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Journal

IEEE Transactions on Signal Processing, 39(5)

ISSN

1053-587X

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Publication Date

1991-05-01

DOI

10.1109/78.80958

Peer reviewed

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A Note on the Cramer-Rao Bound for 2-D Direction Finding Based on 2-D Array

Yingbo Hua and Tapan K. Sarkar

Abstract—We present explicit expressions of the Cramer-Rao bound (CRB) for estimating the two-dimensional (2-D) direction of single source based on a 2-D array of identical omnidirectional sensors. The two often used models, random wave model and unknown wave model, are compared. It is shown that the CRB's for the two models have the same dependency on the array structure. A specialization of the CRB for two orthogonal uniform linear arrays (ULA's) is discussed. It is found that the joint CRB's of the direction angles based on the two orthogonal ULA's can be as low as one quarter (for the random wave model with large number of snapshots and low SNR) or one half (for both models with high SNR) of the CRB's based on each ULA.

I. INTRODUCTION

The Cramer-Rao bound (CRB) [1] has been analyzed for array processing and the time series problem [2]-[8]. Studying CRB helps us not only to find a way of reducing the time of computing CRB but also to gain an insight into the accuracy potential of any given estimation problem.

An important class of signals is represented by

$$\mathbf{x}_t = \mathbf{A}\mathbf{s}_t + \mathbf{n}_t \quad t = 1, 2, \dots, N \quad (1.1)$$

where $\mathbf{x}_t \in \mathbb{C}^{r \times 1}$ which is the data vector, $\mathbf{A} \in \mathbb{C}^{r \times p}$ which depends on a set of desired parameters, $\mathbf{s}_t \in \mathbb{C}^{p \times 1}$ which may be random vector or deterministic unknown vector, and $\mathbf{n}_t \in \mathbb{C}^{r \times 1}$ which is the noise vector. The model is called deterministic if $\mathbf{s}_t$ are assumed to be deterministic unknowns, and called random if $\mathbf{s}_t$ are assumed

to be random. The choice of one of the two models depends on applications. The deterministic model is often used [2]-[6] when N and p are small while the random model [7], [8] is more appropriate when N or p is large.

It is known [1] that the CRB of the random model is achievable when N or SNR is large while the CRB of the deterministic model is achievable only when SNR is large. The reason why the CRB of the deterministic model is not achievable by increasing N is because the number of unknowns increases with N .

A question is, how are the CRB's of the two models related? No general complete answer to this question seems to exist. The previous works [2]-[8] on CRB always focus on one choice of the two models. In this correspondence, we study a special case of 2-D direction finding based on a 2-D array to gain some clues to the above question. Specifically, we will present the CRB expression for both random and deterministic models under the following assumptions:

- 1) Single plane wave arrives at the array so that $\mathbf{A}$ becomes a (steering) vector $\mathbf{a}$, and $\mathbf{s}_t$ becomes a scalar s_t .
- 2) The 2-D array consists of identical omnidirectional sensors so that the i th element of $\mathbf{a}$ is

$$a_i = \exp [j(2\pi/\lambda)(x_i \cos \alpha + y_i \cos \beta)] \quad (1.2)$$

where λ is the wavelength, (x_i, y_i) is the coordinate of the i th sensor, and α and β are the angles against the x and y axes, respectively (see Fig. 1).

- 3) The noise $\mathbf{n}_t$ is additive white Gaussian (and independent of $\mathbf{s}_t$) with the covariance matrix $2\sigma^2 \mathbf{I}_{r \times r}$.
- 4) For the random model, $\mathbf{s}_t$ is assumed to be Gaussian with zero mean and variance P .

The probability density function (pdf) of each model is as follows:

For the random waveform model:

$$p_r(\mathbf{x}_1, \dots, \mathbf{x}_N) = 1/(\pi^{Nr} |\mathbf{R}_s|^N) \exp \left[-\sum_{t=1}^N \mathbf{x}_t^H \mathbf{R}_s^{-1} \mathbf{x}_t \right] \quad (1.3)$$

where

$$\mathbf{R}_s = E\{\mathbf{x}_t \mathbf{x}_t^H\} = P \mathbf{a} \mathbf{a}^H + 2\sigma^2 \mathbf{I}_r \quad (1.4)$$

$E\{\cdot\}$ denotes the expectation, and ^{H} the conjugate transpose.

For the unknown waveform model:

$$p_u(\mathbf{x}_1, \dots, \mathbf{x}_N) = 1/(\pi^{Nr} (2\sigma^2)^{Nr}) \exp \left[-1/2\sigma^2 \sum_{t=1}^N \|\mathbf{x}_t - s_t \mathbf{a}\|^2 \right] \quad (1.5)$$

where $\|\cdot\|$ denotes the 2-norm.

Note that $p_r(\mathbf{x}_1, \dots, \mathbf{x}_N)$ is parameterized by only 4 real unknowns: α , β , P , and $2\sigma^2$, and $p_u(\mathbf{x}_1, \dots, \mathbf{x}_N)$ by $3 + 2N$ real unknowns: α , β , $s_1, \dots, s_N$ and $2\sigma^2$. (Each s_t consists of the two real unknowns: the amplitude $|s_t|$ and the phase ϕ_t .)

In the next section, the CRB's for both models will be developed. It will be shown that the two different models do not change the relation between the CRB's of the direction angles and the array structure $\mathbf{a}$. Our result is also consistent with an observation in [9] that the random waveform model yields higher CRB's than the unknown waveform model. For high SNR, the two models yield the same CRB's.

After presenting the CRB expression for an arbitrary array, we will apply it, in Section III, to the special 2-D array: two orthogonal uniform linear arrays (ULA's). The ULA has a great impor-

Manuscript received January 4, 1990; revised July 11, 1990.
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IEEE Log Number 9042692.

Fig. 1. A coordinate system of 2-D direction of arrival, which is one-to-one transformation of the usual (azimuth and elevation) coordinates.

tance both in theory and application due to its special structure. There are many algorithms available for the ULA problem (e.g., see [10] and the references therein). Hence, for finding multiple 2-D directions based on two orthogonal ULA's, good initial estimates can be easily found by considering each ULA first. It will be shown that by using two orthogonal ULA's jointly, one can obtain the 2-D direction-of-arrival estimates that are four times (for the random wave form model with large N and low SNR) or twice (for both models with high SNR) as accurate as those obtained from each ULA.

II. THE CRAMER-RAO BOUND

The CRB's [1] are the diagonal elements of the inverted Fisher information matrix $\mathbf{F}^{-1}$. The (i, j) th element of $\mathbf{F}$ is defined by

$$F_{i,j} = -E \left\{ \frac{\partial^2 \log(p(\cdot))}{\partial \theta_i \partial \theta_j} \right\} = E \left\{ \frac{\partial \log p(\cdot)}{\partial \theta_i} \frac{\partial \log p(\cdot)}{\partial \theta_j} \right\} \quad (2.1)$$

where $\partial/\partial \theta_i$ denotes the partial derivative w.r.t. the parameter θ_i , and $p(\cdot)$ the pdf under consideration.

For the random waveform model: We denote $(\alpha, \beta, P, 2\sigma^2)$ by $(\theta_1, \theta_2, \theta_3, \theta_4)$. Given the pdf of (1.3), it is known [7], [8] that

$$F_{i,j} = N \text{tr} \left\{ \mathbf{R}_x^{-1} \frac{\partial \mathbf{R}_x}{\partial \theta_i} \mathbf{R}_x^{-1} \frac{\partial \mathbf{R}_x}{\partial \theta_j} \right\} \quad (2.2)$$

where $\text{Tr} \{ \cdot \}$ denotes the trace operator. From (1.4)

$$\mathbf{R}_x^{-1} = \frac{1}{2\sigma^2} \mathbf{I}_r - \frac{P}{(Pr + 2\sigma^2)2\sigma^2} \mathbf{a} \mathbf{a}^H. \quad (2.3)$$

Substituting (1.4) and (2.3) into (2.2) and carrying out a lengthy algebraic manipulation, $\mathbf{F}$ can be derived straightforwardly. In particular, F_{13} , F_{14} , F_{23} , and F_{24} are found to be zero. Due to this result, $\mathbf{F}$ consists of two 2×2 matrices on its diagonal axis. Hence, $\mathbf{F}$ is easy to invert. The first two diagonal elements of $\mathbf{F}^{-1}$ are found to be

$$\text{CRB}(\alpha)_r = (\mathbf{F}^{-1})_{11} = G_r \frac{A\alpha}{B} \quad (2.4)$$

$$\text{CRB}(\beta)_r = (\mathbf{F}^{-1})_{22} = G_r \frac{A\beta}{B} \quad (2.5)$$

where the subscript r denotes the random waveform model, and

$$G_r = \frac{r}{2N} \frac{\text{SNR}_r + 1}{\text{SNR}_r^2} \quad (2.6)$$

$$\text{SNR}_r = \frac{rP}{2\sigma^2} \quad (2.7)$$

$$\begin{aligned} B &= r(\mathbf{a}'_\alpha \mathbf{a}'_\alpha) (\mathbf{a}'_\beta \mathbf{a}'_\beta) + (\mathbf{a}'_\alpha \mathbf{a}'_\alpha) (\mathbf{a}'_\beta \mathbf{a}'_\beta)^2 \\ &\quad + (\mathbf{a}'_\beta \mathbf{a}'_\beta) (\mathbf{a}'_\alpha \mathbf{a}'_\alpha)^2 - r(\mathbf{a}'_\alpha \mathbf{a}'_\beta)^2 \\ &\quad - 2(\mathbf{a}'_\alpha \mathbf{a}'_\alpha) (\mathbf{a}'_\beta \mathbf{a}'_\beta) (\mathbf{a}'_\alpha \mathbf{a}'_\beta)^2 \end{aligned} \quad (2.8)$$

$$A_\alpha = r(\mathbf{a}'_\alpha \mathbf{a}'_\alpha) + (\mathbf{a}'_\alpha \mathbf{a}'_\alpha)^2 \quad (2.9)$$

$$A_\beta = r(\mathbf{a}'_\beta \mathbf{a}'_\beta) + (\mathbf{a}'_\beta \mathbf{a}'_\beta)^2 \quad (2.10)$$

where $\mathbf{a}'_\alpha$ and $\mathbf{a}'_\beta$ are the derivatives of $\mathbf{a}$ with respect to α and β , respectively. (In [7], the CRB is derived for much simpler 2-D arrays.)

The CRB's of P and $2\sigma^2$ can also be derived and they are found to be independent of the structure $\mathbf{a}$ of the array.

For the unknown waveform model: We denote, in this case, the real unknowns $(\alpha, \beta, \phi_1, \phi_2, \dots, \phi_N, |s_1|, |s_2|, \dots, |s_N|, 2\sigma^2)$ by $(\theta_1, \theta_2, \dots, \theta_{3+2N})$. The corresponding $(3 + 2N) \times (3 + 2N)$ Fisher information matrix $\mathbf{F}$ can then be derived from a lengthy algebraic manipulation of (1.5) and (2.1). In particular, we found that $F(\phi_i, \phi_j) = 0$ for i not equal to j ; $F(\theta, |s_i|) = 0$ for θ not equal to $|s_i|$; and $F(\theta, 2\sigma^2) = 0$ for θ not equal to $2\sigma^2$. Due to the sparseness in $\mathbf{F}$, the inverse of $\mathbf{F}$ can be obtained analytically. The CRB's of $|s_i|$ and $2\sigma^2$ are, respectively σ^2/r and $(2\sigma^2)^2/Nr$, which are independent of the array structure $\mathbf{a}$. To find the CRB's of α and β , the $(N + 2) \times (N + 2)$ submatrix of $\mathbf{F}$, associated with $\alpha, \beta, \phi_1, \dots, \phi_N$, needs to be inverted. This requires another lengthy derivation. The results are

$$\text{CRB}(\alpha)_u = G_u \frac{A_\alpha}{B} \quad (2.11)$$

$$\text{CRB}(\beta)_u = G_u \frac{A_\beta}{B} \quad (2.12)$$

where the subscript u denotes the unknown waveform model, and

$$G_u = \frac{r}{2N \text{SNR}_u} \quad (2.13)$$

$$\text{SNR}_u = \frac{r \sum_{i=1}^N |s_i|^2}{2\sigma^2}. \quad (2.14)$$

B , A_α , and A_β are given in (2.8)–(2.10). (The above results can also be derived from a more general CRB in [5]. The results in [2]–[4], [6] are valid only for $N = 1$.)

Comparing (2.4) and (2.5) to (2.11) and (2.12) yields the following:

$$\frac{\text{CRB}(\alpha)_r}{\text{CRB}(\alpha)_u} = \frac{\text{CRB}(\beta)_r}{\text{CRB}(\beta)_u} = \frac{G_r}{G_u} = \frac{(\text{SNR}_r + 1) \text{SNR}_u}{\text{SNR}_r^2}. \quad (2.15)$$

If we let $\text{SNR}_r = \text{SNR}_u = \text{SNR}$, it follows that the random waveform model has higher CRB than the unknown waveform model, and they become equal when SNR is much larger than one.

An interesting observation (unexpected before the results were derived) from (2.4), (2.5), (2.11), and (2.12) is that the CRB's of α and β for both models have the same dependency on the array structure $\mathbf{a}$.

III. THE JOINT ACCURACY OF TWO ORTHOGONAL ULA'S

We now consider the two types (Fig. 2) of two orthogonal ULA's: a) the conventional cross array, and b) the shifted cross array, which is formed by shifting each ULA in the cross array

Fig. 2. Two orthogonal ULA's: (a) cross array, and (b) shifted cross array.

along its x or y axis by the same distance d . In the orthogonal ULA's considered here, the symmetry in the x and y directions is imposed purposely so that $\text{CRB}(\alpha)$ has the same properties as $\text{CRB}(\beta)$. In the following, only $\text{CRB}(\alpha)$ is considered.

We consider only the random waveform model in this section since the CRB for the unknown waveform model is similar.

A. The Cross Array

For this array, we have

$$\mathbf{a}_\alpha^H \mathbf{a}'_\alpha = 0 \quad (3.1)$$

$$\mathbf{a}_\alpha^H \mathbf{a}'_\beta = 0 \quad (3.2)$$

$$\mathbf{a}_\alpha^H \mathbf{a}'_\beta = 0. \quad (3.3)$$

Substituting (3.1)-(3.3) into (2.8)-(2.10) and then into (2.4) leads to

$$\text{CRB}(\alpha)_r = G_r \frac{A_\alpha}{B} = \frac{G_r}{\mathbf{a}_\alpha^H \mathbf{a}'_\alpha}. \quad (3.4)$$

It is clear that α can be estimated based on the ULA on the x axis (as opposed to the two orthogonal ULA's). It can be shown that the CRB of α (denoted by $\text{CRB}(\alpha)_{r, \text{ULA}}$) based on the ULA of $r' = (r + 1)/2$ sensors is given by (3.4) only with r replaced by r' . Then, it follows that

$$\frac{\text{CRB}(\alpha)_r}{\text{CRB}(\alpha)_{r, \text{ULA}}} = \frac{G_r}{G_{r, \text{ULA}}} = \frac{r'(rP/2\sigma^2 + 1)}{r(r'P/2\sigma^2 + 1)} \quad (3.5)$$

which is 1 for $r'P \gg 2\sigma^2$, or $r'/r \cong 1/2$ for $rP \ll 2\sigma^2$.

Equation (3.5) implies that the optimum estimation of α based on the cross array has the same accuracy as that based on the ULA on the x axis if SNR is high, and twice as accurate if N is large and SNR is low.

Fig. 3. CRB of the shifted cross array versus the separation.

B. The Shifted Cross Array

It should be noted that the shifted cross array with an overlapping (or common) sensor consists of odd number of sensors while the shifted cross array without any overlapping sensor consists of even number of sensors. Hence, there is a slight difference in α for the two cases. But for $r \gg 1$, the effect of the difference on the CRB can be shown to be negligible.

For the shifted cross array, (3.3) is still true but (3.1) and (3.2) are not. A lengthy but straightforward derivation leads to

$$\text{CRB}(\alpha)_r = G_r \frac{48}{(2\pi/\lambda\tau \sin \alpha)^2 r^3} \frac{48d^2 + 2\tau^2 r^2}{48d^2 + \tau^2 r^2} \quad (3.6)$$

where d is the distance of each ULA shifted from the center of the cross array along its axis. In deriving (3.6), $r \gg 1$ is assumed.

It can be seen from (3.6) (Fig. 3) that $\text{CRB}(\alpha)_r$ is decreased by 50% as d increases from zero to infinite. But for $d = r\tau/4$, which corresponds to an L-shaped array [11], $\text{CRB}(\alpha)$ is already decreased by the factor 3/8 with respect to the cross array. In [11], a computationally efficient MLE algorithm is presented for the L-shaped array with multiple impinging waves.

From (3.5) and (3.6), it is clear that for high SNR, the estimated angles (α, β) based on the shifted array can be twice as accurate as those based on each of the two independent ULA's, and for large N and low SNR, they can be four times as accurate.

Increasing the distance (an aperture measure) between the two ULA's in the shifted cross array decreases the CRB. However, it is not the case if the CRB is thought to decrease to zero as the shifted distance d is increased to infinity. Equation (3.6) shows the result which is contradictory to the above intuitive thinking. This peculiar phenomenon can also be observed from the 2-D beamformer (its maximizing arguments are the maximum likelihood estimates of α and β):

$$B(\alpha_0, \beta_0) = 1/N \sum_{i=1}^N |\mathbf{x}_i^H \mathbf{a}_0|^2 \quad (3.7)$$

where $\mathbf{a}_0$ is the steering vector directed to (α_0, β_0) . If $\text{SNR} \gg 1$ or $N \gg 1$, then (3.7) is proportional to (easy to show)

$$|\mathbf{a}^H \mathbf{a}_0|^2 = f^2(\alpha_0) + f^2(\beta_0) + 2f(\alpha_0)f(\beta_0) \cdot \exp \left[\frac{2\pi d}{\lambda} ((\cos \alpha - \cos \alpha_0) - (\cos \beta - \cos \beta_0)) \right] \quad (3.8)$$

where

$$f(\alpha_0) = \frac{\sin\left(\frac{\pi\tau}{2\lambda}(\cos\alpha - \cos\alpha_0)\right)}{\sin\left(\frac{\pi\tau}{\lambda}(\cos\alpha - \cos\alpha_0)\right)} \quad (3.9)$$

and $f(\beta_0)$ is similarly defined. The 2-D beamwidth of (3.8) is affected by d only through the nondominant cross term.

IV. CONCLUSIONS

We have studied the CRB's for the 2-D array consisting of identical omnidirectional sensors upon which a planewave impinges. It has been shown that for the two often used models, random wave form and unknown wave form, the CRB's of the wave direction have the identical dependency on the array structure $\mathbf{a}$ (steering vector) regardless the value of SNR or N . Our results are in consistency with the argument [9] that the unknown waveform model yields more conservative lower bound (which is reachable only for high SNR) than the random wave form model does (which is reachable for high SNR as well as for large N).

For the orthogonal ULA's (a direct 2-D extension from the 1-D ULA), we have shown that the joint accuracy of the 2-D array can be four times (for large N and low SNR) or twice (for high SNR) as good as the accuracy of each of two independent orthogonal ULA's.

Finally, we mention that the CRB of direction estimation is generally known [12] to be a decreasing function of array aperture. For the ULA, the array aperture cannot be arbitrarily increased simply by increasing the spacing between sensors without introducing angle ambiguity. It is also known [12] that large aperture nonuniform arrays can be used to reduce the CRB without angle ambiguity. The orthogonal ULA's represent a nonuniform 2-D array which, however, preserves the uniformity along x and y axes so that good initial estimates can be easily [10] obtained from each ULA.

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Threshold Bounds in SVD and a New Iterative Algorithm for Order Selection in AR Models

Konstantinos Konstantinides

Abstract—The problem of order determination of AR models using singular value decomposition is reexamined from a statistical point of view. Thresholds for distinguishing between significant and nonsignificant singular values are derived and a new iterative algorithm for order selection in AR models is presented. Simulation results show the new technique to be very effective even when a small number of samples is available.

I. INTRODUCTION

In many communication, control, and signal processing problems there is a fundamental interest in fitting the correct autoregressive (AR) model to an observed data sequence. For example, in spectral estimation, the accuracy of the frequency estimation strongly depends on the order of the prediction filter [1]. In recent years, singular value decomposition (SVD) is known to be a basic and computationally efficient method for determining the rank of a matrix and the order of AR models [2]-[4]. In SVD techniques, first an $m \times n$ (where $m \geq n$, and n is larger than the estimated order of the system) matrix is formed using either the sampled data themselves or the sample autocorrelation coefficients. Then, the order of the system is the effective rank of that matrix. A fundamental problem in using SVD with noisy data is that of distinguishing between significantly small and insignificantly large singular values (SV's). Furthermore, little is known on how the number of available data samples or the selection of n (the first estimate on the order) affects the performance of these techniques. In [5], confidence regions for the perturbed singular values of matrices with noisy data were derived based on the theories of perturbation of singular values and statistical significance test. Based on those results, in this work the problem of order determination in AR models is reexamined from a statistical point of view and new threshold bounds are derived. Those bounds are applied to a new iterative algorithm for determining the order of AR models. Simulation results show the technique to determine the correct order regardless of the initial choice of n , and even when only a small number of data samples is available.

Manuscript received July 31, 1989; revised December 5, 1990.
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