we are working with $\mathbb{Z}^w$-graded modules M, unless otherwise noted, by M^* we will always mean the *graded dual*, that is $M^* = \bigoplus_{\lambda \in \mathbb{Z}^w} \operatorname{Hom}_k(M_\lambda, k)$, which is naturally again a $\mathbb{Z}^w$-graded vector space (where elements of $\operatorname{Hom}_k(M_\lambda, k)$ have degree $-\lambda$). Note that when M is locally finite, the graded dual M^* remains locally finite and the usual isomorphism $M^{**} \cong M$ holds.

Proposition 2.7. *Let A be a locally finite graded left H-module algebra for a Hopf algebra H, and let $B = A\#H$. Let M be a locally finite H_{S^i}-equivariant A-bimodule, which we make into a right H-module as well as in* (E2.4.1)*. Let M^* be the graded k-linear dual of M.*

(a) *M^* is a graded $H_{S^{-i-2}}$-equivariant A-bimodule.*
(b) *Let H be finite-dimensional, and assume that $i = 0$. Then*

$$(M\#H)^* \cong (H^*)\#M^* \cong {}^1H^\sigma \# M^*$$

as B-bimodules, where $\sigma = S^2 \circ \Xi^r_{\int^r}$. Here, $M\#H$ is the B-bimodule constructed in Lemma 2.5(a) and ${}^1H^\sigma\#M^*$ is the B-bimodule constructed in Lemma 2.5(b).

Proof. (a) Write $l_h|_M$ for the map $M \to M$ given by left multiplication by h. Similarly, $r_h|_M$ means the right multiplication by h. By assumption, the left and right H-actions on M satisfy

$$r_h|_M = l_{S^{-i-1}(h)}|_M$$

for all $h \in H$. Note that $r_h|_{M^*} = (l_h|_M)^*$ and similarly $(r_h|_M)^* = l_h|_{M^*}$. Then

$$r_h|_{M^*} = (l_h|_M)^* = (r_{S^{i+1}(h)}|_M)^* = l_{S^{i+1}(h)}|_{M^*}$$

for all $h \in H$. Clearly M^* is still an A-bimodule and a B-module on both sides, and so it is an $H_{S^{-i-2}}$-equivariant A-bimodule by Lemma 2.4.

(b) M^* is an $H_{S^{-2}}$-equivariant A-bimodule by part (1), and since $\sigma = S^2 \circ \Xi^r_{\int^r}$ satisfies (E2.5.2) by Lemma 2.5(c), ${}^1H^\sigma \# M^*$ is a well-defined B-bimodule. Since $M \# H$ is a B-bimodule, $(M\#H)^*$ is also a B-bimodule.

Since H is finite-dimensional, we can identify $(M \otimes H)^*$ with $H^* \otimes M^*$ as a k-vector space. Since we are taking graded duals and M is locally finite, the bilinear form $\langle \cdot, \cdot \rangle$ given by the natural evaluation map $(H^* \# M^*) \otimes (M \# H) \to k$ is a perfect pairing. We can identify H^* with ${}^1H^\sigma$ as H-bimodules, by Lemma 2.6, where $\mathfrak{u} \in H^*$, the left integral of H^*, is a right and left generator with $\mathfrak{u}h = \sigma(h)\mathfrak{u}$ for all $h \in H$. Thus to complete the proof, we need only verify that under these identifications, the left and right B-module structures on $(M\#H)^*$ and ${}^1H^\sigma \# M^*$ agree. For this it is enough to prove that the left and right A-module and H-module structures all agree. Note that it is most natural to think of B as the right smash product $H \# A$ in the proof below.

Considering the left H-structure, choose $h\mathfrak{u}\#\phi \in {}^1H^\sigma \# M^*$, $(v\#w) \in M\#H$, and $k \in H$. We have

$$\langle (k(h\mathfrak{u}\#\phi), v\#w\rangle = \langle (h\mathfrak{u}\#\phi), (v\#w)k\rangle = \langle (h\mathfrak{u}\#\phi), (v\#wk)\rangle = h\mathfrak{u}(wk)\phi(v),$$

while

$$\langle ((k\#1)(h\mathfrak{u}\#\phi), v\#w\rangle = \langle ((kh\mathfrak{u}\#\phi), v\#w\rangle = kh\mathfrak{u}(w)\phi(v).$$

These are the same because $kh\mathfrak{u}(w) = h\mathfrak{u}(wk)$ by the definition of the left H-action on H^*.

The agreement of the right A-structures is similarly straightforward and we leave the proof to the reader.

For the right H-structure, we have

$$\begin{aligned}\langle ((h\mathfrak{u}\#\phi)k, v\#w\rangle = \langle ((h\mathfrak{u}\#\phi), k(v\#w)\rangle &= \langle ((h\mathfrak{u}\#\phi), (k_1(v)\#k_2w)\rangle \\ &= h\mathfrak{u}(k_2w)\phi(k_1(v)),\end{aligned}$$

while

$$\begin{aligned}\langle ((h\mathfrak{u}\#\phi)(k\#1), v\#w\rangle &= \langle ((h\mathfrak{u}k_2\#\phi^{k_1}), (v\#w)\rangle = h\mathfrak{u}k_2(w)\phi^{k_1}(v) \\ &= h\mathfrak{u}(k_2w)\phi(k_1(v)).\end{aligned}$$

Since $(h\mathfrak{u}\otimes\phi) = (h\#1)(\mathfrak{u}\otimes\phi)$, and we have proved that the left H-actions agree already, it is enough to show that the left A-actions agree when acting on an element of the special form $(\mathfrak{u}\#\phi)$, because of the formula $(1\#a)(h\#1) = (h_2\#1)(1\#a^{h_1})$ which holds in the right smash product. In this case we have

$$\begin{aligned}\langle a(\mathfrak{u}\otimes\phi), v\#w\rangle &= \langle \mathfrak{u}\otimes\phi, (v\#w)a\rangle = \langle \mathfrak{u}\otimes\phi, vw_1(a)\#w_2\rangle \\ &= \phi(vw_1(a))\mathfrak{u}(w_2) = \phi(v\mathfrak{u}(w_2)w_1(a)) \\ &= \phi(v\mathfrak{u}(w)a) \qquad \text{by} \quad \text{(E2.6.1)} \\ &= \phi(va)\mathfrak{u}(w),\end{aligned}$$

while

$$\langle(1\#a)(\mathfrak{u}\otimes\phi), v\#w\rangle = \langle(\mathfrak{u}\otimes a\phi), v\#w\rangle = (a\phi)(v)\mathfrak{u}(w) = \phi(va)\mathfrak{u}(w).$$

This completes the proof. □

The next lemma concerns automorphisms of $A\#H$ that are determined by automorphisms of A and H.

Lemma 2.8. *Let A be a left H-module algebra. Suppose that $\mu \in \mathrm{Aut}(A)$ and $\phi \in \mathrm{Aut}(H)$. Define a k-linear map $\mu\#\phi : A\#H \to A\#H$ by*

$$(\mu\#\phi)(a\#h) = \mu(a)\#\phi(h)$$

for all $a \in A$ and all $h \in H$, and define $\phi\#\mu : H\#A \to H\#A$ similarly. Then the following hold.

(a) *$\mu\#\phi$ is an algebra automorphism of $A\#H$ if and only if $\phi\#\mu$ is an algebra automorphism of $H\#A$.*

(b) *$\mu\#\phi$ is an algebra automorphism of $A\#H$ if and only if*

$$\phi(h)_1(\mu(b))\#\phi(h)_2 = \mu(h_1(b))\#\phi(h_2) \tag{E2.8.1}$$

for all $b \in A$ and all $h \in H$.

(c) *If $\phi = S^{2n} \circ \Xi^l_\eta \circ \Xi^r_\gamma$ for some algebra homomorphisms $\eta, \gamma : H \to k$, then* (E2.8.1) *holds if and only if*

$$(\Xi^l_\eta \circ S^{2n})(h)(\mu(b)) = \mu(\Xi^r_\eta(h)(b)) \tag{E2.8.2}$$

for all $h \in H$ and $b \in A$.

Proof. (a) Recall from Lemma 2.1 that the algebra isomorphisms $\Psi : A\#H \to H\#A$ and $\Psi^{-1} : H\#A \to A\#H$ are defined by $\Psi(a\#h) = h_2\#a^{h_1}$ and by $\Psi^{-1}(g\#b) = g_1(b)\#g_2$. Suppose that $\mu\#\phi$ is an automorphism of $H\#A$. Then $\Psi^{-1}(\mu\#\phi)\Psi$ is an automorphism of $A\#H$, and it is easy to see that $\phi\#\mu = \Psi^{-1}(\mu\#\phi)\Psi$, since this formula holds for elements of A and elements of H. The converse is similar.

(b) By definition, for all $a\#h, b\#g \in A\#H$,

$$\begin{aligned}(\mu\#\phi)(a\#h)(\mu\#\phi)(b\#g) &= (\mu(a)\#\phi(h))(\mu(b)\#\phi(g))\\ &= \mu(a)\phi(h)_1(\mu(b))\#\phi(h)_2\phi(g)\end{aligned}$$

and

$$\begin{aligned}(\mu\#\phi)((a\#h)(b\#g)) &= (\mu\#\phi)(ah_1(b)\#h_2 g)\\ &= \mu(ah_1(b))\#\phi(h_2 g) = \mu(a)\mu(h_1(b))\#\phi(h_2)\phi(g).\end{aligned}$$

The assertion follows by comparing these two equations.

(c) By definition,

$$\phi(h) = \eta(h_1)S^{2n}(h_2)\gamma(h_3)$$

for all $h \in H$. Since S^{2n} is a Hopf algebra automorphism of H, we have

$$\Delta(\phi(h)) = \eta(h_1)S^{2n}(h_2)\otimes S^{2n}(h_3)\gamma(h_4)$$

and

$$h_1\otimes\phi(h_2) = h_1\otimes\eta(h_2)S^{2n}(h_3)\gamma(h_4) = \Xi^r_\eta(h_1)\otimes S^{2n}(h_2)\gamma(h_3).$$

Then (E2.8.1) in this case is the condition

$$(\Xi^l_\eta\circ S^{2n})(h_1)(\mu(b))\#S^{2n}(h_2)\gamma(h_3) = \mu(\Xi^r_\eta(h_1)(b))\#S^{2n}(h_2)\gamma(h_3)$$

for all $h \in H$ and $b \in A$. Clearly then (E2.8.2) implies (E2.8.1). Conversely, if (E2.8.1) holds, then applying the winding automorphism $\Xi^r_{\gamma\circ S}$ to both sides of the previous displayed equation and then applying $1\#\epsilon$ gives (E2.8.2). □

Remark 2.9. Recall from [BZ] that the Nakayama automorphism of a noetherian AS Gorenstein Hopf algebra H has the form $\phi = S^{-2}\circ\Xi^r_\gamma$. Using this automorphism of H in the previous result, (E2.8.2) becomes

$$S^{-2}(h)(\mu(b)) = \mu(h(b))$$

for all $b \in A$ and $h \in H$. This formula is related to (E3.10.2) and Lemma 5.3(a) below.

We conclude this section with a result that will allow us to better understand the bimodule $(M\#H)^*$ constructed in Proposition 2.7, in the special case that $M^* \cong {}^\mu A^1$ for an automorphism μ.

Lemma 2.10. *Let A be a left H-module algebra and let $B = A\#H$. Let $\mu : A \to A$ be an algebra automorphism of A and suppose that the A-bimodule $N = {}^\mu A^1$ has a left H-action making it a H_{S^i}-equivariant A-bimodule for some even integer i. Make N a right H-module as in* (E2.4.1)*. Suppose further that N has a left and right A-module generator $\mathfrak{e}$ which is H-stable in the sense that $h(\mathfrak{e}) \subseteq k\mathfrak{e}$ for all $h \in H$, and define $\eta : H \to k$ by $h(\mathfrak{e}) = \eta(h)\mathfrak{e}$. Suppose that $\sigma : H \to H$ is an automorphism satisfying* (E2.5.2)*, and let $\mathfrak{u}$ be a left and right generator for ${}^1H^\sigma$, so that $\mathfrak{u}h = \sigma(h)\mathfrak{u}$.*

(a) *The element $\mathfrak{u}\#\mathfrak{e}$ is a left and right B-module generator of ${}^1H^\sigma\#^\mu A^1$, where this is the B-bimodule constructed in Lemma 2.5(b).*

(b) *We have ${}^1H^\sigma\#^\mu A^1 \cong {}^\rho B^1$ as B-bimodules, where the automorphism ρ of B has the following formula:*

$$\rho(a\#h) = \mu(a)\#\Xi^l_\eta\circ\sigma^{-1}(h).$$

Proof. (a) We work with the right smash product $H\#A$ in the proof, since the formulas for the B-bimodule structure on ${}^1H^\sigma\#^\mu A^1$ are given in terms of it.

It is straightforward to check the identity

$$\begin{aligned}(\text{E2.10.2})\qquad (\mathfrak{u}\#\mathfrak{e})(g\#1) &= \mathfrak{u}g_2\#\mathfrak{e}^{g_1} = \mathfrak{u}g_2\#S^{-i-1}(g_1)(\mathfrak{e})\\ &= \mathfrak{u}g_2\#\eta(S^{-i-1}(g_1))\mathfrak{e} = \mathfrak{u}g_2\eta(S^{-i-1}(g_1))\#\mathfrak{e}\\ &= \mathfrak{u}\Xi^l_{\eta\circ S}(g)\#\mathfrak{e} \qquad \text{using (E1.2.2).}\end{aligned}$$

Also, $(\mathfrak{u}\#\mathfrak{e})(1\#a) = (\mathfrak{u}\#\mathfrak{e}a)$ and $(g\#a)(\mathfrak{u}\#\mathfrak{e}) = (g\mathfrak{u}\#a\mathfrak{e})$, so it follows that $\mathfrak{u}\#\mathfrak{e}$ is a left and right B-module generator. The same formulas easily imply that no nonzero element of B kills $\mathfrak{u}\#\mathfrak{e}$ on either side, so ${}^1H^\sigma\#^\mu A^1$ is a free B-module of rank 1 on each side. Then this bimodule is isomorphic to ${}^\rho B^1$ for some automorphism $\rho : B \to B$, by Lemma 1.7.

(b) Since $\Xi^l_{\eta\circ S} = (\Xi^l_\eta)^{-1}$, we calculate for any $h \in H$ that

$$\begin{aligned}(h\#1)(\mathfrak{u}\#\mathfrak{e}) &= h\mathfrak{u}\#\mathfrak{e}\\ &= \mathfrak{u}\sigma^{-1}(h)\#\mathfrak{e}\\ &= (\mathfrak{u}\#\mathfrak{e})(\Xi^l_\eta\circ\sigma^{-1}(h)\#1) \qquad \text{by} \quad \text{(E2.10.2).}\end{aligned}$$

This shows that $\rho(h\#1) = \Xi^l_\eta\circ\sigma^{-1}(h)\#1$.

For any $a \in A$,

$$(1\#a)(\mathfrak{u}\#\mathfrak{e}) = \mathfrak{u}\#a\mathfrak{e} = \mathfrak{u}\#\mathfrak{e}\mu(a) = (\mathfrak{u}\#\mathfrak{e})(1\#\mu(a))$$

Hence $\rho(1\#a) = 1\#\mu(a)$.

Thus we know how ρ acts on elements of A and in H, and so $\rho = \Xi^l_\eta \circ \sigma^{-1}\#\mu$ as an automorphism of $H\#A$. By Lemma 2.8(a) and its proof, we see that as an automorphism of the left smash product $A\#H$, we have the formula $\rho = \mu\#\Xi^l_\eta \circ \sigma^{-1}$ as required. □

3. AS Gorenstein algebras and local cohomology

In this section, we introduce the main technical tool of our approach in this paper, which is the local cohomology of graded algebras. We also define a generalization of the AS Gorenstein condition to not necessarily connected graded algebras, and discuss the homological determinant in this setting.

Let A be a locally finite $\mathbb{N}$-graded algebra and $\mathfrak{m}_A$ be the graded ideal $A_{\geq 1}$. Let $A-\mathrm{GrMod}$ denote the category of $\mathbb{Z}$-graded left A-modules. Similarly, if A and C are graded algebras, then $(A, C)-\mathrm{GrMod}$ is the category of $\mathbb{Z}$-graded (A, C)-bimodules. For each n and each graded left A-module M, we define

$$\Gamma_{\mathfrak{m}_A}(M) = \{x \in M \mid A_{\geq n}x = 0 \text{ for some } n \geq 1\} = \lim_{n\to\infty} \mathrm{Hom}_A(A/A_{\geq n}, M)$$

and call this the $\mathfrak{m}_A$-*torsion* submodule of M. It is standard that the functor $\Gamma_{\mathfrak{m}_A}$ is a left exact functor from $A-\mathrm{GrMod}$ to itself. Since this category has enough injectives, the right derived functors $R^i\Gamma_{\mathfrak{m}_A}$ are defined and called the local cohomology functors. Explicitly, one has $R^i\Gamma_{\mathfrak{m}_A}(M) = \lim_{n\to\infty} \mathrm{Ext}^i_A(A/A_{\geq n}, M)$. See [AZ] for more background.

We also consider $\mathbb{Z}^w$-graded algebras for a positive integer w. In the $\mathbb{Z}^w$-graded setting, the degree of any homogeneous element a is denoted by $\mid a \mid= (n_1, \cdots, n_w) \in \mathbb{Z}^w$. Define a new $\mathbb{Z}$-grading by $\|\, a \,\|= \sum_{s=1}^w n_s$. In most cases, we will assume that A is noetherian and a locally finite $\mathbb{N}$-graded algebra with respect to the $\|\ \|$-grading. But in a few occasions, we consider more general $\mathbb{Z}^w$-gradings (for example, in the discussion of Frobenius algebras at the end of this section). The local cohomology functors for $\mathbb{Z}^w$-graded algebras A are defined using the $\|\ \|$-grading and the same torsion functor $\Gamma_{\mathfrak{m}_A}$. This is an endofunctor of the category of $\mathbb{Z}^w$-graded modules and thus the local cohomology modules are also in this category. There is a forgetful functor from the category of $\mathbb{Z}^w$-graded A-modules to the category of $\mathbb{Z}$-graded A-modules. It is easy to check that the local cohomological functors $\Gamma_{\mathfrak{m}_A}$ and the derived functors $R^d\Gamma_{\mathfrak{m}_A}$ commute with the forgetful functor, so we use the same notation for local cohomological functors in either graded category.

In the next two lemmas we consider the local cohomology of (bi)-modules over a smash product.

Lemma 3.1. *Let A be a locally finite $\mathbb{N}$-graded left H-module algebra for a finite-dimensional Hopf algebra H, and let $B = A\#H$. Let $i \geq 0$ be an integer and let C be another graded algebra. Then as endofunctors of* (B, C)-GrMod,

$$R^i\Gamma_{\mathfrak{m}_A}(-) \cong R^i\Gamma_{\mathfrak{m}_B}(-).$$

Proof. Note first that B is a flat right A-module. In fact, it is obvious from thinking of B as a right smash product $H\#A$, as in Lemma 2.1, that B is a free right A-module. Then every (graded) injective left B-module is a (graded) injective left A-module; see [La, Lemma 3.5], or see [KKZ, Lemma 5.1] for a graded version.

Given a graded (B,C)-bimodule M, consider a graded injective (B,C)-bimodule resolution $I^\bullet$ of M. In other words, we take an injective resolution of M in the category $(B \otimes_k C^{op}) - \mathrm{GrMod}$, where $B \otimes_k C^{op}$ is also a free right B-module and hence the left modules I^i are injective in both $B - \mathrm{GrMod}$ and $A - \mathrm{GrMod}$. Thus we can use $I^\bullet$ to calculate either functor. It is easy to check using the formula (E2.0.1) that for any left B-module M, $\Gamma_{\mathfrak{m}_A}(M)$ is a $\mathfrak{m}_B$-torsion B-submodule of M. It easily follows that for any (B,C)-bimodule M, we have $\Gamma_{\mathfrak{m}_A}(M) = \Gamma_{\mathfrak{m}_B}(M)$, and that these are (B,C)-sub-bimodules of M. This implies that $\Gamma_{\mathfrak{m}_A}(-) = \Gamma_{\mathfrak{m}_B}(-)$ as endofunctors of (B,C)-GrMod. The assertion easily follows by applying these functors to $I^\bullet$ and taking homology. □

Lemma 3.2. *Let A be a locally finite $\mathbb{N}$-graded left H-module algebra, and let $B = A\#H$. Suppose that M is a graded H-equivariant A-bimodule.*

(a) *For any integer $d \geq 0$, $R^d\Gamma_{\mathfrak{m}_A}(M)$ has a natural left H-action and is an H-equivariant A-bimodule.*

(b) *Assume H is finite dimensional. Then as B-bimodules,*

$$R^d\Gamma_{\mathfrak{m}_B}(M\#H) \cong R^d\Gamma_{\mathfrak{m}_A}(M\#H) \cong R^d\Gamma_{\mathfrak{m}_A}(M)\#H,$$

where the bimodule structure of $R^d\Gamma_{\mathfrak{m}_A}(M)\#H$ is as in Lemma 2.5(a).

Proof. (a) We have seen in Remark 2.3 that to be a H-equivariant A-bimodule is equivalent to being a left module over the algebra $R = A^e \rtimes H$ of Kaygun. So M is a left R-module, and clearly in fact R is a graded algebra (with H in degree 0) and M is a graded left R-module. Then $\Gamma_{\mathfrak{m}_A}(M)$ is a left B-submodule and a right A-submodule of M, by the same proof as in Lemma 3.1, and it follows that $\Gamma_{\mathfrak{m}_A}(M)$ is an R-submodule of M.

Take a graded injective left R-module resolution $I^\bullet$ of M. Since we have in R that $(1 \otimes b \otimes 1)(a \otimes 1 \otimes h) = (a \otimes b \otimes h)$, for all $a, b \in A$ and $h \in H$, it is easy to see that R is a free right A-module. Thus $I^\bullet$ is also a graded injective left A-module resolution of M, by the same argument as in Lemma 3.1. Thus we may use this resolution to calculate $R^d\Gamma_{\mathfrak{m}_A}(M)$. We conclude that $R^d\Gamma_{\mathfrak{m}_A}(M)$ retains a left R-module structure, in other words, it is a graded H-equivariant A-bimodule. In fact, it is then easy to see that to obtain the left R-module structure on $R^d\Gamma_{\mathfrak{m}_A}(M)$, we can use any resolution of M by a complex of graded R-modules each of which is graded injective as an A-module (in any words, the modules need not be injective as R-modules.

(b) The first isomorphism follows from Lemma 3.1. We note that given an left R-module homomorphism $\phi : M \to N$, we can take the smash product of each module with H as in Lemma 2.5(a) to obtain a map $\phi\#1 : M\#H \to N\#H$. Smashing with H is easily seen to be functorial in the sense that $\phi\#1$ is a B-bimodule map.

Again choosing a graded injective left R-module resolution $I^\bullet$ of M, we can smash this resolution with H; this gives a complex $I^\bullet\#H$ of B-bimodules which is a resolution of $A\#H$. Note that $I^\bullet\#H$ is a complex of graded injective left A-modules, since each I^i is graded injective over A as in part (a) and $I^i\#H$ is isomorphic as a left A-module to a finite direct sum of copies of I^i (recall that H is finite dimensional).

Now it is easy to check that for any graded left R-module N, $\Gamma_{\mathfrak{m}_A}(N\#H) = \Gamma_{\mathfrak{m}_A}(N)\#H$ as subspaces of $N\#H$, and by functoriality since $\Gamma_{\mathfrak{m}_A}(N)$ is an R-submodule of N, it follows that $\Gamma_{\mathfrak{m}_A}(N)\#H$ is a B-sub-bimodule of $N\#H$. Finally, using the injective resolution $I^\bullet\#H$ to compute the derived functors, it follows that $R^d\Gamma_{\mathfrak{m}_A}(M\#H) \cong R^d\Gamma_{\mathfrak{m}_A}(M)\#H$ as B-bimodules, for each d. □

We would like to consider a generalization of AS Gorenstein algebras to the non-connected case, but where the algebra is still locally finite. In this paper we propose the following definition.

Definition 3.3. Let A be a $\mathbb{Z}^w$-graded algebra, for some $w \geq 1$, such that it is locally finite and $\mathbb{N}$-graded with respect to the $|| \quad ||$-grading. We say A is a *generalized AS Gorenstein algebra* if

(a) A has injective dimension d.
(b) A is noetherian and satisfies the χ condition (see [AZ, Definition 3.7]), and the functor $\Gamma_{\mathfrak{m}_A}$ has finite cohomological dimension.
(c) There is an A-bimodule isomorphism $R^d\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^{\mu}A^1(-\mathfrak{l})$ (where this is the graded dual), for some $\mathfrak{l} \in \mathbb{Z}^w$ (called the *AS index*) and for some graded algebra automorphism μ of A (called the *Nakayama automorphism*).

Remark 3.4. Other notions of generalized AS regular algebras were introduced for not necessarily connected graded algebras by Martinez-Villa and Solberg in [M-VS] and by Minamoto and Mori in [MM]. Even if A is noetherian of finite global dimension, our definition above is slightly stronger than the one in [M-VS].

One of our main motivations for introducing the definition of generalized AS Gorenstein is that when A is a connected graded AS Gorenstein algebra which is a graded H-module algebra for some finite-dimensional Hopf algebra H, then $A\#H$ will be generalized AS Gorenstein (Theorem 4.1(b)). The hypotheses in our definition of generalized AS Gorenstein are chosen to make sure that the theory of dualizing complexes will work as usual. We discuss this in the following lemma, which justifies calling μ in Definition 3.3(c) a Nakayama automorphism of A.

Lemma 3.5. *Let A be generalized AS Gorenstein, and let $\mu \in \operatorname{Aut}_{\mathbb{Z}}(A)$ such that*

$$R^d\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^{\mu}A^1(-\,\mathfrak{l})$$

for some $\mathfrak{l} \in \mathbb{Z}$; namely, μ is a Nakayama automorphism in the sense of Definition 3.3. Then we have

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d \\ {}^1A^{\mu}(\mathfrak{l}) & i = d \end{cases};$$

namely, μ is a Nakayama automorphism in the sense of Definition 0.1.

Proof. Van den Bergh's paper [VdB1] works with connected graded algebras only. However, one can check that the results of [VdB1, Sections 3-8] hold with no essential change for a locally finite $\mathbb{N}$-graded algebra. This generalization is similar to the semi-local complete case which is worked out explicitly in [WZ].

Now [VdB1, Theorem 6.3] shows that $R := R^d\Gamma_{\mathfrak{m}_A}(A)^*[d] \cong {}^{\mu}A^1(-\mathfrak{l})[d]$ is a balanced dualizing complex for A. By [VdB1, Proposition 8.2], it is also a rigid dualizing complex, and this implies in particular by [VdB1, Proposition 8.4] that $R^{-1} = \operatorname{RHom}_{A^e}(A, A^e)$, where R^{-1} is the inverse of R under derived tensor.

Equivalently, since the inverse of the bimodule ${}^{\mu}A^{1}$ is isomorphic to ${}^{1}A^{\mu}$, we have $R^{-1} \cong {}^{1}A^{\mu}(\mathfrak{l})[-d]$ and thus

$$\operatorname{Ext}^i_{A^e}(A, A^e) \cong \begin{cases} 0 & i \neq d \\ {}^{1}A^{\mu}(\mathfrak{l}) & i = d \end{cases},$$

as claimed. □

We note in the following remark that generalized AS Gorenstein algebras satisfy generalized versions of properties (b,c) of Definition 1.1.

Remark 3.6. Suppose that A is generalized AS Gorenstein of injective dimension d and AS index $\mathfrak{l}$. Then we claim that for a finite-dimensional graded left A-module S concentrated in degree 0, we have that $\operatorname{Ext}^i(S, A) = 0$ if $i \neq d$, and that $\operatorname{Ext}^d(S, A)$ is a finite-dimensional k-space concentrated in graded degree $-\mathfrak{l}$.

To see this, note that A satisfies the hypotheses of (the locally finite version of) [VdB1, Theorem 5.1], and so we have the local duality formula

$$R\Gamma_{\mathfrak{m}_A}(M)^* = R\operatorname{Hom}_A(M, R\Gamma_{\mathfrak{m}_A}(A)^*)$$

for any graded left A-module M. Since we have $R\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^{\mu}A^{1}(-\mathfrak{l})[d]$ in this case, taking $M = S$ the claim follows as long as $R\Gamma_{\mathfrak{m}_A}(S)^*$ is finite-dimensional and concentrated in graded degree 0. But by the locally finite version of [VdB1, Lemma 4.4], since S is finite-dimensional we have $R\Gamma_{\mathfrak{m}_A}(S)^* = S^*$.

Clearly, one also has a right module analog of the comments above. Note that this implies in particular that a connected graded generalized AS Gorenstein algebra is a (noetherian) AS Gorenstein algebra in the usual sense.

The homological determinant has been an important tool for understanding the theory of connected graded AS Gorenstein algebras, especially the invariant theory of group and Hopf algebra actions. Next, we develop a theory of homological determinant that will apply, in certain cases, to generalized AS Gorenstein algebras.

Definition 3.7. Let A be a $\mathbb{Z}^w$-graded generalized AS Gorenstein algebra. Let A be a graded left H-module algebra for some Hopf algebra H. Since A is an H-equivariant A-bimodule, then $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is a graded $H_{S^{-2}}$-equivariant A-bimodule by Lemma 3.2 and Proposition 2.7. Suppose further that there is a nonzero element $\mathfrak{e} \in R^d\Gamma_{\mathfrak{m}_A}(A)^*$ of degree $\mathfrak{l}$ such that

(i) $\mathfrak{e}$ is a left and right generator of the A-bimodule $R^d\Gamma_{\mathfrak{m}_A}(A)^*$; and
(ii) $k\mathfrak{e}$ is a left H-submodule.

(a) The element $\mathfrak{e}$ is called *an H-stable generator* of $R^d\Gamma_{\mathfrak{m}_A}(A)^*$.
(b) Under the condition (i) alone, there is a graded algebra automorphism μ such that $a\mathfrak{e} = \mathfrak{e}\mu(a)$ for all $a \in A$. We call such a μ the $\mathfrak{e}$-*Nakayama automorphism* of A. Note that by Lemma 1.7, any other Nakayama automorphism will differ by an inner automorphism.
(c) We define an algebra homomorphism $\operatorname{hdet} : H \to k$ by

$$\operatorname{hdet}(h)\ \mathfrak{e} = h(\mathfrak{e}) \tag{E3.7.1}$$

for all $h \in H$ and call the map hdet the $\mathfrak{e}$-*homological determinant* of the H-action.

Note that the $\mathfrak{e}$-homological determinant only depends on the $|| \ ||$-grading, and is independent of possible choices of $\mathbb{Z}^w$-gradings that induce the same $|| \ ||$-grading, as long as the choice of $\mathfrak{e}$ is fixed. When A is connected graded AS Gorenstein, then any bimodule generator $\mathfrak{e}$ of $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is contained in the 1-dimensional degree-$\mathfrak{l}$ piece and so is automatically H-stable since the H-action respects the grading. The $\mathfrak{e}$-Nakayama automorphism is the unique choice of Nakayama automorphism in this case, and the homological determinant is independent of $\mathfrak{e}$. On the other hand, there is no obvious reason why an arbitrary generalized AS Gorenstein algebra should have an H-stable generator. In future work, we hope to generalize some of our theorems below to work without this assumption.

Remark 3.8. In [KKZ, Definition 3.3], the homological determinant hdet : $H \to k$ is defined, given a finite dimensional Hopf algebra H and a connected graded AS Gorenstein algebra which is a graded left H-module algebra. In this case, we claim that the definition of hdet we gave above coincides with the definition in [KKZ]. Both definitions depend on first putting a left H-action on $R^d\Gamma_{\mathfrak{m}_A}(A)$. We did this in Lemma 3.2 by taking a graded injective resolution $I^\bullet$ of A over the Kaygun algebra $R = A^e \rtimes H$ and noting that the terms of this resolution are graded injective over A also. In [KKZ], a $B = A\#H$-module graded injective resolution of A is used. Similarly as in the proof of Lemma 3.2, any B-module resolutions in which the terms are graded injective over A must induce the same B-module structures on the local cohomology, so we get the same B-action on $R^d\Gamma_{\mathfrak{m}_A}(A)$, and in particular the same H-action, in either case.

Then $(R^d\Gamma_{\mathfrak{m}_A}(A))^*$ obtains a right H-action. Choosing a generator $\mathfrak{e}$ of degree $\mathfrak{l}$ gives a map $\eta' : H \to k$ given by $\mathfrak{e}^h = \mathfrak{e}\eta'(h)$ and the homological determinant is defined in [KKZ, Definition 3.3] to be $\eta = \eta' \circ S$. In this paper, we use that the right H-action of $(R^d\Gamma_{\mathfrak{m}_A}(A))^*$ induces a left H-action satisfying $\mathfrak{e}^h = S(h) \cdot \mathfrak{e}$ as in (E2.4.1), since $(R^d\Gamma_{\mathfrak{m}_A}(A))^*$ is a $H_{S^{-2}}$-equivariant bimodule. Since $\eta \circ S^2 = \eta$ by (E1.2.2), the two definitions agree. Moreover, it has already been commented in [KKZ, Remark 3.4] that the definition of homological determinant in [KKZ] agrees in the case $H = kG$ is a group algebra with the original definition given in [JoZ].

In general it is not easy to actually compute the homological determinant. We mention a few examples of connected graded AS Gorenstein algebras for which the answer is known.

Example 3.9. (a) If A is the commutative polynomial ring $k[V]$ where $A_1 = V$ is finite dimensional, then $\operatorname{hdet}\sigma = \det\sigma\mid_V$ for all $\sigma \in \operatorname{Aut}_{\mathbb{Z}}(A)$ [JoZ, p. 322].

(b) If A is a graded down-up algebra (which is a special kind of AS regular algebra of global dimension 3 generated by 2 degree 1 elements), then $\operatorname{hdet}\sigma = (\det\sigma\mid_{A_1})^2$ by a result of Kirkman-Kuzmanovich [KK, Theorem 1.5].

(c) Let A be the skew polynomial ring $k_{-1}[x, y]$ and let $\sigma \in \operatorname{Aut}_{\mathbb{Z}}(A)$ map x to y and y to x. Then $\operatorname{hdet}\sigma = -\det\sigma\mid_{A_1} = 1$.

(d) If A is $\mathbb{Z}^w$-graded and σ is an automorphism which acts on each graded piece by a scalar, then we calculate $\operatorname{hdet}(\sigma)$ in Lemma 5.3 below.

(e) Let A be noetherian AS Gorenstein and $\sigma \in \operatorname{Aut}_{\mathbb{Z}}(A)$. If z is a normal nonzerodivisor such that $\sigma(z) = \lambda z$ for some $\lambda \in k^\times$, then by [JiZ, Proposition 2.4],

$$\operatorname{hdet}\sigma\mid_A = \lambda \operatorname{hdet}\sigma\mid_{A/(z)} . \tag{E3.9.1}$$

On the other hand, it is unclear how to calculate $\operatorname{hdet}\sigma$ for an automorphism σ of an arbitrary AS regular algebra.

In the next result, we see that there is a useful restriction on the interaction between a Nakayama automorphism μ of a generalized AS Gorenstein algebra and a Hopf action on the algebra.

Lemma 3.10. *Let A be a generalized AS Gorenstein algebra which is a graded left H-module algebra for some Hopf algebra H. Then $R^d\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^{\mu}A^1(-\mathfrak{l})$ is naturally a graded $H_{S^{-2}}$-equivariant A-bimodule by Proposition 2.7. Assume further that $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ has an H-stable generator $\mathfrak{e}$, let μ be the $\mathfrak{e}$-Nakayama automorphism, and let* hdet *be the $\mathfrak{e}$-homological determinant.*

Then the identity

$$(\Xi^l_{\text{hdet}} \circ S^{-2})(h)(\mu(a)) = \mu(\Xi^r_{\text{hdet}}(h)(a)) \tag{E3.10.1}$$

holds for all $h \in H$ and all $a \in A$. As a consequence, if H is cocommutative or if $\text{hdet} = \epsilon$, *then*

$$S^{-2}(h)(\mu(a)) = \mu(h(a)) \tag{E3.10.2}$$

for all $a \in A$ and $h \in H$.

Proof. Write $\eta = \text{hdet} : H \to k$ for convenience and recall that this is an algebra map. Applying h to $\mathfrak{e}\mu(a) = a\mathfrak{e}$ and using that ${}^{\mu}A^1(-\mathfrak{l})$ is an $H_{S^{-2}}$-equivariant A-bimodule, we have

$$\eta(h_1)\mathfrak{e}S^{-2}(h_2)(\mu(a)) = h_1(a)\eta(h_2)\mathfrak{e} = \mathfrak{e}\mu(h_1(a)\eta(h_2)).$$

This implies that

$$(\Xi^l_\eta \circ S^{-2})(h)(\mu(a)) = \eta(h_1)S^{-2}(h_2)(\mu(a)) = \mu(h_1(a)\eta(h_2)) = \mu(\Xi^r_\eta(h)(a)),$$

where we use $\eta \circ S^{-2} = \eta$ by (E1.2.2). This is (E3.10.1).

If $\eta = \epsilon$ then $\Xi^l_\eta = \Xi^r_\eta = Id$, while if H is cocommutative, then $\Xi^l_\eta = \Xi^r_\eta$ and $S^2 = 1$. In either case, (E3.10.2) is equivalent to (E3.10.1). □

The following interesting result is an immediate corollary.

Theorem 3.11. *Let A be noetherian connected graded AS Gorenstein. Then μ_A is in the center of* $\operatorname{Aut}_{\mathbb{Z}}(A)$.

Proof. This follows from applying Lemma 3.10 to the action of $H = \operatorname{Aut}_{\mathbb{Z}}(A)$ on A, and noting that the group algebra H is cocommutative. □

For the rest of this section we consider Frobenius algebras. Let E be a finite dimensional $\mathbb{Z}^w$-graded algebra. We say E is a Frobenius algebra if there is a nondegenerate associative bilinear form $\langle -,- \rangle : E \times E \to k$, which is graded of degree $-\mathfrak{l} \in \mathbb{Z}^w$. This is equivalent to the existence of an isomorphism $E^* \cong E[-\mathfrak{l}]$ as graded left (or right) E-modules. As a consequence, the injective dimension of E is zero. The vector $\mathfrak{l} \in \mathbb{Z}^w$ is called the AS index of E. There is a *classical Nakayama automorphism* $\mu \in \operatorname{Aut}_{\mathbb{Z}^w}(E)$ such that $\langle a, b\rangle = \langle \mu(b), a\rangle$ for all $a, b \in E$. Let $\mathfrak{e} = \langle 1, -\rangle = \langle -, 1\rangle \in E^*$. Then $\mathfrak{e}$ is a generator of E^* such that $E^* \cong {}^{\mu}E^1(-\mathfrak{l})$ as graded E-bimodules. Note that μ and $\mathfrak{e}$ are dependent on the choices of the bilinear form, so they are not necessarily unique. See [Mu] for further background.

It is easy to see that if E is $\mathbb{N}$-graded with respect to the $\| \, \|$-grading, then E is generalized AS Gorenstein, with $R^0\Gamma_{\mathfrak{m}_E}(E)^* = E^* \cong {}^{\mu}E^1(-\mathfrak{l})$. Even if E is not $\mathbb{N}$-graded, we still have that $E^* \cong {}^{\mu}E^1(-\mathfrak{l})$; taking G to be the subgroup of

$\mathrm{Aut}_{\mathbb{Z}^w}(E)$ generated by μ, then E is naturally a graded left kG-module algebra, and so E^* is a graded kG-equivariant E-bimodule. Then Definition 3.7 can be interpreted for $\mathfrak{e} \in E^*$ and we have the following.

Lemma 3.12. *Keep the notation above. Then $\mathfrak{e}$ is kG-stable and* $\mathrm{hdet}\,\mu = 1$*, where* hdet *is the $\mathfrak{e}$-homological determinant.*

Proof. For any $b \in E$,

$$\mu(\mathfrak{e})(b) = \mathfrak{e}(\mu^{-1}(b)) = \langle 1, \mu^{-1}(b)\rangle = \langle b, 1\rangle = \mathfrak{e}(b)$$

which means that $\mu(\mathfrak{e}) = \mathfrak{e}$. Since $G = \langle \mu \rangle$, $\mathfrak{e}$ is G-stable. The assertion follows by the definition of hdet (E3.7.1). □

4. Proof of identity (HI1)

The aim of this section is to prove homological identity (HI1) by computing the Nakayama automorphism of the smash product of an AS Gorenstein algebra A with a finite dimensional Hopf algebra H action. This generalizes and partially recovers a result of Le Meur [LM, Theorem 1], who studies the differential graded case, and a result of Liu-Wu-Zhu [LiWZ, Theorem 2.12], where A is assumed to be N-Koszul and H is involutory (although not necessarily finite-dimensional). Other papers where similar problems are considered include [Fa, IR, WZhu]. Our approach differs from the previous ones in several ways: we do not assume finite global dimension, but rather the weaker Gorenstein condition, and our methods emphasize the techniques of local cohomology.

Theorem 4.1. *Suppose that A is a generalized AS Gorenstein algebra of injective dimension d which is a graded left H-module algebra for a finite-dimensional Hopf algebra H. Let $B = A\#H$. Suppose that there is an H-stable generator $\mathfrak{e} \in R^d\Gamma_{\mathfrak{m}_A}(A)^*$. Let D_B and D_A be the rigid dualizing complexes over B and over A, respectively.*

(a) *We have*

$$D_B[-d] \cong R^d\Gamma_{\mathfrak{m}_B}(B)^* \cong H^*\#R^d\Gamma_{\mathfrak{m}_A}(A)^* = {}^1H^\sigma\#D_A[-d],$$

as B-bimodules, where $\sigma = \mu_H^{-1} = S^2 \circ \Xi^r_{\int^r}$.

(b) *B is generalized AS Gorenstein, with Nakayama automorphism*

$$\mu_B = \mu_A\#(\Xi^l_{\mathrm{hdet}} \circ \mu_H),$$

where hdet *is the $\mathfrak{e}$-homological determinant.*

(c) *If A is connected graded AS regular and H is semisimple, then B is skew CY.*

Proof. (a) Recall that as mentioned in the proof of Lemma 3.5, the results in [VdB1] hold for not-necessarily connected but locally finite $\mathbb{N}$-graded algebras; in particular, they hold for the algebra B. Since A is generalized AS Gorenstein, by [VdB1, Theorem 6.3] we have that $D_A[-d] = R^d\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^\mu A^1(\mathfrak{l})$, as complexes concentrated in degree 0, and $R^i\Gamma_{\mathfrak{m}_A}(A) = 0$ for $i \neq d$.

Now for any $i \geq 0$ we have

$$R^i\Gamma_{\mathfrak{m}_B}(B)^* = R^i\Gamma_{\mathfrak{m}_A}(A\#H)^* = (R^i\Gamma_{\mathfrak{m}_A}(A)\#H)^*$$

as B-bimodules, where we have used Lemma 3.2, and the fact that A is an H-equivariant A-bimodule. Thus $R^i\Gamma_{\mathfrak{m}_B}(B) = 0$ for $i \neq d$, and $R^d\Gamma_{\mathfrak{m}_B}(B)^* \cong (R^d\Gamma_{\mathfrak{m}_A}(A)\#H)^*$. Now we use Proposition 2.7 to identify

$$(R^d\Gamma_{\mathfrak{m}_A}(A)\#H)^* \cong {}^1H^\sigma\#R^d\Gamma_{\mathfrak{m}_A}(A)^* = {}^1H^\sigma\#D_A[-d]$$

as B-bimodules, where $\sigma = \mu_H^{-1} = S^2 \circ \Xi^r_{\int^r}$ by Lemma 2.6. Since $D_A[-d] = R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is an $H_{S^{-2}}$-equivariant A-bimodule, the final term is a well-defined B-bimodule as in Lemma 2.5(b).

Note that since H is finite-dimensional, B is a finitely generated left and right A-module, so B is noetherian also. The algebra B satisfies the χ condition and has finite cohomological dimension, since these properties also pass to a finite ring extension [AZ, Theorem 8.3 and Corollary 8.4]. Thus the hypotheses of [VdB1, Theorem 6.3] also hold for B, so the rigid dualizing complex for B exists and equals $R\Gamma_{\mathfrak{m}_B}(B)^*$. Finally, this means that we have $D_B[-d] = R^d\Gamma_{\mathfrak{m}_B}(B)^*$.

(b) By Lemma 2.10, taking $\mathfrak{u}$ to be a bimodule generator of ${}^1H^\sigma$, then $\mathfrak{u}\#\mathfrak{e}$ is a generator of ${}^1H^\sigma\#^\mu A^1(-\mathfrak{l})$, and we have

$${}^1H^\sigma\#^\mu A^1(-\mathfrak{l}) \cong {}^\rho B^1$$

as B-bimodules, where $\rho = \mu_A\#(\Xi^l_{\text{hdet}} \circ \sigma^{-1})$. This implies that $D_B \cong {}^\rho B^1[d](-\mathfrak{l})$, and thus $\mu_B = \rho$ has the claimed formula, and we have verified Definition 3.3(c) for B. We already checked in part (a) that B is noetherian, satisfies χ, and has finite cohomological dimension. By definition a dualizing complex has finite injective dimension in the derived category of left or right modules [VdB1, Definition 6.1]. Since the dualizing complex for B is isomorphic to (a shift of) B as a right or left B-module, it follows that B has finite injective dimension on both sides. Thus all parts of the definition of generalized AS Gorenstein hold for B.

(c) If A is AS regular and H is semisimple, then A is homologically smooth by Lemma 1.2 and H is homologically smooth since it is semisimple. By [LiWZ, Proposition 2.11], $B = A\#H$ is homologically smooth. The rest of the definition of skew CY for B follows from part (b) and Lemma 3.5. $\square$

Theorem 0.2 (namely, (HI1)) follows immediately from the previous result. As mentioned earlier, one of the motivations behind our study of the result above was to better understand examples such as Example 1.6, where A is only skew-CY, but some skew group algebra $A \rtimes G$ becomes CY. The following corollary, which gives a special case of Corollary 0.6, explains this phenomenon.

Corollary 4.2. *Let A be noetherian connected $\mathbb{N}$-graded AS regular algebra with Nakayama automorphism μ. Suppose that μ has finite order. If σ is a graded algebra automorphism of A such that $\sigma^n = \mu$ for some n, and $\operatorname{hdet}\sigma = 1$, then $A\#kG$ is CY, where $G = \langle\sigma\rangle$ is the subgroup of $\operatorname{Aut}_{\mathbb{Z}}(A)$ generated by σ.*

Proof. Note that σ has finite order. Let $H = kG$, and let $B = A\#H$, which is the same as the skew group algebra $A\rtimes G$. The algebra B is skew CY by Theorem 4.1(c). By Theorem 4.1(b), $\mu_B = \mu_A\#(\Xi^l_{\text{hdet}} \circ \mu_H)$. Since $\operatorname{hdet}\sigma = 1$, the homological determinant $\operatorname{hdet} : kG \to k$ is trivial, and as a consequence, $\Xi^l_{\text{hdet}} = Id_H$. Since H is semisimple, $\mu_H = Id_H$ [Lo, 1.7(a)]. Thus $\mu_B = \mu_A\#Id = \sigma^n\#Id$, which is inner since it is conjugation in B by $1\#\sigma^n$. So B is CY. $\square$

One interesting open question is whether Theorem 4.1(c) holds in a more general setting.

Question 4.3. Let H be a Hopf algebra and let A be a left H-module algebra, neither of which is necessarily graded. If A and H are skew CY, then is $A\#H$ skew CY? If so, what is the Nakayama automorphism $\mu_{A\#H}$, in terms of μ_A and μ_H?

5. Proof of identity (HI2)

The goal of this section is to prove homological identity (HI2), and we consider a slightly more general setting. Let A be a $\mathbb{Z}^w$-graded generalized AS Gorenstein algebra. Let G be a subgroup of $\operatorname{Aut}_{\mathbb{Z}^w}(A)$, so that A is a left kG-module algebra where $\sigma(a)$ has its usual meaning for $\sigma \in G \subseteq \operatorname{Aut}_{\mathbb{Z}^w}(A)$. In this context, a kG-eqivariant A-bimodule is an A-bimodule M with left G-action, denoted by $\alpha_\sigma : M \to M$ for each $\sigma \in G$, such that

$$\alpha_\sigma(amb) = \sigma(a)\alpha_\sigma(m)\sigma(b) \tag{E5.0.1}$$

for all $a, b \in A$, all $m \in M$ and all $\sigma \in G$. For a fixed σ, any morphism α of A-bimodules satisfying (E5.0.1) is also called a σ-linear A-bimodule morphism.

We review the definition of graded twists of $\mathbb{Z}^w$-graded algebras and $\mathbb{Z}^w$-graded modules [Zh]. For simplicity, we only consider the graded twists by automorphisms of the algebra. Let $\sigma := \{\sigma_1, \cdots, \sigma_w\} \subset \operatorname{Aut}_{\mathbb{Z}^w}(A)$ be a sequence of commuting $\mathbb{Z}^w$-graded automorphisms of A. Recall that $|m|$ denotes the $\mathbb{Z}^w$-degree of a homogeneous element m in a $\mathbb{Z}^w$-graded module M. Let v be an integral vector $(v_1, \cdots, v_w)$. Write $\sigma^v = \sigma_1^{v_1} \cdots \sigma_w^{v_w}$. Define the twisting system associated to σ to be the set

$$\tilde{\sigma} = \{\sigma^v \mid v \in \mathbb{Z}^w\}.$$

A (left) graded twist of A associated to $\tilde{\sigma}$ is a new graded algebra, denoted by $A^{\tilde{\sigma}}$, such that $A^{\tilde{\sigma}} = A$ as a $\mathbb{Z}^w$-graded vector space, and where the new multiplication $\circledcirc$ of $A^{\tilde{\sigma}}$ is given by

$$a \circledcirc b = \sigma^{|b|}(a)b \tag{E5.0.2}$$

for all homogeneous elements $a, b \in A$. We note that the paper [Zh] works primarily with right graded twists, but left graded twists are more convenient in our setting. Given a left graded A-module N, a left graded twist of N is defined by the same formula (E5.0.2) for all homogeneous $a \in A$, $b \in N$ and denoted by $N^{\tilde{\sigma}}$. Then $N^{\tilde{\sigma}}$ is naturally a left $A^{\tilde{\sigma}}$-module, and the functor $N \mapsto N^{\tilde{\sigma}}$ gives an equivalence of graded module categories A-GrMod $\simeq A^{\tilde{\sigma}}$-GrMod [Zh, Theorem 3.1].

Next, we define the left twist of a graded kG-equivariant A-bimodule M. Continue to write $\sigma = \{\sigma_1, \cdots, \sigma_w\}$, and assume now that each σ_i is in the center of G. (Since the σ_i are assumed to pairwise commute, this additional assumption can be effected if necessary by replacing G with the subgroup generated by the σ_i.) If v is an integral vector $(v_1, \cdots, v_w)$, write α_σ^v for $\prod_{s=1}^w \alpha_{\sigma_s}^{v_s} = \alpha_{\sigma^v}$. The left graded twist of M associated to σ, denoted by $M^{\tilde{\sigma}}$, is defined as follows: as a $\mathbb{Z}^w$-graded k-space, $M^{\tilde{\sigma}} = M$, and the left and right $A^{\tilde{\sigma}}$-multiplication is defined by

$$a \circledcirc m \circledcirc b = \sigma^{|m|+|b|}(a)\alpha_\sigma^{|b|}(m)b \tag{E5.0.3}$$

for all homogeneous elements $a, b \in A = A^{\tilde{\sigma}}$ and $m \in M$. It is routine to check that $M^{\tilde{\sigma}}$ is a left kG-equivariant $\mathbb{Z}^w$-graded $A^{\tilde{\sigma}}$-bimodule, where $g \in G$ acts by the same map $\alpha_g : M \to M$ of the underlying k-space. It is also easy to check that the bimodule twist is functorial, in the sense that if $\phi : M \to N$ is a graded

kG-equivariant A-bimodule map, then the same underlying set map gives a kG-equivariant $A^{\tilde{\sigma}}$-bimodule map $\phi : M^{\tilde{\sigma}} \to N^{\tilde{\sigma}}$.

Lemma 5.1. *Assume that A is finitely graded with respect to the $\| \ \|$-grading. Let M be a left kG-equivariant $\mathbb{Z}^w$-graded A-bimodule. For any $d \geq 0$, $R^d\Gamma_{\mathfrak{m}_{A^{\tilde{\sigma}}}}(M^{\tilde{\sigma}}) \cong R^d\Gamma_{\mathfrak{m}_A}(M)^{\tilde{\sigma}}$ as graded left kG-equivariant $A^{\tilde{\sigma}}$-bimodules.*

Proof. The case $d = 0$ is easy. Namely, $\Gamma_{\mathfrak{m}_{A^{\tilde{\sigma}}}}(M^{\tilde{\sigma}}) = \Gamma_{\mathfrak{m}_A}(M)^{\tilde{\sigma}}$ since both can be identified with the same subset of $M^{\tilde{\sigma}}$.

Now if $R = A^e \rtimes kG$ is the Kaygun algebra of Remark 2.3, we may find an injective resolution $M \to I^\bullet$ in the category of $\mathbb{Z}^w$-graded left R-modules. As we have noted in the proof of Lemma 3.2, this is also a graded injective left A-module resolution. Now since the bimodule twist is functorial, we can twist the entire complex as in (E5.0.3) to get an exact complex $M^{\tilde{\sigma}} \to (I^\bullet)^{\tilde{\sigma}}$ of kG-equivariant graded $A^{\tilde{\sigma}}$-bimodules, where the maps are the same underlying vector space maps. Since the bimodule twist (E5.0.3) restricts to the usual left twist as left A-modules, $(I^\bullet)^{\tilde{\sigma}}$ is a a graded injective resolution of $M^{\tilde{\sigma}}$ as left $A^{\tilde{\sigma}}$-modules by [Zh, Theorem 3.1], so we can use it to calculate $R^d\Gamma_{\mathfrak{m}_{A^{\tilde{\sigma}}}}(M^{\tilde{\sigma}})$. Since $\Gamma_{\mathfrak{m}_{A^{\tilde{\sigma}}}}((I^i)^{\tilde{\sigma}}) = \Gamma_{\mathfrak{m}_A}(I^i)^{\tilde{\sigma}}$ for each i, the result follows. □

Lemma 5.2. *Let M be a kG-equivariant graded A-bimodule. Then there is an isomorphism $(M^{\tilde{\sigma}})^* \cong (M^*)^{\tilde{\sigma}}$ of left kG-equivariant graded $A^{\tilde{\sigma}}$-bimodules.*

Proof. Note that M^* is naturally a kG-equivariant graded A-bimodule by Proposition 2.7 (since $S^2 = 1$), so $(M^*)^{\tilde{\sigma}}$ is a kG-equivariant graded $A^{\tilde{\sigma}}$-bimodule. Similarly, $(M^{\tilde{\sigma}})^*$ is a kG-equivariant graded $A^{\tilde{\sigma}}$-bimodule.

By definition, as graded k-vector spaces,

$$(M^{\tilde{\sigma}})^* = M^*,$$

and we identify these below. We calculate the left and right $A^{\tilde{\sigma}}$-action on $(M^{\tilde{\sigma}})^*$. Let $x \in (M^{\tilde{\sigma}})^*$, $m \in M$ and $a \in A^{\tilde{\sigma}}$. Let $\circ$ denote the left and right $A^{\tilde{\sigma}}$-actions on $(M^{\tilde{\sigma}})^*$, and let $\langle \ \ , \ \ \rangle$ be the canonical bilinear form $M^* \times M \to k$. Then

$$\begin{aligned}\langle x \circ a, m\rangle &= \langle x, a \circledcirc m\rangle = \langle x, \sigma^{|m|}(a)m\rangle = \langle x\sigma^{|m|}(a), m\rangle \\ &= \langle x\sigma^{-|x|-|a|}(a), m\rangle, \qquad \text{since both sides are zero if } |m| + |x| + |a| \neq 0.\end{aligned}$$

Hence $x \circ a = x\sigma^{-|x|-|a|}(a)$. For the left action, we have

$$\begin{aligned}\langle a \circ x, m\rangle &= \langle x, m \circledcirc a\rangle = \langle x, \alpha_\sigma^{|a|}(m)a\rangle = \langle x, \alpha_\sigma^{|a|}(m\sigma^{-|a|}(a))\rangle \\ &= \langle (\alpha_\sigma^*)^{|a|}(x), m\sigma^{-|a|}(a)\rangle = \langle \sigma^{-|a|}(a)(\alpha_\sigma^*)^{|a|}(x), m\rangle.\end{aligned}$$

The natural kG-action on M^*, denoted by $\alpha_\gamma \mid_{M^*}$ for all $\gamma \in G$, satisfies

$$\alpha_\gamma \mid_{M^*} = (\alpha_{S(\gamma)})^* = [(\alpha_\gamma)^*]^{-1}$$

since $S(\gamma) = \gamma^{-1}$. Then

$$\langle \sigma^{-|a|}(a)(\alpha_\sigma^*)^{|a|}(x), m\rangle = \langle \sigma^{-|a|}(a)(\alpha_\sigma \mid_{M^*})^{-|a|}(x), m\rangle$$

and consequently,

$$a \circ x = \sigma^{-|a|}(a)(\alpha_\sigma \mid_{M^*})^{-|a|}(x).$$

Combining the two calculations above, it follows that the $A^{\tilde{\sigma}}$-bimodule $(M^{\tilde{\sigma}})^*$ has left and right $A^{\tilde{\sigma}}$ actions satisfying the rule

$$a \circ n \circ b = \sigma^{-|a|}(a)\alpha_\sigma^{-|a|}(n)\sigma^{-|a|-|n|-|b|}(b) \tag{E5.2.1}$$

for all homogeneous elements $a, b \in A = A^{\tilde{\sigma}}$ and homogeneous element $n \in N = M^*$. It is now straightforward to check that the function $\phi : n \to \alpha_\sigma^{-|n|}(n)$ satisfies $\phi(a \circledcirc n \circledcirc b) = a \circ \phi(n) \circ b$ for all homogeneous $n \in N$, $a, b \in A$. Thus ϕ gives an isomorphism from $(M^*)^{\tilde{\sigma}}$ to $(M^{\tilde{\sigma}})^*$ (identifying the underlying k-space of each with M^*). It is also clear that ϕ respects the left kG-action, since the σ_i are in the center of G. Thus ϕ is the required isomorphism of kG-equivariant $A^{\tilde{\sigma}}$-bimodules. $\square$

For any $\delta = (\delta_1, \cdots, \delta_w) \in (k^\times)^w$ and any $v = (v_1, \cdots, v_w) \in \mathbb{Z}^w$, write δ^v for $\prod_{s=1}^w \delta_s^{v_s}$. Given a $\delta \in (k^\times)^w$, a graded algebra automorphism ξ_δ of A is defined by

$$\xi_\delta(a) = \delta^{|a|} a$$

for all homogeneous elements $a \in A$. Note that ξ_δ is in the center of $\mathrm{Aut}_{\mathbb{Z}^w}(A)$. If $\delta = (c, c, \cdots, c)$, then $\xi_\delta = \xi_c$ as defined in the introduction.

Lemma 5.3. *Let A be $\mathbb{Z}^w$-graded generalized AS Gorenstein and suppose that G is some subgroup of $\mathrm{Aut}_{\mathbb{Z}^w}(A)$. Retain the notation as above.*

(a) *Suppose G consists of automorphisms of the form ξ_δ. Then every homogeneous generator $\mathfrak{e}$ of $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is kG-stable and the $\mathfrak{e}$-homological determinant satisfies $\mathrm{hdet}\, \xi_\delta = \delta^{\mathfrak{l}}$.*

(b) *If $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ has a kG-stable generator $\mathfrak{e}$, then every element of G commutes with the $\mathfrak{e}$-Nakayama automorphism μ (in $\mathrm{Aut}_{\mathbb{Z}^w}(A)$).*

Proof. (a) Every $\mathbb{Z}^w$-graded A-bimodule M has a canonical kG-equivariant bimodule structure where $\xi_\delta(x) = \delta^{|x|}x$ for all homogeneous x. If we take any graded injective resolution of $I^\bullet$ of A as left A^e-modules, then making each I^i into a kG-equivariant bimodule in the canonical way, it is easy to see that the morphisms in the complex automatically respect the kG action. Then $I^\bullet$ is already a graded R-module resolution of A, where $R = A^e \rtimes kG$ is the Kaygun algebra, and so $I^\bullet$ is also an injective resolution in $A - \mathrm{GrMod}$. As noted in the proof of Lemma 3.2, we can use $I^\bullet$ to calculate the R-module structure on $M := R^d\Gamma_{\mathfrak{m}_A}(A)$. Thus we see that the induced kG-action on M is also the canonical one.

For an element $g \in G$, the left g-action on $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is induced by the right g action on $R^d\Gamma_{\mathfrak{m}_A}(A)$, which is the same as the left $S(g) = g^{-1}$-action. Since also elements of $\mathrm{Hom}_k(M_v, k)$ have degree $-v$, we see that the left kG-action on $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is also the canonical one. Now let $\mathfrak{e}$ be any homogeneous generator of $(R^d\Gamma_{\mathfrak{m}}(A))^*$. Then the degree of $\mathfrak{e}$ is $\mathfrak{l}$ and the kG action on $\mathfrak{e}$ is the canonical one given by

$$\xi_\delta \cdot (\mathfrak{e}) = \delta^{\mathfrak{l}} \mathfrak{e}$$

for each $\xi_\delta \in G$. Thus $\mathfrak{e}$ is G-stable and $\mathrm{hdet}\, \xi_\delta = \delta^{\mathfrak{l}}$.

(b) This is a special case of (E3.10.2). $\square$

We are now ready to prove our main result determining the Nakayama automorphism of a graded twist.

Theorem 5.4. *Let A be $\mathbb{Z}^w$-graded generalized AS Gorenstein of AS index $\mathfrak{l}$. Let $G \subset \mathrm{Aut}_{\mathbb{Z}^w}(A)$, and assume that $\mathfrak{e}$ is a kG-stable generator of $R^d\Gamma_{\mathfrak{m}_A}(A)^*$. Let hdet_A be the $\mathfrak{e}$-homological determinant, and let μ_A be the $\mathfrak{e}$-Nakayama automorphism. Suppose that $\sigma = (\sigma_1, \ldots, \sigma_w)$ is a collection of graded automorphisms in the center of G, and let $\mathrm{hdet}(\sigma)$ denote the vector $(\mathrm{hdet}(\sigma_1), \cdots, \mathrm{hdet}(\sigma_w)) \in (k^\times)^w$. For convenience of notation write $\mathfrak{m} = \mathfrak{m}_A$ and $\mathfrak{m}^{\tilde{\sigma}} = \mathfrak{m}_{A^{\tilde{\sigma}}}$.*

(a) *The algebra $A^{\tilde{\sigma}}$ is generalized AS Gorenstein. Under the isomorphism*

$$\psi : (R^d\Gamma_{\mathfrak{m}^{\tilde{\sigma}}}(A^{\tilde{\sigma}}))^* \cong (R^d\Gamma_{\mathfrak{m}}(A)^{\tilde{\sigma}})^* \cong ((R^d\Gamma_{\mathfrak{m}}(A))^*)^{\tilde{\sigma}}$$

coming from Lemmas 5.1 and 5.2, $\mathfrak{e}' = \psi^{-1}(\mathfrak{e})$ is a kG-stable generator of $(R^d\Gamma_{\mathfrak{m}^{\tilde{\sigma}}}(A^{\tilde{\sigma}}))^*$. If $\mu_{A^{\tilde{\sigma}}}$ is the $\mathfrak{e}'$-Nakayama automorphism, then

$$\mu_{A^{\tilde{\sigma}}} = \mu_A \circ \sigma^{\mathfrak{l}} \circ \xi^{-1}_{\mathrm{hdet}(\sigma)}.$$

(b) *Let* $\mathrm{hdet}_{A^{\tilde{\sigma}}}$ *be the $\mathfrak{e}'$-homological determinant. Then* $\mathrm{hdet}_A(\tau) = \mathrm{hdet}_{A^{\tilde{\sigma}}}(\tau)$ *for all $\tau \in G$.*

(c) *If $\mu_A \in G$, then* $\mathrm{hdet}_{A^{\tilde{\sigma}}}\, \mu_{A^{\tilde{\sigma}}} = \mathrm{hdet}_A\, \mu_A$.

Proof. (a) Since the isomorphism $(R^d\Gamma_{\mathfrak{m}^{\tilde{\sigma}}}(A^{\tilde{\sigma}})) \cong (R^d\Gamma_{\mathfrak{m}}(A)^{\tilde{\sigma}})$ in Lemma 5.1 is an isomorphism of kG-equivariant bimodules, taking duals we also get an isomorphism respecting the kG-action. The isomorphism of Lemma 5.2 also respects the kG-action, so both isomorphisms making up ψ are isomorphisms of graded kG-equivariant bimodules. Thus e' must be a kG-stable generator of $(R^d\Gamma_{\mathfrak{m}^{\tilde{\sigma}}}(A^{\tilde{\sigma}}))^*$, of degree $\mathfrak{l}$.

By the generalized AS Gorenstein property for A, we have $R^d\Gamma_{\mathfrak{m}}(A)^* \cong {}^{\mu_A}A^1(-\mathfrak{l})$, and so ${}^{\mu_A}A^1(-\mathfrak{l})$ has a kG-stable generator corresponding to $\mathfrak{e} \in R^d\Gamma_{\mathfrak{m}}(A)^*$, which we also call $\mathfrak{e}$. Combining this isomorphism with ψ we have

$$(R^d\Gamma_{\mathfrak{m}^{\tilde{\sigma}}}(A^{\tilde{\sigma}}))^* \cong ({}^{\mu_A}A^1(-\mathfrak{l}))^{\tilde{\sigma}},$$

where $\mathfrak{e}'$ corresponds to the kG-stable generator $\mathfrak{e}$ of $({}^{\mu_A}A^1(-\mathfrak{l}))^{\tilde{\sigma}}$.

We calculate the left and right action on $({}^{\mu_A}A^1(-\mathfrak{l}))^{\tilde{\sigma}}$. For every $a \in A^{\tilde{\sigma}}$, we have

$$a \circledcirc \mathfrak{e} = \sigma^{\mathfrak{l}}(a)\mathfrak{e} = \mathfrak{e}\mu_A(\sigma^{\mathfrak{l}}(a))$$

and

$$\mathfrak{e} \circledcirc a = \alpha_\sigma^{|a|}(\mathfrak{e})a = \alpha_{\sigma^{|a|}}(\mathfrak{e})a = \mathfrak{e}\,\mathrm{hdet}(\sigma^{|a|})a$$

since by definition the action of α_σ on $\mathfrak{e}$ is given by the homological determinant. Hence $({}^{\mu_A}A^1(-\mathfrak{l}))^{\tilde{\sigma}}$ is isomorphic to ${}^{\phi}(A^{\tilde{\sigma}})^1$ where $\phi(\mathrm{hdet}(\sigma^{|a|})a) = \mu_A(\sigma^{\mathfrak{l}}(a))$. Thus

$$\phi = \mu_A \circ \sigma^{\mathfrak{l}} \circ \xi^{-1}_{\mathrm{hdet}(\sigma)}.$$

The formula for $\mu_{A^{\tilde{\sigma}}}$ follows. In particular, part (c) of the definition of generalized AS Gorenstein holds for $A^{\tilde{\sigma}}$, and the other parts of the definition are easy to check using the properties of graded twists [Zh].

(b) This follows immediately from the fact that $\mathfrak{e}'$ and $\mathfrak{e}$ correspond under an isomorphism of kG-equivariant bimodules.

(c) Since $\xi_{\mathrm{hdet}(\sigma)}$ is in the center of $\mathrm{Aut}_{\mathbb{Z}^w}(A)$ and $\mathfrak{e}$ is always $\xi_{\mathrm{hdet}(\sigma)}$-stable, we can replace G with the group generated by $\xi_{\mathrm{hdet}(\sigma)}$ and G without changing the hypotheses. Then by parts (a,b) and the fact that

$$\mathrm{hdet}(\sigma^{\mathfrak{l}} \circ \xi^{-1}_{\mathrm{hdet}(\sigma)}) = (\mathrm{hdet} \textstyle\prod_s \sigma_s^{\mathfrak{l}_s})(\prod_s \mathrm{hdet}(\sigma_s)^{\mathfrak{l}_s})^{-1} = 1,$$

the assertion follows. □

The proof of homological identity (HI2) is now an easy special case of the preceding theorem.

Proof of Theorem 0.3. Let G be the subgroup of $\mathrm{Aut}(A)$ generated by σ. Then G is abelian, so in particular σ is in the center of G. Since A is connected graded, any generator $\mathfrak{e}$ of $R^d\Gamma_{\mathfrak{m}_A}(A)^*$ is G-stable. The assertion follows from Theorem 5.4(a). $\square$

We close this section with a simple example.

Example 5.5. Let $A := k_{p_{ij}}[x_1, \cdots, x_w]$ be the skew polynomial ring generated by $x_1, \cdots, x_n$ subject to the relations

$$x_j x_i = p_{ij} x_i x_j$$

for all $i < j$, where $\{p_{ij}\}_{1\le i<j\le w}$ is a set of nonzero scalars. Then one checks immediately that A is isomorphic to the $\mathbb{Z}^w$-graded twist of the commutative polynomial ring $B = k[x_1, \ldots, x_w]$ by $\sigma = (\sigma_1, \cdots, \sigma_w)$ where σ_i is defined by

$$\sigma_i(x_s) = \begin{cases} p_{is} x_s & i < s \\ x_s & i \ge s \end{cases}, \quad \text{for all } s.$$

It is easy to see that $\mathfrak{l} = (1, 1, \ldots, 1)$, and of course $\mu_B = 1$. We calculate that $\sigma^{\mathfrak{l}}(x_s) = \prod_i \sigma_i(x_s) = \prod_{a<s} p_{as} x_s$ and using Lemma 5.3(a), we get that $\xi_{\mathrm{hdet}(\sigma)}(x_s) = \mathrm{hdet}(\sigma_s) x_s = \prod_{b>s} p_{sb} x_s$. Combining these calculations, by Theorem 5.4(a) we have

$$\mu_A(x_s) = (\prod_{a<s} p_{as} \prod_{b>s} p_{sb}^{-1}) x_s$$

for all $s = 1, 2, \cdots, w$.

If $w = 2$, $k_{p_{12}}[x_1, x_2]$ is a $\mathbb{Z}$-graded twist of $k[x_1, x_2]$. For $w \ge 3$, $k_{p_{ij}}[x_1, \cdots, x_w]$ is not in general a $\mathbb{Z}$-graded twist of any commutative ring. In fact, it is easy to check that $k_{p_{ij}}[x_1, \cdots, x_w]$ is a $\mathbb{Z}$-graded twist of B if and only if there is a set of nonzero scalars $\{p_1, \cdots, p_w\}$ such that $p_{ij} = p_i p_j^{-1}$ for all i, j. In particular, this example demonstrates why it is useful to consider the generality of $\mathbb{Z}^w$-graded twists as we have done in this section.

6. Proof of Identity (HI3)

The goal of this section is to prove homological identity (HI3), which we recall states that the homological determinant of the Nakayama automorphism is 1. We only consider the connected graded case for simplicity, and we prove it only for Koszul AS regular algebras in this paper. We do show, however, how our formula for the Nakayama automorphism of a graded twist from the previous section allows one to reduce to the case of an algebra with simple Nakayama automorphism of the form ξ_c. This same method may be useful to prove the result in general.

We start with a general lemma about tensor products of Gorenstein algebras. Let $\mathfrak{l}(A)$ denote the AS index of an AS Gorenstein algebra A.

Lemma 6.1. *Let A and B be noetherian connected graded AS Gorenstein algebras. Suppose that $A \otimes B$ is noetherian.*

(a) *The algebra $A \otimes B$ is AS Gorenstein and we have $\mu_{A\otimes B} = \mu_A \otimes \mu_B$ and $\mathfrak{l}(A \otimes B) = \mathfrak{l}(A) + \mathfrak{l}(B)$ as $\mathbb{N}$-graded algebras. The algebra $A \otimes B$ is also $\mathbb{Z}^2$-graded, where the (i, j) graded piece is $A_i \otimes B_j$, and with this grading, $\mathfrak{l}(A \otimes B) = (\mathfrak{l}(A), \mathfrak{l}(B))$.*

(b) *Let $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(A)$, $\tau \in \mathrm{Aut}_{\mathbb{Z}}(B)$. Then $\mathrm{hdet}_{A\otimes B}(\sigma\otimes\tau) = (\mathrm{hdet}_A \sigma)(\mathrm{hdet}_B \tau)$.*

Proof. (a) Let $d_1 = \operatorname{injdim} A$ and $d_2 = \operatorname{injdim} B$. By [VdB1, Theorem 7.1],

$$R^{d_1+d_2}\Gamma_{\mathfrak{m}_{A\otimes B}}(A\otimes B) \cong R^{d_1}\Gamma_{\mathfrak{m}_A}(A)\otimes R^{d_2}\Gamma_{\mathfrak{m}_B}(B), \tag{E6.1.1}$$

and this is a rigid dualizing complex for $A\otimes B$. Since A and B are AS Gorenstein, we have $R^{d_1}\Gamma_{\mathfrak{m}_A}(A)^* \cong {}^{\mu_A}A^1(-\mathfrak{l}(A))$ and $R^{d_2}\Gamma_{\mathfrak{m}_B}(B)^* \cong {}^{\mu_B}B^1(-\mathfrak{l}(B))$, and thus $R^{d_1+d_2}\Gamma_{\mathfrak{m}_{A\otimes B}}(A\otimes B)^* \cong {}^{\mu_A\otimes\mu_B}(A\otimes B)^1(-\mathfrak{l}(A)-\mathfrak{l}(B))$. Now we note that $A\otimes B$ satisfies Definition 3.3, since the χ condition for $A\otimes B$ is part of the existence of the dualizing complex ([VdB1, Theorem 6.3]), and $A\otimes B$ must have finite injective dimension since a dualizing complex always does by definition. Thus $A\otimes B$ is generalized AS Gorenstein and so is also AS Gorenstein in the usual sense by Remark 3.6. We have already calculated its Nakayama automorphism and AS index above. The proof of the multigraded result is the same, since (E6.1.1) also holds as $\mathbb{Z}^2$-graded modules.

(b) Let $G_1 = \langle\sigma\rangle$ and $G_2 = \langle\tau\rangle$. Then $G = G_1\times G_2$ is naturally a subgroup of $\operatorname{Aut}_{\mathbb{Z}}(A\otimes B)$. Recall that the kG structure on $R^{d_1+d_2}\Gamma_{\mathfrak{m}_{A\otimes B}}(A\otimes B)$ can be calculated using an $(A\otimes B)\#kG$-injective resolution of $A\otimes B$, as in Remark 3.8. If $I^\bullet$ is an $A\#kG_1$ graded injective resolution of A and $J^\bullet$ is an $B\#kG_2$ graded injective resolution of B, then the tensor product complex $I^\bullet\otimes_k J^\bullet$ is a resolution of $A\otimes B$ in the category of graded $(A\otimes B)\#k(G_1\times G_2)$-modules. By [VdB1, Lemma 4.5], we have $R\Gamma_{\mathfrak{m}_{A\otimes B}} = R\Gamma_{\mathfrak{m}_A}\circ R\Gamma_{\mathfrak{m}_B}$. Also, since we assume A and B are noetherian, direct sums of injective modules over these rings are injective. Then each term of $I^\bullet\otimes_k J^\bullet$ is an injective B-module. Applying $\Gamma_{\mathfrak{m}_B}$ the resulting complex $I^\bullet\otimes\Gamma_{\mathfrak{m}_B}(J^\bullet)$ consists of injective A-modules and so applying $\Gamma_{\mathfrak{m}_A}$ we get that $\Gamma_{\mathfrak{m}_A}(I^\bullet)\otimes\Gamma_{\mathfrak{m}_B}(J^\bullet)$ is equal to $R\Gamma_{\mathfrak{m}_{A\otimes B}}(A\otimes B)$. Now by construction, the action of $G_1\times G_2$ on the complex

$$\Gamma_{\mathfrak{m}_A}(I^\bullet)\otimes\Gamma_{\mathfrak{m}_B}(J^\bullet)\cong {}^{\mu_A}A^1(-\mathfrak{l}(A))[-d_1]\otimes {}^{\mu_B}B^1(-\mathfrak{l}(B))[-d_2]$$

simply comes from the obvious action induced by G_1 acting on the first tensor component and G_2 acting on the second tensor component. Taking k-linear duals, the result follows from the definition of hdet. □

We now show how to pass from an arbitrary AS Gorenstein algebra to a closely related one with simpler Nakayama automorphism.

Lemma 6.2. *Let A be a connected graded AS Gorenstein algebra. Possibly after replacing the base field k with a finite extension, there is a connected graded AS Gorenstein algebra B such that* $\operatorname{hdet}\mu_A = \operatorname{hdet}\mu_B$*, and where $\mu_B = \xi_c$ for some $c\in k$. Moreover, B is a multigraded twist of a commutative polynomial extension of A.*

Proof. Consider first a noetherian connected $\mathbb{N}^2$-graded generalized AS Gorenstein algebra C with AS index $\mathfrak{l} = (l,1)\neq(-1,1)$, and let μ_C be the Nakayama automorphism of C. Define $\sigma = \{Id, \mu_C^{-1}\}$. By Theorem 5.4(a), $\mu_{C^{\tilde\sigma}} = \xi^{-1}_{\operatorname{hdet}\sigma} = \xi_{1,c}$ where $c = \operatorname{hdet}\mu_C$. Since $l\neq -1$, write $c = d^{l+1}$ for some $d\in k$, which we can do after replacing k by a finite extension field if necessary. (It is not hard to see that all properties of the algebras are preserved by a finite extension of the base field.) Since hdet is independent of the grading, we now view $D = C^{\tilde\sigma}$ as a connected $\mathbb{N}$-graded algebra. Let $\tau = \xi_{1,d^{-1}}$. Then by Theorem 5.4(a), $\mu_{D^{\tilde\tau}} = \xi^{-1}_{\operatorname{hdet}\tau} = \xi_{c'}$ for some $c'\in k^\times$. Finally, by Theorem 5.4(c), $\operatorname{hdet}\mu_{D^{\tilde\tau}} = \operatorname{hdet}\mu_D = \operatorname{hdet}\mu_C$.

Now suppose that A is a noetherian connected graded AS Gorenstein algebra with AS index $\mathfrak{l} \neq -1$. Let $C = A \otimes k[x]$, which is a noetherian $\mathbb{Z}^2$-graded generalized AS Gorenstein algebra with AS index $(\mathfrak{l}, 1) \neq (-1, 1)$, by Lemma 6.1(a). By the previous paragraph, there is an algebra B (which is a sequence of graded twists of C) such that $\operatorname{hdet} \mu_B = \operatorname{hdet} \mu_A$ and $\mu_B = \xi_c$ for some c. Note that B is still AS Gorenstein, since polynomial extensions preserve this property by Lemma 6.1(a), and it is standard that graded twists do also.

If instead A is noetherian connected graded AS Gorenstein algebra with AS index $\mathfrak{l} = -1$, then first replace A by $A[y]$ which is connected graded AS Gorenstein with AS index $\mathfrak{l}' = \mathfrak{l}+1 \neq -1$, and has $\operatorname{hdet} \mu_{A[y]} = \operatorname{hdet} \mu_A$ by Lemma 6.1(b), since obviously $\mu_{k[y]} = 1$. Then proceed as in the previous paragraph.

In all cases, we see that B exists as stated, where B is a series of (multi)-graded twists of a polynomial extension of A in one or two variables. □

Now we are ready to prove homological identity (HI3). The main idea is that for Koszul AS regular algebras A, the Nakayama automorphism of A is related in a known way to the Nakayama automorphism of its Ext algebra E, which is Frobenius, and so Lemma 3.12 applies.

Theorem 6.3. *Let A be a noetherian connected graded Koszul AS regular algebra. Then* $\operatorname{hdet} \mu_A = 1$.

Proof. Let d be the global dimension of A. Let $E = E(A) = \bigoplus_{i \geq 0} \operatorname{Ext}^i_A(k, k)$ be the Ext algebra of A. Since A is Koszul, we may write$A = T(V)/(R)$ for a space of relations $R \subseteq V^{\otimes 2}$, and E is isomorphic to the Koszul dual $A^!$ of A, $T(V^*)/(R^\perp)$, which is a finite-dimensional algebra generated in degree 1. Moreover, since A is AS regular, it is known that the algebra E is a Frobenius k-algebra [Sm]; Let ν denote its (classical) Nakayama automorphism. Then by [VdB1, Theorem 9.2] and Lemma 3.5, the AS index of A is $\mathfrak{l} = d$ and one has

$$\mu_A|_V = \xi_{-1}^{d+1} \circ (\nu|_{V^*})^*.$$

By Lemma 6.2, by extending the base field if necessary and replacing A with a multigraded twist of a polynomial extension of A, we may reduce to the case that $\mu_A = \xi_c$ for some c. (Note that all of these changes preserve the AS regular Koszul hypothesis.) Then by the formula above, since $\mu_A|_V$ is multiplication by c, we get that $\nu|_{V^*}$ is multiplication by $(-1)^{d+1}c$. Since E is generated in degree 1, ν acts on the 1-dimensional degree $d = \mathfrak{l}$ piece by the scalar $((-1)^{d+1}c)^d = (-1)^{d(d+1)}c^d = c^d$. By Lemma 3.12, $\operatorname{hdet} \nu = c^d = 1$. On the other hand, we also have that $\operatorname{hdet} \mu_A = c^{\mathfrak{l}} = c^d$ by Lemma 5.3, so $\operatorname{hdet} \mu_A = 1$. □

As already alluded to in the introduction, we conjecture that the following more general result holds.

Conjecture 6.4. *Let A be a noetherian connected graded AS Gorenstein algebra. Then* $\operatorname{hdet} \mu_A = 1$.

7. Applications

We explore some applications of our results above in this section, including Corollaries 0.6 and 0.7. We concentrate here on connected $\mathbb{Z}^w$-graded AS Gorenstein algebras A. Some of these results depend on knowing that $\operatorname{hdet} \mu_A = 1$, as in Conjecture 6.4.

We start with some applications to the calculation of the homological determinant. Note that Corollary 0.5 is a special case of part (b) of the next result.

Lemma 7.1. *Suppose that Conjecture 6.4 holds. Let A be a noetherian connected graded AS Gorenstein algebra.*

(a) *Suppose that z is a homogeneous μ_A-normal nonzerodivisor of positive degree in A, so that $\mu_A(z) = cz$ for some $c \in k^\times$, and let $\tau \in \mathrm{Aut}_{\mathbb{Z}}(A)$ be the automorphism such that $za = \tau(a)z$ for all $a \in A$. Then*
$$\operatorname{hdet} \tau = c.$$
In particular, if A has trivial Nakayama automorphism, then $\operatorname{hdet} \tau = 1$.

(b) *Let $\varphi \in \mathrm{Aut}_{\mathbb{Z}}(A)$. Then*
$$\operatorname{hdet} \varphi = \mu_{A[t;\varphi]}(t)\ t^{-1}.$$

Proof. (a) First note that $\tau(z) = z$. By Lemma 1.5, $\mu_{A/(z)} = (\mu_A \circ \tau) \mid_{A/(z)}$. Since $A/(z)$ is also AS Gorenstein, by assumption we have that $\operatorname{hdet}_{A/(z)} \mu_{A/(z)} = \operatorname{hdet}_{A/(z)}(\mu_A \circ \tau) = 1$. By (E3.9.1),
$$\operatorname{hdet}_A(\mu_A \circ \tau) = c \operatorname{hdet}_{A/(z)}(\mu_A \circ \tau) \mid_{A/(z)} = c$$
as $(\mu_A \circ \tau)(z) = \mu_A(z) = cz$. Since $\operatorname{hdet}_A \mu_A = 1$, we have
$$\operatorname{hdet}_A \tau = \operatorname{hdet}_A(\mu_A \circ \tau) = c.$$
The last sentence is a special case.

(b) Let $B = A[t; \varphi]$. Note that B is $\mathbb{Z}^2$-graded, with degree (i, j)-piece $A_i t^j$. Thus μ_B is a $\mathbb{Z}^2$-graded automorphism, so $\mu_B(t) = ct$ for some $c \in k^\times$. Let $\tau \in \mathrm{Aut}_{\mathbb{Z}}(B)$ be defined by $\tau(at^n) = \varphi(a)t^n$ for all $a \in A$ and $n \geq 0$. Thus $tb = \tau(b)t$ for all $b \in B$. By (E3.9.1),
$$\operatorname{hdet}_B \tau = 1 \operatorname{hdet}_A \varphi = \operatorname{hdet}_A \varphi$$
since $\tau(t) = 1t$. Now applying part (a), one sees that
$$\operatorname{hdet}_A \varphi = \operatorname{hdet}_B \tau = c = \mu_B(t)t^{-1},$$
as claimed. □

One of our goals for future work is to explore how one might define homological determinant for automorphisms of not necessarily connected graded algebras. This suggests the following question.

Question 7.2. Do Theorems 0.2 and 0.3 suggest a way to define the homological determinant hdet in a more general setting? For example, $\mu_{A\#H}(\mu_A\#\mu_H)^{-1}$ should be $1\#\Xi^l_{\mathrm{hdet}}$. If A is local (not graded), this could be a way of defining hdet.

Another possibility is to use the formula in Proposition 7.5(c),
$$\operatorname{hdet} \varphi = \mu_{A[t;\varphi]}(t)t^{-1}.$$
In the ungraded case, one can show that $\mu_{A[t;\varphi]}(t)t^{-1} \in A^\times$ for any $\varphi \in \mathrm{Aut}(A)$.

We next study several constructions which help to produce algebras with a trivial Nakayama automorphism, in particular, CY algebras.

Proposition 7.3. *Let A be a noetherian connected graded AS regular algebra with* $\operatorname{hdet} \mu_A = 1$. *Then the algebras $B = A[t; \mu_A]$ and $B' = A[t^{\pm 1}; \mu_A]$ are CY algebras.*

Proof. Since A is AS regular, it is well-known that the Ore extension $B = A[t;\mu_A]$ is also. So $A[t;\mu_A]$ is skew CY by Lemma 1.2. Let $C = A[t]$, which as in the proof of Lemma 7.1(b) is $\mathbb{Z}^2$ graded. Let $\sigma = (Id_C, \mu_C^{-1})$. Then the AS index of C is $\mathfrak{l}(C) = (\mathfrak{l}(A), 1)$ by Lemma 6.1(a), and $\operatorname{hdet}\mu_C = \operatorname{hdet}\mu_A = 1$, by Lemma 6.1(b). Now by Theorem 5.4(a),

$$\mu_{C^{\tilde{\sigma}}} = \mu_C \circ \sigma^{\mathfrak{l}} \circ \xi^{-1}_{\operatorname{hdet}(\sigma)} = Id_C.$$

Thus $C^{\tilde{\sigma}}$ is CY. Finally, one may check that the function $\psi : C^{\tilde{\sigma}} \to A[t;\mu_A]$ given by the formula $\sum_i a_i t^i \mapsto \sum_i \mu^i(a_i)t^i$ is an isomorphism of rings.

(b) The ring B' is the localization of B at the set of powers of the normal nonzerodivisor t. It is standard that such a localization of a CY algebra remains CY. □

Proof of Corollary 0.6. (a) This is Proposition 7.3(a).

(b) If μ_A has infinite order, the assertion is equivalent to Proposition 7.3(b). If μ_A has finite order, then the assertion follows from Corollary 4.2 by taking $n = 1$.

(c) Since $\mu = \mu_A$ has finite order and $\operatorname{hdet}\mu = 1$, the invariant subring A^G is AS Gorenstein by [JoZ, Theorem 3.3], where G is the finite group $\langle\mu\rangle$. Let $C = A^G$ and let $\int = \frac{1}{|G|}\sum_{g\in G} g$. By [KKZ, (E3.1.1), Lemmas 3.2 and 3.5(d)],

$$R^d\Gamma_{\mathfrak{m}_C}(C)^* = R^d\Gamma_{\mathfrak{m}}(A)^* \cdot \textstyle\int = (R^d\Gamma_{\mathfrak{m}}(A)^*)^G = ({}^{\mu}A^1(-\mathfrak{l}))^G$$

as C-bimodules, where $\mathfrak{l} = \mathfrak{l}(A)$. Note that the restriction of μ to C is the identity. Hence

$$({}^{\mu}A^1(-\mathfrak{l}))^G = {}^1C^1(-\mathfrak{l})$$

and therefore the Nakayama automorphism of C is trivial by Lemma 3.5. □

We remark that if A is PI, then Corollary 0.6(a) is a special case of a result of Stafford-Van den Bergh [SVdB, Proposition 3.1].

The results above strongly suggest the following question.

Question 7.4. Do the conclusions of Corollary 0.6 hold for ungraded skew CY algebras A? In particular, when A is skew CY, is $A \rtimes \langle\mu_A\rangle$ always CY?

We have already seen in the proof of Lemma 6.2 how our formula for the Nakayama automorphism of a graded twist may allow one to pass to a twist equivalent algebra with simpler Nakayama automorphism. We now offer several similar results about how close one might be able to get to a CY algebra through twisting.

Proposition 7.5. *Let A be a noetherian connected graded AS Gorenstein algebra.*

(a) *Assume k is algebraically closed. Suppose that A is $\mathbb{Z}^w$-graded with a set of generators $x_1, \cdots, x_w$ such that $\deg x_i$ is the ith unit vector in $\mathbb{Z}^w$. If $\mathfrak{l}(A) \neq 0$, then there is an automorphism $\sigma \in \operatorname{Aut}_{\mathbb{Z}^w}(A)$ such that $\mu_{A^{\tilde{\sigma}}} = \xi_c$ for some $c \in k^\times$.*

(b) *Let $\varphi \in \operatorname{Aut}_{\mathbb{Z}}(A)$ and assume that $\operatorname{hdet}\mu_A = 1$. Then there is a $\mathbb{Z}^2$-graded twist of $A[t;\varphi]$ that has trivial Nakayama automorphism.*

Proof. (a) Since A is $\mathbb{Z}^w$-graded, its Nakayama automorphism μ_A is $\mathbb{Z}^w$-graded. Thus $\mu_A(x_i) = a_i x_i$ for all $i = 1, \cdots, w$. Viewing A as a connected graded algebra we assume that $\mathfrak{l} := \mathfrak{l}(A) \neq 0$. Since k is algebraically closed, there are $\delta_i \in k^\times$ such that $\delta_i^{-\mathfrak{l}} = a_i$ for all i. Then $\sigma^{-\mathfrak{l}} = \mu_A$ where $\sigma = \xi_\delta \in \operatorname{Aut}_{\mathbb{Z}^w}(A) \subset \operatorname{Aut}_{\mathbb{Z}}(A)$. The assertion follows from Theorem 5.4(a) (with $w = 1$).

(b) By Proposition 7.3, $A[t;\mu_A]$ is CY. Since μ_A is in the center of $\mathrm{Aut}_{\mathbb{Z}}(A)$ by Theorem 3.11, a similar argument as in the proof of Proposition 7.3 shows that $A[t;\mu_A]$ is a $\mathbb{Z}^2$-graded twist of $A[t;\varphi]$. □

For the rest of this section we prove Corollary 0.7. From now on we assume that k is algebraically closed of characteristic 0. We use [Bor] as a reference for algebraic groups. The following two lemmas are presumably known, but we sketch the proofs since we lack a reference.

Lemma 7.6. *Let A be a finitely generated $\mathbb{Z}^w$-graded algebra, which is locally finite and connected $\mathbb{N}$-graded with respect to the $||\ \ ||$-grading. Then $\mathrm{Aut}_{\mathbb{Z}^w}(A)$ is an affine algebraic group over k.*

Proof. Let A be generated as an algebra by homogeneous elements $\{a_i\}_{i=1}^s$, where $a_i \in A_{\alpha_i}$. Choose a graded presentation $F/I \cong A$, where F is a $\mathbb{Z}^w$-graded free algebra with homogeneous generators x_i which map to the a_i. There is a natural injection $i : \mathrm{Aut}_{\mathbb{Z}^w}(A) \to \prod_{i=1}^s \mathrm{GL}(A_{\alpha_i})$, where $\tau \in \mathrm{Aut}_{\mathbb{Z}^w}(A)$ corresponds to the product of the bijections it induces of the graded subspaces containing the generating set, since any $\tau \in \mathrm{Aut}_{\mathbb{Z}^w}(A)$ is determined by the elements $\tau(a_i)$. Conversely, choosing arbitrary $b_i \in A_{\alpha_i}$, there is an automorphism τ in $\mathrm{Aut}_{\mathbb{Z}^w}(A)$ with $\tau(a_i) = b_i$ if and only if for every relation $r = \sum d_{j_1,\dots,j_k} x_{j_1} \dots x_{j_k} \in I$, we have $\sum d_{j_1,\dots,j_k} b_{j_1} \dots b_{j_k} = 0$ in A. It is easy to see that for any relation r this gives a closed condition on the choice of b_i, and intersecting over all relations (or a generating set of the ideal of relations) we get that i is a closed regular embedding. The map i is clearly a group homomorphism, so $\mathrm{Aut}_{\mathbb{Z}^w}(A)$ is an affine algebraic group. □

Lemma 7.7. *Let $d \neq 0$. Let G be an abelian affine algebraic group, and let G^0 be the connected component of the identity.*

(a) *If G is connected, then for any $x \in G$, there exists $y \in G$ such that $y^d = x$.*
(b) *For any $x \in G$, there exists $z \in G$ of finite order and $y \in G^0$ such that $xz \in G^0$ and $xz = y^d$.*

Proof. (a) Since G is abelian, the map $\phi : G \to G$ defined by $\phi(y) = y^d$ is a homomorphism of algebraic groups. The image H is a closed algebraic subgroup of G. Hence the quotient algebraic group $P = G/H$ exists, and this is again affine [Bor, Theorem 6.8]. It is also still abelian and connected since G is. Note that $z^d = e$ for all $z \in P$. We claim that P is the trivial group $\{e\}$. Since the elements of P are all dth roots of e, it follows that P consists of semisimple elements under any embedding of P in a matrix group. Then P can be embedded in a torus $(k^\times)^n$ for some n, by [Bor, Proposition 8.4]. Since P is connected and a torus contains only finitely many elements of order d, we must have that P is a point. In other words, ϕ is surjective and the result follows.

(b) We just have to produce an element z of finite order such that $xz \in G^0$. The existence of y follows from part (a). Let H be the connected component containing x. Choose any $w \in H$ and $m \geq 1$ such that $w^m \in G^0$. By part (a), there is $v \in G^0$ such that $v^m = w^m$. Then $z = vw^{-1}$ satisfies $z^m = e$, so z has finite order. Moreover, clearly $xz \in G^0$, since $x \in H$, $w \in H$, and $v \in G^0$. □

Theorem 7.8. *Let A be a noetherian connected graded AS Gorenstein algebra with $\mathfrak{l}(A) \neq 0$ and with $\mathrm{hdet}\, \mu_A = 1$. Then there is a $\sigma \in \mathrm{Aut}_{\mathbb{Z}}(A)$ such that $\mu_{A^{\tilde{\sigma}}}$ is of finite order.*

Proof. By Theorem 3.11, μ_A is in the center Z of $\mathrm{Aut}_{\mathbb{Z}}(A)$. Note that $\mathrm{Aut}_{\mathbb{Z}}(A)$ is an affine algebraic group by Lemma 7.6 and that Z is a closed algebraic subgroup of $\mathrm{Aut}_{\mathbb{Z}}(A)$. It suffices to work in Z for the rest of the proof.

Let $x = \mu_A^{-1}$. By Lemma 7.7(b), there is an $z \in Z$ of finite order and $y \in Z^0$, such that $xz \in Z^0$ and $y^{\mathfrak{l}} = xz$ where $\mathfrak{l} = \mathfrak{l}(A)$. Let $\sigma = y$. Then by Theorem 5.4(a),

$$\mu_{A^{\bar{\sigma}}} = \mu_A \circ \sigma^{\mathfrak{l}} \circ \xi^{-1}_{\mathrm{hdet}(\sigma)} = z \circ \xi^{-1}_{\mathrm{hdet}(\sigma)}.$$

Since $y^{\mathfrak{l}} = xz$, we have

$$\mathrm{hdet}(\sigma)^{\mathfrak{l}} = \mathrm{hdet}(y^{\mathfrak{l}}) = \mathrm{hdet}\, x \,\mathrm{hdet}\, z = \mathrm{hdet}\, z$$

since $\mathrm{hdet}\, x = (\mathrm{hdet}\, \mu_A)^{-1} = 1$ by assumption. Since z has finite order, $\mathrm{hdet}\, z$ has finite order. Thus $\mathrm{hdet}(\sigma)$ has finite order. Finally, we see that $\mu_{A^{\bar{\sigma}}}$ has finite order. □

We conclude the paper with the proof of the last corollary from the introduction.

Proof of Corollary 0.7. Since A is AS regular, $\mathfrak{l}(A) > 0$ [SteZh, Proposition 3.1]. The assertion follows from Theorem 7.8. □

References

[ASc] M. Artin and W.F. Schelter, *Graded algebras of global dimension 3*, Adv. in Math. **66** (1987), no. 2, 171–216.

[AZ] M. Artin and J.J. Zhang, *Noncommutative projective schemes*, Adv. Math. **109** (1994), no. 2, 228–287.

[BM] R. Berger and N. Marconnet, *Koszul and Gorenstein properties for homogeneous algebras*, Algebr. Represent. Theory **9** (2006), no. 1, 67–97.

[Boc] R. Bocklandt, *Graded Calabi Yau algebras of dimension 3*, J. Pure Appl. Algebra **212** (2008), no. 1, 14–32.

[BSW] R. Bocklandt, T. Schedler and M. Wemyss, *Superpotentials and higher order derivations*, J. Pure Appl. Algebra **214** (2010), no. 9, 1501–1522,

[Bor] A. Borel, *Linear algebraic groups*, second ed., Springer-Verlag, New York, 1991.

[Br] K.A. Brown, *Noetherian Hopf algebras*, Turkish J. Math. **31** (2007), suppl., 7–23.

[BZ] K.A. Brown and J.J. Zhang, *Dualising complexes and twisted Hochschild (co)homology for Noetherian Hopf algebras*, J. Algebra **320** (2008), no. 5, 1814–1850.

[CWZ] K. Chan, C. Walton and J.J. Zhang, *Hopf actions and Nakayama automorphisms*, preprint (2012), arXiv:1210.6432v1.

[Fa] M. Farinati, *Hochschild duality, localization, and smash products*, J. Algebra **284** (2005), no. 1, 415–434.

[DNR] S. Dăscălescu, C. Năstăsescu, and Ş. Raianu, "Hopf Algebras: An Introduction", Monographs and Textbooks in Pure and Applied Mathematics, vol. 235, Marcel Dekker, Inc., New York, 2001.

[DV] M. Dubois-Violette, *Multilinear forms and graded algebras*, J. Algebra **317** (2007), no. 1, 198–225.

[FMS] D. Fischman, S. Montgomery and H.-J. Schneider, Frobenius extensions of subalgebras of Hopf algebras, Trans. Amer. Math. Soc. **349** (1997), no. 12, 4857–4895.

[FV] Gunnar Fløystad and Jon Eivind Vatne, *Artin-Schelter regular algebras of dimension five*, Algebra, geometry and mathematical physics, Banach Center Publ., vol. 93, Polish Acad. Sci. Inst. Math., Warsaw, 2011, pp. 19–39.

[Gi] V. Ginzburg, *Calabi-Yau algebras*, arXiv:math/0612139 v3, 2006.

[IR] O. Iyama and I. Reiten, *Fomin-Zelevinsky mutation and tilting modules over Calabi-Yau algebras*, Amer. J. Math. **130** (2008), no. 4, 1087–1149.

[JiZ] N. Jing and J.J. Zhang, *Gorensteinness of invariant subrings of quantum algebras*, J. Algebra **221** (1999), no. 2, 669–691.

[JoZ] P. Jorgensen and J.J. Zhang, *Gourmet's guide to Gorensteinness*, Adv. Math. **151** (2000), no. 2, 313–345.

[Kay] A. Kaygun, *Hopf-Hochschild (co)homology of module algebras*, Homology, Homotopy Appl. **9** (2007), no. 2, 451–472.

[Ke] B. Keller, *Calabi-Yau triangulated categories*, Trends in representation theory of algebras and related topics, 467-489, EMS Ser. Congr. Rep., *Eur. Math. Soc., Zürich*, 2008.

[KK] E. Kirkman and J. Kuzmanovich, *Fixed subrings of Noetherian graded regular rings*, J. Algebra **288** (2005), no. 2, 463–484.

[KKZ] E. Kirkman, J. Kuzmanovich and J.J. Zhang, *Gorenstein subrings of invariants under Hopf algebra actions*, J. Algebra **322** (2009), no. 10, 3640–3669.

[Ko] M. Kontsevich, *Triangulated categories and geometry*, Course at the École Normale Supérieure, Paris, Notes taken by J. Bellaïche, J.-F. Dat, I. Marin, G. Racinet and H. Randriambololona, 1998.

[La] T. Y. Lam, *Lectures on Modules and Rings*, Graduate Texts in Mathematics, 189, Springer-Verlag, New York, 1999.

[LM] P. Le Meur, *Crossed productd of Calabi-Yau algebras by finite groups*, preprint, (2010) arXiv:1006.1082.

[LiWZ] L.-Y. Liu, Q.-S. Wu and C. Zhu, *Hopf Action on Calabi-Yau algebras*, New trends in noncommutative algebra, 189–209, Contemp. Math., 562, Amer. Math. Soc., Providence, RI, 2012.

[Lo] Martin Lorenz, *Representations of finite-dimensional Hopf algebras*, J. Algebra **188** (1997), no. 2, 476–505.

[LuWZ1] D.-M. Lu, Q.-S. Wu and J.J. Zhang, *Hopf algebras with rigid dualizing complexes*, Israel J. Math. **169** (2009), 89–108.

[LuWZ2] D.-M. Lu, Q.-S. Wu and J.J. Zhang, *Homological integral of Hopf algebras*, Trans. Amer. Math. Soc. **359** (2007), no. 10, 4945–4975.

[Lu] V. A. Lunts, *Categorical resolution of singularities*, J. Algebra **323** (2010), no. 10, 2977–3003.

[M-VS] R. Martinez-Villa and O. Solberg, *Artin-Schelter regular algebras and categories*, J. Pure Appl. Algebra **215** (2011), no. 4, 546–565.

[MM] H. Minamoto and I. Mori, *The structure of AS-Gorenstein algebras*, Adv. Math. **226** (2011), no. 5, 4061–4095.

[Mo] S. Montgomery, "Hopf Algebras and Their Actions on Rings", CBMS Reg. Conf. Ser. Math., vol. 82, Amer. Math. Soc., Providence, RI, 1993.

[Mu] W. Murray, *Bilinear forms on Frobenius algebras*, J. Algebra **293** (2005), no. 1, 89–101.

[RRZ] M. Reyes, D. Rogalski and J.J. Zhang, *Homological determinant for non-connected skew Calabi-Yau algebras*, in preparation.

[Sm] S. Paul Smith, *Some finite-dimensional algebras related to elliptic curves*, in "Representation theory of algebras and related topics (Mexico City, 1994)," 315–348, CMS Conf. Proc. 19, Amer. Math. Soc., Providence, RI, 1996.

[SVdB] J.T. Stafford and M. Van den Bergh, *Noncommutative resolutions and rational singularities*, Special volume in honor of Melvin Hochster, Michigan Math. J. **57** (2008), 659–674.

[SteZh] Darin R. Stephenson and James J. Zhang, *Growth of graded Noetherian rings*, Proc. Amer. Math. Soc. **125** (1997), no. 6, 1593–1605.

[VdB1] M. Van den Bergh, *Existence theorems for dualizing complexes over non-commutative graded and filtered rings*, J. Algebra **195** (1997), no. 2, 662–679.

[VdB2] M. Van den Bergh, *A relation between Hochschild homology and cohomology for Gorenstein rings* Proc. Amer. Math. Soc. 126 (1998), no. 5, 1345–1348; *Erratum to: "A relation between Hochschild homology and cohomology for Gorenstein rings"*, Proc. Amer. Math. Soc. 130 (2002), no. 9, 2809-2810.

[VdB3] M. Van den Bergh, *Calabi-Yau algebras and superpotentials*, preprint, (2010), arXiv:1008.0599.

[WZ] Q.-S. Wu and J. J. Zhang, *Dualizing complexes over noncommutative local rings*, J. Algebra **239** (2001), no. 2, 513–548.

[WZhu] Q.-S. Wu and C. Zhu, *Skew group algebras of Calabi-Yau algebras*, J. Algebra **340** (2011), 53–76.

[Ye] A. Yekutieli, *The rigid dualizing complex of a universal enveloping algebra*, J. Pure Appl. Algebra **150** (2000), no. 1, 85–93.

[YeZ1] A. Yekutieli and J.J. Zhang, *Rings with Auslander dualizing complexes*, J. Algebra **213** (1999), no. 1, 1–51.

[YeZ2] A. Yekutieli and J.J. Zhang, *Homological transcendence degree*, Proc. London Math. Soc. (3) **93** (2006), no. 1, 105–137.

[YuZ1] X.L. Yu and Y.H. Zhang, *Calabi-Yau pointed Hopf algebras of finite Cartan type,* preprint (2011), arXiv:1111.3342v2.

[YuZ2] X.L. Yu and Y.H. Zhang, *The Calabi-Yau property of Hopf algebras and braided Hopf algebras*, preprint (2011), arXiv:1111.4201.

[Zh] J.J. Zhang, *Twisted graded algebras and equivalences of graded categories,* Proc. London Math. Soc. (3) **72** (1996), no. 2, 281–311.

(Reyes) Bowdoin College, Department of Mathematics, 8600 College Station, Brunswick, ME 04011-8486

E-mail address: reyes@bowdoin.edu

(Rogalski) UCSD, Department of Mathematics, 9500 Gilman Dr. #0112, La Jolla, CA 92093-0112, USA.

E-mail address: drogalsk@math.ucsd.edu

(Zhang) University of Washington, Department of Mathematics, Box 354350, Seattle, WA 98195-4350, USA.

E-mail address: zhang@math.washington.edu