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Orbit Equivalence and Von Neumann Rigidity for Actions of Wreath Product Groups

A dissertation submitted in partial satisfaction of the
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James Owen Sizemore

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Abstract of the Dissertation

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Professor Sorin Popa, Chair

We use deformation-rigidity theory in the von Neumann algebra framework to study probability measure preserving actions by wreath product groups and their associated von Neumann algebras. In particular, we single out large families of wreath product groups satisfying various types of orbit equivalence (OE) rigidity.For instance, we show that whenever H, K, Γ, Λ are icc, property (T) groups such that $H \wr \Gamma$ and $K \wr \Lambda$ admit stably orbit equivalent action σ and ρ such that $\sigma|_\Gamma$, $\rho|_\Lambda$, $\sigma|_{H^\Gamma}$, and $\rho|_{K^\Lambda}$ are ergodic, then automatically σ_Γ is stably orbit equivalent to ρ_Λ and $\sigma|_{H^\Gamma}$ is stably orbit equivalent to $\rho|_{K^\Lambda}$. Rigidity results for von Neumann algebras arising from certain actions of such groups (i.e. W*-rigidity results) are also obtained.

The dissertation of James Owen Sizemore is approved.

Dimitri Shlyakhtenko

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To my parents

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- I. Chifan and S. Popa and J. O. Sizemore, Some OE and W^*-rigidity results for actions by wreath product groups. (2011), Preprint, arXiv:1110.2151v1.
- J.O. Sizemore, A. Winchester, *A Unique Prime Decomposition Result for Wreath Product Factors.* (2011), Preprint, arXiv:1110.3389v1

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T. Mattman, J.O. Sizemore, *Bounds on the Crosscap Number of Torus Knots.* Journal of Knot Theory and its Ramifications. 16 (2007), no. 8, 1043-1051.

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J.O. Sizemore, A. Winchester, *A Unique Prime Decomposition Result for Wreath Product Factors.* (2011), Preprint, arXiv:1110.3389v1, Submitted to Pacific Journal of Mathematics.

CHAPTER 1

Introduction and Notations

The purpose of this work is to study rigidity phenomena in von Neumann factors of type II_1 and orbit equivalence relations arising from actions of wreath product groups on probability measure spaces, by using deformation/rigidity methods.

Rigidity in von Neumann algebras (or $\mathrm{W}^* -$ *rigidity*) occurs whenever the mere isomorphism of two *group measure space* II_1 *factors* $L^\infty(X) \rtimes \Gamma \simeq L^\infty(Y) \rtimes \Lambda$ (or of two *group factors* $L(\Gamma) \simeq L(\Lambda)$), constructed from free, ergodic, measure preserving actions of countable groups on probability spaces, $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ (respectively from infinite conjugacy class groups Γ, Λ), forces the groups/actions to share some common properties. The similar type of phenomena in orbit equivalence, (OE), ergodic theory, which derives common properties of the actions $\Gamma \curvearrowright X$, $\Lambda \curvearrowright Y$ from the isomorphism of their orbit equivalence relations, is called *OE-rigidity*. These two types of results are in fact closely related, as any OE of actions implements an isomorphism of their associated group measure space von Neumann algebras (cf. [Sin55]), i.e. a W^*-equivalence of the actions. In other words, orbit equivalence is a stronger notion of equivalence for group actions than W^*-equivalence, thus making W^*-rigidity results more challenging to establish than OE-rigidity. The ultimate purpose for studying such phenomena is, of course, the classification of group measure space II_1 factors and equivalence relations in terms of their building data $\Gamma \curvearrowright X$. In this respect, the "rigidity" point of view offers a more suggestive and nuanced terminology, and a far more intuitive set up.

W^* and OE rigidity can only occur for non-amenable groups, as by a celebrated result of Connes ([Con76]) all II_1 factors $L^\infty(X) \rtimes \Gamma$ with Γ amenable are approximately finite dimensional and thus isomorphic to the so-called *hyperfinite* factor R. Similarly, all measure preserving (m.p.) ergodic actions of amenable groups on the standard probability space are OE ([OW80], [CFW81]). Moreover, non-amenable groups give rise to non-hyperfinite II_1 factors and orbit equivalence relations. It has been known for some time that non-amenable groups can produce many classes of non-isomorphic II_1 factors and orbit equivalence relations ([MV43],[Dye63],[McD69],[Con75],[Con80],[Zim80], [Pop86],[CH89]), indicating a very complex picture, and a rich and deep underlying rigidity theory. But it was during the last ten years that this subject really took off, with an avalanche of surprising rigidity results being obtained on both OE and W* sides.

Much of this is due to the emergence of *deformation/rigidity theory* ([Pop06c], [Pop06b], [Pop06d], [Pop07]), a set of techniques that exploits the tension between "soft" and "rigid" parts of a group measure space II_1 factor $M = L^\infty(X) \rtimes \Gamma$, in order to recapture the initial data $\Gamma \curvearrowright X$, or part of it. This approach is based on the discovery that if the group action has both a "relatively soft" part and a "relatively rigid" part, complementing one another, then the overall rigidity of the resulting II_1 factor M is considerably enhanced. The "soft spots" of an algebra M are gauged by *deformations* by completely positive maps, a prototype of which being *malleable deformations*, that for instance Bernoulli and Gaussian actions have.

It is due to such a combination/complementarity of "soft" and "rigid" parts that wreath product groups $G = H \wr \Gamma$ have soon been recognized to be "exceptionally rigid" in the von Neumann algebra context. Indeed, it was already shown in [Pop06d] that any isomorphism between group II_1 factors $L(G) \simeq L(G')$, with $G = H \wr \Gamma$, $G' = H' \wr \Gamma'$ wreath product groups, H, H' abelian and Γ, Γ' having property (T) of Kazhdan, forces the groups Γ, Γ' to be isomorphic. The same was in fact shown to be true if Γ, Γ' are non-amenable product groups ([Pop08]) and for certain amalgamated free product groups Γ (with Γ' arbitrary!) in [PV10], while in [Ioa07] it is shown that for non-amenable ICC groups H, H' and amenable groups Γ, Γ', the isomorphism $L(G) \simeq L(G')$ implies $\Gamma \simeq \Gamma'$. Also, II_1 factors $L(G)$ arising from wreath products $G = H \wr \Gamma$ with H amenable and Γ non-amenable were shown to be prime in [Pop08], a fact that was later strengthened significantly, in two ways: a relative solidity result for such $L(G)$ is proved in [CI10], while a unique prime decomposition result for tensor products of such factors is obtained in [SW11]. Finally, let us mention that in [IPV11], a large class of generalized wreath product groups G were shown to be $\mathrm{W}^* -$ *superrigid*, i.e. any isomorphism between $L(G)$ and the II_1 factor $L(G')$ of an arbitrary group G', forces $G \simeq G'$.

It has been suggested that a group measure space factor and orbit equivalence relation arising from ANY action $G \curvearrowright X$ of a wreath product group $G = H \wr \Gamma$ may exhibit a certain level of rigidity. This has been confirmed at the OE-level by Hiroki Sako in [Sak09], who was able to prove that for a large class of groups Γ, the OE class of an action $H \wr \Gamma \curvearrowright X$ is completely determined by the OE-class of its restriction $\Gamma \curvearrowright X$. More precisely, he showed that, if two actions by wreath products groups are orbit equivalent, $H \wr \Gamma \cong_{OE} K \wr \Lambda$, where H, K are amenable and Γ, Λ are products of non-amenable, exact groups, then $\Gamma \cong_{OE} \Lambda$. His methods rely on Ozawa's techniques involving class $\mathcal{S}$ groups ([Oza04],[Oza06]) thus being C^*-algebraic in nature and depending crucially on exactness of the groups involved.

In turn, in this paper we use a deformation/rigidity approach to this problem. This will allow us to exhibit several large classes of groups for which the OE rigidity phenomenon described above holds. It will also allow us to obtain some W^*-rigidity results of a similar type.

In order to state our OE rigidity result in its full generality, we recall the following terminology (see e.g. [Gab05], [Fur99]): Two groups Γ, Λ are *stably orbit equivalent*, or *measure equivalent* (*ME*), if there exist free ergodic probability measure preserving actions $\Gamma \curvearrowright^{\sigma} (X, \mu)$, $\Lambda \curvearrowright^{\rho} (Y, \nu)$, subsets of positive measure $X_0 \subset X$, $Y_0 \subset Y$ and an isomorphism of the corresponding probability spaces $\theta : (X_0, \mu_0) \simeq (Y_0 \nu_0)$ (where $\mu_0 = \mu/\mu(X_0)$, $\nu_0 = \nu/\nu(Y_0)$), such that $\theta(\Gamma t \cap X_0) = \Lambda(\theta(t))$, for almost all $t \in X_0$. We then write $\Gamma \cong_{ME} \Lambda$ for the groups and $\sigma \simeq_{SOE} \rho$ for the actions.

We consider the following three families of groups: for each $k = 1, 2, 3$, we denote by $\mathbf{WR}(k)$ the collection of all generalized wreath product groups $H \wr_I \Gamma$ with Γ icc, I a Γ-set with finite stabilizers and satisfying the condition:

1. Γ has property (T) and H has Haagerup's property;

2. Γ and H have property (T) and H is icc;

3. Γ is a non-amenable product of infinite groups and H is amenable.

With this notation, we obtain the following:

Theorem 1.1. *Let $H \wr_I \Gamma, K \wr_J \Lambda \in \boldsymbol{WR}(k)$ for some $k = 1, 2, 3$. If a measure preserving action, σ, of $H \wr_I \Gamma$ is stably orbit equivalent to a measure preserving action, ρ, of $K \wr_J \Lambda$, and both $\sigma|_\Gamma$ and $\rho|_\Lambda$ are ergodic then $\sigma|_\Gamma$ follows stably orbit equivalent to $\rho|_\Lambda$. Moreover, if $H \wr_I \Gamma, K \wr_J \Lambda \in \boldsymbol{WR}(2)$ and both $\sigma|_{H^I}$ and $\rho|_{K^J}$ are ergodic, then we additionally have that $\sigma|_{H^I} \cong_{SOE} \rho|_{K^J}$.*

To prove the above result, we exploit the fact that the group measure space von Numann algebra M associated to an action of a wreath product group $H \wr \Gamma$ is "distinctly soft" on its $H^{(\Gamma)}$-part, independently of the action. In turn, the fact that Γ acts in a very mixing way on $H^{(\Gamma)}$ makes Γ "strongly singular" (or "malnormal") in M. When combined with rigidity assumptions on Γ, this allows us to first extract information about the associated crossed product von Neumann algebra regardless of how the group acts, then finally deducing the above OE rigidity result.

On the other hand, if we now assume that Γ acts compactly on the probability space X, then we can distinguish the subalgebra $L(H^{(\Gamma)})$ on which Γ acts mixingly from the subalgebra $L^\infty(X)$ on which it acts compactly. This allows us to obtain the following *strong W^*-rigidity* result:

Theorem 1.2. *Let H, K be amenable groups and Γ, Λ groups with the property (T). Assume that $H \wr \Gamma \curvearrowright^\sigma X$ and $K \wr \Lambda \curvearrowright^\rho Y$ are free, measure preserving action such that $\sigma_{|\Gamma}$ is compact, ergodic and $\rho_{|\Lambda}$ is ergodic. If $L^\infty(X) \rtimes (H \wr \Gamma) \simeq L^\infty(Y) \rtimes (K \wr \Lambda)$, then $\Gamma \curvearrowright^{\sigma_{|\Gamma}} X$ is virtually conjugate to $\Lambda \curvearrowright^{\rho_{|\Lambda}} Y$.*

In the last few chapters we adapt these techniques to von Neumann algebras arising from actions of direct products of wreath product groups and this allows us to get a uniqueness of tensor product decomposition type theorem as described below.

Theorem 1.3. *Let $A_1, \dots, A_n$ be non-trivial amenable groups; $H_1, \dots, H_n$ be non-amenable groups; and $Q_1, \dots, Q_k$ be diffuse von Neumann algebras such that*

$$M = L(A_1 \wr H_1)\overline{\otimes} \dots \overline{\otimes} L(A_n \wr H_n) = Q_1 \overline{\otimes} \dots \overline{\otimes} Q_k$$

If $k \geq n$, then $n = k$, and after permutation of indices we have that $L(A_i \wr H_i) \simeq Q_i^{t_i}$ for some positive numbers $t_1, t_2, \dots t_n$ whose product is 1.

Organization of the paper. In the second chapter we describe the von Neumann algebras we will be studying and the deformation that we will be using. In the third chapter we collect various intertwining results concerning subalgebras of von Neumann algebras arising from actions of wreath product groups. The fourth and fifth chapter are dedicated to the heart of the deformation/rigidity arguments of the paper, and focus on locating the malnormal, rigid subgroup Γ of a wreath product $H \wr \Gamma$. Chapter six deals with the application to orbit equivalence theory while chapter seven deals with application to W* strong rigidity. In chapter eight we explain the relevant background material for the uniqueness of tensor product result. The last two chapters deal with the proof of preliminary results as well as the proof of the final uniqueness theorem.

Notations. Throughout this paper all finite von Neumann algebras M that we consider are equipped with a normal faithful tracial state denoted by τ. This trace induces a norm on N by letting $\|x\|_2 = \tau(x^*x)^{\frac{1}{2}}$ and $L^2(M)$ denotes the $\|\cdot\|_2$-completion of M. A Hilbert space $\mathcal{H}$ is a $M - $ *bimodule* if it carries commuting left and right Hilbert M-module structures.

Given a von Neumann subalgebra $Q \subset M$ we denote by $E_Q : M \to M$ the unique τ-preserving *conditional expectation* onto Q. If e_Q is the orthogonal projection of $L^2(M)$ onto $L^2(Q)$ then $\langle M, e_Q\rangle$ denotes the *basic construction*, i.e., the von Neumann algebra generated by M and e_Q in $\mathcal{B}(L^2(M))$. The span of $\{xe_Qy \mid x, y \in M\}$ forms a dense $*$-subalgebra of $\langle M, e_Q\rangle$ and there exists a semifinite trace $Tr : \langle N, e_Q\rangle \to \mathbb{C}$ given by the formula $Tr(xe_Qy) = \tau(xy)$ for all $x, y \in M$. We denote by $L^2\langle M, E_Q\rangle$ the Hilbert space obtained with respect to this trace.

The *normalizer of* Q *inside* M, denoted $\mathcal{N}_M(Q)$, consists of all unitary elements $u \in \mathcal{U}(M)$ satisfying $uQu^* = Q$. A maximal abelian selfadjoint subalgebra A of M, abbreviated MASA, is called a *Cartan subalgebra* if the von Neumann algebra generated by its normalizer in M, $\mathcal{N}_M(A)''$ is equal to M.

If $\Gamma \curvearrowright^\sigma A$ is a trace preserving action by automorphisms of a countable group Γ on a finite von Neumann algebra A we denote by $M = A \rtimes_\sigma \Gamma$ the crossed product von Neumann algebra associated with the action. When no confusion will arise we will drop the symbol σ. Given a subset $F \subset \Gamma$, we will denote by P_F the orthogonal projection onto the closure of the span of $\{au_\gamma \mid a \in A; \gamma \in F\}$.

Given ω a free ultrafilter on $\mathbb{N}$ and (M, τ) a finite von Neumann algebra we denote by (M^ω, τ^ω) its ultrapower algebra, i.e., $M^\omega = \ell^\infty(\mathbb{N}, M)/\mathcal{I}$ where the trace is defined as $\tau^\omega((x_n)_n) = \lim_{n\to\omega} \tau(x_n)$ and $\mathcal{I}$ is the ideal consisting of all $x \in \ell^\infty(\mathbb{N}, M)$ such that $\tau^\omega(x^*x) = 0$. Notice that M embeds naturally into M^ω by considering constant sequences. Many times when working with $M = A \rtimes \Gamma$ we will consider the subalgebra $A^\omega \rtimes \Gamma$ of M^ω.

For all other notations and terminology, that we may have omitted to explain in the paper, we refer the reader to [Pop08], [PV10], [Vae10].

CHAPTER 2

Malleable Deformations of Factors Associated to Wreath Product Groups

Let H and Γ be two countable discrete groups and assume that I is a Γ-set. We denote by $H^I = \bigoplus_I H$ the infinite direct sum of H indexed by the elements of I, which can also be viewed as the group of finitely supported H-valued functions on I, with pointwise multiplication. Next consider Γ acting on H^I by the generalized Bernoulli shift i.e. $\rho_g((s_\iota)_{\iota\in I}) = (s_{g^{-1}\iota})_{\iota\in I}$ for every $g \in \Gamma$. The corresponding semidirect product $H^I \rtimes_\rho \Gamma = H \wr_I \Gamma$ is called the *generalized wreath product* of H and Γ along I. Throughout this paper, for every $\iota \in I$ we denote its stabilizing group by $\Gamma_\iota = \{g \in \Gamma \mid g\iota = \iota\}$.

Given (A, τ) a finite von Neumann algebra, let $H \wr_I \Gamma \curvearrowright^\sigma (A, \tau)$ be a trace preserving action and denote by $M = A \rtimes_\sigma (H \wr_I \Gamma)$ the corresponding crossed product von Neumann algebra. One important feature of these algebras is that they admit *$s-$malleable deformations*, in the general sense of [Pop07]. More specifically, this is obtained as a combination of the Bernoulli-type malleable deformation in [Pop06c], [Pop06d] and the free malleable deformations in [Pop06c], [IPP08], being very similar with the malleable deformation considered in [Ioa07]. The detailed construction is as follows.

Denote by $\tilde{H} = H * \mathbb{Z}$ and then extend σ to an action, still denoted by σ, $\tilde{H} \wr_I \Gamma \curvearrowright^\sigma (A, \tau)$ by letting the generator u of $\mathbb{Z}$ to act trivially on (A, τ). This gives rise to a crossed product von Neumann algebra $\tilde{M} = A \rtimes_\sigma (\tilde{H} \wr_I \Gamma)$ and observe that $M \subset \tilde{M}$.

Seen as an element of $L\mathbb{Z}$, u is a Haar unitary and therefore one can find a selfadjoint element $h \in L\mathbb{Z}$ such that $u = \exp(ih)$. For every $t \in \mathbb{R}$, denote by $u^t = \exp(ith) \in L\mathbb{Z}$

and observe that $\mathrm{Ad}(u^t) \in \mathrm{Aut}(L\tilde{H})$. We further consider the tensor product automorphism $\theta_t = \otimes_I \mathrm{Ad}(u^t) \in \mathrm{Aut}(L\tilde{H}^I)$ and since θ_t commutes with ρ then it can be extended to an automorphism of $\tilde{M}$ which acts identically on the subalgebra $A \rtimes_\sigma \Gamma$.

From the definitions one can easily see that $\lim_{t\to 0} \|u^t - 1\|_2 = 0$ and hence we have $\lim_{t\to 0} \|\theta_t(x) - x\|_2 = 0$ for all $x \in \tilde{M}$. Therefore, the path $(\theta_t)_{t\in\mathbb{R}}$ is a deformation by automorphisms of $\tilde{M}$.

Next we show that θ_t admits a "symmetry", i.e. there exists an automorphism β of $\tilde{M}$ satisfying the following relations:

$$\beta^2 = 1,\ \beta_{|_M} = id_{|_M},\ \beta\theta_t\beta = \theta_{-t},\ \text{for all } t \in \mathbb{R}. \tag{2.1}$$

To see this, first define $\beta_{|_{LH^I}} = id_{|_{LH^I}}$ and then for every $\iota \in I$ we let $(u)_\iota$ to be the element in $L\tilde{H}^I$ whose ι^{th}-entry is u and 1 otherwise. On elements of this form we define $\beta((u)_\iota) = (u^*)_\iota$, and since β commutes with ρ, it extends to an automorphism of $L(\tilde{H} \wr_I \Gamma)$ by acting identically on $L\Gamma$. Finally, the automorphism β extends to an automorphism of $\tilde{M}$, still denoted by β, which acts trivially on A. Verifying relations (2.1) is a straightforward computation and we leave it to the reader.

For further use, we recall that all malleable deformations admitting a symmetry (i.e. *s-malleable* deformations) satisfy the following "transversality" property:

Theorem 2.1 ([Pop08]). *For all $t \in \mathbb{R}$ and all $x \in M$ we have that*

$$\|\theta_{2t}(x) - x\|_2 \leqslant 2\|\theta_t(x) - E_M \circ \theta_t(x)\|_2.$$

CHAPTER 3

Intertwining Techniques for Subalgebras of Wreath Product Factors

We review here the techniques of intertwining subalgebras in [Pop06a], [Pop06d], which are an essential part of deformation/rigidity theory. Given a projection $p_0 \in M$ and two subalgebras $P \subset M$ and $Q \subset p_0Mp_0$ one says that *a corner of P can be embedded into Q inside M* if there exist nonzero projections $p \in P$ $q \in Q$, nonzero partial isometry $v \in M$ and a $*$-homomorphism $\psi : pPp \to qQq$ such that $vx = \psi(x)v$, for all $x \in pPp$. Throughout this paper we denote by $P \prec_M Q$ whenever this property holds and by $P \nprec_M Q$ otherwise.

Theorem 3.1 (Popa, [Pop06d]). *Let (M, τ) be a finite von Neumann algebra with $P \subset M$, $Q \subset M$ two subalgebras and consider the following properties:*

1. *$P \prec_M Q$.*
2. *Given any subgroup $\mathcal{G} \subset \mathcal{U}(P)$ such that $\mathcal{G}'' = P$ then for all $x_1, x_2, ..., x_n \in M$ and every $\epsilon > 0$ there exists $u \in \mathcal{G}$ such that*

$$\|E_Q(x_i u x_j)\|_2 < \epsilon, \ \ \textit{for every } 1 \leqslant i, j \leqslant n.$$

3. *Given any subgroup $\mathcal{G} \subset \mathcal{U}(P)$ such that $\mathcal{G}'' = P$ there exists a sequence $u_n \in \mathcal{G}$ such that*

$$\lim_{n\to\infty} \|E_Q(x u_n y)\|_2 \to 0, \ \ \textit{for every } x, y \in M.$$

Then one has the following equivalences:

$$non(1) \Leftrightarrow (2) \Leftrightarrow (3)$$

Based on this criterion, we present below a few intertwining lemmas needed in the coming sections. The first result we prove deals with embedding of normalizers and will be used quite extensively in Section 5. Roughly speaking, given Q a regular subalgebra of M with $Q \subseteq N \subseteq M$ and $\mathcal{G}$ a subgroup of normalizers of Q in M, if there exits a nonzero partial isometry intertwining $\mathcal{G}''$ into N then one can find a nonzero partial isometry in M intertwining the (possibly larger) algebra $(\mathcal{U}(Q)\mathcal{G})''$ into N. The precise statement is the following:

Lemma 3.2. *Let $Q \subseteq N \subseteq M$ be finite von Neumann algebras such that $\mathcal{N}_M(Q)'' = M$. If $\mathcal{G} \subset \mathcal{N}_M(Q)$ is a subgroup such that $\mathcal{G}'' \prec_M N$ then $(\mathcal{U}(Q)\mathcal{G})'' \prec_M N$.*

Proof. Suppose by contradiction that we have $(\mathcal{U}(Q)\mathcal{G})'' \nprec_M N$. Therefore, by Theorem 3.1, there exists an infinite sequence $x_n = a_n u_n \in \mathcal{U}(Q)\mathcal{G}$ with $a_n \in \mathcal{U}(Q)$ and $u_n \in \mathcal{G}$ such that

$$\lim_{n\to\infty} \|E_N(x x_n y)\|_2 = 0 \text{ for all } x, y \in M. \tag{3.1}$$

Taking $x = y = 1$ in (3.1) it is immediate that the sequence $(u_n)_n$ must be infinite. Below we prove that

$$\lim_{n\to\infty} \|E_N(x u_n y)\|_2 = 0 \text{ for all } x, y \in M. \tag{3.2}$$

Fix two arbitrary unitaries $x, y \in \mathcal{N}_M(Q)$. Then for all a_n we have $x a_n x^* \in \mathcal{U}(Q) \subset N$ and using (3.1) we deduce that:

$$\lim_{n\to\infty} \|E_N(x u_n y)\|_2 = \lim_{n\to\infty} \|x a_n x^* E_N(x u_n y)\|_2 =$$
$$= \lim_{n\to\infty} \|E_N(x a_n x^* x u_n y)\|_2 = \lim_{n\to\infty} \|E_N(x x_n y)\|_2 = 0.$$

The above convergence extends to all elements x, y that are finite linear combinations of unitaries in $\mathcal{N}_M(Q)$ and furthermore, using $\|\cdot\|_2$-approximations, to all elements x, y belonging to $\mathcal{N}_M(Q)''$. Since $\mathcal{N}_M(Q)'' = M$, this completes the proof of (3.2).

Finally, by Theorem 3.1 convergence (3.2) implies that $\mathcal{G}'' \nprec_M N$ thus leading to a contradiction. □

The next lemma is more specialized, providing a criterion for intertwining certain subalgebras inside von Neumann algebras arising from actions by wreath product groups. In essence the result is a translation of Theorem 3.1 in the setting of ultrapower algebras and we include a proof only for the sake of completeness. The reader may also consult Section 3 in [Pop06e] or Proposition 2.1 in [CP10], for a similar arguments.

Lemma 3.3. *Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on a finite von Neumann algebra A. Denote by $M = A \rtimes (H \wr_I \Gamma)$ and let $P \subset M$ be a II_1 subfactor such that $\mathcal{N}_M(P)' \cap M = \mathbb{C}1$. If $S \subset I$ is a subset, then $P \prec_M A \rtimes H^S$ implies $P^\omega \subseteq (A \rtimes H^S)^\omega \vee M$. When assuming $S = I$ the two conditions are actually equivalent.*

Proof. Assume $P \prec_M A \rtimes H^S$. Therefore one can find nonzero projections $p \in P$, $q \in A \rtimes H^S$, a $*$-homomorphism $\psi : pPp \to q(A \rtimes H^S)q$ and nonzero partial isometry $v \in M$ such that $v\psi(x) = xv$ for all $x \in pPp$. The last equation implies that $vv^* \in (pPp)' \cap pMp$ and therefore we have the following

$$pPpvv^* = v\psi(pPp)v^* \subseteq v(A \rtimes H^S)v^*. \tag{3.3}$$

We notice that there exists nonzero projection $p' \in P' \cap M$ such that $vv^* = pp'$ and combining this with (3.3) we obtain

$$(pPp)^\omega p' \subseteq (A \rtimes H^S)^\omega \vee M. \tag{3.4}$$

Since P is a II_1 factor then after shrinking the projection p if necessary one may assume that p has trace $\frac{1}{k}$, for some positive integer k. Also, for every $1 \leqslant i, j \leqslant k$ there exist partial isometries $e_{ij} \in P$ such that $e_{11} = p$, $e_{ij}^* = e_{ji}$, $e_{ij}e_{ji} = e_{ii} \in \mathcal{P}(P)$ and $\sum_i e_{ii} = 1$. If $(x_n)_n \in P^\omega$ then using the above relations in combination with $p' \in P' \cap M$ we have

$$\begin{aligned}(x_n)_n(p')_n &= (x_np')_n = (\sum_{i,j} e_{ii}x_ne_{jj}p')_n = \sum_{i,j}(e_{i1}e_{1i}x_ne_{j1}e_{1j}p')_n \\ &= \sum_{i,j}(e_{i1})_n(e_{1i}x_ne_{j1})_n(p')_n(e_{1j})_n. \end{aligned}\tag{3.5}$$

One can easily see that $(e_{1i}x_ne_{j1})_n \in (pPp)^\omega$ and combining this with (3.4) and (3.5) we conclude that $(x_n)_n(p')_n \in (A \rtimes H^S)^\omega \vee M$, thus showing that $P^\omega p' \subseteq (A \rtimes H^S)^\omega \vee M$.

Conjugating by $u \in \mathcal{N}_M(P) \subseteq \mathcal{N}_M(P' \cap M)$ we obtain $P^\omega up'u^* \subseteq (A \rtimes H^S)^\omega \vee M$, for all $u \in \mathcal{N}_M(P)$, and hence $P^\omega p_0 \subseteq (A \rtimes H^S)^\omega \vee M$ where $p_0 = \vee_{u \in \mathcal{N}_M(P)} up'u^* \in P' \cap M$. It is clear that p_0 commutes with $\mathcal{N}_M(P)$ and thus it belongs to $\mathcal{N}_M(P)' \cap (P' \cap M)$. By assumption we have $\mathcal{N}_M(P)' \cap M = \mathbb{C}1$ which forces $p_0 = 1$ and therefore $P^\omega \subseteq (A \rtimes H^S)^\omega \vee M$.

For the converse we proceed by contraposition, i.e., assuming $S = I$ we show that $P \nprec_M A \rtimes H^I$ implies $P^\omega \nsubseteq (A \rtimes H^I)^\omega \vee M$. If $P \nprec_M A \rtimes H^I$, by Theorem 3.1, there exists a sequence of unitaries $a_n \in \mathcal{U}(P)$ such that for all $x, y \in M$ we have $\|E_{A \rtimes H^I}(x a_n y)\|_2 \to 0$ as $n \to \infty$. This implies $a \perp M(A \rtimes H^I)^\omega M$, where $a = (a_n)_n \in P^\omega$ and since $M(A \rtimes H^I)^\omega M = (A \rtimes H^I)^\omega \vee M$ we conclude that $P^\omega \nsubseteq (A \rtimes H^I)^\omega \vee M$. $\square$

In the following lemma we collect three situations when we have good control over intertwiners between certain subalgebras of von Neumann algebras arising from actions of wreath product groups. The result is a mild extension of Theorem 3.1 in [Pop06d], and has exactly the same proof, which however we include here for the reader's convenience.

Lemma 3.4. *Let $H \wr_I \Gamma \curvearrowright^\sigma (A, \tau)$ be a trace preserving action on a finite algebra A. Denote by $\tilde{M} = A \rtimes_{\tilde{\sigma}} (\tilde{H} \wr_I \Gamma)$, $M = A \rtimes_\sigma (H \wr_I \Gamma)$ and $P = A \rtimes \Gamma$.*

1. *Let $q \in M$ be a projection and $Q \subset qMq$ be a von Neumann subalgebra. Assume that for every $\iota \in I$ one has $Q \nprec_M A \rtimes (H \wr_I \Gamma_\iota)$. If $0 \neq \xi \in L^2(q\tilde{M})$ satisfies $Q\xi \subset L^2(\sum_i \xi_i M)$ for some $\xi_1, ..., \xi_n \in L^2(\tilde{M})$ then $\xi \in L^2(M)$; in particular we have $Q' \cap q\tilde{M}q \subseteq \mathcal{N}_{q\tilde{M}q}(Q)'' \subseteq M$.*

   If I has finite stabilizers and $Q \subset qMq$ such that $Q \nprec_M A \rtimes H^I$ then we have $Q' \cap q\tilde{M}q \subseteq \mathcal{N}_{q\tilde{M}q}(Q) \subseteq qMq$.

2. *Let $q \in P$ be a projection and $Q \subset qPq$ be a von Neumann subalgebra. Assume that for every $\iota \in I$ one has $Q \nprec_P A \rtimes \Gamma_\iota$. If $0 \neq \xi \in L^2(qM)$ satisfies $Q\xi \subset L^2(\sum_i \xi_i P)$ for some $\xi_1, ..., \xi_n \in L^2(M)$ then $\xi \in L^2(P)$; in particular we have $Q' \cap qMq \subseteq \mathcal{N}_{qMq}(Q) \subseteq P$.*

   If I has finite stabilizers and $Q \subset qPq$ such that $Q \nprec_P A$ then we have $Q' \cap qMq \subseteq \mathcal{N}_{qMq}(Q)'' \subseteq qPq$.

3. *Assume that I has finite stabilizers and let $F \subset I$ be a finite subset. If $Q \subset A \rtimes H^F$ is a subalgebra such that $Q \nprec_M A$ then we have*

$$\mathcal{N}_M(Q)'' \prec_M A \rtimes H^I,$$

Proof. Let p denote the orthogonal projection of $L^2(M)$ onto the Hilbert subspace $\overline{Q\xi M}^{\|\cdot\|_2} \subset L^2(\tilde{M})$. Note that $p \in Q' \cap q\langle \tilde{M}, e_M\rangle q$ and $0 \neq Tr(p) < \infty$, where Tr denotes the canonical trace on $\langle \tilde{M}, e_M\rangle$. To prove that $\xi \in M$ it is sufficient to show that $p \leq e_M$ or, equivalently, $(1-e_M)p(1-e_M) = 0$.

By taking spectral projections, to show that $(1-e_M)p(1-e_M) = 0$ it is in fact sufficient to show that if $f \in Q' \cap \langle M, e_P\rangle$ is a projection such that $0 \neq Tr(f) < \infty$ and $f \leqslant 1 - e_M$, then $f = 0$. To this end, we will show that $\|f\|_{2,Tr}$ is arbitrarily small.

Thus, let $\tilde{\eta}_0 = e$ and let $\tilde{\eta}_1, ..., \tilde{\eta}_n, ...$ be an enumeration of elements in $(\tilde{H} \setminus H)^I$ which are representatives for left cosets of $H \wr_I \Gamma$ in $\tilde{H} \wr_I \Gamma$. Next if we denote by $f_n = \sum_{i=1}^n u_{\tilde{\eta}_i} e_M u_{\tilde{\eta}_i^{-1}}$ then, as f has finite trace and $f \leqslant 1 - e_M = \sum_{i=1}^{\infty} u_{\tilde{\eta}_i} e_M u_{\tilde{\eta}_i^{-1}}$, there exists $n \in \mathbb{N}$ such that $\|f_n f - f\|_{2,Tr} < \epsilon \|f\|_{2,Tr}$. Thus, if $u \in \mathcal{U}(Q)$ then

$$Tr(f_n u f_n u^*) \geqslant Tr(f f_n f u f_n u^*) - |Tr(f f_n (1-f) u f_n u^*)| - |Tr((1-f) f_n u f_n u^*)|. \quad (3.6)$$

Since $f_n f$ is $\epsilon-$close to f in the norm $\|\cdot\|_{2,Tr}$ and f commutes with $u \in Q$ we deduce:

$$Tr(f f_n f u f_n u^*) = Tr(f_n f u f_n f u^*) \geqslant (1 - 2\epsilon - \epsilon^2)\|f\|_{2,Tr}^2. \quad (3.7)$$

Similarly, we have:

$$|Tr(f f_n (1-f) u f_n u^*)| + |Tr((1-f) f_n f u f_n u^*)| \leqslant 2\epsilon(1+\epsilon)\|f\|_{2,Tr}^2.$$

Combining this with (3.6) and (3.7) we get that for all $u \in \mathcal{U}(Q)$ we have

$$Tr(f_n u f_n u^*) \geqslant (1-4\epsilon-3\epsilon^2)\|f\|_{2,Tr}^2. \tag{3.8}$$

On the other hand a straight forward computation shows that

$$Tr(f_n u f_n u^*) = Tr(\sum_{i,j} u_{\tilde{\eta}_i} e_M u_{\tilde{\eta}_i^{-1}} u u_{\tilde{\eta}_j} e_M u_{\tilde{\eta}_j^{-1}} u^*) = \sum_{i,j} \|E_M(u_{\tilde{\eta}_i} u u_{\tilde{\eta}_j^{-1}})\|_2^2. \tag{3.9}$$

Thus, in order to prove that $\|f\|_{2,Tr}$ is small, it suffices to show that for every $\tilde{\eta}_1, \dots, \tilde{\eta}_n \in (\tilde{H} \setminus H)^I$ and every $\epsilon > 0$ there exists $u \in \mathcal{U}(Q)$ such that for all $1 \leqslant i, j \leqslant n$ we have

$$\|E_M(u_{\tilde{\eta}_i} u u_{\tilde{\eta}_j^{-1}})\|_2 \leqslant \epsilon. \tag{3.10}$$

Fix an $\varepsilon > 0$ and an arbitrary set $\{\tilde{\eta}_1, \tilde{\eta}_2, \tilde{\eta}_3, \dots, \tilde{\eta}_n\}$. For every $1 \leqslant i \leqslant n$ denote by $\tilde{F}_i$ the support of η_i and let $\mathcal{F} = \bigcup_{i=1}^n \tilde{F}_i \subset I$. It is easy to see that for every $1 \leqslant i, j \leqslant n$ we have the following containment

$$\{g \in \Gamma \mid g\tilde{F}_j = \tilde{F}_i\} \subseteq \bigcup_{\kappa,\ell \in \mathcal{F}} \{g \in \Gamma \mid g\kappa = \ell\}. \tag{3.11}$$

Furthermore, observe that $\{g \in \Gamma \mid g\kappa = \ell\}$ is either empty or equal to $g_{\kappa,\ell}\Gamma_\kappa$ for a fixed element $g_{\kappa,\ell} \in \Gamma$ satisfying $g_{\kappa,\ell}\kappa = \ell$. When combined with (3.11) it implies that for every $1 \leqslant i, j \leqslant n$ we have

$$\{g \in \Gamma \mid g\tilde{F}_j = \tilde{F}_i\} \subseteq \bigcup_{\kappa,\ell \in \mathcal{F}} g_{\kappa,\ell}\Gamma_\kappa,$$

which further implies that

$$\{\eta g \in H \wr_I \Gamma \mid \eta \in H^I; g\tilde{F}_j = \tilde{F}_i\} \subseteq \bigcup_{\kappa,\ell \in \mathcal{F}} g_{\kappa,\ell}(H \wr_I \Gamma_\kappa). \tag{3.12}$$

Since $\mathcal{F}$ is a finite set and for every $\kappa \in \mathcal{F}$ we assumed that $Q \nprec_P A \rtimes_\sigma (H \wr_I \Gamma_\kappa)$ then by Theorem 3.1 there exists a unitary $u_{\mathcal{F},\varepsilon} \in \mathcal{U}(Q)$ such that, for all $\kappa, \ell \in \mathcal{F}$ we have

$$\|E_{A \rtimes_\sigma H^I \rtimes \Gamma_\kappa}(u_{g_{\kappa,\ell}^{-1}} u_{\mathcal{F},\varepsilon})\|_2 \leqslant \frac{\varepsilon}{|\mathcal{F}|}.$$

Using the Fourier expansion $u_{\mathcal{F},\varepsilon} = \sum_{\eta g \in H \wr_I \Gamma} a_{\eta g} u_{\eta g}$, a little computation shows that the above inequality is equivalent to

$$\sum_{\eta g \in g_{\kappa,\ell}(H \wr_I \Gamma_\kappa)} \|a_{\eta g}\|_2^2 \leqslant \frac{\varepsilon^2}{|\mathcal{F}|^2} \text{ for all } \kappa, \ell \in \mathcal{F}. \tag{3.13}$$

Next we show that the unitary $u_{\mathcal{F},\varepsilon} \in Q$ found above satisfies (3.10). Indeed, employing the formula for the conditional expectation, we obtain

$$\|E_M(u_{\tilde{\eta}_i} u_{\mathcal{F},\varepsilon} u_{\tilde{\eta}_j^{-1}})\|_2^2 = \sum_{\{\eta g | \tilde{\eta}_i \eta \rho_g(\tilde{\eta}_j^{-1}) \in H^I\}} \|\tilde{\sigma}_{\tilde{\eta}_i}(a_{\eta g})\|_2^2 \leqslant \sum_{\{\eta g | \eta \in H^I; g\tilde{F}_j = \tilde{F}_i\}} \|a_{\eta g}\|_2^2,$$

and combining this with (3.12) and (3.13) we have

$$\|E_P(u_{\eta_i} u u_{\eta_j^{-1}})\|_2^2 \leqslant \sum_{\kappa,\ell \in \mathcal{F}} \sum_{\eta g \in g_{\kappa,\ell}(H \wr_I \Gamma_\kappa)} \|a_{\eta g}\|^2 \leqslant \sum_{\kappa,\ell \in \mathcal{F}} \frac{\varepsilon^2}{|\mathcal{F}|^2} = \varepsilon^2,$$

which finishes the proof of (1).

The proof of the part (2) is very similar with the first one and it will be omitted.

Below we prove part (3). Let $\tilde{K} = \{g \in \Gamma | \exists x, y \in F \text{ such that } gx = y\}$.

First observe that since $Q \nprec_M A$ by Theorem 3.1, there exists a sequence of unitaries $x_l \in Q$ such that for all $z, t \in M$ we have

$$\lim_{l \to \infty} \|E_A(z x_l t)\|_2 = 0$$

Using Fourier expansion we have $x_l = \sum_{\eta \in H^F} b_{l,\eta} u_\eta \in Q$ and therefore the above convergence is equivalent to the following

$$\|b_{l,\eta}\|_2^2 \to 0 \text{ for every } \eta \in H^F. \tag{3.14}$$

Next we prove that for all $c, d \in A \rtimes_\sigma (H^I)$, $g \in \Gamma \setminus \tilde{K}$, and $\gamma \in \Gamma$ we have

$$\lim_{l \to \infty} \|E_{A \rtimes H^F}(c u_g x_l u_{\gamma^{-1}} d)\|_2 = 0.$$

Using $\|\cdot\|_2$- approximations it suffices to show our claim only for elements of the form $c = c_1u_{\eta_1}$, $d = c_2u_{\eta_2}$, where $c_{1,2} \in A$ and $\eta_{1,2} \in H^I$. Therefore, using the expansion $x_l = \sum_\eta b_{l,\eta}u_\eta$, we have that:

$$\|E_{A\rtimes H^F}(cu_gx_lu_{\gamma^{-1}}d)\|_2 =$$

$$\begin{aligned} &= \|\sum_\eta E_{A\rtimes H^F}(c_1u_{\eta_1}u_gb_{l,\eta}u_\eta u_{\gamma^{-1}}c_2u_{\eta_2})\|_2 \\ &= \|\sum_{\substack{\eta\in H^F \\ \eta_1 g\eta\gamma^{-1}\eta_2\in H^F}} c_1u_{\eta_1}u_gb_{l,\eta}u_\eta u_{\gamma^{-1}}c_2u_{\eta_2}\|_2 \end{aligned}$$

Since $\eta \in H^F$ and $\eta_1, \eta_2 \in H^I$, we observe that condition $\eta_1 g\eta\gamma^{-1}\eta_2 \in H^F$ is equivalent to $g\gamma^{-1} = e$ and thus we have $\eta_1 g\eta\gamma^{-1}\eta_2 = \eta_1\rho_g(\eta)\eta_2 \in H^F$. Since $g \in \Gamma \setminus \tilde{K}$, then the latter condition is equivalent to the following: There exist at most finitely many η_k^1, subwords of η_1 and finitely many η_m^2 subwords of η_2, such that $\eta_k^1 rho_g(\eta)\eta_m^2 = e$. This is furthermore equivalent with $\eta = \rho_{g^{-1}}((\eta_k^1)^{-1}(\eta_l^2)^{-1}) = \rho_{g^{-1}}((\eta_m^2\eta_k^1)^{-1})$ and hence the above sum is equal to:

$$\begin{aligned} &\|\sum_{\eta=\rho_{g^{-1}}((\eta_m^2\eta_k^1)^{-1});k,m} c_1u_{\eta_1}u_gb_{l,\eta}u_\eta u_{g^{-1}}c_2u_{\eta_2}\|_2^2 \\ \leq\ &\|c\|^2\|d\|^2\|\sum_{\eta=\rho_{g^{-1}}((\eta_m^2\eta_k^1)^{-1});k,m} b_{l,\eta}u_\eta\|_2^2 \end{aligned}$$

$$= \|c\|^2\|d\|^2\sum_{k,m}\|b_{l,\rho_{g^{-1}}((\eta_m^2\eta_k^1)^{-1})}\|_2^2 \tag{3.15}$$

Since η_k^1 and η_l^2 are finite sets depending only on c, d, g, γ (which were fixed!) then by (3.14) the sum (3.15) converges to 0 when $l \to \infty$ thus finishing the proof of the claim.

Now we continue with the proof. We proceed by contradiction so assume that $\mathcal{N}_M(Q)'' \not\prec_M A \rtimes H^I$. Fix $\varepsilon > 0$ and by Theorem 3.1 there exists a unitary $u = \sum_{g\in\Gamma} a_gu_g \in \mathcal{N}_M(Q)$, with $a_g \in A \rtimes H^I$, such that $\sum_{g\in\tilde{K}} \|a_g\|_2^2 < \epsilon$. Furthermore, we can find a finite set $K \subset \Gamma$ such that $\sum_{g\in\Gamma\setminus K} \|a_g\|_2^2 < \epsilon$.

Denoting by $v = \sum_{g\in K\setminus\tilde{K}} a_g u_g$ the above inequalities imply that $\|ux_l u^* - vx_l u^*\|_2^2 < 2\epsilon$. Using this in combination with $x_l \in A \rtimes H^F$ and $u \in \mathcal{N}_M(Q)$, a straight forward computation shows that

$$\|E_{A\rtimes H^F}(vx_l u^*)\|_2^2 > 1 - 2\epsilon. \tag{3.16}$$

On the other hand we have

$$\|E_{A\rtimes H^F}(vx_l u^*)\|_2 = \|\sum E_{A\rtimes H^F}(a_g u_g b_{l,\eta} u_\eta u_\gamma^* a_\gamma^*)\|_2.$$

Notice that, if a term in the sum above is nonzero we must have that $g\gamma^{-1} = e$, where $g \in K \setminus \tilde{K}$. Since K is finite, this means that only finitely many g will contribute to the sum. By our claim above for each $g \in K \setminus \tilde{K}$ we know the above norm converges to 0. Since there are only finitely many such g we get that

$$\|E_{A\rtimes H^F}(vx_l u^*)\|_2 \to 0.$$

This however, contradicts (3.16) when letting ϵ to be sufficiently small.

$\square$

CHAPTER 4

Rigid Subalgebras of M

In this section we come to the heart of the deformation/rigidity arguments of the paper. The central idea, as usual in Popa's deformation/rigidity theory, is to use deformations to reveal the position of rigid subalgebras of von Neumann algebras M arising from actions by wreath product groups. More precisely, our main result shows that if the deformation θ_t introduced in the first section converges uniformly to the identity on the unit ball of a diffuse subalgebra Q then one can completely determine the position of Q inside M. One consequence we derive from this is Theorem 4.5 describing all rigid diffuse subalgebras of M.

This result is very much in the spirit of Theorem 4.1 of [Pop06d] and Theorem 3.6 of [Ioa07] and in fact most of our proofs resemble the proofs of these results. Roughly speaking, the methods we use, employ averaging arguments in combination with the intertwining techniques described in the previous section.

The following technical result can be seen as a criterion for locating subalgebras inside von Neumann algebras M arising from actions by wreath product groups.

Theorem 4.1. *Let H, Γ be countable groups and let I a Γ-set with finite stabilizers. Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on a finite algebra A and denote by $M = A \rtimes (H \wr_I \Gamma)$. If $Q \subset pMp$ is a diffuse subalgebra such that $\theta_t \to id$ uniformly on the unit ball of Q, then one of the following alternatives holds:*

1. *$Q \prec_M A \rtimes \Gamma$,*

2. *There exists a finite set $F \subset I$ such that $Q \prec_M A \rtimes H^F$.*

The proof of this theorem will result from a sequence of lemmas. The first one is taken from [Pop06c], [Pop06d], but we include a proof for completeness.

Lemma 4.2. *Let H, Γ be countable groups and let I a Γ-set with finite stabilizers. Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on a finite algebra A and denote by $M = A \rtimes (H \wr_I \Gamma)$. If $Q \subset pMp$ is a diffuse subalgebra such that $\theta_t \to id$ uniformly on the unit ball of Q, then one of the following alternatives holds:*

1. *There exists a nonzero partial isometry $w \in \tilde{M}$ such that $\theta_1(x)w = wx$ for all $x \in Q$.*
2. *$Q \prec_M A \rtimes H^I$.*

Proof. Since $\theta_t \to id$ uniformly on the unit ball of Q we can find $n \geqslant 1$ such that

$$\|\theta_{1/2^n}(u) - u\|_2 \leqslant 1/2, \text{ for all } u \in \mathcal{U}(Q).$$

Let v be the minimal $\|.\|_2$ element of $\mathcal{K} = \overline{co}^w\{\theta_{1/2^n}(u)u^* | u \in \mathcal{U}(Q)\}$. Since $\|\theta_{1/2^n}(u)u^* - 1\|_2 \leqslant 1/2$, for all $u \in \mathcal{U}(Q)$, we get that $\|v - 1\|_2 \leqslant 1/2$, thus $v \neq 0$. Also, since $\theta_{1/2^n}(u)\mathcal{K}u^* = \mathcal{K}$ and $\|\theta_{1/2^n}(u)xu^*\|_2 = \|x\|_2$, for all $u \in \mathcal{U}(Q)$, the uniqueness of v implies that $\theta_{1/2^n}(u)v = vu$ for all $u \in \mathcal{U}(Q)$ and hence

$$\theta_{1/2^n}(x)v = vx, \text{ for all } x \in Q. \tag{4.1}$$

Assume that (2) is false, then $Q \nprec_M A \rtimes H^I$. Since I has finite stabilizers this implies that for every $\iota \in I$ we have $Q \nprec_M A \rtimes (H \wr_I \Gamma_\iota)$. Therefore part (1) of Lemma 3.4 implies that $Q' \cap \tilde{M} \subset M$. On the other hand, since θ_t is a s-malleable deformation then combining (4.1) with the procedure from [Pop06d] of patching up intertwiners, one can find a non-zero partial isometry $w \in \tilde{M}$ such that $\theta_1(u)w = wu$, for all $u \in \mathcal{U}(Q)$, which proves (1). □

Our second lemma is a refinement of arguments in Section 4 of [Pop06d].

Lemma 4.3. *Let M and $\tilde{M}$ be as above. Assume $Q \subset pMp$ is a von Neumann subalgebra such that there exists a nonzero partial isometry $v \in \tilde{M}$ satisfying that $\theta_1(x)v = vx$, for all $x \in Q$. Then one of the following alternatives holds:*

1. $Q \prec_M A \rtimes \Gamma$,

2. $Q \prec_M A \rtimes H^I$.

Proof. Let us assume that neither of the above conclusions hold. Then $Q \not\prec_M A \rtimes \Gamma$ and $Q \not\prec_M A \rtimes H^I$. By our assumption it suffices to show that $Q \not\prec_{\tilde{M}} \theta_1(M)$. Since $Q \not\prec_M A \rtimes \Gamma$ and $Q \not\prec_M A \rtimes H^I$, we have a sequence of unitaries $(v_n) \in Q$ such that

$$\|E_{A\rtimes\Gamma}(xv_ny)\|_2 \to 0 \text{ and } \|E_{A\rtimes H^I}(xv_ny)\|_2 \to 0$$

for all $x, y \in M$. Now we would like to show that

$$\|E_{\theta_1(M)}(xv_ny)\|_2 \to 0$$

for all $x, y \in \tilde{M}$. Approximating x, y by finite sums of elements of the form au_{sg} with $a \in A$, $s \in \tilde{H}^I$ and $g \in \Gamma$, we assume that $x = u_s$ and $y = u_t^*$.

Let $F_s = \{i \in I | s(i) \neq e\}$ and similarly for F_t. Then we know that F_s, F_t, and $\{g \in \Gamma | gF_s \cap F_t \neq \emptyset\}$ are all finite sets. Define $s_0 \in H^I$ by $s_0(i)$ is the last letter of $s(i) \in \tilde{H} = H * \mathbb{Z}$ if this letter is in H, $s_0(i) = e$ otherwise. Define t_0 similarly.

If $g \in \Gamma$ and $r \in H^I$ are such that $u_{srgt^{-1}} \in \theta_1(M)$, then either $gF_s \cap F_t \neq \emptyset$ or $s_0 r \alpha_g(t_0^{-1}) = e$

Now if we write the Fourier decomposition of v_n as $v_n = \sum_{r,g} a_{n,rg} u_{rg}$, then we can see that

$$\begin{aligned}
\|E_{\theta_1(M)}(u_s v_n u_t^*)\|_2^2 &= \sum_{r,g,u_{srgt^{-1}} \in \theta_1(M)} \|a_{n,rg}\|_2^2 \\
&\leqslant \sum_{r,g,s_0 r\alpha_g(t_0^{-1})=e} \|a_{n,rg}\|_2^2 + \sum_{g,gF_s\cap F_t\neq\emptyset} \sum_{r\in H^I} \|a_{n,rg}\|_2^2 \\
&= \|E_{A\rtimes\Gamma}(u_{s_0} v_n u_{t_0^{-1}})\|_2^2 + \sum_{g,gF_s\cap F_t\neq\emptyset} \|E_{A\rtimes H^I}(v_n u_{g^{-1}})\|_2^2 \\
&\to 0.
\end{aligned}$$

This shows that $Q \not\prec_{\tilde{M}} \theta_1(M)$, as desired. $\square$

So for the proof of the main theorem, if $\theta_t \to id$ converges uniformly on the unit ball of Q, then by the above two lemmas we have that either $Q \prec_M A \rtimes \Gamma$ or $Q \prec_M A \rtimes H^I$. In the second case, we can view Q as embedded in a corner of $A \rtimes H^I$ and since θ_t converges uniformly to id_Q then an averaging argument shows that θ_t must be implemented by a partial isometry v in $\tilde{M}$. Looking closely, it would seem that v would have to conjugate each coordinate of H^I by u, since this is exactly what θ_t does. However, the only way for this to happen would be if the algebra Q would be supported on H^F, for some finite set $F \subset I$. In fact we show below this is indeed the case. So we finish the proof of Theorem 4.5 with the following lemma whose proof is an adaption of the proof of Theorem 3.6 (ii) in [Ioa07].

Lemma 4.4. *Let $\tilde{M}$ and M as in Theorem 4.5 and let $N \subset p(A \rtimes H^I)p$ be a subalgebra such that $\theta_t \to id$ uniformly on $(N)_1$. Then one can find a finite set $F \subset I$ such that $N \prec_M A \rtimes H^F$.*

Proof. Notice that, since $\theta_t \to id$ uniformly on $(N)_1$ then by arguing as in Lemma 4.2 we find that there exists $t > 0$ and a nonzero partial isometry $v \in \tilde{M}$ such that

$$\theta_t(x)v = vx \text{ for all } x \in N. \tag{4.2}$$

Consider the Fourier expansion $v = \sum_{\tilde{\eta}g \in \tilde{H} \wr_I \Gamma} a_{\tilde{\eta}g} u_{\tilde{\eta}g}$ and letting $v_g = \sum_{\tilde{\eta} \in \tilde{H}^I} a_{\tilde{\eta}g} u_{\tilde{\eta}} \in A \rtimes_\sigma \tilde{H}^I$ we have that $v = \sum_{g \in \Gamma} v_g u_g$.

Fix $g \in \Gamma$ such that $v_g \neq 0$.

We know we can find a finite set $F \subset I$ and an element $v'_g \in A \rtimes \tilde{H}^F$, such that $\|v_g - v'_g\|_2 < \epsilon$. If we identify the u_g coefficient on both sides of equation (4.2), we have

$$\theta_t(x)v_g = v_g \sigma_g(x) \text{ for all } x \in N.$$

Combining this with the above inequality we obtain

$$\|\theta_t(x)v'_g - v'_g\sigma_g(x)\|_2 < 2\epsilon \text{ for all } x \in (N)_1. \tag{4.3}$$

Since $\theta_t(x)v'_g \in \mathcal{H} = L^2(\theta_t(A \rtimes H^{I\setminus F}))\overline{\otimes}L^2(A \rtimes \tilde{H}^F)$, if we let T be the orthogonal projection onto $\mathcal{H}$, then combining the above with triangle inequality we obtain

$$\|T(v'_g\sigma_g(x)) - v'_g\sigma_g(x)\|_2 < 4\epsilon \text{ for all } x \in (N)_1.$$

On the other hand for every $x \in L(H)$ we have $E_{\theta_t(L(H))}(x) = |\tau(u_t)|^2\theta_t(x)$ and therefore a little computation shows that for all $\xi \in L^2(A \rtimes H^{I\setminus F})\overline{\otimes}L^2(A \rtimes \tilde{H}^F)$ we have

$$\|T(\xi)\|_2^2 \leqslant |\tau(u_t)^4|\|\xi\|_2^2 + (1 - |\tau(u_t)|^4)\|E_{A\rtimes\tilde{H}^F}(\xi)\|_2^2.$$

Using the last inequality for $v'_g\sigma_g(x)$ in combination with (4.3) we get that for all $x \in \mathcal{U}(N)$ we have

$$\begin{aligned}
& \|E_{A\rtimes\tilde{H}^F}(v'_g\sigma_g(x))\|_2^2 \\
\geqslant\ & \left(1 - |\tau(u_t)|^4\right)^{-1}\left[(\|v'_g\sigma_g(x)\|_2 - 4\epsilon)^2 - |\tau(u_t)|^4\|v'_g\sigma_g(x)\|_2^2\right] \\
=\ & \|v'_g\sigma_g(x)\|_2^2 - (1 - |\tau(u_t)|^4)^{-1}(8\epsilon\|v'_g\sigma_g(x)\|_2 - 16\epsilon^2) \\
\geqslant\ & (\|v_g\sigma_g(x)\|_2 - \epsilon)^2 - (1 - |\tau(u_t)|^4)^{-1}(8\|v'_g\sigma_g(x)\|_2\epsilon - 16\epsilon^2).
\end{aligned}$$

Choosing ϵ sufficiently small, one can find a element $g \in \Gamma$ and a constant $c > 0$ such that for all $x \in \mathcal{U}(N)$ we have

$$\begin{aligned}
\|E_{A\rtimes\tilde{H}^F}(\theta_t(x)vu_g^*)\|_2 &= \|E_{A\rtimes\tilde{H}^F}(vxu_g^*)\|_2 \\
&= \|E_{A\rtimes\tilde{H}^F}(v_g\sigma_g(x))\|_2 \geqslant c.
\end{aligned}$$

This implies $\|E_{A\rtimes\tilde{H}^F}(x\theta_{-t}(v)u_g^*)\|_2 \geqslant c$ and by expanding F to a larger finite set if necessary, we can find $v' \in A \rtimes \tilde{H}^F$ with v' close to v in $\|\cdot\|_2$ such that

$$\|E_{A\rtimes\tilde{H}^F}(x\theta_{-t}(v'))\|_2 \geqslant \frac{c}{2}.$$

Now if we further truncate v' such that it is supported on elements of $\tilde{H}^F$ with bounded word length in $\tilde{H}$ then we can find elements $a_1, ..., a_n \in M$ with

$$\sum_i \|E_{A\rtimes H^F}(xa_i)\|_2 \geqslant \frac{c}{4},$$

and therefore by Theorem 3.1 we have $N \prec_M A \rtimes H^F$. □

Applying Theorem 4.1 in the context of rigid, i.e. property (T), subalgebras of M we obtain the following structural result

Theorem 4.5. *Let H, Γ countable groups and let I a Γ-set with finite stabilizers. Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on a finite algebra A and denote by $M = A \rtimes (H \wr_I \Gamma)$. If $Q \subset pMp$ is a diffuse rigid subalgebra then one of the following alternatives holds:*

1. *$Q \prec_M A \rtimes \Gamma$,*
2. *There exists a finite set $F \subset I$ such that $Q \prec_M A \rtimes H^F$.*

Proof. Since $Q \subset pMp$ is rigid, we know that $Q \subset p\tilde{M}p$ is rigid as well. Thus, we have that $\theta_t \to id$ uniformly on the unit ball of Q and the conclusion follows from Theorem 4.1.

□

Also, for further use, we point out the following consequence of the above theorem:

Theorem 4.6. *Let H be a group with Haagerup's property and I a Γ-set with finite stabilizers. Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on an abelian algebra A and denote by $M = A \rtimes (H \wr_I \Gamma)$. If $Q \subset M$ is a diffuse property (T) subalgebra then $Q \prec_M A \rtimes \Gamma$.*

Proof. Notice that, by Theorem 4.5, we only need to show that $Q \not\prec_M A \rtimes H^I$. Below we proceed by contradiction to show this is indeed the case.

So assuming $Q \not\prec_M A \rtimes H^I$, without loosing any generality, we may actually suppose that $Q \subset A \rtimes H^I$ is a possibly non-unital subalgebra.

Since H has Haagerup property it follows that H^I also has the Haagerup property. Therefore one can find a sequence, $\{\phi_n\} \in c_o(H^I)$, of positive definite functions that converge to the constant function 1 pointwise. It is well known that the corresponding multipliers $m_n = m_{\phi_n} : A \rtimes H^I \to A \rtimes H^I$ given by $m_n(\sum a_g u_g) = \sum \phi_n(g) a_g u_g$ form a sequence of completely positive maps converging pointwise to the identity. Since Q has property (T), they must converge uniformly on the unit ball of Q. Thus there is a finite set $F \subset H^I$ such that if $x = \sum_{g \in H^I} x_g u_g \in (Q)_1$ then $\| \sum_{g \in F} x_g u_g \|_2 > \frac{1}{2}$ for all $x \in (Q)_1$.

This implies that $\sum_{g \in F} \|E_A(x u_g^*)\|_2 > \frac{1}{2}$ and by Theorem 3.1 we obtain $Q \prec_M A$, which is a contradiction because A is abelian while Q has property (T). $\square$

CHAPTER 5

Commuting Subalgebras of M

In this section we study commuting subalgebras of von Neumann algebras arising from actions by wreath product groups. Our main result is a general theorem describing the position of all subalgebras of M having large commutant. The first result in this direction was obtained by the second named author in [Pop08], in the context of von Neumann algebras arising from Bernoulli actions. For similar results the reader may consult [Oza06], [CI10].

Theorem 5.1. *Let H, Γ be countable groups with H amenable and let I be a Γ-set with finite stabilizers. Let $H \wr_I \Gamma \curvearrowright A$ be a trace preserving action on an amenable algebra A and denote by $M = A \rtimes (H \wr_I \Gamma)$. Let $p \in M$ be a projection and $P \subset pMp$ be a subalgebra with no amenable direct summand. If we denote by $Q = P' \cap pMp$ then we have that $Q \prec_M A \rtimes \Gamma$.*

Moreover, if we also assume that $A \rtimes \Gamma$ is a factor and $Q \not\prec_M A \rtimes \Gamma$ then there exists a unitary $u \in M$ such that $u^* \mathcal{N}_M(Q)'' u \subseteq A \rtimes \Gamma$.

Our proof is again based on deformation/rigidity technology, resembling the proof of Theorem 4.5. The main difference however is that, instead of property (T), we will use the "spectral gap rigidity" argument from [Pop08] to show that the deformation θ_t converges uniformly to the identity on the unit ball of Q. For the proof of Theorem 5.1 we need the following preliminary result.

Lemma 5.2. *Let M and $\tilde{M}$ as above and let ω be a free ultrafilter on $\mathbb{N}$. If $P \subset M \subset \tilde{M}$ is a subalgebra with no amenable direct summand then $P' \cap \tilde{M}^\omega \subset M^\omega$.*

Proof. The first step is to decompose the M-bimodule $L^2(\tilde{M}) \ominus L^2(M)$ as a direct sum of cyclic M-bimodules. It is a straightforward exercise for the reader to see that the above

M-bimodule can be written as a direct sum of M-bimodules $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$, where the cyclic vectors $\tilde{\eta}_s$ correspond to an enumeration of all elements of $\tilde{H}^I$ whose non-trivial coordinates start and end with non-zero powers of u.

Next, for every s, we denote by η_s the element of H^I that remains from $\tilde{\eta}_s$ after deleting all nontrivial powers of u. Also for every s let Δ_s be the support of $\tilde{\eta}_s$ in I and observe that if $Stab_\Gamma(\tilde{\eta}_s)$ denotes the stabilizing group of $\tilde{\eta}_s$ inside Γ then we have $Stab_\Gamma(\tilde{\eta}_s)(I \setminus \Delta_s) \subset (I \setminus \Delta_s)$. Hence we can consider the von Neumann algebra $K_s = A \rtimes_\sigma (H \wr_{I\setminus\Delta_s} Stab_\Gamma(\tilde{\eta}_s))$ and using similar computations as in Lemma 5 of [CI10], one can easily check that the map $x\tilde{\eta}_s y \to x\eta_s e_{K_s} y$ implements an M-bimodule isomorphism between $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$ and $L^2(\langle M, e_{K_s}\rangle)$.

Therefore, as M-bimodules, we have the following isomorphism

$$L^2(\tilde{M}) \ominus L^2(M) \cong \bigoplus_s L^2(\langle M, e_{K_s}\rangle). \tag{5.1}$$

Notice that, since I is a Γ-set with amenable, in fact finite, stabilizers if follows that $Stab_\Gamma(\tilde{\eta}_s)$ are amenable for all s. Also, since H is an amenable group and A is an amenable algebra, we conclude that the algebra K_s is amenable for all s and therefore the bimodule in (5.1) is weakly contained in a multiple of the coarse bimodule $L^2(M)\overline{\otimes}L^2(M)$, which in turn shows that P has a non-trivial amenable direct summand. □

We can now proceed with the proof of Theorem 5.1.

Proof. First we use the spectral gap argument to show that the deformation θ converges to the identity uniformly on $(Q)_1$. Indeed, exactly as in [Pop08], since P has no amenable direct summand, Lemma 5.2 implies that $P' \cap \tilde{M}^\omega \subset M^\omega$. Hence, for any $\epsilon > 0$ there exist $\delta_\epsilon > 0$ and $\mathcal{F} \in \mathcal{U}(P)$ a finite set, such that whenever $x \in \tilde{M}$ satisfies $\|[x,u]\|_2 \leqslant \delta_\epsilon$ for all $u \in \mathcal{F}$ we have that $\|x - E_M(x)\|_2 \leqslant \epsilon$.

If we let $t_\epsilon > 0$ such that $\|\theta_{t_\epsilon}(u) - u\| \leqslant \frac{\delta_\epsilon}{2}$ for all $u \in \mathcal{F}$ then the triangle inequality implies that for every $0 \leqslant t \leqslant t_\epsilon$ and every $x \in (Q)_1$ we have

$$\|[\theta_t(x), u]\|_2 \leqslant 2\|\theta_t(u) - u\| \leqslant \delta_\epsilon.$$

Therefore by the above we obtain that $\|\theta_t(x) - E_M(\theta_t(x))\|_2 \leqslant \epsilon$ and using the transversality of θ_t (Theorem 2.1) we conclude that $\|\theta_{2t}(x) - x\|_2 \leqslant 2\epsilon$ for all $x \in (Q)_1$ and $0 \leqslant t \leqslant t_\epsilon$.

In conclusion deformation θ_t converges uniformly on $(Q)_1$ and hence, by applying Theorem 4.1, we have the following two alternatives: either $Q \prec_M A \rtimes \Gamma$ or there exist a finite set F such that $Q \prec_M A \rtimes H^F$.

Next we show that the second case, together with the assumption $Q \not\prec_M A$ will lead to a contradiction. By these assumptions, using [Vae08], one can find nonzero projections $q \in Q$, $p \in A \rtimes H^F$, a $*$-homomorphism $\phi : qQq \to p(A \rtimes H^F)p$ and a partial isometry $w \in M$ such that $\phi(x)w = wx$ for all $x \in qQq$ and $\phi(qQq) \not\prec_{A \rtimes H^F} A$ for all $j \in I$.
Since $\phi(qQq)$ is a diffuse subalgebra of $p(A \rtimes H^F)p$ then part (3) of Lemma 3.4 implies that

$$\phi(qQq)' \cap pMp \subset \sum_{s \in \tilde{K}} [A \rtimes H^I] u_s. \tag{5.2}$$

On the other hand $P \subset Q' \cap M$ and hence by (5.2) we have $wPw^* \subset \sum_{s\in\tilde{K}}[A \rtimes H^I]u_s$. Since $\tilde{K}$ is finite then by using intertwining by bimodule techniques this implies that $P \prec_M A \rtimes H^I$. However, this is impossible because $A \rtimes H^I$ is amenable while P has no amenable direct summand.

Therefore the only possibility is $Q \prec_M A \rtimes \Gamma$ and the remaining part of the conclusion follows proceeding in the same way as in Theorem 4.4 *ii)* of [Pop06d]. $\square$

An algebra N is called *solid* if for every $A \subset N$ diffuse subalgebra $A' \cap N$ is amenable. As a consequence of previous theorem we obtain the following stability property similar with Corollary 8 in [CI10].

Corollary 5.3. *Let (A, τ) be an amenable von Neumann algebra and H be an amenable group. Assume that $(H \wr \Gamma) \curvearrowright A$ is a trace preserving action such that $M = A \rtimes (H \wr \Gamma)$ and $A \rtimes \Gamma$ are factors and for every diffuse $Q \subset A$ the relative commutant $Q' \cap M$ is amenable. Then $A \rtimes (H \wr \Gamma)$ is a solid if and only if $A \rtimes \Gamma$ is solid.*

Proof. Notice that the proof follows once we show that $A \rtimes \Gamma$ is solid implies $A \rtimes (H \wr \Gamma)$ is a solid. Hence assume that $A \rtimes \Gamma$ is solid and let $B \subset M = A \rtimes (H \wr \Gamma)$ be a diffuse von Neumann subalgebra. If we assume by contradiction that the commutant $P = B' \cap M$ is non-amenable, then we can find a non-zero projection $z \in \mathcal{Z}(P)$ such that Pz has no amenable direct summand. Since $[Bz, Pz] = 0$ then $Bz \prec_M A \rtimes \Gamma$ and by the hypothesis assumption we have that $Bz \nprec_M A$. Therefore, since $A \rtimes \Gamma$ is a factor then by the second part of Theorem 5.1 one can find a unitary $u \in M$ such that $u(Bz \vee Pz)u^* \subset A \rtimes \Gamma$. This however contradicts the solidity of $A \rtimes \Gamma$ and we are done.

□

Remark 5.4. *It is immediate from Theorem 5.1 that if H is an amenable group then for any non-amenable group Γ and any free, ergodic, measure preserving action $H \wr \Gamma \curvearrowright (X, \mu)$ the II_1 factor $L^\infty(X, \mu) \rtimes (H \wr \Gamma)$ is prime, i.e. it cannot be decomposed as a tensor product of two diffuse factors.*

CHAPTER 6

OE-rigidity results

Sako showed in [Sak09] that a measure equivalence between two wreath products groups $H \wr \Gamma$ and $K \wr \Lambda$, where H, K are amenable and Γ, Λ are products of non-amenable exact groups, implies the measure equivalence of the malnormal subgroups Γ and Λ. Further he showed that, given two stably orbit equivalent actions, σ and ρ, of such groups with $\sigma|_\Gamma$ and $\rho|_\Lambda$ ergodic, one has $\sigma|_\Gamma$ and $\rho|_\Lambda$ are stably orbit equivalent. He was also able to prove a similar measure equivalence rigidity for certain classes of direct products and amalgamated free products, thus obtaining rigidity results á la Monod-Shalom [MS06], as well as of Bass-Serre type [IPP08], [AG08], [CH10]. His methods rely on Ozawa's techniques [Oza04], [Oza06] involving the class $\mathcal{S}$ of groups, being C^*-algebraic in nature and depending crucially on exactness of the groups involved.

In this section we apply the results from the previous section to show that this type of orbit equivalence rigidity for wreath products holds true for much larger classes of groups (Corollary 6.3 below). The techniques we use in the proof are purely von Neumann algebra, using Popa's deformation/rigidity theory.

The Classes WR(k). Recall from the introduction that for each $k = 1, 2, 3$, we denote by $\mathbf{WR}(k)$ the class of all generalized wreath product groups $H \wr_I \Gamma$ with Γ i.c.c., I a Γ-set with finite stabilizers and satisfying the corresponding condition from below:

1. Γ has property (T) and H has Haagerup's property;
2. Γ and H have property (T) and H is i.c.c.;
3. Γ is a non-amenable product of infinite groups and H is amenable.

Theorem 6.1. *Let $H\wr_I\Gamma, K\wr_J\Lambda \in$ $\boldsymbol{WR}(k)$ and suppose that $(H\wr_I\Gamma) \curvearrowright^{\sigma} A$ and $(K\wr_J\Lambda) \curvearrowright^{\rho} B$ are free, trace preserving actions on diffuse, abelian algebras such that $\sigma|_\Gamma$ and $\rho|_\Lambda$ are ergodic. Denote by $M = A \rtimes (H \wr_I \Gamma)$, $N = B \rtimes (K \wr_J \Lambda)$, let $t > 0$ and assume that $\phi : M \to N^t$ is a $*$-isomorphism such that $\phi(A) = B^t$.*

Then one can find a unitary $u \in \mathcal{N}_{N^t}(B^t)$ such that $u^*\phi(A \rtimes \Gamma)u = (B \rtimes \Lambda)^t$.

Proof. Denote by $P = A \rtimes \Gamma$, $Q = B \rtimes \Lambda$ and observe that $A \subset P \subset M$ and $B \subset Q \subset N$. To simplify the technicalities we will assume without loosing any generality that $t = 1$. Since Γ either has property (T) or is a non-amenable product of infinite groups and ϕ is an isomorphism it follows that either $\phi(L\Gamma)$ is a property (T) subalgebra of M or $\phi(L\Gamma)$ is a non-amenable tensor product of two diffuse factors.

Below, we argue that for all cases (1)-(3) covered in the definition of the classes $\mathbf{WR}(k)$ we have

$$\phi(L\Gamma) \prec_N Q. \tag{6.1}$$

For case (1) this follows directly from Corollary 4.6 while for case (3) it follows from Theorem 5.1. Therefore it only remains to treat case (2), i.e. when all groups H, K, Γ, Λ have property (T).

Applying Theorem 4.5 we have that either $\phi(L\Gamma) \prec_N Q$ or there exists a finite subset $T \subset J$ such that $\phi(L\Gamma) \prec_N B \rtimes K^T$ and therefore to finish the proof of (6.1) it suffices to show that the second possibility leads to a contradiction.

Notice that since $\phi^{-1}(LK^T)$ is a property (T) subalgebra of M then Theorem 4.5 again implies that either $\phi^{-1}(LK^T) \prec_M P$ or there exists a finite subset $S \subset I$ such that $\phi^{-1}(LK^T) \prec_M A \rtimes H^S$. Next we show that both situations are leading to a contradiction.

Assuming the first situation, since LK^T and P are a factors, then proceeding as in the proof of Theorem 5.1 in [IPP08] one can find a nonzero projection $p_1 \in LK^{J\setminus T}$ and a unitary $u_1 \in M$ such that $u_1^*(\phi^{-1}((LK^T)p_1))u_1 \subset P$. Using Lemma 3.4, this implies that

$u_1^*(\phi^{-1}(p_1(LK^J)p_1))u_1 \subset P$. Moreover, since P is a factor, we have that $u_1^*(\phi^{-1}(L(K^J))u_1 \subset P$ and therefore Lemma 3.4 implies that $u_1^*(\phi^{-1}(L(K \wr_J \Lambda))u_1 \subset P$. However, since $\phi^{-1}(B) = A$ then by Lemma 3.2 again we have that $M = \phi^{-1}(N) \prec_M P$, which is obviously a contradiction.

Assuming the second situation, since $\phi^{-1}(B) = A$, then Lemma 3.2 gives that $\phi^{-1}(B \rtimes K^T) \prec_M A \rtimes H^S$. From the initial assumptions $B \rtimes K^T$ is a factor and therefore Lemma 3.3 implies that $\phi^{-1}(B \rtimes K^T)^\omega \subset (A \rtimes H^S)^\omega \vee M$ or equivalently

$$(B \rtimes K^T)^\omega \subset (\phi(A \rtimes H^S))^\omega \vee N. \tag{6.2}$$

Also, since $\phi(L\Gamma) \prec_N B \rtimes K^T$, the same argument as above shows that

$$(\phi(L\Gamma))^\omega \subset (B \rtimes K^T)^\omega \vee N,$$

and combining this with (6.2) we obtain that $(\phi(L\Gamma))^\omega \subset (\phi(A \rtimes H^I))^\omega \vee N$. Therefore the second part of Lemma 3.3 implies $L\Gamma \prec_M A \rtimes H^I$ but one can easily see this is again impossible.

Hence we proved (6.1) and, moreover, since $\phi(A) = B$ then Lemma 3.2 implies that

$$\phi(P) \prec_N Q. \tag{6.3}$$

Next we show that the intertwining above can be extended to unitary conjugacy preserving the Cartan subalgebra B.

By (6.3) one can find nonzero projections $p \in P$, $q \in Q$, a nonzero partial isometry $w \in M$ and a unital isomorphism $\psi : \phi(pPp) \to qQq$ such that

$$w\psi(x) = xw \text{ for all } x \in \phi(pPp). \tag{6.4}$$

The previous relation automatically implies that $ww^* \in \phi(pPp)' \cap \phi(p)N\phi(p)$ and $w^*w \in \psi(\phi(pPp))' \cap qMq$. Since P is a factor then Lemma 3.4 gives that $\phi(pPp)' \cap \phi(p)N\phi(p) = \mathbb{C}\phi(p)$ and therefore $ww^* = \phi(p)$.

Similarly, since $\psi(\phi(pPp))$ is a II_1 factor and $B \rtimes Stab_\Lambda(j)$ is a type I algebra for all $j \in J$ then $\psi(\phi(pPp)) \nprec_Q B \rtimes Stab_\Lambda(j)$ and by Lemma 3.4 we have that $\psi(\phi(pPp))' \cap qNq \subset Q$. When this is combined with the above we obtain $w^*w \in Q$ and hence relation (6.4) implies that

$$w^*\phi(P)w = w^*w\psi(\phi(pPp)) \subseteq Q. \tag{6.5}$$

Letting $v_0 \in N$ to be a unitary such that $w = ww^*v_0$, the previous relation rewrites as $v_0^*\phi(pPp)v_0 \subseteq Q$ and since Q is a factor one can find a unitary $v \in N$ such that

$$v\phi(P)v^* \subseteq Q. \tag{6.6}$$

Next we claim that $vBv^* \prec_Q B$. To see this, suppose by contradiction that $vBv^* \nprec_Q B$. Since $Stab_\Lambda(j)$ is finite for all $j \in J$ this is equivalent to $vBv^* \nprec_Q B \rtimes Stab_\Lambda(j)$. Therefore Lemma 3.4 implies that $\mathcal{N}_N(vBv^*)'' \subseteq Q$ and because vBv^* is a Cartan subalgebra of N one gets that $N \subset Q$. However this is impossible and hence we proved our claim.

Furthermore, since vBv^* and B are Cartan subalgebras of Q satisfying $vBv^* \prec_Q B$, Theorem A.1. in [Pop06a] shows that there exists a unitary $v_1 \in Q$ such that $v_1vBv^*v_1^* = B$. Therefore $u = v_1v \in \mathcal{N}_N(B)$ and combining this with (6.6) we obtain that

$$u\phi(P)u^* \subseteq Q. \tag{6.7}$$

In the remaining part of the proof we show that the two algebras above coincide. Indeed, applying the same reasoning as before for the isomorphism ϕ^{-1}, one can find a unitary $u_o \in \mathcal{N}_M(A)$ such that

$$u_o\phi^{-1}(Q)u_o^* \subseteq P,$$

and combining this with (6.7) we obtain

$$u_o\phi^{-1}(u)P\phi^{-1}(u^*)u_o^* \subseteq u_o\phi^{-1}(Q)u_o^* \subseteq P. \tag{6.8}$$

However, Lemma 3.4 implies that $u_o\phi^{-1}(u) \in P$ and therefore relation (6.8) became $u_o\phi^{-1}(u)P\phi^{-1}(u^*)u_o^* = u_o\phi^{-1}(Q)u_o^* = P$, which in particular entails that $u\phi(P)u^* = Q$. $\square$

Theorem 6.2. *Let $H \wr_I \Gamma$, $K \wr_J \Lambda$ be generalized wreath product groups such that H, K are i.c.c. groups with property (T) and I, J have finite stabilizers. Suppose that $(H \wr_I \Gamma) \curvearrowright^{\sigma} A$ and $(K \wr_J \Lambda) \curvearrowright^{\rho} B$ are free, trace preserving actions on diffuse, abelian algebras as above. Additionally, assume that $\sigma|_{H^I}$ and $\rho|_{K^J}$ are ergodic. Denote by $M = A \rtimes (H \wr_I \Gamma)$, $N = B \rtimes (K \wr_J \Lambda)$.*

If $t > 0$ and $\phi : M \to N^t$ is a $*$-isomorphism such that $\phi(A) = B^t$ then one can find a unitary $x \in \mathcal{N}_{N^t}(B^t)$ such that $x\phi(A \rtimes H^I)x^* = (B \rtimes K^J)^t$.

Proof. To simplify the technicalities we assume that $t = 1$. Since H has property (T) then $\phi(LH)$ is a rigid subalgebra of N and therefore by Theorem 4.5 we have that either $\phi(LH) \prec_N B \rtimes \Lambda$ or there exits a finite subset $T \subset J$ such that $\phi(LH) \prec_N B \rtimes K^T$. Using the same arguments as in the proof of Theorem 6.1 one can easily show that the first possibility will lead to a contradiction. Therefore we have that $\phi(LH) \prec_N B \rtimes K^T$ and by applying Lemma 3.4 we get that $\phi(LH^I) \prec_N B \rtimes K^J$. Applying Lemma 3.2 this further implies that $\phi(A \rtimes H^I) \prec_N B \rtimes K^J$ and therefore there exists a $A \rtimes H^I$-$B \rtimes K^J$ bimodule $\mathcal{H}$ with finite dimension over $B \rtimes K^J$.

A similar argument for ϕ^{-1} shows that $B \rtimes K^J \prec_N \phi(A \rtimes H^I)$ and hence one can find a nonzero $B \rtimes K^J$-$A \rtimes H^I$ bimodule $\mathcal{K}$ with finite dimension over $B \rtimes K^J$. Since Γ, Λ are i.c.c. and $B \rtimes K^J$ and $\phi(A \rtimes \Gamma)$ are irreducible, regular subfactors of N then, by Theorem 8.4 in [IPP08], there exists a unitary $u \in N$ such that $u\phi(A \rtimes H^I)u^* = B \rtimes K^J$. Denoting by $\psi_u = Ad(u)$ this further implies that $\psi_u \circ \phi$ is an isomorphism from $A \rtimes H^I$ onto $B \rtimes K^J$ which satisfies

$$\psi_u \circ \phi(a)u = u\phi(a),$$

for all $a \in A$. Next we consider the Fourier decomposition $u = \sum_{\lambda \in \Lambda} y_\lambda v_\lambda$ with $y_\lambda \in B \rtimes K^J$ and using the above equation there exists a nonzero element $y_\lambda \in B \rtimes K^J$ such that for all $a \in A$ we have

$$\psi_u \circ \phi(a)y_\lambda = y_\lambda \rho_\lambda(\phi(a)). \tag{6.9}$$

Note that since $B = \phi(A)$ is a maximal abelian subalgebra of N then (6.9) implies that

$y_\lambda^* y_\lambda \in B$. Furthermore taking the polar decomposition $y_\lambda = w_\lambda |y_\lambda|$ with w_λ partial isometry in (6.9) we conclude that

$$\psi_u \circ \phi(a) w_\lambda = w_\lambda \rho_\lambda(\phi(a)),$$

for all $a \in A$.

This shows in particular $\psi_u(B) \prec_{B \rtimes K^J} B$ and since B and $\psi_u(B)$ are Cartan subalgebras of $B \rtimes K^J$ then by Theorem A.1 [Pop06a] there exists a unitary $u_o \in B \rtimes K^J$ such that $u_o \psi_u(B) u_o^* = B$. Finally the conclusion follows by letting $x = u_o u \in \mathcal{N}_N(B)$. $\square$

We now have the following immediate corollary of Theorem 6.1:

Corollary 6.3. *Given* $1 \leqslant k \leqslant 3$ *let* $H \wr_I \Gamma, K \wr_J \Lambda \in \boldsymbol{WR}(k)$. *Let* σ *and* ρ *be stably orbit equivalent actions of* $H \wr_I \Gamma$ *and* $K \wr_J \Lambda$, *respectively. If one assumes that* $\sigma|_\Gamma$ *and* $\rho|_\Lambda$ *are ergodic then we have* $\sigma|_\Gamma \cong_{SOE} \rho|_\Lambda$.

A natural question one may ask is to try classifying *all* groups Γ and H for which the above orbit equivalence rigidity phenomena holds. This however remains widely open as for the moment it is unclear what general condition one may be formulate at the level of groups Γ and H to insure this type of rigidity. For instance even when assuming Γ has property (T) it is not obvious what are all groups H for which this rigidity holds.

Another interesting problem is to find situations when a similar orbit equivalence rigidity can be upgraded also at the level of the "core" groups H^I and K^J. A desirable result in this direction would be that an orbit equivalence between actions of $H \wr \Gamma$ and $K \wr \Lambda$ induces an orbit equivalence not only between the malnormal groups Γ and Λ but also between the normal groups H^Γ and K^Λ. Notice that combining Theorems 6.3 and 6.2 above we obtain one instance of this phenomenon.

Corollary 6.4. *If* $H \wr_I \Gamma, K \wr_J \Lambda \in \boldsymbol{WR}(2)$ *and* σ *and* ρ *are as above. If we additionally assume that* $\sigma|_{H^I}$ *and* $\rho|_{K^J}$ *are ergodic then we also have that* $\sigma|_{H^I} \cong_{SOE} \rho|_{K^J}$.

CHAPTER 7

W*-rigidity results

Some of the technical results obtained in the previous sections can be pushed to slightly more general situations. For instance rather than studying commuting subalgebras algebras of von Neumann algebras arising from actions by wreath product groups one can study *weakly compact embeddings.*

This notion was introduced by Ozawa and Popa and it was triggered by their discovery that in a free group factor M the normalizing group $\mathcal{N}_M(P)$ of any amenable algebra P acts on P by conjugation in a "compact" way [OP10]. This was a key ingredient which allowed the authors to prove that in a free group factor the normalizing algebra of any amenable subalgebra is still amenable. For reader's convenience, we recall the following definition from [OP10]:

Definition. Let $\Lambda \overset{\sigma}{\curvearrowright} P$ where P is a finite von Neumann algebra. The action σ is called *weakly compact* if there exist a net (η_α) of unit vectors in $L^2(P\bar{\otimes}\bar{P})_+$ such that:

$$\|\eta_\alpha - (v \otimes \bar{v})\eta_\alpha\|_2 \to 0 \qquad \text{for all } v \in \mathcal{U}(P); \tag{7.1}$$

$$\|\eta_\alpha - \sigma_g \otimes \bar{\sigma}_g(\eta_\alpha)\|_2 \to 0 \qquad \text{for all } g \in \Gamma; \tag{7.2}$$

$$\langle (x \otimes 1)\eta_\alpha, \eta_\alpha\rangle = \tau(x) = \langle \eta_\alpha, (1 \otimes \bar{x})\eta_\alpha\rangle \qquad \text{for all } \alpha \text{ and } x \in P. \tag{7.3}$$

More generally, if $P \subset M$ is a subalgebra such that the action by conjugation of the normalizing group $\mathcal{N}_M(P)$ on P is weakly compact then we say that the inclusion $P \subset M$ is a *weakly compact embedding.* It is straightforward from the definitions that every compact action $\Lambda \overset{\sigma}{\curvearrowright} P$ is automatically weakly compact and hence every profinite action [Ioa11] is also weakly compact.

In the main result of this section we describe all weakly compact embeddings in cross-products algebras of type $M = A \rtimes (H \wr \Gamma)$ with A an amenable algebra and H an amenable group. Roughly speaking, we obtain a dichotomy result asserting that every weakly compact embedding in M, either has "small" normalizing algebra or "lives" inside $A \rtimes \Gamma$. This should be seen as analogous to Theorem 4.9 in [OP10]. In fact our proof follows the same recipe as the proof of Theorem 4.9 in [OP10]. The main difference at the technical level is that instead of working with the malleable deformation for actions of free groups we will work with the deformation described in the first section. Therefore the compactness argument used in the proof of Theorem 4.9 in [OP10] will be replaced by the transversality property from Theorem 2.1. Most of the arguments used in [OP10] apply verbatim in our situation and we include some details only for reader's convenience.

Theorem 7.1. *Let (A, τ) be an amenable von Neumann algebra and H be an amenable group. Assume that $H \wr \Gamma \curvearrowright A$ is a trace preserving action and denote by $M = A \rtimes (H \wr \Gamma)$. If we assume that $P \subset M$ is a diffuse subalgebra and $\mathcal{G} \subset \mathcal{N}_M(P)$ is such that $\mathcal{G}$ act weakly compactly on P and such that $\mathcal{N}_M(P)' \cap M = \mathbb{C}1$ then one of the following must hold true:*

1. *There exists a nonzero projection $p \in P$ such that $p(P \cap \mathcal{G})''p$ is amenable.*
2. *$P \prec_M A \rtimes \Gamma$.*

If we assume in addition that $P \subset M$ is a Cartan subalgebra then we have that $P \prec_M A$.

Proof. Let $\mathcal{G} \subset \mathcal{N}_M(P)$ be a subgroup that acts weakly compactly on P and assume that $\mathcal{U}(P) \subset \mathcal{G}$. First we will show that, when we view $P \subset \tilde{M}$, if θ_t does not converge uniformly on $(P)_1$ then $\mathcal{G}''$ is amenable.

So let us assume that θ_t does not converge uniformly on $(P)_1$. Therefore by transversality of θ_t, Theorem 2.1, one can find a constant $0 < c < 1$, and infinite sequences $t_k \in \mathbb{R}$, $u_k \in \mathcal{U}(P)$ such that $t_k \to 0$ and

$$\|\theta_{t_k}(u_k) - E_M(\theta_{t_k}(u_k))\|_2 \geqslant c.$$

Since $\|\theta_{t_k}(u_k)\|_2 = 1$ then Pythagorean theorem further implies that

$$\|E_M(\theta_{t_k}(u_k))\|_2 \leqslant \sqrt{1-c^2}. \tag{7.4}$$

Now we fix $\epsilon > 0$ and $F \subset \mathcal{G}$ a finite set. Then we choose $\delta > 0$ satisfying $1-2\delta > \sqrt{1-c^2}$ and k sufficiently large such that for all $u \in F$ we have

$$\|u - \theta_{t_k}(u)\|_2 \leqslant \frac{\epsilon}{6}.$$

For the rest of the proof we denote by $\theta = \theta_{t_k}$ and $v = u_k$ and let (η_α) be as in the definition of weak compactness. Then we consider the following nets

$$\begin{aligned}
&\tilde{\eta}_\alpha = (\theta \otimes 1)(\eta_\alpha) \in L^2(\tilde{M})\overline{\otimes} L^2(\overline{M}),\\
&\zeta_\alpha = (e_M \otimes 1)(\tilde{\eta}_\alpha) \in L^2(M)\overline{\otimes} L^2(\overline{M}),\\
&\zeta_\alpha^\perp = \tilde{\eta}_\alpha - \zeta_\alpha \in (L^2(\tilde{M}) \ominus L^2(M))\overline{\otimes} L^2(\overline{M}).
\end{aligned}$$

Using the identity $\|(x \otimes 1)\tilde{\eta}_\alpha\|_2^2 = \tau(E_M(\theta^{-1}(x^*x))) = \|x\|_2^2$ then for every $u \in F$ and a sufficiently large α we obtain the following inequalities

$$\|[u \otimes \overline{u}, \zeta_\alpha^\perp]\|_2 \leqslant \|[u \otimes \overline{u}, \tilde{\eta}_\alpha]\|_2 \leqslant \|(\theta \otimes 1)([u \otimes \overline{u}, \eta_\alpha])\|_2 + 2\|u - \theta(u)\|_2 \leqslant \frac{\epsilon}{2}.$$

Below we proceed by contradiction to show the following inequality

$$\mathrm{Lim}_\alpha \|\zeta_\alpha^\perp\|_2 > \delta. \tag{7.5}$$

Assuming (7.5) does not hold we get the following estimations:

$$\begin{aligned}
\mathrm{Lim}_\alpha\|\tilde{\eta}_\alpha - (E_M(\theta(v)) \otimes \overline{v})\zeta_\alpha\|_2 &\leqslant \mathrm{Lim}_\alpha\|\tilde{\eta}_\alpha - ((E_M \otimes id)(\theta(v) \otimes \overline{v})\zeta_\alpha\|_2\\
&\leqslant \mathrm{Lim}_\alpha\|\tilde{\eta}_\alpha - (e_M \otimes id)((\theta(v) \otimes \overline{v})\zeta_\alpha)\|_2\\
&\leqslant \mathrm{Lim}_\alpha\|(e_M \otimes id)\tilde{\eta}_\alpha - (e_M \otimes id)(\theta(v) \otimes \overline{v})\tilde{\eta}_\alpha\|_2 + 2\delta\\
&\leqslant \mathrm{Lim}_\alpha\|(e_M \otimes 1)(\theta \otimes id)(\eta_\alpha - (v \otimes \overline{v})\eta_\alpha)\|_2 + 2\delta\\
&= 2\delta
\end{aligned}$$

Now using the above inequalities we obtain

$$\|E_M(\theta(v))\|_2 \geqslant \mathrm{Lim}_\alpha\|((E_M(\theta(v)))\otimes\overline{v})\zeta_\alpha\|_2 \geqslant \mathrm{Lim}_\alpha\|\tilde{\eta}_\alpha\|_2 - 2\delta \geqslant \sqrt{1-c^2},$$

which obviously contradicts (7.4). Thus we have shown that $\mathrm{Lim}_\alpha\|\zeta_\alpha^\perp\| > \delta$.

For large enough α, the vector $\zeta = \zeta_\alpha^\perp \in \mathcal{H}$ satisfies $\|\zeta\|_2 \geqslant \delta$ and $\|[u\otimes u, \zeta]\|_2 \leqslant \frac{\varepsilon}{2}$, for all $u \in F$. Also, for every $x \in M$ we have that

$$\begin{aligned}\|(x\otimes 1)\zeta\|_2 &= \|(x\otimes 1)(e_M^\perp \otimes 1)\tilde{\eta}_\alpha\|_2\\ &= \|(e_M^\perp\otimes 1)(x\otimes 1)\tilde{\eta}_\alpha\|_2\\ &\leqslant \|(x\otimes 1)\tilde{\eta}_\alpha\|_2 = \|x\|_2.\end{aligned}$$

Using Lemma 5.2 we can view ζ as a vector in $(\bigoplus_i L^2(\langle M, e_{K_i}\rangle))\overline{\otimes}L^2(M)$. Since K_i is amenable then $L^2(\langle M, e_{K_i}\rangle)$ is weakly contained in the coarse bimodule $L^2(M)\overline{\otimes}L^2(M)$. Therefore we can assume $\zeta = (\zeta_i)_i$, with $\zeta_i \in (L^2(M)\overline{\otimes}L^2(M))\overline{\otimes}L^2(M)$. Define $\zeta_i' = ((\mathrm{id}\otimes\tau)(\zeta_i\zeta_i^*))^{\frac{1}{2}} \in L^2(M)\overline{\otimes}L^2(M)$ and $\zeta' = (\zeta_i')_i \in \bigoplus_{i=1}^\infty(L^2(M)\overline{\otimes}L^2(M))$. By proceeding exactly as in the last part of the proof of Theorem 4.9. in [OP10], one derives that $\|x\zeta'\|_2 \leqslant \|x\|_2$ for all $x \in M$, $\|[u,\zeta']\|_2 \leqslant \varepsilon$ for all $u \in F$ and $\|\zeta'\|_2 \geqslant \delta$. But then Corollary 2.3. in [OP10] shows that $\mathcal{G}''$ is amenable.

So now we are left to deal with the case when θ_t does converges uniformly on $(P)_1$. In this case Theorem 4.1 implies that $P \prec_M A\rtimes\Gamma$ or $P \prec_M A \rtimes H^F$ for some finite set $F \subset \Gamma$. Since the first case already gives one of the conclusions of our theorem, for the remaining part we assume that $P \not\prec_M A\rtimes\Gamma$ and $P \prec_M A\rtimes H^F$.

Since $P \not\prec_M A\rtimes\Gamma$ then $P \not\prec_M A$. Since $P \prec_M A\rtimes H^F$, after cutting by a projection, p, and applying a homomorphism we can assume $pPp \subset A\rtimes H^F$. Now we can apply part (3) of Lemma 3.4 to get that

$$(\mathcal{N}_{pMp}(pPp))'' \prec_M A\rtimes H^\Gamma.$$

Since $A\rtimes H^\Gamma$ is amenable, we get that $(\mathcal{N}_{pMp}(pPp))''$ and thus $p(P\cap\mathcal{G})''p$ is amenable as well.

□

When combined with results from previous section, this technical result allows us to derive a strong W^*-rigidity result for compact actions of certain wreath product groups. To introduce the result let us recall first the following definition.

Definition. Let $\Gamma \curvearrowright X$ and $\Lambda \curvearrowright Y$ be two free, ergodic actions. We say that they are *virtually conjugate* if one can find finite index subgroups, $\Gamma_1 \subset \Gamma$ and $\Lambda_1 \subset \Lambda$, positive measure subsets $X_1 \subset X$ and $Y_1 \subset Y$ with X_1 being Γ_1-invariant and Y_1 being Λ_1-invariant such that the restrictions $\Gamma_1 \curvearrowright X_1$ and $\Lambda_1 \curvearrowright Y_1$ are conjugate.

Theorem 7.2. *Let H, K be amenable groups and Γ, Λ groups with the property (T). Assume that $H \wr \Gamma \curvearrowright^{\sigma} X$ and $K \wr \Lambda \curvearrowright^{\rho} Y$ are free, measure preserving action such that $\sigma_{|\Gamma}$ is compact, ergodic and $\rho_{|\Lambda}$ is ergodic. If $L^\infty(X) \rtimes (H \wr \Gamma) \simeq L^\infty(Y) \rtimes (K \wr \Lambda)$, then $\Gamma \curvearrowright^{\sigma_{|\Gamma}} X$ is virtually conjugate to $\Lambda \curvearrowright^{\rho_{|\Lambda}} Y$.*

Proof. Denote by $M = L^\infty(X) \rtimes_\sigma (H \wr \Gamma)$ and $N = L^\infty(Y) \rtimes_\rho (K \wr \Lambda)$. By assumption there exists a $*$-isomorphism θ between M and N and since $\sigma|_\Gamma$ is compact, and thus weakly compact, then we can apply the previous result. Noticing that $\theta(L^\infty(X))$ is regular in N, the second part of the Theorem 7.1 implies that $\theta(L^\infty(X)) \prec_N L^\infty(Y)$. Furthermore, since both $\theta(L^\infty(X))$ and $L^\infty(Y)$ are Cartan subalgebras of N, one can find a unitary $u \in N$ such that $u\theta(L^\infty(X))u^* = L^\infty(Y)$. In particular, we have obtained that $H \wr \Gamma \curvearrowright^{\sigma} X \cong_{OE} K \wr \Lambda \curvearrowright^{\rho} Y$ which, by Theorem 6.1, implies that $\Gamma \curvearrowright^{\sigma_{|\Gamma}} X \cong_{OE} \Lambda \curvearrowright^{\rho_{|\Lambda}} Y$. Finally, the conclusion follows by applying Ioana's Cocycle Superrigidity Theorem from [Ioa11]. □

Remark 7.3. *Note that the requirements that Γ have property (T) and that σ be compact on Γ in the previous theorem, forces Γ to be residually finite. Indeed, first note that since Γ has property (T), it is finitely generated. Also recall that if the action $\Gamma \curvearrowright (X, \mu)$ is compact then the associated unitary representation on $L^2(X, \mu)$ decomposes as a direct sum of finite dimensional representations, which we denote $\bigoplus_{i \in I}(\pi_i, \mathcal{H}_i)$. So if the action is faithful (which is the case, because it is free), then given $g \in \Gamma$ we can chose $i \in I$ such* *that* $\pi_i(g)$ *is nontrivial. Since the image of* Γ *under* π_i *is finite dimensional and* Γ *is finitely generated, by a theorem of Mal'cev (see [Mal40]), the group* $\pi_i(\Gamma)$ *is residually finite. Thus there is a finite group* $G_{i,g}$ *and a homomorphism* $\phi_{i,g} : \pi_i(\Gamma) \to G_{i,g}$ *such that* $\phi_{i,g} \circ \pi_i(g)$ *is non trivial. Thus* Γ *has a finite quotient* $\phi_{i,g} \circ \pi_i(\Gamma)$ *in which the image of* g *is non-trivial, showing that* Γ *is residually finite. Note also that if* H *is a residually finite abelian group (e.g. if it is finitely generated abelian), then* $H \wr \Gamma$ *follows residually finite as well (see e.g. [Gru57]). Finally, in order to see that there are many actions of wreath product groups verifying the conditions in 6.4, note that if* $H \wr \Gamma$ *is residually finite then it has profinite (thus compact) actions. Altogether, we can take* Γ *to be any "classic" Kazhdan group, like* $SL(n, \mathbb{Z})$, $n \geq 3$, *and* H *to be any finitely generated abelian group, like* $\mathbb{Z}^k$, $(\mathbb{Z}/m\mathbb{Z})^k$, *etc.*

CHAPTER 8

On Uniquenesss of Tensor Products Decomposition and Relative Amenability.

In the remainder of this work we will focus on proving the uniqueness of tensor product decomposition for wreath product factors. A major goal of the study of II_1 factors is the classification of these algebras based on the "input data" that goes into their construction. A significant landmark was the result, due to Connes [Con76], that all amenable II_1 factors are isomorphic. However, in the non-amenable realm there is a much greater variety, and a striking classification theory has developed.

One thrust of this research is to determine if some algebra which, *a priori*, is constructed in one manner, can be obtained in some other manner. For example, if we have a II_1 factor that we know to be a free product of two II_1 factors, is it also possible to be the tensor product of two (possibly different) II_1 factors?

In this vein we study whether certain factors can be written as a tensor product in two distinct ways. Such results go back to the study of prime factors, (ie. a factor which cannot be written as the tensor product of two other II_1 factors.) The first result was obtained by Popa in, [Pop83], where he showed that the group von Neumann algebra of an uncountable free group is prime.

Later, in [Ge98], Ge proves that all group factors coming from finitely generated free groups are prime. Using C^* techniques this was greatly generalized by Ozawa, [Oza04], to show that all i.c.c. Gromov hyperbolic groups give rise to prime factors. Also, using his deformation/rigidity theory, Popa showed in [Pop08] that all II_1 factors arising from the

Bernoulli actions of nonamenable groups are prime. Further, Peterson used his derivation approach to deformation/rigidity ([Pet09]) to prove that any II_1 factor coming from a countable group with positive first l^2-betti number is also prime. Finally we should also note that using Popa's deformation/rigidity theory, Chifan and Houdayer, [CH10], gave many more examples of prime II_1-factors coming from amalgamated free products.

A natural question about prime factors is whether a tensor product of a finite number of such factors $P_1, P_2, ..., P_n$, has a "unique prime factor decomposition", i.e., if $P_1\overline{\otimes}...\overline{\otimes}P_n = Q_1\overline{\otimes}....\overline{\otimes}Q_m$, for some other prime factors Q_j, forces $n = m$ and P_i unitary conjugate to Q_i, modulo some permutation of indices and modulo some "rescaling" by appropriate amplifications of the prime factors involved. A first such result was obtained by Ozawa and Popa in [OP04], where a combination of C^* techniques from [Oza04] and intertwining techniques from [Pop06d] is used to show that any II_1 factor arising from a tensor product of hyperbolic group factors has such a unique tensor product decomposition. A similar prime factor decomposition is in fact obtained in [Pet09], for tensor products of II_1 factors arising from groups with positive first l^2-betti number.

In this paper we prove an analogous unique prime factor decomposition result for tensor products of wreath product II_1 factors. More precisely, we prove the following result:

Theorem 8.1. *Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; and $Q_1, \ldots, Q_k$ be diffuse von Neumann algebras such that*

$$M = L(A_1 \wr H_1)\overline{\otimes} \ldots \overline{\otimes}L(A_n \wr H_n) = Q_1\overline{\otimes} \ldots \overline{\otimes}Q_k$$

If $k \geq n$, then $n = k$, and after permutation of indices we have that $L(A_i \wr H_i) \simeq Q_i^{t_i}$ for some positive numbers $t_1, t_2, \ldots t_n$ whose product is 1.

Also we have a natural generalization of this theorem to unique decomposition for orbit equivalence relations coming from finite products of wreath product groups. Such results are reminiscent of measure equivalence results that were achieved for products of groups of the class $\mathcal{C}_{reg}$ by Monod and Shalom (Theorem 1.16 in [MS06]), for products of bi-exact groups by Sako (Theorem 4 in [Sak09]), and for products of groups in $\mathcal{QH}_{reg}$ by Chifan and Sinclair (Corollary C in [CS11].)

Theorem 8.2. *Let $A_1, \ldots, A_n$ be non-trivial amenable groups; $H_1, \ldots, H_n$ be non-amenable groups; Let σ be a free ergodic action of $A_1 \wr H_1 \times \cdots \times A_n \wr H_n$, such that $\sigma|_{A_i \wr H_i}$ is ergodic for all i. Also let $K_1, \ldots, K_m$ be groups and ρ be an action of $K_1 \times \cdots \times K_m$ such that σ is stably orbit equivalent to ρ.*

If $m \geq n$, then $n = m$, and after permutation of indices we have that $\sigma|_{A_i \wr H_i} \simeq_{SOE} \rho|_{K_i}$.

We prove these results by using deformation/rigidity theory. More precisely, we use the same malleable deformation for wreath product group factors as above, combined with Popa's spectral gap rigidity and intertwining by bimodules techniques.

Following [OP10] we have the following definition:

Definition. Let $P, Q \subset M$ be finite von Neuman algebras. We say that P *is amenable over Q inside M*, which we denote $P \lessdot_M Q$, if there is a P-central state, φ, on $\langle M, e_Q \rangle$ such that $\varphi|_M = \tau$, where τ is the trace on M.

Let us note that by Theorem 2.1 in [OP10] $P \lessdot_M Q$ is equivalent to $L^2(P) \prec \bigoplus L^2(\langle M, e_Q \rangle)$ as P-bimodules. Further, if $P \prec_M Q$ then $L^2(M)$ contains a sub P-Q-module, $\mathcal{H}$, that is finitely generated as a right Q module. Therefore, the projection onto this module will commute with the right action of Q and will have finite trace. Therefore, it will be a vector in $L^2(\langle M, e_N \rangle)$. Further, it will also commute with P, so if we look at $L^2(\langle M, e_N \rangle)$ as a P-bimodule, it will contain a central vector. Since strong containment implies weak containment we get the following observation.

Proposition 8.3. *Let $P, Q \subset M$ be von Neumann algebras. If $P \prec_M Q$ then $P \lessdot_M Q$.*

CHAPTER 9

Generalization of Intertwining Techniques for Wreath Products

In this section we generalize earlier intertwining results for wreath products proven above. While this leads to some repetition we have tried to keep this to a minimum, and thus try to reference the above proofs and indicate where changes are needed in order to arrive at the new results that are necessary in order to prove our desired uniqueness of of tensor product decomposition.

The following proposition is a relative version of Lemma 5.2 above, and will follow a similar proof.

Proposition 9.1. *Let N be a finite von Neumann algebra. Let A, H be groups with A non-trivial amenable and H non-amenable. Let $Q \subset N \rtimes A \wr H = M$ be an inclusion of von Neumann algebras. Assume Q is not amenable over N inside M then $Q' \cap \tilde{M}^\omega \subseteq M^\omega$.*

Proof. As mentioned above this proof follows closely the proof of Lemma 5.2 above as well as Lemma 5.1 in [Pop08] and other similar results in the literature.

We will prove the contrapositive so let us assume that $Q' \cap \tilde{M}^\omega \nsubseteq M^\omega$ Then proceeding as in Lemma 5.1 in [Pop08] We see that

$$L^2(Q) \prec L^2(\tilde{M}) \ominus L^2(M)$$

as Q-bimodules. Now we decompose $L^2(\tilde{M}) \ominus L^2(M)$ as an M-bimodule.

One can see that the above M-bimodule can be written as a direct sum of M-bimodules

$\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$, where the cyclic vectors $\tilde{\eta}_s$ correspond to an enumeration of all elements of $\tilde{A}^H$ whose non-trivial coordinates start and end with non-zero powers of u.

Next, for every s, we denote by η_s the element of A^H that remains from $\tilde{\eta}_s$ after deleting all nontrivial powers of u. Also for every s let $\Delta_s \subset H$ be the support of $\tilde{\eta}_s$ and observe that if $Stab_H(\tilde{\eta}_s)$ denotes the stabilizing group of $\tilde{\eta}_s$ inside H then we have $Stab_H\,(\tilde{\eta}_s)(H \setminus \Delta_s) \subset H \setminus \Delta_s$.

Hence we can consider the von Neumann algebra $K_s = N \rtimes (A \wr_{H\setminus\Delta_s} Stab_H\,(\tilde{\eta}_s))$ and using similar computations as in Lemma 5.1 of [Pop08], one can easily check that the map $x\tilde{\eta}_s y \to x\eta_s e_{K_s} y$ implements an M-bimodule isomorphism between $\overline{M\tilde{\eta}_s M}^{\|\cdot\|_2}$ and $L^2(\langle M, e_{K_s}\rangle)$.

Therefore, as M-bimodules, we have the following isomorphism

$$L^2(\tilde{M}) \ominus L^2(M) = \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Thus we can get the following weak containment of Q-bimodules

$$L^2(Q) \prec \bigoplus L^2(\langle M, e_{K_s}\rangle).$$

Notice that, since Δ_s is finite, and the action of H on itself is free, then $Stab_H(\tilde{\eta}_s)$ is finite for all s. Also, since A is an amenable group we have that $K_s \lessdot_N N$ for all s. Thus for all s we have the following weak containment of K_s-bimodules

$$L^2(K_s) \prec \bigoplus L^2(\langle K_s, e_N\rangle) \simeq \bigoplus L^2(K_s) \otimes_N L^2(K_s)$$

Now if we induce to M-bimodules and restrict to Q-bimodules and use continuity of weak containment under induction and restriction we get the following inclusions of Q-bimodules:

$$\begin{aligned} L^2(Q) &\prec \bigoplus L^2(\langle M, e_{K_s}\rangle) \\ &\simeq \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_{K_s} L^2(M) \\ &\prec \bigoplus L^2(M) \otimes_{K_s} L^2(K_s) \otimes_N L^2(K_s) \otimes_{K_s} L^2(M) \\ &\simeq \bigoplus L^2(M) \otimes_N L^2(M) \\ &\simeq \bigoplus L^2(\langle M, e_N\rangle) \end{aligned}$$

Thus $Q \lessdot_M N$

□

We finish this section with a final theorem which allows us to locate regular subfactors with large commutant.

Theorem 9.2. *Let N be a finite von Neumann algebra. Let A and H be groups with A non-trivial amenable and H non-amenable. Let $Q \subset N \rtimes A \wr H = M$ be a subalgebra that is not amenable over N. Let $P = Q' \cap M$. If P is a regular subfactor of M then $P \prec_M N$.*

Proof. Applying Proposition 9.1 and following the proof of Theorem 5.1 above we see that the deformation θ_t converges uniformly on the unit ball of P, and thus by Theorem 4.1 above we have that $P \prec_M N \rtimes A^H$ or $P \prec_M N \rtimes H$.

Following the same argument as Theorem 5.1 if we assume that $P \prec_M N \rtimes A^H$ and $P \not\prec_M N$ then we get $Q \prec_M N \rtimes A \wr H_0$ for some finite subgroup $H_0 \subset H$. Since A is amenable and H_0 is finite then $N \rtimes A \wr H_0 \lessdot_M N$. So since $Q \prec_M N \rtimes A \wr H_0$ then by Proposition 8.3 we have $Q \lessdot_M N \rtimes A \wr H_0$. Then by part 3 of Proposition 2.4 in [OP10] we have that $Q \lessdot_M N$ contradicting our assumption.

Thus $P \prec_M N \rtimes H$. Therefore, by Theorem 3.1, there exists nonzero projections $p \in P, q \in N \rtimes H$, a nonzero partial isometry $v \in M$, and a *-homomorphism $\varphi : pPp \to q(N \rtimes H)q$ such that $vx = \varphi(x)v, \forall x \in pPp$. Furthermore we have that $v^*v = p$ and

$vv^* = \hat{q} \in \varphi(pPp)' \cap qMq$. Also, by Lemma 3.5 in [Pop06d] we know that pPp is a regular subalgebra of pMp.

Then for all $u \in \mathcal{N}_{pMp}(pPp)$ let us calculate:

$$\begin{aligned} \varphi(x)vuv^* &= vxuv^* \\ &= vu(u^*xu)v^* \\ &= vuv^*v(u^*xu)v^* \\ &= vuv^*\varphi(u^*xu)vv^* \\ &= vuv^*\varphi(u^*xu) \end{aligned}$$

Now assume that $P \not\prec_M N$, then by part (2) of Lemma 3.4 in above we have that $vuv^* \in N \rtimes H$. Since pPp is regular in pMp we would then get that $M \prec_M N \rtimes H$. However, this is impossible since the fact that A is nontrivial implies that $[M : N \rtimes H] = \infty$.

□

CHAPTER 10

Proof of Uniqueness of Tensor Product Decomposition

In this section we prove our main theorem. Our main technical tool is the following, which is proposition 2.7 in [PV11]. Before we state the result let us recall that two von Neumann subalgebras $M_1, M_2 \subset M$ of a finite von Neumann algebra M are said to form a commuting square if $E_{M_1}E_{M_2} = E_{M_2}E_{M_1}$.

Theorem 10.1 (Popa-Vaes, [PV11])**.** *Let (M, τ) be a tracial von Neumann algebra with von Neumann subalgebras $M_1, M_2 \subset M$. Assume that M_1 and M_2 form a commuting square and that M_1 is regular in M. If a von Neumann subalgebra $Q \subset pMp$ is amenable relative to both M_1 and M_2, then Q is amenable relative to $M_1 \cap M_2$.*

Notice that this theorem allows us to eliminate the case where Q is amenable over M_1. More specifically we have the following observation.

Proposition 10.2. *Let G_1 and G_2 be groups. Let A be a finite amenable von neumann algebra with an action of $G_1 \times G_2$, and let $Q \subset A \rtimes G_1 \times G_2$ be a nonamenable subalgebra. Then there exists an i such that Q is not amenable over $A \rtimes G_i$.*

Proof. If we let $A \rtimes G_i = M_i$ then it is easy to see that $M_1, M_2 \subset M$ form a commuting square. So if Q is amenable over both M_i we would have that it would be amenable over the intersection, which is A. Since A is amenable this would imply that Q is amenable. □

Finally combining the above results we can prove our main theorem (Theorem 8.1).

Proof. First let us mention that for the case $n = 1$, this is equivalent to the primeness of II_1-factors arising from Bernoulli shifts, which was proven in [Pop08].

Now notice that we can write M as $M = N_i \rtimes_\sigma A_i \wr H_i$, where $N_i = L(A_1 \wr H_1)\overline{\otimes} \dots \overline{\otimes} L(A_{i-1} \wr H_{i-1})\overline{\otimes} L(A_{i+1} \wr H_{i+1})\overline{\otimes} \dots \overline{\otimes} L(A_n \wr H_n)$ and σ is the trivial action.

Let us define $\widehat{Q_i} = (Q_i)' \cap M = Q_1 \overline{\otimes} \dots \overline{\otimes} Q_{i-1} \overline{\otimes} Q_{i+1} \overline{\otimes} \dots \overline{\otimes} Q_k$. Since $H_i \wr \Gamma_i$ does not have property Gamma for all i this implies, in particular, that Q_1 is non-amenable. By proposition 10.2, where we let $A = \mathbb{C}$, we know that there is an i such that Q_1 is not amenable over N_i

Since $\widehat{Q_1}$ is a regular subalgebra of M, then by Theorem 9.2 we get that $\widehat{Q_1} \prec_M N$.

We complete the argument by following Proposition 12 and the induction argument of the proof of Theorem 1 in [OP04].

□

Now we have the proof of the final result (8.2.)

Proof. Let $A_1 \wr H_1, ..., A_n \wr H_n$, $K_1, ..., K_m, \sigma, \rho$ be as above. Thus we know that since $A_i \wr H_i$ is nonamenable for all i, then K_j is nonamenable for all j.

Now we know that there are actions on $L^\infty(X)$ such that $M = L^\infty(X) \rtimes A_1 \wr H_1 \times \cdots \times A_n \wr H_n \simeq (L^\infty(X) \rtimes K_1 \times \cdots \times K_m)^t$. We may assume that $t = 1$.

Let $N_i = A \rtimes A_1 \wr H_1 \times \dots A_{i-1} \wr H_{i-1} \times A_{i+1} \wr H_{i+1} \times \cdots \times A_n \wr H_n$, so that we have $M = N_i \rtimes A_i \wr H_i$. As in the proof of the previous theorem, since K_i is nonamenable, there is an i such that $L(K_1)$ is nonamenable over N_i. Now by the proof of Theorem 9.2 this implies that $L(K_1)' \cap M = L(K_2 \times \cdots \times K_m) \prec N_i \rtimes H_i$. Thus by Lemma 3.2 above we have that $A \rtimes K_2 \times \cdots \times K_m \prec N_i \rtimes H_i$. Now since $A \rtimes K_2 \times \cdots \times K_m$ is a regular subalgebra we have by Theorem 9.2 that $A \rtimes K_2 \times \cdots \times K_m \prec N_i$.

Notice that now we can follow exactly as in the proof of Corolarry C in [CS11] to get our desired result.

□

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