Math. Ann. (2009) 344:801–846
DOI 10.1007/s00208-008-0328-y

**Mathematische Annalen**

# Spectra and semigroup smoothing for non-elliptic quadratic operators

**Michael Hitrik · Karel Pravda-Starov**

Received: 15 January 2008 / Revised: 23 September 2008 / Published online: 13 January 2009


**Abstract** We study non-elliptic quadratic differential operators. Quadratic differential operators are non-selfadjoint operators defined in the Weyl quantization by complex-valued quadratic symbols. When the real part of their Weyl symbols is a non-positive quadratic form, we point out the existence of a particular linear subspace in the phase space intrinsically associated to their Weyl symbols, called a singular space, such that when the singular space has a symplectic structure, the associated heat semigroup is smoothing in every direction of its symplectic orthogonal space. When the Weyl symbol of such an operator is elliptic on the singular space, this space is always symplectic and we prove that the spectrum of the operator is discrete and can be described as in the case of global ellipticity. We also describe the large time behavior of contraction semigroups generated by these operators.

**Mathematics Subject Classification (2000)** 47A10 · 47D06 · 35P05

## 1 Introduction

### 1.1 Miscellaneous facts about quadratic differential operators

Since the classical work by J. Sjöstrand [12], the study of spectral properties of quadratic diffrential operators has played a basic rôle in the analysis of partial differential operators with double characteristics. Roughly speaking, if we have, say, a

M. Hitrik (✉)
Department of Mathematics, University of California, Los Angeles, CA 90095-1555, USA
e-mail: hitrik@math.ucla.edu

K. Pravda-Starov
Mathematics Department, Imperial College London, London SW7 2AZ, UK
e-mail: k.pravda-starov@imperial.ac.uk

classical pseudodifferential operator $p(x,\xi)^w$ on $\mathbb{R}^n$ with the Weyl symbol $p(x,\xi) = p_m(x,\xi) + p_{m-1}(x,\xi) + \ldots$ of order $m$, and if $X_0 = (x_0, \xi_0) \in \mathbb{R}^{2n}$ is a point where $p_m(X_0) = dp_m(X_0) = 0$ then it is natural to consider the quadratic form $q$ which begins the Taylor expansion of $p_m$ at $X_0$. The study of a priori estimates for $p(x,\xi)^w$, such as hypoelliptic estimates of the form

$$||\, u \,||_{m-1} \leq C_K \left(||\, p(x,\xi)^w u \,||_0 + ||\, u \,||_{m-2}\right), \quad u \in C_0^\infty(K),\ K \subset\subset \mathbb{R}^n,$$

then often depends on the spectral analysis of the quadratic operator $q(x,\xi)^w$. See also [7], as well as Chapter 22 of [8] together with further references given there. In [12], the spectrum of a general quadratic differential operator has been determined, under the basic assumption of global ellipticity of the associated quadratic form.

Now there exist many situations where one is naturally led to consider non-selfadjoint quadratic differential operators whose symbols are not elliptic but rather satisfy certain weaker conditions. An example particularly relevant to the following discussion is obtained if one considers the Kramers–Fokker–Plank operator with a quadratic potential [3,4]. The corresponding (complex-valued) symbol is not elliptic, but nevertheless, the operator has discrete spectrum and the associated heat semigroup is well behaved in the limit of large times (see [5]).

The purpose of the present paper is to provide a proof of a number of fairly general results concerning the spectral and semigroup properties for the class of quadratic differential operators in the case when the global ellipticity fails. Specifically, and as alluded to above, we shall consider the class of pseudodifferential operators defined by the Weyl quantization formula,

$$q(x,\xi)^w u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^{2n}} e^{i(x-y).\xi} q\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi, \tag{1.1.1}$$

for some symbols $q(x,\xi)$, where $(x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n$ and $n \in \mathbb{N}^*$, which are complex-valued quadratic forms. Since the symbols are quadratic forms, the corresponding operators in (1.1.1) are in fact differential operators. Indeed, the Weyl quantization of the quadratic symbol $x^\alpha \xi^\beta$, with $(\alpha,\beta) \in \mathbb{N}^{2n}$ and $|\alpha + \beta| \leq 2$, is the differential operator

$$\frac{x^\alpha D_x^\beta + D_x^\beta x^\alpha}{2},\ D_x = i^{-1}\partial_x.$$

Let us also notice that since the Weyl symbols in (1.1.1) are complex-valued, the quadratic differential operators are a priori formally non-selfadjoint.

In this paper, we shall first study the properties of contraction semigroups generated by quadratic differential operators whose Weyl symbols have a non-positive real part,

$$\operatorname{Re} q \leq 0. \tag{1.1.2}$$

Our first goal is to point out the existence of a linear subspace $S$ in $\mathbb{R}_x^n \times \mathbb{R}_\xi^n$, which will be called the singular space and which is defined in terms of the Hamilton map of the Weyl symbol $q$, such that when $S$ has a *symplectic structure*, the associated heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - q(x,\xi)^w u(t,x) = 0 \\ u(t,\cdot)|_{t=0} = u_0 \in L^2(\mathbb{R}^n), \end{cases} \tag{1.1.3}$$

is smoothing in every direction of the orthogonal complement $S^{\sigma\perp}$ of $S$ with respect to the canonical symplectic form $\sigma$ on $\mathbb{R}^{2n}$,

$$\sigma\left((x,\xi),(y,\eta)\right) = \xi.y - x.\eta,\ (x,\xi) \in \mathbb{R}^{2n}, (y,\eta) \in \mathbb{R}^{2n}. \tag{1.1.4}$$

We shall also describe the large time behavior of contraction semigroups

$$e^{tq(x,\xi)^w}, \quad t \geq 0,$$

associated to (1.1.3). When the Weyl symbol $q$ satisfies (1.1.2) and an assumption of *partial* ellipticity, namely when $q$ is elliptic on the singular space $S$ in the sense that

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0, \tag{1.1.5}$$

then $S$ is automatically symplectic, and we prove that the spectrum of the quadratic differential operator $q(x,\xi)^w$ is only composed of a countable number of eigenvalues of finite multiplicity, with its structure similar to the one known in the case of global ellipticity [12].

It seems to us that the singular space $S$ introduced in this paper plays a basic rôle in the understanding of non-elliptic quadratic differential operators. Its study may therefore be also particularly relevant in the analysis of general pseudodifferential operators with double characteristics, when the ellipticity of their quadratic approximations fails.

Before giving the precise statements of these results, let us begin by recalling some facts and notation about quadratic differential operators. Let

$$\begin{aligned} q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n &\rightarrow \mathbb{C} \\ (x,\xi) &\mapsto q(x,\xi), \end{aligned}$$

be a complex-valued quadratic form with a non-positive real part,

$$\text{Re}\, q(x,\xi) \leq 0, \quad (x,\xi) \in \mathbb{R}^{2n}, n \in \mathbb{N}^*. \tag{1.1.6}$$

We know from [9] (p. 425) that the maximal closed realization of the operator $q(x,\xi)^w$, i.e., the operator on $L^2(\mathbb{R}^n)$ with the domain

$$\{u \in L^2(\mathbb{R}^n) : q(x,\xi)^w u \in L^2(\mathbb{R}^n)\},$$

coincides with the graph closure of its restriction to $\mathcal{S}(\mathbb{R}^n)$,

$$q(x,\xi)^w : \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{S}(\mathbb{R}^n),$$

and that every quadratic differential operator whose Weyl symbol has a non-positive real part, generates a contraction semigroup. The Mehler formula proved by L. Hörmander in [9] gives an explicit expression for the Weyl symbols of these contraction semigroups.

Associated to the quadratic symbol $q$ is the numerical range $\Sigma(q)$ defined as the closure in the complex plane of all its values,

$$\Sigma(q) = \overline{q(\mathbb{R}_x^n \times \mathbb{R}_\xi^n)}. \tag{1.1.7}$$

We also recall [8] that the Hamilton map $F \in M_{2n}(\mathbb{C})$ associated to the quadratic form $q$ is the map uniquely defined by the identity

$$q\left((x,\xi);(y,\eta)\right) = \sigma\left((x,\xi), F(y,\eta)\right), \ (x,\xi) \in \mathbb{R}^{2n}, (y,\eta) \in \mathbb{R}^{2n}, \tag{1.1.8}$$

where $q\,(\cdot;\cdot)$ stands for the polarized form associated to the quadratic form $q$. It follows directly from the definition of the Hamilton map $F$ that its real part Re $F$ and its imaginary part Im $F$ are the Hamilton maps associated to the quadratic forms Re $q$ and Im $q$, respectively. Next, (1.1.8) shows that a Hamilton map is always skew-symmetric with respect to $\sigma$. This is just a consequence of the properties of skew-symmetry of the symplectic form and symmetry of the polarized form,

$$\forall X, Y \in \mathbb{R}^{2n}, \ \sigma(X, FY) = q(X;Y) = q(Y;X) = \sigma(Y, FX) = -\sigma(FX, Y). \tag{1.1.9}$$

Let us now consider the elliptic case, i.e., the case of quadratic differential operators whose Weyl symbols are *globally* elliptic in the sense that

$$(x,\xi) \in \mathbb{R}^{2n}, \ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0. \tag{1.1.10}$$

In this case, the numerical range of a quadratic form can only take very particular shapes. J. Sjöstrand proved in [12] (Lemma 3.1) that if $q$ is a complex-valued elliptic quadratic form on $\mathbb{R}^{2n}$, with $n \geq 2$, then there exists $z \in \mathbb{C}^*$ such that $\mathrm{Re}(zq)$ is a positive definite quadratic form. If $n = 1$, the same result is fulfilled if we assume besides that $\Sigma(q) \neq \mathbb{C}$. This shows that the numerical range of an elliptic quadratic form can only take two shapes. The first possible shape is when $\Sigma(q)$ is equal to the whole complex plane. This case can only occur in dimension $n = 1$. The second possible shape is when $\Sigma(q)$ is equal to a closed angular sector with a vertex in 0 and an aperture strictly less than $\pi$ (see [11] for more details).

We also know that elliptic quadratic differential operators define Fredholm operators (see Lemma 3.1 in [7] or Theorem 3.5 in [12]),

$$q(x,\xi)^w + z : B \to L^2(\mathbb{R}^n), \tag{1.1.11}$$

where $B$ is the Hilbert space

$$\begin{aligned} B &= \left\{u \in L^2(\mathbb{R}^n) : q(x,\xi)^w u \in L^2(\mathbb{R}^n)\right\} \\ &= \left\{u \in L^2(\mathbb{R}^n) : x^\alpha D_x^\beta u \in L^2(\mathbb{R}^n) \text{ if } |\alpha+\beta| \leq 2\right\}, \end{aligned} \tag{1.1.12}$$

with the norm

$$\|u\|_B^2 = \sum_{|\alpha+\beta|\leq 2} \|x^\alpha D_x^\beta u\|^2_{L^2(\mathbb{R}^n)}.$$

Moreover, the index of the operator (1.1.11) is independent of $z$ and is equal to 0 when $n \geq 2$. In the case $n = 1$, the index can take the values $-2$, 0 or 2. It vanishes as soon as $\Sigma(q) \neq \mathbb{C}$.

When $\Sigma(q) \neq \mathbb{C}$, J. Sjöstrand has proved in Theorem 3.5 of [12] (see also Lemma 3.2 and Theorem 3.3 in [7]) that the spectrum of an elliptic quadratic differential operator

$$q(x,\xi)^w : B \to L^2(\mathbb{R}^n),$$

is only composed of eigenvalues with finite multiplicity,

$$\sigma\left(q(x,\xi)^w\right) = \left\{ \sum_{\substack{\lambda \in \sigma(F),\\ -i\lambda \in \Sigma(q)\setminus\{0\}}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{1.1.13}$$

where $F$ is the Hamilton map associated to the quadratic form $q$ and $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$.

Let us also recall the result proved in [11] about contraction semigroups generated by elliptic quadratic differential operators whose Weyl symbols have a non-positive real part. This result shows that, as soon as the real part of their Weyl symbols is a non-zero quadratic form, the norm of contraction semigroups generated by these operators decays exponentially in time.

In this paper, we study the case when the ellipticity fails. Our second result (Theorem 1.2.2) extends the description of the spectra (1.1.13) to the case of quadratic differential operators whose Weyl symbols are partially elliptic, but not necessarily globally so. To get this result, we only require that these symbols have a non-positive real part and are elliptic on their associated singular spaces. We also prove a result on the exponential decay in time for the norm of contraction semigroups generated by non-elliptic quadratic differential operators.

Let us now define this singular space. The *singular* space $S$ associated to the symbol $q$ is defined as the following intersection of the kernels,

$$S = \left(\bigcap_{j=0}^{+\infty} \operatorname{Ker}\left[\operatorname{Re} F(\operatorname{Im} F)^j\right]\right) \cap \mathbb{R}^{2n}, \tag{1.1.14}$$

where the notation Re $F$ and Im $F$ stands respectively for the real part and the imaginary part of the Hamilton map associated to $q$. Notice that the Cayley-Hamilton theorem applied to Im $F$ shows that

$$(\mathrm{Im}\ F)^k X \in \mathrm{Vect}\Big(X, \dots, (\mathrm{Im}\ F)^{2n-1}X\Big), \quad X \in \mathbb{R}^{2n},\ k \in \mathbb{N},$$

where $\mathrm{Vect}\big(X, \dots, (\mathrm{Im}\ F)^{2n-1}X\big)$ is the vector space spanned by the vectors $X, \dots,$ $(\mathrm{Im}\ F)^{2n-1}X$, and therefore the singular space is actually equal to the following finite intersection of the kernels,

$$S = \left(\bigcap_{j=0}^{2n-1} \mathrm{Ker}\Big[\mathrm{Re}\ F(\mathrm{Im}\ F)^j\Big]\right) \cap \mathbb{R}^{2n}. \tag{1.1.15}$$

The subspace $S$ obviously satisfies the two following properties,

$$(\mathrm{Re}\ F)S = \{0\} \quad \text{and} \quad (\mathrm{Im}\ F)S \subset S. \tag{1.1.16}$$

We can now give the statements of the main results contained in this paper.

## 1.2 Statement of the main results

In the following statements, we consider a complex-valued quadratic form

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C},$$

with a non-positive real part,

$$\mathrm{Re}\, q(x,\xi) \leq 0,\ (x,\xi) \in \mathbb{R}^{2n},\ n \in \mathbb{N}^*, \tag{1.2.1}$$

and we denote by $S$ the singular space defined in (1.1.14) or (1.1.15).

Our first result states that when the singular space $S$ has a symplectic structure, in the sense that the restriction of $\sigma$ to $S$ is nondegenerate, the heat equation (1.1.3) associated to the operator $q(x,\xi)^w$ is smoothing in every direction of its orthogonal complement $S^{\sigma\perp}$ with respect to the canonical symplectic form in $\mathbb{R}^{2n}$.

**Theorem 1.2.1** *Let us assume that the singular space $S$ has a symplectic structure. If $(x',\xi')$ are some linear symplectic coordinates on the symplectic space $S^{\sigma\perp}$, then for all $t > 0$, $N \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^n)$,*

$$\Big((1+|x'|^2+|\xi'|^2)^N\Big)^w e^{tq(x,\xi)^w} u \in L^2(\mathbb{R}^n). \tag{1.2.2}$$

Let us mention that the assumption about the symplectic structure of $S$ is always fulfilled by any quadratic symbol $q$ elliptic on $S$, i.e.,

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0. \tag{1.2.3}$$

This assumption is therefore always fulfilled for elliptic quadratic differential operators. We will see that it is also the case for instance for the Kramers–Fokker–Planck operator with a quadratic potential, which is a non-elliptic operator.

When $q$ is a complex-valued quadratic form with a non-positive real part verifying (1.2.3), we can give another description of the singular space in terms of the eigenspaces of $F$ associated to its real eigenvalues. Under these assumptions, the set of real eigenvalues of the Hamilton map $F$ can be written as

$$\sigma(F) \cap \mathbb{R} = \{\lambda_1, \dots, \lambda_r, -\lambda_1, \dots, -\lambda_r\},$$

with $\lambda_j \neq 0$ and $\lambda_j \neq \pm\lambda_k$ if $j \neq k$. The singular space is then the direct sum of the symplectically orthogonal spaces

$$S = S_{\lambda_1} \oplus^{\sigma\perp} S_{\lambda_2} \oplus^{\sigma\perp} \dots \oplus^{\sigma\perp} S_{\lambda_r}, \tag{1.2.4}$$

where $S_{\lambda_j}$ is the symplectic space

$$S_{\lambda_j} = \big(\mathrm{Ker}(F - \lambda_j) \oplus \mathrm{Ker}(F + \lambda_j)\big) \cap \mathbb{R}^{2n}. \tag{1.2.5}$$

These facts will be proved in Sect. 1.4.

Our second result deals with the structure of the spectra for non-elliptic quadratic differential operators. This result extends the description of the spectra (1.1.13) proved by J. Sjöstrand in [12] (Theorem 3.5) for elliptic quadratic differential operators to the case of quadratic differential operators which are only partially elliptic. To get this description, we only require in addition to the assumption (1.2.1) the property of partial ellipticity (1.2.3) for their Weyl symbols.

**Theorem 1.2.2** *If $q$ is a complex-valued quadratic form with a non-positive real part and if $q$ is elliptic on $S$,*

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0,$$

*then the spectrum of the quadratic differential operator $q(x,\xi)^w$ is only composed of eigenvalues of finite multiplicity,*

$$\sigma\left(q(x,\xi)^w\right) = \left\{ \sum_{\substack{\lambda\in\sigma(F),\\ -i\lambda\in\mathbb{C}_-\cup(\Sigma(q|_S)\setminus\{0\})}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{1.2.6}$$

*where $F$ is the Hamilton map associated to the quadratic form $q$, $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$,*

$$\Sigma(q|_S) = \overline{q(S)} \quad \textit{and} \quad \mathbb{C}_- = \{z \in \mathbb{C} : \mathrm{Re}\, z < 0\}.$$

Since the singular space $S$ is distinct from the whole phase space as soon as the real part of $q$ is not identically equal to zero, Theorem 1.2.2 is a generalization of the result proved by J. Sjöstrand for elliptic quadratic differential operators.

Finally, we give a result concerning the large time behavior of contraction semigroups generated by non-elliptic quadratic differential operators, which extends the result obtained by the second author in [11].

**Theorem 1.2.3** *Let us consider a complex-valued quadratic form*

$$q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \to \mathbb{C},\ n \in \mathbb{N}^*,$$

*with a non-positive real part, such that its singular space $S$ has a symplectic structure. Then, the following assertions are equivalent*:

(i) *The norm of the contraction semigroup generated by the operator $q(x,\xi)^w$ decays exponentially in time,*

$$\exists M > 0, \exists a > 0, \forall t \geq 0,\ \|e^{tq(x,\xi)^w}\|_{\mathcal{L}(L^2)} \leq Me^{-at}.$$

(ii) *The real part of the symbol $q$ is a non-zero quadratic form*

$$\exists (x_0,\xi_0) \in \mathbb{R}^{2n},\ \text{Re}\, q(x_0,\xi_0) \neq 0.$$

(iii) *The singular space is distinct from the whole phase space $S \neq \mathbb{R}^{2n}$.*

Since the assumption about the symplectic structure of the singular space $S$ is always fulfilled when the symbol $q$ verifies (1.2.3), Theorem 1.2.3 is a generalization of the result proved in [11] for elliptic quadratic differential operators.

*Remark* It follows from [11] that if the quadratic form $q$, satisfying (1.1.2), is such that Re $F$ is not nilpotent, then the statement (i) in Theorem 1.2.3 holds. On the other hand, if Re $F$ is nilpotent then necessarily $(\text{Re}\, F)^2 = 0$, see [11]. In this case the assumption about the symplectic structure of the singular space in Theorem 1.2.3 cannot be dropped completely. Indeed, let us consider the quadratic differential operator defined in the Weyl quantization by the symbol

$$q(x,\xi) = -x^2.$$

This operator is just the operator of multiplication by $-x^2$, which generates the contraction semigroup

$$e^{tq(x,\xi)^w}u = e^{-tx^2}u,\ t \geq 0,\ u \in L^2(\mathbb{R}^n),$$

whose norm is identically equal to 1,

$$\|e^{tq(x,\xi)^w}\|_{\mathcal{L}(L^2)} = 1,\ t \geq 0.$$

*Remark* Let us mention that our proof will show in particular that when $q$ is a complex-valued quadratic form on $\mathbb{R}^{2n}$, $n \geq 1$, with a non-positive real part and a zero singular space $S = \{0\}$, then

$$e^{tq(x,\xi)^w} = e^{tq(x,\xi)^w}\Pi_a + \mathcal{O}_a(e^{-at}),\ t \geq 0,$$

in the space $\mathcal{L}(L^2)$ of bounded operators on $L^2(\mathbb{R}^n)$, for any $a > 0$ such that

$$\begin{aligned}&\sigma\left(q(x,\xi)^w\right) \cap \{z \in \mathbb{C} : \operatorname{Re} z = -a\}\\ &= \left\{\sum_{\substack{\lambda\in\sigma(F),\\ \operatorname{Re}(-i\lambda)<0}} (r_\lambda + 2k_\lambda)\,(-i\lambda) : k_\lambda \in \mathbb{N}\right\} \cap \{z \in \mathbb{C} : \operatorname{Re} z = -a\} = \emptyset,\end{aligned}$$

where $\Pi_a$ stands for the finite rank spectral projection associated to the following eigenvalues of the operator $q(x,\xi)^w$,

$$\begin{aligned}&\sigma\left(q(x,\xi)^w\right) \cap \{z \in \mathbb{C} : -a \leq \operatorname{Re} z\}\\ &= \left\{\sum_{\substack{\lambda\in\sigma(F),\\ \operatorname{Re}(-i\lambda)<0}} (r_\lambda + 2k_\lambda)\,(-i\lambda) : k_\lambda \in \mathbb{N}\right\} \cap \{z \in \mathbb{C} : -a \leq \operatorname{Re} z\},\end{aligned}$$

where $F$ is the Hamilton map associated to the quadratic form $q$ and $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$.

Let us now explain the key arguments in our proofs of these theorems.

### 1.3 Structure of the proof

Our main assumption about the symplectic structure of the singular space $S$ fulfilled in the assumptions of all the three theorems allows us to find some symplectic coordinates $(x',\xi')$ in $S^{\sigma\perp}$ and $(x'',\xi'')$ in $S$ such that the complex-valued quadratic form $q$ verifying (1.2.1) can be written as the sum of two quadratic forms with a tensorization of the variables $(x',\xi')$ and $(x'',\xi'')$,

$$q = q|_S + q|_{S^{\sigma\perp}},\ (x,\xi) = (x',x'';\xi',\xi'') \in \mathbb{R}^{2n},$$

where the first quadratic form $q|_S$ is equal to

$$q|_S = i\tilde{q}|_S,$$

with $\tilde{q}|_S$ a real-valued quadratic form; and where the second quadratic form $q|_{S^{\sigma\perp}}$ is a complex-valued quadratic form with a non-positive real part. This real part is not in general negative definite (unless the real part of $q$ is). However, it follows from the definition of the singular space $S$ that the average of the real part of the quadratic form $q|_{S^{\sigma\perp}}$ by the flow generated by the Hamilton vector field of its imaginary part, $H_{\mathrm{Im} q|_{S^{\sigma\perp}}}$,

$$\langle \mathrm{Re}\, q|_{S^{\sigma\perp}} \rangle_T (X') = \frac{1}{2T} \int_{-T}^{T} \mathrm{Re}\, q|_{S^{\sigma\perp}} (e^{tH_{\mathrm{Im} q|_{S^{\sigma\perp}}}} X') dt, \ T > 0, \ X' = (x', \xi'),$$

is negative definite. Studying the contraction semigroup

$$e^{tq|^w_{S^{\sigma\perp}}}, \ t \geq 0, \tag{1.3.1}$$

generated by the operator $q|^w_{S^{\sigma\perp}}$, on the FBI-Bargmann transform side, we prove, using the averaging property just mentioned, that (1.3.1) is compact and strongly regularizing for every $t > 0$. This compactness result is really the key point in our proofs of the three theorems, and their complete statements then follow from a small additional amount of work.

## 1.4 Some examples

In this section, we prove that if a quadratic symbol $q$ is elliptic on its singular space then the singular space always has a symplectic structure. We also check that this property is fulfilled for the Kramers–Fokker–Plank operator with a quadratic potential.

### *1.4.1 Partially elliptic quadratic differential operators*

Let us consider the case of quadratic differential operators whose Weyl symbols are elliptic on their singular spaces. Let

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \rightarrow \mathbb{C}, \ n \in \mathbb{N}^*,$$

be a complex-valued quadratic form, which is elliptic on its singular space $S$,

$$(x, \xi) \in S, \ q(x, \xi) = 0 \Rightarrow (x, \xi) = 0. \tag{1.4.1}$$

We want to prove that

$$S = \left( \bigcap_{j=0}^{2n-1} \mathrm{Ker}\Big[ \mathrm{Re}\, F (\mathrm{Im}\, F)^j \Big] \right) \cap \mathbb{R}^{2n},$$

has a symplectic structure. This fact follows from some arguments similar to those used in [11] (Lemma 3).

We can assume that $S \neq \{0\}$ since the space $\{0\}$ is obviously symplectic. Let us therefore consider $X_0 \in S\backslash\{0\}$. We define

$$\begin{cases} e_1 = X_0 \\ \varepsilon_1 = -\dfrac{1}{\mathrm{Im}\, q(X_0)}\mathrm{Im}\, F X_0. \end{cases} \tag{1.4.2}$$

This is possible, since from (1.1.8) and (1.1.16), we have

$$\mathrm{Re}\, q(X_0) = \sigma(X_0, \mathrm{Re}\, F X_0) = 0,$$

and the ellipticity of $q$ on $S$ implies that

$$\mathrm{Im}\, q(X_0) \neq 0,$$

as $X_0 \in S\backslash\{0\}$. By using the skew-symmetry of the Hamilton map Im $F$ [see (1.1.9)], it follows that

$$\sigma(\varepsilon_1, e_1) = \sigma\Big(-(\mathrm{Im}\, q(X_0))^{-1}\mathrm{Im}\, F X_0, X_0\Big) = (\mathrm{Im}\, q(X_0))^{-1}\sigma(X_0, \mathrm{Im}\, F X_0) = 1,$$

which shows that the system $(e_1, \varepsilon_1)$ is symplectic. We get from (1.1.16) and (1.4.2) that

$$\mathrm{Vect}(e_1, \varepsilon_1) \subset S.$$

If $S = \mathrm{Vect}(e_1, \varepsilon_1)$, the singular space $S$ is symplectic. If it is not the case, so that,

$$S \neq \mathrm{Vect}(e_1, \varepsilon_1),$$

we can continue our construction of a symplectic basis for $S$ by considering

$$X_1 \in S\backslash\mathrm{Vect}(e_1, \varepsilon_1)$$

and

$$\tilde{X}_1 = X_1 + \sigma(X_1, \varepsilon_1)e_1 - \sigma(X_1, e_1)\varepsilon_1 \in S\backslash\mathrm{Vect}(e_1, \varepsilon_1). \tag{1.4.3}$$

Let us set

$$\begin{cases} e_2 = \tilde{X}_1 \\ \varepsilon_2 = -\dfrac{1}{\mathrm{Im}\, q(\tilde{X}_1)}\Big(\mathrm{Im}\, F\tilde{X}_1 + \sigma(\mathrm{Im}\, F\tilde{X}_1, \varepsilon_1)e_1 - \sigma(\mathrm{Im}\, F\tilde{X}_1, e_1)\varepsilon_1\Big), \end{cases} \tag{1.4.4}$$

which is again possible according to (1.1.16) and the assumption of ellipticity on $S$ since

$$\mathrm{Re}\, q(\tilde{X}_1) = \sigma(\tilde{X}_1, \mathrm{Re}\, F\tilde{X}_1) = 0,$$

because $\tilde{X}_1 \in S\backslash\{0\}$. Then, we can directly verify by using (1.4.3) and (1.4.4) that $(e_1, e_2, \varepsilon_1, \varepsilon_2)$ is a symplectic system. By using (1.1.16) again, we get

$$\text{Vect}(e_1, e_2, \varepsilon_1, \varepsilon_2) \subset S.$$

If $S = \text{Vect}(e_1, e_2, \varepsilon_1, \varepsilon_2)$, then $S$ is symplectic. If it is not the case, then

$$S \neq \text{Vect}(e_1, e_2, \varepsilon_1, \varepsilon_2),$$

and we can again iterate the preceding construction. After a finite number of such iterations, we obtain with this process a symplectic basis of $S$, proving its symplectic structure.

Let us now consider a complex-valued quadratic form

$$q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \to \mathbb{C},\ n \in \mathbb{N}^*,$$

with a non-positive real part

$$\text{Re}\, q \leq 0,$$

such that (1.4.1) is fulfilled and denote by $F$ its Hamilton map. We know from Proposition 4.4 in [9] that the kernel $\text{Ker}(F + \lambda)$ is the complex conjugate of the kernel $\text{Ker}(F - \lambda)$ for every $\lambda \in \mathbb{R}$, and that the spaces

$$\text{Ker}(F - \lambda) \oplus \text{Ker}(F + \lambda),$$

where $\lambda \in \mathbb{R}^*$, and $\text{Ker}\, F$, are the complexifications of their intersections with $\mathbb{R}^{2n}$.

Let us set

$$S_0 = (\text{Ker}\, F) \cap \mathbb{R}^{2n} \tag{1.4.5}$$

and

$$S_\lambda = (\text{Ker}(F - \lambda) \oplus \text{Ker}(F + \lambda)) \cap \mathbb{R}^{2n}, \tag{1.4.6}$$

for $\lambda \in \mathbb{R}^*$. Proposition 4.4 in [9] also shows that

$$\text{Re}\, F\ \text{Ker}(F \pm \lambda) = \{0\},$$

for all $\lambda \in \mathbb{R}$. This implies that

$$(\text{Re}\, F)S_\lambda = \{0\} \quad \text{and} \quad (\text{Im}\, F)S_\lambda \subset S_\lambda, \tag{1.4.7}$$

and proves in view of (1.1.15) that for all $\lambda \in \mathbb{R}$,

$$S_\lambda \subset S. \tag{1.4.8}$$

If $0 \in \sigma(F) \cap \mathbb{R}$, this would imply that $S_0 \neq 0$. Since from (1.4.5),

$$q(X) = \sigma(X, FX) = 0,$$

for all $X \in S_0$, the inclusion (1.4.8) would then contradict our assumption of ellipticity on the singular space (1.4.1). This proves that the set of real eigenvalues of the Hamilton map $F$ can be written as

$$\sigma(F) \cap \mathbb{R} = \{\lambda_1, \dots, \lambda_r, -\lambda_1, \dots, -\lambda_r\}, \tag{1.4.9}$$

with $\lambda_j \neq 0$ and $\lambda_j \neq \pm\lambda_k$ if $j \neq k$.

Let us now check that the spaces $S_{\lambda_j}$, $j = 1, \dots, r$, are symplectic. Let $X_0$ be in $S_{\lambda_j}$ such that for all $Y \in S_{\lambda_j}$,

$$\sigma(X_0, Y) = 0.$$

It follows that for all $Y$ and $Z$ in $S_{\lambda_j}$,

$$\sigma(X_0, Y + iZ) = 0,$$

which induces that

$$\forall X \in \operatorname{Ker}(F - \lambda_j) \oplus \operatorname{Ker}(F + \lambda_j),\ \sigma(X_0, X) = 0,$$

because $\operatorname{Ker}(F - \lambda_j) \oplus \operatorname{Ker}(F + \lambda_j)$ is a complexification of $S_{\lambda_j}$. On the other hand, since $X_0 \in S_{\lambda_j}$, we have $FX_0 \in \operatorname{Ker}(F - \lambda_j) \oplus \operatorname{Ker}(F + \lambda_j)$, which implies that

$$q(X_0) = \sigma(X_0, FX_0) = 0.$$

We then deduce from the ellipticity of $q$ on the singular space (1.4.1) and (1.4.8) that $X_0 = 0$, which proves the symplectic structure of the space $S_{\lambda_j}$.

Let us now assume that there exists another real eigenvalue $\lambda_k$ of $F$ distinct from $\lambda_j$ and $-\lambda_j$. We already know that this eigenvalue $\lambda_k$ is necessarily non-zero. Let

$$X \in \operatorname{Ker}(F - \varepsilon_1\lambda_j) \quad \text{and} \quad Y \in \operatorname{Ker}(F - \varepsilon_2\lambda_k),$$

with $\varepsilon_1, \varepsilon_2 \in \{\pm 1\}$, we obtain from the skew-symmetry property of the Hamilton map $F$ with respect to $\sigma$ that

$$\sigma(X, Y) = \sigma(X, \varepsilon_2^{-1}\lambda_k^{-1}FAY) = -\frac{1}{\varepsilon_2\lambda_k}\sigma(FX, Y) = -\frac{\varepsilon_1\lambda_j}{\varepsilon_2\lambda_k}\sigma(X, Y).$$

Since

$$\left|\frac{\varepsilon_1\lambda_j}{\varepsilon_2\lambda_k}\right| \neq 1,$$

because $\lambda_j$ and $\lambda_k$ are real numbers such that $\lambda_k \notin \{\lambda_j, -\lambda_j\}$, we finally deduce that

$$\sigma(X, Y) = 0,$$

which proves that the spaces $S_{\lambda_j}$ and $S_{\lambda_k}$ are symplectically orthogonal, and we get from (1.4.8) and (1.4.9) that

$$S_{\lambda_1} \oplus^{\sigma\perp} S_{\lambda_2} \oplus^{\sigma\perp} \ldots \oplus^{\sigma\perp} S_{\lambda_r} \subset S. \tag{1.4.10}$$

Let us prove that the singular space is actually exactly equal to this direct sum of symplectic spaces. We recall that from (1.1.16),

$$(\mathrm{Re}\, F)S = \{0\} \quad \text{and} \quad (\mathrm{Im}\, F)S \subset S.$$

Since

$$q(X) = \sigma(X, FX) = i\sigma(X, \mathrm{Im}\, FX),\ X \in S,$$

we deduce from (1.4.1) and the lemma 18.6.4 in [8] that we can find new symplectic basis $(\tilde{e}_1, \ldots, \tilde{e}_m, \tilde{\varepsilon}_1, \ldots, \tilde{\varepsilon}_m)$ in the symplectic space $S$ such that

$$q(X) = i\varepsilon \sum_{j=1}^{m} \mu_j(\tilde{\xi}_j^2 + \tilde{x}_j^2),\ X = \tilde{x}_1\tilde{e}_1 + \cdots + \tilde{x}_m\tilde{e}_m + \tilde{\xi}_1\tilde{\varepsilon}_1 + \cdots + \tilde{\xi}_m\tilde{\varepsilon}_m, \tag{1.4.11}$$

where $\varepsilon \in \{\pm 1\}$ and $\mu_j > 0$ for all $j = 1, \ldots, m$. Indeed, this is linked to the fact that a real-valued elliptic quadratic form must be positive definite or negative definite. By computing $F$ from (1.4.11), we get that

$$FX_j = -\varepsilon\mu_j X_j \quad \text{and} \quad F\tilde{X}_j = \varepsilon\mu_j\tilde{X}_j, \tag{1.4.12}$$

if $X_j = \tilde{e}_j + i\tilde{\varepsilon}_j$ and $\tilde{X}_j = \tilde{e}_j - i\tilde{\varepsilon}_j$ for all $j = 1, \ldots, m$. The identities (1.4.12) prove that the singular space is actually equal to the direct sum of the symplectic spaces $S_{\lambda_j}$ defined in (1.4.6),

$$S = S_{\lambda_1} \oplus^{\sigma\perp} S_{\lambda_2} \oplus^{\sigma\perp} \ldots \oplus^{\sigma\perp} S_{\lambda_r}.$$

*1.4.2 Kramers–Fokker–Plank operator with a quadratic potential*

Let us consider the Kramers–Fokker–Plank operator [4],

$$K = -\Delta_v + \frac{v^2}{4} - \frac{1}{2} + v.\partial_x - (\partial_x V(x)).\partial_v,\ (x, v) \in \mathbb{R}^2,$$

with a quadratic potential

$$V(x) = \frac{1}{2}ax^2,\ a \in \mathbb{R}^*.$$

We can write

$$K = -q(x, v, \xi, \eta)^w - \frac{1}{2}, \tag{1.4.13}$$

with

$$q(x, v, \xi, \eta) = -\eta^2 - \frac{1}{4}v^2 - i(v\xi - ax\eta). \tag{1.4.14}$$

This symbol $q$ is a non-elliptic complex-valued quadratic form with a non-positive real part and a numerical range equal to the half-plane

$$\Sigma(q) = \{z \in \mathbb{C} : \text{Re } z \leq 0\}.$$

We can directly check that its Hamilton map $F$ for which

$$q(x, v, \xi, \eta) = \sigma\left((x, v, \xi, \eta), F(x, v, \xi, \eta)\right),$$

is given by

$$F = \begin{pmatrix} 0 & -\frac{1}{2}i & 0 & 0 \\ \frac{1}{2}ai & 0 & 0 & -1 \\ 0 & 0 & 0 & -\frac{1}{2}ai \\ 0 & \frac{1}{4} & \frac{1}{2}i & 0 \end{pmatrix}, \tag{1.4.15}$$

and that the singular space

$$S = \left(\bigcap_{j=0}^{3} \text{Ker}\left[\text{Re } F(\text{Im } F)^j\right]\right) \cap \mathbb{R}^4,$$

is reduced to the trivial symplectic space $\{0\}$.

As a simple application of Theorem 1.2.2, let us also describe explicitly the spectrum of the quadratic operator $q(x, v, \xi, \eta)^w$ (see also [3], [5]). We are interested in eigenvalues of the Hamilton map $F$ in (1.4.15) such that $\text{Im } \lambda < 0$. Now a computation shows that $\lambda \in \mathbb{C}$ is an eigenvalue of $2F$ precisely when

$$\left(\frac{a}{\lambda} - \lambda\right)^2 + 1 = 0,$$

and we easily see that when $a < 0$, the eigenvalues $\lambda$ of $2F$ with $\text{Im } \lambda < 0$ are given by

$$\lambda_1 = \frac{-i - i\sqrt{1-4a}}{2}, \quad \lambda_2 = \frac{i - i\sqrt{1-4a}}{2}.$$

When $a > 0$, we get the eigenvalues

$$\lambda_1 = \frac{-i + i\sqrt{1-4a}}{2}, \quad \lambda_2 = \frac{-i - i\sqrt{1-4a}}{2}.$$

According to Theorem 1.2.2, the spectrum of $q(x, v, \xi, \eta)^w$ is given by

$$\left\{ \left(\frac{1}{2} + k_1\right)\frac{\lambda_1}{i} + \left(\frac{1}{2} + k_2\right)\frac{\lambda_2}{i}, \quad k_j \in \mathbb{N} \right\}.$$

In particular, when $a > 0$, we observe that the lowest eigenvalue of the spectrum of the operator $K$ in (1.4.13) is 0 (see also [5]).

*Remark* Starting from the quadratic Kramers–Fokker–Plank operator, we may construct examples of quadratic forms with non-positive real parts, for which the singular space $S$ is non-trivial. Indeed, when $q_1(x', \xi')$ is a quadratic form defined as in (1.4.14), we let

$$q(x', x'', \xi', \xi'') = q_1(x', \xi') + iq_2(x'', \xi''),$$

where $q_2$ is a real valued quadratic form in the variables $(x'', \xi'')$. It is then easy to see that the singular space associated to the quadratic form $q$ is given by $S = \{(x', x'', \xi', \xi'') : x' = \xi' = 0\}$.

## 2 Symplectic decomposition of the symbol

In this section, we explain how the main assumption about the symplectic structure of the singular space $S$ fulfilled in the hypotheses of all our three theorems allows us to tensor the variables in the symbol $q$ by writing it as a sum of two quadratic forms where the first one is purely imaginary-valued and where the second one verifies the averaging property of its real part by the flow defined by the Hamilton vector field of its imaginary part.

Let us consider a complex-valued quadratic form $q$ verifying (1.2.1) and let us assume that the singular space $S$ defined in (1.1.15) is symplectic. Let us recall that it is indeed the case when $q$ verifies (1.2.3). Then, we can find $\chi$, a real linear symplectic transformation of $\mathbb{R}^{2n}$, such that

$$(q \circ \chi)(x, \xi) = q_1(x', \xi') + iq_2(x'', \xi''), \; (x, \xi) = (x', x''; \xi', \xi'') \in \mathbb{R}^{2n}, \quad (2.0.1)$$

where $q_1$ is a complex-valued quadratic form on $\mathbb{R}^{2n'}$ with a non-positive real part

$$\text{Re}\, q_1 \leq 0, \qquad (2.0.2)$$

and $q_2$ is a real-valued quadratic form verifying the following properties:

**Proposition 2.0.1** *The two quadratic forms $q_1$ and $q_2$ satisfy the following properties*:

(i) *For all* $T > 0$*, the average of the real part of the quadratic form* $q_1$ *by the flow defined by the Hamilton vector field of* $\text{Im } q_1$*,*

$$\langle \text{Re } q_1 \rangle_T (X') = \frac{1}{2T} \int_{-T}^{T} \text{Re } q_1(e^{tH_{\text{Im}q_1}} X')dt, \ \ X' = (x', \xi') \in \mathbb{R}^{2n'},$$

*is negative definite.*

(ii) *The quadratic form*

$$\sum_{j=0}^{2n-1} \text{Re } q_1 \left( (\text{Im } F_1)^j X' \right), \ \ X' = (x', \xi') \in \mathbb{R}^{2n'},$$

*where* $F_1$ *stands for the Hamilton map of* $q_1$*, is negative definite.*

(iii) *If the symbol* $q$ *fulfills an additional assumption of ellipticity on* $S$*,*

$$(x, \xi) \in S, \ q(x, \xi) = 0 \Rightarrow (x, \xi) = 0, \tag{2.0.3}$$

*then we can assume that*

$$q_2(x'', \xi'') = \varepsilon \sum_{j=1}^{n''} \lambda_j (\xi_j''^2 + x_j''^2),$$

*where* $\varepsilon \in \{\pm 1\}$ *and* $\lambda_j > 0$ *for all* $j = 1, \dots, n''$*.*

To prove these results, we begin by considering $S^{\sigma\perp}$, the orthogonal complement of $S$ in $\mathbb{R}^{2n}$ with respect to the symplectic form and $F$ the Hamilton map of $q$. The space $S^{\sigma\perp}$ is symplectic because it is the case for $S$. Moreover, since according to (1.1.16), $S$ is stable by the maps Re $F$ and Im $F$, its orthogonal complement also fulfills these properties. Indeed, let $X$ be in $S^{\sigma\perp}$. By using (1.1.16) and the skew-symmetry of any Hamilton map with respect to $\sigma$, we get for all $Y \in S$,

$$\sigma(Y, \text{Re } FX) + i\sigma(Y, \text{Im } FX) = -\sigma(\text{Re } FY, X) - i\sigma(\text{Im } FY, X) = 0,$$

because $(\text{Re } F)Y \in S$ and $(\text{Im } F)Y \in S$. This induces that for all $Y \in S$,

$$\sigma(Y, \text{Re } FX) = \sigma(Y, \text{Im } FX) = 0,$$

and proves that $(\text{Re } F)X \in S^{\sigma\perp}$ and $(\text{Im } F)X \in S^{\sigma\perp}$.

We can then write the phase space $\mathbb{R}^{2n}$ as a direct sum of two symplectically orthogonal real symplectic spaces stable by the maps Re $F$ and Im $F$,

$$\mathbb{R}^{2n} = S_1 \oplus^{\sigma\perp} S_2, \ (\text{Re } F)S_j \subset S_j, \ (\text{Im } F)S_j \subset S_j, \tag{2.0.4}$$

for $j \in \{1, 2\}$ with

$$S_1 = S^{\sigma\perp} \quad \text{and} \quad S_2 = S. \tag{2.0.5}$$

Let us now consider a symplectic basis $(e_{1,j}, \dots, e_{N_j,j}, \varepsilon_{1,j}, \dots, \varepsilon_{N_j,j})$ of $S_j$. By collecting these two bases, we get a symplectic basis of $\mathbb{R}^{2n}$, which allows by using the stability and the orthogonality properties of the spaces $S_j$ to obtain the following decomposition of $q$,

$$\begin{aligned} q(x,\xi) &= \sigma\left(\sum_{\substack{1\le j\le 2,\\ 1\le k\le N_j}} (x_{k,j}e_{k,j} + \xi_{k,j}\varepsilon_{k,j}),\, F\left(\sum_{\substack{1\le j\le 2,\\ 1\le k\le N_j}} (x_{k,j}e_{k,j} + \xi_{k,j}\varepsilon_{k,j})\right)\right)\\ &= \sum_{1\le j\le 2} \sigma\left(\sum_{1\le k\le N_j} (x_{k,j}e_{k,j}+\xi_{k,j}\varepsilon_{k,j}),\, F\left(\sum_{1\le k\le N_j} (x_{k,j}e_{k,j}+\xi_{k,j}\varepsilon_{k,j})\right)\right). \end{aligned}$$

This implies that we can find symplectic coordinates

$$(x, \xi) = (x', x''; \xi', \xi'') \in \mathbb{R}^{2n},$$

where $(x', \xi')$ and $(x'', \xi'')$ are some symplectic coordinates in $S^{\sigma\perp}$ and $S$ respectively, such that

$$q(x,\xi) = q_1(x',\xi') + q_2(x'',\xi''), \tag{2.0.6}$$

with

$$q_1(x',\xi') = \sigma\left((x',\xi'), F|_{S^{\sigma\perp}}(x',\xi')\right) \tag{2.0.7}$$

and

$$q_2(x'',\xi'') = \sigma\left((x'',\xi''), F|_S(x'',\xi'')\right). \tag{2.0.8}$$

Since from (1.1.16),

$$(\mathrm{Re}\, F)S = \{0\},$$

the quadratic form $q_2$ is purely imaginary-valued and can be written as

$$q_2 = i\tilde{q}_2, \tag{2.0.9}$$

where $\tilde{q}_2$ is the real-valued quadratic form

$$\tilde{q}_2(x'',\xi'') = \sigma\left((x'',\xi''), \mathrm{Im}\, F|_S(x'',\xi'')\right). \tag{2.0.10}$$

When the additional assumption (2.0.3) is fulfilled, this quadratic form $\tilde{q}_2$ must be elliptic on $\mathbb{R}^{2n''}$. Since a real-valued elliptic quadratic form is necessarily a positive definite or negative definite quadratic form, we deduce from the lemma 18.6.4 in [8] that we can find new symplectic coordinates $(x'', \xi'')$ in $S$ and $\varepsilon \in \{\pm 1\}$ such that

$$\tilde{q}_2(x'', \xi'') = \varepsilon \sum_{j=1}^{n''} \lambda_j (\xi_j''^2 + x_j''^2), \tag{2.0.11}$$

where $\lambda_j > 0$ for all $j = 1, \dots, n''$. This proves (iii) in Proposition 2.0.1.

Let us now study the properties of the quadratic form $q_1$. We denote by $F_1$ its Hamilton map

$$F_1 = F|_{S^{\sigma\perp}}, \tag{2.0.12}$$

and define the following quadratic form

$$r(X') = \sum_{j=0}^{2n-1} \operatorname{Re} q_1 \left( (\operatorname{Im} F_1)^j X' \right), \ X' = (x', \xi') \in S^{\sigma\perp}. \tag{2.0.13}$$

Since from (1.2.1), (2.0.6) and (2.0.9), $\operatorname{Re} q_1$ is a non-positive quadratic form, we already know that $r$ is a non-positive quadratic form. We now prove that $r$ is actually a negative definite quadratic form. Let us consider $X_0' \in S^{\sigma\perp}$ such that

$$r(X_0') = 0.$$

The non-positivity of the quadratic form $\operatorname{Re} q_1$ induces that for all $j = 0, \dots, 2n-1$,

$$\operatorname{Re} q_1 \left( (\operatorname{Im} F_1)^j X_0' \right) = 0. \tag{2.0.14}$$

Let us denote by $\operatorname{Re} q_1(X'; Y')$ the polarized form associated to $\operatorname{Re} q_1$. We deduce from the Cauchy-Schwarz inequality, (1.1.8) and (2.0.14) that for all $j = 0, \dots, 2n-1$ and $Y' \in S^{\sigma\perp}$,

$$\begin{aligned} |\operatorname{Re} q_1 \left( Y'; (\operatorname{Im} F_1)^j X_0' \right)|^2 &= |\sigma \left( Y', \operatorname{Re} F_1 (\operatorname{Im} F_1)^j X_0' \right)|^2 \\ &\leq [-\operatorname{Re} q_1(Y')][-\operatorname{Re} q_1 \left( (\operatorname{Im} F_1)^j X_0' \right)] = 0. \end{aligned}$$

It follows that for all $j = 0, \dots, 2n-1$ and $Y' \in S^{\sigma\perp}$,

$$\sigma \left( Y', \operatorname{Re} F_1 (\operatorname{Im} F_1)^j X_0' \right) = 0,$$

which implies that for all $j = 0, \dots, 2n-1$,

$$\operatorname{Re} F_1 (\operatorname{Im} F_1)^j X_0' = 0, \tag{2.0.15}$$

because from (2.0.4), (2.0.5) and (2.0.12), $\operatorname{Re} F_1 (\operatorname{Im} F_1)^j X_0' \in S^{\sigma\perp}$ and $S^{\sigma\perp}$ is a symplectic vector space. Since $X_0' \in S^{\sigma\perp}$, we deduce from (1.1.15), (2.0.4), (2.0.5), (2.0.12) and (2.0.15) that $X_0' \in S \cap S^{\sigma\perp} = \{0\}$, which proves that $r$ is a negative definite quadratic form. This proves (ii) in Proposition 2.0.1.

*Remark* According to the previous proof, let us notice that the property (ii) implies that for all $X \in \mathbb{R}^{2n'}$, $X \neq 0$, there exists $j_0 \in \{0, \ldots, 2n-1\}$ such that

$$\forall\, 0 \leq j \leq j_0 - 1,\ \text{Re}\, F_1(\text{Im}\, F_1)^j X = 0,\ \text{Re}\, F_1(\text{Im}\, F_1)^{j_0} X \neq 0. \tag{2.0.16}$$

Let us now prove that for all $T > 0$, the average of the real part of the quadratic form $q_1$ by the flow defined by the Hamilton vector field of Im $q_1$,

$$\langle \text{Re}\, q_1 \rangle_T (X') = \frac{1}{2T} \int_{-T}^{T} \text{Re}\, q_1(e^{tH_{\text{Im} q_1}} X') dt,$$

is negative definite. Let us notice that this flow is globally defined since the symbol Im $q_1$ is quadratic. Let us consider $X_0'$ in $\mathbb{R}^{2n'}$ such that

$$\langle \text{Re}\, q_1 \rangle_T (X_0') = 0. \tag{2.0.17}$$

Since Re $q_1$ is a non-positive quadratic form, it follows from (2.0.17) that

$$\text{Re}\, q_1(e^{tH_{\text{Im} q_1}} X_0') = 0,$$

for all $-T \leq t \leq T$. This implies in particular that for all $k \in \mathbb{N}$,

$$\frac{d^k}{dt^k} \Big(\text{Re}\, q_1(e^{tH_{\text{Im} q_1}} X_0')\Big)\Big|_{t=0} = H^k_{\text{Im} q_1} \text{Re}\, q_1(X_0') = 0. \tag{2.0.18}$$

If $X_0' \neq 0$, we deduce from (ii) that there exists $j_0 \in \{0, \ldots, 2n-1\}$ such that

$$\forall\, 0 \leq j \leq j_0 - 1,\ \text{Re}\, q_1 \Big((\text{Im}\, F_1)^j X_0'\Big) = 0,\ \text{Re}\, q_1 \Big((\text{Im}\, F_1)^{j_0} X_0'\Big) < 0. \tag{2.0.19}$$

Let us check that it would imply that

$$H^{2j_0}_{\text{Im} q_1} \text{Re}\, q_1(X_0') \neq 0, \tag{2.0.20}$$

and contradict (2.0.18). To prove (2.0.20), we use some arguments already used in [11] together with the following lemma also proved in [11].

**Lemma 2.0.1** *If $q_1$ and $q_2$ are two complex-valued quadratic forms on $\mathbb{R}^{2n}$, then the Hamilton map associated to the complex-valued quadratic form defined by the Poisson bracket*

$$\{q_1, q_2\} = \frac{\partial q_1}{\partial \xi} . \frac{\partial q_2}{\partial x} - \frac{\partial q_1}{\partial x} . \frac{\partial q_2}{\partial \xi},$$

*is* $-2[F_1, F_2]$ *where* $[F_1, F_2]$ *stands for the commutator of* $F_1$ *and* $F_2$*, the Hamilton maps of* $q_1$ *and* $q_2$.

We deduce from the previous lemma that the Hamilton map associated to the quadratic form $H_{\mathrm{Im} q_1}^{2j_0} \mathrm{Re}\, q_1$ is

$$4^{j_0}[\mathrm{Im}\, F_1, [\mathrm{Im}\, F_1, [\ldots, [\mathrm{Im}\, F_1, \mathrm{Re}\, F_1]\ldots], \tag{2.0.21}$$

with exactly $2j_0$ terms $\mathrm{Im}\, F_1$ appearing in the formula. We can write

$$\begin{aligned} &4^{j_0}[\mathrm{Im}\, F_1, [\mathrm{Im}\, F_1, [\ldots, [\mathrm{Im}\, F_1, \mathrm{Re}\, F_1]\ldots] \\ &\quad = \sum_{j=0}^{2j_0} (-1)^j c_j (\mathrm{Im}\, F_1)^j \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{2j_0-j}, \end{aligned} \tag{2.0.22}$$

with $c_j > 0$ for all $j = 0, \ldots, 2j_0$. Indeed, by using the following identity

$$[P, [P, Q]] = P^2 Q - 2PQP + QP^2,$$

we can prove by induction that for all $n \in \mathbb{N}^*$, there exist some positive constants $d_{n,j}$, $j = 0, \ldots, 2n$ such that

$$[P, [P, [\ldots, [P, Q]\ldots] = \sum_{j=0}^{2n} (-1)^j d_{n,j} P^j Q P^{2n-j},$$

if there are exactly $2n$ terms $P$ in the left-hand side of the previous identity. It follows from (2.0.21) and (2.0.22) that

$$\begin{aligned} H_{\mathrm{Im} q_1}^{2j_0} \mathrm{Re}\, q_1(X_0') = &(-1)^{j_0} c_{j_0} \sigma\left(X_0', (\mathrm{Im}\, F_1)^{j_0} \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{j_0} X_0'\right) \\ &+ \sum_{j=0}^{j_0-1} (-1)^j c_j \sigma\left(X_0', (\mathrm{Im}\, F_1)^j \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{2j_0-j} X_0'\right) \\ &+ \sum_{j=0}^{j_0-1} (-1)^{2j_0-j} c_{2j_0-j} \sigma\left(X_0', (\mathrm{Im}\, F_1)^{2j_0-j} \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^j X_0'\right). \end{aligned} \tag{2.0.23}$$

Now on the one hand,

$$\begin{aligned} \sigma\left(X_0', (\mathrm{Im}\, F_1)^{j_0} \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{j_0} X_0'\right) &= (-1)^{j_0} \sigma\left((\mathrm{Im}\, F_1)^{j_0} X_0', \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{j_0} X_0'\right) \\ &= (-1)^{j_0} \mathrm{Re}\, q_1\left((\mathrm{Im}\, F_1)^{j_0} X_0'\right), \end{aligned}$$

by the skew-symmetry of the Hamilton map Im $F_1$. On the other hand, using (1.1.8), (2.0.19), and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\left|\sigma\left(X_0', (\mathrm{Im}\, F_1)^j \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{2j_0-j} X_0'\right)\right| \\
&\quad = \left|\sigma\left((\mathrm{Im}\, F_1)^j X_0', \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{2j_0-j} X_0'\right)\right| \\
&\quad = -\mathrm{Re}\, q_1\left((\mathrm{Im}\, F_1)^j X_0'; (\mathrm{Im}\, F_1)^{2j_0-j} X_0'\right) \\
&\quad \leq [-\mathrm{Re}\, q_1((\mathrm{Im}\, F_1)^j X_0')]^{\frac{1}{2}} [-\mathrm{Re}\, q_1((\mathrm{Im}\, F_1)^{2j_0-j} X_0')]^{\frac{1}{2}} = 0
\end{aligned}$$

and

$$\begin{aligned}
&\left|\sigma\left(X_0', (\mathrm{Im}\, F_1)^{2j_0-j} \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{j} X_0'\right)\right| \\
&\quad = \left|\sigma\left((\mathrm{Im}\, F_1)^{2j_0-j} X_0', \mathrm{Re}\, F_1 (\mathrm{Im}\, F_1)^{j} X_0'\right)\right| \\
&\quad = -\mathrm{Re}\, q_1\left((\mathrm{Im}\, F_1)^{2j_0-j} X_0'; (\mathrm{Im}\, F_1)^{j} X_0'\right) \\
&\quad \leq [-\mathrm{Re}\, q_1((\mathrm{Im}\, F_1)^{2j_0-j} X_0')]^{\frac{1}{2}} [-\mathrm{Re}\, q_1((\mathrm{Im}\, F_1)^{j} X_0')]^{\frac{1}{2}} = 0
\end{aligned}$$

if $j = 0, \dots, j_0 - 1$. It follows from (2.0.19) and (2.0.23) that

$$H_{\mathrm{Im} q_1}^{2j_0} \mathrm{Re}\, q_1(X_0') = c_{j_0} \mathrm{Re}\, q_1\left((\mathrm{Im}\, F_1)^{j_0} X_0'\right) < 0,$$

because $c_{j_0} > 0$. This proves (2.0.20) and ends the proof (i). □

*Remark* Let us notice that we have actually proved that the symbol $q_1$ has a finite order $\tau$,

$$1 \leq \tau \leq 4n - 2, \tag{2.0.24}$$

in every point of the set $q_1(\mathbb{R}^{2n'})\backslash\{0\}$. We recall that the order $k(x_0, \xi_0)$ of a symbol $p(x, \xi)$ at a point $(x_0, \xi_0) \in \mathbb{R}^{2n}$ (see section 27.2, Chapter 27 in [8]) is the element of $\mathbb{N} \cup \{+\infty\}$ defined by

$$k(x_0, \xi_0) = \sup\{j \in \mathbb{Z} : p_I(x_0, \xi_0) = 0,\ \forall\, 1 \leq |I| \leq j\}, \tag{2.0.25}$$

where $I = (i_1, i_2, \dots, i_k) \in \{1, 2\}^k$, $|I| = k$ and $p_I$ stands for the iterated Poisson brackets

$$p_I = H_{p_{i_1}} H_{p_{i_2}} \dots H_{p_{i_{k-1}}} p_{i_k},$$

where $p_1$ and $p_2$ are respectively the real and the imaginary part of the symbol $p$, $p = p_1 + ip_2$. The order of a symbol $q$ at a point $z$ is then defined as the maximal order of the symbol $p = q - z$ at every point $(x_0, \xi_0) \in \mathbb{R}^{2n}$ verifying

$$p(x_0, \xi_0) = q(x_0, \xi_0) - z = 0.$$

## 3 Proofs of the main results

### 3.1 Heat semigroup smoothing for non-elliptic quadratic operators

In this section, we prove Theorem 1.2.1. Let us consider a complex-valued quadratic form

$$q : \mathbb{R}_x^n \times \mathbb{R}_\xi^n \to \mathbb{C},\ n \in \mathbb{N}^*,$$

with a non-positive real part

$$\text{Re}\, q \leq 0,$$

such that its singular space $S$ has a symplectic structure. We recall that this assumption is actually fulfilled in the assumptions of Theorems 1.2.1, 1.2.2 and 1.2.3 since this singular space is always symplectic when the symbol is elliptic on $S$.

We can then use the symplectic decomposition of the symbol obtained in Sect. 2. We deduce from (2.0.1) and (2.0.2) that there exists $\chi$, a real linear symplectic transformation of $\mathbb{R}^{2n}$, such that

$$(q \circ \chi)(x, \xi) = q_1(x', \xi') + iq_2(x'', \xi''),\ (x, \xi) = (x', x''; \xi', \xi'') \in \mathbb{R}^{2n}, \quad (3.1.1)$$

where $q_1$ is a complex-valued quadratic form on $\mathbb{R}^{2n'}$ with a non-positive real part

$$\text{Re}\, q_1 \leq 0, \quad (3.1.2)$$

and $q_2$ is a real-valued quadratic form verifying the properties stated in Proposition 2.0.1. The key point in our proof of Theorems 1.2.1, 1.2.2 and 1.2.3 is to prove the following proposition.

**Proposition 3.1.1** *If $n' \geq 1$, then the spectrum of the quadratic differential operator $q_1(x', \xi')^w$ is only composed of eigenvalues with finite multiplicity*

$$\sigma\left(q_1(x', \xi')^w\right) = \left\{ \sum_{\substack{\lambda \in \sigma(F_1), \\ \text{Re}(-i\lambda) < 0}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\},$$

*where $F_1$ is the Hamilton map associated to the quadratic form $q_1$ and $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F_1$ in $\mathbb{C}^{2n'}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$. Moreover, the operator $q_1(x', \xi')^w$ generates a contraction semigroup such that*

$$e^{tq_1(x', \xi')^w} u \in \mathcal{S}(\mathbb{R}^{n'}),$$

*for any $t > 0$ and $u \in L^2(\mathbb{R}^{n'})$.*

*Remark* It will be clear from the proof that Proposition 3.1.1 extends to the vector-valued case, so that if $\mathcal{H}$ is a complex Hilbert space and $u \in L^2(\mathbb{R}^{n'}; \mathcal{H})$ then for any $t > 0$ we have $e^{tq_1(x',\xi')^w} u \in \mathcal{S}(\mathbb{R}^{n'}; \mathcal{H})$.

Theorem 1.2.1 directly follows from Proposition 3.1.1 together with the preceding remark. Indeed, by using the symplectic invariance of the Weyl quantization given by the theorem 18.5.9 in [8], we can find a metaplectic operator $U$, which is a unitary transformation on $L^2(\mathbb{R}^n)$ and an automorphism of $\mathcal{S}(\mathbb{R}^n)$ such that

$$(q \circ \chi)(x, \xi)^w = U^{-1} q(x, \xi)^w U. \tag{3.1.3}$$

This implies at the level of the generated semigroups that

$$e^{t(q\circ\chi)(x,\xi)^w} = U^{-1} e^{tq(x,\xi)^w} U,\ t \geq 0. \tag{3.1.4}$$

Since from the tensorization of the variables (3.1.1),

$$e^{t(q\circ\chi)(x,\xi)^w} = e^{tq_1(x',\xi')^w} e^{itq_2(x'',\xi'')^w},$$

we directly deduce from (2.0.7), (3.1.4), Proposition 3.1.1 together with the following remark, and the symplectic invariance of the Weyl quantization that if $(x', \xi')$ are some symplectic coordinates on the symplectic space $S^{\sigma\perp}$ then for all $t > 0$, $N \in \mathbb{N}$ and $u \in L^2(\mathbb{R}^n) = L^2(\mathbb{R}^{n'}; L^2(\mathbb{R}^{n''}))$, we have

$$\Big((1 + |x'|^2 + |\xi'|^2)^N\Big)^w e^{tq(x,\xi)^w} u \in L^2(\mathbb{R}^n),$$

which proves Theorem 1.2.1.

Let us now prove Proposition 3.1.1. For convenience, we drop the index and we consider a complex-valued quadratic form

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C},\ n \in \mathbb{N}^*,$$

with a non-positive real part

$$\mathrm{Re}\, q \leq 0, \tag{3.1.5}$$

such that for all $T > 0$, the average of the real part of the quadratic form $q$ by the flow defined by the Hamilton vector field of $\mathrm{Im}\, q$,

$$\langle \mathrm{Re}\, q\rangle_T(X) = \frac{1}{2T}\int_{-T}^{T} \mathrm{Re}\, q(e^{tH_{\mathrm{Im} q}} X) dt,\ X = (x, \xi) \in \mathbb{R}^{2n}, \tag{3.1.6}$$

is negative definite. We also know from (2.0.16) that for all $X \in \mathbb{R}^{2n}$, $X \neq 0$, there exists $j_0 \in \mathbb{N}$ verifying

$$\forall\, 0 \leq j \leq j_0 - 1,\ \operatorname{Re} F(\operatorname{Im} F)^j X = 0,\ \operatorname{Re} F(\operatorname{Im} F)^{j_0} X \neq 0, \tag{3.1.7}$$

if $F$ stands for the Hamilton map of the quadratic form $q$.

Let us denote by

$$Q = q(x, \xi)^w, \tag{3.1.8}$$

the quadratic differential operator defined by the Weyl quantization of the symbol $q$. When proving Proposition 3.1.1, we shall work with the metaplectic FBI-Bargmann transform

$$Tu(x) = C \int_{\mathbb{R}^n} e^{i\varphi(x,y)} u(y)\, dy,\ x \in \mathbb{C}^n,\ C > 0, \tag{3.1.9}$$

where we may choose

$$\varphi(x, y) = \frac{i}{2}(x - y)^2,$$

as in the standard Bargmann transform. Other quadratic phase functions $\varphi$ such that $\operatorname{Im} \varphi''_{yy} > 0$ and $\det \varphi''_{xy} \neq 0$, are also possible (see Sect. 1 of [15]). It is well known that for a suitable choice of $C > 0$, $T$ defines a unitary transformation

$$T : L^2(\mathbb{R}^n) \to H_{\Phi_0}(\mathbb{C}^n),$$

where

$$H_{\Phi_0}(\mathbb{C}^n) = \operatorname{Hol}(\mathbb{C}^n) \cap L^2\left(\mathbb{C}^n, e^{-2\Phi_0(x)} L(dx)\right), \tag{3.1.10}$$

with

$$\Phi_0(x) = \sup_{y \in \mathbb{R}^n} -\operatorname{Im} \varphi(x, y) = \frac{1}{2} (\operatorname{Im} x)^2,$$

and $L(dx)$ being the Lebesgue measure in $\mathbb{C}^n$.

*Remark* Let us recall (see, e.g. section 3 of [16]) that the same definitions apply in the vector-valued case, so that we have a unitary operator

$$T : L^2(\mathbb{R}^n; \mathcal{H}) \to H_{\Phi_0}(\mathbb{C}^n; \mathcal{H}),$$

where $\mathcal{H}$ is a complex Hilbert space.

We recall next from [15] that

$$TQu = Q_0 Tu,\ u \in \mathcal{S}(\mathbb{R}^n), \tag{3.1.11}$$

where $Q_0$ is a quadratic differential operator on $\mathbb{C}^n$ whose Weyl symbol $q_0$ satisfies

$$q_0 \circ \kappa_T = q. \tag{3.1.12}$$

Here

$$\kappa_T : \mathbb{C}^{2n} \ni \Big(y, -\varphi'_y(x, y)\Big) \mapsto \big(x, \varphi'_x(x, y)\big) \in \mathbb{C}^{2n}, \tag{3.1.13}$$

is the complex linear canonical transformation associated to $T$. From [15], we recall next that if we define

$$\Lambda_{\Phi_0} = \left\{ \left( x, \frac{2}{i} \frac{\partial \Phi_0}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\}, \tag{3.1.14}$$

then we have

$$\Lambda_{\Phi_0} = \kappa_T(\mathbb{R}^{2n}). \tag{3.1.15}$$

When

$$\sigma = \sum_{j=1}^{n} d\xi_j \wedge dx_j,$$

is the complex symplectic (2,0)-form on $\mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$, then the restriction $\sigma_{\Lambda_{\Phi_0}}$ of $\sigma$ to $\Lambda_{\Phi_0}$ is real and nondegenerate. The map $\kappa_T$ in (3.1.13) can therefore be viewed as a canonical transformation between the real symplectic spaces $\mathbb{R}^{2n}$ and $\Lambda_{\Phi_0}$.

Continuing to follow [15], let us recall next that when realizing $Q_0$ as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$, we may use first the contour integral representation

$$Q_0 u(x) = \frac{1}{(2\pi)^n} \int\limits_{\theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right)} e^{i(x-y)\cdot\theta} q_0\left(\frac{x+y}{2}, \theta\right) u(y)\, dy\, d\theta,$$

and then, using that the symbol $q_0$ is holomorphic, by a contour deformation we obtain the following formula for $Q_0$ as an unbounded operator on $H_{\Phi_0}(\mathbb{C}^n)$,

$$Q_0 u(x) = \frac{1}{(2\pi)^n} \int\limits_{\theta = \frac{2}{i} \frac{\partial \Phi_0}{\partial x}\left(\frac{x+y}{2}\right) + it\overline{(x-y)}} e^{i(x-y)\cdot\theta} q_0\left(\frac{x+y}{2}, \theta\right) u(y)\, dy\, d\theta, \tag{3.1.16}$$

for any $t > 0$.

We shall now discuss certain IR-deformations of the real phase space $\mathbb{R}^{2n}$, where the averaging procedure along the flow defined by the Hamilton vector field of Im $q$ (see (3.1.6)) plays an important rôle. To that end, let $G = G_T$ be a real-valued quadratic form on $\mathbb{R}^{2n}$ such that

$$H_{\mathrm{Im} q} G = -\mathrm{Re}\, q + \langle \mathrm{Re}\, q \rangle_T. \tag{3.1.17}$$

As in [6], we solve (3.1.17) by setting

$$G(X) = \int\limits_{\mathbb{R}} k_T(t) \mathrm{Re}\, q(e^{tH_{\mathrm{Im} q}} X) dt, \tag{3.1.18}$$

where $k_T(t) = k(t/2T)$ and $k \in C(\mathbb{R}\backslash\{0\})$ is the odd function given by

$$k(t) = 0 \quad \text{for} \quad |t| \geq \frac{1}{2} \quad \text{and} \quad k'(t) = -1 \quad \text{for} \quad 0 < |t| < \frac{1}{2}.$$

Let us notice that $k$ and $k_T$ have a jump of size 1 at the origin. Associated with $G$ there is a linear IR-manifold, defined for $0 \leq \varepsilon \leq \varepsilon_0$, with $\varepsilon_0 > 0$ small enough,

$$\Lambda_{\varepsilon G} = e^{i\varepsilon H_G}(\mathbb{R}^{2n}) \subset \mathbb{C}^{2n}, \tag{3.1.19}$$

where $e^{i\varepsilon H_G}$ stands for the flow generated by the linear Hamilton vector field $i\varepsilon H_G$ taken at the time 1. It is then well-known and easily checked (see, for instance, sections 3 and 5 in [5]), that

$$\kappa_T(\Lambda_{\varepsilon G}) = \Lambda_{\tilde{\Phi}_\varepsilon} := \left\{ \left( x, \frac{2}{i} \frac{\partial \tilde{\Phi}_\varepsilon}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\}, \tag{3.1.20}$$

where $\tilde{\Phi}_\varepsilon$ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, such that

$$\tilde{\Phi}_\varepsilon(x) = \Phi_0(x) + \varepsilon G(\operatorname{Re} x, -\operatorname{Im} x) + \mathcal{O}(\varepsilon^2 |x|^2). \tag{3.1.21}$$

Associated with the function $\tilde{\Phi}_\varepsilon$ is the weighted space of holomorphic functions $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$ defined as in (3.1.10). The operator $Q_0$ can also be defined as an unbounded operator

$$Q_0 : H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n) \to H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n),$$

if we make a new contour deformation in (3.1.16) and set

$$Q_0 u(x) = \frac{1}{(2\pi)^n} \int_{\theta = \frac{2}{i} \frac{\partial \tilde{\Phi}_\varepsilon}{\partial x}\left(\frac{x+y}{2}\right) + it\overline{(x-y)}} e^{i(x-y)\cdot\theta} q_0 \left( \frac{x+y}{2}, \theta \right) u(y)\, dy\, d\theta, \tag{3.1.22}$$

for any $t > 0$. By coming back to the real side by the FBI-Bargmann transform, the operator $Q_0$ can be viewed as an unbounded operator on $L^2(\mathbb{R}^n)$ with the Weyl symbol

$$\widetilde{q}(X) = q\left(e^{i\varepsilon H_G} X\right), \tag{3.1.23}$$

and here the real part of this expression is easily seen to be equal to

$$\operatorname{Re} \widetilde{q}(X) = \operatorname{Re} q(X) + \varepsilon H_{\operatorname{Im} q} G(X) + \mathcal{O}(\varepsilon^2 |X|^2).$$

It follows therefore from (3.1.5), (3.1.17) and the assumption that the quadratic form (3.1.6) is negative definite, that

$$-\operatorname{Re} \widetilde{q}(X) \geq \frac{\varepsilon}{C} |X|^2, \; C > 1, \quad X \in \mathbb{R}^{2n}, \tag{3.1.24}$$

for $0 < \varepsilon \ll 1$. We may therefore apply Theorem 3.5 of [12] to the operator $Q_0$ viewed as an unbounded operator on $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$.

**Lemma 3.1.1** *Let us consider $Q_0$ as an unbounded operator on $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$, for $0 < \varepsilon \leq \varepsilon_0$, with $\varepsilon_0 > 0$ sufficiently small. The spectrum of the operator $Q_0$ is only composed of eigenvalues with finite multiplicity*

$$\sigma(Q_0) = \left\{ \sum_{\substack{\lambda \in \sigma(F),\\ \mathrm{Re}(-i\lambda)<0}} (r_\lambda + 2k_\lambda)(-i\lambda) : \ k_\lambda \in \mathbb{N} \right\}, \tag{3.1.25}$$

*where $F$ is the Hamilton map associated to the quadratic form $q$ and $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$.*

To get the statement of this lemma, it suffices to combine Theorem 3.5 of [12] together with the observation that the Hamilton maps $F$ and $\widetilde{F}$ of the quadratic forms $q$ and $\widetilde{q}$, respectively, are isospectral since from (3.1.23) the symbols $q$ and $\widetilde{q}$ are related by a canonical transformation.

Having determined the spectrum of $Q_0$ in the weighted space $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$, $0 < \varepsilon \ll 1$, we now come to the proof of Proposition 3.1.1. In doing so, we shall study the spectral properties of the holomorphic quadratic differential operator $Q_0$ acting on $H_{\Phi_0}(\mathbb{C}^n)$.

We shall consider the heat evolution equation associated to the operator $Q_0$. Let us notice explicitly that we got this idea by studying Remark 11.7 in [5], and indeed, the following argument can be seen as a natural continuation of some ideas sketched in that remark. Using Fourier integral operators with quadratic phase in the complex domain, we may describe the heat semigroup $e^{tQ_0}$ for $0 \leq t \leq t_0$, when $t_0 > 0$ is small enough. More precisely, we are interested in solving

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) - Q_0 u(t,x) = 0 \\ u(t,\cdot)|_{t=0} = u_0 \in H_{\Phi_0}(\mathbb{C}^n). \end{cases}$$

Let $\varphi(t,x,\eta)$ be the quadratic form in the variables $x$, $\eta$, depending smoothly on $t$, $0 \leq t \leq t_0 \ll 1$, and solving the Hamilton-Jacobi equation

$$\begin{cases} i\dfrac{\partial \varphi}{\partial t}(t,x,\eta) - q_0\left(x, \dfrac{\partial \varphi}{\partial x}(t,x,\eta)\right) = 0 \\ \varphi(t,x,\eta)|_{t=0} = x \cdot \eta. \end{cases}$$

We know that for $0 \leq t \leq t_0 \ll 1$, $\varphi(t,x,\eta)$ can be obtained as a generating function of the complex canonical transformation

$$e^{itH_{q_0}} : \left(\varphi'_\eta(t,x,\eta), \eta\right) \mapsto \left(x, \varphi'_x(t,x,\eta)\right).$$

Then for $t \geq 0$ small enough, the operator $e^{tQ_0}$ acting on $H_{\Phi_0}(\mathbb{C}^n)$ has the form

$$e^{tQ_0}u = \frac{1}{(2\pi)^n} \int_{\Gamma_x} e^{i(\varphi(t,x,\eta)-y\cdot\eta)} a(t,x,y,\eta)u(y)dy\,d\eta,$$

where $a(t,x,y,\eta)$ is a suitable amplitude which we need not specify here, and, following the general theory of [13], we take $\Gamma_x$ to be a suitable contour passing through the critical point of the function

$$(y,\eta) \mapsto -\mathrm{Im}\,(\varphi(t,x,\eta) - y\cdot\eta) + \Phi_0(y).$$

We then know from the general theory that the operator $e^{tQ_0}$ is bounded

$$e^{tQ_0} : H_{\Phi_0}(\mathbb{C}^n) \to H_{\Phi_t}(\mathbb{C}^n), \tag{3.1.26}$$

where $\Phi_t$ is a strictly plurisubharmonic quadratic form on $\mathbb{C}^n$, depending smoothly on $t$, such that if

$$\Lambda_{\Phi_t} = \left\{ \left( x, \frac{2}{i}\frac{\partial \Phi_t}{\partial x}(x) \right) : x \in \mathbb{C}^n \right\}, \tag{3.1.27}$$

then

$$\Lambda_{\Phi_t} = \exp\left( t\widehat{H_{-\frac{1}{i}q_0}} \right)\left(\Lambda_{\Phi_0}\right). \tag{3.1.28}$$

Here, when $f$ is a holomorphic function on $\mathbb{C}^{2n} = \mathbb{C}^n_x \times \mathbb{C}^n_\xi$, $H_f$ is the standard Hamilton field of $f$, of type (1,0), given by the usual formula

$$H_f = \sum_{j=1}^n \left( \frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial x_j} - \frac{\partial f}{\partial x_j}\frac{\partial}{\partial \xi_j} \right),$$

and $\widehat{H_f} = H_f + \overline{H_f}$ is the corresponding real vector field.

It follows from the classical Hamilton-Jacobi theory applied with respect to the real symplectic form $\mathrm{Im}\,\sigma$, where

$$\sigma = \sum_{j=1}^n d\xi_j \wedge dx_j,$$

is the complex symplectic (2,0)-form on $\mathbb{C}^{2n}$, that the quadratic form $\Phi(t,x) = \Phi_t(x)$ introduced in (3.1.26), satisfies the eikonal equation

$$\begin{cases} \dfrac{\partial \Phi}{\partial t}(t,x) - \mathrm{Re}\left[ q_0\left( x, \dfrac{2}{i}\dfrac{\partial \Phi}{\partial x}(t,x) \right) \right] = 0 \\ \Phi(t,\cdot)|_{t=0} = \Phi_0. \end{cases} \tag{3.1.29}$$

See also [14] for a detailed (and much more general) discussion of this point.

Instrumental in the proof of Proposition 3.1.1 is the following result.

**Lemma 3.1.2** *For each $T_0 > 0$ small enough, there exists $\alpha = \alpha(T_0) > 0$ such that*

$$\Phi_{T_0}(x) \leq \Phi_0(x) - \alpha\, |x|^2\,, \quad x \in \mathbb{C}^n. \tag{3.1.30}$$

Once Lemma 3.1.2 has been established, it is easy to finish the proof of the first result in Proposition 3.1.1. Indeed, elementary arguments together with (3.1.10) and (3.1.30) show that the natural embedding $H_{\Phi_t}(\mathbb{C}^n) \rightarrow H_{\Phi_0}(\mathbb{C}^n)$ is compact for $t > 0$ small, and hence by using the semigroup property, we deduce that the semigroup

$$e^{tQ_0} : H_{\Phi_0}(\mathbb{C}^n) \rightarrow H_{\Phi_0}(\mathbb{C}^n), \tag{3.1.31}$$

is compact for each $t > 0$. An application of Theorem 2.20 in [2] then shows that the spectrum of the operator $Q_0$ acting on $H_{\Phi_0}(\mathbb{C}^n)$ consists of a countable discrete set of eigenvalues each of finite multiplicity.

When deriving the explicit description of the spectrum of $Q_0$ on $H_{\Phi_0}(\mathbb{C}^n)$, we argue in the following way. Let us assume that $\lambda \in \mathbb{C}$ is an eigenvalue of $Q_0$ on $H_{\Phi_0}(\mathbb{C}^n)$ and let $u_0 \in H_{\Phi_0}(\mathbb{C}^n)$ be a corresponding eigenvector,

$$Q_0 u_0 = \lambda u_0.$$

We deduce from (3.1.26) that

$$e^{tQ_0} u_0 \in H_{\Phi_t}(\mathbb{C}^n),$$

and since

$$e^{tQ_0} u_0 = e^{t\lambda} u_0,$$

it follows from Lemma 3.1.2 that

$$u_0 \in H_{\Phi_0 - \delta|x|^2}(\mathbb{C}^n),$$

for some $\delta > 0$. In particular, we obtain from (3.1.21) that $u_0 \in H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$ for $\varepsilon > 0$ small enough, and hence that $\lambda$ is in the spectrum of the operator $Q_0$ acting on $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$, which has been described in Lemma 3.1.1.

On the other hand, if $\lambda$ is in the spectrum of $Q_0$ acting on $H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$ and $u_0 \in H_{\tilde{\Phi}_\varepsilon}(\mathbb{C}^n)$ is a corresponding eigenvector, with $\varepsilon > 0$ sufficiently small, then we have

$$e^{tQ_0} u_0 = e^{t\lambda} u_0 \in H_{\widetilde{\Phi}_{\varepsilon,t}}(\mathbb{C}^n),$$

where $\widetilde{\Phi}_{\varepsilon,t}$ is a quadratic form on $\mathbb{C}^n$ depending smoothly on $t \geq 0$ and $\varepsilon \geq 0$, for $\varepsilon$ sufficiently small, which satisfies the eikonal equation (3.1.29) along with the initial condition

$$\widetilde{\Phi}_{\varepsilon,t}(x)|_{t=0} = \widetilde{\Phi}_\varepsilon(x).$$

It follows from (3.1.21) and (3.1.29) that

$$\widetilde{\Phi}_{\varepsilon,t}(x) = \Phi_t(x) + \mathcal{O}(\varepsilon\, |x|^2),$$

where the implicit constant is uniform in $0 \leq t \leq t_0$, for $t_0 > 0$ small enough. By taking $T_0 > 0$ small enough but fixed such that $0 < T_0 \leq t_0$ and (3.1.30) holds, we get

$$u_0 = e^{-T_0\lambda} e^{T_0 Q} u_0 \in H_{\tilde{\Phi}_{\varepsilon,T_0}}(\mathbb{C}^n) = H_{\Phi_{T_0}+\mathcal{O}(\varepsilon|x|^2)}(\mathbb{C}^n).$$

In view of (3.1.30), we can choose $\varepsilon_0 > 0$ small enough such that for all $0 < \varepsilon \leq \varepsilon_0$,

$$u_0 \in H_{\Phi_0 - \tilde{\delta}|x|^2}(\mathbb{C}^n) \subset H_{\Phi_0}(\mathbb{C}^n),$$

where $\tilde{\delta}$ is a positive constant. It follows that $\lambda$ is also in the spectrum of the operator $Q_0$ acting on $H_{\Phi_0}(\mathbb{C}^n)$. Altogether, this shows that the spectrum of $Q_0$ acting on $H_{\Phi_0}(\mathbb{C}^n)$ is equal to the spectrum of $Q_0$ acting on $H_{\widetilde{\Phi}_\varepsilon}(\mathbb{C}^n)$, for $\varepsilon > 0$ sufficiently small, and furthermore, that the algebraic multiplicities agree. We have therefore identified the spectrum of $Q_0$ on $H_{\Phi_0}(\mathbb{C}^n)$ and also the spectrum of the operator $Q$ on $L^2(\mathbb{R}^n)$ by coming back to the real side.

*Remark* In the argument above we have worked with the eigenfunctions of the operator $Q_0$ acting on $H_{\widetilde{\Phi}_\varepsilon}(\mathbb{C}^n)$ for some sufficiently small but fixed $\varepsilon > 0$. It may be interesting to notice that the (generalized) eigenfunctions of the operator $Q_0$ do not depend on $\varepsilon > 0$, for $0 < \varepsilon \leq \varepsilon_0$, with $\varepsilon_0 > 0$ small enough. See also Remark 11.7 in [5]. While refraining from providing a detailed proof of this statement, let us mention that its validity relies crucially upon the fact that the quadratic symbol $q_0$ is elliptic on $\Lambda_{\widetilde{\Phi}_\varepsilon}$, $0 < \varepsilon \leq \varepsilon_0$. In the terminology of [13], this is an instance of the principle of non-characteristic deformations.

We shall now prove Lemma 3.1.2. Integrating (3.1.29) for $t = 0$ to $t = T$, we obtain

$$\Phi_T(x) - \Phi_0(x) = \int_0^T \mathrm{Re}\left[q_0\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t,x)\right)\right] dt,\ T > 0.$$

Here the integral in the right hand side is a real-valued quadratic form on $\mathbb{C}^n$, and Lemma 3.1.2 is implied by the claim that it is negative definite. When proving the claim, we shall use the following general relation, explained for instance in [10],

$$\widehat{H_{if}} = H^{-\mathrm{Im}\sigma}_{-\mathrm{Re} f}, \tag{3.1.32}$$

valid for a holomorphic function $f$ on $\mathbb{C}^{2n}$.

It follows from (3.1.32) that $\mathrm{Re}\, q_0$ is constant along the flow of the Hamilton vector field

$$\widehat{H_{-\frac{1}{i}q_0}} = H^{-\mathrm{Im}\sigma}_{-\mathrm{Re}\, q_0}.$$

We therefore deduce from (3.1.5), (3.1.12), (3.1.15) and (3.1.28) that

$$\mathrm{Re}\left[q_0|_{\Lambda_{\Phi_t}}\right] \leq 0. \tag{3.1.33}$$

According to (3.1.27), (3.1.29) implies therefore that

$$\frac{\partial \Phi}{\partial t}(t,x) \leq 0,$$

so that the function

$$t \mapsto \Phi_t(x),$$

is decreasing. Let us assume that there exists $x_0 \in \mathbb{C}^n$, $x_0 \neq 0$, such that

$$\int_0^T \mathrm{Re}\left[q_0\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t,x_0)\right)\right] dt = 0.$$

Since from (3.1.27) and (3.1.33), the integrand is non-positive, it follows that for all $0 \leq t \leq T$,

$$\mathrm{Re}\left[q_0\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t,x_0)\right)\right] = 0. \tag{3.1.34}$$

Therefore, in view of (3.1.29), we get

$$\frac{\partial \Phi}{\partial t}(t,x_0) = 0,$$

for all $0 \leq t \leq T$. Here, the quadratic form

$$f_t(x) = \frac{\partial \Phi}{\partial t}(t,x) = \mathrm{Re}\left[q_0\left(x, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t,x)\right)\right] \leq 0,$$

is non-positive and such that $f_t(x_0) = 0$ for all $0 \leq t \leq T$. It follows that

$$\nabla_{\mathrm{Re}x,\mathrm{Im}x} f_t(x_0) = 0,$$

for all $0 \leq t \leq T$, and therefore

$$\frac{\partial f_t}{\partial x}(x_0) = \frac{\partial^2 \Phi}{\partial x \partial t}(t,x_0) = 0,$$

for all $0 \leq t \leq T$. Hence the function

$$t \mapsto \frac{\partial \Phi}{\partial x}(t, x_0),$$

does not depend on $t$ for $0 \leq t \leq T$, so that

$$\frac{\partial \Phi}{\partial x}(t, x_0) = \frac{\partial \Phi}{\partial x}(0, x_0) = \frac{\partial \Phi_0}{\partial x}(x_0), \tag{3.1.35}$$

for all $0 \leq t \leq T$. Since from (3.1.27), the point

$$\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t, x_0)\right) = \left(x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0)\right), \tag{3.1.36}$$

belongs to $\Lambda_{\Phi_t}$ for all $0 \leq t \leq T$, we obtain from (3.1.14) and (3.1.28) that there exists $y_0(t) \in \mathbb{C}^n$ such that

$$\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t, x_0)\right) = \exp\left(t\widehat{H_{-\frac{1}{i}q_0}}\right)\left(y_0(t), \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\,(y_0(t))\right). \tag{3.1.37}$$

It follows from (3.1.34) that for all $0 \leq t \leq T$,

$$\mathrm{Re}\left[q_0\left(x_0, \frac{2}{i}\frac{\partial \Phi}{\partial x}(t, x_0)\right)\right] = \mathrm{Re}\left[q_0\left(y_0(t), \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\,(y_0(t))\right)\right] = 0, \tag{3.1.38}$$

because $\mathrm{Re}\, q_0$ is constant along the flow of the Hamilton vector field

$$\widehat{H_{-\frac{1}{i}q_0}} = H^{-\mathrm{Im}\sigma}_{-\mathrm{Re}q_0}.$$

Let us now set

$$L_0 = \left\{\tilde{X} \in \Lambda_{\Phi_0} : \mathrm{Re}[q_0(\tilde{X})] = 0\right\}. \tag{3.1.39}$$

We can notice by using similar arguments as in (2.0.14) and (2.0.15) that

$$\{X \in \mathbb{R}^{2n}, \mathrm{Re}\, \tilde{q}(X) = 0\} = \mathrm{Ker}(\mathrm{Re}\, \tilde{F}) \cap \mathbb{R}^{2n},$$

for any complex-valued quadratic form $\tilde{q}$ with a non-positive real part if $\mathrm{Re}\, \tilde{F}$ is the Hamilton map of the quadratic form $\mathrm{Re}\, \tilde{q}$. We therefore deduce from (3.1.12) and (3.1.15) that

$$L_0 = \kappa_T\left(\mathrm{Ker}(\mathrm{Re}\, F) \cap \mathbb{R}^{2n}\right). \tag{3.1.40}$$

We get from (3.1.14), (3.1.36), (3.1.37), (3.1.38), (3.1.39) and (3.1.40) that

$$\left(y_0(t), \frac{2}{i}\frac{\partial \Phi_0}{\partial x}\,(y_0(t))\right) = \exp\left(t\widehat{H_{\frac{1}{i}q_0}}\right)\left(x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0)\right) \in L_0, \tag{3.1.41}$$

for all $0 \leq t \leq T$, and therefore

$$\operatorname{Re} F\left(\kappa_T^{-1}\left(\exp(t\widehat{H_{\frac{1}{i}q_0}})\left(x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0)\right)\right)\right) = 0, \tag{3.1.42}$$

for all $0 \leq t \leq T$. In view of (3.1.14) and (3.1.15), we may write

$$\left(x_0, \frac{2}{i}\frac{\partial \Phi_0}{\partial x}(x_0)\right) = \kappa_T(X_0), \tag{3.1.43}$$

for some $X_0 \in \mathbb{R}^{2n}$, $X_0 \neq 0$ because $x_0 \neq 0$. We can now deduce from (3.1.7) that there exists an integer $m \in \mathbb{N}$ such that

$$\operatorname{Re} F\,(\operatorname{Im} F)^j\, X_0 = 0, \quad 0 \leq j < m, \tag{3.1.44}$$

while

$$\operatorname{Re} F\,(\operatorname{Im} F)^m\, X_0 \neq 0. \tag{3.1.45}$$

On the other hand, we get from (3.1.42) and (3.1.43) that

$$\operatorname{Re} F\left(\kappa_T^{-1}\left(\widehat{H_{\frac{1}{i}q_0}}\right)^j \kappa_T(X_0)\right) = 0, \tag{3.1.46}$$

for all $j \in \mathbb{N}$ because $q_0$ is a quadratic form. We shall establish the following result.

**Lemma 3.1.3** *For all* $0 \leq j \leq m$, *we have*

$$\left(\widehat{H_{\frac{1}{i}q_0}}\right)^j \kappa_T(X_0) = \left(H_{\operatorname{Im} q_0}^{\sigma_{\Lambda_{\Phi_0}}}\right)^j \kappa_T(X_0).$$

*Proof* When proving this lemma, we shall argue by induction with respect to $j$, and start with the case $j = 0$, which is of course fulfilled. Let us recall from (3.1.41) and (3.1.43) that $\kappa_T(X_0) \in L_0$, and notice that, as recalled for example in section 11 (Remark 11.7) in [5] (see also [10]), we have at the points of $L_0$,

$$\widehat{H_{\frac{1}{i}q_0}} = H_{\operatorname{Im} q_0}^{\sigma_{\Lambda_{\Phi_0}}}. \tag{3.1.47}$$

Let us now check that for all $0 \leq j \leq m-1$,

$$\left(H_{\operatorname{Im} q_0}^{\sigma_{\Lambda_{\Phi_0}}}\right)^j \kappa_T(X_0) \in L_0. \tag{3.1.48}$$

Let us consider $0 \leq j \leq m-1$ and notice from (3.1.12) that

$$\kappa_T(H_{\operatorname{Im} q})\kappa_T^{-1} = H_{\operatorname{Im} q_0}^{\sigma_{\Lambda_{\Phi_0}}}. \tag{3.1.49}$$

Since a direct computation using (1.1.8) shows that

$$H_{\mathrm{Im}\, q} = 2\mathrm{Im}\, F, \tag{3.1.50}$$

we obtain by using (1.1.8), (3.1.12), (3.1.44) and (3.1.49) that

$$\begin{aligned}\mathrm{Re}\Big[q_0\Big((H^{\sigma_{\Lambda_{\Phi_0}}}_{\mathrm{Im}\, q_0})^j\kappa_T(X_0)\Big)\Big] &= \mathrm{Re}\Big[q_0\Big(\kappa_T H^j_{\mathrm{Im}\, q}(X_0)\Big)\Big]\\ = \mathrm{Re}\, q\Big(2^j(\mathrm{Im}\, F)^j X_0\Big) &= 2^{2j}\sigma\Big((\mathrm{Im}\, F)^j X_0, \mathrm{Re}\, F(\mathrm{Im}\, F)^j X_0\Big) = 0,\end{aligned}$$

for any $0 \leq j \leq m-1$. Thus we have verified (3.1.48), and by an application of (3.1.47), we get that for all $0 \leq j \leq m-1$,

$$\widehat{H_{\frac{1}{i}q_0}}\Big((\widehat{H_{\frac{1}{i}q_0}})^j\kappa_T(X_0)\Big) = H^{\sigma_{\Lambda_{\Phi_0}}}_{\mathrm{Im}\, q_0}\Big((\widehat{H_{\frac{1}{i}q_0}})^j\kappa_T(X_0)\Big) = (H^{\sigma_{\Lambda_{\Phi_0}}}_{\mathrm{Im}\, q_0})^{j+1}\kappa_T(X_0),$$

if

$$(\widehat{H_{\frac{1}{i}q_0}})^j\kappa_T(X_0) = (H^{\sigma_{\Lambda_{\Phi_0}}}_{\mathrm{Im}\, q_0})^j\kappa_T(X_0).$$

This proves by induction Lemma 3.1.3. □

It is now easy to finish the proof of Lemma 3.1.2. By using (3.1.46) when $j=m$, (3.1.49), (3.1.50) and applying Lemma 3.1.3, we get

$$0 = \mathrm{Re}\, F\Big(\kappa_T^{-1}(H^{\sigma_{\Lambda_{\Phi_0}}}_{\mathrm{Im}\, q_0})^m\kappa_T(X_0)\Big) = \mathrm{Re}\, F(H^m_{\mathrm{Im}\, q}X_0) = 2^m\mathrm{Re}\, F(\mathrm{Im}\, F)^m X_0,$$

which contradicts (3.1.45) and completes the proof of Lemma 3.1.2.

Let us finally notice that the semigroup $e^{tQ}$, $t>0$, is strongly regularizing on $L^2(\mathbb{R}^n)$. We actually deduce from (3.1.26), (3.1.30) and the fundamental property of semigroups that for all $t>0$, there exists $\delta>0$ such that

$$\forall u \in H_{\Phi_0}(\mathbb{C}^n),\ e^{tQ_0}u \in H_{\Phi_0-\delta|x|^2}(\mathbb{C}^n),$$

on the FBI transform side. By using the fact that a holomorphic function $U$ on $\mathbb{C}^n$ is of the form $Tu$ for some $u \in \mathcal{S}(\mathbb{R}^n)$, if and only if

$$\forall N \in \mathbb{N},\ \int_{\mathbb{C}^n}|U(x)|^2 e^{-2\Phi_0(x)}\langle x\rangle^N L(dx) < +\infty,$$

(see for instance [15]) we finally obtain that

$$\forall t>0, \forall u \in L^2(\mathbb{R}^n),\ e^{tQ}u \in \mathcal{S}(\mathbb{R}^n),$$

which ends the proof of Proposition 3.1.1.

### 3.2 Large time behavior of contraction semigroups

In this section, we prove Theorem 1.2.3. Let us consider a complex-valued quadratic form

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \to \mathbb{C},\ n \in \mathbb{N}^*,$$

with a non-positive real part

$$\text{Re}\, q \leq 0,$$

such that its singular space $S$ has a symplectic structure.

Let us assume that the real part of the symbol $q$ is a non-zero quadratic form

$$\exists X_0 \in \mathbb{R}^{2n},\ \text{Re}\, q(X_0) \neq 0.$$

This implies that the singular space is distinct from the whole phase space $S \neq \mathbb{R}^{2n}$ because from (1.1.8) and (1.1.14),

$$\forall X \in S,\ \text{Re}\, q(X) = \sigma(X, \text{Re}\, FX) = 0.$$

It proves that (ii) implies (iii) in Theorem 1.2.3.

Let us now assume that $S \neq \mathbb{R}^{2n}$ and prove (i). We deduce from (2.0.1) and (2.0.7) that there exists $\chi$, a real linear symplectic transformation of $\mathbb{R}^{2n}$, such that

$$(q \circ \chi)(x,\xi) = q_1(x',\xi') + iq_2(x'',\xi''),\ (x,\xi) = (x',x'';\xi',\xi'') \in \mathbb{R}^{2n}, \quad (3.2.1)$$

where $q_1$ is a complex-valued quadratic form on $\mathbb{R}^{2n'}$, $n' \geq 1$, with a non-positive real part and $q_2$ is a real-valued quadratic form verifying the properties stated in Proposition 2.0.1. By using the symplectic invariance of the Weyl quantization given by the theorem 18.5.9 in [8], we can find a metaplectic operator $U$, which is a unitary transformation on $L^2(\mathbb{R}^n)$ and an automorphism of $\mathcal{S}(\mathbb{R}^n)$ such that

$$(q \circ \chi)(x,\xi)^w = U^{-1} q(x,\xi)^w U.$$

This implies at the level of the generated semigroups that

$$e^{t(q\circ\chi)(x,\xi)^w} = U^{-1} e^{tq(x,\xi)^w} U,\ t \geq 0$$

and

$$\|e^{t(q\circ\chi)(x,\xi)^w}\|_{\mathcal{L}(L^2)} = \|e^{tq(x,\xi)^w}\|_{\mathcal{L}(L^2)},\ t \geq 0, \quad (3.2.2)$$

because $U$ is a unitary operator on $L^2(\mathbb{R}^n)$. Since both operators $iq_2(x'',\xi'')^w$ and $-iq_2(x'',\xi'')^w$ generate contraction semigroups verifying

$$\left(e^{tiq_2(x'',\xi'')^w}\right)^{-1} = e^{t(-iq_2(x'',\xi'')^w)},\ t \geq 0,$$

the semigroup $e^{itq_2(x'',\xi'')^w}$ is unitary for all $t \geq 0$. It follows from the tensorization of the variables (3.2.1),

$$e^{t(q\circ\chi)(x,\xi)^w} = e^{tq_1(x',\xi')^w} e^{itq_2(x'',\xi'')^w},$$

and (3.2.2) that

$$\|e^{tq(x,\xi)^w}\|_{\mathcal{L}(L^2)} = \|e^{tq_1(x',\xi')^w}\|_{\mathcal{L}(L^2)},\ t \geq 0.$$

For proving (i), it is therefore sufficient to prove the exponential decay in time for the norm of the contraction semigroup generated by the operator $q_1(x',\xi')^w$. We have proved in Proposition 3.1.1 [see also (2.0.7)] that the spectrum of the operator $q_1(x',\xi')^w$ is only composed of the following eigenvalues

$$\sigma\left(q_1(x',\xi')^w\right) = \left\{ \sum_{\substack{\lambda\in\sigma(F_1),\\ \mathrm{Re}(-i\lambda)<0}} (r_\lambda + 2k_\lambda)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{3.2.3}$$

where $F_1 = F|_{S^{\sigma\perp}}$ is the Hamilton map associated to the quadratic form $q_1$ and $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F_1$ in $\mathbb{C}^{2n'}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$. We have also seen in the proof of Proposition 3.1.1 that the contraction semigroup generated by the operator $q_1(x',\xi')^w$ is compact for all $t > 0$. We proved this fact in (3.1.31) on the FBI transform side. This allows us to apply Theorem 2.20 in [2] to obtain the following description of the spectrum,

$$\sigma(e^{tq_1(x',\xi')^w}) = \{0\} \cup \left\{e^{t\mu} : \mu \in \sigma\left(q_1(x',\xi')^w\right)\right\}.$$

Its spectral radius is therefore given by

$$\mathrm{rad}\left(e^{tq_1(x',\xi')^w}\right) = e^{-at}, \tag{3.2.4}$$

where

$$a = \inf\left\{ \sum_{\substack{\lambda\in\sigma(F_1),\\ \mathrm{Re}(-i\lambda)<0}} (r_\lambda + 2k_\lambda)(-\mathrm{Re}(-i\lambda)) : k_\lambda \in \mathbb{N} \right\}. \tag{3.2.5}$$

It follows that the constant $a$ is positive. Since from Theorem 1.22 in [2], we have

$$-a = \lim_{t\to+\infty} \frac{1}{t} \log \|e^{tq_1(x',\xi')^w}\|_{\mathcal{L}(L^2)},$$

we obtain that there exists $M > 0$ such that

$$\|e^{tq_1(x',\xi')^w}\|_{\mathcal{L}(L^2)} \leq M e^{-\frac{a}{2}t}, \tag{3.2.6}$$

for all $t \geq 0$. This proves that (iii) implies (i). Finally, the fact that (i) implies (ii) is a consequence of a property that we have already mentioned, namely, when the real part of the symbol $q$ is identically equal to zero then the contraction semigroup

$$e^{tq(x,\xi)^w},$$

is unitary for all $t \geq 0$. This ends the proof of Theorem 1.2.3.

### 3.3 Spectra of non-elliptic quadratic operators

In this section, we prove Theorem 1.2.2. Let us consider a complex-valued quadratic form

$$q : \mathbb{R}^n_x \times \mathbb{R}^n_\xi \rightarrow \mathbb{C},\ n \in \mathbb{N}^*,$$

with a non-positive real part

$$\operatorname{Re} q \leq 0,$$

which is elliptic on its singular space $S$,

$$(x,\xi) \in S,\ q(x,\xi) = 0 \Rightarrow (x,\xi) = 0. \tag{3.3.1}$$

Let us recall that this assumption of partial ellipticity on the singular space ensures that the singular space has a symplectic structure. We can therefore resume the beginning of our reasoning explained in Sect. 3.1:
By using the symplectic decomposition of the symbol obtained in Sect. 2, we deduce from (2.0.1) and (2.0.2) that there exists $\chi$, a real linear symplectic transformation of $\mathbb{R}^{2n}$, such that

$$(q \circ \chi)(x,\xi) = q_1(x',\xi') + iq_2(x'',\xi''),\ (x,\xi) = (x',x'';\xi',\xi'') \in \mathbb{R}^{2n}, \tag{3.3.2}$$

where $q_1$ is a complex-valued quadratic form on $\mathbb{R}^{2n'}$ with a non-positive real part

$$\operatorname{Re} q_1 \leq 0, \tag{3.3.3}$$

and $q_2$ is a real-valued quadratic form verifying the properties stated in Propositions 2.0.1 and 3.1.1.

To obtain the result of Theorem 1.2.2, let us notice from Proposition 2.0.1 that when the symbol $q$ is elliptic on $S$, we can assume that

$$q_2(x'',\xi'') = \varepsilon \sum_{j=1}^{n''} \lambda_j (\xi_j''^2 + x_j''^2), \tag{3.3.4}$$

where $\varepsilon \in \{\pm 1\}$ and $\lambda_j > 0$ for all $j = 1,\dots,n''$.

By using again the symplectic invariance of the Weyl quantization given by the theorem 18.5.9 in [8], we can find a metaplectic operator $U$, which is a unitary transformation on $L^2(\mathbb{R}^n)$ and an automorphism of $\mathcal{S}(\mathbb{R}^n)$ such that

$$(q \circ \chi)(x,\xi)^w = U^{-1} q(x,\xi)^w U. \tag{3.3.5}$$

Since the quadratic form $q_2$ is elliptic on $\mathbb{R}^{2n''}$, we deduce from the theorem 3.5 in [12] that the spectrum of the operator $iq_2(x'',\xi'')^w$ is only composed of eigenvalues with finite multiplicity

$$\sigma\left(iq_2(x'',\xi'')^w\right) = \left\{ \sum_{\substack{\lambda\in\sigma(iF_2),\\ -i\lambda\in\Sigma(iq_2)\setminus\{0\}}} \left(r''_\lambda + 2k_\lambda\right)(-i\lambda) : k_\lambda \in \mathbb{N} \right\}, \tag{3.3.6}$$

where $F_2$ is the Hamilton map associated to the quadratic form $q_2$ and $r''_\lambda$ is the dimension of the space of generalized eigenvectors of $iF_2$ in $\mathbb{C}^{2n''}$ belonging to the eigenvalue $\lambda \in \mathbb{C}$. We notice from (2.0.6), (2.0.7), (2.0.8) and (2.0.9) that

$$F_1 = F|_{S^{\sigma\perp}} \quad \text{and} \quad F_2 = \frac{1}{i} F|_S. \tag{3.3.7}$$

Let us notice that if $\lambda$ is an eigenvalue of $F_1$, such that $\mathrm{Re}(-i\lambda) \leq 0$, then we necessarily have

$$\mathrm{Re}(-i\lambda) < 0,$$

because if we had $\mathrm{Re}(-i\lambda) = 0$, it would imply that the Hamilton map $F_1$ has a real eigenvalue and induce, as we saw in (1.4.8), that the singular space of the symbol $q_1$ is not reduced to $\{0\}$. However, this singular space is trivial by construction (see (ii) in Proposition 2.0.1). This proves that

$$\mathrm{Re}(-i\lambda) < 0.$$

By using now that when the numerical range of a quadratic form $\tilde{q}$ is contained in a closed angular sector $\Gamma$ with a vertex in 0 and an aperture strictly less than $\pi$ then $\lambda$ is an eigenvalue of its Hamilton map $\tilde{F}$ if and only if $-\lambda$ is an eigenvalue of $\tilde{F}$, and

$$-i\lambda \in \Gamma \text{ or } i\lambda \in \Gamma,$$

(see section 3 in [7]), we obtain from (2.0.6), (2.0.7), (2.0.8) and (2.0.9) that

$$\begin{aligned}\{\lambda \in \mathbb{C} : \lambda \in \sigma(F), -i\lambda \in \mathbb{C}_- \cup (\Sigma(q|_S)\setminus\{0\})\} = \{\lambda \in \mathbb{C} : \lambda \in \sigma(F_1),\\ \mathrm{Re}(-i\lambda) < 0\} \sqcup \{\lambda \in \mathbb{C} : \lambda \in \sigma(iF_2), -i\lambda \in \Sigma(iq_2)\setminus\{0\}\},\end{aligned} \tag{3.3.8}$$

where $F$ is the Hamilton map associated to the quadratic form $q$,

$$\Sigma(q|_S)=\overline{q(S)} \quad \text{and} \quad \mathbb{C}_-=\{z\in\mathbb{C} : \operatorname{Re} z<0\}.$$

We shall now deduce from the tensorization of the variables (3.3.2), (3.3.5) and (3.3.7) that the spectrum of the quadratic differential operator $q(x,\xi)^w$ is only composed of eigenvalues with finite multiplicity

$$\sigma\big(q(x,\xi)^w\big)=\left\{\sum_{\substack{\lambda\in\sigma(F),\\ -i\lambda\in\mathbb{C}_-\cup(\Sigma(q|_S)\setminus\{0\})}}(r_\lambda+2k_\lambda)(-i\lambda) : k_\lambda\in\mathbb{N}\right\}, \tag{3.3.9}$$

where $r_\lambda$ is the dimension of the space of generalized eigenvectors of $F$ in $\mathbb{C}^{2n}$ belonging to the eigenvalue $\lambda\in\mathbb{C}$. This result is trivial when the singular space is equal to the whole phase space because in that case the quadratic form $q_1$ is identically equal to 0 and $n''=n$. We therefore assume in the following that

$$S\neq\mathbb{R}^{2n}. \tag{3.3.10}$$

Let us begin by recalling that we know from (3.2.3) that the spectrum of the operator $q_1(x',\xi')^w$ is only composed of eigenvalues with finite multiplicity

$$\sigma\big(q_1(x',\xi')^w\big)=\left\{\sum_{\substack{\lambda\in\sigma(F_1),\\ \operatorname{Re}(-i\lambda)<0}}\big(r'_\lambda+2k_\lambda\big)(-i\lambda) : k_\lambda\in\mathbb{N}\right\}, \tag{3.3.11}$$

where $r'_\lambda$ is the dimension of the space of generalized eigenvectors of $F_1$ in $\mathbb{C}^{2n'}$ belonging to the eigenvalue $\lambda\in\mathbb{C}$. It follows from (3.3.11) that when a positive constant $a$ verifies

$$\sigma\big(q_1(x',\xi')^w\big)\cap\{z\in\mathbb{C} : \operatorname{Re} z=-a\}=\emptyset, \tag{3.3.12}$$

the operator $q_1(x',\xi')^w$ only has a finite number of eigenvalues in the half-plane

$$\{z\in\mathbb{C} : -a\leq \operatorname{Re} z\}. \tag{3.3.13}$$

In the following, we besides assume that the singular space is not reduced to zero

$$S\neq\{0\}, \tag{3.3.14}$$

because the description (3.3.9) is then a direct consequence of (3.3.11). We now need some estimates for the resolvent of the operator $q_1(x',\xi')^w$ to obtain the description (3.3.9) for the spectrum of the operator $q(x,\xi)^w$.

**Proposition 3.3.1** *For all $a > 0$ such that*

$$\sigma\left(q_1(x',\xi')^w\right) \cap \{z \in \mathbb{C} : \text{Re}\, z = -a\} = \emptyset,$$

*there exists $C_a > 0$ such that*

$$\left\| \left(z - q_1(x',\xi')^w\right)^{-1} \right\| \leq C_a, \tag{3.3.15}$$

*for all $z \in \mathbb{C}$ with $-a < \text{Re}\, z$ and $|\text{Im}\, z| \geq C_a$. Here the norm is the operator norm on $L^2$.*

*Proof* When proving Proposition 3.3.1, we first recall from (3.2.6) and (3.3.10) that there exist $\tilde{a} > 0$ and $M > 0$ such that

$$||\, e^{tq_1(x',\xi')^w}\, || \leq M e^{-\tilde{a}t}, \quad t \geq 0. \tag{3.3.16}$$

By using Theorem 2.8 in [2], we can write that for all $z \in \mathbb{C}$ such that Re $z > -\tilde{a}$,

$$\left(z - q_1(x',\xi')^w\right)^{-1} = \int_0^{+\infty} e^{-zt}\, e^{tq_1(x',\xi')^w} dt,$$

and we deduce from (3.3.16) that

$$\left\| \left(z - q_1(x',\xi')^w\right)^{-1} \right\| \leq \frac{M}{\tilde{a} + \text{Re}\, z}, \tag{3.3.17}$$

for all $z \in \mathbb{C}$, $-\tilde{a} < \text{Re}\, z$. This proves the estimate (3.3.15) when the positive constant $a$ is small enough. To prove the result in the general case, we shall follow an argument used by L.S. Boulton in [1]. Let us consider a positive constant $a$ verifying (3.3.12). We have already seen that the operator $q_1(x',\xi')^w$ has only a finite number of eigenvalues with finite multiplicity in the half-plane

$$\{z \in \mathbb{C} : -a \leq \text{Re}\, z\}.$$

We can therefore consider $\Pi_a$, the finite-rank spectral projection associated to the eigenvalues

$$\sigma\left(q_1(x',\xi')^w\right) \cap \{z \in \mathbb{C} : -a \leq \text{Re}\, z\},$$

and write for all $z \in \mathbb{C}$ with $z \notin \sigma\left(q_1(x',\xi')^w\right)$ that

$$\left(z - q_1(x',\xi')^w\right)^{-1} = \left(z - q_1(x',\xi')^w\right)^{-1} \Pi_a + \left(z - q_1(x',\xi')^w\right)^{-1} (1 - \Pi_a). \tag{3.3.18}$$

Here

$$\big(z - q_1(x',\xi')^w\big)^{-1}(1-\Pi_a) = \big(z - q_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}\big)^{-1}(1-\Pi_a), \tag{3.3.19}$$

and we can write by using Theorems 1.22 and 2.8 in [2] that

$$\big(z - q_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}\big)^{-1} = \int_0^{+\infty} e^{-zt} e^{tq_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}} dt, \tag{3.3.20}$$

for all $z \in \mathbb{C}$ with $-b < \mathrm{Re}\, z$ if we set

$$-b = \lim_{t\to+\infty} \frac{1}{t} \log \|e^{tq_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}}\|. \tag{3.3.21}$$

Let us now notice that the contraction semigroup

$$e^{tq_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}} = e^{tq_1(x',\xi')^w}(1-\Pi_a),$$

is compact for any $t > 0$ since we have seen in Sect. 3.2 that the contraction semigroup generated by the operator $q_1(x',\xi')^w$ is compact for any $t > 0$. Since on the other hand the spectrum of the operator $q_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}$ is equal to

$$\sigma\left(q_1(x',\xi')^w\right) \cap \{z \in \mathbb{C} : \mathrm{Re}\, z \leq -a\},$$

we deduce from Theorems 1.22 and 2.20 in [2] that $-b < -a$, which implies in view of (3.3.21) that there exists $\tilde{M} > 0$ such that

$$\|e^{tq_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}}\| \leq \tilde{M} e^{-at}, \quad t \geq 0. \tag{3.3.22}$$

We then deduce from (3.3.20) and (3.3.22) that

$$\Big\|\big(z - q_1(x',\xi')^w|_{\mathrm{Ran}(1-\Pi_a)}\big)^{-1}(1-\Pi_a)\Big\| \leq \frac{\tilde{M}}{a + \mathrm{Re}\, z}\|1-\Pi_a\|, \tag{3.3.23}$$

for all $z \in \mathbb{C}$ with $-a < \mathrm{Re}\, z$. Since on the other hand

$$\big(z - q_1(x',\xi')^w\big)^{-1}\Pi_a = \big(z - q_1(x',\xi')^w|_{\mathrm{Ran}\,\Pi_a}\big)^{-1}\Pi_a,$$

and the vector space $\mathrm{Ran}\,\Pi_a$ is finite-dimensional, we therefore have

$$\Big\|\big(z - q_1(x',\xi')^w\big)^{-1}\Pi_a\Big\| = \mathcal{O}_a(1), \tag{3.3.24}$$

for any $z \in \mathbb{C}$ when $|\mathrm{Im}\, z|$ is large enough depending on $a$. We finally deduce the result of Proposition 3.3.1 from (3.3.18), (3.3.19), (3.3.23) and (3.3.24). □

*Remark* Let us notice that the previous proof actually shows that when $q$ is a complex-valued quadratic form on $\mathbb{R}^{2n}$, $n \geq 1$, with non-positive real part and a zero singular space

$$S = \{0\},$$

then

$$e^{tq(x,\xi)^w} = e^{tq(x,\xi)^w}\Pi_a + \mathcal{O}_a(e^{-at}), \quad t \geq 0,$$

in $\mathcal{L}(L^2)$ for any $a > 0$ such that

$$\sigma\left(q(x,\xi)^w\right) \cap \{z \in \mathbb{C} : \mathrm{Re}\, z = -a\} = \emptyset.$$

We can now resume our proof of Theorem 1.2.2. In doing so, we recall that any quadratic differential operator $\tilde{q}(x,\xi)^w$ whose symbol has a non-positive real part, is defined by the maximal closed realization on $L^2(\mathbb{R}^n)$ with the domain

$$\{u \in L^2(\mathbb{R}^n) : \tilde{q}(x,\xi)^w u \in L^2(\mathbb{R}^n)\},$$

which coincides with the graph closure of its restriction to $\mathcal{S}(\mathbb{R}^n)$,

$$\tilde{q}(x,\xi)^w : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

By noticing from (3.3.5) that

$$\sigma\left((q \circ \chi)(x,\xi)^w\right) = \sigma\left(q(x,\xi)^w\right), \tag{3.3.25}$$

we deduce from (3.3.2), (3.3.6), (3.3.8) and (3.3.11) that it is sufficient for obtaining (3.3.9) and ending the proof of Theorem 1.2.2 to establish that

$$\sigma\left(q(x,\xi)^w\right) = \sigma\left(q_1(x',\xi')^w\right) + \sigma\left(iq_2(x'',\xi'')^w\right). \tag{3.3.26}$$

We then notice that since the spectra of the operators $q_1(x',\xi')^w$ and $q_2(x'',\xi'')^w$ are only composed of eigenvalues, we directly get the first inclusion

$$\sigma\left(q_1(x',\xi')^w\right) + \sigma\left(iq_2(x'',\xi'')^w\right) \subset \sigma\left(q(x,\xi)^w\right),$$

by considering the functions

$$u(x',x'') = u_1(x')u_2(x'') \in L^2(\mathbb{R}^n),$$

where $u_1$ and $u_2$ are respectively some eigenvectors of the operators $q_1(x',\xi')^w$ and $q_2(x'',\xi'')^w$, since these functions are eigenvectors for the operator $q(x,\xi)^w$. We shall now prove the opposite inclusion. Let us consider $z \in \mathbb{C}$ such that

$$z \notin \sigma\left(q_1(x',\xi')^w\right) + \sigma\left(iq_2(x'',\xi'')^w\right). \tag{3.3.27}$$

In view of (3.3.25), it is sufficient to prove that the map

$$(q \circ \chi)(x,\xi)^w - z : \{u \in L^2(\mathbb{R}^n) : (q \circ \chi)(x,\xi)^w u \in L^2(\mathbb{R}^n)\} \to L^2(\mathbb{R}^n),$$

is bijective to obtain the second inclusion. We denote by

$$\varphi_\alpha(x) = H_\alpha(x)e^{-x^2/2}, \ \alpha \in \mathbb{N}^n,$$

the orthonormal basis of $L^2(\mathbb{R}^n)$ composed by Hermite functions. Here the Hermite polynomials $H_\alpha(x)$ satisfy

$$H_\alpha(x) = \prod_{j=1}^{n} H_{\alpha_j}(x_j),$$

and therefore we write

$$\varphi_\alpha(x) = \varphi_{\alpha'}(x')\varphi_{\alpha''}(x''), \ \alpha' \in \mathbb{N}^{n'}, \ \alpha'' \in \mathbb{N}^{n''}. \tag{3.3.28}$$

Let us consider the following equation with $u$ and $v$ in $L^2(\mathbb{R}^n)$,

$$(q \circ \chi)(x,\xi)^w u - zu = v. \tag{3.3.29}$$

We can write

$$u(x) = \sum_{\alpha',\alpha''} a_{\alpha'\alpha''}\varphi_{\alpha'}(x')\varphi_{\alpha''}(x''), \ v(x) = \sum_{\alpha',\alpha''} b_{\alpha'\alpha''}\varphi_{\alpha'}(x')\varphi_{\alpha''}(x''), \tag{3.3.30}$$

where the two sums are taken for $(\alpha',\alpha'') \in \mathbb{N}^{n'} \times \mathbb{N}^{n''}$. By using from (3.3.2) that

$$(q \circ \chi)(x,\xi)^w = q_1(x',\xi')^w + iq_2(x'',\xi'')^w,$$

we obtain from (3.3.4) that

$$\begin{aligned}(q \circ \chi)(x,\xi)^w u - zu \\ = \sum_{\alpha',\alpha''} a_{\alpha'\alpha''}\left[q_1(x',\xi')^w\varphi_{\alpha'}(x') + i\mu_{\alpha''}\varphi_{\alpha'}(x') - z\varphi_{\alpha'}(x')\right]\varphi_{\alpha''}(x''),\end{aligned} \tag{3.3.31}$$

with

$$\mu_{\alpha''} = \varepsilon\sum_{j=1}^{n''}\lambda_j\left(2\alpha_j'' + 1\right),$$

since

$$q_2(x'', \xi'')^w \varphi_{\alpha''}(x'') = \mu_{\alpha''} \varphi_{\alpha''}(x'').$$

By setting for any $\alpha'' \in \mathbb{N}^{n''}$,

$$v_{\alpha''}(x') = \sum_{\alpha' \in \mathbb{N}^{n'}} b_{\alpha'\alpha''} \varphi_{\alpha'}(x') \in L^2(\mathbb{R}^{n'}),$$

so that according to (3.3.30),

$$v(x) = \sum_{\alpha'' \in \mathbb{N}^{n''}} v_{\alpha''}(x') \varphi_{\alpha''}(x''), \tag{3.3.32}$$

we deduce from (3.3.31) that for solving the equation (3.3.29), we have to solve all the equations

$$\big(q_1(x', \xi')^w + i\mu_{\alpha''} - z\big)\, u_{\alpha''}(x') = v_{\alpha''}(x'),\ \alpha'' \in \mathbb{N}^{n''}, \tag{3.3.33}$$

where

$$u_{\alpha''}(x') = \sum_{\alpha' \in \mathbb{N}^{n'}} a_{\alpha'\alpha''} \varphi_{\alpha'}(x') \in L^2(\mathbb{R}^{n'}).$$

We deduce from (3.3.27) that there is a unique solution $u_{\alpha''}(x')$ in $L^2(\mathbb{R}^{n'})$ for each of the equations (3.3.33). This proves that for every $v \in L^2(\mathbb{R}^n)$, there is at most one solution to the equation (3.3.29). Let us denote by $u_{\alpha''}$ the solutions to the equations (3.3.33) and

$$u = \sum_{\alpha'' \in \mathbb{N}^{n''}} u_{\alpha''}(x') \varphi_{\alpha''}(x''). \tag{3.3.34}$$

The equation (3.3.29) will have a unique solution in $L^2(\mathbb{R}^n)$ for every $v \in L^2(\mathbb{R}^n)$ if we prove that the function $u$ defined in (3.3.34) actually belongs to $L^2(\mathbb{R}^n)$. This is the case. Indeed, we obtain from (3.3.32) and (3.3.33) that

$$\begin{aligned}\|u\|^2_{L^2(\mathbb{R}^n)} &= \sum_{\alpha'' \in \mathbb{N}^{n''}} \|u_{\alpha''}\|^2_{L^2(\mathbb{R}^{n'})} = \sum_{\alpha'' \in \mathbb{N}^{n''}} \Big\| \big(q_1(x', \xi')^w + i\mu_{\alpha''} - z\big)^{-1} v_{\alpha''} \Big\|^2_{L^2(\mathbb{R}^{n'})} \\ &\leq C \sum_{\alpha'' \in \mathbb{N}^{n''}} \|v_{\alpha''}\|^2_{L^2(\mathbb{R}^{n'})} = C\|v\|^2_{L^2(\mathbb{R}^n)} < +\infty,\end{aligned}$$

because we deduce from Proposition 3.3.1 and (3.3.27) that the quantities

$$\Big\| \big(q_1(x', \xi')^w + i\mu_{\alpha''} - z\big)^{-1} \Big\|,$$

are bounded with respect to the parameter $\alpha''$ in $\mathbb{N}^{n''}$. This ends our proof of Theorem 1.2.2.

**Acknowledgments** The research of the first author is supported in part by the National Science Foundation under grant DMS–0653275 and the Alfred P. Sloan Research Fellowship. He would also like to thank Joe Viola for a stimulating discussion. The second author is very grateful to Bernard Helffer and Francis Nier for their interest in his thesis work and for their very stimulating questions during the defense that we have tried to answer in the present work. We would also like to thank the referee for helpful comments leading to the improved presentation.



## References

1. Boulton, L.S.: Non-self-adjoint harmonic oscillator semigroups and pseudospectra. J. Oper. Theory **47**, 413–429 (2002)
2. Davies, E.B.: One-Parameter Semigroups. Academic Press, London (1980)
3. Helffer, B., Nier, F.: Hypoelliptic Estimates and Spectral Theory for Fokker–Planck Operators and Witten Laplacians, SLN 1862. Springer, Heidelberg (2005)
4. Hérau, F., Nier, F.: Isotropic hypoellipticity and trend to equilibrium for the Fokker–Planck equation with a high degree potential. Arch. Ration. Mech. Anal. **171**, 151–218 (2004)
5. Hérau, F., Sjöstrand, J., Stolk, C.: Semiclassical analysis for the Kramers–Fokker–Planck equation. Comm. PDE **30**(4–6), 689–760 (2005)
6. Hérau, F., Hitrik, M., Sjöstrand, J.: Tunnel effect for Kramers–Fokker–Planck type operators. Ann. Henri Poincaré (2008, to appear)
7. Hörmander, L.: A class of hypoelliptic pseudodifferential operators with double characteristics. Math. Ann. **217**, 165–188 (1975)
8. Hörmander, L.: The Analysis of Linear Partial Differential Operators, vol. I–IV. Springer, Heidelberg (1985)
9. Hörmander, L.: Symplectic classification of quadratic forms, and general Mehler formulas. Math. Z. **219**, 413–449 (1995)
10. Melin, A., Sjöstrand, J.: Determinants of pseudodifferential operators and complex deformations of phase space. Methods Appl. Anal. **9**(2), 177–237 (2002)
11. Pravda-Starov, K.: Contraction semigroups of elliptic quadratic differential operators. Math. Z. (2007, to appear)
12. Sjöstrand, J.: Parametrices for pseudodifferential operators with multiple characteristics. Ark. för Mat. **12**, 85–130 (1974)
13. Sjöstrand, J.: Singularités analytiques microlocales, Astérisque. Soc. Math. France, Paris **95**, 1–166 (1982)
14. Sjöstrand, J.: Analytic wavefront sets and operators with multiple characteristics. Hokkaido Math. J. **12**(3), part 2, 392–433 (1983)
15. Sjöstrand, J.: Function Spaces Associated to Global I-Lagrangian Manifolds, Structure of Solutions of Differential Equations, Katata/Kyoto, 1995. World Sci. Publ., River Edge (1996)
16. Sjöstrand, J., Zworski, M.: The complex scaling method for scattering by strictly convex obstacles. Ark. för Mat. **33**, 135–172 (1995)